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Systems of Linear First Order Partial Differential Equations Admitting a Bilinear Multiplication of Solutions

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Abstract

The Cauchy–Riemann equations admit a bilinear multiplication of solutions, since the product of two holomorphic functions is again holomorphic. This multiplication plays the role of a nonlinear superposition principle for solutions, allowing for construction of new solutions from already known ones, and it leads to the exceptional property of the Cauchy–Riemann equations that all solutions can locally be built from power series of a single solution $z = x + \mathrm{i}y \in \mathbb{C}$.

In this thesis we have found a differential algebraic characterization of linear first order systems of partial differential equations admitting a bilinear $*$-multiplication of solutions, and we have determined large new classes of systems having this property. Among them are the already known quasi-Cauchy–Riemann equations, characterizing integrable Newton equations, and the gradient equations $\nabla f = M \nabla g$ with constant matrices M. A systematic description of linear systems of PDEs with variable coefficients have been given for systems with few independent and few dependent variables.

An important property of the $*$-multiplication is that infinite families of solutions can be constructed algebraically as power series of known solutions. For the equation $\nabla f = M \nabla g$ it has been proved that the general solution, found by Jodeit and Olver, can be locally represented as convergent power series of a single simple solution similarly as for solutions of the Cauchy–Riemann equations.

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Populärvetenskaplig sammanfattning

Multiplikation av lösningar till system av partiella differentialekvationer

En partiell differentialekvation är en ekvation som beskriver en relation mellan funktioner av flera variabler och deras derivator. Partiella differentialekvationer används för att konstruera modeller av verkliga fenomen såväl inom de naturvetenskapliga disciplinerna som inom ekonomi. Några välkända exempel på differentialekvationer och deras tillämpningar är Navier–Stokes ekvationer inom strömningsmekanik, Black–Scholes ekvation inom finansmatematik, värmeledningsekvationen, vågekvationen, och Maxwells elektromagnetiska ekvationer.

I allmänhet är det omöjligt att explicit beskriva alla lösningar till en differentialekvation, vilket medför att olika metoder för att konstruera speciella lösningar spelar en central roll när man vill beskriva lösningsrummet till en differentialekvation. I detta forskningsprojekt har vi studerat en speciell metod, kallad $*$-multiplikation, för att generera nya lösningar från redan kända lösningar till en stor klass av system av partiella differentialekvationer. Denna "multiplikation" av lösningar tillskriver på ett rent algebraiskt vis, till varje par V, W av lösningar, en ny lösning genom bildandet av $*$-produkten $V * W$. Speciellt kan denna metod användas för att konstruera oändliga följder av lösningar från en given enkel lösning V genom bildandet av $*$-potenser

$$V_*^n = \underbrace{V * V * \cdots * V}_{n \text{ faktorer}},$$

där n är ett godtyckligt positivt heltal.

De välkända Cauchy–Riemanns ekvationer utgör ett exempel på system med $*$-multiplikation. Dessa ekvationer används för att karakterisera analytiska (mycket reguljära) komplexa funktioner, vilka ingår i många beskrivningar av fysikaliska processer och tekniska tillämpningar. Ett annat exempel som vi studerat i denna avhandling är kvasi-Cauchy–Riemannska ekvationer. Dessa ekvationer beskriver viktiga klasser av mekaniska system på Newtonform vilka kan lösas genom separation av variabler.

Contents

Introduction

This dissertation consists of four research papers [9, 11, 12, 10], preceded by an introduction.

1 Background

When studying systems of partial differential equations (PDEs), the first question a mathematician is concerned about is the conditions guaranteeing existence of solutions. There are many fundamental results about existence of solutions for analytic PDEs [2].

When existence is established, the next goal is to describe all solutions of the system of PDEs and, possibly, a general solution containing all other solutions. Such general solutions usually depend on arbitrary functions. For the majority of PDEs and systems of PDEs, this is an impossible task since there is no general description of solutions in terms of quadratures.

For this reason, when studying properties of solutions, we usually have to be satisfied when it is possible to find particular solutions, possibly fulfilling some additional conditions (initial, boundary, etc.) relevant for the model where they are used. There are several techniques of searching for special solutions of systems of PDEs such as separation of variables [19], symmetry methods [20], and certain classes of equations admit superposition of solutions.

For linear systems of PDEs, any linear combination of solutions is again a solution, and this property (called the linear superposition principle) is the basis of the Fourier method of solving linear PDEs like the heat equation, the wave equation, and many other equations of mathematical physics.

For a nonlinear PDE, a linear combination of solutions is not a solution, but there are many known equations admitting a nonlinear superposition of solutions that allows for construction of new solutions from already known ones. The best known examples are soliton equations such as the sine–Gordon equation and the KdV equation [13].

In this dissertation, we study systems of linear PDEs which, in addition to the linear superposition principle, admits a special kind of bilinear superposition principle, here called $*$-multiplication. If such a superposition exists, then one can build complex solutions as polynomials and power series of certain simple solutions, and in this way describe a large subset of the whole solution space.

A prototype example are the Cauchy–Riemann equations for which all solutions are obtained from the single solution (x, y) through power series of the complex function $z = x + \mathrm{i}y$.

2 Nonlinear superposition principles for differential equations

We shall start with presenting a few examples of known differential equations admitting nonlinear superposition of solutions.

