Lectures on the Geometric Group Theory

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Contents

1 Preliminaries

1.1 Introduction

This book is based upon a set lecture notes for a course that I was teaching at the University of Utah in Fall of 2002. Our main goal is to describe various tools of *quasi-isometric rigidity* and to give (essentially self-contained) proofs of several fundamental theorems in this area: Gromov's theorem on groups of polynomial growth and Schwartz's quasi-isometric rigidity theorem for nonuniform lattices in the real-hyperbolic spaces. We conclude with a survey of the quasi-isometric rigidity theory.

The main idea of the geometric group theory is to treat finitely-generated groups as *geometric objects*: with each finitely-generated group G we will associate a metric space, the *Cayley graph* of G. One of the main issues of the geometric group theory is to recover as much as possible algebraic information about G from the geometry of the Cayley graph. A primary obsticle for this is the fact that the Cayley graph depends not only on G but on a particular choice of a generating set of G. Cayley graphs associated with different generating sets are not isometric but *quasi-isometric*. One of the fundamental questions which we will try to address in this book is:

- If G, G' are quasi-isometric groups, to which extent G and G' share the same algebraic properies?

The best one can hope here is to recover the group G *up to weak commensurability* from its geometry. The equivalence relation of weak commensurability is generated by the two operations:

1. Passing to a finite index subgroup (this leads to the *commensurability equivalence relation*).

2. Taking finite kernel extensions G of a group Γ:

$$1 \to F \to G \to \Gamma \to 1$$

is a short exact sequence so that F is finite.

Weak commensurability implies quasi-isometry but, in general, the converse is false. One of the easiest examples is the following: Pick two matrices $A, B \in SL(2,\mathbb{Z})$ so that $A^n \neq B^m$ for all $n, m \in \mathbb{Z} \setminus \{0\}$. Define two actions of $\mathbb{Z}$ on $\mathbb{Z}^2$ so that the generator $1 \in \mathbb{Z}$ acts by the automorphisms given by A and B respectively. Then the semidirect products $G := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, $G' := \mathbb{Z}^2 \rtimes_B \mathbb{Z}$ are quasi-isometric but not weakly commensurable. Observe that both groups G, G' are *polycyclic*. The following is unknown even for the group G above:

Problem 1. Suppose that Γ is a group quasi-isometric to a polycyclic group G. Is Γ commensurable to a polycyclic group?

An example when quasi-isometry implies weak commensurability is given by the following theorem due to R. Schwartz:

Theorem 2. *Suppose that G is a nonuniform lattice acting on the hyperbolic space $\mathbb{H}^n, n \geq 3$. Then for each group Γ quasi-isometric to G, the group Γ is weakly commensurable with G.*

We will present a proof of this theorem in chapter 7. Another example of *quasi-isometric rigidity* is the following corollary from Gromov's theorem on groups of polynomial growth:

Corollary 3. *Suppose that G is a group quasi-isometric to a nilpotent group. Then G itself is* virtually nilpotent, *i.e. contains a nilpotent subgroup of finite index.*

Gromov's theorem and its corollary will be proven in chapter 5.

Proving these theorems are the main objectives of this course. Along the way we will introduce several tools of the geometric group theory: coarse topology, ultralimits and quasiconformal mappings.

1.2 Cayley graphs of finitely generated groups

Let Γ be a finitely generated group with the generating set $S = \{s_1, ..., s_n\}$, we shall assume that the identity does not belong to S. Define the *Cayley graph* $C = C(\Gamma, S)$ as follows: The vertices of C are the elements of Γ. Two vertices $g, h \in \Gamma$ are connected by an edge if an only if there is a generator $s_i \in S$ such that $h = gs_i$. Then C is a locally finite graph. Define the *word metric* d on C by assuming that each edge has the unit length, this defines the length of finite PL-paths in C, finally the distance between points $p, q \in C$ is the infimum (same as minimum) of the lengths of PL-paths in C connecting p to q. For $g \in G$ the *word length* $\ell(g)$ is just the distance $d(1, g)$ in C. It is clear that the left action of the group Γ on the metric space (C, d) is isometric.

Below are two simple examples of Cayley graphs.

Example 4. Let Γ be free Abelian group on two generators s_1, s_2. Then $S = \{s_i, i = 1, 2\}$. The Cayley graph $C = C(\Gamma, S)$ is the square grid in the Euclidean plane: The vertices are points with integer coordinates, two vertices are connected by an edge if and only if exactly only two of their coordinates are distinct and they differ by ± 1.

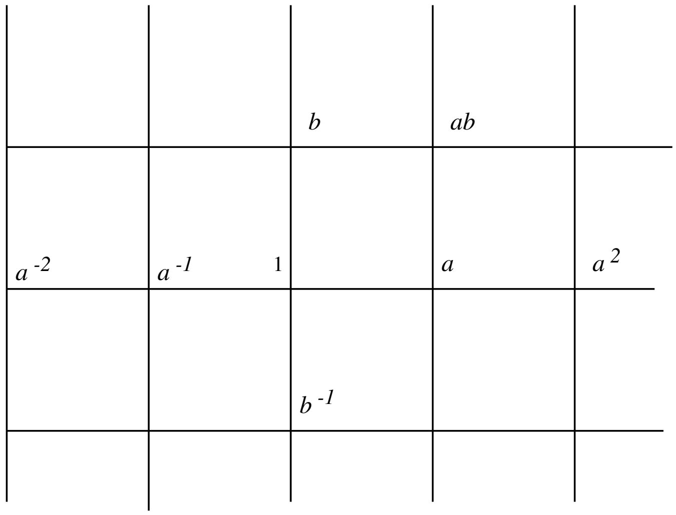

Figure 1: *Free abelian group.*

Example 5. Let Γ be the free group on two generators s_1, s_2. Take $S = \{s_i, i = 1, 2\}$. The Cayley graph $C = C(\Gamma, S)$ is the 4-valent tree (there are four edges incident to each vertex).

See Figures 1, 2.

1.3 Quasi-isometries

Let X be a metric space. We will use the notation $N_R(A)$ to denote R-neighborhood of a subset $A \subset X$, i.e. $N_R(A) = \{x \in X : d(x, A) < R\}$. Recall that *Hausdorff distance* between subsets $A, B \subset X$ is defined as

$$d_{Haus}(A, B) := \inf\{R : A \subset N_R(B), B \subset N_R(A)\}.$$

Two subsets of X are called *Hausdorff-close* if they are within finite Hausdorff distance from each other.

Definition 6. Let X, Y be complete metric spaces. A map $f : X \to Y$ is called (L, A)-**coarse Lipschitz** if

$$d_Y(f(x), f(x')) \leq L d_X(x, x') + A \tag{7}$$

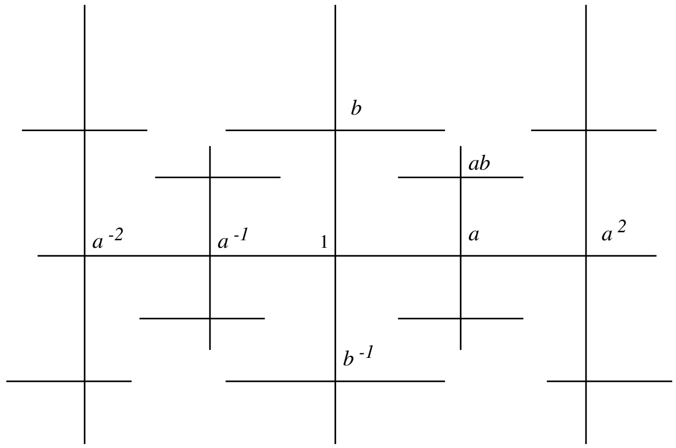

Figure 2: *Free group.*

for all $x, x' \in X$. A map $f : X \to Y$ is called a (L, A)-**quasi-isometric embedding** if

$$L^{-1} d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq L d_X(x, x') + A \tag{8}$$

for all $x, x' \in X$. Note that a quasi-isometric embedding does not have to be an embedding in the usual sense, however distant points have distinct images.

An (L, A)-quasi-isometric embedding is called an (L, A)-**quasi-isometry** if it admits a **quasi-inverse** map $\bar{f} : Y \to X$ which is a (L, A)-quasi-isometric embedding so that:

$$d_X(\bar{f} f(x), x) \leq A, \quad d_Y(f \bar{f}(y), y) \leq A \tag{9}$$

for all $x \in X, y \in Y$.

We will abbreviate *quasi-isometry*, *quasi-isometric* and *quasi-isometrically* to QI.

In the most cases the *quasi-isometry constants* L, A do not matter, so we shall use the words *quasi-isometries* and *quasi-isometric embeddings* without specifying constants. If X, Y are spaces such that there exists a quasi-isometry $f : X \to Y$ then X and Y are called *quasi-isometric.* In applications X and Y will be nonempty, however, by working with *relations* instead of maps one can modify this definition so that the empty set is quasi-isometric to any bounded metric space.

Exercise 10. If $f : X \to Y$ is a quasi-isometry and g is within finite distance from f (i.e. $\sup d(f(x), g(x)) < \infty$) then g is also a quasi-isometry.

Exercise 11. A subset S of a metric space X is said to be r-**dense** in X if the Hausdorff distance between S and X is at most r. Show that if $f : X \to Y$ is a quasi-isometric embedding such that $f(X)$ is r-dense in X for some $r < \infty$ then f is a quasi-isometry. Hint: Construct a quasi-inverse $\bar{f}$ to the map f by mapping point $y \in Y$ to $x \in X$ such that

$$d_Y(f(x), y) \leq d_Y(f(X), y) + 1.$$

For instance, the cylinder $X = \mathbb{S}^n \times \mathbb{R}$ is quasi-isometric to $Y = \mathbb{R}$; the quasi-isometry is the projection to the second factor.

Exercise 12. Show that quasi-isometry is an equivalence relation between (nonempty) metric spaces.

A separated net in a metric space X is a subset $Z \subset X$ which is r-dense for some $r < \infty$ and such that there exists $\epsilon > 0$ for which $d(z, z') \geq \epsilon$, $\forall z \neq z' \in Z$.

Alternatively, one can describe quasi-isometric spaces as follows.

Lemma 13. *Metric spaces X and Y are quasi-isometric iff there are separated nets $Z \subset X, W \subset Y$, constants L and C, and L-Lipschitz maps*

$$f : Z \to Y, \bar{f} : W \to X,$$

so that $d(\bar{f} \circ f, id) \leq C, d(f \circ \bar{f}, id) \leq C$.

