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Exterior Differential Systems and Cartan–Kähler Theory

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Abstract. We give an elementary introduction to exterior differential systems and the Cartan–Kähler theorem. No proofs are given, but the results are illustrated by means of examples.

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1. Introduction

The theory of exterior differential systems provides a geometric approach to differential equations which is particularly well suited to the study of problems arising in differential geometry and geometric mechanics. In the exterior differential systems setting, the solutions of a differential equation are constructed as mappings between manifolds pulling back a set of differential forms to zero.

The most general existence theorem for analytic exterior differential systems is the Cartan–Kähler theorem. This theorem is a generalization of the Cauchy–Kovalevskaia theorem which gives conditions for the existence of solutions. Furthermore it gives a local description of the degree of generality of the solutions in terms of arbitrary constants and arbitrary functions of a certain number of variables. There are also existence theorems for smooth exterior differential systems, such as the Frobenius theorem, which rely on properties of ordinary differential equations. These apply to a restricted, but nevertheless very important class of differential systems.

Our objective in this expository "tutorial" paper is to give an introduction to exterior differential systems, putting some emphasis on the Cartan–Kähler theorem. This paper is definitely not a survey, nor is it aimed at experts. It also makes no claims to originality and is devoid of any substantial proofs. On the contrary, the exposition is purposely kept at an elementary level, and illustrated by many examples. It is our hope that this paper will encourage the reader to learn more about the fascinating subject of exterior differential systems, and to further explore their geometric applications. Our main references are the classical treatise by Cartan [7], and the more recent book by Bryant, Chern, Gardner, Goldschmidt and Griffiths [3].

2. Exterior Differential Systems

Our purpose in this section is to review some of the basic definitions of the theory of exterior differential systems.

We consider an n-dimensional manifold M_n of class C^∞ and denote by

$$\Omega^*(M_n) = \bigoplus_{i=0}^{n} \Omega^i(M_n) \tag{1}$$

the graded $C^\infty(M_n; \mathbb{R})$ module of differential forms.

DEFINITION 1. An exterior differential system is a finitely generated ideal

$$\mathcal{I} \subset \Omega^*(M_n), \tag{2}$$

which is closed under the exterior differential d. An exterior differential system generated as a differential ideal by 1-forms is called a Pfaffian system.

If $\mathcal{I}$ is a Pfaffian system, then locally there exists an integer s and linearly independent 1-forms θ^a, $1 \leqslant a \leqslant s \leqslant n-1$, such that

$$\mathcal{I} = \{\theta^1, \ldots, \theta^s, \mathrm{d}\theta^1, \ldots, \mathrm{d}\theta^s\}, \tag{3}$$

where we use the brackets to denote the module closure of a set of differential forms. To a Pfaffian system $\mathcal{I}$ we can associate a submodule I of the $C^\infty(M_n; \mathbb{R})$-module of sections of T^*M_n, defined by

$$I = \{\theta^1, \ldots, \theta^s\} \subset \Omega^1(M_n). \tag{4}$$

The integer s is called the *dimension* of I.

DEFINITION 2. An *integral manifold of dimension m* of an exterior differential system $\mathcal{I}$ is a C^∞ immersion $h\colon W_m \to M_n$, such that

$$h^*\omega = 0, \quad \forall \omega \in \mathcal{I}. \tag{5}$$

The following example shows that locally the integral manifolds of a Pfaffian differential system correspond to solutions of systems of partial differential equations:

EXAMPLE 3. On $M \simeq \mathbb{R}^n$, consider the Pfaffian system (3), where

$$\theta^a = \sum_{i=1}^{n} A_i^a(x^1, \ldots, x^n)\,\mathrm{d}x^i, \quad 1 \leqslant a \leqslant s. \tag{6}$$

An immersion $f\colon U \subset \mathbb{R}^p \to \mathbb{R}^n$ given by

$$(u^1, \dots, u^p) \mapsto (x^1 = f^1(u^1, \dots, u^p), \dots, x^n = f^n(u^1, \dots, u^p)), \tag{7}$$

will be an integral manifold of $\mathcal{I}$ if and only if

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{p} A_i^a(f^1(u^1, \dots, u^p), \dots, f^n(u^1, \dots, u^p))\frac{\partial f^i}{\partial u^\alpha} = 0, \\ 1 \leqslant a \leqslant s,\ 1 \leqslant \alpha \leqslant p. \tag{8}$$

We now give some more examples, including one which is not Pfaffian:

EXAMPLE 4. Let $\mathbb{R}^4$ be endowed with coordinates (x, y, z, u), and let M be the open subset $\mathbb{R}^4$ defined by $u > 0$, $y > 0$, $x + z > 0$. On M consider the Pfaffian system $\mathcal{I}$ defined by

$$\mathcal{I} = \{\theta^1, \theta^2, \mathrm{d}\theta^1, \mathrm{d}\theta^2\}, \tag{9}$$

where

$$\theta^1 = u^2(x + z)(\mathrm{d}x + 2\,\mathrm{d}z), \qquad \theta^2 = y^4(\mathrm{d}y + u\,\mathrm{d}u). \tag{10}$$

The surfaces given as the intersection of the parabolic cylinders $2y + u^2 = c_1$ with the hyperplanes $x + 2z = c_2$ in M, where c_1 and c_2 are real constants, are the integral manifolds of maximal dimension of $\mathcal{I}$.

EXAMPLE 5. On $M = \mathbb{R}^3$ with coordinates (x, y, p), consider the Pfaffian system

$$\mathcal{I} = \{\mathrm{d}y - p\,\mathrm{d}x,\ \mathrm{d}p \wedge \mathrm{d}x\}. \tag{11}$$

The curves $h(t) = (x(t), y(t), p(t))$ on which $y' - px' \neq 0$ are integral manifolds of $\mathcal{I}$. An integral curve on which $x' \neq 0$ can be re-parametrized as a graph $h(t) = (t, f(t), f'(t))$, where f is an arbitrary function of one variable.