2.1 The Riccati equation

Consider the nonlinear first order ordinary differential equation

$$V' + a(x)V^2 + b(x)V + c(x) = 0, \tag{1}$$

known as the Riccati equation. If V_1 is a particular solution, a change of dependent variable, $W = (V - V_1)^{-1}$, transforms the equation (1) into the linear equation

$$W' + \tilde{a}(x)W + \tilde{b}(x) = 0, \tag{2}$$

where $\tilde{a} = -2V_1 - b$ and $\tilde{b} = -a$. If V_2 is another particular solution of the Riccati equation, the function $W_1 = (V_2 - V_1)^{-1}$ is a particular solution of the linearized equation (2). Thus, with two particular solutions available, the problem of describing the general solution of (1) is reduced to the problem of solving the linear homogeneous equation

$$W' + \tilde{a}(x)W = 0. \tag{3}$$

The general solution of (2) can be written as $W = cf_1(x) + f_2(x)$, where f_1 and f_2 are fixed functions and c is an arbitrary constant. Therefore, by transforming back to the original dependent variable V, we see that the general solution of the Riccati equation (1) can be written as

$$V(x) = V_1(x) + \frac{1}{cf_1(x) + f_2(x)} = \frac{cf_3(x) + f_4(x)}{cf_1(x) + f_2(x)},$$

where f_3 and f_4 are also fixed functions. Thus, since the anharmonic ratio is invariant under Möbius transformations, any four particular solutions V_1, V_2, V_3, V_4, with corresponding constants c_1, c_2, c_3, c_4, satisfy the relation

$$\frac{V_4 - V_1}{V_4 - V_2} \Big/ \frac{V_3 - V_1}{V_3 - V_2} = \frac{c_4 - c_1}{c_4 - c_2} \Big/ \frac{c_3 - c_1}{c_3 - c_2}.$$

Hence, the general solution of the Riccati equation is obtained from any three particular solutions V_1, V_2, V_3 by solving the algebraic equation

$$\frac{V - V_1}{V - V_2} \bigg/ \frac{V_3 - V_1}{V_3 - V_2} = c, \tag{4}$$

where c is an arbitrary constant.

Thus, for the Riccati equation, when one particular solution is known, the general solution can be obtained with the use of two quadratures by solving the inhomogeneous first order equation (2). When two particular solutions are known, the general solution can be obtained with the use of one quadrature by solving the homogeneous first order equation (3). But when three particular solutions are known, the general solution is determined without any quadrature by the nonlinear superposition formula (4). We say that the Riccati equation admits a nonlinear superposition of solutions. A similar superposition principle is established for a large class of systems of ODEs, known as matrix Riccati equations [7].

2.2 Bäcklund transformations as nonlinear superposition principle

A Bäcklund transformation for a second order PDE is a system of first order PDEs, which relates each solution of the original PDE with a solution of another differential equation. A rigorous and elementary survey of Bäcklund transformations can be found in [5]. Sometimes Bäcklund transformation equations can be used in order to find the general solution of an equation by linking it to a simpler equation that can be solved. One such example is the Liouville equation.

Example 1. *The Liouville equation $\partial^2 V/\partial x \partial y = \exp V$, is associated with the wave equation $\partial^2 \tilde{V}/\partial x \partial y = 0$ through the Bäcklund transformation*

$$\begin{aligned} \frac{\partial V}{\partial x} - \frac{\partial \tilde{V}}{\partial x} &= a \exp\left(\frac{V + \tilde{V}}{2}\right) \\ \frac{\partial V}{\partial y} + \frac{\partial \tilde{V}}{\partial y} &= -\frac{2}{a} \exp\left(\frac{\tilde{V} - V}{2}\right), \end{aligned} \tag{5}$$

where a is an arbitrary constant. By inserting the general solution $\tilde{V} = \phi(x) + \psi(y)$ of the wave equation in the Bäcklund transformation, the resulting overdetermined system for V can be integrated and the general solution

$$\exp V = 2 \frac{\phi'(x)\psi'(y)}{(\phi(x) + \psi(y))^2}$$

of the Liouville equation is obtained [15].

Bäcklund transformations can also map solutions of a given equation to different solutions of the same equation. These so-called auto-Bäcklund transformations can be useful since they give a method for constructing new solutions from known particular solutions.

Example 2. *The sine–Gordon equation is the nonlinear second order partial differential equation*

$$\frac{\partial^2 V}{\partial x \partial y} = \sin V \tag{6}$$

for the unknown function $V(x, y)$. *This equation was first studied in differential geometry, where it is related to surfaces of constant curvature [4]. The sine–Gordon equation can be integrated with inverse scattering methods [15], but it also allows a nonlinear superposition of solutions. The Bäcklund transformation equations for the sine–Gordon equation are*

$$\begin{aligned} \frac{\partial V}{\partial x} - \frac{\partial \tilde{V}}{\partial x} &= 2a \sin\left(\frac{V + \tilde{V}}{2}\right) \\ \frac{\partial V}{\partial y} + \frac{\partial \tilde{V}}{\partial y} &= \frac{2}{a} \sin\left(\frac{V - \tilde{V}}{2}\right) \end{aligned} \tag{7}$$

where a *is an arbitrary non-zero constant. We note that if* $(V, \tilde{V})$ *is a solution of* (7), *then*

$$\begin{aligned} \frac{\partial^2 V}{\partial x \partial y} &= \frac{\partial^2}{\partial x \partial y}\left(\frac{V - \tilde{V}}{2} + \frac{V + \tilde{V}}{2}\right) \\ &= a \frac{\partial}{\partial y} \sin\left(\frac{V + \tilde{V}}{2}\right) + \frac{1}{a}\frac{\partial}{\partial x} \sin\left(\frac{V - \tilde{V}}{2}\right) \\ &= \cos\left(\frac{V + \tilde{V}}{2}\right) \sin\left(\frac{V - \tilde{V}}{2}\right) + \cos\left(\frac{V - \tilde{V}}{2}\right) \sin\left(\frac{V + \tilde{V}}{2}\right) \\ &= \sin V. \end{aligned}$$

