Invariant Modules and the Reduction of Nonlinear Partial Differential Equations to Dynamical Systems

Niky Kamran[†]
Department of Mathematics
McGill University
Montréal, Québec H3A 2K6
CANADA
nkamran@math.mcgill.ca

Robert Milson[‡]
Department of Mathematics
Dalhousie University
Halifax, Nova Scotia B3H 3J5
CANADA
milson@mscs.dal.ca

Peter J. Olver[§]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.
olver@math.umn.edu
http://www.math.umn.edu/∼olver

Abstract. We completely characterize all nonlinear partial differential equations leaving a given finite-dimensional vector space of analytic functions invariant. Existence of an invariant subspace leads to a reduction of the associated dynamical partial differential equations to a system of ordinary differential equations, and provide a nonlinear counterpart to quasi-exactly solvable quantum Hamiltonians. These results rely on a useful extension of the classical Wronskian determinant condition for linear independence of functions. In addition, new approaches to the characterization of the annihilating differential operators for spaces of analytic functions are presented.

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1. Introduction.

The construction of explicit solutions to partial differential equations by symmetry reduction dates back to the original work of Sophus Lie, [**30**]. The reduction of partial differential equations to ordinary differential equations was generalized by Clarkson and Kruskal, [**6**], in their direct method, which was later shown, [**35**], to be included in the older Bluman and Cole nonclassical symmetry reduction approach, [**3**]. A survey of these methods as of 1992 can be found in [**33**]. Meanwhile, Galaktionov, [**12**], introduced the method of nonlinear separation that reduces a partial differential equation to a system of ordinary differential equations, developed in further depth in [**12**, **13**, **14**, **15**]. Similar ideas appear in the work of King, [**26**], and the "antireduction" methods introduced by Fushchych and Zhdanov, [**10**, **9**]. Svirshchevskii, [**49**, **50**, **51**], made the important observation that one could, in the one-dimensional case, characterize in terms of higher order (or generalized) symmetries those nonlinear ordinary differential operators that admit a given invariant subspace leading to nonlinear separation.

In quantum mechanics, linear differential operators with invariant subspaces form the foundation of the theory of quasi-exactly solvable (QES) quantum models as initiated by Turbiner, Shifman, Ushveridze, and collaborators, [**41**, **42**, **43**, **52**]. The basic idea is that a Hamiltonian operator which leaves a finite-dimensional subspace of functions invariant can be restricted to this subspace, resulting in an eigenvalue problem which can be solved by linear algebraic techniques. The Lie algebraic approach to quasi-exactly solvable problems requires that the subspace in question be invariant under a Lie algebra of differential operators $\mathfrak{g}$, in which case the Hamiltonian belongs to the universal enveloping algebra of $\mathfrak{g}$; see [**16**, **17**, **38**, **47**, **52**] for details and [**23**, **29**] for applications to molecular spectroscopy, nuclear physics, and so on. Zhdanov, [**56**], indicated how one could characterize quasi-exactly solvable operators using higher order symmetry methods. We should also mention Hel–Or and Teo, [**21**], who have applied group-invariant subspaces in computer vision, naming their elements "steerable functions".

Motivated by a problem of Bochner, [**4**], to characterize differential operators having orthogonal polynomial solutions, Turbiner, [**44**], initiated the study of differential operators leaving a polynomial subspace invariant. In one-dimension, the remarkable result is that the operators leaving the entire subspace of degree $\leq n$ polynomials invariant are the quasi-exactly solvable operators constructed by Lie algebra methods. These results were further developed for multidimensional and matrix differential operators, and difference operators by Turbiner, Post and van den Hijligenberg, [**39**, **40**, **45**, **46**], and Finkel and Kamran, [**8**].

In this paper, we broaden the general theory of nonlinear separation to include partial differential operators, and argue that it constitutes the proper nonlinear generalization of quasi-exactly solvable linear operators. Our theory provides an explicit characterization of all nonlinear differential operators that leave a given subspace of functions invariant. In the time-independent case, solutions lying in the subspace are obtained by solving a system of nonlinear algebraic equations. For evolution equations and certain other dynamical partial differential equations, we show how explicit solutions in the given subspace are obtained by reducing the partial differential equations to a finite-dimensional dynamical system. We illustrate our method with a number of significant examples.

Our methods can also be compared and contrasted with the more algebraic theory of D-modules, cf. [**2**, **7**, **31**, **32**]. In particular, we describe new algorithms for determining the annihilator of a given finite-dimensional subspace, based on the study of generalized Wronskian matrices and their ranks. These methods derive their justification from the general theory of prolonged group transformations developed in [**36**, **37**].

The first section of the paper outlines the basic setting of our methods — finite-dimensional spaces of analytic functions defined on an open subset of real Euclidean space. The case of analytic functions of several complex variables is more subtle, and requires cohomological or geometric restrictions on the domain. Section 3 presents the basic tools in our study: a multi-dimensional generalization of the classical Wronskian condition for linearly independence of functions. In fact, these results form a very particular case of a general study of orbit dimensions of prolonged group actions formulated in [**37**]. The key concept is the notion of a "regular" space of analytic functions, and only for such subspaces is one able to characterize the differential operators that annihilate the subspace or leave it invariant. While the Hilbert basis theorem is not generally applicable to ideals of analytic differential operators, one can nevertheless algorithmically determine a finite generating set of annihilating differential operators when the subspace is regular. Section 4 contains our basic approach to the annihilator, that relies on a useful extension which we name an "affine annihilator", which is a differential operator that maps every function in the subspace to a constant. The construction ultimately rests on an interesting lemma characterizing analytic solutions to a system of variable coefficient linear equations of constant rank; it is at this critical point that the distinction between the real and complex analytic situations becomes evident. We include several examples illustrating our construction, including a convenient characterization of the affine annihilators of simplicial subspaces of monomials. Section 5 applies these constructions to characterize all linear and nonlinear differential operators that leave a given regular subspace invariant. In the scalar case, Svirshchevskii, [**49**, **50**, **51**], characterized these differential operators using generalized symmetries, and we prove that Svirshchevskii's symmetry operators coincide with our affine annihilators, thereby establishing the generalization of Svirshchevskii's methods to analytic functions of several variables. Finally, section 6 outlines how our results can be applied to the construction of explicit solutions to linear and nonlinear partial differential equations based on the method of nonlinear separation.

2. Function Spaces.

Let $X \subset \mathbb{R}^m$ be an open, connected subset of Euclidean space, with coordinates $(x_1, \ldots, x_m)$. Our basic set of allowable functions will the space $\mathcal{F} = \mathcal{C}^\omega(X)$ of analytic real-valued functions $f\colon X \to \mathbb{R}$. We may regard $\mathcal{F}$, depending on the circumstances, as either a real vector space, or as an algebra over the reals.

Even though we shall primarily restrict our attention to real domains and real analytic functions, much of the exposition can be adapted to other contexts. For instance, a much easier situation is when $\mathcal{F} = \mathcal{O}_a$ is the space of all germs of analytic functions at a single point $a = (a_1, \ldots, a_m) \in X$, or, equivalently, the space $\mathbb{C}[\![x_1 - a_1, \ldots, x_m - a_m]\!]$ of convergent power series at the point a. Another interesting example, studied extensively in D-module theory, is when $\mathcal{F}$ is the field of meromorphic functions on X, or, more generally, any differential field, cf. [**27**]. The case when $\mathcal{F} = \mathcal{C}^\infty(X)$ consists of smooth functions on X is also quite interesting, but much more difficult to treat owing to a number of pathologies that do not appear in the analytic context. With the proper restrictions, our methods and results can also be made to apply to various function spaces in the complex-analytic category. We briefly indicate how this may be done in Section 4, below.

Once we have fixed the function space $\mathcal{F}$, our primary object of study are finite-dimensional subspaces $\mathcal{M} \subset \mathcal{F}$. In particular, given functions $f_1, \ldots, f_k \in \mathcal{F}$, we let $\mathcal{M} = \{f_1, \ldots, f_k\}$ denote the subspace spanned thereby. We shall use r to denote the dimension of $\mathcal{M}$, and often use $f_1, \ldots, f_r \in \mathcal{F}$ to denote a basis. In applications to quasi-exactly solvable quantum problems, $\mathcal{M}$ is a finite dimensional module for the action of a transformation group G on X, cf. [**16, 17**].

Let $\mathcal{D} = \mathcal{D}(\mathcal{F})$ denote the space of linear differential operators whose coefficients belong to the function space $\mathcal{F}$. The multiplication of differential operators is defined in the usual manner, making $\mathcal{D}$ into an associative, but noncommutative algebra over $\mathcal{F}$ — the latter acting by left multiplication. Furthermore, the operators in $\mathcal{D}$ define linear maps $L: \mathcal{F} \to \mathcal{F}$, and so $\mathcal{F}$ will also be regarded as a $\mathcal{D}$-module. There is a natural filtration on $\mathcal{D}$, with $\mathcal{D}^n$ denoting the $\mathcal{F}$-submodule consisting of differential operators of order $\leq n$. The domains X considered here are such that every linear differential operator $L \in \mathcal{D}^n$ is realized as a finite sum

$$L = \sum_I h_I(x)\, \partial_I, \qquad \text{where} \qquad \partial_I = \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}, \tag{2.1}$$

and the coefficients $h_I \in \mathcal{F}$. The sum in (2.1) is over symmetric multi-indices $I = (i_1, \ldots, i_k)$ of orders $0 \leq k = \#I \leq n$. Since $\mathcal{D}^{n+1} = \mathcal{D}^1 \cdot \mathcal{D}^n$, we may identify the quotient $\mathcal{D}^n / \mathcal{D}^{n-1}$ with the $\mathcal{F}$-module of homogeneous differential operators of order n.

We note that there are $q_n = \binom{m+n-1}{n}$ different symmetric multi-indices I of order $\#I = n$, which is the number of different n^{th} order partial derivatives ∂_I. Similarly, there are

$$q^{(n)} = q_0 + q_1 + \ldots + q_n = \binom{m+n}{n} \tag{2.2}$$

different symmetric multi-indices I of order $\#I \leq n$. An n^{th} order differential operator is uniquely determined by its $q^{(n)}$ different coefficients $h_I(x)$, $\#I \leq n$. Therefore, we can identify $\mathcal{D}^n$ with the space $\mathcal{F}^{\times q^{(n)}}$ — the Cartesian product of $q^{(n)}$ copies of $\mathcal{F}$. Explicitly, the isomorphism $\sigma_n: \mathcal{D}^n \xrightarrow{\sim} \mathcal{F}^{\times q^{(n)}}$ maps a linear differential operator (2.1) to the column vector

$$\sigma_n(L) = \mathbf{h}(x) = (\, \ldots, h_I(x), \ldots\,)^T \tag{2.3}$$

whose entries, indexed by the multi-indices of order $\#I \leq n$, are the coefficients of L.