EXAMPLE 6. Let $M \simeq \mathbb{R}^{2n}$, let $(q^1, \dots, q^n, p^1, \dots, p^n)$ denote global coordinates on M, and consider the exterior differential system

$$\mathcal{I} = \left\{\Omega := \sum_{i=1}^{n} \mathrm{d}p^i \wedge \mathrm{d}q^i\right\}. \tag{12}$$

The maximal integral manifolds of $\mathcal{I}$ are of dimension n, and correspond to the Lagrangian submanifolds of $\mathbb{R}^{2n}$, viewed as a symplectic manifold when endowed with the symplectic form Ω. The Lagrangian submanifolds of (M, Ω) are locally parametrized by one smooth function of n variables. For example, the immersion $h_f(u^1, \dots, u^n) = (u^1, \dots, u^n, \frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^n})$ defines a Lagrangian submanifold of $\mathbb{R}^{2n}$ for any choice of the C^∞ function $f(u^1, \dots, u^n)$.

3. Some Existence Theorems for Integral Manifolds of Smooth Pfaffian Systems

Before discussing the Cartan–Kähler theorem, we first present some local existence theorems for integral manifolds of special Pfaffian systems of class C^∞. The simplest and most important such theorem is the Frobenius theorem:

THEOREM 7. *Let M_n be a C^∞ manifold and let*

$$\mathcal{I} = \{\theta^1, \dots, \theta^s, \mathrm{d}\theta^1, \dots, \mathrm{d}\theta^s\} \tag{13}$$

be a C^∞ Pfaffian system on M_n. If

$$\{\theta^1, \dots, \theta^s, \mathrm{d}\theta^1, \dots, \mathrm{d}\theta^s\} = \{\theta^1, \dots, \theta^s\}, \tag{14}$$

that is

$$\mathrm{d}\theta^a \wedge \theta^1 \wedge \dots \wedge \theta^s = 0, \quad 1 \leqslant a \leqslant s, \tag{15}$$

then there exist local coordinates $(u^1, \dots, u^n)$ such that

$$\mathcal{I} = \{\mathrm{d}u^1, \dots, \mathrm{d}u^s\}. \tag{16}$$

Pfaffian systems satisfying the Frobenius condition (15) are said to be *completely integrable*. The Frobenius theorem implies that the local integral manifolds of $\mathcal{I}$ of maximal dimension are the joint level sets $u^1 = c_1, \dots, u^s = c_s$ of the first integrals. In terms of differential equations, we can think of the solutions of a Frobenius system as being parametrized by arbitrary constants. This can be seen explicitly on the example (4), which is a Frobenius system.

The following theorem, known as the *Pfaff normal form*, provides a description of the integral manifolds of maximal dimension of a smooth Pfaffian system which is generated as a differential ideal by a single one-form ($s = 1$), and which is not completely integrable.

THEOREM 8. *Let M_n be a C^∞ manifold and let*

$$\mathcal{I} = \{\omega, \mathrm{d}\omega\} \tag{17}$$

be a C^∞ Pfaffian system on M_n. Suppose that there exists an integer $r \geqslant 0$ such that we have

$$(\mathrm{d}\omega)^r \wedge \omega \neq 0, \qquad (\mathrm{d}\omega)^{r+1} \wedge \omega = 0, \tag{18}$$

on M_n. Then there exist local coordinates $(x^1, \dots, x^r, z, p^1, \dots, p^r, u^{2r+2}, \dots, u^n)$ such that

$$\mathcal{I} = \left\{\mathrm{d}z - \sum_{i=1}^{r} p_i\, \mathrm{d}x^i, \sum_{i=1}^{r} \mathrm{d}p_i \wedge \mathrm{d}x^i\right\}. \tag{19}$$

The proof of the Pfaff normal form is based on the Frobenius theorem. The integral manifolds of maximal dimension of $\mathcal{I}$ are given locally by immersions $(u^1, \dots, u^r) \mapsto (x^1 = u^1, \dots, x^r = u^r, z = f, p_1 = \frac{\partial f}{\partial u^1}, \dots, p_r = \frac{\partial f}{\partial u^r})$. They are thus locally parametrized by one arbitrary C^∞ function of r variables. The Pfaffian system (5) is an example of an exterior differential system satisfying the Pfaff normal form.

The Goursat normal form theorem, which we now present, applies to a class of Pfaffian systems which are generated as a differential ideal by more than a single one-form, and which are not completely integrable.

THEOREM 9. *Let M_n be a C^∞ manifold and let*

$$\mathcal{I} = \{\omega^1, \dots, \omega^r, \mathrm{d}\omega^1, \dots, \mathrm{d}\omega^r\}, \tag{20}$$

be a Pfaffian system of class C^∞ on M_n. Suppose that there exist 1*-forms α and π, where α and π are not congruent to zero modulo $\mathcal{I}$, such that,*

$$\mathrm{d}\omega^1 \equiv \omega^2 \wedge \pi, \mathrm{mod}\{\omega^1\}, \tag{21}$$

$$\mathrm{d}\omega^2 \equiv \omega^3 \wedge \pi, \mathrm{mod}\{\omega^1, \omega^2\}, \tag{22}$$

$$\vdots$$

$$\mathrm{d}\omega^{r-1} \equiv \omega^r \wedge \pi, \mathrm{mod}\{\omega^1, \dots, \omega^{r-1}\}, \tag{23}$$

$$\mathrm{d}\omega^r \equiv \alpha \wedge \pi, \mathrm{mod}\{\omega^1, \dots \omega^r\}. \tag{24}$$

Then there exist local coordinates $(x, y, y^1, \dots, y^r)$ such that

$$\mathcal{I} = \{\mathrm{d}y - y^1\,\mathrm{d}x, \dots, \mathrm{d}y^{r-1} - y^r\,\mathrm{d}x, \mathrm{d}y^1 \wedge \mathrm{d}x, \dots, \mathrm{d}y^r \wedge \mathrm{d}x\}. \tag{25}$$

It is easily seen that the integral manifolds of maximal dimension of a system satisfying the Goursat normal form are locally parametrized by one arbitrary C^∞ function of one variable.

