Course Notes for Math 753: Differential Forms and Algebraic Topology

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Chapter 1

Introduction

This course attempts to describe the many interesting and important connections between differential forms and algebraic topology. On the whole, we follow some of the main ideas contained in the book "Differential Forms in Algebraic Topology" by *Raoul Bott* and *Loring Tu*. Sometimes we shall skip some of their material, either in order to get at other topics more quickly or to present a slightly more sophisticated viewpoint that relies on the assumption that students have already seen background material earlier. This chapter surveys some background material. It gives a brief overview of the general algebraic-topological and smooth-manifold contexts within which we shall be studying differential forms, particularly de Rham cohomology. The following short list of references can be consulted in case some concepts need further clarification.

William Boothby, "An Introduction to Differentiable Manifolds and Riemannian Geometry."
Allen Hatcher, "Algebraic Topology."
Saunders MacLane, "Homology."

1.1 Graded algebras

Let R be a commutative ring (with identity). By a *graded R-module* we mean an indexed collection of (left) R-modules $\mathbf{A} = \{A_n | n \in \mathbb{Z}\}$. We may write $\mathbf{A}(n)$ for A_n. $\mathbf{A}$ is said to be *non-negative* if $\mathbf{A}(n) = 0$ for all negative n, and is said to be *concentrated in degree k* if $\mathbf{A}(n) = 0$ for $n \neq k$. If A is an R-module, it is sometimes convenient to identify it with the graded R-module concentrated in degree k which equals A in that degree. We denote this graded module by $A < k >$.

If $\mathbf{B} = \{B_n\}$ is a graded R-module such that $A_n \subseteq B_n$ for all n, we say that $\mathbf{A}$ is a *submodule* of $\mathbf{B}$, and we may write $\mathbf{A} \leq \mathbf{B}$ to indicate this. A graded *quotient module* $\mathbf{B}/\mathbf{A}$ is then defined by analogy with the ungraded case. If a family $\mathbf{A}_\alpha$ of submodules of a fixed module is given, it is clear how to define the intersection submodule $\bigcap\limits_\alpha \mathbf{A}_\alpha$.

If $\mathbf{A}$ and $\mathbf{B}$ are graded modules, the *tensor product* $\mathbf{A} \otimes \mathbf{B}$ is defined by

$$(\mathbf{A} \otimes \mathbf{B})(n) = \bigoplus_{i+j=n} \mathbf{A}(i) \otimes \mathbf{B}(j),$$

where the tensor products on the right are taken over the ring R. The direct sum $\mathbf{A} \bigoplus \mathbf{B}$ is defined by summing in each degree. Both of these definitions can be extended to n-fold tensor products and sums, in fact to tensor products and sums of arbitrary families of graded modules. At most, we'll be applying this to a sequence of modules.

Similarly, we can define $\mathbf{Hom}(\mathbf{A}, \mathbf{B})$ as follows:

$$\mathbf{Hom}(\mathbf{A}, \mathbf{B})(n) = \prod_{p=-\infty}^{\infty} Hom_R(A_p, B_{p-n}).$$

(Note that the standard convention is to define this as $\mathbf{Hom}(\mathbf{A}, \mathbf{B})(-n)$.) If $\mathbf{B}$ is concentrated in degree 0, say $\mathbf{B} = B < 0 >$, then this definition gives a natural identification $\mathbf{Hom}(\mathbf{A}, \mathbf{B})(n) = Hom_R(A(n), B)$.

A *degree k map* of graded modules $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a collection $f_n : \mathbf{A}(n) \to \mathbf{B}(n+k)$ of R-module homomorphisms. When $k = 0$, we call $\mathbf{f}$ simply a *map* (of graded modules). As examples of such maps, say that $\mathbf{A} \leq \mathbf{B}$. Then the canonical inclusion and projection maps of R-modules $\mathbf{A}(n) \to \mathbf{B}(n) \to \mathbf{B}(n)/\mathbf{A}(n)$ define "inclusion" and "projection" maps $\mathbf{A} \to \mathbf{B} \to \mathbf{B}/\mathbf{A}$. For another example, a homomorphism of R-modules $f : A \to B$ determines a map of graded modules $A < 0 > \to B < 0 >$ in an obvious way. In accord with the convention already mentioned, we may denote this graded map again by $f : A \to B$.

Linear combinations of degree k maps and the composition of a degree k map and a degree l map are defined as expected and enjoy the usual properties.

If a (degree 0) map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ has an inverse, it is called an isomorphism. This defines an equivalence relation $\approx$.

Given a degree k map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, graded versions of kernel and image are defined for $\mathbf{f}$ by analogy with the ungraded case and are denoted $\mathbf{ker}(\mathbf{f})$ and $\mathbf{im}(\mathbf{f})$, respectively. As usual, we have $\mathbf{A}/\mathbf{ker}(\mathbf{f}) \approx \mathbf{im}(\mathbf{f})$.

If $\mathbf{d} : \mathbf{A} \to \mathbf{A}$ is a map of degree -1 such that $\mathbf{d}^2 = \mathbf{0}$, we call it a differential on $\mathbf{A}$, and call the pair $(\mathbf{A}, \mathbf{d})$ a *chain complex*. In this case, $\mathbf{im}(\mathbf{d}) \leq \ker(\mathbf{d})$, so we can form the *homology of* $(\mathbf{A}, \mathbf{d})$, that is, quotient module $\mathbf{ker}(\mathbf{d})/\mathbf{im}(\mathbf{d})$, which we denote $\mathbf{H}_*(\mathbf{A}, \mathbf{d})$. We usually write $\mathbf{H}_*(\mathbf{A}, \mathbf{d})(n)$ as $H_n(\mathbf{A}, \mathbf{d})$ and omit reference to $\mathbf{d}$ when no confusion will result. The same considerations can be applied in case $\mathbf{d}$ has degree 1. However, since it is sometimes convenient to distinguish the two cases, we shall call a pair $(\mathbf{A}, \mathbf{d})$ a *cochain complex* when

$\mathbf{d}$ has degree 1. Also in this case, we'll call the homology $\mathbf{H}(\mathbf{A}, \mathbf{d})$ the *cohomology* of $(\mathbf{A}, \mathbf{d})$ and denote it (resp., $\mathbf{H}(\mathbf{A}, \mathbf{d})(n)$) by $\mathbf{H}^*(\mathbf{A}, \mathbf{d})$ (resp., $H^n(\mathbf{A}, \mathbf{d})$).

Given chain complexes $(\mathbf{A}, \mathbf{d})$ and $(\mathbf{B}, \mathbf{e})$ a *chain map* $\mathbf{f} : (\mathbf{A}, \mathbf{d}) \to (\mathbf{B}, \mathbf{e})$ is a map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that $\mathbf{fd} = \mathbf{ef}$. Such an $\mathbf{f}$ sends $\mathbf{im}(\mathbf{d})$ (resp., $\mathbf{ker}(\mathbf{d})$) to $\mathbf{im}(\mathbf{e})$ (resp., $\mathbf{ker}(\mathbf{e})$), hence inducing a map of graded homology modules, which we denote either $\mathbf{f}_*$ or $\mathbf{H}_*(\mathbf{f})$. It is clear that the identity map is a chain map and that chain maps are closed under composition. Completely analogous definitions apply to cochain complexes, yielding cochain maps.