3. Wronskians and Stabilization.

The most basic necessary and sufficient condition for the linear independence of solutions to a homogeneous linear scalar ordinary differential equation is the nonvanishing of their Wronskian determinant, cf. [**20**]. Many of our results will rely on a significant multi-dimensional generalization of this classical Wronskian lemma, which can be applied to *any* collection of analytic functions. We refer to [**37**] for details on the following definitions and results, including extensions to both smooth and vector-valued functions.

Definition 3.1. The n^{th} order *Wronskian matrix* of the functions $f_1, \ldots, f_r \in \mathcal{F}$, is the $r \times q^{(n)}$ matrix

$$\mathbf{W}_n(x) = \mathbf{W}_n[f_1, \ldots, f_r](x) = \begin{pmatrix} f_1(x) & \ldots & \partial_I f_1(x) & \ldots \\ \vdots & \ddots & \vdots & \ddots \\ f_r(x) & \ldots & \partial_I f_r(x) & \ldots \end{pmatrix}, \tag{3.1}$$

whose entries consist of the partial derivatives of the f_κ's with respect to the x_i's of all orders $0 \leq \# I \leq n$.

In the scalar case $X \subset \mathbb{R}$, the standard Wronskian determinant coincides with the determinant of the $(r-1)^{\text{st}}$ order Wronskian matrix $\mathbf{W}_{r-1} = \mathbf{W}_{r-1}[f_1, \ldots, f_r]$, which happens to be a square matrix.

To this end, we define the *Wronskian matrix rank* function

$$\rho_n(x) = \operatorname{rank} \mathbf{W}_n[f_1, \ldots, f_r](x). \tag{3.2}$$

Note that $\rho_n(x)$ only depends only on the subspace $\mathcal{M} = \{f_1, \ldots, f_r\}$ spanned by the given functions, and not on the particular generators or basis. In particular, $0 \leq \rho_n(x) \leq \dim \mathcal{M}$. Moreover, $\rho_n(x)$ is lower semi-continuous: if $\rho_n(x_0) = k$, then $\rho_n(x) \geq k$ for all x in a sufficiently small neighborhood of x_0. The *generic Wronskian rank* of order n is

$$\rho_n^* = \max \{\, \rho_n(x) \mid x \in X \,\}.$$

The first key result is the following:

Theorem 3.2. *Let $\mathcal{M} \subset \mathcal{F}$ be an r-dimensional subspace and ρ_n^*, $n \geq 0$, be the sequence of its generic Wronskian ranks. Then $\rho_n^* = r = \dim \mathcal{M}$ for $n \gg 0$ sufficiently large. Moreover, if we define the (generic)* stabilization order *$s = \min\{\, n \mid \rho_n^* = r \,\}$ of $\mathcal{M}$ to be the minimal such n, then we find*

$$\rho_0^* < \rho_1^* < \cdots < \rho_{s-1}^* < \rho_s^* = \rho_{s+1}^* = \rho_{s+2}^* = \cdots = r = \dim \mathcal{M}. \tag{3.3}$$

In other words, once the generic Wronskian ranks become equal, they stabilize, and achieve a value equal to the dimension of the subspace. In particular, the ranks cannot "pseudostabilize", [**36, 37**], and so *must* strictly increase before stabilization sets in. It is entirely possible, though, that they increase only by 1 at each order — an evident example occurs when all the functions only depend on a single variable.

While knowledge of the stabilization order reduces the amount of work required to compute the order of the generic Wronskian rank, one can, in all cases, replace s by $r-1$. Thus, one has:

Corollary 3.3. *The analytic functions $f_1(x), \ldots, f_r(x)$ are linearly independent if and only if the generic rank of their $(r-1)^{\text{st}}$ order Wronskian matrix $\mathbf{W}_{r-1}(x) = \mathbf{W}_{r-1}[f_1, \ldots, f_r](x)$ is equal to r.*

Proof: Indeed, Theorem 3.2 implies that $f_1, \ldots, f_r$ are linearly independent if and only if $\rho_n^* = r$ for any $n \geq s$ greater than or equal to the stabilization order. Moreover, (3.3) implies that the stabilization order of an r-dimensional subspace $\mathcal{M} \subset \mathcal{F}$ is always bounded by $s \leq r-1$. Thus, one typically does not need to compute the rank of the order $n = r-1$ Wronskian to detect linear independence — checking at the stabilization order is sufficient. Another way of stating this result is the following: if the generic ranks of the Wronskians $\mathbf{W}_k(x)$ and $\mathbf{W}_{k+1}(x)$ are equal, then this rank is the same as the dimension of $\mathcal{M}$; moreover, this equality first occurs when $k = s \leq r-1$. *Q.E.D.*

Kolchin, [**27**; p. 86], states and proves a version of Corollary 3.3 assuming that the functions belong to a differential field, e.g., the field of meromorphic functions. It would be of interest to adapt our constructions to this case.

It is noteworthy that Theorem 3.2 is a particular case of a general theorem governing the orbit geometry of prolonged transformation groups acting on jet bundles, described in detail in [**36**]. In the present situation, the relevant group is the elementary r-parameter abelian group

$$(x, u) \longmapsto \left(x, u + \sum_{\kappa=1}^{r} t_\kappa f_\kappa(x) \right), \tag{3.4}$$

acting on $E = X \times \mathbb{R}$. The dimension of the prolonged group orbits contained in the n^{th} order jet fiber $\mathrm{J}^n E|_x$ equals the Wronskian rank $\rho_n(x)$. Moreover, in the analytic category, the group (3.4) acts effectively if and only if the subspace $\mathcal{M} = \{f_1, \ldots, f_r\}$ has dimension r. The extensions of this result to more general smooth group actions are also treated in [**37**], and can be applied to the more subtle case of subspaces $\mathcal{M} \subset \mathcal{C}^\infty$ containing smooth functions. For brevity, we shall refrain from discussing this more complicated case here.

Although, for an r-dimensional subspace $\mathcal{M}$, the $(r-1)^{\text{st}}$ order Wronskian has generic rank r, it may certainly have lower rank at particular points in the domain X. A simple example is provided by the functions $f_1(x) = 1$, $f_2(x) = e^{x^2}$, which have first order Wronskian determinant $\det \mathbf{W}_1(x) = 2x e^{x^2}$, which is singular at $x = 0$. (The classical Wronskian lemma implies that these functions cannot be common solutions to a regular, homogeneous, second order linear differential equation.) Our applications typically require that, for some n sufficiently large, the Wronskian rank $\rho_n(x)$ be equal to r at every $x \in X$, and so we need to determine when this occurs. In the preceding example,

$$\mathbf{W}_2[f_1, f_2](x) = \begin{pmatrix} f_1 & f_1' & f_1'' \\ f_2 & f_2' & f_2'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ e^{x^2} & 2x e^{x^2} & (2+4x^2) e^{x^2} \end{pmatrix}$$

has rank 2 everywhere, as do all the higher order Wronskian matrices. Therefore, it is of interest in understanding, not just how the generic Wronskian ranks behave, but also how the $\rho_n(x)$ behave at a single point.

Theorem 3.4. *If* $\dim \mathcal{M} = r$ *and* $x \in X$ *is any point, then there exists a* finite *integer* $n = n(x)$ *such that* $\rho_k(x) = r$ *for all* $k \geq n$. *We call* n *the* Wronskian order *of the point* x, *and denote it by*

$$\omega(x) = \min\{\, n \mid \rho_n(x) = r \,\}. \tag{3.5}$$

For example, if we take $f_1(x) = 1$, $f_2(x) = e^{x^k}$, for $k \geq 2$ an integer, then the Wronskian order of $x = 0$ is equal to $\omega(0) = k$. Therefore, the Wronskian order of a point can be arbitrarily large.

Again, Theorem 3.4 is a corollary of a more general theorem about the geometry of prolonged transformation groups. In the language of [**37**], a point at which $\rho_k(x) < r$ for *all* k would be known as a "totally singular point" for the associated transformation group (3.4). However, Theorem 6.4 of [**37**] states that an analytic transformation group admits no totally singular points. In fact, for the elementary group (3.4), this result is not hard to prove directly — it is a consequence of the basic result that a nonzero analytic function cannot have all zero derivatives at a single point. And this is the main reason why we must restrict our attention to analytic, as opposed to smooth, functions.

Given an r-dimensional subspace $\mathcal{M} = \{f_1, \ldots, f_r\}$, Corollary 3.3 implies that most — meaning those belonging to a dense open subset of X — points have Wronskian order $r - 1$ or less.† The exceptions are the points x where $\mathbf{W}_{r-1}(x)$ has less than maximal rank. This permits us to formulate a basic estimate on the Wronskian order of the space $\mathcal{M}$. In general, given an ordered r-tuple $\mathbf{I} = (I_1, \ldots, I_r)$ consisting of symmetric multi-indices, we define its order to be $\#\mathbf{I} = \max\{\, \#I_\nu \mid \nu = 1, \ldots, r \,\}$. Let $\mathbf{W_I}(x)$ denote the associated $r \times r$ submatrix of $\mathbf{W}_k(x)$ whose columns are indexed by the multi-indices I_ν in $\mathbf{I}$. More prosaically, the entries of $\mathbf{W_I}(x)$ are the partial derivatives $\partial_{I_\nu} f_\mu$, $\mu, \nu = 1, \ldots, r$, indicated by $\mathbf{I}$. Note that $\mathbf{W_I}(x)$ depends on at most $\#\mathbf{I}^{\text{th}}$ order derivatives of the functions $f_1, \ldots, f_r$. The $\mathbf{I}^{\text{th}}$ *Wronskian minor* is then

$$\mathbf{M_I}(x) = \det \mathbf{W_I}(x). \tag{3.6}$$

Clearly, the Wronskian order of a point x is the smallest multi-index order with nonvanishing Wronskian minor, i.e.,

$$\omega(x) = \min\{\, \#\mathbf{I} \mid \mathbf{M_I}(x) \neq 0 \,\}. \tag{3.7}$$

Given a non-zero analytic function $h(x) \not\equiv 0$, we define its *vanishing order* $\varpi_h(x)$ at a point to be the order of its first nonzero terms in the power series expansion at x. Alternatively, we can define the vanishing order by differentiation:

$$\varpi_h(x) = \min\{\, \#I \mid \partial_I h(x) \neq 0 \,\}. \tag{3.8}$$

In particular, $\varpi_h(x) = 0$ if $h(x) \neq 0$. Note that the vanishing order of a nonzero analytic function at a point is always finite.