We now introduce the very important notion of the *derived flag* of a Pfaffian system. The derived flag measures the degree to which a Pfaffian system fails to be completely integrable. Let

$$\mathcal{I} = \{\theta^1, \dots, \theta^s, \mathrm{d}\theta^1, \dots, \mathrm{d}\theta^s\}, \tag{26}$$

be a Pfaffian system and let

$$I = \{\theta^1, \dots, \theta^s\} \subset \Omega^1(M_n) \tag{27}$$

denote the corresponding $C^\infty(M_n; \mathbb{R})$-module of sections of T^*M_n. Likewise, let $\{I\}$ be the ideal generated by I in the algebra $\Omega^*(M_n)$ of C^∞ differential forms on M_n. The exterior differential

$$\mathrm{d}\colon\ I \to \Omega^2(M_n), \tag{28}$$

induces a map

$$\delta\colon\ I \to \Omega^2(M_n)/(\{I\} \cap \Omega^2(M_n)). \tag{29}$$

The *first derived system* $I^{(1)}$ of I is defined as

$$I^{(1)} = \ker \delta. \tag{30}$$

It follows that $\mathcal{I}$ is completely integrable if and only if

$$I^{(1)} = I. \tag{31}$$

The derived flag of I is defined recursively by

$$\cdots \subset I^{(k)} \subset I^{(k-1)} \subset \cdots \subset I^{(1)} \subset I, \tag{32}$$

where

$$I^{(k)} = (I^{(k-1)})^{(1)}. \tag{33}$$

Since d and pull-backs commute, a diffeomorphism $f\colon\ M_n \to \tilde{M}_n$ such that

$$f^*\tilde{I} = I, \tag{34}$$

will satisfy

$$f^*\tilde{I}^{(k)} = I^{(k)}, \tag{35}$$

for all k. In other words the derived flag is a diffeomorphism invariant of Pfaffian systems. Note that if a Pfaffian system $\mathcal{I}$ satisfies the Goursat normal form conditions, then its derived flag is given by

$$I = \{\omega^1, \dots, \omega^r\}, \tag{36}$$

$$I^{(1)} = \{\omega^1, \dots, \omega^{r-1}\}, \tag{37}$$

$$\vdots$$

$$I^{(r-1)} = \{\omega^1\}, \tag{38}$$

$$I^{(r)} = \{0\}, \tag{39}$$

with dimensions

$$\dim I^{(k)} = r - k, \quad 0 \leqslant k \leqslant r. \tag{40}$$

The Goursat normal form has an interesting application to the following underdetermined ordinary differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \tag{41}$$

known as the *Hilbert–Cartan equation* [1]. This equation is obviously integrable by quadratures in terms of one arbitrary function of one variable, i.e.

$$v(x) = \int^x \left(\frac{\mathrm{d}^2 g}{\mathrm{d}t^2}\right)^2 \mathrm{d}t, \tag{42}$$

where g is arbitrary. One may ask if the integration sign in (42) is necessary, that is whether (41) admits solutions of the form

$$x = X(t, w(t), w'(t), \ldots, w^{(r)}(t)), \tag{43}$$

$$u = U(t, w(t), w'(t), \ldots, w^{(r)}(t)), \tag{44}$$

$$v = W(t, w(t), w'(t), \ldots, w^{(r)}(t)), \tag{45}$$

for some $r < \infty$, where $w(t)$ is an arbitrary function of class C^∞. Let us call these *parametric solutions of finite rank*. Hilbert [9] and Cartan [5] proved that this is impossible:

THEOREM 10. *The Hilbert–Cartan equation*

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right)^2, \tag{46}$$

does not admit parametric solutions of finite rank.

Cartan's proof is given in [5] and is based on the diffeomorphism invariance of the derived flag structure of a Pfaffian system of Goursat type. One first notes that an under-determined equation

$$v' = F(x, u, v, u', u'') \tag{47}$$

will have parametric solutions of finite rank if $r = 3$ and the Pfaffian system $\mathcal{I}$ determined by

$$I = \{\mathrm{d}u - u'\,\mathrm{d}x, \mathrm{d}u' - u''\,\mathrm{d}x, \mathrm{d}v - F\,\mathrm{d}x\}, \tag{48}$$

is a Goursat system with $r = 3$. Let us see how the structure of the derived flag of I depends on F. We have

$$I^{(1)} = \left\{\mathrm{d}u - u'\,\mathrm{d}x, \mathrm{d}v - \frac{\partial F}{\partial u''}\,\mathrm{d}u' - \left(F - u''\frac{\partial F}{\partial u''}\right)\mathrm{d}x\right\}, \tag{49}$$

and $\dim I^{(2)} = 1$ if and only if

$$\frac{\partial^2 F}{(\partial u'')^2} = 0. \tag{50}$$

This linearity condition is violated by the Hilbert–Cartan equation, and it therefore follows that it does not admit parametric solutions of finite rank. On the other hand, the ordinary differential equation

$$v' = u^m u'', \tag{51}$$

will admit parametric solutions of finite rank, [4], which can be given explicitly in terms of one arbitrary function of one variable and its derivatives of order one and two:

$$x(t) = -2tf''(t) - f'(t), \tag{52}$$

$$u(t) = [(m+1)^2 t^3 (f''(t))^2]^{\frac{1}{m+1}}, \tag{53}$$

$$v(t) = (m-1)t^2 f''(t) - mtf'(t) + mf(t). \tag{54}$$

4. Involutive Systems and the Cartan–Kähler Theorem

We are now ready to embark on the topic of the Cartan–Kähler theorem. As was pointed out in the introduction, this theorem applies only to analytic differential systems. Accordingly, all the exterior differential systems considered from now on will be assumed to be of class C^ω.