Proof. Observe that if a map $f : X \to Y$ is coarse Lipschitz then its restriction to each separated net in X is Lipschitz. Conversely, if $f : Z \to Y$ is a Lipschitz map from a separated net in X then f admits a coarse Lipschitz extension to X. ☐

In some cases it suffices to check a weaker version of (9) to show that f is a quasi-isometry.

Let X, Y be topological spaces. Recall that a (continuous) map $f : X \to Y$ is called proper if the inverse image $f^{-1}(K)$ of each compact in Y is a compact in X. A metric space X is called *proper* if each closed and bounded subset of X is compact. Equivalently, the distance function $f : X \to \mathbb{R}_+$, $f(x) = d(x, o)$ is a proper function. (Here $o \in X$ is a base-point.)

Definition 14. A map $f : X \to Y$ between proper metric spaces is called *uniformly proper* if f is coarse Lipschitz and there exists a *distortion function* $\psi(R)$ such that $\mathrm{diam}(f^{-1}(B(y, R))) \leq \psi(R)$ for each $y \in Y, R \in \mathbb{R}_+$. In other words, there exists a proper function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that whenever $d(x, x') \geq r$, we have $d(f(x), f(x')) \geq \eta(r)$.

To see an example of a map which is proper but not uniformly proper consider the biinfinite curve Γ embedded in $\mathbb{R}^2$ (Figure 3):

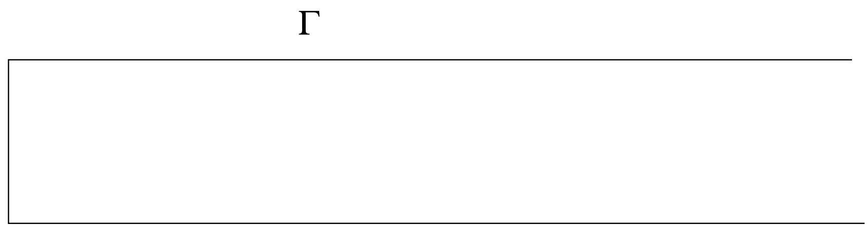

Figure 3:

Lemma 15. *Suppose that Y is a geodesic metric space, $f : X \to Y$ is a uniformly proper map whose image is r-dense in Y for some $r < \infty$. Then f is a quasi-isometry.*

Proof. Let's construct a quasi-inverse to the map f. Given a point $y \in Y$ pick a point $\bar{f}(y) := x \in X$ such that $d(f(x), y) \leq r$. Let's check that $\bar{f}$ is coarse Lipschitz. Since Y is a geodesic metric space it suffices to verify that there is a constant A such that for all $y, y' \in Y$ with $d(y, y') \leq 1$, one has:

$$d(\bar{f}(y), \bar{f}(y')) \leq A.$$

Pick $t > 1$ which is in the image of the distortion funcion η. Then take $A \in \eta^{-1}(t)$.

It is also clear that $f, \bar{f}$ are quasi-inverse to each other. $\square$

Lemma 16. *Let X be a proper geodesic metric space. Let G be a group acting isometrically properly discontinuously cocompactly on X. Pick a point $x_0 \in X$. Then the group G is finitely generated; for some choice of finite generating set S of the group G the map $f : G \to X$, given by $f(g) = g(x_0)$, is a quasi-isometry. Here G is given the word metric induced from $C(G, S)$.*

Proof. Our proof follows [24, Proposition 10.9]. Let $B = B_R(x_0)$ be the closed ball of radius R in X with the center at x_0 such that $B_{R-1}(x_0)$ projects onto X/G. Since the action of G is properly discontinuous, there are only finitely many elements $s_i \in G - \{1\}$ such that $B \cap s_i B \neq \emptyset$. Let S be the subset of G which consists of the above elements s_i (it is clear that s_i^{-1} belongs to S iff s_i does). Let

$$r := \inf\{d(B, g(B)), g \in G - (S \cup \{1\})\}.$$

Clearly $r > 0$. We claim that S is a generating set of G and that for each $g \in G$

$$\ell(g) \leq d(x_0, g(x_0))/r + 1 \tag{17}$$

where ℓ is the word length on G (with respect to the generating set S). Let $g \in G$, connect x_0 to $g(x_0)$ by the shortest geodesic γ. Let m be the smallest integer so that $d(x_0, g(x_0)) \leq mr + R$. Choose points $x_1, ..., x_{m+1} = g(x_0) \in \gamma$, so that $x_1 \in B$, $d(x_j, x_{j+1}) < r$, $1 \leq j \leq m$. Then each x_j belongs to $g_j(B)$ for some $g_j \in G$. Let $1 \leq j \leq m$, then $g_j^{-1}(x_j) \in B$ and $d(g_j^{-1}(g_{j+1}(B)), B) \leq d(g_j^{-1}(x_j), g_j^{-1}(x_{j+1})) < r$. Thus the balls $B, g_j^{-1}(g_{j+1}(B))$ intersect, which means that $g_{j+1} = g_j s_{i(j)}$ for some $s_{i(j)} \in S \cup \{1\}$. Therefore

$$g = s_{i(1)} s_{i(2)} .... s_{i(m)}.$$

We conclude that S is indeed a generating set for the group G. Moreover,

$$\ell(g) \leq m \leq (d(x_0, g(x_0)) - R)/r + 1 \leq d(x_0, g(x_0))/r + 1.$$

The word metric on the Cayley graph $C = C(G, S)$ of the group G is left-invariant, thus for each $h \in G$ we have:

$$d(h, hg) = d(1, g) \leq d(x_0, g(x_0))/r + 1 = d(h(x_0), hg(x_0))/r + 1.$$

Hence for any $g_1, g_2 \in G$

$$d(g_1, g_2) \leq d(f(g_1), f(g_2))/r + 1.$$

On the other hand, the triangle inequality implies that

$$d(x_0, g(x_0)) \leq t\ell(g)$$

where $d(x_0, s(x_0)) \leq t \leq 2R$ for all $s \in S$. Thus

$$d(f(g_1), f(g_2))/t \leq d(g_1, g_2).$$

We conclude that the map $f : G \to X$ is a quasi-isometric embedding. Since $f(G)$ is R-dense in X, it follows that f is a quasi-isometry. $\square$

Corollary 18. *Let S_1, S_2 be finite generating sets for a finitely generated group G and d_1, d_2 be the word metrics on G corresponding to S_1, S_2. Then the identity map $(G, d_1) \to (G, d_2)$ is a quasi-isometry.*

Proof. The group G acts isometrically cocompactly on the proper metric space

$$(C(G, S_2), d_2).$$

Therefore the map $id : G \to C(G, S_2)$ is a quasi-isometry. $\square$

Lemma 19. *Let X be a locally compact path-connected topological space, let G be a group acting properly discontinuously cocompactly on X. Let d_1, d_2 be two proper geodesic metrics on X (consistent with the topology of X) both invariant under the action of G. Then the group G is finitely generated and the identity map $id : (X, d_1) \to (X, d_2)$ is a quasi-isometry.*

Proof. The group G is finitely generated by Lemma 16, choose a word metric d on G corresponding to any finite generating set (according to the previous corollary it does not matter which one). Pick a point $x_0 \in X$, then the maps

$$f_i : (G, d) \to (X, d_i), \quad f_i(g) = g(x_0)$$

are quasi-isometries, let $\bar{f}_i$ denote their quasi-inverses. Then the map $id : (X, d_1) \to (X, d_2)$ is within finite distance from the quasi-isometry $f_2 \circ \bar{f}_1$. $\square$

A (k, c)-quasigeodesic segment in a metric space X is a (k, c)-quasi-isometric embedding $f : [a, b] \to X$; similarly, a complete (k, c)-quasigeodesic is a (k, c)-quasi-isometric embedding $f : \mathbb{R} \to X$. By abusing notation we will refer to the image of a (k, c)-quasigeodesic as a *quasigeodesic*.

Corollary 20. *Let d_1, d_2 be as in Lemma 19. Then any (complete) geodesic γ with respect to the metric d_1 is also a quasigeodesic with respect to the metric d_2.*

1.4 Gromov-hyperbolic spaces

Roughly speaking, Gromov-hyperbolic spaces are the ones which exhibit "tree-like behavior", at least if we restrict to finite subsets.

Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhood of the union of two other sides. An *R-fat triangle* is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (Rips was the first to introduce this definition) if each geodesic triangle in Z is δ-thin. A finitely generated group is said to be Gromov-hyperbolic if its Cayley graph is Gromov-hyperbolic.

Notation 21. For a subset S in a metric space X we will use the notation $N_R(S)$ for the metric R-neighborhood of S in X.

Below is an alternative definition of δ-hyperbolicty due to Gromov.

Let X be a metric space (which is no longer required to be geodesic). Pick a base-point $p \in X$. For each $x \in X$ set $|x|_p := d(x,p)$ and define the *Gromov product*

$$(x,y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x,y)).$$

Note that the triangle inequality implies that $(x,y)_p \geq 0$ for all x, y, p; the Gromov product measures how far the triangle inequality if from being an equality.

Exercise 22. Suppose that X is a metric tree. Then $(x,y)_p$ is the distance $d(p,\gamma)$ from p to the segment $\gamma = \overline{xy}$.

In general we observe that for each point $z \in \gamma = \overline{xy}$

$$(p,x)_z + (p,y)_z = |z|_p - (x,y)_p. \tag{23}$$

In particular, $d(p,\gamma) \geq (x,y)_p$.

Suppose now that X is δ-hyperbolic in the sense of Rips. Then the Gromov product is "comparable" with $d(p,\gamma)$:

Lemma 24.

$$(x,y)_p \leq d(p,\gamma) \leq (x,y)_p + 2\delta.$$

Proof. The inequality $(x,y)_p \leq d(p,\gamma)$ was proven above; so we have to establish the other inequality. Note that since the triangle $\Delta(pxy)$ is δ-thin, for each point $z \in \gamma = \overline{xy}$ we have

$$\min\{(x,p)_z, (y,p)_z\} \leq \min\{d(z,\overline{px}), d(z,\overline{py})\} \leq \delta.$$

By continuity, there exists a point $z \in \gamma$ such that $(x,p)_z, (y,p)_z \leq \delta$. By applying the equality (23) we get:

$$|z|_p - (x,y)_p = (p,x)_z + (p,y)_z \leq 2\delta.$$

Since $|z|_p \leq d(p,\gamma)$, we conclude that $d(p,\gamma) \leq (x,y)_p + 2\delta$. $\square$

Now define a number $\delta_p \in [0,\infty]$ as follows:

$$\delta_p := \inf_{\delta\in[0,\infty]} \{\delta | \forall x,y,z \in X, (x,y)_p \geq \min((x,z)_p, (y,z)_p) - \delta\}.$$

Exercise 25. Suppose that X is a geodesic metric space. Show that X is zero-hyperbolic (in the sense of Rips or Gromov) iff X is a metric tree.