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† If $\dim X = 1$, then the Wronskian order is never less than $r-1$. More generally, if $\dim X = m$, then the minimal Wronskian order l of any subspace is bounded from below by the inequality $q^{(l)} = \binom{m+l}{l} \geq r$.

Definition 3.5. Given a subspace $\mathcal{M}$, let $\varpi_{\mathbf{I}}(x) = \varpi_{\mathbf{M}_{\mathbf{I}}}(x)$ denote the vanishing order of its $\mathbf{I}^{\text{th}}$ Wronskian minor $\mathbf{M}_{\mathbf{I}}(x)$. The *Wronskian vanishing order* of $\mathcal{M}$ is defined as the minimal vanishing order of the minors of $\mathbf{W}_{r-1}(x)$, so

$$\varpi(x) = \min \{ \varpi_{\mathbf{I}}(x) \mid \#\mathbf{I} \leq r-1 \} . \tag{3.9}$$

The Wronskian vanishing order of a subspace provides an immediate bound on its Wronskian order at a point.

Proposition 3.6. *Let* $\dim \mathcal{M} = r$ *and let* $x \in X$. *If the Wronskian vanishing order of* $\mathcal{M}$ *at* X *is* $\varpi(x)$, *then the Wronskian order of* x *is bounded by*

$$\omega(x) \leq r - 1 + \varpi(x). \tag{3.10}$$

Proof: It suffices to note that if J is any multi-index of order $\#J = j$, then the J^{th} order derivative of a Wronskian minor $\mathbf{M}_{\mathbf{I}}(x)$ of order $\#\mathbf{I} = i$ can be written as a finite linear combination of Wronskian minors

$$\partial_J \mathbf{M}_{\mathbf{I}}(x) = \sum_{\mathbf{K}} \mathbf{M}_{\mathbf{K}}(x)$$

of orders $\#\mathbf{K} \leq i+j$. Hence, if J and $\mathbf{I}$ are such that $\partial_J \mathbf{M}_{\mathbf{I}}(x) \neq 0$, then there must exist a $\mathbf{K}$ with $\#\mathbf{K} \leq i+j$ such that $\mathbf{M}_{\mathbf{K}}(x) \neq 0$. *Q.E.D.*

Example 3.7. Consider the three-dimensional subspace $\mathcal{M} = \{1, \cos x, \cos 2x\}$. The Wronskian determinant of these three functions is $\det \mathbf{W}_2(x) = -4(\sin x)^3$, and hence $\mathcal{M}$ has Wronskian vanishing order 3 at $x = n\pi$, $n \in \mathbb{Z}$. The fourth order Wronskian matrix $\mathbf{W}_4(x)$ is found to have rank 3 for all $x \in \mathbb{R}$, and hence the Wronskian order at the singular points $x = n\pi$ is 4.

If x_0 has Wronskian order n, then, by continuity, $\rho_n(x) = r$ for all x in some neighborhood of x_0. However, a global bound on the Wronskian order may not exist. We therefore introduce the following important definition.

Definition 3.8. A subspace $\mathcal{M}$ is called *regular* if it has uniformly bounded Wronskian order at each point, i.e., there exists a finite n such that $\rho_n(x) = r$ for *all* $x \in X$. The minimal such n is called the *order* of $\mathcal{M}$.

Many of the basic results in this paper require the underlying regularity of the subspace.

Example 3.9. Not every subspace is regular. For example, let $X = \mathbb{R}$. Consider the one-dimensional subspace spanned by an analytic function $f(x)$ which has a zero of order k at x_k, for $k = 1, 2, 3, \ldots$, where $x_k \to \infty$ as $k \to \infty$; such a function can be constructed using a Weierstrass product expansion, cf. [**1**; p. 194]. Then $\operatorname{rank} \mathbf{W}_n[f](x_k) = 0$ for $n < k$, while $\operatorname{rank} \mathbf{W}_n[f](x) = 1$ for $n \geq k$ and x in any neighborhood of x_k that does not contain $x_{k+1}, x_{k+2}, \ldots$. In other words, the Wronskian order of each x_k is equal to k.

We know two straightforward mechanisms for proving that a given subspace is regular. The key is to avoid an infinite sequence of points whose common Wronskian order is unbounded, as in the preceding example. The first method relies Proposition 3.6 for proving regularity.

Proposition 3.10. *If $\mathcal{M}$ is a subspace with uniformly bounded Wronskian vanishing order, so $\varpi(x) \leq n$ for all $x \in X$, then X is regular.*

Alternatively, one can generate regular subspaces by restricting the domain of the functions and using the basic properties of compactness.

Proposition 3.11. *If $\mathcal{M} \subset \mathcal{C}^\omega(X)$ is any subspace, and $Y \Subset X$ is an open subset with compact closure in X, then the subspace $\widehat{\mathcal{M}} = \mathcal{M} \mid_Y \subset \mathcal{C}^\omega(Y)$ obtained by restriction to Y is a regular subspace.*

4. Annihilators.

Our characterization of differential operators preserving spaces of analytic functions relies on several intermediate constructions of independent interest. The first order of business is to characterize the differential operators that annihilate all the elements of our subspace $\mathcal{M}$. We begin with the linear annihilating operators; their nonlinear counterparts will be treated in the following section.

Definition 4.1. The *annihilator* $\mathcal{A} = \mathcal{A}(\mathcal{M})$ of a subspace $\mathcal{M} \subset \mathcal{F}$ is the set of all linear differential operators that annihilate every function in $\mathcal{M}$, so

$$\mathcal{A}(\mathcal{M}) = \{\, K \in \mathcal{D} \mid \ K[f] = 0 \ \text{ for all } \ f \in \mathcal{M} \,\}.$$

We will refer to the elements of $\mathcal{A}(\mathcal{M})$ as *annihilating operators*. We note that $\mathcal{A} \subset \mathcal{D}$ is, in fact, a left ideal, since if $K[f] = 0$ and $L \in \mathcal{D}$ is any linear differential operator, then clearly $L[K[f]] = 0$ and so $L \cdot K \in \mathcal{A}$.

Example 4.2. Let $X \subset \mathbb{R}$. Let $\mathcal{M} \subset \mathcal{C}^\omega(\mathbb{R})$ be an r-dimensional subspace. A classical construction, cf. [**54**], produces an r^{th} order annihilating operator

$$K_r = h_r(x)\partial^r + h_{r-1}(x)\partial^{r-1} + \cdots + h_0(x). \tag{4.1}$$

Indeed, introducing a basis $f_1, \ldots, f_r$ of $\mathcal{M}$, then the conditions $K_r[f_\nu] = 0$, $\nu = 1, \ldots, r$, forms a homogeneous system of r linear equations for the $r+1$ coefficients of K_r. Cramer's rule produces the solution

$$h_k(x) = (-1)^{r-k}\, \mathbf{M}_k(x), \tag{4.2}$$

where $\mathbf{M}_k(x) = \det \mathbf{W}_{01\ldots k-1,k+1\ldots r}(x)$ denotes the $r \times r$ Wronskian minor obtained by deleting the k^{th} column of the $r \times (r+1)$ Wronskian matrix $\mathbf{W}_r(x)$. In particular, the leading term $h_r(x) = \mathbf{M}_r(x) = \det \mathbf{W}_{r-1}(x)$ is the classical Wronskian determinant, and hence K_r is a nonsingular differential operator if and only if the classical Wronskian never vanishes, which implies that $\mathcal{M}$ is regular of the minimal possible order $r-1$.

If $h_r(x) \neq 0$ is never zero, then the annihilator $\mathcal{A}$ is generated by K_r. Indeed, to prove that every other annihilating operator has the form $K = L \cdot K_r$, we note that every linear ordinary differential operator can be written (uniquely) in the form $T = Q \cdot K_r + R$, where $Q \in \mathcal{D}$ and $R \in \mathcal{D}^{r-1}$ has order at most $r-1$. (If T has order $\leq r-1$, then $Q = 0$.) Then $T \in \mathcal{A}(\mathcal{M})$ if and only if $R \in \mathcal{A}(\mathcal{M})$ is an annihilating operator. But the dimension of the kernel of a linear differential operator of order k is at most† k, and so $R = 0$.

On the other hand, if $h_r(x)$ vanishes at points $x \in X$, then K_r does *not* provide a basis for the annihilator of $\mathcal{M}$. For example, the annihilator of the one-dimensional subspace spanned by the function $f_1(x) = x$ is generated by the two differential operators $x\partial - 1$ and ∂^2. (Although $\partial^2 = \frac{1}{x}\partial \cdot (x\partial - 1)$, the operator $\frac{1}{x}\partial$ is not analytic, and so not allowed.) More complicated cases are discussed below.

Example 4.3. Let $X \subset \mathbb{R}^m$ and consider a one-dimensional subspace $\mathcal{M} = \{f\}$ where $f \not\equiv 0$. The linear operators $K_i = f\partial_i - f_i$, $i = 1, \ldots, m$, where $f_i = \partial_i f = \partial f / \partial x_i$, clearly belong to $\mathcal{A}^1$. If $f(x) \neq 0$ never vanishes, then $K_1, \ldots, K_m$ form a basis for $\mathcal{A}$. The proof of this fact is similar to the ordinary differential operator result of Example 4.2. We first note that any differential operator can be written in the form $T = \sum Q_i \cdot K_i + g$, where $Q_1, \ldots, Q_m \in \mathcal{D}$ are differential operators, and $g \in \mathcal{D}^0 \simeq \mathcal{C}^\omega$ is a multiplication operator. Clearly $T \in \mathcal{A}(\mathcal{M})$ if and only if $g \equiv 0$, proving the result. The case when f vanishes on a subvariety of X is more subtle — see below. Extensions to higher dimensional subspaces are also discussed below.

The standard filtration of $\mathcal{D}$ induces a filtration of the annihilator, and we let $\mathcal{A}^n = \mathcal{A} \cap \mathcal{D}^n$ denote the subspace of annihilating operators of order at most n.

