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SYMMETRY OF AFFINE SYSTEMS*

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INTRODUCTION

We consider a nonlinear dynamical control system (DCS)

$$\dot{x} = f(t, x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control, $f:\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth (C^∞) function, $\dot{x} \equiv dx/dt$. A solution of this system is any pair of C^∞ functions $(x(t), u(t))$ defined on some interval $T \subset \mathbb{R}^1$ that reduce (1) to an identity on T. System (1) can be interpreted as an "underdetermined" system of ordinary differential equations with m missing equations of the form $\dot{u} = g(t, x, u)$. This interpretation [1] enables us to study system (1) in the framework of geometrical theory [2-3] and group analysis [4-6] of differential equations. The DCS (1) is treated as a submanifold in the manifold of 1-jets $J^1(\pi)$ of the smooth N-dimensional vector fibering $\pi:E \to \mathbb{R}^1$, $N = n + m$ of co-dimension less than N. A Cartan distribution is defined on the manifold $J^1(\pi)$, and the transformations of $J^1(\pi)$ preserving this distribution are called Lie transformations [2]. Lie transformations preserving the DCS (1) (i.e., taking the specified submanifold in $J^1(\pi)$ into itself) are called symmetries of the DCS. In the terminology of [2], these are classical external symmetries. Their particular cases are the symmetries previously studied in [7-11]. In the present article, we mainly focus on infinitesimal symmetries of affine dynamical control systems (ADCS). These symmetries play an important role in the problem of decomposition of ADCS [1].

INFINITESIMAL SYMMETRIES

On the manifold $J^1(\pi)$ we define the local coordinates $(t, x, u, p, q) = (t, x_1, \ldots, x_n, u_1, \ldots, u_m, p_1, \ldots, p_n, q_1, \ldots, q_m)$, where p corresponds to $\dot{x}(t)$ and q corresponds to $\dot{u}(t)$. Then the DCS (1) is written as the system of equations $p = f(t, x, u)$ and thus represents a submanifold

$$\Phi = \left\{(t, x, p, q) \in J^1(\pi) : \tilde{f}(t, x, u, p, q) = 0\right\}$$

in $J^1(\pi)$ of co-dimension n, where $\tilde{f}(t, x, u, p, q) = f(t, x, u) - p$. Each section $(x(t), u(t))$, $t \in T \subset \mathbb{R}^1$ has its continuation – a curve l_{xu} in $J^1(\pi)$ of the form $t \to (t, x(t), u(t), \dot{x}(t), \dot{u}(t))$. The section $(x(t), u(t))$ is a solution of the DCS (1) if and only if $l_{xu} \subset \Phi$.

The Cartan distribution on $J^1(\pi)$ in these local coordinates is defined by a system of 1-forms

$$\omega_i = dx_i - p_i dt, \quad i = 1, 2, \ldots, n,$$
$$\tau_j = du_j - q_j dt, \quad j = 1, 2, \ldots, m.$$

The curves l_{xu} obtained by continuation of the sections $(x(t), u(t))$ are the integral curves of the Cartan distribution. We know that a locally maximal one-dimensional integral submanifold of the Cartan distribution (with the exception of the singular points of the projection $\pi_1:J^1(\pi) \to \mathbb{R}^1$) locally has the form of continued curves l_{xu}. These submanifolds are called R-manifolds [2]. R-manifolds contained in Φ will be called generalized solutions of the DCS (1). The Lie transformations of the space $J^1(\pi)$ preserve the Cartan distribution, and thus take R-manifolds into R-manifolds. If a Lie transformation takes the manifold Φ into itself (and thus takes each generalized solution into a generalized solution), then it is called a (classical

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external) symmetry of the DCS (1). Since $N = n + m > 1$, every Lie transformation is a continuation of some transformation of the manifold $J^0(\pi) = E$ [2-5].

The preceding discussion is extended to one-parameter groups of Lie transformations corresponding to their infinitesimal generators. These are vector fields, and they are called Lie fields. In our case, since the Lie transformations are lifted from the manifold E, the Lie vector fields are also a lifting of vector fields defined on E. Specifically, if the vector field on E is defined in the coordinates (t, x, u) by the formula

$$X = \xi(t,x,u)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(t,x,u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \varphi_j(t,x,u)\frac{\partial}{\partial u_j}, \tag{2}$$

then its lifting $X^{(1)}$ on $J^1(\pi)$ is the field [2-5]

$$X^{(1)} = X + \sum_{i=1}^{n} \zeta_i(t,x,u,p,q)\frac{\partial}{\partial p_i} + \sum_{j=1}^{m} \psi_j(t,x,u,p,q)\frac{\partial}{\partial q_j}, \tag{3}$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)^T$ and $\psi = (\psi_1, \ldots, \psi_m)^T$ are obtained from the formulas

$$\zeta = D\eta - pD\xi, \quad \psi = D\varphi - qD\xi, \tag{4}$$

and $\eta = (\eta_1, \ldots, \eta_n)^T$, $\varphi = (\varphi_1, \ldots, \varphi_m)^T$, $p = (p_1, \ldots, p_n)^T$, $q = (q_1, \ldots, q_m)^T$,

$$D = \frac{\partial}{\partial t} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} q_j \frac{\partial}{\partial u_j}$$

is the total derivative operator with respect to the independent variable t.