Analogously, $\tilde{V}$ *is also a solution of the sine–Gordon equation. Conversely, one can prove that for any solution* V *of* (6), *there exists a unique "conjugate" solution* $\tilde{V}$ *such that* $(V, \tilde{V})$ *is a solution of* (7). *In fact, the Bäcklund transformations for both the sine–Gordon equation and the Liouville equation are special cases of a more general class of first order system of PDEs which are proven in [2] to be equivalent to single second order equations for one unknown function. Thus, given a particular solution* V_0 *(for instance we can choose the trivial solution* $V_0 = 0$*) of the sine–Gordon equation, a second solution* V_1 *can be found by quadrature by solving the Bäcklund transformation equations for any choice of constant* $a = a_1$ *and with* $\tilde{V} = V_0$ *held* *fixed. A third solution can then be obtained by integrating* (7) *with* $a = a_2$ *and* $\tilde{V} = V_1$. *By continuing this process, an infinite family of solutions is obtained by quadrature (figure* 1*).*

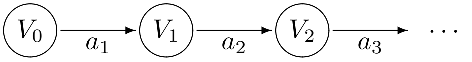

Figure 1: Generating an infinite sequence of solutions from one solution V_0.

However, like for the Riccati equation, there is also a way to obtain particular solutions of the sine–Gordon equation without using quadrature. This is done by relating different solutions through the so called theorem of permutability. Starting with a solution V_0 *and going two steps in the process illustrated in figure* 1*, with the constants* a_1 *and* a_2*, will give the same result as using the constants in reversed order. This is illustrated by the diagram in figure* 2*. By algebraic manipulation of the corresponding*

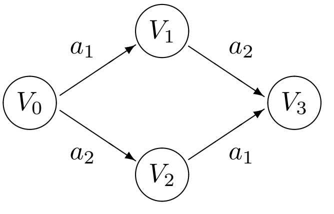

Figure 2: The same solution V_3 is obtained from V_0 by using the constants a_1, a_2, regardless of in which order the constants are taken.

Bäcklund transformations (7)*, one finds [14] that the solutions in figure* 2 *satisfy the relation*

$$\tan\left(\frac{V_3 - V_0}{4}\right) = \frac{a_1 + a_2}{a_1 - a_2} \tan\left(\frac{V_1 - V_2}{4}\right). \tag{8}$$

The formula (8) *is known as the theorem of permutability for the sine–Gordon equation. Thus, for any given three solutions* V_0, V_1, V_2*, a fourth solution* V_3 *may be constructed algebraically from* (8)*. By constructing* single soliton solutions *from the trivial solution* $V_0 = 0$ *through integration of the corresponding Bäcklund transformation for different constants* a*, more complex* multisoliton solutions *can be constructed algebraically by repeated use of the theorem of permutability. This nonlinear superposition principle is illustrated in figure* 3 *below. By starting with several single soliton solutions, higher order multisoliton solutions can be constructed in the same way.*

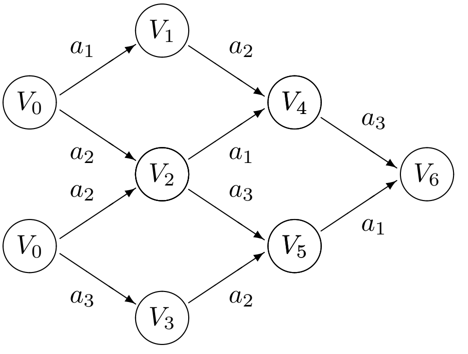

Figure 3: The single soliton solutions V_1, V_2, V_3 are found by quadrature from the trivial solution $V_0 = 0$ and the two-soliton solutions V_4, V_5 and the three-soliton solution V_6 can then be obtained algebraically by the theorem of permutability.

Both the sine–Gordon equation and the Riccati equation are well known examples of nonlinear differential equations having a nonlinear superposition principle. We give now an example of a linear system of PDEs, admitting a nonlinear superposition of solutions

2.3 The Cauchy–Riemann equations

The Cauchy–Riemann equations (CR)

$$\frac{\partial V}{\partial x} = \frac{\partial W}{\partial y}$$
$$\frac{\partial V}{\partial y} = -\frac{\partial W}{\partial x}$$

are a system of two linear homogeneous first order PDEs for two unknown functions $V(x, y)$ and $W(x, y)$. Since the system is linear, it admits the linear superposition principle. But in addition, as a consequence of the multiplication of holomorphic functions, CR also admits a bilinear superposition principle.