Proposition 4.4. *Let the analytic functions $f_1, \ldots, f_r$ span a finite-dimensional subspace $\mathcal{M} = \{f_1, \ldots, f_r\} \subset \mathcal{F}$. Let $\mathbf{W}_n$ the associated n^{th} order Wronskian matrix. The map σ_n defined by* (2.3) *defines an isomorphism*

$$\sigma_n \colon \mathcal{A}^n \xrightarrow{\sim} \ker \mathbf{W}_n \,, \tag{4.3}$$

between the n^{th} order annihilator of $\mathcal{M}$ and the common kernel

$$\ker \mathbf{W}_n = \left\{ \mathbf{h} \in \mathcal{F}^{\times q^{(n)}} \;\middle|\; \mathbf{W}_n \cdot \mathbf{h} = 0 \right\}.$$

Proof: It is sufficient to note that for every $K \in \mathcal{A}^n(\mathcal{M})$ and corresponding vector $\mathbf{h} = \sigma_n(K)$ with analytic entries, the κ^{th} entry of the matrix product $\mathbf{W}_n \cdot \mathbf{h}$ is equal to $K[f_\kappa]$. *Q.E.D.*

Had we taken $\mathcal{F}$ to be the algebra of power series (or analytic germs), then a straightforward adaptation of the classical Hilbert basis theorem, [**22**, **55**] would prove that the annihilator $\mathcal{A}(\mathcal{M})$ of every finite-dimensional subspace $\mathcal{M} \subset \mathcal{F}$ is finitely generated. This is because the power series algebra is Noetherian, cf. [**55**]. However, the algebra $\mathcal{F} = \mathcal{C}^\omega(X)$ of globally defined analytic functions is *not* Noetherian, and so the Hilbert basis theorem does not apply. The following example shows that not every ideal of $\mathcal{F} = \mathcal{C}^\omega(X)$ is finitely generated.

† Operators with degenerate symbols may not admit enough analytic solutions to span a full k-dimensional kernel. For example, the functions annihilated by the first order operator $L = x\partial_x + 1$ are multiples of the non-analytic function $1/x$, and so L has a zero dimensional (analytic) kernel.

Example 4.5. Consider again the analytic function $f(x)$ introduced in Example 3.9. Let $f^{(k)}$, $k = 0, 1, 2, \ldots$ denote the corresponding derivatives and note that $f\,\partial - f^{(k)}$ is an annihilating operator for all k. It isn't hard to see that the ideal of $\mathcal{F} = \mathcal{C}^{\omega}(X)$ that is generated by all the $f^{(k)}$ is not finitely generated, and therefore, in this case, the annihilator $\mathcal{A}$ of $\mathcal{M} = \{f\}$ is not a finitely generated ideal of $\mathcal{D}$.

Later we will prove that the annihilator ideal of a *regular* subspace is finitely generated. to this end, we introduce a useful extension of the notion of the annihilator.

Definition 4.6. The *affine annihilator* $\mathcal{K}(\mathcal{M}) \subset \mathcal{D}$ of a subspace $\mathcal{M} \subset \mathcal{F}$ is the subspace of those linear differential operators that map every function in $\mathcal{M}$ to a constant function:

$$\mathcal{K}(\mathcal{M}) = \{\, L \in \mathcal{D} \mid \; L[f] = c \in \mathbb{R}, \quad \text{for all} \quad f \in \mathcal{M} \,\}.$$

Note that, as defined, the affine annihilator $\mathcal{K}(\mathcal{M})$ is a vector space, rather than an $\mathcal{F}$-module. Considered as a vector space, the annihilator $\mathcal{A}(\mathcal{M})$ is evidently a subspace of $\mathcal{K}(\mathcal{M})$. The difference between the two subspaces has a natural interpretation.

Definition 4.7. The *operator dual* to the subspace $\mathcal{M}$ is the quotient vector space

$$\mathcal{M}^* = \mathcal{K}(\mathcal{M})/\mathcal{A}(\mathcal{M}). \tag{4.4}$$

Since we are quotienting by the annihilator, there is a natural action of $\mathcal{M}^*$ on $\mathcal{M}$ itself induced by the action of $\mathcal{K}(\mathcal{M})$. In this fashion, if $\mathcal{M}$ is finite-dimensional, then there is a natural linear injection from $\mathcal{M}^*$ into the abstract dual of $\mathcal{M}$. As the following theorem will demonstrate, regularity implies that this injection is, in fact, an isomorphism.

Theorem 4.8. *If $\mathcal{M} \subset \mathcal{F}$ is a regular r-dimensional subspace of order s, then its operator dual $\mathcal{M}^*$ is also r-dimensional. Moreover, if $f_1, \ldots, f_r$ forms a basis of $\mathcal{M}$, then there exists a dual basis for $\mathcal{M}^*$ represented by differential operators $L_1, \ldots, L_r$ of order at most s such that*

$$L_i(f_j) = \delta^i_j, \qquad i, j = 1, \ldots, r. \tag{4.5}$$

Example 4.9. Let us consider the "triangular" or *simplicial* polynomial subspaces

$$\mathcal{T}_n = \left\{\, x^i y^j \;\middle|\; 0 \le i + j \le n \,\right\}. \tag{4.6}$$

We remark that $\mathcal{T}_n$ forms a module for the standard representation of the Lie algebra $\mathfrak{sl}(3)$ by first-order differential operators, cf. [**16**], and so plays an important role in the theory of quasi-exactly solvability and orthogonal polynomials, cf. [**44, 45**]. It is not hard to see that $\mathcal{T}_n$ forms a regular subspace of order $s = n$.

To construct the dual basis, we look for a set of differential operators $L_{ij}, 0 \le i+j \le n$, such that

$$L_{ij}(x^k y^l) = \delta_{ik}\delta_{jl}, \qquad \text{for all} \qquad 0 \le i + j \le n, \quad 0 \le k + l \le n. \tag{4.7}$$

Let

$$S = x\,\partial_x + y\,\partial_y$$

denote the degree or scaling operator. Let

$$p_k(x) = \frac{(-1)^k}{k!}\,(x-1)(x-2)\cdots(x-k)$$

be the unique polynomial of degree k such that $p_k(0) = 1$ and $p_k(j) = 0$ for $j = 1, \ldots, k$. Then dual basis operators are given by

$$L_{ij} = \frac{1}{i!\,j!}\,p_{n-i-j}(S)\cdot\partial_x^i\,\partial_y^j. \tag{4.8}$$

Indeed, the two last factors annihilate any monomial of degree $\leq i+j$ except for $x^i y^j$, whereas the polynomial in S will annihilate the higher degree monomials. For $n = 2$, the dual basis is explicitly given by

$$\begin{aligned}
&L_{20} = \tfrac{1}{2}\partial_x^2, \qquad L_{11} = \partial_{xy}, \qquad L_{02} = \tfrac{1}{2}\partial_y^2,\\
&L_{10} = (-S+1)\cdot\partial_x = (-x\partial_x - y\partial_y + 1)\cdot\partial_x,\\
&L_{01} = (-S+1)\cdot\partial_y = (-x\partial_x - y\partial_y + 1)\cdot\partial_y,\\
&L_{00} = \tfrac{1}{2}S^2 - \tfrac{3}{2}S + 1 = \tfrac{1}{2}x^2\partial_x^2 + xy\partial_{xy} + \tfrac{1}{2}y^2\partial_y^2 - x\partial_x - y\partial_y + 1.
\end{aligned} \tag{4.9}$$

Formula (4.8) readily generalizes to the simplicial modules in m variables, i.e., the polynomial subspaces generated by the monomials

$$x^I = x_1^{i_1}\cdots x_k^{i_k}, \qquad \#I = \sum_l i_l \leq n. \tag{4.10}$$

We note that this subspace forms a finite-dimensional module for the standard representation of $\mathfrak{sl}(n)$ by first-order differential operators.

The proof of Theorem 4.8 ultimately rests on the following technical Lemma.

Lemma 4.10. *Let $A(x)$ be a real-analytic, $r\times n$ matrix-valued function, defined for $x \in X \subset \mathbb{R}^m$. Suppose* $\operatorname{rank} A(x) = r$ *for all $x \in X$. Let $b\colon X \to \mathbb{R}^r$ be an analytic vector-valued function. Then there exists a solution $h\colon X \to \mathbb{R}^n$ to the matrix equation $A(x)h(x) = b(x)$ which is analytic for all x.*

Proof: Clearly, it suffices to prove the result when $b(x) = e_j$ is constant, equal to the j^{th} basis vector. Let $h_j(x)$ denote the sought-after solution with this particular choice of the right hand side. Let $v_1(x), \ldots, v_r(x)$ denote the rows of A, considered as vector-valued functions $v_k\colon X \to \mathbb{R}^n$. Let $*\colon \bigwedge^k\mathbb{R}^n \to \bigwedge^{n-k}\mathbb{R}^n$ denote the flat space Hodge star operation in the exterior algebra $\bigwedge^*\mathbb{R}^n$, cf. [**53**; p. 79]. In particular, $*1 = e_1\wedge\cdots\wedge e_n \in \bigwedge^n\mathbb{R}^n$ is the volume form. One can write the matrix system $A(x)h_j(x) = e_j$ in the equivalent exterior form

$$v_i(x)\wedge\,*\,h_j(x) = \begin{cases} 0, & i \neq j,\\ *\,1, & i = j, \end{cases} \tag{4.11}$$

obtained by applying $*$ to each equation and using the fact that $v\cdot w = *\,(v\wedge *\,w)$. An evident solution to the first $r-1$ equations in (4.11) is

$$h_j(x) = *\,\left[\, v_1(x)\wedge\cdots\wedge v_{j-1}(x)\wedge v_{j+1}(x)\wedge\cdots\wedge v_r(x)\wedge\gamma(x)\,\right], \tag{4.12}$$

where $\gamma\colon X \to \bigwedge^{n-r}\mathbb{R}^n$ is an arbitrary analytic map. The final equation in (4.11) leads to

$$(-1)^{n+j}\, v_1(x) \wedge \cdots \wedge v_r(x) \wedge \gamma(x) = \,*\, 1. \tag{4.13}$$

This is a linear equation in the coefficients $g_1(x), \ldots, g_k(x)$ of $\gamma(x)$, where $k = \binom{n}{r}$, and as such has the form

$$a_1(x) g_1(x) + \cdots + a_k(x) g_k(x) \equiv 1, \qquad (4.14).$$

Here $a_1(x), \ldots, a_k(x)$ are analytic functions explicitly determined by $v_1(x), \ldots, v_r(x)$; indeed, up to sign, they are just the rank r minors of $A(x)$. Moreover, these minors cannot simultaneously vanish since $v_1(x), \ldots, v_r(x)$ are assumed to be everywhere linearly independent. Since we are dealing with real-valued functions, an evident solution to (4.14) is

$$g_\nu(x) = \frac{a_\nu(x)}{a_1(x)^2 + \cdots + a_k(x)^2}\,, \qquad \nu = 1, \ldots, k. \tag{4.15}$$

Substituting (4.15) into (4.12) and using the fact that $*(\omega \wedge *\omega) = \|\, \omega\, \|^2$ for any $\omega \in \bigwedge^*\mathbb{R}^n$, produces an explicit analytic solution to (4.11) in the form

$$h_j(x) = (-1)^{n+j} * \left[\frac{v_1(x) \wedge \cdots \wedge v_{j-1}(x) \wedge v_{j+1}(x) \wedge \cdots \wedge v_r(x) \wedge *\big(v_1(x) \wedge \cdots \wedge v_r(x)\big)}{\|\, v_1(x) \wedge \cdots \wedge v_r(x)\, \|^2} \right] \tag{4.16}$$

Note that the denominator is nowhere zero since the rank of $A(x)$ equals r. *Q.E.D.*

Lemma 4.10 plays a critical role in our theory. It relies on the fact that there exists an analytic solution $a_1(x), \ldots, a_k(x)$ of (4.14), which, in the real category considered here, is constructed in (4.15). The advantage of choosing the reals as the ground field lies in the fact that our results are valid for arbitrary domains. However, it is important to keep in mind that the solutions to equation (4.14) given by (4.15) may fail to be analytic in the complex category, and indeed, Lemma 4.10 is not true for all complex domains X.