If the Lie field (3) is tangent to the submanifold Φ, then the corresponding local one-parameter Lie transformation group takes Φ into itself. In this case, the field (3) is called an infinitesimal (classical external) symmetry of the DCS (1). A necessary and sufficient condition of tangency of the field (3) to Φ is

$$X^{(1)}(\tilde{f}_i)|_\Phi = 0, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $(\tilde{f}_1, \ldots, \tilde{f}_n)^T = \tilde{f}$.

In what follows DCS symmetries are infinitesimal classical external symmetries of the form (3). Since the vector field (2) uniquely defines the lifting (3) and conversely, we do not distinguish between the names of these fields, and (2) is also called a symmetry if (3) is a symmetry.

DEFINING EQUATIONS FOR DCS SYMMETRIES

Condition (5) combined with (2)-(4) is transformed in coordinate form to

$$\xi\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\varphi - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}f - \frac{\partial \eta}{\partial u}q + f(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}f + \frac{\partial \xi}{\partial u}q) = 0 \tag{6}$$

This equality should hold for all (t, x, u, q). System (6) is linear in q, and it therefore decomposes into two systems

$$\xi\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\varphi - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}f + f\frac{\partial \xi}{\partial t} + f\frac{\partial \xi}{\partial x}f = 0, \tag{7}$$

$$\frac{\partial \eta}{\partial u} - f\frac{\partial \xi}{\partial u} = 0. \tag{8}$$

Equations (7)-(8) are called the defining equations for the symmetries of the DCS (1). They can be written in a more compact form by utilizing the concepts of vector field and commutator of vector fields.

The local coordinates (t, x, u, p, q) on the manifold $J^1(\pi)$ are induced by the local coordinates (t, x, u) of the manifold $J^0(\pi)$. The DCS (1) can be interpreted as a vector field F on $J^0(\pi)$ which in these local coordinates has the form

$$F = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad (f_1, \ldots, f_n)^T = f. \tag{9}$$

We introduce the vector field

$$H = \xi \frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i}, \tag{10}$$

and rewrite the symmetry X in the form

$$X = H + \sum_{j=1}^{m} \varphi_j \frac{\partial}{\partial u_j}.$$

For arbitrary vector fields V and W defined on some manifold, we denote by $[V, W]$ their Lie bracket (or commutator). If the fields V and W are written in the local coordinates $(z_1, \ldots, z_s)$ in the form

$$V = \sum_{i=1}^{s} v_i \frac{\partial}{\partial z_i}, \qquad W = \sum_{i=1}^{s} w_i \frac{\partial}{\partial z_i},$$

then their commutator in these coordinates has the form

$$[V, W] = \sum_{i=1}^{s} \left(\sum_{l=1}^{s} v_l \frac{\partial w_l}{\partial z_i} - w_l \frac{\partial v_l}{\partial z_i} \right) \frac{\partial}{\partial z_i}. \tag{11}$$

Let

$$F_j = \left[\frac{\partial}{\partial u_j}, F \right], \quad j = 1, \ldots, m, \tag{12}$$

where $\partial/\partial u_j$ are coordinate vector fields on $J^0(\pi)$. Then system (7) can be written in the form [1]

$$[F, H] - F(\xi)F = \sum_{j=1}^{m} \varphi_j F_j. \tag{13}$$

Condition (13) implies that the field $[F, H] - F(\xi)F$ is a linear combination of the fields $F_1, \ldots, F_m$ with the coefficients $\varphi_1, \ldots, \varphi_m$. This field is therefore contained in the distribution $\mathcal{F}_u$ on $J^0(\pi)$ generated by the vector fields $F_1, \ldots, F_m$, i.e.,

$$[F, H] - F(\xi)F \in \mathcal{F}_u. \tag{14}$$

Using the vector fields introduced above, we rewrite condition (8) in the form

$$\left[\frac{\partial}{\partial u_j}, H \right] - \frac{\partial \xi}{\partial u_j} F = 0, \quad j = 1, \ldots, m. \tag{15}$$

Conditions (13), (15) are merely an alternative form of the defining equations (7)-(8). Condition (14) is equivalent to (7) in the following sense: if the vector field H satisfies (14), then it defines at least one vector field X in $J^0(\pi)$ that is projected into the field H. Thus, the problem of describing the symmetries of the DCS (1) reduces to the determination of vector fields H of the form (1) that satisfy relationships (14), (15).