The Cartan–Kähler theorem gives necessary and sufficient conditions for a subspace of the tangent space at a point of M_n to be the tangent space of an integral manifold. It is well-known that exterior algebra provides a very convenient way of parametrizing the set of all subspaces of a given vector space, and we therefore begin with some simple preliminaries from exterior algebra.

Let V be an n-dimensional real vector space. The pairing between V and its dual V^* will be denoted by $\langle \cdot, \cdot \rangle$. There is an induced pairing between the exterior algebras $\bigwedge(V)$ and $\bigwedge(V^*)$, also denoted by $\langle \cdot, \cdot \rangle$, which is conveniently described in terms of bases. Let $\{e_1, \dots, e_n\}$ denote a basis of V and $\{e^{*1}, \dots, e^{*n}\}$ denote the dual basis of V^*. If

$$v = \frac{1}{p!} \sum_{1 \leqslant i_1 \dots i_p \leqslant n} v^{i_1 \dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \tag{55}$$

and

$$\phi = \frac{1}{p!} \sum_{1 \leqslant i_1 \dots i_p \leqslant n} \phi_{i_1 \dots i_p} e^{*i_1} \wedge \cdots \wedge e^{*i_p} \tag{56}$$

then we have

$$\langle v, \phi \rangle = \frac{1}{p!} \sum_{1 \leqslant i_1 \dots i_p \leqslant n} v^{i_1 \dots i_p} \phi_{i_1 \dots i_p}. \tag{57}$$

If on the other hand we have $v \in \bigwedge^p(V)$ and $\phi \in \bigwedge^q(V^*)$, with $p \neq q$, then

$$\langle v, \phi \rangle = 0. \tag{58}$$

Monomials in the exterior algebra of a vector space are very useful for the representation of subspaces. Recall that a k-dimensional subspace $W \subset V$ determines a decomposable k-vector $v_W = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k(V)$, where $\{v_1, \ldots, v_k\}$ is a basis of W. This monomial of degree k determines the subspace W, because we have $w \in W$ if and only if $w \wedge v_W = 0$. The correspondence between decomposable k-vectors and k-dimensional subspaces is not one-to-one, because two decomposable k-vectors which are non-zero multiples of each other represent the some subspace. This is taken care of by thinking of equivalence classes of nonzero k-vectors. We will thus denote by $[W] \in \mathbb{P}(\bigwedge^k(V))$ the equivalence class of any k-vector representing W.

Let $\mathcal{I}$ be an exterior differential system on a manifold M_n. If an immersion $h\colon W_m \to M_n$ is an integral manifold of $\mathcal{I}$, then

$$\langle h_* T_u W_m, \omega\rangle = 0, \tag{59}$$

for all $u \in W_m$ and $\omega \in \mathcal{I}$. This gives an obvious necessary condition for a subspace of the tangent space $T_x M_n$ to be tangent to an integral manifold:

DEFINITION 11. Let $x \in M_n$ and let $E^p \subset T_x M_n$ be a p-dimensional subspace of the tangent space to M_n at x. The pair (x, E^p) is a p-dimensional integral element of $\mathcal{I}$ if $\langle E^p, \omega|_x\rangle = 0$ for all $\omega \in \mathcal{I}$.

The principle of the Cartan–Kähler theorem is to construct integral manifolds of maximal dimension by extending lower-dimensional integral manifolds. This is first done infinitesimally by first trying to extend tangent spaces to integral manifolds into integral elements of higher dimension. To this effect, we need to introduce the concept of the polar space of an integral element.

DEFINITION 12. The polar space $H(E^p)$ of an integral element (x, E^p) of $\mathcal{I}$ is the set of all tangent vectors $v \in T_x M_n$ such that

$$\langle [E^p] \wedge v, \omega|_x\rangle = 0, \tag{60}$$

for all $\omega \in \mathcal{I}$.

Note that every integral element is contained in its own polar space,

$$E^p \subseteq H(E^p). \tag{61}$$

The rank of the polar equations of E_p leads to a numerical invariant $\sigma_{p+1} \geqslant -1$ defined by

$$\dim H(E^p) = p + 1 + \sigma_{p+1}. \tag{62}$$

We will be interested in flags of integral elements which are in general position. To this effect, we introduce the Grassmann bundle $\pi\colon G_p(M_n) \to M_n$, whose fiber at $x \in M_n$ is the Grassmann manifold of p-dimensional subspaces of $T_x M_n$, and consider the variety $\mathcal{V}_p(\mathcal{I}) \subset G_p(M_n)$ of p-dimensional integral elements of $\mathcal{I}$.

DEFINITION 13. An integral element (x, E^p) is said to be Kähler-regular if the rank of the polar equations of a p-dimensional integral element is constant in a neighborhood of (x, E^p), and if there exist, in a neighborhood of (x, E^p), s independent p-forms $\beta^1, \ldots, \beta^s$ such that $\mathcal{V}_p(\mathcal{I})$ is given in that neighborhood as the set of p-planes $[F^p]$ such that

$$\langle [F^p], \beta^1 \rangle = 0, \ldots, \langle [F^p], \beta^1 \rangle = 0. \tag{63}$$

It is clear from the above that the theory of exterior differential systems treats all local coordinate systems on the same footing, and in particular that it does not distinguish a set of independent variables. One can nevertheless introduce some measure of independence by prescribing a transversality condition which restricts the set of integral manifolds under consideration. We are thus led to define an *exterior differential system with independence condition* to be an exterior differential system $\mathcal{I}$ endowed with a decomposable p-form Ω which is not congruent to zero modulo $\mathcal{I}$. We have the following definition:

DEFINITION 14. An integral element of dimension p on which $\Omega \neq 0$ is called *admissible*. A p-dimensional integral element on which $\Omega \neq 0$ is called admissible.