Exercise 26. If $\delta_p \le \delta$ for some p then $\delta_q \le 2\delta$ for all $q \in X$

X is said to be *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \ge \delta_p$ for all $p \in X$. The advantage of this definition is that it does not require X to be geodesic and this notion is manifestly QI-invariant:

If X, X' are quasi-isometric and X is δ-hyperbolic in the sense of Gromov then X' is δ'–hyperbolic in the sense of Gromov. In contrast, QI invariance of Rips-hyperbolicity is not a priori obvious. We will prove QI invariance of Rips-hyperbolicity in the corollary 70 as a corollary of Morse lemma.

Lemma 27 (See [28], section 6.3C.). *If X a geodesic metric space which is δ-hyperbolic in Gromov's sense then X is 4δ-hyperbolic in the sense of Rips and vice-versa.*

In what follows, we will refer to δ-hyperbolic spaces in the sense of Rips as being δ-hyperbolic.

Here are some examples of Gromov-hyperbolic spaces.

1. Let $X = \mathbb{H}^n$ be the hyperbolic n-space. Then X is δ-hyperbolic for appropriate δ. The reason for this is that the "largest" triangle in X is an ideal triangle, i.e. a triangle all whose three vertices are on the boundary sphere of $\mathbb{H}^n$. All such triangles are congruent to each other since $Isom(\mathbb{H}^n)$ acts transitively on triples of distinct points in S^{n-1}. Thus it suffices to verify thinness of a single ideal triangle in $\mathbb{H}^2$, the triangle with the ideal vertices $0, 2, \infty$. I claim that for each point x on the arc between 0 and m the distance to the side γ is < 1. Indeed, since dilations with center at zero are hyperbolic isometries, the maximal distance from x to γ is realized at the point $m = 1 + i$. Computing the hyperbolic length of the horizontal segment between m and $i \in \gamma$ we conclude that it equals 1. Hence $d(x, \gamma) \le d(m, \gamma) < 1$. See Figure 4.

Remark 28. By making more careful computation with the hyperbolic distances one can conclude that $\sinh(d(m, \gamma)) = 1$.

2. Suppose that X is a complete Riemannian manifold of sectional curvature $\le \kappa < 0$. Then X is Gromov-hyperbolic. This follows from Rauch-Toponogov comparison theorem. Namely, let Y be the hyperbolic plane with the curvature normalized to be $= \kappa < 0$. Then Y is δ-hyperbolic. Let $\Delta = \Delta(xyz)$ be a geodesic triangle in X. Construct the *comparison triangle* $\Delta' := \Delta(x'y'z') \subset Y$ whose sides have the same length as for the triangle Δ. Then the triangle Δ' is δ-thin. Pick a pair

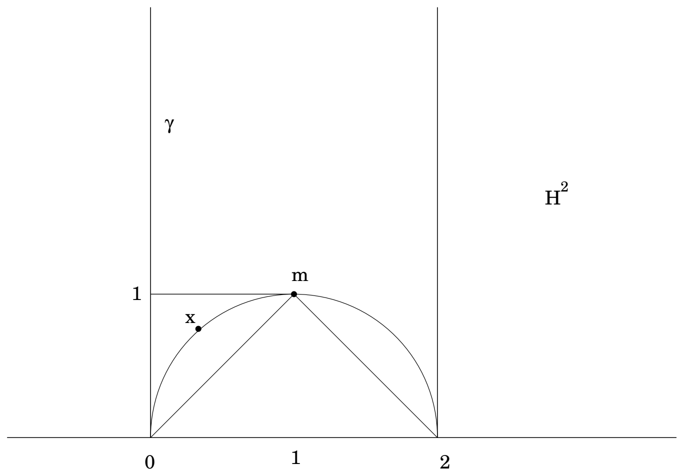

Figure 4: *Ideal triangle* $\Delta(0,2,\infty)$ *in the hyperbolic plane:* $d(x,\gamma) \le d(m,\gamma) < 1$.

of points $p \in \overline{xy}, q \in \overline{yz}$ and the corresponding points $p' \in \overline{x'y'}, q' \in \overline{y'z'}$ so that $d(x,p) = d(x',p'), d(y,q) = d(y',q')$. Then Rauch-Toponogov comparison theorem implies that $d(p,q) \le d(p',q')$. It immediately follows that the triangle Δ is δ-thin.

1.5 Ideal boundaries

Suppose that X is a proper geodesic metric space. Introduce an equivalence relation on the set of geodesic rays in X by declaring $\rho \sim \rho'$ iff they are *asymptotic* i.e. are within finite distance from each other. Given a geodesic ray ρ we will denote by $\rho(\infty)$ its equivalence class. Define the *ideal boundary* of X as the collection $\partial_\infty X$ of equivalence classes of geodesic rays in X. Our next goal is to topologize $\partial_\infty X$. Note that the space of geodesic rays (parameterized by arc-length) in X has a natural compact-open topology (we regard geodesic rays as maps from $[0,\infty)$ into X). Thus we topologize $\partial_\infty X$ by giving it the quotient topology τ.

We now restrict our attention to the case when X is δ-hyperbolic.

Then for each geodesic ray ρ and a point $p \in X$ there exists a geodesic ray ρ' with the initial point p such that $\rho(\infty) = \rho'(\infty)$: Consider the sequence of geodesic segments $\overline{p\rho(n)}$ as $n \to \infty$. Then the thin triangles property implies that these segments are contained in a δ-neighborhood of $\rho \cup \overline{p\rho(0)}$. Properness of X implies that this sequence subconverges to a geodesic ray ρ' as required.

Lemma 29. *(Asymptotic rays are uniformly close). Let* ρ_1, ρ_2 *be asymptotic geodesic* *rays in X such that $\rho_1(0) = \rho_2(0) = p$. Then for each t,*

$$d(\rho_1(t), \rho_2(t)) \le 2\delta.$$

Proof. Suppose that the raus ρ_1, ρ_2 are within distance $\le C$ from each other. Take $T \gg t$. Then (since the rays are asymptotic) there is $\tau \in \mathbb{R}_+$ such that

$$d(\rho_1(T), \rho_2(\tau)) \le C.$$

By δ-thinness of the triangle $\Delta(p\rho_1(T)\rho_2(\tau))$, the point $\rho_1(t)$ is within distance $\le \delta$ from a point either on $\overline{p\rho_2(\tau)}$ or on $\overline{\rho_1(T)\rho_2(\tau)}$. Since the length of $\overline{\rho_1(T)\rho_2(\tau)}$ is $\le C$ and $T \gg t$, it follows that there exists t' such that

$$d(\rho_1(t), \rho_2(t')) \le \delta.$$

By the triangle inequality, $|t - t'| \le \delta$. It follows that $d(\rho_1(t), \rho_2(t)) \le 2\delta$. ☐

Pick a base-point $p \in X$. Given a number $k > 2\delta$ define a topology τ_k on $\partial_\infty X$ with the basis of neighborhoods of a point $\rho(\infty)$ given by

$$U_{k,n}(\rho) := \{\rho' : d(\rho'(t), \rho(t)) < k, t \in [0, n]\}, n \in R_+$$

where the rays ρ' satisfy $\rho'(0) = p = \rho(0)$.

Lemma 30. *Topologies τ and τ_k coincide.*

Proof. 1. Suppose that ρ_j is a sequence of rays emanating from p such that $\rho_j \notin U_{k,n}(\rho)$ for some n. If $\lim_j \rho_j = \rho'$ then $\rho' \notin U_{k,n}$ and by the previous lemma, $\rho'(\infty) \ne \rho(\infty)$.

2. Conversely, if for each n, $\rho_j \in U_{k,n}(\rho)$ (provided that j is large enough), then the sequence ρ_j subconverges to a ray ρ' which belongs to each $U_{k,n}(\rho)$. Hence $\rho'(\infty) = \rho(\infty)$. ☐

Example 31. Suppose that $X = \mathbb{H}^n$ is the hyperbolic n-space realized in the unit ball model. Then the ideal boundary of X is S^{n-1}.

Lemma 32. *Let X be a proper geodesic Gromov-hyperbolic space. Then for each pair of distinct points $\xi, \eta \in \partial_\infty X$ there exists a geodesic γ in X which is asymptotic to both ξ and η.*

Proof. Consider geodesic rays ρ, ρ' emanating from the same point $p \in X$ and asymptotic to ξ, η respectively. Since $\xi \neq \eta$, for each $R < \infty$ the set

$$K(R) := \{x \in X : d(x, \rho) \leq R, d(x, \rho') \leq R\}$$

is compact. Consider the sequences $x_n := \rho(n), x_n' := \rho'(n)$ on ρ, ρ' respectively. Since the triangles $\Delta p x_n x_n'$ are δ-thin, each segment $\gamma_n := \overline{x_n x_n'}$ contains a point within distance $\leq \delta$ from both $\overline{px_n}, \overline{px_n'}$, i.e. $\gamma_n \cap K(\delta) \neq \emptyset$. Therefore the sequence of geodesic segments γ_n subconverges to a complete geodesic γ in X. Since $\gamma \subset N_\delta(\rho \cup \rho')$ it follows that γ is asymptotic to ξ and η. $\square$

Definition 33. We say that a sequence $x_n \in X$ converges to a point $\xi = \rho(\infty) \in \partial_\infty X$ in the **cone topology** if there is a constant C such that $x_n \in N_C(\rho)$ and the geodesic segments $\overline{x_1 x_n}$ converge to a geodesic ray asymptotic to ξ.

For instance, suppose that $X = \mathbb{H}^m$ in the upper half-space model, $\xi = 0 \in \mathbb{R}^{m-1}$, L is the vertical geodesic from the origin. Then a sequence $x_n \in X$ converges ξ in the cone topology iff all the points x_n belong to the Euclidean cone with the axis L and the Euclidean distance from x_n to 0 tends to zero. See Figure 5. This explains the name *cone topology*.