In general, to determine the fields H from conditions (14), (15) we have to solve some generalization of the problem of integrating a system of first-order partial differential equations with a common principal part [12]. In some cases, the DCS symmetries can be described explicitly [1].

DEFINING EQUATIONS FOR AFFINE SYSTEMS

An affine DCS (ADCS) is a DCS (1) linear in control. In this case, it can be written as

$$\dot{x} = a(t, x) + \sum_{j=1}^{m} b_j(t, x) u_j, \tag{16}$$

where a, b_1, ..., b_m: $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions. To system (16) correspond unique vector fields

$$A = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \qquad B_j = \sum_{i=1}^{n} b_{ij} \frac{\partial}{\partial x_i}, \quad j = 1, \ldots, m,$$

where $(a_1, \ldots, a_n)^{\mathrm{T}} = a(t, x)$, $(b_{1j}, \ldots, b_{nj})^{\mathrm{T}} = b_j(t, x)$. The vector field F associated to the DCS (1) of general form by formula (9) has the form

$$F = A + \sum_{j=1}^{m} u_j B_j. \tag{17}$$

The vector fields F_j are identical with B_j, $j = 1, 2, \ldots, m$, and the distribution $\mathcal{F}_u$ is generated by the fields B_1, ..., B_m (to emphasize this fact, we introduce an alternative notation $\mathcal{B}$ for this distribution).

For the ADCS (16) condition (15) is written in the form

$$\left[\frac{\partial}{\partial u_k}, H\right] - \frac{\partial \xi}{\partial u_k}\left(A + \sum_{j=1}^{m} u_j B_j\right) = 0, \quad k = 1, \ldots, m. \tag{18}$$

Consider condition (13). It implies that

$$[F, H] - F(\xi) F = \sum_{j=1}^{m} \varphi_j B_j. \tag{19}$$

Substituting representation (17) in (19), we obtain

$$[A, H] + \sum_{j=1}^{m} [u_j B_j, H] - A(\xi) A - \sum_{j=1}^{m} u_j B_j(\xi) A - \sum_{j=1}^{m} u_j A(\xi) B_j - \sum_{j,k=1}^{m} u_j u_k B_j(\xi) B_k = \sum_{j=1}^{m} \varphi_j B_j \in \mathcal{B},$$

which gives

$$[A, H] - A(\xi) A + \sum_{j=1}^{m} u_j \{[B_j, H] - B_j(\xi) A\} \in \mathcal{B}. \tag{20}$$

The vector fields A and B_j, $j = 1, 2, \ldots, m$, are independent of the controls. If ξ is also independent of the controls, i.e., $d_u \xi \equiv 0$, then it follows from (15) that the vector field H is independent of the controls. In this case, condition (20) is linear in u_j and thus decomposes into the following system:

$$[A, H] - A(\xi) A \in \mathcal{B}, \tag{21}$$

$$[B_j, H] - B_j(\xi) A \in \mathcal{B}, \quad j = 1, \ldots, m. \tag{22}$$

In fact, the supplementary condition of control independence of ξ is not essential, and the system of relationships (18), (20) is equivalent to the system (18), (21), (22). Relationship (20) obviously follows from (21), (22). We will show that if the field H satisfies conditions (18), (20), then it also satisfies conditions (21), (22). Note that $[\partial/\partial u_j, \mathcal{B}] \subset \mathcal{B}$, $j = 1, \ldots, m$, because the distribution $\mathcal{B}$ is generated by the control-independent fields B_j. Therefore from condition (19) we have

$$\left[\frac{\partial}{\partial u_j},[F,H]\right]-\left[\frac{\partial}{\partial u_j},F(\xi)F\right]\in\mathcal{B}. \tag{23}$$

Using the identities

$$[A,\alpha B]=A(\alpha)B+\alpha[A,B]$$
$$[A,[B,C]]=[B,[A,C]]+[[A,B],C],$$

we transform (23) to the form

$$\left[F,\left[\frac{\partial}{\partial u_j},H\right]\right]+\left[\left[\frac{\partial}{\partial u_j},F\right],H\right]-\left(\frac{\partial}{\partial u_j}F(\xi)\right)F-F(\xi)\left[\frac{\partial}{\partial u_j},F\right]\in\mathcal{B}.$$

Since $[\partial/\partial u_j, F] = B_j$, $j = 1, \dots, m$, we use (15), (17) to obtain successively

$$\left[F,\frac{\partial\xi}{\partial u_j}F\right]+[B_j,H]-\left(\frac{\partial}{\partial u_j}F(\xi)\right)F\in\mathcal{B},$$

or

$$[B_j,H]+\left(F\frac{\partial}{\partial u_j}(\xi)\right)F-\left(\frac{\partial}{\partial u_j}F(\xi)\right)F\in\mathcal{B},$$

i.e.,

$$[B_j,H]+\left[F,\frac{\partial}{\partial u_j}\right](\xi)F\in\mathcal{B}.$$

We finally obtain

$$[B_j,H]-B_j(\xi)A\in\mathcal{B},$$

because $F-A\in\mathcal{B}$.