There is a $1-1$ correspondence between continuously differentiable solutions of CR and holomorphic functions of one complex variable, so that for each holomorphic function $f = V + \mathrm{i}W$, the pair (V, W) is a solution of CR. Since the product $fg = (V + \mathrm{i}W)(\tilde{V} + \mathrm{i}\tilde{W}) = (V\tilde{V} - W\tilde{W}) + \mathrm{i}(V\tilde{W} + W\tilde{V})$ of two holomorphic functions is again holomorphic, solutions of CR can also be “multiplied” as

$$(V, W) * (\tilde{V}, \tilde{W}) := (V\tilde{V} - W\tilde{W},\ V\tilde{W} + W\tilde{V})$$

prescribing a new solution $(V,W) * (\tilde{V},\tilde{W})$ to any two solutions. By using the linear and the bilinear superposition, new solutions of CR can be built as convergent power series of simple solutions. Since any holomorphic function can be expressed (locally) as a power series of $z = x + \mathrm{i}y \in \mathbb{C}$, it follows that the whole solution space of CR can, in fact, be described in terms of these power series. In a neighborhood of origin, every solution of CR can be built by forming power series of the simple solution (x, y)

$$(V,W) = \sum_{r=0}^{\infty} a_r (x,y)_*^r, \quad \text{where} \quad (x,y)_*^r = \underbrace{(x,y) * (x,y) * \cdots * (x,y)}_{r \text{ factors}}, \tag{9}$$

and a_r are real constants.

2.4 The quasi-Cauchy–Riemann equations

Another example of a linear system of PDEs having a bilinear superposition of solutions is the quasi-Cauchy–Riemann equation (QCR), defined on a Riemannian manifold Q with a metric (g_{ij}), of the form

$$\frac{J}{\det J}\nabla V = \frac{\tilde{J}}{\det \tilde{J}}\nabla W, \tag{10}$$

where J and $\tilde{J}$ are special conformal Killing tensors and ∇ is the gradient operator ($(\nabla V)^i = g^{ij}\partial_j V$). In Euclidean space with Cartesian coordinates q^i, J and $\tilde{J}$ are square matrices, with quadratic entries, of the form

$$J = \alpha q \otimes q + \beta \otimes q + q \otimes \beta + \gamma, \quad \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^n,\ \gamma \in \mathbb{R}^{n\times n},$$

where $q = [q^1 \quad q^2 \quad \cdots \quad q^n]^T$. For fixed tensors J and $\tilde{J}$, the QCR equation (10) is a linear first order system of PDEs for two unknown functions $V(q)$ and $\tilde{V}(q)$, and the solutions characterize all cofactor pair systems

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = F^i, \quad \text{where} \quad F = -\frac{J}{\det J}\nabla V = -\frac{\tilde{J}}{\det \tilde{J}}\nabla W.$$

Cofactor pair systems constitute an important class of Newton equations [21, 17, 18, 22, 3, 1], containing all classical separable potential systems. A generic cofactor pair system is equivalent, in the sense of Levi-Civita, to a potential system which is separable in the Hamilton–Jacobi sense [1].

A recursive formula for certain cofactor pair systems was found in [24], and later generalized to all cofactor pair systems [21, 17]. This recursion allows, for any given solution of a QCR equation, construction of an infinite family of solutions. Lundmark [16] realized later that the recursion was only a special case of a multiplicative structure on the solution space, possessed by any QCR equation. For $n = 2$, this multiplication formula is particularly simple. Given any two solutions (V,W) and $(\tilde{V},\tilde{W})$ of (10), a new solution is defined by

$$(V,W)*(\tilde{V},\tilde{W})=\Big(V\tilde{V}-\det(\tilde{J}^{-1}J)W\tilde{W},\ V\tilde{W}+W\tilde{V}-\operatorname{tr}(\tilde{J}^{-1}J)W\tilde{W}\Big).$$

The infinite family of solutions that is generated from a given solution (V,W) through the recursion formula can also be expressed through the multiplication by forming "products"

$$(V,W)*\underbrace{(0,1)*(0,1)*\cdots*(0,1)}_{r\text{ factors}}$$

for different powers of the trivial solution $(0,1)$. When $n>2$, both the recursion and multiplication exist but they are defined for a related parameter dependent equation

$$\frac{J+\mu\tilde{J}}{\det(J+\mu\tilde{J})}\nabla V_\mu=\frac{\tilde{J}}{\det\tilde{J}}\nabla W, \tag{11}$$

where μ is a real parameter and the unknown function V_μ is a polynomial of degree $n-1$ in μ. There is a $1-1$ correspondence between solutions of (11) and solutions of the original equation (10). Equation (11) can also be written as

$$\Big(\tilde{J}^{-1}J+\mu I\Big)\nabla V_\mu=\det(\tilde{J}^{-1}J+\mu I)\nabla W,$$

which in turn can be expressed as a congruence equation

$$\Big(\tilde{J}^{-1}J+\mu I\Big)\nabla V_\mu\equiv 0\quad\Big(mod\ \det(\tilde{J}^{-1}J+\mu I)\Big), \tag{12}$$

which means that V_μ is a solution if the vector $(\tilde{J}^{-1}J+\mu I)\nabla V_\mu$ can be written as a product of the function $\det(\tilde{J}^{-1}J+\mu I)$ and a vector which does not depend on μ. The $*$-product of two solutions V_μ and W_μ is then defined as the remainder of the ordinary product $V_\mu W_\mu$ after polynomial division by $\det(\tilde{J}^{-1}J+\mu I)$.