The intuitive content of the notion of involutivity is that one should be able to construct admissible integral manifolds of an exterior differential system with independence condition one dimension at a time by successive applications of the Cauchy–Kovalevskaia theorem. Such a system should admit a p-dimensional admissible integral element which sits at the end of a flag of regular integral elements. This is precisely the content of the next definition:

DEFINITION 15. An admissible integral element (x, E^p) of an exterior differential system with independence condition $(\mathcal{I}, \Omega)$ is said to be ordinary if there exists a flag

$$E^1 \subset E^2 \subset \cdots \subset E^{p-1} \subset E^p, \tag{64}$$

such that each pair (x, E^i), $1 \leqslant i \leqslant p-1$ is a Kähler-regular integral element.

The following definition is extremely important, because it will lead through the Cartan–Kähler theorem to a precise characterization of those admissible integral elements of an exterior differential system with independence condition which are tangent to an integral manifold.

DEFINITION 16. An exterior differential system with independence condition $(\mathcal{I}, \Omega)$ is said to be in involution if there exists an ordinary integral element (x, E^p) through each $x \in M_n$.

With these definitions at hand, we are now ready to state the Cartan–Kähler theorem in its general form:

THEOREM 17. *If* $(\mathcal{I}, \Omega)$ *is in involution and* (x, E^p) *is an admissible integral element, then there exists an admissible integral manifold* $W_p \subset M_n$ *through* x*, such that* $T_x W_p = E^p$.

The statement of this theorem is powerful in its simplicity, but it is difficult to apply in practice if one does not have a manageable criterion for determining if an exterior differential system with independence condition is in involution. There is such a criterion, known as E. Cartan's involutivity test. In what follows we shall present this test in the case of quasi-linear Pfaffian systems, and refer the reader to [7] and [3] for a detailed treatment of the general case. Quasi-linear systems are more manageable, because the variety of admissible integral elements $\mathcal{V}_p(\mathcal{I}, \Omega)$ is an affine bundle over M_n.

We start from a Pfaffian system with independence condition $(\mathcal{I}, \Omega)$, where

$$\mathcal{I} = \{\theta^1, \dots, \theta^s, \mathrm{d}\theta^1, \dots, \mathrm{d}\theta^s\}, \tag{65}$$

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p, \tag{66}$$

and let $\pi^1, \dots, \pi^l$, $l = n - s - p$, be 1-forms such that $\theta^1 \wedge \cdots \wedge \theta^s \wedge \omega^1 \wedge \cdots \wedge \omega^p \wedge \cdots \wedge \pi^1 \wedge \cdots \wedge \pi^l \neq 0$. We have, for $1 \leqslant i \leqslant s$,

$$\mathrm{d}\theta^i \equiv \sum_{\alpha=1}^{l} \sum_{b=1}^{p} A^i_{\alpha b} \pi^\alpha \wedge \omega^b + \frac{1}{2} \sum_{a,b=1}^{p} B^i_{ab} \omega^a \wedge \omega^b + \frac{1}{2} \sum_{\alpha,\beta=1}^{l} C^i_{\alpha\beta} \pi^\alpha \wedge \pi^\beta \quad \operatorname{mod} \theta^1, \dots, \theta^s. \tag{67}$$

We assume that the condition

$$C^i_{\alpha\beta} = 0, \quad 1 \leqslant i \leqslant s,\ 1 \leqslant \alpha, \beta \leqslant l \tag{68}$$

is satisfied. Such Pfaffian systems are said to be *quasi-linear*. The admissible integral elements (x, E^p) can be taken in the form

$$\left.\left(\pi^\alpha - \sum_{b=1}^{p} t^\alpha_b \omega^b\right)\right|_{E^p} = 0, \tag{69}$$

where $1 \leqslant \alpha \leqslant l$, and the polar equations of E^p are given by

$$\sum_{\alpha=1}^{l} (A^i_{\alpha b} t^\alpha_c - A^i_{\alpha c} t^\alpha_b) + B^i_{bc} = 0, \tag{70}$$

where $1 \leqslant b, c \leqslant p$ and $1 \leqslant i \leqslant s$. These equations are indeed linear. We denote the dimension of their solution space by d. We now define the *reduced characters* $\sigma'_1, \dots, \sigma'_r$, $r \leqslant p$ of $(\mathcal{I}, \Omega)$ by

$$\sigma'_1 + \cdots + \sigma'_r = \max_{v_1 \dots v_r \in \mathbb{R}^l} \operatorname{rk} \begin{pmatrix} \sum_{\alpha=1}^{l} v^\alpha_1 A^i_{\alpha b} \\ \vdots \\ \sum_{\alpha=1}^{l} v^\alpha_r A^i_{\alpha b} \end{pmatrix}. \tag{71}$$

Necessary and sufficient conditions for involutivity are given by the following theorem, known as *Cartan's involutivity test*:

THEOREM 18. *We have*

$$d \leqslant \sum_{i=1}^{p} i\sigma_i', \tag{72}$$

with equality if and only if the system $(\mathcal{I}, \Omega)$ *is in involution. If* $\sigma_q' = k \neq 0$ *with* q *maximal, then the admissible local integral manifolds are parametrized by* k C^ω *functions of* q *variables.*

The remainder of this paper is devoted to illustration of Cartan's test on a series of examples. We start with a trivial example.