This proves relationship (22). Relationship (21) follows directly from (20) and (22).

Condition (8) on the symmetries of the DCS (1), and therefore condition (18) on the symmetries of the ADCS (16), imply for $\dim\mathcal{B} = m > 1$ that the vector field H is control-independent [1]. Using this fact, we can state our results in the form of the following assertions.

THEOREM 1. The vector field H defines a symmetry of the form (2) for the ADCS (16) with $m > 1$ controls if and only if this field is control-independent and satisfies conditions (21), (22).

THEOREM 2. The vector field H defines a symmetry of the form (2) for the ADCS

$$\dot{x}=a(t,x)+b(t,x)u,\qquad x\in\mathbb{R}^n,u\in\mathbb{R}^1 \tag{24}$$

with scalar control ($m = 1$) if and only if this field satisfies the conditions

$$[A,H]-A(\xi)A\in\mathcal{B}, \tag{25}$$

$$[B,H]-B(\xi)A\in\mathcal{B}, \tag{26}$$

$$\left[\frac{\partial}{\partial u},H\right]-\frac{\partial\xi}{\partial u}(A+uB)=0, \tag{27}$$

where

$$A = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i(t,x)\frac{\partial}{\partial x_i}, \quad (a_1(t,x),\dots,a_n(t,x))^{\mathrm{T}} = a(t,x),$$
$$B = \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i}, \quad (b_1(t,x),\dots,b_n(t,x))^{\mathrm{T}} = b(t,x). \tag{28}$$

TIME-DEPENDENT SYMMETRIES

If the DCS (1) has a symmetry for which the time coordinate is control-dependent, i.e., $d_u\xi(t_0, x_0, u_0) \neq 0$ at some point $M_0(t_0, x_0, u_0)$, then in the neighborhood of this point the system actually has a single control: there exists a function $h(t, x, v)$ such that $f(t, x, u) \equiv h(t, x, v)$, where $v = \xi(t, x, u)$. This follows from the fact that $\mathcal{F}_u \leq 1$ [1].

All the questions considered here are of local character. We naturally assume that the dimensions of the relevant distributions are locally constant, because the set of points where this condition breaks down is nowhere dense. In what follows, we indeed adopt this assumption.

The condition $\dim\mathcal{F}_u \leq 1$ (with the previous assumption) implies that either $\mathcal{F}_u \equiv 0$, or $\dim\mathcal{F}_u \equiv 1$. In the first case, this is a system without control, i.e., a system of ordinary differential equations. In the second case, the system actually has a single independent control.

These conclusions remain valid for the affine system (16). If the ADCS has a symmetry with a control-dependent time coordinate, then $\dim \mathcal{B} \leq 1$. If we ignore the case without any control, then the ADCS can be written in the form (24) after a change of control. This is the case that we consider here.

THEOREM 3. The affine system (24) has a symmetry of the form (2) that satisfies the condition $d_u\xi(t_0, x_0, u_0) \neq 0$ if and only if the vector fields A and B (28) corresponding to the system satisfy in some neighborhood of the point $M_0(t_0, x_0, u_0)$ the condition

$$[A, B] \in \mathcal{B}, \tag{29}$$

where $\mathcal{B} = \operatorname{span}\{B\}$.

Proof. Necessity. Commuting relationship (26) (which is true for any symmetry) with the vector field $\partial/\partial u$, we obtain

$$\left[\frac{\partial}{\partial u}, [B, H]\right] - \left[\frac{\partial}{\partial u}, B(\xi)A\right] \in \mathcal{B}.$$

After identical transformations this can be written as

$$\left[B, \left[H, \frac{\partial}{\partial u}\right]\right] + \frac{\partial}{\partial u}(B(\xi))A \in \mathcal{B}.$$

Using (27) and noting that $[B, \xi_u uB] \in \mathcal{B}$, we obtain

$$\left[B, \frac{\partial\xi}{\partial u}A\right] - \frac{\partial}{\partial u}(B(\xi))A \in \mathcal{B}, \tag{30}$$

whence

$$\left[\frac{\partial}{\partial u}, B\right](\xi)A + \frac{\partial\xi}{\partial u}[B, A] \in \mathcal{B}.$$

But $[\partial/\partial u, B] = 0$, and thus

$$\frac{\partial \xi}{\partial u}[B, A] \in \mathcal{B}.$$

This relationship is true for any symmetry of the ADCS (24). Therefore, if there are symmetries with $\xi_u \neq 0$, then condition (29) necessarily holds.