Example 4.11. A simple counterexample is provided by the case $X = \mathbb{C}^2 \setminus \{(0,0)\}$, and the 1×2 matrix function $A(x,y) = (x,y)$, which has rank 1 for all $(x,y) \in X$. There is no complex-analytic vector-valued function $h(x,y) = (h_1(x,y), h_2(x,y))^T$ such that

$$A(x,y)\, h(x,y) = x\, h_1(x,y) + y\, h_2(x,y) \equiv 1, \tag{4.17}$$

for all $(x,y) \neq (0,0)$. This stems from the fact that X is not a domain of holomorphy, and so any complex-analytic function on X can be extended to a complex analytic function on all of $\mathbb{C}^2$, cf. [**28**; §0.3.1]. Extending $h_1(x,y)$ and $h_2(x,y)$ in this manner, by continuity (4.17) would also have to hold at $x = y = 0$, which is clearly impossible.

The preceding example makes clear that the complex case is more subtle, and requires additional assumptions. One could, for instance, restrict oneself to the case where $\mathcal{F}$ is the algebra of complex-analytic germs (i.e., convergent power series). In this case, the natural analogue of the order of an r-dimensional module would be the smallest n such that the $r \times n$ matrix formed by the constant terms of the n^{th} order Wronskian $\mathbf{W}_n$ has rank r. For such an n, one of the rank r minors of $\mathbf{W}_n$ would be a unit, thereby making Lemma 4.10 true.

The utilization of more general domains $X \subset \mathbb{C}^n$ would require a cohomological assumption in order that (4.14) admit an analytic solution. One could, for instance, demand the vanishing of every H^1 whose coefficients lie in a coherent sheaf, or more generally that X be a Stein manifold, [**19**]. Alternatively, one could impose a geometric restriction on the domain X. For example, Corollary 7.2.6 in Krantz, [**28**; p. 293], states that if $X \subset \mathbb{C}^m$ is a *pseudoconvex* subdomain, then (4.14) has a complex analytic solution $g_1(x), \ldots, g_k(x)$ provided the a_ν's do not simultaneously vanish. Therefore, Lemma 4.10, and its consequences, would hold provided X is pseudoconvex. To keep matters simple, though, we shall not return to the complex-analytic situation.

Proof of Theorem 4.8: Recall first that we are seeking dual basis operators of order s. To construct an i^{th} dual basis operator, we use the isomorphism (2.3) to rewrite equation (4.5) in the equivalent matrix form

$$\mathbf{W}_s(x)\,\mathbf{h}_i(x) = e_i, \tag{4.18}$$

where $\mathbf{h}_i(x) = \sigma_s(L_i) \in \mathcal{F}^{\times q^{(s)}}$ is the vector of coefficients of the operator L_i, and e_i is the standard i^{th} basis vector for $\mathbb{R}^r$. Since s is the order of the subspace $\mathcal{M}$, we have $\operatorname{rank} \mathbf{W}_s(x) = r$ for all x, and hence Lemma 4.10 shows that there exists a solution $\mathbf{h}_i(x)$ to (4.18) which is analytic for all $x \in X$. This simple construction produces the required basis for the affine annihilating operators. *Q.E.D.*

Having proven the existence of a dual operator basis, we can return to the examination of the annihilator ideal. Thus, for the remainder of this section we assume that $\mathcal{M}$ is a regular r-dimensional subspace of functions with basis $f_1, \ldots, f_r$, and fix a dual basis $L_1, \ldots, L_r \in \mathcal{K}(\mathcal{M})$ of operators of order s or less. It is important to keep in mind that $\mathcal{M}^*$, as defined, is a vector space, and not an $\mathcal{F}$-module. When the need arises, we will use $\widetilde{\mathcal{M}}^*$ to denote the $\mathcal{F}$-module generated by $L_1, \ldots, L_r$. Our results rely on the following key observation, whose straightforward proof is left to the reader.

Proposition 4.12. *The mapping $\mathfrak{a}\colon \mathcal{D} \to \mathcal{A}$ given by*

$$\mathfrak{a}(T) = T - \sum_{\kappa=1}^{r} T[f_\kappa]\, L_\kappa, \qquad\qquad T \in \mathcal{D}, \tag{4.19}$$

defines an $\mathcal{F}$-module homomorphism that projects an arbitrary differential operator T onto an annihilator $\mathfrak{a}(T) \in \mathcal{A}$. Furthermore, $\mathfrak{a}$ is a left inverse of the inclusion homomorphism $\iota\colon \mathcal{A} \to \mathcal{D}$, thereby yielding the $\mathcal{F}$-module decomposition $\mathcal{D} \simeq \mathcal{A} \oplus \widetilde{\mathcal{M}}^*$.

At this point it is also important to note that a basis $f_1, \ldots, f_r$ of $\mathcal{M}$ does not uniquely determine the dual basis $L_1, \ldots, L_r$; one can obtain other bases by adding elements of $\mathcal{A}^s$ to the operators L_κ. For this reason the projection $\mathfrak{a}\colon \mathcal{D} \to \mathcal{A}$ is not a natural object, but rather depends on the choice of the dual basis.

Remark: On the other hand, for a simplicial module $\mathcal{T}_n$ described in Example 4.9, there are no annihilators of degree less than $n+1 = s+1$, and hence the dual module $\mathcal{M}^*$ is *uniquely* determined, or equivalently, $\mathcal{A}^s = 0$. Thus, the splitting of $\mathcal{D} \simeq \mathcal{A} \oplus \widetilde{\mathcal{M}}^*$ described in Proposition 4.12 is canonical in this case. Subspaces having this property will be called *saturated*, and form an interesting class worth further investigation.

For each multi-index I let us denote the "basic" annihilator

$$K_I = \mathfrak{a}(\partial_I) \tag{4.20}$$

obtained by projecting the basis differential operators ∂_I onto the annihilator $\mathcal{A}$. It is important to note that since the L_ν have orders at most s, if $\#I \geq s+1$, then the leading order term (or symbol) of K_I is just ∂_I.

Corollary 4.13. *Any differential operator $T \in \mathcal{D}$ can be written as a finite linear combination*

$$T = \sum_{\nu=1}^{r} g_\nu(x)\, L_\nu + \sum_I b_I(x)\, K_I, \tag{4.21}$$

of the dual affine annihilators and the basic annihilators.

Proof: Since the operators ∂_I generate $\mathcal{D}$ as an $\mathcal{F}$-module, we may use Proposition 4.12 to infer that $L_1, \ldots, L_r$ along with the operators K_I also generate $\mathcal{D}$; and that the affine annihilators along with all K_I such that $\#I \leq s$ generate $\mathcal{D}^s$. It follows immediately that for $n \geq s$, the K_I such that $\#I \leq n$ generate $\mathcal{A}^n$. *Q.E.D.*

Remark: The linear combination (4.21) is not necessarily unique. However, one can eliminate precisely r of the K_I's in order to suppress any redundancy and thereby yield a well-defined $\mathcal{F}$-module basis $\{L_\nu, K_I\}$ for $\mathcal{D}$.

Theorem 4.14. *Let $\mathcal{M} \subset \mathcal{F}$ be a regular r-dimensional subspace of order s. Then its annihilator ideal $\mathcal{A}(\mathcal{M})$ is finitely generated by differential operators of order at most $s+1$.*

Proof: We shall prove that the operators K_I such that $\#I \leq s+1$ generate $\mathcal{A}$ as an ideal of $\mathcal{D}$. Clearly all of $\mathcal{A}^{s+1}$ can be so generated. Let J be a multi-index whose order is $\#J > s+1$, and choose multi-indices I, N with $\#I = s+1$ and such that $\partial_J = \partial_N \partial_I$. According to the remark at the end of the proof of Corollary 4.13, K_J and $\partial_N \cdot K_I$ have the same leading term, and therefore the order of the difference $K_J - \partial_N \cdot K_I$ of the two operators will be smaller than the order of J. The desired conclusion now follows by induction. *Q.E.D.*

Remark: The set of generators K_I constructed above is typically not minimal. However, since the set in question is finite, minimal generating sets do exist.

Remark: In the theory of D-modules, one is interested in subspaces generated by rational functions $f(x) = p(x)/q(x)$, with polynomial p, q. However, the annihilating operators of interest are required to have polynomial coefficients, and so the set-up is a bit different from that considered in this paper. See [**32**] for applications of powerful techniques from Gröbner bases and the theory of D-modules towards the determination of the annihilators of such rational subspaces. It would be interesting to see whether our techniques have anything to add to this theory.

Example 4.15. Consider the two-dimensional subspace $\mathcal{M} = \{1, \cos x\}$. The second order Wronskian matrix is

$$\mathbf{W}_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ \cos x & -\sin x & -\cos x \end{pmatrix},$$

and hence $\mathcal{M}$ is regular of order 2. Applying the algorithm of Lemma 4.10 leads to the dual operators

$$L_1 = \cos^2 x\, \partial_x^2 + \cos x \sin x\, \partial_x + 1, \qquad L_2 = -\cos x\, \partial_x^2 - \sin x\, \partial_x.$$

It's important to mention that often the algorithm does not produce the most efficient answer. Indeed, for this particular $\mathcal{M}$, a more stream-lined basis of dual operators is given by

$$L_1 = \partial_x^2 + 1, \qquad L_2 = -\cos x\, \partial_x^2 - \sin x\, \partial_x.$$

The annihilator ideal, $\mathcal{A}(\mathcal{M})$ is generated as a ring by the following operators:

$$A_1 = -\sin x\, \partial_x^2 + \cos x\, \partial_x, \quad A_2 = \partial_x^3 + \partial_x.$$

This is a consequence of Theorem 4.14, which states that $\mathcal{A}(\mathcal{M})$ is generated by the projections of the order 3 basic differential operators via (4.19). Indeed the projected operators are:

$$\mathfrak{a}(1) = \sin x\, A_1, \quad \mathfrak{a}(\partial_x) = \cos x\, A_1, \quad \mathfrak{a}(\partial_x^2) = -\sin x\, A_1, \quad \mathfrak{a}(\partial_x^3) = A_2 - \cos x\, A_1$$

Therefore A_1 and A_2 suffice to generate all of $\mathcal{A}(\mathcal{M})$.