EXAMPLE 19. Consider on $\mathbb{R}^3$ with coordinates (x, u, p) the Pfaffian system with independence condition $(\mathcal{I}, \Omega)$ where

$$\mathcal{I} = \{\theta, \mathrm{d}\theta\}, \qquad \Omega = \mathrm{d}x, \tag{73}$$

and

$$\theta = \mathrm{d}u - p\,\mathrm{d}x. \tag{74}$$

We have

$$\mathrm{d}\theta = \mathrm{d}p \wedge \mathrm{d}x = \pi \wedge \Omega. \tag{75}$$

The admissible integral elements are of the form

$$\pi - t\Omega, \tag{76}$$

$$d = 1, \qquad \sigma_1' = 1. \tag{77}$$

The involutivity test is therefore satisfied and the admissible integral manifolds are one-dimensional, as expected. They are parametrized by one arbitrary function of one variable. Of course, we could have seen this immediately by direct inspection. The next example is less trivial. It is taken from [2].

EXAMPLE 20. We consider a scalar partial differential equation

$$F\left(x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 0, \quad 1 \leqslant i, j \leqslant p, \tag{78}$$

where F is assumed to be C^ω in all its arguments. We shall see using the Cartan–Kähler theorem that the local solutions of the partial differential equation (78) are parametrized by two analytic functions of $p - 1$ variables.

To the partial differential equation (78), we associate on $\mathbb{R}^{\frac{p(p+5)}{2}+1}$, with local coordinates (x^i, u, u_i, u_{ij}), $1 \leqslant i, j \leqslant p$, the zero set of the function F of the same arguments, and we assume that

$$\det\left(\frac{\partial F}{\partial u_{ij}}\right)\bigg|_{F=0} \neq 0. \tag{79}$$

The partial differential equation (78) is now reformulated as an exterior differential system with independence condition $(\mathcal{I}, \Omega)$, where $\mathcal{I}$ is the differential ideal of $\Omega^*(\mathbb{R}^{\frac{p(p+5)}{2}+1})$ generated by

$$F(x^i, u, u_i, u_{ij}) = 0, \qquad \theta_0 = \mathrm{d}u - \sum_{i=1}^{p} \mathrm{d}x^i, \tag{80}$$

$$\theta_i = \mathrm{d}u_i - \sum_{j=1}^{n} u_{ij}\,\mathrm{d}x^j, \quad 1 \leqslant i \leqslant p, \tag{81}$$

where

$$\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p. \tag{82}$$

The structure equations of $(\mathcal{I}, \Omega)$ are easily determined. We have

$$\mathrm{d}F = 0, \qquad \mathrm{d}\theta_0 \equiv 0, \qquad \mathrm{d}\theta_i \equiv \sum_{j=1}^{n} \pi_{ij} \wedge \mathrm{d}x^j, \quad \operatorname{mod} \theta_0, \ldots, \theta_p, \tag{83}$$

where $\pi_{ij} := -\mathrm{d}u_{ij}$. It is not difficult to compute the reduced characters of $(\mathcal{I}, \Omega)$ once we use the nondegeneracy condition (79) to put the above structure equations in normal form. Under a change of coframe of the form

$$\bar{\omega}^i = \sum_{j=1}^{p} a^i{}_j\,\mathrm{d}x^j, \qquad \bar{\theta}^i = \sum_{j=1}^{p} (a^{-1})_i{}^j\,\theta_j, \tag{84}$$

we have

$$\bar{\pi}_{ij} = -\sum_{k,l=1}^{p} \mathrm{d}u_{kl}\,a^k_{\ i} a^l{}_j. \tag{85}$$

Using the rank condition (79), we can locally choose $a^i{}_j$ so as to have

$$\sum_{i,j=1}^{p} \frac{\partial F}{\partial u_{ij}}\,(a^{-1})_i{}^k (a^{-1})_j^{\ l} = \delta_{ij}\varepsilon_i, \tag{86}$$

where $\varepsilon_i^2 = 1$. The structure equation $\mathrm{d}F = 0$ then takes the form

$$\sum_{i=1}^{p} \varepsilon_i \bar{\pi}_{ii} + \sum_{k=1}^{p} b_k \bar{\omega}^k \equiv 0, \quad \mathrm{mod}\, \bar{\theta}_0, \ldots, \bar{\theta}_p, \tag{87}$$

for some functions b_k, $1 \leqslant k \leqslant p$. We now define

$$\bar{\bar{\pi}}_{ii} = \bar{\pi}_{ii} + \varepsilon_i b_i \bar{\omega}_i, \qquad \bar{\bar{\pi}}_{ij} = \bar{\pi}_{ij}, \quad 1 \leqslant i \neq j \leqslant p. \tag{88}$$

Dropping bars, the structure equations (83) become

$$\mathrm{d}\theta_0 \equiv 0, \qquad \mathrm{d}\theta_i \equiv \sum_{j=1}^{p} \pi_{ij} \wedge \omega^j, \quad \mathrm{mod}\, \theta_0, \ldots, \theta_p, \tag{89}$$

where

$$\sum_{i=1}^{p} \varepsilon_i \pi_{ii} \equiv 0, \qquad \pi_{ij} \equiv \pi_{ji}, \quad \mathrm{mod}\, \theta_0, \ldots, \theta_p, \tag{90}$$

and where $1 \leqslant i, j \leqslant p$. The reduced characters are computed using (71). We have

$$\sigma_1' = p, \sigma_2' = p - 1, \ldots, \sigma_{p-1} = 2, \; \sigma_p = 0, \tag{91}$$

where the final drop in the value of the characters from 2 to 0 is due to the trace condition (90). We thus have

$$\sum_{i=1}^{p} i\sigma_i' = \frac{p(p+1)(p+2)}{6} - p. \tag{92}$$

In order to apply the involutivity test, we consider the admissible integral elements (x, E^p) of $(\mathcal{I}, \Omega)$, which are given by

$$\theta_0|_{E^p} = 0, \qquad \theta^i|_{E^p} = 0, \qquad \left.\left(\pi_{ij} - \sum_{k=1}^{p} L_{ijk}\omega^k\right)\right|_{E^p} = 0, \tag{93}$$

where $1 \leqslant i, j, k \leqslant p$, and where we have

$$L_{ijk} = L_{jik} = L_{ikj}, \qquad \sum_{i=1}^{p} \varepsilon_i L_{iik} = 0. \tag{94}$$