Sufficiency. If condition (29) holds, then we can choose functions ξ and h so that the vector field

$$H = \xi A + hB \tag{31}$$

satisfies the conditions of Theorem 2. Indeed,

$$\begin{aligned}
[A, H] - A(\xi)A &= [A, \xi A + hB] - A(\xi)A = A(\xi)A + A(h)B + h[A, B] - A(\xi)A = \\
&= A(h)B + h[A, B] \in \mathcal{B}, \\
[B, H] - B(\xi)A &= [B, \xi A + hB] - B(\xi)A = B(\xi)A + \xi[B, A] + B(h)B - B(\xi)A = \\
&= B(h)B + \xi[B, A] \in \mathcal{B},
\end{aligned}$$

because conditions (25), (26) hold for any functions ξ and h. It remains to check condition (27). Substituting (31) in (27), we obtain

$$\left[\frac{\partial}{\partial u}, \xi A + hB\right] - \frac{\partial \xi}{\partial u}(A + uB) = \frac{\partial \xi}{\partial u}A + \frac{\partial h}{\partial u}B - \frac{\partial \xi}{\partial u}A - \frac{\partial \xi}{\partial u}uB = \left(\frac{\partial h}{\partial u} - u\frac{\partial \xi}{\partial u}\right)B = 0.$$

Thus, condition (27) is satisfied if the functions ξ and h satisfy the relationship $h_u' - u\xi_u' = 0$. This relationship holds, for instance, for the functions $\xi \equiv 2u$, $h \equiv u^2$.

Remark. It is shown in [1] that if the ADCS (24) has a symmetry (2) with $d_u(\xi)(M_0) \neq 0$, and also $B(M_0) \neq 0$, then in some neighborhood of the point $M_0(t_0, x_0, u_0)$ we can choose a coordinate system (t, y, u), $y = (y_1, \ldots, y_n)$ such that in the new system the ADCS has the form

$$\begin{aligned}
\dot{y}_i &= \alpha_i(t, y_1, \ldots, y_{n-1}), \qquad i = 1, \ldots, n-1, \\
\dot{y}_n &= \alpha_n(t, y) + \beta(t, y)u.
\end{aligned}$$

Now this follows also from Theorem 2, because condition (29) implies A, B-invariance of the distribution $\mathcal{B}$ [13].

ADCS WITH SCALAR CONTROL AND RANK CONTROLLABILITY CONDITION

For the ADCS (24) consider the vector fields

$$\mathrm{ad}_A^0 B = B, \quad \mathrm{ad}_A^1 B = [A, B], \quad \mathrm{ad}_A^{r+1} B = [A, \mathrm{ad}_A^r B], \quad r > 0.$$

If at some point (t, x) the distribution generated by the vector fields

$$\mathrm{ad}_A^0 B, \ldots, \mathrm{ad}_A^{n-1} B, \tag{32}$$

is n-dimensional (i.e., of the same dimension as the state space), we say that the ADCS (24) satisfies the rank condition of controllability [13]. This condition holds in some neighborhood of the point (t, x).

THEOREM 4. Assume that the ADCS (24) satisfies the rank condition of controllability at the point (t, x). The vector field H (11) defines a symmetry X of the form (2) in the neighborhood of the point (t, x) if and only if this field is representable in the form

$$H = \xi A + \sum_{i=0}^{n-1} h_i \,\mathrm{ad}_A^i B, \tag{33}$$

and the functions $\xi = \xi(t, x)$, $h_i = h_i(t, x)$, $i = 0, \ldots, n-1$ satisfy the equations

$$A(h_j) = -h_{j-1} - h_{n-1}\alpha_j, \quad j = 1, 2, \ldots, n-1, \tag{34}$$

$$B(h_j) + \sum_{k=1}^{n-1} \beta_{jk} h_k = 0, \quad j = 2, 3, \ldots, n-1, \tag{35}$$

$$\xi = B(h_1) + \sum_{k=1}^{n-1} \beta_{1k} h_k, \tag{36}$$

where $\alpha_i = \alpha_i(t, x)$ and $\beta_{jk} = \beta_{jk}(t, x)$ are the coefficients of the expansions

$$\operatorname{ad}_A^n B = \sum_{i=0}^{n-1} \alpha_i \operatorname{ad}_A^i B, \tag{37}$$

$$[B, \operatorname{ad}_A^k B] = \sum_{j=0}^{n-1} \beta_{jk} \operatorname{ad}_A^j B, \quad k = 1, \ldots, n-1. \tag{38}$$