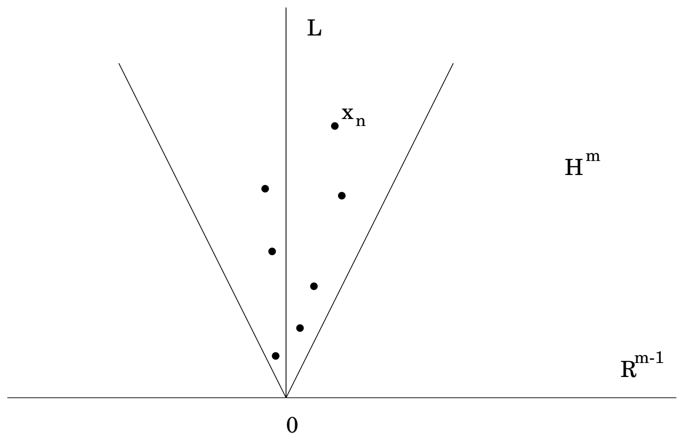

Figure 5: *Convergence in the cone topology.*

Theorem 34. *1. Suppose that G is a hyperbolic group. Then $\partial_\infty G$ consists of 0, 2 or continuum of points.*

2. The group G acts by homeomorphisms on $\partial_\infty G$ as a **uniform convergence group**, *i.e. the action of G on $Trip(\partial_\infty G)$ is properly discontinuous and cocompact, where $Trip(\partial_\infty G)$ consists of triples of distinct points in $\partial_\infty G$.*

2 Coarse topology

The goal of this section is to provide tools of algebraic topology for studying quasi-isometries and other concepts of the geometric group theory. The class of *bounded geometry metric cell complexes* provides a class of spaces for which application of algebraic topology is possible.

A metric space X *has bounded geometry* if there is a function $\phi(r)$ such that each ball $B(x,r) \subset X$ contains at most $\phi(r)$ points. For instance, if G is a finitely generated group with word metric then G has bounded geometry.

A *metric cell complex* is a cell complex X together with a metric.

A metric cell complex X^0 is said to have *bounded geometry* if:

(a) Each ball $B(x,r) \subset X$ intersects at most $\phi(r,k)$ cells of dimension $\leq k$.

(b) Diameter of each k-cell is at most c_k, $k = 1,2,3,\ldots$..

Example 35. Let M be a compact simplicial complex. Metrize each simplex to be isometric to the standard simplex with unit edges in the Euclidean space. Note that for each m-simplex σ^m and its face σ^k, the inclusion $\sigma^k \to \sigma^m$ is an isometric embedding. This allows us to define a path-metric on M so that each simplex is isometrically embedded in M. Lift this metric to a cover X of M gives X structure of a metric cell complex of bounded geometry.

Recall that quasi-isometries are not necessarily continuous. We therefore have to approximate quasi-isometries by continuous maps.

Lemma 36. *Suppose that X, Y are bounded geometry metric cell complexes, Y is uniformly contractible, and $f : X \to Y$ is a coarse (L,A)-Lipschitz map. Then there exists a (continuous) cellular map $g : X \to Y$ such that $d(f,g) \leq Const$, where $Const$ depends only on (L,A) and the geometric bounds on X and Y.*

Proof. The proof of this lemma is a prototype of most of the proofs presented in this section. We construct g by induction on skeleta of X. First, of all, for each vertex $x \in X^{(0)}$ we let $g(x)$ denote a point in $Y^{(0)}$ which is nearest to $f(x)$. It is clear that $d(f(x),g(x)) \leq const_0$, where $const_0$ is an upper bound on the diameter of the top-dimensional cells in Y. Note that if x, x' belong to the boundary of a 1-cell in X then $d(g(x),g(x')) \leq LConst_1 + A + 2const_0$, where $Const_1$ is an upper bound on the diameter of 1-cells in X.

Inductively, assume that g was constructed on $X^{(k)}$. Let σ denote a $k+1$-cell in X. Then, inductively, $\mathrm{diam}(g(\partial\sigma)) \leq C_k$ and $d(f, g|X^{(k)}) \leq C'_k$. Then, using uniform contractibility of Y, we extend g to σ so that $\operatorname{diam}(g(\sigma)) \leq C'_{k+1}$. Then $d(f, g|X^{(k+1)})) \leq C'_{k+1} + LConst_k + A$. Since X is finite-dimensional the induction terminates after finitely many steps. $\square$

2.1 Ends of spaces

In this section we review the (historically the first) coarse topological notion. Let X be a locally compact connected topological space (e.g. a proper geodesic metric space). Given a compact subset $K \subset X$ we consider its complement K^c. Then the system of sets $\pi_0(K^c)$ is an inverse system:

$$K \subset L \Rightarrow \pi_0(L^c) \to \pi_0(K^c).$$

Then the set of ends $\epsilon(X)$ is defined as the inverse limit

$$\lim_{K \subset X} \pi_0(K^c).$$

The elements of $\epsilon(X)$ are called *ends of* X. Analogously, one can define "higher homotopy groups" $\pi_i^\infty(X, x_\bullet)$ at infinity of X by considering inverse systems of higher homotopy groups: This requires a choice of a system of base-points $x_k \in K^c$ representing a single element of $\epsilon(X)$. The inverse limit of this sequence of base-points, $x_\bullet \in \epsilon(X)$, serves as a "base-point" for the homotopy group $\pi_i^\infty(X, x_\bullet)$.

Here is a more down-to-earth description of the ends of X. Consider a nested sequence of compacts $K_i \subset X, i \in \mathbb{N}$ (for instance, if X is a proper metric space take $K_R := B_R(p)$ for fixed $p \in X$). For each i pick a connected component $U_i \subset K_i^c$ so that $U_i \supset U_{i+1}$. Then the nested sequence (U_i) represents a single point in $\epsilon(X)$. Even more concretely, pick a point $x_i \in U_i$ for each i and connect x_i, x_{i+1} by a curve $\gamma_i \subset U_i$. The concatenation of the curves γ_i defines a proper map $\gamma : [0, \infty) \to X$. Call two proper curves $\gamma, \gamma' : \mathbb{R}_+ \to X$ equivalent if for each compact $K \subset X$ there are points $x \in \gamma(\mathbb{R}_+), x' \in \gamma'(\mathbb{R}_+)$ which belong to the same connected component of K^c. The equivalence classes of such curves are in bijective correspondence with the ends of X, the map $(U_i) \mapsto \gamma$ was described above.

See Figure 6 as an example. The space X in this picture has 5 visibly different ends: $\epsilon_1, ..., \epsilon_5$. We have $K_1 \subset K_2 \subset K_3$. The compact K_1 separates the ends ϵ_1, ϵ_2. The next compact K_2 separates ϵ_3 from ϵ_4. Finally, the compact K_3 separates ϵ_4 from ϵ_5.

Topology on $\epsilon(X)$. Let $\eta \in \epsilon(X)$ be represented by a nested sequence (U_i). Each U_i defines a neighborhood $N_i(\eta)$ of η consisting of all $\eta' \in \epsilon(X)$ which are represented by nested sequences (U'_j) such that $U'_j \subset U_i$ for all but finitely many $j \in \mathbb{N}$.

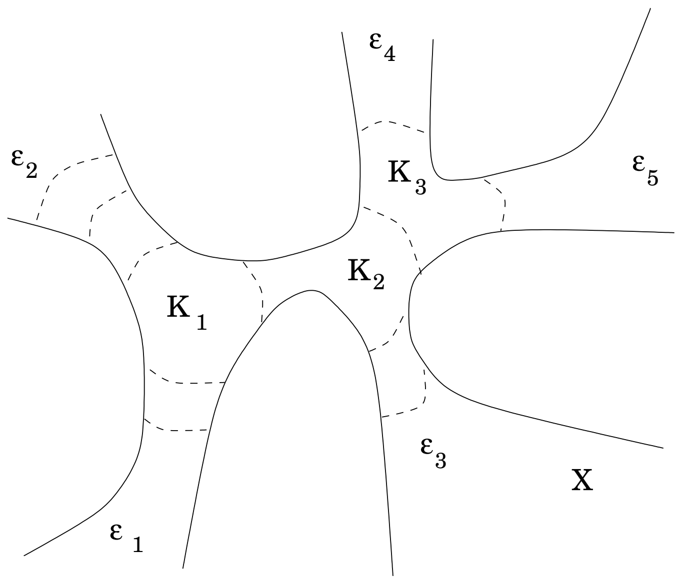

Figure 6: *Ends of X.*

Lemma 37. *If $f : X \to Y$ is an (L, A)-quasi-isometry of proper geodesic metric spaces then f induces a homeomorphism $\epsilon(X) \to \epsilon(Y)$.*

Proof. Note that for each bounded subset $B \subset Y$ the inverse image $f^{-1}(B)$ is again bounded. Although for a connected subset $C \subset X$ the preimage $f(C)$ is not necessarily connected, the $R := L + A$-neighborhood $N_R(f(C))$ is connected. Thus we define a map $f_* : \epsilon(X) \to \epsilon(Y)$ as follows. Suppose that $\eta \in \epsilon(X)$ is represented by a nested sequence (U_i). Without loss of generality we may assume that for each i, $N_R(U_i) \subset U_{i-1}$. Thus we get a nested sequence of connected subsets $N_R(f(U_i)) \subset Y$ each of which is contained in a connected component V_i of the complement to the bounded subset $f(K_{i-1}) \subset Y$. Thus we send η to $f_*(\eta)$ represented by (V_i). It follows from the construction that By considering the quasi-inverse $\bar{f}$ to f it is clear that f_* has inverse map $(\bar{f})_*$. It is also clear that both f_* and $(\bar{f})_*$ are continuous. $\square$

If G is a finitely generated group then the space of ends $\epsilon(G)$ is defined to be the set of ends of its Cayley graph. The previous lemma implies that $\epsilon(G)$ does not depend on the choice of a finite generating set.

Theorem 38. *Properties of $\epsilon(X)$:*

1. *$\epsilon(X)$ is compact, Hausdorff and totally disconnected.*

2. Suppose that G is a finitely-generated group. Then $\epsilon(G)$ consists of 0, 1, 2 points or of continuum of points. In the latter case the set $\epsilon(G)$ is perfect: Each point is a limit point.

3. $\epsilon(G)$ is empty iff G is finite. $\epsilon(G)$ consists of 2-points iff G is virtually (infinite) cyclic.

4. $|\epsilon(G)| > 1$ iff G splits nontrivially over a finite subgroup.

All the properties listed above are relatively trivial except for the last one: if $|\epsilon(G)| > 1$ then G splits nontrivially over a finite subgroup, which is a theorem of Stallings [52]. For the proof of the rest see for instance [5, Theorem 8.32].

Corollary 39. *1. Suppose that G is quasi-isometric to $\mathbb{Z}$ then G contains $\mathbb{Z}$ as a finite index subgroup.*

2. Suppose that G splits nontrivially as $A * B$ and G' is quasi-isometric to G. Then G' splits nontrivially as $H *_F E$ (amalgamated product) or as $H*_F$ (HNN splitting) where F is a finite group.