Example 3. *Consider the QCR equation* (10) *on a* 3*-dimensional Euclidean space with Cartesian coordinates* (x,y,z) *where*

$$J=\begin{bmatrix}1&0&x\\0&0&y\\x&y&2z\end{bmatrix}$$

and $\tilde{J}=I$ *is the identity matrix. Written out in components, we get the* *following overdetermined system of PDEs (when $y \neq 0$):*

$$\begin{aligned} 0 &= \frac{\partial V}{\partial x} + x\frac{\partial V}{\partial z} + y^2\frac{\partial W}{\partial x} \\ 0 &= y\frac{\partial V}{\partial z} + y^2\frac{\partial W}{\partial y} \\ 0 &= x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + 2z\frac{\partial V}{\partial z} + y^2\frac{\partial W}{\partial z}. \end{aligned} \tag{13}$$

The related parameter dependent equation (12) *is then given by*

$$\begin{bmatrix} 1+\mu & 0 & x \\ 0 & \mu & y \\ x & y & 2z+\mu \end{bmatrix} \nabla(V + U\mu + W\mu^2) \equiv 0 \quad (mod\ Z_\mu), \tag{14}$$

where $Z_\mu = -y^2 + (2z - y^2)\mu + (1+2z)\mu^2 + \mu^3$, and the $*$-product of two solutions $V + U\mu + W\mu^2$ and $\tilde{V} + \tilde{U}\mu + \tilde{W}\mu^2$ is given by the expression

$$\begin{aligned} & V\tilde{V} + y^2(V\tilde{U} + U\tilde{V}) - y^2(1+2z)W\tilde{W} \\ & + \mu\Big(V\tilde{U} + U\tilde{V} + (x^2 + y^2 - 2z)(U\tilde{W} + W\tilde{U}) \\ & \qquad + (4z^2 - x^2 - 2z(x^2 + y^2 - 1))W\tilde{W}\Big) \\ & + \mu^2\Big(V\tilde{W} + W\tilde{V} + U\tilde{U} - (1+2z)(U\tilde{W} + W\tilde{U}) \\ & \qquad + (2z(2z+1) + 1 + x^2 + y^2)W\tilde{W}\Big) \end{aligned}$$

For instance, the $*$-product of the trivial solutions $\mu + \mu^2$ and μ is given by the non-trivial solution

$$(\mu + \mu^2) * \mu = y^2 + (x^2 + y^2 - 2z)\mu + 2z\mu^2.$$

For any solution $V + U\mu + W\mu^2$ of the parameter dependent QCR equation (14)*, (V, W) is a solution of the original QCR equation* (13)*.*

A detailed study of this peculiar multiplication, defined on the solution space of every QCR equation, has been presented in [9].

3 Systems of linear first order homogeneous partial differential equations

The purpose of this dissertation is study of bilinear multiplication of solutions (like for QCR equations) for systems of homogeneous linear partial differential equations of first order of the form

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ijk}(x_1, x_2, \ldots, x_n)\frac{\partial V_i}{\partial x_j} = 0, \quad k = 1, 2, \ldots, r, \tag{15}$$

where $x_1, x_2, \ldots, x_n$ are independent variables (real or complex) and $V_1, V_2, \ldots, V_m$ are dependent variables, and a_{ijk} are given functions of (at least) class $\mathcal{C}^1$. In this section we shall recall [2, 6] a few properties of such systems of PDEs. Let $r' \leq r$ be the maximal number of linearly independent equations of (15). The system is *overdetermined* if $m < r'$, *determined* if $m = r'$, and *underdetermined* if $m > r'$.

The case when (15) has only one unknown function ($m = 1$) is considerably simpler than the general case when there are several unknown functions. This special case of (15) is discussed in detail in [6], and we present below the main facts for these systems.

3.1 $m = 1$, one dependent variable

When $m = 1$, (15) reduces to a system

$$\begin{aligned} X_1(V) &:= a_{11}\frac{\partial V}{\partial x_1} + a_{12}\frac{\partial V}{\partial x_2} + \cdots + a_{1n}\frac{\partial V}{\partial x_n} = 0 \\ X_2(V) &:= a_{21}\frac{\partial V}{\partial x_1} + a_{22}\frac{\partial V}{\partial x_2} + \cdots + a_{2n}\frac{\partial V}{\partial x_n} = 0 \\ &\vdots \\ X_r(V) &:= a_{r1}\frac{\partial V}{\partial x_1} + a_{r2}\frac{\partial V}{\partial x_2} + \cdots + a_{rn}\frac{\partial V}{\partial x_n} = 0, \end{aligned} \tag{16}$$

where $X_i = a_{i1}\frac{\partial}{\partial x_1} + a_{i2}\frac{\partial}{\partial x_2} + \cdots + a_{in}\frac{\partial}{\partial x_n}$ denote vector fields acting on the dependent variable V. The coefficients a_{ij} are assumed to be analytic functions of the independent variables.

When $r = 1$, the system (16) reduces to a single equation

$$a_1\frac{\partial V}{\partial x_1} + a_2\frac{\partial V}{\partial x_2} + \cdots + a_n\frac{\partial V}{\partial x_n} = 0. \tag{17}$$

Solving the equation (17) is equivalent to solving the system

$$\frac{\mathrm{d}x_1}{\mathrm{d}s} = a_1, \quad \frac{\mathrm{d}x_2}{\mathrm{d}s} = a_2, \quad \ldots, \quad \frac{\mathrm{d}x_n}{\mathrm{d}s} = a_n \tag{18}$$

of ordinary differential equations. The general solution of (17) can be written as $\phi(f_1, f_2, \ldots, f_{n-1})$, where ϕ is an arbitrary function and $f_1, f_2, \ldots, f_{n-1}$ are functionally independent integrals of motion of (18), i.e., they satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}s} f_i(x_1, \ldots, x_n) = \frac{\partial f_i}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s} + \cdots + \frac{\partial f_i}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}s} = 0$$

for any solution $x_1(s), \ldots, x_n(s)$ of (18).