If $(\mathbf{A}, \mathbf{d})$ is a chain complex and C is an R-module, then $(\mathbf{A} \otimes C, \mathbf{d} \otimes 1)$ is again a chain complex, whose homology we denote by $\mathbf{H}_*(\mathbf{A}; C)$. In this case, we may describe this homology group as the homology of $\mathbf{A}$ with *coefficients* in C. If $\mathbf{f}$ is a chain map $(\mathbf{A}, \mathbf{d}) \to (\mathbf{B}, \mathbf{e})$, then $\mathbf{f} \otimes 1$ is a chain map $(\mathbf{A} \otimes C, \mathbf{d} \otimes 1) \to (\mathbf{B} \otimes C, \mathbf{e} \otimes 1)$, inducing a homology homomorphism, which we usually still denote $\mathbf{f}_*$.

Next note that if $\mathbf{d} : \mathbf{A} \to \mathbf{A}$ is a differential of degree -1, and C is an R-module, we can define $\mathbf{d}^* : \mathbf{Hom}(\mathbf{A}, C) \to \mathbf{Hom}(\mathbf{A}, C)$ to be the 'adjoint' of $\mathbf{d}$, and this is a differential of degree 1. Therefore, the Hom functor can be used to transform a chain complex into a cochain complex. The corresponding cohomology may be described as the cohomology of $\mathbf{A}$ with coefficents in C. Similarly for chain maps. Starting with a chain map $\mathbf{f} : (\mathbf{A}, \mathbf{d}) \to (\mathbf{B}, \mathbf{e})$, we can form a cochain map $\mathbf{Hom}(\mathbf{f},1) : (\mathbf{Hom}(\mathbf{B}, C), \mathbf{e}^*) \to \mathbf{Hom}(\mathbf{A}, C), \mathbf{d}^*)$—note the 'changed' direction of the arrow. This last induces a cohomology homomorphism, which we usually denote $\mathbf{f}^*$.

A *graded R-algebra* is a graded R-module $\mathbf{A}$ equipped with a multiplication map

$$\mu : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A},$$

which, of course, is required to satisfy the usual associative law. A graded subalgebra of $\mathbf{A}$ is defined in the obvious way. If $\mathbf{A}(0)$ contains a copy of the ring R such that $\mu | R \otimes R$ gives the usual multiplication in R and μ-multiplication by $1 \in R$ is the identity map of $\mathbf{A}$, then we call $\mathbf{A}$ a *graded R-algebra with identity.*

If $\mathbf{A}$ and $\mathbf{B}$ are graded R-algebras, a map of graded R-modules $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is called a *homomorphism* if $\mathbf{f}$ respects multiplication. It is an isomorphism if it has degree zero and admits an inverse (which is then automatically a homomorphism). It is easy to check that $\mathbf{im}(\mathbf{f})$ is a subalgebra of $\mathbf{B}$.

By an *ideal* in a graded algebra $\mathbf{A}$, we mean a submodule $\mathbf{I} \leq \mathbf{A}$ such that the composition of μ with the inclusion $\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$ factors through the inclusion $\mathbf{I} \to \mathbf{A}$. It is not hard to check that an arbitrarily indexed intersection of ideals is again an ideal. Thus, if $\mathbf{S}$ is a graded subset of $\mathbf{A}$, there is a unique smallest ideal in $\mathbf{A}$ containing $\mathbf{S}$, said to be the *ideal generated by* $\mathbf{S}$. If $\mathbf{I}$ is an ideal in $\mathbf{A}$, then the multiplication of $\mathbf{A}$ induces an algebra multiplication on the quotient module $\mathbf{A}/\mathbf{I}$, as in the ungraded case. It is easy to check that if $\mathbf{f}:\mathbf{A}\to\mathbf{B}$ is a homomorphism, then $\mathbf{ker}(\mathbf{f})$ is an ideal in $\mathbf{A}$.

Let $\mathbf{t}:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}\otimes\mathbf{A}$ be the R-module map defined by the equations $\mathbf{t}(n)(x\otimes y) = (-1)^{pq}y\otimes x$, for all $x\in A(p)$ and $y\in A(q), p+q=n$. We say that $\mathbf{A}$ is *commutative* if $\mu\mathbf{t}=\mu$. For any algebra $\mathbf{A}$, we may form the sets $\mathbf{S}(n)=\{xy-(-1)^{pq}yx|x\in\mathbf{A}(p), y\in\mathbf{A}(q), p+q=n\}$ and let $\mathbf{I}$ be the ideal generated by $\mathbf{S}$. The quotient algebra $\mathbf{C}(\mathbf{A})=\mathbf{A}/\mathbf{I}$ is clearly commutative. The reader is invited to make precise the observation that $\mathbf{C}(\mathbf{A})$ is the largest commutative image of $\mathbf{A}$.

Suppose that the graded algebra $\mathbf{A}$ admits a degree-one differential $\mathbf{d}$ that satisfies the following *derivation* property:

$$\mathbf{d}(n)(xy)=\mathbf{d}(p)(x)y+(-1)^{p}x\mathbf{d}(q)(y),$$

for all x,y,p,q such that $x\in\mathbf{A}(p)$, $y\in\mathbf{A}(q)$, $p+q=n$. Then we call $(\mathbf{A},\mathbf{d})$ a graded, differential algebra (DGA). In this case, one easily verifies that $\mathbf{im}(\mathbf{d})$ is an ideal in $\mathbf{ker}(\mathbf{d})$, and so

$$\mathbf{H}^*(\mathbf{A},\mathbf{d})=\mathbf{ker}(\mathbf{d})/\mathbf{im}(\mathbf{d})$$

inherits the structure of a graded algebra. If $\mathbf{A}$ is commutative, so is $\mathbf{H}^*(\mathbf{A},\mathbf{d})$

1.2 Examples

1.2.1 The tensor algebra

Suppose that $\mathbf{M}$ is a positive graded R-module. For each $n>0$, the n-fold tensor product of copies of $\mathbf{M}$ is a positive graded R-module that we denote by $\mathbf{M}^{\otimes\mathbf{n}}$. Now define the graded R-module

$$\mathbf{T}(\mathbf{M})=R\oplus\mathbf{M}^{\otimes\mathbf{1}}\oplus\mathbf{M}^{\otimes\mathbf{2}}\oplus\dots\ .$$

There is an obvious map of graded R-modules $\mu:\mathbf{T}(\mathbf{M})\bigotimes\mathbf{T}(\mathbf{M})\to\mathbf{T}(\mathbf{M})$ that makes $\mathbf{T}(\mathbf{M})$ into a graded R-algebra with identity. This is called the *tensor algebra* of $\mathbf{M}$. The identity map $\mathbf{M}\to\mathbf{M}=\mathbf{M}^{\otimes\mathbf{1}}$ yields a map if $\mathbf{M}$ into $\mathbf{T}(\mathbf{M})$, which we call the canonical inclusion and by which we identify $\mathbf{M}$ with a submodule of $\mathbf{T}(\mathbf{M})$.