Example 4.16. The subspace $\mathcal{M} = \{1, \cos x, \cos 2x\}$ considered in Example 3.7 is considerably more difficult, since one needs to use the fourth order Wronskian to implement the algorithm. The explicit formulae are quite complicated. Set

$$\nu(x) = 144 - 432 \sin^2 x + 432 \sin^4 x + 400 \sin^6 x.$$

Indeed, $\nu(x)$ is the sum of the squares of the rank 3 minors of the fourth order Wronskian. The reciprocal of $\nu(x)$ will therefore be a factor in the expressions for the dual operator basis obtained from the algorithm of Lemma 4.10.

The dual operators are given by

$$\begin{aligned}
\nu(x)L_1 &= \nu(x) + (180\cos x\sin x - 612\cos x\sin^3 x + 696\cos x\sin^5 x)\partial_x + \\
&\quad + (180 - 720\sin^2 x + 1164\sin^4 x - 616\sin^6 x)\partial_x^2 + \\
&\quad + (36\cos x\sin x + 180\cos x\sin^3 x - 456\cos x\sin^5 x)\partial_x^3 + \\
&\quad + (36 - 144\sin^2 x - 60\sin^4 x + 136\sin^6 x)\partial_x^4, \\
\nu(x)L_2 &= (-192\sin x + 656\sin^3 x - 736\sin^5 x)\partial_x + \\
&\quad + (-192\cos x + 576\cos x\sin^2 x - 656\cos x\sin^4 x)\partial_x^2 + \\
&\quad + (-48\sin x - 176\sin^3 x + 496\sin^5 x)\partial_x^3 + \\
&\quad + (-48\cos x + 144\cos x\sin^2 x + 176\cos x\sin^4 x)\partial_x^4, \\
\nu(x)L_3 &= (12\cos x\sin x - 20\cos x\sin^3 x)\partial_x + (12 - 24\sin^2 x + 20\sin^4 x)\partial_x^2 + \\
&\quad + (12\cos x\sin x + 20\cos x\sin^3 x)\partial_x^3 + (12 - 24\sin^2 x - 20\sin^4 x)\partial_x^4,
\end{aligned}$$

Once again, since we are not dealing with a saturated module, the choice of the dual operators is not canonical, and there exists a more stream-lined dual basis:

$$\begin{aligned}
L_1 &= \tfrac{1}{4}\,\partial_x^4 + \tfrac{5}{4}\,\partial_x^2 + \partial_x, \\
L_2 &= -\tfrac{1}{3}\cos x\,\partial_x^4 - \tfrac{1}{3}\sin x\,\partial_x^3 - \tfrac{4}{3}\cos x\,\partial_x^2 - \tfrac{4}{3}\sin x\,\partial_x \\
L_3 &= \tfrac{1}{12}\partial_x^4 + \tfrac{1}{6}\sin x\,\cos x\,\partial_x^3 + \left(\tfrac{1}{12} + \tfrac{1}{2}\sin^2 x\right)\partial_x^2 - \tfrac{1}{3}\sin x\,\cos x\,\partial_x
\end{aligned}$$

The annihilator ideal, $\mathcal{A}(\mathcal{M})$ is generated as a ring by the following operators:

$$\begin{aligned}
A_1 &= 4(\cos^2 x - 1)\,\partial_x^3 + 12\sin x\cos x\,\partial_x^2 - 4(2\cos^2 x + 1)\,\partial_x \\
A_2 &= -\sin x\,\partial_x^4 + \cos x\,\partial_x^3 - 4\sin x\,\partial_x^2 + 4\cos x\,\partial_x \\
A_3 &= \partial_x^5 + 5\,\partial_x^3 + 4\,\partial_x
\end{aligned}$$

To confirm this one needs to check that A_1, A_2, A_3 generate the projections $\mathfrak{a}(\partial_x^n)$, $n = 0, \ldots, 5$. Indeed the projected operators are given by:

$$\begin{aligned}
\mathfrak{a}(1) &= -\tfrac{1}{24}\sin 2x\,A_1 + \tfrac{1}{6}\sin x\,A_2, & \mathfrak{a}(\partial_x) &= -\tfrac{1}{12}\cos 2x\,A_1, \\
\mathfrak{a}(\partial_x^2) &= \tfrac{1}{6}\sin 2x\,A_1 + \tfrac{1}{3}\sin x\,A_2, & \mathfrak{a}(\partial_x^3) &= \tfrac{1}{3}\cos 2x\,A_1 + \cos x\,A_2, \\
\mathfrak{a}(\partial_x^4) &= -\tfrac{2}{3}\sin 2x\,A_1 - \tfrac{7}{3}\sin x\,A_2, & \mathfrak{a}(\partial_x^5) &= -\tfrac{4}{3}\cos 2x\,A_1 - 5\cos x\mathcal{A}_2 + A_3
\end{aligned}$$

5. Nonlinear Operators and Invariant Subspaces.

Let us next turn to the characterization of nonlinear annihilating and affine annihilating operators. We let $\mathcal{E} = \mathcal{E}(X)$ denote the space of nonlinear differential operators. More specifically, in the case of analytic functions, $\mathcal{F} = \mathcal{C}^\omega(X)$, the space $\mathcal{E}$ consists of all analytic differential functions, cf. [**36**], meaning analytic functions $F\colon \mathrm{J}^n(X,\mathbb{R}) \to \mathbb{R}$ defined (globally) on the n^{th} order jet space of real-valued functions on X. Here $n \geq 0$ is the order of the differential function F. We write $F[u] = F(x, u^{(n)})$ for such an operator, where the square brackets indicate that F depends on x, u and derivatives of u.

Our main goal is to prove a structure theorem for those differential operators, both linear and nonlinear, which preserve a given subspace $\mathcal{M} \subset \mathcal{F}$. Throughout this section we assume that $\mathcal{M}$ is a regular r-dimensional subspace of order s, with basis $f_1, \ldots, f_r$ and dual basis $L_1, \ldots, L_r$. Let $\mathcal{N}(\mathcal{M}) \subset \mathcal{E}$ denote the set of nonlinear annihilating operators, i.e., those operators that map every function in $\mathcal{M}$ to zero. Using our basic annihilating operators (4.20), we can readily construct the most general nonlinear annihilating differential operator for our subspace.

Theorem 5.1. *Every operator $F \in \mathcal{N}(\mathcal{M})$ can be written as a finite sum*

$$F[u] = \sum_I G_I[u] \cdot K_I[u], \tag{5.1}$$

where the G_I are arbitrary elements of $\mathcal{E}$ and $K_I = \mathfrak{a}(\partial_I)$ are the basic annihilating operators.

Proof: From Corollary 4.13, we know that there is an analytic function $G(x, y, z)$, where $y = (y_1, \ldots, y_r)$, $z = (\ldots, z_I, \ldots)$, such that

$$F[u] = G(x, L_1[u], \ldots, L_r[u], \ldots, K_I[u], \ldots), \tag{5.2}$$

depending on finitely many of the K_I. If $u = \sum c_i f_i(x) \in \mathcal{M}$, then $L_i[u] = c_i \in \mathbb{R}$, and so substituting u into (5.2) yields

$$F[u] = G(x, c, 0) = 0, \qquad \text{where} \qquad c = (c_1, \ldots, c_r). \tag{5.3}$$

Equation (5.3) will hold for all x, c if and only if

$$G(x, y, z) = \sum_I G_I(x, y, z)\, z_I.$$

Therefore,

$$G(x, L_1[u], \ldots, L_r[u], \ldots, K_I[u], \ldots) = \sum_{\nu=1}^{q} G_I[u]\, K_I[u],$$

where the coefficients $G_I[u]$ are differential functions. *Q.E.D.*

Let $K_1, \ldots, K_l$ be a (minimal if desired) generating set for the annihilator $\mathcal{A}$. For example, according to Theorem 4.14, one can choose the K_ν from among the basic annihilators K_I of orders $\# I \leq s + 1$. Then we can immediately simplify (5.1) to only use the generating annihilating operators.

To be precise, let $\mathcal{G} = \mathcal{G}(X)$ denote the $\mathcal{E}$-module consisting of differential operators whose coefficients are differential functions. Such an operator $Z \in \mathcal{G}$ is given by a finite sum

$$Z = \sum_I G_I[u]\, \partial_I = \sum_I G_I(x, u^{(n)})\, \partial_I\,. \tag{5.4}$$

Note that the operator $G\colon \mathcal{E} \to \mathcal{E}$ maps differential functions to differential functions.

Corollary 5.2. *Every nonlinear annihilating operator $F \in \mathcal{N}(\mathcal{M})$ has the form*

$$F = \sum_{\nu=1}^{l} Z_\nu \cdot K_\nu, \tag{5.5}$$

where $Z_1, \ldots, Z_l$ are arbitrary elements of $\mathcal{G}$.

We are now in a position to prove the main result of the paper. Let

$$\mathcal{P}(\mathcal{M}) = \{ \, P \in \mathcal{D} \mid \; P[\mathcal{M}] \subset \mathcal{M} \, \}, \qquad \mathcal{Q}(\mathcal{M}) = \{ \, Q \in \mathcal{E} \mid \; Q[\mathcal{M}] \subset \mathcal{M} \, \} \tag{5.6}$$

denote, respectively, the left ideals consisting of all linear, respectively nonlinear, differential operators that preserves the given subspace $\mathcal{M}$.

Theorem 5.3. *Let $\mathcal{M} \subset \mathcal{F}$ be a regular r-dimensional subspace of analytic functions of order s. Let $L_1, \ldots, L_r \in \mathcal{K}(\mathcal{M})$ be a dual basis for its affine annihilator, and $K_1, \ldots, K_l \in \mathcal{A}(\mathcal{M})$ a generating set of annihilating operators. Then every nonlinear operator $Q \in \mathcal{Q}(\mathcal{M})$ that preserves $\mathcal{M}$ can be written in the form*

$$Q[u] = \sum_{i=1}^{r} \, f_i(x) \, H_i(L_1[u], \ldots, L_r[u]) + \sum_{\nu=1}^{l} \, Z_\nu[u] \cdot K_\nu[u], \tag{5.7}$$

where the $H_i \in \mathcal{C}^\omega(\mathbb{R}^r)$ are arbitrary analytic functions, and where the $Z_\nu \in \mathcal{G}(X)$ are arbitrary operators.