Proof. Since the rank controllability condition holds, every vector field H of the form (11) defining an ADCS symmetry can be expanded in the sum (33), where the coefficients ξ and h_i in general are control-dependent. Note that the t coordinate of every field $\operatorname{ad}_A{}^i B$ is 0. Therefore, in the representation (33) the function ξ is the time coordinate of the field H. Apply Theorem 2 to the expansion (33). Using (33) for H, we obtain

$$\begin{aligned}
[A, H] - A(\xi)A &= A(\xi)A + \sum_{i=0}^{n-1} \left(A(h_i) \operatorname{ad}_A^i B + h_i \operatorname{ad}_A^{i+1} B\right) - A(\xi)A = \\
&= \sum_{i=0}^{n-1} A(h_i) \operatorname{ad}_A^i B + \sum_{j=1}^{n-1} h_{j-1} \operatorname{ad}_A^j B + h_{n-1} \sum_{i=0}^{n-1} \alpha_i \operatorname{ad}_A^i B = \\
&= \sum_{i=1}^{n-1} (A(h_i) + h_{i-1} + h_{n-1}\alpha_i) \operatorname{ad}_A^i B + (A(h_0) + h_{n-1}\alpha_0) B.
\end{aligned}$$

Thus, relationship (34) is necessary and sufficient for condition (25) to be true.

Similarly,

$$\begin{aligned}
[B, H] - B(\xi)A &= B(\xi)A + \xi[B, A] + \sum_{i=0}^{n-1} \left(B(h_i) \operatorname{ad}_A^i B + h_i [B, \operatorname{ad}_A^i B]\right) - B(\xi)A = \\
&= -\xi \operatorname{ad}_A B + \sum_{i=0}^{n-1} B(h_i) \operatorname{ad}_A^i B + \sum_{k=1}^{n-1} h_k \sum_{i=0}^{n-1} \beta_{ik} \operatorname{ad}_A^i B = \\
&= -\xi \operatorname{ad}_A B + \sum_{i=0}^{n-1} \left(B(h_i) + \sum_{k=1}^{n-1} h_k \beta_{ik} \right) \operatorname{ad}_A^i B.
\end{aligned}$$

Therefore condition (26) holds if and only if

$$\xi = B(h_1) + \sum_{k=1}^{n-1} h_k \beta_{1k},$$

which is equivalent to (36), and

$$B(h_i) + \sum_{k=1}^{n-1} h_k \beta_{ik} = 0, \quad i = 2, \ldots, n-1,$$

which in turn is identical with (35).

Finally, the rank controllability condition implies that rank$(B, \mathrm{ad}_A B) = 2$. But then $\xi_u = 0$ by Theorem 3. Thus, condition (27) for the field H implies that $[\partial/\partial u, H] = 0$, and since the field H is in the form (33), we obtain

$$\sum_{i=0}^{n-1} h'_{iu}\,\mathrm{ad}^i_A\, B = 0.$$

Thus, $h_{iu}' = 0$, $i = 0, 1, \dots, n-1$. Q.E.D.

The system of equations (34)-(35) can be regarded as a system of equations for h_j, $j = 2, \dots, n-1$, with the same principal part [12], which parametrically depends on the function $h_1(t, x)$. Consistency of this system is a condition on the function h_1. Given all these functions h_j, we can find all the remaining coefficients of expansion (33) from (34), (36).

ADCS WITH SCALAR CONTROL OF SMALL DIMENSION

Two-Dimensional Systems

If a system with a two-dimensional state space and scalar control ($m = 1$, $n = 2$) satisfies the rank controllability condition, then Theorem 4 takes a simpler form. In this case, conditions (34)-(35) are trivial and everything is determined by an arbitrary function $h_1(t, x)$, because $h_0 = -A(h_1) - h_1\alpha_1$, $\xi = B(h_1) + \beta_{11}h_1$.

Assume that a two-dimensional ADCS does not satisfy the rank controllability condition. If the relevant distributions are of locally constant dimensions, then either rank$\{B, \mathrm{ad}_A B\} \equiv 0$ or rank$\{B, \mathrm{ad}_A B\} \equiv 1$. In the first case, the system is control-independent. In the second case we may assume, again locally, that $B \neq 0$. Then condition (29) is satisfied and the ADCS has symmetries that satisfy the condition $\xi_u(M) \neq 0$. Therefore the choice of a new system of coordinates locally reduces the ADCS to the form

$$\dot{y}_1 = \alpha_1(t, y_1),$$
$$\dot{y}_2 = \alpha_2(t, y_1, y_2) + \beta(t, y_1, y_2)u.$$

Three-Dimensional Systems

If an ADCS with a three-dimensional state space and scalar control ($m = 1$, $n = 3$) satisfies the rank controllability condition, then conditions (34)-(35) on the vector field H take the form

$$A(h_1) + h_0 + h_2\alpha_1 = 0,$$
$$A(h_2) + h_1 + h_2\alpha_2 = 0,$$
$$B(h_2) + \beta_{21}h_1 + \beta_{22}h_2 = 0,$$

whence

$$h_1 = -A(h_2) - h_2\alpha_2, \tag{39}$$

where

$$B(h_2) - \beta_{21}\big(A(h_2) + h_2\alpha_2\big) + \beta_{22}h_2 = 0. \tag{40}$$