We can assume that equations (16) are linearly independent, since otherwise we could discard equations which are linear combinations of the other equations. There can be at most n independent linear equations, so $r \leq n$. Since there can be no solutions except the trivial constant solutions when $r = n$, we can assume that $r < n$.

All commutator equations

$$[X_i, X_j](V) := X_i\left(X_j(V)\right) - X_j\left(X_i(V)\right) = 0, \quad 1 \leq i < j \leq r \tag{19}$$

are also linear homogeneous first order equations for V, which are satisfied for any solution of (16). Thus, by extending the system (16) with the maximal number of equations (19) so that the resulting equations are still linearly independent, we obtain a new system of the form (16) consisting of $r' \geq r$ independent equations with the same solution space as the original system. By repeating this procedure of adding equations (19) so that the equations in the extended system are independent, we will, after a finite number of iterations, reach a new system of the form (16) for which all commutator equations (19) are linear combinations of the equations $X_i(V) = 0$. Such a system is said to be *complete*. Henceforth, we will assume that the system (16) is complete.

A system $Y_1(V) = 0, Y_2(V) = 0, \ldots, Y_r(V) = 0$, defined by linear combinations

$$Y_i(V) = \lambda_{1i} X_1(V) + \lambda_{2i} X_2(V) + \cdots + \lambda_{ri} X_r(V), \quad i = 1, 2, \ldots, r,$$

where $\lambda_{ij} = \lambda_{ij}(x_1, x_2, \ldots, x_n)$ are functions of the independent variables such that $\det(\lambda_{ij}) \neq 0$, is *equivalent* to (16).

Suppose now that $y_2, y_3, \ldots, y_n$ are functionally independent integrals of the first equation $X_1(V) = 0$ (which is an equation of the form (17)), and choose a function y_1 in such a way that $y_1, y_2, \ldots, y_n$ define new independent variables. The equation $X_1(V) = 0$ then reduces to $\partial V/\partial y_1 = 0$, and the system (16) can be replaced with an equivalent system of the form

$$\begin{aligned} Y_1(V) &= \frac{\partial V}{\partial y_1} = 0 \\ Y_2(V) &= \frac{\partial V}{\partial y_2} + b_{21}\frac{\partial V}{\partial y_{r+1}} + \cdots + b_{2,n-r}\frac{\partial V}{\partial y_n} = 0 \\ &\vdots \\ Y_r(V) &= \frac{\partial V}{\partial y_r} + b_{r1}\frac{\partial V}{\partial y_{r+1}} + \cdots + b_{r,n-r}\frac{\partial V}{\partial y_n} = 0. \end{aligned} \tag{20}$$

Completeness for a system is an invariant property under both equivalence of systems and under changes of independent variables. Therefore, (20) is again a complete system, which can only be the case if the vector fields $Y_1, Y_2, \ldots Y_r$ commute, i.e., $[Y_i, Y_j] = 0$. (since they will only contain the derivatives $\partial V/\partial y_{r+1}, \ldots, \partial V/\partial y_n$). Especially, we have

$$[Y_1, Y_i](V) = \frac{\partial b_{i1}}{\partial y_1}\frac{\partial V}{\partial y_{r+1}} + \cdots + \frac{\partial b_{i,n-r}}{\partial y_1}\frac{\partial V}{\partial y_n} \equiv 0, \quad i = 2, 3, \ldots, r,$$

from which we conclude that the coefficients b_{ij} are independent of y_1. Hence, the equations $Y_2(V) = 0, Y_3(V) = 0, \ldots, Y_r(V) = 0$ form a complete system of $r-1$ equations in $n-1$ independent variables, which in turn can be reduced to a complete system of $r-2$ equations in $n-2$ variables. By repeating this procedure we conclude that any complete system (16) can be reduced to a single equation in $n-r+1$ independent variables, and that the general solution therefore is an arbitrary function of $n-r$ independent particular solutions that are integrals of motions of the related dynamical system (18). We note especially that, by using this procedure, it is possible to determine without quadrature whether a system (16) admits non-trivial solutions.

When the system (15) contains more than one dependent variable ($m > 1$), there is no general procedure, like the one described above, for obtaining the general solution. Therefore, the discovery of other methods of constructing exact solutions becomes highly important.

4 Multiplication of solutions

Most of this thesis is devoted to study of a bilinear superposition principle, which we call $*$-*multiplication*, that generates new solutions for systems of the kind (15). In other words, if we let S denote the solution space for a certain system of the form (15), we consider an operation

$$\begin{gathered} * : S \times S \to S \\ \Big((V_1, \ldots, V_m), (W_1, \ldots, W_m)\Big) \mapsto (V_1, \ldots, V_m) * (W_1, \ldots, W_m), \end{gathered}$$

which, as a consequence of the required bilinearity, must have the form

$$(V_1, \ldots, V_m) * (W_1, \ldots, W_m) = \left(\sum_{i,j=1}^{m} f_{ij}^1 V_i W_j, \ldots, \sum_{i,j=1}^{m} f_{ij}^m V_i W_j \right) \tag{21}$$

where the coefficients $f_{ij}^k = f_{ij}^k(x_1, x_2, \ldots, x_n)$ are functions of the independent variables. Since the system (15) is linear, the presence of a multiplication turns the solution space S into an algebra over the current field ($\mathbb{R}$ or $\mathbb{C}$). Not every system (15) admits a non-trivial $*$-multiplication, and the question about existence of a multiplication leads to a number of complicated differential relations among the functions f_{ij}^k. For instance, since a constant vector $(c_1, \ldots, c_m)$ is a solution of (15), all $*$-products of constant solutions must again be solutions

$$(c_1, \ldots, c_m) * (d_1, \ldots, d_m) = \sum_{i,j=1}^{m} c_i d_j (f_{ij}^1, \ldots, f_{ij}^m).$$