Theorem 40. *Suppose that G is a hyperbolic group. Then there exists a continuous equivariant surjection*

$$\sigma : \partial_\infty G \to \epsilon(G)$$

such that the preimages $\sigma^{-1}(\xi)$ are connected components of $\partial_\infty G$.

2.2 Rips complexes and coarse connectedness

Let X be a metric space of bounded geometry, $R \in \mathbb{R}_+$. Then the *R-Rips complex* $\mathrm{Rips}_R(X)$ is the simplicial complex whose vertices are points of X; vertices $x_1, ..., x_n$ span a simplex iff $d(x_i, x_j) \le R$ for each i, j. Note that the system of Rips complexes of X is a direct system $\mathrm{Rips}_\bullet(X)$ of simplicial complexes:

For each pair $0 \le r \le R < \infty$ we have a natural embedding $\iota_{r,R} : \mathrm{Rips}_r(X) \to \mathrm{Rips}_R(X)$ and $\iota_{r,\rho} = \iota_{R,\rho} \circ \iota_{r,R}$ provided that $r \le R \le \rho$.

One can metrize $\mathrm{Rips}_R(X)$ by declaring each simplex to be isometric to a regular Euclidean simplex with unit edges. Note that the assumption that X has bounded geometry implies that $\mathrm{Rips}_R(X)$ is finite-dimensional for each R. Moreover, $\mathrm{Rips}_R(X)$ is a metric cell complex of bounded geometry.

The following simple observation explains why Rips complexes are useful for analyzing quasi-isometries:

Lemma 41. *Let $f : X \to Y$ be an L-Lipschitz map. Then f induces a (continuous) simplicial map* $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld}(Y)$ *for each $d \geq 0$.*

Proof. Consider an $(m-1)$-simplex σ in $\mathrm{Rips}_d(X)$, the vertices of σ are points $x_1, ..., x_m$ within distance $\leq R$ from each other. Since f is L-Lipschitz, the points $f(x_1), ..., f(x_m)$ are within distance $\leq LR$ from each other, hence they span a simplex σ' of dimension $\leq m-1$ in $\mathrm{Rips}_{Ld}(Y)$. The map f sends vertices of σ to vertices of σ', extend this map linearly to the simplex σ. It is clear that this extension defines a (continuous) simplicial map of simplicial complexes $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld}(Y)$. $\square$

Definition 42. A metric space X is **coarsely k-connected** if for each r there exists $R \geq r$ so that the mapping $\mathrm{Rips}_r(X) \to \mathrm{Rips}_R(X)$ induces a trivial map of π_i for $0 \leq i \leq k$.

For instance, X is coarsely 0-connected if there exists a number R such that each pair of points $x, y \in X$ can be connected by an R-chain of points $x_i \in X$, i.e. a chain of points where $d(x_i, x_{i+1}) \leq R$ for each i. Note that for $k \leq 1$ coarse k-connectedness of X is equivalent to the property that $\mathrm{Rips}_R(X)$ is k-connected for sufficiently large R.

Properties of the direct system of Rips complexes:

Lemma 43. *Let $r, C < \infty$, then each simplicial spherical cycle σ of diameter $\leq C$ in* Rips_r *bounds a disk of diameter $\leq C + d$ within* Rips_{r+C}.

Proof. Pick a point $x \in \sigma$. Then Rips_{r+C} contains a simplicial cone $\beta(\sigma)$ over σ with the origin at x. Clearly $\sim (\beta) \leq r + C$. $\square$

Corollary 44. *Let*

$$f, g : \mathrm{Rips}_{d_1}(X) \to \mathrm{Rips}_{d_2}(Y)$$

be L-Lipschitz within distance $\leq C$ from each other. Then there exists $d_3 \geq d_2$ such that the maps $f, g :$ $\mathrm{Rips}_{d_1} \to \mathrm{Rips}_{d_3}(Y)$ *are homotopic via a homotopy whose tracks have lengths $\leq C' = C'(C, d_1, d_2, L)$.*

Proof. Construct the homotopy via induction on skeleta using the previous lemma. $\square$

We will refer to the maps f, g above as being *coarsely homotopic*. In the same way one defines coarse homotopy equivalence between the direct systems of Rips complexes.

Corollary 45. *Suppose that $f, g : X \to Y$ be L-Lipschitz maps within finite distance from each other. Then they induce coarsely homotopic maps* $\mathrm{Rips}_d(X) \to \mathrm{Rips}_{Ld}(Y)$ *for each* $d \geq 0$.

Corollary 46. *if $f : X \to Y$ is a quasi-isometry, then f induces a coarse homotopy-equivalence of the Rips complexes:* $\mathrm{Rips}_\bullet(X) \to \mathrm{Rips}_\bullet(Y)$.

Corollary 47. *Coarse k-connectedness is a QI invariant.*

Proof. Suppose that X' is coarsely k-connected and $f : X \to X'$ is an L-Lipschitz quasi-isometry with L-Lipschitz quasi-inverse $\bar{f} : X' \to X$. Let γ be a spherical i-cycle in $\mathrm{Rips}_d(X)$, $0 \leq i \leq k$. Then we have the induced spherical i-cycle $f(\gamma) \subset \mathrm{Rips}_{Ld}(X')$. Since X' is coarsely k-connected, there exists $d' \geq Ld$ such that $f(\gamma)$ bounds a singular $i+1$-disk β within $\mathrm{Rips}_{d'}(X')$. Consider now $\bar{f}(\beta) \subset \mathrm{Rips}_{L^2 d}(X)$. The boundary of this singular disk is a singular i-sphere $\bar{f}(\gamma)$. Since $\bar{f} \circ f$ is homotopic to id within $\mathrm{Rips}_{d''}(X)$, $d'' \geq L^2 d$, there exists a singular cylinder σ in $\mathrm{Rips}_{d''}(X)$ which cobounds γ and $\bar{f}(\gamma)$. Note that d'' does not depend on γ. By combining σ and $\bar{f}(\beta)$ we get a singular $i+1$-disk in $\mathrm{Rips}_{d''}(X)$ whose boundary is γ. Hence X is coarsely k-connected. ∎

Our next goal is to find a large supply of examples of metric spaces which are coarsely k-connected.

Definition 48. A bounded geometry metric cell complex X is said to be *uniformly k-connected* if there is a function $\psi(k, r)$ such that for each $i \leq k$, each singular i-sphere of diameter $\leq r$ in $X^{(i+1)}$ bounds a singular $i+1$-disk of diameter $\leq \psi(k, r)$.

For instance, if X is a finite-dimensional contractible complex which admits a co-compact cellular group action, then X is uniformly k-connected for each k.

Here is an example of a simply-connected complex which is not uniformly simply-connected. Take $S^1 \times \mathbb{R}_+$ with the product metric and attach to this complex a 2-disk along the circle $S^1 \times \{0\}$.

Theorem 49. *Suppose that X is a metric cell complex of bounded geometry such that X is uniformly n-connected. Then $Z := X^{(0)}$ is coarsely n-connected.*

Proof. Let $\gamma : S^k \to \mathrm{Rips}_R(Z)$ be a spherical m-cycle in $\mathrm{Rips}_R(Z)$, $0 \leq k \leq n$. Without loss of generality (using simplicial approximation) we can assume that γ is a simplicial cycle, i.e. the sphere S^k is given a triangulation τ so that γ sends simplices of S^k to simplices in $\mathrm{Rips}_R(Z)$ so that the restriction of γ to each simplex is a linear map. Let Δ_1 be a k-simplex in S^k. Then $\gamma(\Delta_1)$ is spanned by points $x_1, ..., x_{k+1} \in Z$

which are within distance $\leq R$ from each other. Since X is uniformly k-connected, there is a singular k-disk $\gamma_1(\Delta_1)$ containing $x_1, ..., x_{k+1}$ and having diameter $\leq R'$, where R' depends only on R. Namely, we construct γ_1 by induction on skeleta: First connect each pair of points x_i, x_j by a path in X (of length bounded in terms of R), this defines the map γ_1 on the 1-skeleton of Δ_1. Then continue inductively. This construction ensures that if Δ_2 is a k-simplex in S^k which shares an m-face with Δ_1 then γ_2 and γ_1 agree on $\Delta_1 \cap \Delta_2$. As the result, we have "approximated" γ by a singular spherical k-cycle $\gamma' : S^k \to X^{(k)}$ (the restriction of γ' to each Δ_i equals γ_i). See figure 7 in the case $k = 1$.

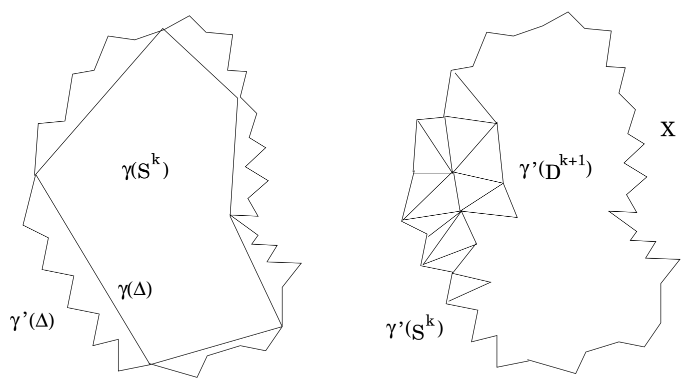

Figure 7:

Since X is k-connected, the map γ' extends to a cellular map $\gamma' : D^{k+1} \to X^{(k+1)}$. Let D denote the maximal diameter of a $k+1$-cell in X. For each simplex $\sigma \subset D^{k+1}$ the diameter of $\gamma'(\sigma)$ is at most D. We therefore can "push" the singular disk $\gamma'(D^{k+1})$ into $\mathrm{Rips}_D(Z)$ by replacing each linear map $\gamma' : \sigma \to \gamma'(\sigma) \subset X$ with the linear map $\gamma'' : \sigma \to \gamma''(\sigma) \subset \mathrm{Rips}_D(Z)$ where $\gamma''(\sigma)$ is the simplex spanned by the vertices of $\gamma'(\sigma)$. This yields a map $\gamma'' : D^{k+1} \to \mathrm{Rips}_D(Z)$. Observe that the map γ'' is a cellular map with respect to a subdivision τ' of the initial triangulation τ of S^k.