Relationship (40) is an equation for the function h_2, which always has a solution. We thus conclude that every symmetry is defined by the function $h_2(t, x)$ that satisfies (40). Given h_2, we obtain h_1 from (39) and the functions ξ and h_0 from the relationships

$$\begin{aligned} h_0 &= -A(h_1) - h_2\alpha_1, \\ \xi &= B(h_1) + \beta_{11}h_1 + \beta_{12}h_2. \end{aligned} \tag{41}$$

Consider an example:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u. \end{cases}$$

This ADCS satisfies the rank controllability condition. Here $\alpha_i = 0$, $\beta_{ij} = 0$. Relationship (40) thus reduces to the equality

$$\frac{\partial h_2}{\partial x_3} = 0.$$

This means that the function h_2 is independent of x_3, $h_2 = h_2(t, x_1, x_2)$. The other functions h_1, h_2, ξ are calculated from Eqs. (39), (41).

If a three-dimensional ADCS does not satisfy the rank controllability condition, then by local constancy of dimensions we may have three cases:

$$\operatorname{rank}\{B, \operatorname{ad}_A B, \operatorname{ad}_A^2 B\} \equiv r, \tag{42}$$

with $r = 0, 1, 2$. The case $r = 0$ arises when the ADCS is control-independent. Let us consider the case $r = 2$.

Assume that

$$\operatorname{rank}\{B, \operatorname{ad}_A B, [B, \operatorname{ad}_A B]\} \equiv 3. \tag{43}$$

Then the vector field H is representable in the form

$$H = \xi A + h_0 B + h_1 \operatorname{ad}_A B + h_2 [B, \operatorname{ad}_A B],$$

and the vector field $\operatorname{ad}_A^2 B$ is decomposed in the fields B and $\operatorname{ad}_A B$:

$$\operatorname{ad}_A^2 B = \alpha_0 B + \alpha_1 \operatorname{ad}_A B. \tag{44}$$

Also consider the decomposition

$$[B, [B, \operatorname{ad}_A B]] = \beta_0 B + \beta_1 \operatorname{ad}_A B + \beta_2 [B, \operatorname{ad}_A B]. \tag{45}$$

From these representations we have

$$\begin{aligned} [A, H] - A(\xi)A = A(\xi)A + A(h_0)B + h_0 \operatorname{ad}_A B + A(h_1) \operatorname{ad}_A B + \\ + h_1 \operatorname{ad}_A^2 B + A(h_2)[B, \operatorname{ad}_A B] + h_2 [A, [B, \operatorname{ad}_A B]] - A(\xi)A. \end{aligned}$$

Since

$$\begin{aligned} [A, [B, \operatorname{ad}_A B]] &= -[\operatorname{ad}_A B, \operatorname{ad}_A B] + [B, \operatorname{ad}_A^2 B] = \\ &= B(\alpha_0)B + B(\alpha_1) \operatorname{ad}_A B + \alpha_1 [B, \operatorname{ad}_A B] + B(\alpha_2)[B, \operatorname{ad}_A B] + \\ &+ \alpha_2 (\beta_0 B + \beta_1 \operatorname{ad}_A B + \beta_2 [B, \operatorname{ad}_A B]), \end{aligned}$$

using (44) we obtain

$$\begin{aligned} [A, H] - A(\xi)A = \big(A(h_0) + h_1\alpha_0 + h_2 B(\alpha_0) + \alpha_2\beta_0\big)B + \\ + \big(h_0 + A(h_1) + h_1\alpha_1 + h_2 B(\alpha_1) + h_2\alpha_2\beta_1\big) \operatorname{ad}_A B + \\ + \big(h_1\alpha_2 + A(h_2) + h_2\alpha_1 + h_2 B(\alpha_2) + h_2\alpha_2\beta_2\big)[B, \operatorname{ad}_A B]. \end{aligned}$$

Thus, condition (25) is satisfied if and only if

$$\begin{aligned} h_0 + A(h_1) + h_1\alpha_1 + h_2 B(\alpha_1) + h_2\alpha_2\beta_1 = 0, \\ h_1\alpha_2 + A(h_2) + h_2\alpha_1 + h_2 B(\alpha_2) + h_2\alpha_2\beta_2 = 0. \end{aligned} \tag{46}$$

Now,

$$[B, H] - B(\xi)A = B(\xi)A - \xi \operatorname{ad}_A B + B(h_0)B + \\ + B(h_1)\operatorname{ad}_A B + h_1[B, \operatorname{ad}_A B] + B(h_2)[B, \operatorname{ad}_A B] + \\ + h_2\big[B, [B, \operatorname{ad}_A B]\big] - B(\xi)A.$$