Proof: Suppose $f(x) = \sum c_j f_j(x) \in \mathcal{M}$. Since $Q[f] \in \mathcal{M}$, for each $i = 1, \ldots, r$ we have $L_i[Q[f]] = H_i(c_1, \ldots, c_r)$ is a constant depending on the coefficients of f. It follows immediately that

$$Q[u] - \sum_{i,j=1}^{r} \sum_{i=1}^{r} \, f_i(x) \, H_i(L_1[u], \ldots, L_r[u]) = 0$$

for all $u \in \mathcal{M}$, and so the result (5.7) follows immediately from (5.5). *Q.E.D.*

In particular, every linear operator $P \in \mathcal{P}(\mathcal{M})$ that preserves $\mathcal{M}$ can be written in the form

$$P = \sum_{i,j=1}^{r} a_{ij} f_i L_j + \sum_{\nu=1}^{l} R_\nu \cdot K_\nu, \tag{5.8}$$

where the $a_{ij} \in \mathbb{R}$ are arbitrary constants, and the $R_\nu \in \mathcal{D}$ are arbitrary linear differential operators. Therefore, we conclude:

Corollary 5.4. *The space of linear differential operators leaving a regular subspace $\mathcal{M} \subset \mathcal{F}$ of analytic functions invariant can be decomposed into a semi-direct product $\mathcal{P}(\mathcal{M}) \simeq \mathrm{End}(\mathcal{M}) \ltimes \mathcal{A}(\mathcal{M})$ of the space $\mathrm{End}(\mathcal{M}) = \{A\colon \mathcal{M} \to \mathcal{M}\} \simeq \mathcal{M} \otimes \mathcal{M}^*$ of linear endomorphisms with the annihilator ideal $\mathcal{A}(\mathcal{M})$.*

In the case when $\mathcal{M}$ is generated by polynomials, this corollary is proved directly by Post and Turbiner, [**39**]. Thus, the class of regular subspaces form the proper analytic generalization of the polynomial subspaces. See also [**8**, **40**].

In the one-dimensional case, the expressions $L_i[f]$ are directly related to the generalized symmetries considered by Svirshchevskii, [**49**]. The following example serves to illustrate this.

Example 5.5. Let $p = 1$ and let $\mathcal{M}$ denote the space of quadratic polynomials $ax^2 + bx + c$ in the scalar variable x, with basis $1, x, x^2$. The annihilator is generated by $K = \partial_x^3$ since the subspace forms the solution space to the linear ordinary differential equation

$$u_{xxx} = 0. \tag{5.9}$$

According to [**49**; Example 3], the fundamental invariants for (5.9) are $I_\nu(x, u^{(2)}) = J_\nu[u]$, where

$$J_1 = \partial_x^2, \qquad J_2 = x\partial_x^2 - \partial_x, \qquad J_3 = x^2\partial_x^2 - 2x\partial_x + 2.$$

Since

$$\begin{array}{lll} J_1[1] = 0, & J_2[1] = 0, & J_3[1] = 2, \\ J_1[x] = 0, & J_2[x] = -1, & J_3[x] = 0, \\ J_1[x^2] = 2, & J_2[x^2] = 0, & J_3[x^2] = 0, \end{array}$$

we see that $L_1 = \frac{1}{2}J_3, L_2 = -J_2, L_3 = \frac{1}{2}J_1$ are the dual operators with respect to the given basis of $\mathcal{M}$. In accordance with the general result, every nonlinear differential operator $Q \in \mathcal{Q}(\mathcal{M})$ leaving $\mathcal{M}$ invariant takes the form

$$Q[u] = A_0[u] + A_1[u]x + A_2[u]x^2 + T[u] \cdot K[u],$$

where

$$A_\nu[u] = H_\nu(J_1[u], J_2[u], J_3[u]) = H_\nu(u_{xx}, xu_{xx} - u_x, x^2u_{xx} - 2xu_x + 2u),$$

with $H_\nu \in \mathcal{C}^\omega(\mathbb{R}^3)$ arbitrary, and where $T \in \mathcal{G}$ is an arbitrary (nonlinear) differential operator.

The precise connection between our approach and that of Svirshchevskii relies on the theory of generalized symmetries of differential equations, as presented, for example, in [**34**; Chapter 5]. We assume that the reader is familiar with this theory for the remainder of this section, and take u to be a scalar variable, although vector-valued generalizations are straightforward.

Proposition 5.6. *Let $\Delta[u] = 0$ be an analytic, homogeneous system of linear differential equations and let $\mathcal{M} \subset \mathcal{F}$ denote the vector space of solutions. Then the generalized vector field $\mathbf{v}_Q = Q[u]\,\partial_u$ is a symmetry of $\Delta = 0$ if and only if the differential operator $Q[u] \in \mathcal{Q}(\mathcal{M})$ leaves $\mathcal{M}$ invariant.*

Note that $Q[u]$ may be nonlinear, although many nondegenerate linear partial differential equations only admit linear generalized symmetries; see [**48**]. Proposition 5.6 is an immediate consequence of the basic definition of generalized symmetry, which requires that the vector field $\mathbf{v}_Q$ leave the solution space to $\Delta = 0$ infinitesimally invariant. Linearity of the differential equation implies that infinitesimal invariance coincides with invariance of the solution space. Thus, if the differential equation is of finite type, [**18**], its solution space is finite-dimensional, and hence we can completely characterize the generalized symmetries of the system using the affine annihilators of the solution space along with Theorem 5.3.

In Svirshchevskii's method, one considers an r-dimensional subspace $\mathcal{M}$ consisting of analytic functions of a single variable x. Assume, for simplicity, that $\mathcal{M}$ is regular of order $r-1$. The linear ordinary differential equation that characterizes $\mathcal{M}$ is given by

$$K_r[u] = 0, \tag{5.10}$$

where K_r is the r^{th} order differential operator (4.1) constructed in Example 4.2. The resulting formulae (5.7) for the generalized symmetries $\mathbf{v}_Q$ of (5.10) is written in [**49**] in terms of the first integrals of (5.10). Recall that a differential function $F(x, u^{(r-1)})$ depending on at most $(r-1)^{\text{st}}$ order derivatives of u is called a *first integral* of the r^{th} order ordinary differential equation (5.10) if and only if its derivative $D_x F = 0$ vanishes on the solutions. The first integrals of a linear ordinary differential equation can be constructed in terms of the solutions to the adjoint equation

$$K_r^*[z] = 0, \tag{5.11}$$

where K_r^* is the usual (formal) adjoint differential operator, cf. [**34**; p. 328]. Indeed, integration by parts shows that if $z(x)$ is any solution to the adjoint equation (5.11), then

$$z\, K_r[u] = D_x F[z, u] \tag{5.12}$$

determines a first integral to (5.10). The explicit formula for the first integral $F[z, u]$ is classical, and described in [**49**].

On the other hand, since a first integral is constant on the solution space $\mathcal{M}$ to (5.10), it *is* an affine annihilator. Choosing a basis $z_1, \ldots, z_r$ for the solution space to the adjoint equation (5.11) produces r linearly independent first integrals, and hence a basis for the operator dual of the subspace. Thus, in this manner we recover Svirshchevskii's construction of the generalized symmetries of homogeneous, linear ordinary differential equations.

Conversely, given a nonlinear ordinary differential equation, $Q[u] = 0$, one can construct the subspaces $\mathcal{M} \subset \mathcal{F}$ it leaves invariant by the following "inverse symmetry procedure". One needs to classify all linear ordinary differential equations which admit $\mathbf{v}_Q$ as a generalized symmetry. For example, it can be shown that if $Q(x, u^{(n)})$ is a *nonlinear* generalized symmetry of a nonsingular linear ordinary differential equation (5.11) of order r, then, necessarily $r \leq 2n+1$. Consequently, any invariant subspace of a nonlinear n^{th} order differential operator has dimension at most $2n+1$. In practice, the determination of the equations of a prescribed order possessing a given generalized symmetry is a straightforward adaptation of the usual infinitesimal computational algorithm for symmetry groups of differential equations, cf. [**34**]. Examples of this procedure appear in [**49, 50, 51**].

For functions depending on more than one independent variable, a similar procedure works, although now one can no longer use a single linear partial differential equation to characterize the subspace, but must employ a linear basis for the annihilator, which will form a linear system of partial differential equations of finite type. The outline of this method is reasonably clear, but full details remain to be worked out. In particular, the existence of corresponding bounds on the dimensions of invariant subspaces for nonlinear partial differential operators is not known.

6. Applications to Differential Equations.

A principal application of our theory is to find explicit solutions to both linear and nonlinear differential equations. In the theory of quasi-exact solvability, one considers an eigenvalue problem

$$Q[u] = \lambda u, \tag{6.1}$$

which, in physical applications, is the stationary Schrödinger equation. A linear differential operator Q is said to be *quasi-exactly solvable* if it leaves a finite-dimensional subspace of wave functions $\mathcal{M} \subset \mathcal{F}$ invariant, so that $Q[u] \in \mathcal{P}(\mathcal{M})$. In this case, restricting to the subspace $\mathcal{M} = \{f_1(x), \ldots, f_r(x)\}$, whereby

$$u(x) = c_1 f_1(x) + \cdots c_r f_r(x), \tag{6.2}$$

reduces the eigenvalue problem (6.1) to a linear eigenvalue problem

$$A\,c = \lambda\,c, \qquad \text{for} \qquad c = (c_1, \ldots, c_r)^T. \tag{6.3}$$

In this way, one reduces the solution of the differential equation (6.1) to an algebraic problem. More generally, one can consider nonlinear differential operators $Q[u] \in \mathcal{Q}(\mathcal{M})$ that leave $\mathcal{M}$ invariant, in which case the restriction of (6.1) to the subspace leads to a system of nonlinear algebraic equations

$$H(c) = \lambda c \tag{6.4}$$

for the coefficients in (6.2).

For evolution equations, the basic idea underlying the method of nonlinear separation goes back to Galaktionov and his collaborators, [**11**–**15**], King, [**26**], and Fushchych and Zhdanov, [**9**, **10**]. The basic idea appears in the following theorem.