Note however that γ and $\gamma''|S^k$ are different maps. Let V denote the vertices of a k-simplex $\Delta \subset S^k$; let V'' denote the set of vertices of τ' within the simplex Δ. Then the diameter of $\gamma''(V'')$ is at most R'. Hence $\gamma(V) \subset \gamma''(V)$ is contained in a simplex in $\mathrm{Rips}_{R+R'}(Z)$. Therefore, by taking $\rho = R + D + R'$ we conclude that the maps $\gamma, \gamma'' : S^k \to \mathrm{Rips}_\rho(Z)$ are homotopic. See Figure 8. Thus the map γ is nil-homotopic within $\mathrm{Rips}_\rho(Z)$. $\square$

Corollary 50. *Suppose that G is a finitely-presented group with the word metric. Then G is coarsely simply-connected.*

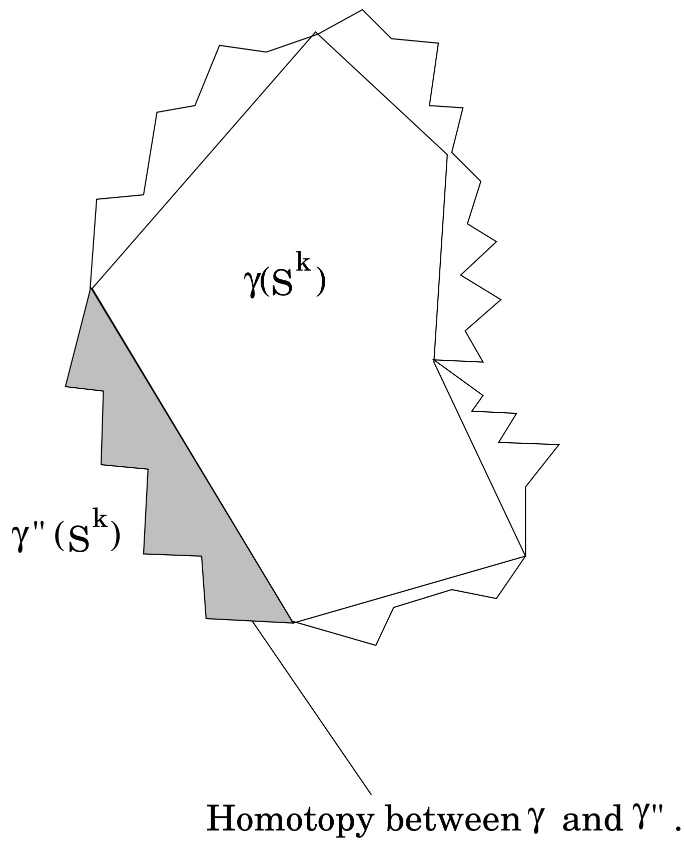

Figure 8:

Corollary 51. *(See for instance [5, Proposition 8.24]) Finite presentability is a QI invariant.*

Proof. It remains to show that each coarsely 1-connected group G is finitely presentable. The Rips complex $X := \mathrm{Rips}_R(G)$ is 1-connected for large R. The group G acts on X properly discontinuously and cocompactly. Therefore G is finitely presentable. $\square$

Definition 52. A group G is said to be of type F_n ($n \leq \infty$) if its admits a cellular action on a cell complex X such that for each $k \leq n$: (1) $X^{(k+1)}/G$ is compact. (2) $X^{(k+1)}$ is k-connected. (3) The action $G \curvearrowright X$ is free.

Example 53. (See [3].) Let $\mathbb{F}_2$ be free group on 2 generators a, b. Consider the group $G = \mathbb{F}_2^n$ which is the direct product of $\mathbb{F}_2$ with itself n times. Define a homomorphism

$\phi : G \to \mathbb{Z}$ which sends each generator a_i, b_i of G to the same generator of $\mathbb{Z}$. Let $K := Ker(\phi)$. Then K is of type F_{n-1} but not of type F_n.

Thus, analogously to Corollary 51 we get:

Theorem 54. *(See [29, 1.C2]) Type F_n is a QI invariant.*

Proof. It remains to show that each coarsely n-connected group has type F_n. The proof below follows [33]. We build the complex X on which G would act as required by the definition of type F_n. We build this complex and the action by induction on skeleta.

(0). $X^{(1)}$, is a Cayley graph of G; the action of G is cocompact, free, cellular.

(i$\Rightarrow$ i+1). Suppose that $X^{(i)}$ has been constructed. Using i-connectedness of $\mathrm{Rips}_{\bullet}(G)$ we construct (by induction on skeleta) a G-equivariant cellular map $f : X^{(i)} \to \mathrm{Rips}_D(G)$ for a sufficiently large D. If G were torsion-free, the action $G \curvearrowright \mathrm{Rips}_D(G)$ is free; this allows one to we construct (by induction on skeleta) a G-equivariant "retraction" $\rho : \mathrm{Rips}_D(G)^{(i)} \to X^{(i)}$, i.e. a map such that the composition $\rho \circ f$ is G-equivariantly homotopic to the identity.

However, if G contains nontrivial elements of finite order, we have to use a more complicated construction.

Suppose that $2 \le i \le n$ and an $i-1$-connected complex $X^{(i)}$ together with a free discrete cocompact action $G \curvearrowright X^{(i)}$ was constructed. Let $x_0 \in X^{(0)}$ be a base-point.

Lemma 55. *There are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_1(X^{(i)})$, in the sense that the normal closure of the cycles $\{g\hat{\sigma}_j : j = 1, ..., k, g \in G\}$ is $\pi_i(X^{(i)})$, where each $\hat{\sigma}_j$ is obtained from σ_j by attaching a "tail" from x_0.*

Proof. Without loss of generality we can assume that $X^{(i)}$ is a (metric) simplicial complex. Let $f : X^{(i)} \to Y := \mathrm{Rips}_D(Z)$ be a G-equivariant continuous map as above.

Here is the construction of σ_j's:

Let $\tau_\alpha : S^i \to Y^{(i)}, \alpha \in \mathbb{N}$, denote the attaching maps of the $i+1$-cells in Y, these maps are just simplicial homeomorphic embeddings from the boundary S^i of the standard $i+1$-simplex into $Y^{(i)}$. Starting with a G-equivariant projection $Y^{(0)} \to X^{(0)}$ one inductively constructs a (non-equivariant!) map $\bar{f} : Y^{(i)} \to X^{(i)}$ so that $f \circ \bar{f} : Y^{(i)} \to Y^{(i+1)}$ is within distance $\le Const$ from the identity. Hence (by coarse connectedness of Z) this composition is homotopic to the identity inclusion within $\mathrm{Rips}_{D'}(Z)$. The homotopy H is such that its tracks have "uniformly bounded complexity", i.e. the compositions

$$H \circ (\tau_\alpha \times id) : S^i \times I \to \mathrm{Rips}_{D'}(Z)$$

are simplicial maps with a uniform upper bound on the number of simplices in a triangulation of $S^i \times I$. Let $B \subset X^{(i)}$ denote a compact subset such that $GB = X^{(i)}$. We let σ_j denote the composition $g_\alpha \circ \bar{f} \circ \tau_\alpha$ where $g_\alpha \in G$ are chosen so that the image of σ_j intersects B. $\square$

We now equivariantly attach $i+1$-cells along G-orbits of the cycles σ_j: for each j and $g \in G$ we attach an $i+1$-cell along $g(\sigma_j)$. Note that if σ_j is stabilized by a subgroup of order $m = m(j)$ in G, then we attach m copies of the $i+1$-dimensional cell along σ_j. We let $X^{(i+1)}$ denote the resulting complex and we extend the G-action to $X^{(i+1)}$ in obvious fashion. It is clear that $G \curvearrowright X^{(i+1)}$ is free, discrete and cocompact. $\square$

2.3 Coarse separation

Suppose that X is a metric cell complex and $Y \subset X$ is a subset. We let $N_R(Y)$ denote the metric R-neighborhood of Y in X. Let C be a complementary component of $N_R(Y)$ in Y. Define the *inradius*, $inrad(C)$, of C to be the supremum of radii of metric balls in X contained in C. A component C is called *shallow* if $inrad(C)$ is $< \infty$ and *deep* if $inrad(C) = \infty$.

Example 56. Suppose that Y is compact. Then deep complementary components of $X \setminus N_R(Y)$ are those components which have infinite diameter.

A subcomplex Y is said to *coarsely separate* X if there is R such that $N_R(Y)$ has at least two distinct deep complementary components.

Example 57. The curve Γ in $\mathbb{R}^2$ does not coarsely separate $\mathbb{R}^2$. A straight line in $\mathbb{R}^2$ coarsely separates $\mathbb{R}^2$.

Theorem 58. *Suppose that Y, X be uniformly contractible metric cell complexes of bounded geometry which are homeomorphic to $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ respectively. Then for each uniformly proper map $f : Y \to X$, the image $f(Y)$ coarsely separates X. Moreover, the number of deep complementary components is 2.*

Proof. Actually, our proof will use the assumption on the topology of Y only weakly: to get coarse separation it suffices to assume that $H^{n-1}_c(Y, \mathbb{R}) \neq 0$.

Let $W := f(Y)$. Given $R \in \mathbb{R}_+$ we define a *retraction* $\rho : N_R(W) \to Y$, so that $d(\rho \circ f, id_Y) \le const$, where *const* depends only on the distortion function of f and on the geometry of X and Y. Here $\mathcal{N}_R(W)$ is the smallest subcomplex in X containing the R-neighborhood of W in X. We define ρ by induction on skeleta of $N_R(W)$. For each vertex $x \in \mathcal{N}_R(W)$ we pick a vertex $\rho(x) := y \in Y$ such that the distance $d(x, f(y))$ is the smallest possible. If there are several such points y, we pick one of them arbitrarily. The fact that f is a uniform proper embedding ensures that

$$d(\rho \circ f, id_{Y^0}) \le const_0.$$

Note also that for any 1-cell σ in $\mathcal{N}_R(W)$, $\mathrm{diam}(\rho(\partial\sigma)) \le Const_0$. Suppose that we have constructed ρ on $\mathcal{N}_R^{(k)}(W)$. Inductively we assume that:

$$d(\rho \circ f, id_{Y^k}) \le const_k, \mathrm{diam}(\rho(\partial\sigma)) \le Const_k, \tag{59}$$

for each $k+1$-cell σ. We extend ρ to the $k+1$-skeleton by using uniform contractibility of Y: For each $k+1$-cell σ there exists a singular disk $\eta : D^{k+1} \to Y$ in Y^{k+1} of diameter $\le \psi(Const_k)$ whose boundary is $\rho(\partial\sigma)$. Then we extend ρ to σ via η. It is clear that the extension satisfies the inequalities (59) with k replaced with $k+1$.

Since Y is uniformly contractible we get a homotopy $\rho \circ f \cong id_Y$, whose tracks are uniformly bounded (construct it by induction on skeleta the same way as before).