Using (45) we obtain

$$[B, H] - B(\xi)A = \big(B(h_0) + h_2\beta_0\big)B + \\ + \big(-\xi + B(h_1) + h_2\beta_1\big)\operatorname{ad}_A B + \big(h_1 + B(h_2) + h_2\beta_2\big)[B, \operatorname{ad}_A B].$$

Thus, condition (26) is satisfied if and only if

$$\begin{aligned} -\xi + B(h_1) + h_2\beta_1 &= 0, \\ h_1 + B(h_2) + h_2\beta_2 &= 0. \end{aligned} \tag{47}$$

Since $\operatorname{rank}\{B, \operatorname{ad}_A B\} = 2$ (this follows, e.g., from (43)), we conclude by Theorem 3 that ξ is control-independent, and thus the vector field H is control-independent. This is equivalent to the conditions $(h_i)_u{}' = 0$, $i = 0, 1, 2$ and $\xi_u{}' = 0$.

Finally, from conditions (46)-(47) we have

$$\begin{aligned} h_1 &= -B(h_2) - h_2\beta_2, \\ \xi &= B(h_1) + h_2\beta_1, \\ h_0 &= -A(h_1) - h_1\alpha_1 - h_2\big(B(\alpha_1) + \alpha_2\beta_1\big), \end{aligned}$$

i.e., the functions ξ, h_0, h_1 are expressed in terms of the function h_2. The function h_2 by the same conditions is the solution of a first-order linear homogeneous partial differential equation

$$A(h_2) - \alpha_2 B(h_2) + h_2\big(\alpha_1 + B(\alpha_2)\big) = 0.$$

Assume that the three-dimensional ADCS satisfies (42) with $r = 2$ in some neighborhood of the point M, and

$$\operatorname{rank}\{B, \operatorname{ad}_A B, [B, \operatorname{ad}_A B]\} = 2.$$

Also assume that the vector fields B and $\operatorname{ad}_A B$ generate a two-dimensional involutive distribution. Then in the given neighborhood of the point M the vector fields A, B, $\operatorname{ad}_A B$ generate a three-dimensional involutive distribution. This distribution in some neighborhood of M_0 has an integral $h(t, x)$ for which $dh(M_0) \neq 0$. At the same time the function $h(t, x)$ is the first integral of the ADCS, because its derivative by the ADCS is a derivative in the direction of the vector field $A + uB$ contained in the above-mentioned three-dimensional distribution. Therefore, the level surfaces $h(t, x) = h_0 = \text{const}$ are invariant manifolds of the ADCS, and the restrictions of the ADCS to these manifolds are h_0-parametric nonstationary two-dimensional ADCS that satisfy the rank controllability conditions.

REFERENCES

1. A. N. Kanatnikov and A. P. Krishchenko, "Symmetry and decomposition of nonlinear systems," Diff. Uravn., **30**, No. 11, 1880-1891 (1994).
2. A. M. Vinogradov, I. S. Krasil'shchik, and V. V. Lychagin, An Introduction to the Geometry of Nonlinear Differential Equations [in Russian], Nauka, Moscow (1986).
3. P. Olver, Application of Lie Groups to Differential Equations [Russian translation], Mir, Moscow (1989).
4. L. V. Ovsyannikov, Group-Theoretical Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
5. N. Kh. Ibragimov, Transformation Groups in Mathematical Physics [in Russian], Nauka, Moscow (1983).

6. Yu. N. Pavlovskii and G. N. Yakovenko, Optimization Methods and Their Applications [in Russian], Nauka, Novosibirsk (1982), pp. 155-189.
7. V. I. Elkin, "Automorphisms and decomposition of affine controlled systems," Dokl. Akad. Nauk SSSR, **316**, No. 1, 30-32 (1991).
8. G. N. Yakovenko, "Solution of the controllability problem using symmetry," in: Applied Mechanics and Control Processes [in Russian], MFTI, Moscow (1991), pp. 17-31.
9. G. N. Yakovenko, "State symmetries in controlled systems," in: Applied Mechanics and Mathematics [in Russian], MFTI, Moscow (1992), pp. 155-176.
10. J. W. Grizzle and S. I. Marcus, "The structure of nonlinear control systems possessing symmetries," IEEE Trans. Autom. Contr., **30**, No. 3, 248-257 (1985).
11. A. J. van der Schaft, "Symmetries and conservation laws for Hamiltonian systems with inputs and outputs: A generalization of Noether's theorem," Syst. Contr. Lett., **1**, 108-115 (1981).
12. V. I. Elkin, "General solution of systems of partial differential equations with the same principal part," Diff. Uravn., **21**, No. 8, 1389-1398 (1985).
13. H. Nijmeijer and A. J. van der Schaft, Nonlinear Dynamical Control Systems, Springer, New York (1990).