This entails that $F_{ab} := (f^1_{ab}, \dots, f^m_{ab})$ must be a solution of (15) for every choice of a and b. One can then go further and study the higher degree polynomials of f^k_{ij}, obtained by forming higher order $*$-products

$$(c^1_1, \dots, c^1_m) * (c^2_1, \dots, c^2_m) * \dots * (c^r_1, \dots, c^r_m), \quad r = 1, 2, \dots, \tag{22}$$

of constant solutions. Since all possible products (22) must be solutions, we obtain an infinite number of restrictive equations for the functions f^k_{ij}. Surprisingly, large classes of systems with $*$-multiplication exist. Both the Cauchy–Riemann equations and the more general quasi-Cauchy–Riemann equations [9] are non-trivial examples of systems of partial differential equations which allow multiplication of solutions.

We give here a simple example of an overdetermined system that admits a $*$-multiplication but is not a QCR equation.

Example 4. *Consider the following system of differential equations for the unknown functions $U(x, y)$, $V(x, y)$ and $W(x, y)$:*

$$\begin{aligned}
0 &= x\frac{\partial U}{\partial x} + (y^2+1)\frac{\partial U}{\partial y} + x\frac{\partial W}{\partial x} \\
0 &= -x\frac{\partial U}{\partial x} - y\frac{\partial U}{\partial y} + x\frac{\partial W}{\partial y} \\
0 &= -\frac{\partial V}{\partial x} + (y+1)\frac{\partial W}{\partial x} + (y+1)\frac{\partial W}{\partial y} \\
0 &= -\frac{\partial V}{\partial y} - x\frac{\partial W}{\partial x} + (1-x)\frac{\partial W}{\partial y}.
\end{aligned}$$

This system admits a multiplication of solutions, where the $*$-product

$$(U, V, W) * (\tilde{U}, \tilde{V}, \tilde{W}) =: (P, Q, R)$$

of two solutions is defined by

$$\begin{aligned}
P &= U\tilde{U} - xV\tilde{W} - xW\tilde{V} + (x + xy - x^2)W\tilde{W} \\
Q &= U\tilde{V} + V\tilde{U} - yV\tilde{W} - yW\tilde{V} + \left(y - x + y^2 - xy\right) W\tilde{W} \\
R &= U\tilde{W} + V\tilde{V} + W\tilde{U} + (x - 1 - y)(V\tilde{W} + W\tilde{V}) \\
&\quad + \left((1 + y - x)^2 - y\right) W\tilde{W}.
\end{aligned}$$

Thus, for example, forming the $*$-product of the two trivial solutions $(0, 1, 0)$ and $(0, 0, -1)$ gives the non-trivial solution

$$(0, 1, 0) * (0, 0, -1) = (x, y, 1 + y - x).$$

For the Cauchy–Riemann equations all coefficients f^k_{ij} in the $*$-multiplication formula (21) are constant ($f^1_{11} = f^2_{12} = f^2_{21} = 1$, $f^1_{22} = -1$, $f^1_{12} = f^1_{21} = f^2_{11} = f^2_{22} = 0$), so that products of trivial (constant) solutions are again trivial. On the other hand, when the coefficients f_{ij}^k are not all constant, which is the case in example 3 and example 4, products of trivial solutions will in general give non-trivial solutions.

By combining the $*$-multiplication with the linear superposition principle of solutions, it is possible to form $*$-*polynomials*

$$\sum_{j=0}^{N} a_j V_*^j, \quad \text{where} \quad V_*^j = \underbrace{V * V * \cdots * V,}_{j \text{ factors}} \tag{23}$$

of any solution $V = (V_1, \ldots, V_m)$. Thus, given a solution V, infinite families of solutions may be constructed by forming $*$-polynomials (23). It is also possible to construct $*$-*power series*

$$\sum_{j=0}^{\infty} a_j V_*^j,$$

which define new solutions.

In papers [9, 11, 12], we study three main problems about $*$-multiplication:

1. The study of relations between the coefficients f_{ij}^k in (21) that must be satisfied for a system (15) to admit $*$-multiplication. For certain types of linear systems of PDEs (15), we give equivalent characterizations of systems admitting $*$-multiplication that leads to determination of explicit families of systems of PDEs having $*$-multiplication. We give also methods for constructing, from known systems, new systems of PDEs admitting $*$-multiplication.

2. The search for a canonical form of systems of linear PDEs having $*$-multiplication. The most ideal situation would be to have a complete set of canonical systems, such that any system admitting a $*$-multiplication could be transformed (for instance by a change of independent variables) into a unique member of this set. We have not been able to give such a classification in the most general situations and it seems to be a difficult problem. Instead, we describe classes of certain generic (typical) systems, into which most systems having $*$-multiplication can be transformed.

3. The study of the operation $*$ as a tool for construction of new solutions from known solutions of system (15). For some systems with $*$-multiplication, interesting infinite families of solutions may be constructed as $*$-polynomials of certain simple solution which is easy to find. We also study the natural question about which solutions are $*$-*analytic*, i.e., about which solutions can be represented locally as $*$-power series of certain simple solutions. Recall that any solution of the Cauchy–Riemann equations is locally a $*$-power series of the

linear solution (x, y) (9). The choice of "simple" solution, used for building $*$-power series, depends on the type of system of linear PDEs. In some cases it is sufficient to construct power series of a constant solution, while for systems with constant coefficients f_{ij}^k (like the CR equations), one has to build power series of suitable non-trivial solutions.