$\mathbf{T}(\mathbf{M})$ has the following universal property: For any map of graded R-modules $\mathbf{f}:\mathbf{M}\to\mathbf{N}$, where $\mathbf{N}$ has the structure of a graded R-algebra, there is a unique homomorphism of algebras $\mathbf{T}(\mathbf{f}):\mathbf{T}(\mathbf{M})\to\mathbf{N}$ that extends $\mathbf{f}$. In particular, if $\mathbf{f}:\mathbf{A}\to\mathbf{B}$ is a map of graded R-modules, there is a unique induced homomorphism $\mathbf{T}(\mathbf{f}):\mathbf{T}(\mathbf{A})\to\mathbf{T}(\mathbf{B})$.

Some important special cases of the above construction occur when $\mathbf{M}=A<k>$ for some R-module A and integer $k\geq 0$. Thus, when $k=0$, $\mathbf{T}(\mathbf{M})$ is concentrated in degree 0 and may be identified with the R-module

$$R \oplus A \oplus (A \otimes A) \oplus \ldots,$$

which is the classical tensor algebra of A. For this course, we consider only the additional case $k = 1$, although other cases are useful in different contexts.

1.2.2 The exterior algebra

Suppose that $\mathbf{M} = A < 1 >$. Then we call the commutative algebra $\mathbf{C}(\mathbf{T}(\mathbf{M}))$ the *exterior algebra of* A, and we denote it by $\bigwedge^*(A)$. We shall write $\bigwedge^*(A)(n)$ as $\bigwedge^n(A)$, in accord with more conventional notation. Note that we may identify $\bigwedge^1(A)$ with A. If $x_1, \ldots, x_n$ are elements of A, so that $x_1 \otimes \ldots \otimes x_n$ is an element of $\mathbf{T}(\mathbf{M})(n)$, we denote the image of $x_1 \otimes \ldots \otimes x_n$ in $\bigwedge^n(A)$ by $x_1 \wedge \ldots \wedge x_n$. It is not hard to check that if A is generated by a family of elements $\{x_j | j \in J\}$, then, for each $n > 0$, $\bigwedge^n A$ is generated by the family $\{x_{j_1} \wedge \ldots \wedge x_{j_n} | j_i \in J\}$. In fact, if A is a free R-module with basis $\{x_j | j \in J\}$, then $\bigwedge^n A$ is free with basis a subset of $\{x_{j_1} \wedge \ldots \wedge x_{jn} | j_i \in J\}$. For example, choose any linear order on J and use the subset of all $x_{j_1} \wedge \ldots \wedge x_{j_n}$ such that $j_1 < \ldots < j_n$. It follows from this that if A is free of rank r, then $\bigwedge^n A$ is free of rank $\binom{r}{n}$.

By construction, the exterior algebra on A is a commutative algebra. It has the following property, which follows immediately from the universal property for tensor algebras: namely, let $f : A \to B$ be a map of R-modules. Then there exists a unique homomorphism of exterior algebras $\bigwedge^*(f) : \bigwedge^*(A) \to \bigwedge^*(B)$ extending f. Clearly, for any elements $x_1, \ldots, x_n \in A$, the homomorphism property of $\bigwedge^*(f)$ produces the familiar formula

$$\bigwedge{}^*(f)(x_1 \wedge \ldots \wedge x_n) = f(x_1) \wedge \ldots f(x_n)$$

.

Exercise 1. Suppose that V is a finite-dimensional real vector space, and let $L_k(V, \mathbb{R})$ denote the vector space of all alternating, k-linear forms on V. Then, $L_1(V, \mathbb{R})$ is the dual V^* of V. Define *canonical* isomorphisms

$$\bigwedge{}^k(V^*) \to (\bigwedge{}^k(V))^* \to L_k(V, \mathbb{R}).$$

1.2.3 Singular homology and cohomology

The *singular chain complex* of a space X is a graded chain complex $(\mathbf{S}_*(X), \partial)$, and singular homology $\mathbf{H}_*(X)$ is just the graded $\mathbb{Z}$-module $\mathbf{H}_*(\mathbf{S}_*(X), \partial)$. If A is any abelian group, we can replace $\mathbf{S}_*(X)$ by $\mathbf{S}_*(X) \otimes A$ and obtain singular homology with coefficients in A, $\mathbf{H}_*(X; A)$. This usually loses information but may simplify computations. Generally, no choice of A allows the introduction of a natural graded algebra structure on $\mathbf{H}_*(X; A)$, although for special X (e.g., topological groups) this can be done. However, 'dualizing' does allow the introduction of an algebra structure. In particular, we apply the $Hom(--, A)$ functor to the singular chain complex $\mathbf{S}_*(X)$, as explained in Section 1.1, obtaining the singular cochain functor (with 'coefficients' in A) $\mathbf{S}^*(X; A)$. Passing to cohomology yields $\mathbf{H}^*(X; A)$, singular cohomology with coefficients in A. At this point, dualizing has not gained us anything. Of course, different A will yield different kinds of cohomology, just as in the case of homology above. But all the abelian-group information contained in these various cohomology groups is already contained in the homology groups. Indeed, this can be made precise and formal by the various Universal Coefficient Theorems (e.g., see *Hatcher* or *MacLane*). However, if A is chosen to have the structure of a commutative ring, then we actually gain more information with cohomology, as we explain below.

First, however, we pause to recall that a continuous map of spaces $f : X \to Y$ induces a chain map $\mathbf{f}_* : \mathbf{S}_*(X) \to \mathbf{S}_*(Y)$ and, hence, for any abelian group A, a homology homomorphism denoted $\mathbf{f}_* : \mathbf{H}_*(X; A) \to \mathbf{H}_*(Y; A)$. Further, this chain map induces a corresponding cochain map $\mathbf{f}^* : \mathbf{S}^*(Y; A)) \to \mathbf{S}^*(X; A)$, as explained earlier, which, in turn, induces a cohomology homomorphism $\mathbf{f}^* : \mathbf{H}^*(Y; A) \to \mathbf{H}^*(X; A)$. One easily verfies that singular homology (resp., cohomology)with coefficients in A is a covariant (resp., contravariant functor) from the category of topological spaces and continuous maps to the category of graded abelian groups.