Recall that we have a system of isomorphisms

$$P : H_c^{n-1}(\mathcal{N}_r) \cong H_1(X, X \setminus \mathcal{N}_r)$$

given by the Poincare duality in $\mathbb{R}^n$. This isomorphism moves support sets of $n-1$-cocycles by a uniformly bounded amount (to support sets of 1-cycles). Let ω be a generator of $H_c^{n-1}(Y)$. Given $R > 0$ consider "retraction" ρ as above and the pull-back $\omega_R := \rho^*(\omega)$. If for some $0 < r < R$ the restriction ω_r of ω_R to $\mathcal{N}_r(W)$ is zero then we get a contradiction, since $f^* \circ \rho^* = id$ on the compactly supported cohomology of Y. Thus ω_r is nontrivial. Applying the Poincare duality operator P to the cohomology class ω_r we get a nontrivial relative homology class

$$P(\omega_r) \in H_1(X, X \setminus \mathcal{N}_r) \cong \tilde{H}_0(X \setminus \mathcal{N}_r).$$

We note that for each $R \ge r$ the class $P(\omega_r) \in H_1(\mathcal{N}_r, \partial\mathcal{N}_r)$ is represented by "restriction" of the class $P(\omega_R) \in H_1(\mathcal{N}_R, \partial\mathcal{N}_R)$ to $\mathcal{N}_r$, see Figure 9. In particular, the images α_r, α_R of $P(\omega_r), P(\omega_R)$ in $\tilde{H}_0(X \setminus \mathcal{N}_r)$, $\tilde{H}_0(X \setminus \mathcal{N}_R)$ are homologous in $\tilde{H}_0(X \setminus \mathcal{N}_r)$. Moreover, α_R restricts nontrivially to $\alpha_1 \in \tilde{H}_0(X \setminus \mathcal{N}_1)$.

Therefore, we get sequences of points

$$x_i, x'_i \in \partial \mathcal{N}_i, i \in \mathbb{N},$$

such that x_i, x'_i belong to the support sets of α_i for each i, x_i, x_{i+1} belong to the same component of $X \setminus \mathcal{N}_i$, x'_i, x'_{i+1} belong to the same component of $X \setminus \mathcal{N}_i$, but the points x_i, x'_i belong to distinct components C, C' of $X \setminus \mathcal{N}_1$. It follows that C, C' are distinct deep complementary components of W. The same argument run in the reverse implies that there are exactly two deep complementary components (although we will not use this fact). $\square$

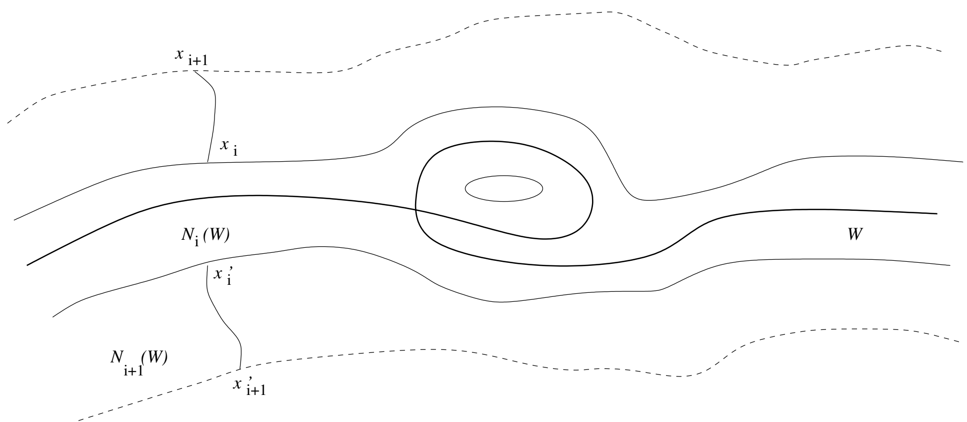

Figure 9: *Coarse separation.*

I refer to [20], [34] for further discussion and generalization of coarse separation and coarse Poincare/Alexander duality.

2.4 Other notions of coarse equivalence

Theorem 60. *(Gromov, [29], see also de la Harpe [12, page 98]) Groups G and Γ are QI iff they admit commuting (i.e. extending to an action of $G \times \Gamma$) proper cocompact topological actions on a locally compact topological space Y.*

Proof. 1. Suppose that there exists an (L, A)-quasi-isometry $G \to \Gamma$. Consider the collection F of all (L, A)-quasi-isometries f from G to Γ, given the compact-open topology. By Arcela-Ascoli, the space F is locally compact. The groups G and Γ act on F by left and right multiplication:

$$g^*(f)(x) = f(g^{-1}(x)), g \in G,$$

$$g_*(f)(x) = \gamma f(x), \gamma \in \Gamma.$$

It is clear that these are commuting topological actions. Since both G, Γ act on themselves properly, both actions $G, \Gamma \curvearrowright F$ are proper. Let $f_j \in F$, then, since the action of Γ on itself is transitive, there exists a sequence $\gamma_j \in \Gamma$ such that $\gamma_j f_j(1) = 1$. Hence, by Arcela-Ascoli theorem, the action $\Gamma \curvearrowright F$ is cocompact. (So far, everything works if instead of QI mappings we use QI embeddings). On the other hand, since for each f_j the image $f_j(G)$ is A-dense in Γ, for each j there exists $x_j \in G$ such that $d(f_j(x_j), 1) \leq A$. Hence the sequence $(x_j^{-1})^* f_j$ is also relatively compact in F. Hence both actions $G, \Gamma \curvearrowright F$ are cocompact.

2. Suppose that $G, \Gamma \curvearrowright Y$ are commuting actions. Pick a compact $K \subset Y$ which maps onto both $Y/G, Y/\Gamma$. Choose a point $k \in K$ and consider the mapping $f : G \to \Gamma$ which sends $g \in G$ to an element $\gamma^{-1} \in \Gamma$ such that $g(k) \in \gamma(K)$. I claim that f is a quasi-isometry. Let's first check that f is Lipschitz. Let $S = \{s_1, ..., s_m\}$ be a finite generating set of G. It suffices to check that f distorts each edge of the corresponding Cayley graph by a uniformly bounded amount. Pick $g \in G, \gamma^{-1} := f(g)$.

Since S is finite, $\widehat{K} := \cup_{s \in S} K$ is compact, hence there exists a finite subset $\Sigma \subset \Gamma$ such that

$$\widehat{K} \subset \widetilde{K} := \cup_{\sigma \in \Sigma} \sigma(K).$$

In addition define a finite set

$$\Sigma' := \{\alpha \in \Gamma : \alpha(K) \cap \widetilde{K} \neq \emptyset\}$$

Set $L := \max\{d_\Gamma(\alpha, 1), \alpha \in \Sigma'\}$.

Recall that the group operation on G is defined so that $h \circ g = gh$. Thus $d(s_i \circ g, g) = 1$ for each $s_i \in S$. We have:

$$s_i \circ g(k) = s_i \circ \gamma(y) = \gamma \circ s_i(y) \in \gamma \circ \sigma(K), \text{ for some } y \in K, \sigma \in \Sigma.$$

Observe that $\gamma' := [f(gs_i)]^{-1}$ also satisfies $s_i \circ g(k) \in \gamma'(K)$. Hence $\gamma^{-1} \circ \gamma'(K) \cap \sigma(K) \neq \emptyset$, i.e $\gamma' \gamma^{-1} \in \Sigma'$. Therefore $d_\Gamma(\gamma' \gamma^{-1}, 1) \leq L$ and hence

$$d_\Gamma(\gamma^{-1}, \gamma'^{-1}) \leq L, d_\Gamma(f(g), f(gs_i)) \leq L.$$

This proves that f is L-Lipschitz. Construct a map $\bar{f} : \Gamma \to G$ in the similar fashion:

$$\bar{f}(\gamma) := g^{-1}, \gamma(k) \in g(K);$$

the same arguments as above show that $\bar{f}$ is L'-Lipschitz for some $L' < \infty$.

Suppose that $f(g) = \gamma^{-1}, \bar{f}(\gamma^{-1}) = h$. Then

$$\gamma(k) \in h^{-1}(K) \iff h(k) \in \gamma^{-1}(K),$$

(since the actions of G and Γ commute). Thus $d(\bar{f} \circ f, id) \le Const$, $d(f \circ \bar{f}, id) \le Const$ for some finite constant. $\square$

Definition 61. Groups G_1, G_2 are said to **have a common geometric model** if there exists a proper geodesic metric space X such that G_i, G_2 both act isometrically, properly discontinuously, cocompactly on X.

In view of Lemma 16, if groups have a common geometric model then they are quasi-isometric. The following theorem shows that the converse is false:

Theorem 62. *(Mosher, Sageev, Whyte, [43]) Let $G_1 := \mathbb{Z}_p * \mathbb{Z}_p, G_2 := \mathbb{Z}_q * \mathbb{Z}_q$, where p, q are distinct primes. Then the groups G_1, G_2 do not have a common geometric model.*

This theorem in particular implies that in Theorem 60 one cannot assume that both group actions are isometric.

Spaces (or finitely generated groups) X_1, X_2 are *bilipschitz equivalent* if there exists a bilipschitz bijection $f : X_1 \to X_2$.

Theorem 63. *(Whyte, [61]) Suppose that G_1, G_2 are non-amenable finitely generated groups which are quasi-isometric. Then G_1, G_2 are bilipschitz equivalent.*

On the other hand, there are examples (Burago, Kleiner, McMullen, [7, 40]) of separated nets in $\mathbb{R}^2$ which are not bi-Lipschitz homeomorphic. I am unaware of examples of amenable grooups which are quasi-isometric but are not bilipschitz equivalent.

3 Ultralimits of Metric Spaces

Let (X_i) be a sequence of metric spaces. One can describe the limiting behavior of the sequence (X_i) by studying limits of sequences of finite subsets $Y_i \subset X_i$. Ultrafilters are an efficient technical device for simultaneously taking limits of all such sequences of subspaces and putting them together to form one object, namely an ultralimit of (X_i).

3.1 Ultrafilters

Let I be an infinite set, $\mathcal{S}$ is a collection of subsets of I. A *filter based on* $\mathcal{S}$ is a nonempty family ω of members of $\mathcal{S}$ with the properties:

- $\emptyset \notin \omega$.
- If $A \in \omega$ and $A \subset B$, then $B \in \omega$.
- If $A_1, \ldots, A_n \in \omega$, then $A_1 \cap \cdots \cap A_n \in \omega$.