5 Overview of research papers

In the first paper [9], we study multiplication for solutions of quasi-Cauchy–Riemann equations (10), that are related to the cofactor pair systems. We give the following characterization of systems admitting $*$-multiplication: any system, on a Riemannian manifold, of the form

$$(X + \mu I)\, \nabla V_\mu \equiv 0 \quad (mod\ \det (X + \mu I))\,,$$

where X is a $(1, 1)$ tensor, admits $*$-multiplication of solutions if and only if

$$(X + \mu I)\, \nabla \det (X + \mu I) \equiv 0 \quad (mod\ \det (X + \mu I))\,. \tag{24}$$

The $*$-product $V_\mu * W_\mu$ of two solutions is defined as the remainder in the polynomial division of the ordinary product $V_\mu W_\mu$ with the divisor $\det (X + \mu I)$.

The characterizing equation (24) is studied and it is proven that it is satisfied by several families of rank two tensors X, beyond the ones which were already known in the theory of quasi-Cauchy–Riemann equations. Especially, it has been shown that any tensor X with vanishing Nijenhuis torsion satisfies (24), due to validity of the following relation:

$$2\Big(X \mathrm{d}(\det X) - \det X \mathrm{d}(\operatorname{tr} X)\Big)_i = (N_X)^k_{ij} C^j_k,$$

where N_X is the Nijenhuis torsion of X and $C = (\det X) X^{-1}$ is the cofactor tensor of X.

Cofactor pair systems are closely related to the concept of equivalence for dynamical systems, and we discuss in [9] which role the multiplication of QCR equations plays in this relation. In particular we give examples of infinite families of separable Lagrangian systems which are generated by $*$-multiplication from a single system.

In the second paper [11], it is shown that the $*$-multiplication is admitted by a much larger class of equations than the one described in [9]. Any system

$$A_\mu \nabla V_\mu \equiv 0 \quad (mod\ Z_\mu)\,, \tag{25}$$

admits a $*$-multiplication whenever the $(1,1)$ tensor A_μ and the smooth function Z_μ (both depending polynomially on the parameter μ) satisfy the relation

$$A_\mu \nabla Z_\mu \equiv 0 \quad (mod\ Z_\mu)\,.$$

We give in [11] a classification of systems admitting $*$-multiplication, depending on the dimension of the manifold, and on the polynomial degrees of A_μ and Z_μ, respectively.

By combining the $*$-multiplication and the ordinary linear superposition principle of solutions, $*$-polynomials

$$\sum_{r=0}^{N} a_r (V_\mu)_*^r, \quad \text{where} \quad (V_\mu)_*^r = \underbrace{V_\mu * V_\mu * \cdots * V_\mu}_{r \text{ factors}}, \tag{26}$$

of a simple solution V_μ are constructed. This means that infinite families of non-trivial solutions are constructed from a simple solution V_μ. Sufficient conditions for a $*$-power series (obtained by letting $N \to \infty$ in (26)), of a constant solution, to converge and to define a new solution have been established with the use of a matrix notation, introduced in [11] specially for this purpose.

In the third paper [12] of this dissertation, we study matrix equations

$$\nabla f = M \nabla g, \tag{27}$$

where M is a $n \times n$ matrix with constant entries, in a open convex domain of a vector space over the real or complex numbers. The general solution of the equation (27) is described in [8] and some further results are also given in [23]. We show that every equation of the form (27) can be extended to a system which admits $*$-multiplication on the solution space. The main result in [12] is that every analytic solution is also $*$-analytic, meaning that it can be expressed locally through power series, with respect to the $*$-multiplication, of simple solutions.

The last paper [10] presents an explicit formula for the remainder of polynomial division, using the companion matrix of the divisor. Let $Z(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial over some commutative ring. For each polynomial $p(x)$, according to the Euclidean algorithm, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = q(x)Z(x) + r(x), \quad \deg r < n.$$

Traditionally, the residue $r(x)$ is constructed through a recursive algorithm. We show that $r(x)$ can be described explicitly as

$$r(x) = \begin{bmatrix} 1 & x & \cdots & x^{n-1} \end{bmatrix} p(C) [1 \quad 0 \quad \cdots \quad 0]^T,$$

where C is the companion matrix of $Z(x)$

$$C = C[Z] := \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

and $p(C)$ is the matrix polynomial obtained by formally substituting the matrix C in the place of the variable x.

This result is a generalization of the method used in [11] to prove convergence of the constructed $*$-power series solutions of the equation (25).

6 Conclusions

In this dissertation we have discovered that large classes of linear systems of first order PDEs admit, beside the linear superposition of solutions, a new kind of bilinear superposition called $*$-multiplication. We show that, by combining these two superpositions, one can construct large families of explicit solutions by forming *$*$-power series* of certain simple solutions. For the subclass of equations of the form $\nabla f = M \nabla g$, where M is a constant matrix, every analytic solution is also *$*$-analytic*, but in more general cases it remains an open question how large part of the whole solution space is generated by $*$-power series of simple solutions.

We have also in this dissertation attempted to classify the systems of PDEs that admit $*$-multiplication. This description of systems is the first of this kind and it is still an interesting future problem to give a more complete characterization of such PDEs.

The $*$-multiplication is a new valuable tool of algebraically constructing large classes of explicit solutions for a wide family of systems of PDEs, where the general solution is not explicitly known.

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