We can now described how the ring structure in cohomology is obtained. We choose any commutative ring R and regard it as an abelian group, so that, for any space X, $\mathbf{H}^*(X; R)$ is defined and has the natural structure of a graded R-module. The so-called 'diagonal' map $\Delta : X \to X \times X$ induces a map of graded R-modules

$$H^*(X \times X; R) \stackrel{\Delta^*}{\to} H^*(X; R), \tag{1.1}$$

which looks as though it's close to an algebra product. What is missing is a natural(Eilenberg-Zilber) map[1]

$$H^*(X; R) \otimes H^*(X; R) \to H^*(X \times X; R), \tag{1.2}$$

so that (1.1) and (1.2) compose to yield

$$H^*(X; R) \otimes H^*(X; R) \stackrel{\mu}{\to} H^*(X; R). \tag{1.3}$$

One verifies that μ endows cohomology with the structure of a graded, commutative R-algebra. To define μ, the ring structure of R is needed since the values of cocycles get multiplied. (We shall not go into these details in this course, but the interested reader can consult *Hatcher* or *MacLane*.) It is noteworthy that the ring structure for de Rham cohomology has a much easier and more direct description, as we'll see later.

The added R-algebra structure on cohomology makes it a finer tool than homology for

[1]Actually, the Eilenberg-Zilber maps are defined on the chain-complex level and involve the product of possibly distinct spaces $X \times Y$. These induce maps on cohomology.

detecting differences between spaces. For example, for any non-trivial commutative ring R,

$$H^*(S^2 \vee S^3 \vee S^6; R) \quad \text{and} \quad H^*(S^2 \times S^3; R)$$

are isomorphic *as graded R-modules* but are not isomorphic as graded R-algebras. (As graded R-modules, both are free on single generators in dimensions $2, 3$, and 6, say, x, y, and z, respectively, but in the former $xy = 0$, whereas in the latter $xy = \pm z$.)

1.2.4 Cohomology and obstructions to extensions

Another virtue of cohomology is that it arises naturally in connection with extension problems. As pointed out long ago by N. Steenrod, most problems in algebraic topology can be formulated as extension problems. That is, in simplest terms, one is given a (reasonable) pair of spaces (X, A) and a (continuous) map $f : A \to Y$, which one seeks to extend to a map $X \to Y$. That this is generally *not* possible is the basic fact underlying the development of most of algebraic topology, which can be regarded as the machinery that measures how hard such extension problems are to solve. Note that if $g : W \to X$ is any map, and we set $B = g^{-1}(A)$, then g induces another extension problem for the pair (W, B) and the map $f(g|B) : B \to Y$. Clearly a solution to the first problem yields a solution to the second. But even if the first cannot be solved, we may still wish to measure the extent of this failure—the "obstruction" to extension— and relate that to the analogous obstruction for the second problem. Whatever "obstruction" means here, it is clear that it should be contravariantly natural. Thus, if any of the usual algebraic-topological invariants comes into play here, it should be something like cohomology.

Indeed, the notion of a *primary obstruction* to the extension problem above can be defined precisely as an element of the cohomology of the pair (X, A) with coefficients in a certain homotopy group of Y. A notion of secondary obstruction can also be defined. A description of these ideas would take us too far afield here, but the interested reader is referred to a readable exposition of obstruction theory in *Factorization and Induced Homomorphisms*, Advances in Mathematics, January 1969, by Paul Olum.

1.2.5 Cohomology and Stokes's Theorem

Let M be a smooth manifold, which is smoothly triangulated with all simplexes oriented. Here, we assume known the notions of smooth differential form and integral of a k-form ω over a simplical k-chain c,

$$\int_c \omega.$$

Then, Stokes's Theorem says:

$$\int_c d\omega = \int_{\partial c} \omega. \tag{1.4}$$

We'll discuss this rigorously in the smooth context later. Here, we interpret it as follows. The rule $c \mapsto \int_c \omega$ gives an element of $Hom(C_k(M), \mathbb{R})$, where C_k denotes simplicial k-chains. We call this element $\underline{\omega}$. Then (1.4) becomes

$$\underline{d(\omega)}(c) = \underline{\omega}(\partial c), \tag{1.5}$$

i.e., $\underline{d(\omega)}$ is just $\delta(\underline{\omega})$, where δ is the adjoint of ∂ (as in the case of singular chains and cochains).

So cohomology arises naturally when looking at smooth forms, i.e., the rule $\omega \mapsto \underline{\omega}$ gives a natural map from the smooth k-forms on M to the $\mathbb{R}$-valued simplicial k-cochains on M, which intertwines d with δ. (A map to singular k-cochains can be defined similarly.) As we shall see in the next chapter, the smooth forms on M, together with d, form a commutative, DGA. Therefore, without further ado — that is, without an Eilenberg-Zilber Theorem or a Künneth Theorem—its homology inherits a commutative algebra structure, as described earlier. It is known as the *de Rham cohomology* of M. One result proved in this course is that the cited map from de Rham cohomology to simplicial or singular cohomology (with real coefficients) induces an isomorphism of cohomology algebras (de Rham's Theorem).

Chapter 2

The de Rham Complex

This chapter gives a summary of requisite definitions and properties but does not give all the proofs. The reader is encouraged to supply details.

2.1 Tangent vectors on $\mathbb{R}^n$

$\mathbb{R}^n$is the real vector space of real n-tuples $u = (u_1, \ldots, u_n)$, and $C^\infty(\mathbb{R}^n)$ is the $\mathbb{R}$-algebra of smooth ($= C^\infty$) functions on $\mathbb{R}^n$. Given $u, v \in \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$, the *derivative of f at u in direction v* is defined by

$$Df(u)(v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}. \tag{2.1}$$

It determines an $\mathbb{R}$-linear map

$$\partial_u(v) : C^\infty(\mathbb{R}^n) \to \mathbb{R} \tag{2.2}$$

which satisfies a so-called *derivation identity*

$$\partial_u(v)(fg) = g(u)\partial_u(v)(f) + f(u)\partial_u(v)(g). \tag{2.3}$$

If $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$, then we may denote $\partial_u(e_i)(f)$ by $\partial_i f(u)$, or, if u is understood, simply by $\partial_i f$.

Any $\mathbb{R}$-linear map $X_u : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ that satisfies (2.3) is called a *tangent vector at u*, and the set of all such is called the *tangent space* to $\mathbb{R}^n$ at u, denoted $T_u\mathbb{R}^n$. Obviously, the set of tangent vectors is closed under real scalar multiplication and addition, and so $T_u\mathbb{R}^n$ is a real vector space.

A function $X \to \bigcup_u T_u\mathbb{R}^n$ satisfying $X(u) \in T_u\mathbb{R}^n$ is called a *vector field* on $\mathbb{R}^n$. For every $f \in C^\infty(\mathbb{R}^n)$, X determines a function $Xf : \mathbb{R}^n \to \mathbb{R}$ by the rule $u \mapsto X(u)(f)$. We say that

X is smooth if Xf is smooth for each such f. We denote the set of all smooth vector fields on $\mathbb{R}^n$ by $\mathcal{X}(\mathbb{R}^n)$. Clearly this has a natural structure as a $C^\infty(\mathbb{R}^n)$-module.

Since (2.1) is well-defined for any smooth f defined in a neighborhood of u, the foregoing definitions apply to any open set $U \subseteq \mathbb{R}^n$ and $f \in C^\infty(U)$, yielding T_uU and $\mathcal{X}(U)$. If $i : U \to V$ is any inclusion of open sets, it induces a restriction map by right-composition, $i^* : C^\infty(V) \to C^\infty(U)$, hence a 'dual' $\mathbb{R}$-linear map $i_* : T_uU \to T_uV$, for each $u \in U$.