The dimension of the solution space of the polar equations of (x, E^p) is thus given by

$$d = \binom{p+2}{p-1} - p = \frac{p(p+1)(p+2)}{6} - p, \tag{95}$$

and the system is in involution, with top character $\sigma_{p-1}' = 2$. The local C^ω solutions are thus parametrized by 2 arbitrary functions of $p - 1$ variables, as claimed.

EXAMPLE 21. We now give our final example, which comes from the geometry of submanifolds and which is taken from [6]. This example is the starting point of the Laplace transformation of submanifolds with conjugate nets of curves [8, 10]. We start with a definition:

DEFINITION 22. A Riemannian manifold (M_n, g) isometrically immersed in $\mathbb{R}^{2n}$ is said to be a Cartan submanifold if the second-order osculating space of M_n is everywhere $2n$-dimensional, and if near every point $x \in M_n$ there exist local coordinates $(x^1, \ldots, x^n)$ such that the net of coordinate curves is conjugate.

This means that if $\mathbf{X}\colon U \subset \mathbb{R}^n \to \mathbb{R}^{2n}$ denotes the local expression of such an immersion in conjugate coordinates, then we have

$$\mathbf{X}_{ij} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \mathbf{X}_k + \sum_{\alpha=1}^{n} \Omega_{ij}^{\alpha} \mathbf{N}_\alpha, \tag{96}$$

where

$$\Omega_{ij}^{\alpha} = 0, \quad 1 \leqslant i \neq j \leqslant n,\ 1 \leqslant \alpha \leqslant n. \tag{97}$$

It is not immediately obvious that Cartan submanifolds of dimension greater than two, which are not flat, should exist at all. We give two examples taken from [10]. We first have the elementary example of the Clifford torus $\mathbb{T}^n \subset \mathbb{R}^{2n}$, given by

$$\mathbf{X}(x^1, \ldots, x^n) = (\cos x^1, \sin x^1, \ldots, \cos x^n, \sin x^n). \tag{98}$$

Another example which is less elementary, is a toroidal submanifold of $S^{2n-1} \subset \mathbb{R}^{2n}$ which is described as follows. Let $c_3, \ldots, c_n$ be real numbers such that

$$\sum_{j=3}^{n} c_j^2 < 1, \quad c_j \neq 0, \tag{99}$$

let

$$\lambda := \left(1 - \sum_{j=3}^{n} c_j^2\right)^{1/2}, \tag{100}$$

and let r and μ be nonzero real numbers such that $r^2 = \mu^2 + 1$. The immersion

$$\begin{aligned}\mathbf{X} \;=\; & (\lambda f_0, \lambda f_1, \lambda f_2 \cos x^2, \lambda f_2 \sin x^2, c_3 \cos x^3, c_3 \sin x^3, \ldots, \\ & c_n \cos x^n, c_n \sin x^n),\end{aligned} \tag{101}$$

where

$$0 < x^1 < \frac{\pi}{2\mu}, \quad 0 < x^j < 2\pi,\ 2 \leqslant j \leqslant n, \tag{102}$$

and where

$$f_0 = \frac{\mu}{r}\sin rx^1 \cos\mu x^1 - \cos rx^1 \sin\mu x^1, \tag{103}$$

$$f_1 = \frac{\mu}{r}\cos rx^1 \cos\mu x^1 - \sin rx^1 \sin\mu x^1, \tag{104}$$

$$f_2 = -\frac{1}{r}\cos\mu x^1, \tag{105}$$

defines a Cartan submanifold.

One would like to know how many immersions $\mathbf{X}\colon U \subset \mathbb{R}^n \to \mathbb{R}^{2n}$ define Cartan submanifolds. It is easy to derive a set of necessary conditions that the Christoffel symbols Γ^k_{ij} of the immersion must satisfy in order for this to happen. We know by definition that

$$\mathbf{X}_{ij} = \sum_{k=1}^{n} \Gamma^k_{ij}\mathbf{X}_k, \quad \text{for } i \neq j. \tag{106}$$

From this, it follows that if $\mathbf{X}\colon U \subset \mathbb{R}^n \to \mathbb{R}^{2n}$ is a Cartan submanifold parametrized by conjugate coordinates, then (106) holds. It follows furthermore that

$$\mathbf{X}_{ij} = \Gamma^i_{ij}\mathbf{X}_i + \Gamma^j_{ij}\mathbf{X}_j, \tag{107}$$

where the Christoffel symbols Γ^k_{ij} satisfy

$$\frac{\partial\Gamma^k_{ik}}{\partial x^l} + \Gamma^k_{ik}\Gamma^k_{kl} - \Gamma^i_{il}\Gamma^k_{ik} - \Gamma^l_{il}\Gamma^k_{lk} = 0, \tag{108}$$

for $1 \leqslant i \neq l \neq k \leqslant n$.