Theorem 6.1. *Consider an evolution equation*

$$u_t = Q[u]. \tag{6.5}$$

Suppose the right hand side $Q[u] \in \mathcal{Q}(\mathcal{M})$ preserves a finite-dimensional subspace $\mathcal{M}$, that is

$$Q[u] = \sum_{i=1}^{r} H_i(L_1[u], \ldots, L_r[u])\, f_i + \sum_{\nu=1}^{l} Z_\nu[u] \cdot K_\nu[u],$$

where the $H_i \in \mathcal{C}^\omega(\mathbb{R}^r)$ are arbitrary analytic functions, and where the $Z_\nu \in \mathcal{G}(X)$ are arbitrary operators. Then there exist "separable solutions" of the evolution equation taking the form

$$u(x,t) = \sum_{i=1}^{n} \varphi_i(t) f_i(x), \tag{6.6}$$

if an only if the coefficients $\varphi_1, \ldots, \varphi_n$ are solutions to the dynamical system

$$\frac{d\varphi_i}{dt} = H_i(\varphi_1, \ldots, \varphi_n), \qquad i = 1, \ldots, n. \tag{6.7}$$

The proof is immediate from Theorem 5.3. More generally, one can replace the evolution equation (6.5) by a time-dependent Schrödinger equation

$$i\, u_t = Q[u],$$

resulting a complex first order dynamical system, or a wave-type equation

$$u_{tt} = Q[u],$$

which reduces to a second order dynamical system, with $d^2\varphi_i/dt^2$ on the left hand side of (6.7). Indeed, one can apply the method to dynamical partial differential equations of the general form

$$T[u] = Q[u], \tag{6.8}$$

where T is a linear ordinary differential operator in t, or, even more generally, a time-dependent linear combination of operators $T_k(\partial^k u/\partial t^k)$ where each $T_k \in \mathcal{Q}(\mathcal{M})$ leaves $\mathcal{M}$ invariant. An interesting class of examples are the equations of *Fuchsian type*, studied in depth by Kichenassamy, [**24**] in connection with blow-up and Painlevé expansions, [**25**], in which $T[u]$ is a constant coefficient polynomial in the scaling operator $S = t\partial_t$.

Remark: See Cherniha, [**5**], for an even more general nonlinear separation ansatz in which the basis functions $f_i(x,t)$ (and hence the module) are also allowed to depend on t.

Example 6.2. We can already give many examples of non-linear QES evolution equations in two space variables by using the generators (4.8). Consider for example the simplicial subspace

$$\mathcal{T}_2 = \{1, x, y, x^2, xy, y^2\}.$$

The dual basis of operators is given in (4.9). Therefore, the most general second order $\mathcal{T}_2$ invariant evolution equation takes the form

$$u_t = A + Bx + Cy + Dx^2 + Exy + Fy^2,$$

where A, B, C, D, E, F are arbitrary (linear or nonlinear) functions of the dual operators

$$u_{xx}, \quad u_{xy}, \quad u_{yy}, \quad u_x - xu_{xx} - yu_{xy},$$
$$u_y - xu_{xy} - yu_{yy}, \quad \tfrac{1}{2}x^2u_{xx} + xyu_{xy} + \tfrac{1}{2}y^2u_{yy} - xu_x - yu_y + u.$$

Higher order evolution equations are obtained by adding in arbitrary annihilators. Replacing u_t by u_{tt} or other types of linear temporal differential operators, leads to wave and more general types of equations that leave the indicated subspace invariant. It follows, for example that the evolution equation

$$u_t = \tfrac{1}{4}\, x^2 u_{xx}^2 - xyu_{xy} - y^2 u_{yy} + \tfrac{1}{8}\, y^2 u_{yy}^3$$

admits the solutions given by

$$u(x,y,t) = \frac{x^2}{k_1 - t} + k_2 e^{-t}\, xy + \frac{\sqrt{2}}{\sqrt{1 - k_3 e^{4t}}}\, y^2 + k_3 + k_4 x + k_5 y,$$

where the k_i are arbitrary constants determined by the initial conditions. This is a very simple example which was chosen such that the dynamical system governing the $c_i(t)$ was decoupled and therefore integrable by quadratures.

Example 6.3. As a more substantial example, consider the following rotationally invariant evolution equation

$$u_t = (\Delta - B)[\nabla(u^\sigma\, \nabla u) - Au^{\sigma+1}] + Cu + d, \quad u = u(\vec{x},t), \qquad \sigma = 1, 2, \tag{6.9}$$

in n space variables $\vec{x} = (x^1, \ldots, x^n)$. We claim that the subspace

$$\mathcal{M} = \{1, u_1 = e^{\vec{\alpha}_1 \cdot \vec{x}}, u_2 = e^{\vec{\alpha}_2 \cdot \vec{x}}\}$$

is invariant for a suitable choice of $\vec{\alpha}_1, \vec{\alpha}_2$. To see this let us define

$$R[u] = \operatorname{grad}(u^\sigma \operatorname{grad} u) - A\, u^{\sigma+1},$$

and note the following identities:

$$R[u+v] = R[u] + R[v] + \begin{cases} (\Delta - A)[uv], & \sigma = 1, \\ (\Delta - A)[uv(u+v)], & \sigma = 2. \end{cases}$$

Also let us note that for $\vec{\alpha} \in \mathbb{R}^n$,

$$R[e^{\vec{\alpha}\cdot\vec{x}}] = \left[(\sigma+1)\|\vec{\alpha}\|^2 - A\right] e^{\vec{\alpha}\cdot\vec{x}}.$$

Hence, taking $\vec{\alpha}_1, \vec{\alpha}_2$ so that $\|\vec{\alpha}_i\|^2 = A/(\sigma+1)$, $i = 1, 2$, we will have

$$R[c_1 u_1] = R[c_2 u_2] = 0, \qquad \text{for all} \qquad c_1, c_2 \in \mathbb{R}.$$

Of course R does not annihilate all of $\mathcal{M}$; we are left with quadratic cross-terms. Indeed, taking $u = c_1 u_1 + c_2 u_2$ we have for $\sigma = 1$

$$R[u] = R[c_1 u_1] + R[c_2 u_2] + c_1 c_2 (\Delta - A)[u_1 u_2].$$

Hence, taking $B = \|\vec{\alpha}_1 + \vec{\alpha}_2\|^2$ will ensure that

$$(\Delta - B)[R[u]] = 0.$$

For $\sigma = 2$, we need to take $B = \|2\vec{\alpha}_1 + \vec{\alpha}_2\|^2$, which is the same as $\|\vec{\alpha}_1 + 2\vec{\alpha}_2\|^2$. Indeed, we have:

$$R[u] = c_1 c_2 (\Delta - A)[u_1 u_2 (u_1 + u_2)] = C_1 (e^{(2\vec{\alpha}_1 + \vec{\alpha}_2)\cdot\vec{x}} + e^{(\vec{\alpha}_1 + 2\vec{\alpha}_2)\cdot x}),$$

and the right hand side is manifestly annihilated by $\Delta - B$. It isn't hard to check that $R[u + c_3]$ differs from $R[u]$ by a linear combination of u_1, u_2, and a constant. Hence, $(\Delta - B)[R[u + c_3]]$ continues to lie in $\mathcal{M}$, and therefore the right hand side of equation (6.9) preserves $\mathcal{M}$.

We could also consider the extended module

$$\mathcal{M} = \{u_0 = 1, u_1 = e^{\vec{\alpha}_1 \cdot \vec{x}}, u_2 = e^{\vec{\alpha}_2 \cdot \vec{x}}, u_{-1} = e^{-\vec{\alpha}_1 \cdot \vec{x}}, u_{-2} = e^{-\vec{\alpha}_2 \cdot \vec{x}}\}.$$

The same reasoning as above can be applied to show that $R[u]$, where u is a general element of $\mathcal{M}$, is the sum of a linear combination of $R[u_i]$, $i \in \{1, 2, -1, -2\}$ and a number of cross-terms of the form constant times $u_i u_j$, $i \neq j$, $i, j \in \{0, 1, 2, -1, -2\}$. We therefore require that $\Delta - B$ annihilate $u_1 u_2$, $u_1 u_{-2}$, and their reciprocals. For $\sigma = 1$, this constraint requires

$$B = 2\|\vec{\alpha}_1\|^2 = 2\|\vec{\alpha}_1\|^2, \qquad \text{and} \qquad \vec{\alpha}_1 \cdot \vec{\alpha}_2 = 0$$

so that $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are perpendicular. For $\sigma = 2$, the conditions are:

$$B = 5\|\vec{\alpha}_1\|^2 = 5\|\vec{\alpha}_1\|^2, \qquad \text{and} \qquad \vec{\alpha}_1 \cdot \vec{\alpha}_2 = 0$$

and the conclusions are identical. We remark that, by making use of the rotational invariance of Q, we can without loss of generality assume that $\vec{\alpha}_1 \cdot \vec{x} = x^1$ and that $\vec{\alpha}_2 \cdot \vec{x} = x^2$.

7. Conclusions.

In this paper, we have developed a general theory of invariant subspaces, and shown how these results can apply to provide nonlinear separation of variables ansätze for a wide variety of linear and nonlinear partial differential equations. These operators further provide a nonlinear generalization of the quasi-exactly solvable operators of importance in the algebraic approach to quantum mechanical systems. A number of interesting problems warrant further development of this method.

(1) A significant mystery is the connection of this method with the Lie algebraic approach to quasi-exactly solvable modules. For example, the linear operators preserving the simplicial subspace (4.6) lie in the universal enveloping algebra of a standard realization of the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, cf. [**16**, **46**]. However, this is not evident from the form of the affine annihilator and annihilator. Thus, a fundamental issue is which subspaces admit such a Lie algebraic interpretation of the space of differential operators that leave them invariant.

(2) The "inverse problem" of characterizing the invariant finite-dimensional subspaces for a given linear or nonlinear operator is of critical importance. The extension of Svirshchevskii's symmetry approach, [**49**], is under investigation.

(3) Applications of our analytical Wronskian methods to the algebraic context of D–module theory, [**2**, **31**, **32**], looks quite promising. In particular, the characterization of the analytic or polynomial annihilators of subspaces of rational functions would be a particularly interesting case.

(4) The formulae for the affine annihilators and annihilators are often extremely complicated, even for relatively simple subspaces. (As an example, the reader is invited

to write down the formulae for $\mathcal{M} = \{x^2, xy, y^2\}$.) Moreover, in the final formulae (5.8), (5.7) for the linear and nonlinear operators leaving the subspace invariant, one may encounter a significant amount of simplification and lowering of order. Thus, it would be important to characterize low order operators in $\mathcal{P}(\mathcal{M})$ and $\mathcal{Q}(\mathcal{M})$, as well as "simple" operators of physically relevant type, e.g., elliptic, constant coefficient, Lorentz invariant, etc.

(5) Extensions of these methods to finite difference operators, building on the work of Turbiner, [**46**], can be profitably pursued.

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