Lemma 2.1.1. *Let B and C be concentric open balls in $\mathbb{R}^n$ such that $\overline{B} \subseteq C$. Then, there exists a smooth function $b : \mathbb{R}^n \to [0, 1]$ such that $\overline{B} = b^{-1}(1)$ and $\mathbb{R}^n \setminus C = b^{-1}(0)$.* □

Corollary 2.1.2. *Let U be an open subset of $\mathbb{R}^n$ and $u \in U$. If $f, g \in C^\infty(U)$ coincide near u, then $X_uf = X_ug$, for any tangent vector $X_u \in T_uU$.*

Proof: It suffices to prove the result when $g = 0$. Choose balls B, C and a function b as in the lemma, such that $u \in B$, $C \subseteq U$, and $f = 0$ on C. Then bf is identically zero, so that $0 = X_u(bf) = f(u)X_ub + b(u)X_uf = X_uf$, as desired. □

The key property that makes the foregoing argument work is the derivation property. We shall have occasion to apply this kind of argument again (cf. 2.3.2).

Corollary 2.1.3. *Given any inclusion of open sets $i : U \to V$ and $u \in U$, $i_* : T_uU \to T_uV$ is an isomorphism.*

Proof: Given any $f \in C^\infty U$, we can find, with the aid of Lemma 2.1.1, a function $\hat{f} \in C^\infty(V)$ which coincides with f near u. This leads immediately to the bijectivity of i_* as follows: (a) Suppose $i_*(X_u) = 0$ and $f \in C^\infty(U)$. Since $\hat{f}|U$ coincides with f near u, we have, by Corollary 2.1.2, $X_uf = X_u(\hat{f}|U) = i_*(X_u)\hat{f} = 0$. Thus, $X_u = 0$. (b) Choose any $Y_u \in T_uV$. For any $f \in C^\infty U$, define $X_uf = Y_u\hat{f}$. This is well-defined by Corollary 2.1.2, and one easily verifies that it defines an element of T_uU. Now choose an arbitrary $g \in C^\infty V$. Since g coincides with $\widehat{g|U}$ near u, we have $i_*(X_u)g = X_u(g|U) = Y_u(\widehat{g|U}) = Y_ug$, so that $i_*X_u = Y_u$. □

Lemma 2.1.4. *Choose any $f \in C^\infty(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. Then, there exist functions $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that*

a. $f(x) = f(u) + (x_1 - u_1)g_1(x) + \ldots + (x_n - u_n)g_n(x)$, *for all* $x \in \mathbb{R}^n$.

b. For each $i = 1, \ldots, n$, $g_i(u) = \partial_i f(u)$. □

Lemma 2.1.5. *For each open $U \subseteq \mathbb{R}^n$ and $u \in U$, the rule $v \mapsto \partial_u(v)$ defines an isomorphism $\mathbb{R}^n \to T_uU$ which is compatible with the inclusion isomorphisms.*

Proof: Clearly the given rule defines an $\mathbb{R}$-linear map compatible with the inclusion-induced isomorphisms. Therefore, it suffices to verify that this map is bijective when $U = \mathbb{R}^n$. Given any X_u, apply it to the equation in Lemma 2.1.4 to obtain a unique representation $X_u = a_1\partial_1 + \ldots + a_n\partial_n$, where $a_i = Xx_i$ (x_i the usual i^{th} coordinate function on $\mathbb{R}^n$). □

This result shows that the tangent vectors $\partial_1(u), \ldots, \partial_n(u)$ form a basis of T_uU. We often omit the u, as in the above proof. We may refer to ∂_i by the more standard

$$\frac{\partial}{\partial x_i}\Big|_u,$$

or simply

$$\frac{\partial}{\partial x_i}.$$

Thus, for $X_u \in T_uU$, there is a unique real, linear combination

$$X_u = \sum_{i=1}^{n} a_i(u)\frac{\partial}{\partial x_i}. \tag{2.4}$$

Exercise 2. Let $(v_1, \ldots, v_n)$ be an ordered basis of $\mathbb{R}^n$ and, accordingly, write

$$X(u) = \sum_{i=1}^{n} b_i(u)\partial_u(v_i).$$

Show that X is smooth if and only if each $b_i(u)$ is a smooth function of u. Find an expression for each b_i in terms of the a_j in (2.4).

2.2 Forms on $\mathbb{R}^n$

Let U be an open subset of $\mathbb{R}^n$, and let V be the vector space $Hom_{\mathbb{R}}(T_uU, \mathbb{R}) = T_u^*U$. We recall the exterior algebra $\bigwedge^*(V)$ defined in Chapter 1. If $(dx_1, \ldots, dx_n)$ is the ordered basis of T_u^*U dual to the standard basis $(\partial_1, \ldots, \partial_n)$, then

$$\{dx_{i_1} \wedge \ldots \wedge dx_{i_k} | 1 \le i_1 < \ldots < i_k \le n\}$$

is a basis of $\bigwedge^k T_u^*U$ (cf. Section 1.2.2). We set $I = (i_1, \ldots, i_k)$ and use this *multi-index* to abbreviate $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ as dx_I. It is sometimes convenient to let $|I| = k$. Accordingly, any element $\alpha \in \bigwedge^k T_u^*U$ can be written uniquely as

$$\sum_{|I|=k} a_I dx_I, \tag{2.5}$$

with $a_I \in \mathbb{R}$ and I ranging over all multi-indices with $1 \leq i_1 < \ldots < i_k \leq n$.

A k-form on U is a function $\alpha : U \to \bigcup_u \bigwedge^k T_u^* U$ such that $\alpha(u) \in \bigwedge^k T_u^* U$, for each $u \in U$. We sometimes write α_u instead of $\alpha(u)$. Thus, $\alpha(u)$ may be written uniquely as in 2.5. We say that it is *smooth* if each a_I is a smooth function of u.

Exercise 3. Let α be a k-form on U. Use the canonical isomorphisms in Exercise 1 to recognize $\alpha(u)$ as an alternating k-linear function on $T_u U$, so that $\alpha(u)(X_1(u), \ldots, X_k(u))$ is a well-defined function of u for all vector fields $X_1, \ldots, X_k$ on U. Show that α is smooth if and only if this function is smooth for all *smooth* $X_1, \ldots, X_k$.

Let $\Omega^k(U)$ denote the $C^\infty(U)$-module of smooth k-forms on U, and let $\Omega^*(U)$ denote the corresponding graded $C^\infty(U)$-module. Wedge-product, pointwise defined, makes this into a graded, commutative algebra over $C^\infty(U)$. Indeed, the representation (2.5) shows that this is an exterior algebra over $C^\infty(U)$. (Note: Later, when we extend these definitions to manifolds, this last assertion is no longer true.) If $i : U \to V$ denotes the inclusion map of open sets, then it induces the restriction map $i^* : \Omega^*(V) \to \Omega^*(U)$, a homomorphism of commutative algebras "over" the homomorphism $i^* : C^\infty(V) \to C^\infty(U)$.