One would like to analyze these constraints to determine the degree of generality of the set of Cartan submanifolds of $\mathbb{R}^{2n}$. This was done by Cartan in [6], using the Cartan–Kähler theorem and the involutivity test as given for quasi-linear exterior differential systems. It is more natural from this point of view to think of Cartan submanifolds as being immersed in real projective space $\mathbb{P}^{2n}$, since the property of admitting a conjugate net of curves is essentially projective. The theorem proved by Cartan for such submanifolds is the following:

THEOREM 23. *Locally, the set of C^ω Cartan submanifolds* $\mathbf{X}\colon U \subset \mathbb{R}^n \to \mathbb{P}^{2n}$ *is parametrized by $n(n-1)$ functions of 2 variables.*

Proof. This is only a sketch of the proof. The structure equations for the bundle of projective frames in $\mathbb{P}^{2n}$ are given by

$$\mathrm{d}\mathbf{A} = \omega_{00}\mathbf{A} + \omega_1\mathbf{A}_1 + \cdots + \omega_{2n}\mathbf{A}_{2n}, \tag{109}$$

$$\mathrm{d}\mathbf{A}_i = \omega_{i0}\mathbf{A} + \omega_{i1}\mathbf{A}_1 + \cdots + \omega_{i2n}\mathbf{A}_{2n}, \tag{110}$$

$$\mathrm{d}\omega_i = \omega_{00} \wedge \omega_i + \sum_{j=1}^{2n} \omega_j \wedge \omega_{ji}, \tag{111}$$

$$\mathrm{d}\omega_{ij} = \omega_{i0} \wedge \omega_j + \sum_{k=1}^{2n} \omega_{ik} \wedge \omega_{kj}, \tag{112}$$

where $1 \leqslant i, j \leqslant 2n$. We adapt the projective frame $(\mathbf{A}, \mathbf{A}_1, \dots, \mathbf{A}_{2n})$ to $\mathbf{X}$, so that at every point of the image of $\mathbf{X}$, we have

$$\omega_{n+1} = \cdots = \omega_{2n} = 0, \qquad \omega_{i\alpha} = \sum_{j=1}^{n} \Omega_{ij}^{\alpha-n} \omega_j, \tag{113}$$

where $1 \leqslant i \leqslant n$, $n + 1 \leqslant \alpha \leqslant 2n$ and

$$\Omega_{ij}^{\alpha-n} = \Omega_{ji}^{\alpha-n}. \tag{114}$$

Consider the pencil of quadratic differential forms

$$\Omega = \sum_{\alpha=n+1}^{2n} \lambda_\alpha \Omega_\alpha, \tag{115}$$

where

$$\Omega_\alpha = \sum_{i=1}^{n} \omega_i \otimes \omega_{i\alpha} = \sum_{i,j=1}^{n} \Omega_{ij}^{\alpha-n} \omega_i \otimes \omega_j. \tag{116}$$

We are looking for all immersions $\mathbf{X}$ such that

$$\Omega = \sum_{i=1}^{n} \nu_i \omega_i^2. \tag{117}$$

By a further adaptation of the projective frame $(\mathbf{A}, \mathbf{A}_1, \dots, \mathbf{A}_{2n})$, this condition reduces to

$$\Omega_{n+i} = \omega_i^2, \tag{118}$$

and our problem thus reduces to constructing the integral manifolds of the Pfaffian system on the bundle of projective frames generated by the 1-forms

$$\omega_\alpha, \quad n + 1 \leqslant \alpha \leqslant 2n, \tag{119}$$

$$\omega_{i,n+i} - \omega_i, \quad 1 \leqslant i \leqslant n, \tag{120}$$

$$\omega_{j,n+i}, \quad 1 \leqslant i \neq j \leqslant n, \tag{121}$$

with independence condition given by $\omega_1 \wedge \cdots \wedge \omega_n$. It follows from the structure equations for the projective frames that this Pfaffian system is quasi-linear. Its reduced characters are easily computed to be

$$\sigma_1' = 0, \qquad \sigma_2' = n(n-1), \qquad \sigma_{2+l}' = 0. \tag{122}$$

On the other hand, the dimension of the space of solutions of the polar equations for an admissible n-dimensional integral element is equal to $2n(n-1)$. The conclusion follows by applying Cartan's involutivity test. □

References

1. Anderson, I., Kamran, N. and Olver, P. J.: Internal, external and generalized symmetries, *Adv. Math.* **100**(1) (1993), 53–100.
2. Bryant, R., Chern, S. S. and Griffiths, P.: Exterior differential systems, In: *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations*, Science Press, Beijing, 1982, pp. 219–338.
3. Bryant, R., Chern, S. S., Gardner, R. B., Goldschmidt, H. and Griffiths, P.: *Exterior Differential Systems*, MSRI Publications 18, Springer-Verlag, New York, 1991.
4. Cartan, E.: Sur quelques quadratures dont l'élément différentiel contient des fonctions arbitraires, *Bull. Soc. Math. France* **29** (1901), 118–130.
5. Cartan, E.: Sur l'intégration de certains systèmes indéterminés d'équations différentielles, *J. Reine Angew. Math.* **145** (1915), 86–91.
6. Cartan, E.: Sur les variétés à courbure constante d'un espace euclidien ou non-euclidien, *Bull. Soc. Math. France* **77** (1919), 125–160.
7. Cartan, E.: *Les systèmes différentiels extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.
8. Chern, S. S.: Laplace transform of a class of higher-dimensional varieties in a projective space of *n* dimensions, *Proc. Nat. Acad. Sci. U.S.A.* **30** (1944), 95–97.
9. Hilbert, D.: Über den Begriff der Klasse von Differentialgleichungen, *Math. Ann.* **73** (1912), 81–93; also in Gesammelte Abhandlungen, Vol. 3, Springer-Verlag, Berlin, 1935.
10. Kamran, N. and Tenenblat, K.: Laplace transformation in higher dimensions, *Duke Math. J.* **84**(1) (1996), 237–266.