Exercise 4. Fix any $u_0 \in U$. Find a canonical isomorphism of commutative algebras

$$C^\infty(U) \otimes_{\mathbb{R}} \bigwedge^* (T_{u_0}^* U) \approx \Omega^*(U).$$

2.3 Exterior derivative

Let U be any open subset of $\mathbb{R}^n$. We define a degree-one $\mathbb{R}$-linear map

$$d = d_U : \Omega^*(U) \to \Omega^*(U)$$

with the following properties:

a. For $f \in \Omega^0(U) = C^\infty(U)$ and $X_u \in T_u U$,

$$df(u)(X_u) = X_u(f).$$

b. For any multi-index I,

$$d(a_I dx_I) = da_I \wedge dx_I.$$

c. For any smooth p-form α and any smooth form β,

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^p \alpha \wedge d(\beta).$$

d. $d^2 = 0$.

Exercise 5. Let $\pi_i : U \to \mathbb{R}$ denote the i^{th} coordinate projection. Verify that $d\pi_i$ equals the smooth 1-form dx_i defined in §2.2 above. Henceforth, we use x_i instead of π_i.

Indeed, to define d, use property (a) to define it for 0-forms, and then use property (b), together with the unique representation (2.5) to define it for all smooth k-forms, $k > 0$. The verification of properties (c) and (d) is left to the reader. The differential, graded $\mathbb{R}$-algebra $(\Omega^*(U), d)$ is called the *de Rham complex* of U.

Lemma 2.3.1. *If an $\mathbb{R}$-linear, degree-one map $d' : \Omega^*(U) \to \Omega^*(U)$ satisfies properties (a),(c), (d), then $d' = d$.*

Proof: By property (a), $d'x_i(\partial_j) = \partial_j x_i = \delta_{ij}$, so $d'x_i = dx_i$. By property (d), $d'(dx_i) = (d')^2(x_i) = 0$, and, by property (c), applied inductively, $d'(dx_I) = 0$, for all I. Therefore, by properties (a) and (c), $d'(a_I dx_I) = d(a_I dx_I)$. So, $d' = d$. □

The proof of the next lemma uses Lemma 2.1.1 similarly to the way it is used in the proof of Corollary 2.1.2. We leave details to the reader.

Lemma 2.3.2. *If the smooth form ω vanishes near $u \in U$, then so does $d\omega$.* □

The point u in this lemma is an interior point of the zero set of ω. The complement of this set of interior points is called the *support* of ω. The lemma may be paraphrased to say: "d decreases supports".

Corollary 2.3.3. *Let $i : U \to V$ be an inclusion of open sets in $\mathbb{R}^n$, and choose any $\omega \in \Omega^k(V)$. Then,*

$$i^* d_V \omega = d_U(i^*\omega).$$

Hint for Proof: This is proved similarly to Corollary 2.1.3. Given any $\lambda \in \Omega^k(U)$ and $u \in U$, show that there exists a $\lambda_u \in \Omega^k(V)$ which coincides with λ near u. Set $d'\lambda(u) = i^* d_V \lambda_u(u)$, and show that d' is well-defined and satisfies properties (a),(c),(d), above. So, $d' = d_U$. Apply this when $\lambda = i^*\omega$. □

We began this section with a fixed open set U and defined a differential d_U on $\Omega^*(U)$, ostensibly depending on U. Corollary 2.3.3 shows that the various d_U are compatible with restriction, so we may omit the subscript U. For any U, we refer to d as the *exterior derivative* on U.

2.4 Induced maps

Let U be an open subset of $\mathbb{R}^m$ and V an open subset of $\mathbb{R}^n$, and let $h : U \to V$ be a smooth map. The derivative Dh may be defined just as in (2.1). Right-composition with h induces an algebra homomorphism

$$h^* : C^\infty(V) \to C^\infty(U),$$

just as in the special case when h is an open inclusion. Further right-composition with h^* induces, for each $u \in U$, a linear transformation

$$h_* : T_u U \to T_{h(u)} V,$$

and further right-composition with this induces an adjoint map

$$h^* : T^*_{h(u)} V \to T^*_u U.$$

This last extends, via the universal property, to a map of exterior algebras

$$\bigwedge^* T^*_{h(u)} V \to \bigwedge^* T^*_u U,$$

again denoted h^*. Finally, these last maps piece together, over all $u \in U$, to yield a map of commutative algebras,

$$h^* : \Omega^*(V) \to \Omega^*(U).$$

Admittedly these conventions sorely overuse the symbol h^*, but the context will indicate the correct interpretation. The symbol upper * is used to indicate the contravariant feature of these maps.

Exercise 6. Recall the canonical isomorphisms $\partial_u : \mathbb{R}^m \to T_u U$ defined in Lemma 2.1.5. Verify that, for each $u \in U$,

$$\partial_{h(u)} \circ Dh(u) = h_* \circ \partial_u.$$

This gives an identification of $h^* : T^*_{h(u)} V \to T^*_u U$ with the adjoint $Dh(u)^* : (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$.

The definitions make clear that these induced maps have the following properties:

- If $h : U \to V$ and $k : V \to W$ are smooth maps, then $(kh)_* = k_* h_*$ and $(kh)^* = h^* k^*$.
- $(id_U)_*$ and $(id_U)^*$ are identity maps.

Note that it follows from these functorial properties that a smooth map h with a smooth inverse (i.e., a diffeomorphism) induces isomorphisms h_* and h^*.

In particular, the rule $U, h \mapsto \Omega^*(U), h^*$ defines a contravariant functor from the category of open subsets of the Euclidean spaces and their smooth maps to the category of graded, commutative algebras and their (degree-0) homomorphisms.

One additional piece of structure needs to be included here: namely, the fact that induced maps commute with exterior derivatives.

Exercise 7. Prove this last assertion.

Thus, the target category above can be taken to be the category of graded, commutative, differential algebras and their homomorphisms. Note that it follows immediately that diffeomorphic open sets have isomorphic de Rham complexes.

2.5 Smooth manifolds

A k-chart (U, h) in a topological space X consists of an open subset U of X and a homeomorphism $h : U \to V$, V open in $\mathbb{R}^k$. A $k-$ *atlas* for X is a collection [1] $\{(U_\alpha, h_\alpha)\}$ of k-charts with $\bigcup\limits_\alpha U_\alpha = X$. It is called *smooth* if all of the *transition mappings*

$$h_{\alpha\beta} = h_\beta h_\alpha^{-1}| : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$$

are smooth. Let $\mathcal{A}$ be a smooth k-atlas. If $\mathcal{A} = \mathcal{A} \cup \{(U, h)\}$ for any chart (U, h) such that $\mathcal{A} \cup \{(U, h)\}$ is a smooth atlas, then we say that $\mathcal{A}$ is *maximal* and call it a *differentiable structure*, or *smoothness structure*, or *smoothing* for (or of) X. Note that every smooth atlas is contained in a unique smoothness structure. Note also that, in general, a space X may have a large number of distinct smoothness structures. When a smoothness structure $\mathcal{S}$ for X is given, then we call the charts in $\mathcal{S}$ *smooth charts*.

In these notes, a *smooth k-manifold* is a 2^{nd}-countable, Hausdorff space M, together with a smoothing $\mathcal{S}$ for M consisting of k-charts. Usually when given a smooth manifold $M, \mathcal{S}$ we omit explicit mention of $\mathcal{S}$, and we reserve the expression *smooth subatlas on* M for an atlas for M contained in $\mathcal{S}$. k is called the dimension of M.

If $N, \mathcal{T}$ is a smooth ℓ-manifold, then a continuous map $f : M \to N$ is called *smooth* if, for every $(U, g) \in \mathcal{S}$ and $(V, h) \in \mathcal{T}$, the composition

$$g(U \cap f^{-1}(V)) \xrightarrow{g^{-1}} U \cap f^{-1}(V) \xrightarrow{f} V \xrightarrow{h} h(V)$$

is smooth, whenever and wherever defined. Notice that the same family of smooth maps is obtained if, in this definition, we replace $\mathcal{S}$ and $\mathcal{T}$ by arbitrary smooth subatlases. This fact can be useful, for example, when verifying that a map is smooth. In particular, the standard

[1] The indices α, β, etc., are used for notational convenience and are in $1-1$ correspondence with the members of the collection.

smoothing of $\mathbb{R}^n$ is the one determined by the singleton atlas $\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$, and we usually use the latter when dealing with questions involving smooth mappings into or out of $\mathbb{R}^n$.

We can now proceed just as we did for open subsets of Euclidean space to define tangent vectors and forms for smooth manifolds in general. Thus, the set of all smooth functions $M \to \mathbb{R}$ forms an $\mathbb{R}$-algebra, which we denote by $C^\infty(M)$, and a smooth map $h : M \to N$ induces an algebra homomorphism $h^* : C^\infty(N) \to C^\infty(M)$ just as before. All the associated structures that derive from the smooth functions (tangent spaces, smooth forms, smooth vector fields, and the corresponding induced maps) are all defined just as before and have analogous properties and notation. Note that for the definitions of smoothness, we use the characterization that does not refer to local representations: thus, a vector field X is smooth if Xf is a smooth function for each smooth f, and analogously for forms (cf. Exercise 3). There are two points, however, that require some special attention: namely, the dimension of the tangent spaces T_pM, for $p \in M$, and the definition of the exterior derivative $d_M = d$.

First, we deal with the tangent spaces T_pM. When U is an open subset of M, then any smoothing of M induces a smoothing of U in the obvious way and, with respect to this smoothing, the inclusion $i : U \to M$ is smooth. The following result is proved in the same way as Corollary 2.1.3:

Lemma 2.5.1. *The inclusion $i : U \to M$ induces an isomorphism $i_* : T_pU \to T_pM$, for every $p \in U$.* $\square$

Now suppose that (U, h) is a smooth chart. Then $h : U \to h(U)$ is a diffeomorphism. and so it induces an isomorphism $h_* : T_pU \to T_{h(p)}h(U)$, for every $p \in U$. Combining this with Lemma 2.5.1 and Lemma 2.1.5, we may conclude that T_pM is a k-dimensional vector space, for all $p \in M$.

We shall define the exterior derivative by defining it locally and then patching together. For this to work, we need some information about the restriction of forms. Let $i : U \to M$ be the smooth inclusion used above. As indicated above, it induces a homomorphism $i^* : \Omega^*(M) \to \Omega^*(U)$ just as before, which we often call a restriction homomorphism. Given any smooth form ω on M, it will often be convenient to denote its image under i^* by $\omega|U$. Clearly, if V is an open subset of U, then $\omega|V = (\omega|U)|V$.

Exercise 8. Suppose that $\mathcal{U}$ is an open cover of M and that for each $U \in \mathcal{U}$ there is a form $\omega_U \in \Omega^k(U)$ such that for any V in $\mathcal{U}$, $\omega_U|U \cap V = \omega_V|U \cap V$. Prove that there exists a unique smooth k-form ω on M such that $\omega|U = \omega_U$ for every $U \in \mathcal{U}$

This exercise shows that smooth forms are determined by their local properties and may be constructed via suitable local definitions. We often have occasion to analyze a smooth form ω by looking at its restrictions to charts $\omega|U$.

To define the exterior derivative for M, we make use of the isomorphism h_* defined for every smooth chart (U, h), more precisely the associated isomorphism $h^* : \Omega^*(h(U)) \to \Omega^*(U)$.

Exercise 9. a. If d is the exterior derivative on $h(U)$, define a degree-one, $\mathbb{R}$-linear map d_U on $\Omega^*(U)$ by conjugating with h^* : $d_U = h^* d(h^*)^{-1}$. Verify that d_U satisfies properties (a), (c), (d) of Section 2.3 and that it is characterized by these properties.

In particular, then, for U given, d_U is independent of the choice of smooth chart (U, h).

b. Verify the analogue of Lemma 2.3.2 for d_U.

c. Verify the analogue of Corollary 2.3.3 for any inclusion $i : U \to V$, where (U, h) and (V, g) are smooth charts for some h and g. That is, conclude that, in this situation, $d_U i^* = i^* d_V$.

Now suppose that $\omega \in \Omega^k(M)$, and let (V, g) be any other smooth chart such that $U \cap V \neq \emptyset$.

Exercise 10. Verify that

$$d_U(\omega|U)|U \cap V = d_V(\omega|V)|U \cap V = d_{U \cap V}(\omega|U \cap V).$$

We can now define d_M as follows. Given $\omega \in \Omega^k(M)$ and $p \in M$, choose any smooth chart (U, h) with $p \in U$. Define $d_M\omega(p) = d_U(\omega|U)(p)$.

Exercise 11. Verify:

a. d_M is a well-defined, degree-one $\mathbb{R}$-linear map $\Omega^*(M) \to \Omega^*(M)$.

b. d_M satisfies properties (a), (c), (d) of Section 2.3 and is characterized by these properties.

Because of the canonical nature of d_M, we usually omit the subscript.

Finally, we verify just as before that induced homomorphisms of de Rham algebras commute with exterior derivatives.

Exercise 12. Let U be an open subset of M. An indexed collection $\xi_i \in C^\infty(U)$, $i = 1, \ldots, k$ is called a set of *local coordinates* for M on U if the map $h : U \to \mathbb{R}^k$ given by $p \mapsto (\xi_1(p), \ldots, \xi_k(p))$ defines a diffeomorphism onto an open subset of $\mathbb{R}^k$, i.e., (U, h) is a smooth chart on M. Suppose this is the case.

a. Show that $d\xi_i = h^* dx_i$, $i = 1, \ldots, k$. Thus, $\Omega^*(U)$ is an exterior algebra over $C^\infty(U)$ with generators $d\xi_1, \ldots, d\xi_k$. Using multi-index notation as before, it follows that $\Omega^m(U)$ is a *free* $C^\infty(U)$ module on the basis $\{d\xi_I| \quad |I| = m\}$. In general, this is not true of $\Omega^m(M)$.

b. †[2] Show that $\Omega^m(M)$ is a projective $C^\infty(M)$-module.

An important, virtually verbatim extension of the foregoing discussion on smooth manifolds is obtained when the model Euclidean spaces $\mathbb{R}^k$ are replaced by the half-spaces $\mathbb{R}^k_+$, which we take to consist of all real k-tuples $u = (u_1, \ldots, u_k)$ for which $u_1 \geq 0$. All of the foregoing definitions, constructions, and properties still hold. The extension has, however, an important additional feature: namely, the notion of the *boundary* of a manifold.

Let M be a manifold modeled on the half-space $\mathbb{R}^k_+$. A point $p \in M$ is called a *boundary point* if it is contained in a smooth chart (U, h) such that $h(p) = 0$. The set of boundary points is called the *boundary* of M and will be denoted by bdM. All other points in M are called *interior* points of M, and the set of these, $intM$ is called the *interior* of M. We shall sometimes call the manifold M a *manifold with boundary*; to distinguish the earlier case, we may call those *manifolds without boundary* or *with empty boundary.*

Exercise 13. a. Let M be a manifold with boundary and $p \in M$. Show that $p \in bdM$ if and only if, for *every* smooth chart (U, h) containing p, $x_1(h(p)) = 0$. (Recall that x_1 is the first coordinate projection $\mathbb{R}^k \to \mathbb{R}$.) Conclude that bdM is a closed subset of M. We topologize it as a subset of M.

b. Let $\mathcal{A}$ be the smoothness structure of the manifold with boundary M. For each $(U, h) \in \mathcal{A}$, define $(\hat{U}, \hat{h})$ as follows: $\hat{U} = U \cap bdM$; if $h(p) = (\xi_1(p), \ldots, \xi_k(p))$, for all $p \in U$, let $\hat{h}(p) = (\xi_2(p), \ldots, \xi_k(p))$. Show that $\{(\hat{U}, \hat{h})| \quad (U, h) \in \mathcal{A}\}$ is a smooth atlas for bdM. This determines a smoothing for bdM which we call the *smoothing induced by* M. Henceforth, we always suppose that bdM is endowed with this induced smoothing.

c. Let $h : M \to N$ be a diffeomorphism of two manifolds with boundary. Prove that $h(bdM) = bdN$ and that $h|bdM$ induces a diffeomorphism $bdM \to bdN$.

d. Suppose that M is as above and that $bdM = \emptyset$. Show that for every smooth chart (U, h), $h(U) \subseteq \{u \in \mathbb{R}^k| \quad u_1 > 0\}$. Conclude that in this case the smoothing of M as a manifold with boundary determines a unique smoothing for M as a manifold without boundary. Thus, in this sense, manifolds without boundary are special cases of manifolds with boundary, and we always assume this in the following.

Exercise 14. a. Show that every connected manifold is path-connected.

b. † Suppose that M is a connected 1-manifold. Show that if $bdM = \emptyset$, then M is diffeomorphic to the unit circle S^1 or to $\mathbb{R}$. Show that if $bdM \neq \emptyset$, then M is diffeomorphic to a closed interval or to a half-closed interval. (If you have problems with this exercise

[2] An exercise with the † symbol is considered particularly challenging and will, in general, require some experience with material not in these notes.

and wish to read a nice proof, look at the Appendix in"Topology from the Differentiable Viewpoint" by John Milnor, The University Press of Virginia, 1965. This is a book well worth owning.)

2.6 De Rham cohomology

Let M be a smooth manifold. The cohomology $\mathbf{H}^*(\Omega^*(M), d) = ker(d)/im(d)$ is known as the *de Rham cohomology* of M and is usually denoted $H^*_{DR}(M)$, or simply $H^*(M)$ when there is no other kind of cohomology being considered and there is no danger of confusion. The elements of $ker(d)$, usually known as *cocycles* when dealing with cohomology, are called *closed forms* in this context. Elements of $im(d)$, usually called *coboundaries* are here called *exact forms*.

A smooth map $h : M \to N$ induces a map of de Rham cochain algebras, as already noted, and hence a (degree-zero) map of de Rham cohomology $H^*_{DR}(N) \to H^*_{DR}(M)$, which we again denote by the somewhat shopworn symbol h^*. The rule $M, h \mapsto \Omega^*(M), h^*$ is a contravariant functor, just as we noted earlier for the induced maps on de Rham complexes, from which it follows that diffeomorphic manifolds have isomorphic de Rham cohomology.

It follows immediately from the definitions that, for a k-manifold M, $H^i(M) = 0$, unless $0 \le i \le k$.

There is a useful and important variation on the foregoing, in which we restrict attention to forms with *compact support*. Recall that the support of a form ω on M consists of all points $p \in M$ for which $\omega_q \neq 0$, for some sequence of q converging to p. This is clearly always a closed set. Since the differential decreases support, $d\omega$ has compact support whenever ω does. Thus, if $\Omega^*(M)_c$ is the (graded) set of all forms of compact support, which is clearly a $C^\infty(M)$ submodule of $\Omega^*(M)$, then it is d-invariant, i.e., the restriction of d to $\Omega^*(M)_c$ (again called d) gives an $\mathbb{R}$-cochain complex whose cohomology is denoted $H^*_{DR,c}(M)$ or simply $H^*_c(M)$.

Exercise 15. Prove that $\Omega^*(M)_c$ is closed under wedge product and that $im(d) \cap \Omega^*(M)_c$ is an ideal in the algebra $ker(d) \cap \Omega^*(M)_c$. Thus, $H^*_{DR,c}(M)$ inherits the structure of a graded $\mathbb{R}$-algebra.

Note, however, that if $h : M \to N$ is an arbitrary smooth map and ω is a smooth form on N with compact support, it does not follow that $h^*\omega$ has compact support. For example, consider the trivial map $h : \mathbb{R}^1 \to \mathbb{R}^0$ and the 0-form $\omega = 1$ on $\mathbb{R}^0$, i.e., the constant function 1. Then $support(\omega) = \mathbb{R}^0$, which is compact, whereas $support(h^*\omega) = \mathbb{R}^1$, which is not. So, forms with compact support do not yield the same kind of cohomology functor as de Rham cohomology. Still, the two interact in important ways. We come back to this later.

Exercise 16. a. Suppose that M is any manifold. Show that $H^0(M)$ is isomorphic to the vector space of all locally-constant, real-valued functions on M, whereas $H_c^0(M)$ is isomorphic to the vector subspace spanned by the characteristic functions of the components of M.

b. Show that $H^i(\mathbb{R}) = \mathbb{R}, 0$ when $i = 0, 1$, respectively, and $H_c^i(\mathbb{R}) = 0, \mathbb{R}$, when $i = 0, 1$, respectively.