If $\mathcal{S}$ consists of *all* subsets of I we will say that ω is a filter on I. Subsets $A \subset I$ which belong to a filter ω are called ω-large. We say that a property (P) holds for ω-*all* i, if (P) is satisfied for all i in some ω-large set. An *ultrafilter* is a maximal filter. The maximality condition can be rephrased as: For every decomposition $I = A_1 \cup \cdots \cup A_n$ of I into finitely many disjoint subsets, the ultrafilter contains exactly one of these subsets.

For example, for every $i \in I$, we have the *principal* ultrafilter δ_i defined as $\delta_i := \{A \subset I \mid i \in A\}$. An ultrafilter is principal if and only if it contains a finite subset. The interesting ultrafilters are of course the non-principal ones. They cannot be described explicitly but exist by Zorn's lemma: Every filter is contained in an ultrafilter. Let $\mathcal{Z}$ be the *Zariski filter* which consists of complements to finite subsets in I. An ultrafilter is a nonprincipal ultrafilter, if and only if it contains $\mathcal{Z}$.

Here is an alternative interpretation of ultrafilters. An ultrafilter is a finitely additive measure defined on all subsets of I so that each subset has measure 0 or 1. An ultrafilter is nonprincipal iff the measure contains no atoms: The measure of each point is zero.

Given an ultrafilter ω on I and a collection of sets $X_i, i \in I$, define the *ultraproduct*

$$\prod_{i \in I} X_i/\omega$$

to be the collection of equivalence classes of maps $f : I \to X$ such that $f \sim g$ iff $f(i) = g(i)$ for ω-all i.

Given a function $f : I \to Y$ (where Y is a topological space) define the ω-limit

$$\omega\text{-}\lim_i f(i)$$

to be a point $y \in Y$ such that for every neighborhood U of y the preimage $f^{-1}U$ belongs to ω.

Lemma 64. *Suppose that Y is compact and Hausdorff. Then for each function $f : I \to Y$ the ultralimit exists and is unique.*

Proof. To prove existence of a limit, assume that there is no point $y \in Y$ satisfying the definition of the ultralimit. Then each point $z \in Y$ possesses a neighborhood U_z such that $f^{-1}U_z \notin \omega$. By compactness, we can cover Y with finitely many of these neighborhoods. It follows that $I \notin \omega$. This contradicts the definition of a filter. Uniqueness of the point y follows, because Y is Hausdorff. $\square$

Note that if y is an accumulation point of $\{f(i)\}_{i\in I}$ then there is a non-principal ultrafilter ω with $\omega\text{-}\lim f = y$, namely an ultrafilter containing the pullback of the neighborhood basis of y.

3.2 Ultralimits of metric spaces

Let $(X_i)_{i\in I}$ be a family of metric spaces parameterized by an infinite set I. For an ultrafilter ω on I we define the ultralimit

$$X_\omega = \omega\text{-}\lim_i X_i$$

as follows. Let $\prod_i X_i$ be the product of the spaces X_i, i.e. it is the space of sequences $(x_i)_{i\in I}$ with $x_i \in X_i$. The distance between two points $(x_i), (y_i) \in \prod_i X_i$ is given by

$$d_\omega\big((x_i),(y_i)\big) := \omega\text{-}\lim\big(i \mapsto d_{X_i}(x_i,y_i)\big)$$

where we take the ultralimit of the function $i \mapsto d_{X_i}(x_i, y_i)$ with values in the compact set $[0,\infty]$. The function d_ω is a pseudo-distance on $\prod_i X_i$ with values in $[0,\infty]$. Set

$$(X_\omega, d_\omega) := (\prod_i X_i, d_\omega)/\sim$$

where we identify points with zero d_ω-distance.

Exercise 65. Let $X_i = Y$ for all i, where Y is a compact metric space. Then $X_\omega \cong Y$ for all ultrafilters ω.

If the spaces X_i do not have uniformly bounded diameter, then the ultralimit X_ω decomposes into (generically uncountably many) components consisting of points of mutually finite distance. We can pick out one of these components if the spaces X_i have base-points x_i^0. The sequence $(x_i^0)_i$ defines a base-point x_ω^0 in X_ω and we set

$$X_\omega^0 := \big\{x_\omega \in X_\omega \mid d_\omega(x_\omega, x_\omega^0) < \infty\big\}.$$

Define the *based ultralimit* as

$$\omega\text{-}\lim_i (X_i, x_i^0) := (X_\omega^0, x_\omega^0).$$

Example 66. For every locally compact space Y with a base-point y_0, we have:

$$\omega\text{-}\lim_i (Y, y_0) \cong (Y, y_0).$$

Lemma 67. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of geodesic δ_i-hyperbolic spaces with δ_i tending to 0. Then for every non-principal ultrafilter ω each component of the ultralimit X_ω is a metric tree.*

Proof. We first verify that between any pair of points $x_\omega, y_\omega \in X_\omega$ there is a unique geodesic segment. Let γ_ω denote the ultralimit of the geodesic segments $\gamma_i := \overline{x_i y_i} \subset X_i$; it connects the points x_ω, y_ω. Suppose that β is another geodesic segment connecting x_ω to y_ω. Pick a point $p_\omega \in \beta$. Then

$$\omega\text{-}\lim_i (x_i, y_i)_{p_i} = \omega\text{-}\lim_i \frac{1}{2}[d(x_i, p_i) + d(y_i, p_i) - d(x_i, y_i)] = 0.$$

Since, by Lemma 24,

$$d(p_i, \gamma_i) \le (x_i, y_i)_{p_i} + 2\delta_i,$$

$$d(p_\omega \gamma_\omega) = 0.$$

Now, suppose that $\Delta(x_\omega y_\omega z_\omega)$ is a geodesic triangle in X_ω. By uniqueness of geodesics in X_ω, this triangle appears as ultralimit of the δ_i-thin triangles $\Delta(x_i y_i z_i)$. It follows that $\Delta(x_\omega y_\omega z_\omega)$ is zero-thin, i.e. each component of X_ω is zero-hyperbolic. $\square$

Exercise 68. If T is a metric tree, $-\infty < a < b < \infty$ and $f : [a, b] \to T$ is a continuous embedding then the image of f is a geodesic segment in T. (Hint: use PL approximation of f to show that the image of f contains the geodesic segment connecting $f(a)$ to $f(b)$.)

Lemma 69. *(Morse Lemma) Let X be a δ–hyperbolic geodesic space, k, c be positive constants, then there is a function $\theta = \tau(k, c)$ such that for any (k, c)-quasi-isometric embedding $f : [a, b] \to X$ the Hausdorff distance between the image of f and the geodesic segment $[f(a)f(b)] \subset X$ is at most θ.*

Proof. Suppose that the assertion of lemma is false. Then there exists a sequence of (k, c)-quasi-isometric embeddings $f_n : [-n, n] \to X_n$ to $CAT(-1)$-spaces X_n such that

$$\lim_{n\to\infty} d_{Haus}(f([-n, n]), [f(-n), f(n)]) = \infty$$

where d_{Haus} is the Hausdorff distance in X_n.

Let $d_n := d_{Haus}(f([-n,n]), [f(-n), f(n)])$. Pick points $t_n \in [-n,n]$ such that $|d(t_n, [f(-n), f(n)]) - d_n| \leq 1$. Consider the sequence of pointed metric spaces $(\frac{1}{d_n}X_n, f_n(t_n))$, $(\frac{1}{d_n}[-n,n], t_n)$. It is clear that $\omega\text{-}\lim n/d_n > 1/k > 0$ (but this ultralimit could be infinite). Let $(X_\omega, x_\omega) = \omega\text{-}\lim(\frac{1}{d_n}X_n, f_n(t_n))$ and $(Y, y) := \omega\text{-}\lim(\frac{1}{d_n}[-n,n], t_n)$. The metric space Y is either a nondegenerate segment in $\mathbb{R}$ or a closed geodesic ray in $\mathbb{R}$ or the whole real line. Note that the Hausdorff distance between the image of f_n in $\frac{1}{d_n}X_n$ and $[f_n(-n), f_n(n)] \subset \frac{1}{d_n}X_n$ is at most $1 + 1/d_n$. Each map

$$f_n : \frac{1}{d_n}[-n,n] \to \frac{1}{d_n}X_n$$

is a $(k, c/n)$-quasi-isometric embedding. Therefore the ultralimit

$$f_\omega = \omega\text{-}\lim f_n : (Y, y) \to (X_\omega, x_\omega)$$

is a $(k, 0)$-quasi-isometric embedding, i.e. it is a k-bilipschitz map:

$$|t - t'|/k \leq d(f_\omega(t), f_\omega(t')) \leq k|t - t'|.$$

In particular this map is a continuous embedding. On the other hand, the sequence of geodesic segments $[f_n(-n), f_n(n)] \subset \frac{1}{d_n}X_n$ also ω-converges to a nondegenerate geodesic $\gamma \subset X_\omega$, this geodesic is either a finite geodesic segment or a geodesic ray or a complete geodesic. In any case the Hausdorff distance between the image L of f_ω and γ is exactly 1, it equals the distance between x_ω and γ which is realized as $d(x_\omega, z) = 1$, $z \in \gamma$. I will consider the case when γ is a complete geodesic, the other two cases are similar and are left to the reader. Then $Y = \mathbb{R}$ and by Exercise 68 the image L of the map f_ω is a complete geodesic in X_ω which is within Hausdorff distance 1 from the complete geodesic γ. This contradicts the fact that X_ω is a metric tree. $\square$

Historical Remark. Morse [42] proved a special case of this lemma in the case of $\mathbb{H}^2$ where the quasi-geodesics in question where geodesics in another Riemannian metric on $\mathbb{H}^2$, which admits a cocompact group of isometries. Busemann, [9], proved a version of this lemma in the case of $\mathbb{H}^n$, where metrics in question were not necessarily Riemannian. A version in terms of quasi-geodesics is due to Mostow [44], in the context of negatively curved symmetric spaces, although his proof is general.

Corollary 70. *Suppose that X, X' are quasi-isometric geodesic metric spaces and X is Gromov-hyperbolic. Then X' is also Gromov-hyperbolic.*

Proof. Let $f : X' \to X$ be a (L, A)-quasi-isometry. Pick a geodesic triangle $\Delta ABC \subset X'$. Its image is a quasi-geodesic triangle whose sides are (L, A)-quasi-geodesic. Therefore each of the quasi-geodesic sides of $f(\Delta ABC)$ is within distance $\leq c = c(L, A)$ from a geodesic connecting the end-points of this side. See Figure 10. The geodesic triangle $\Delta f(A)f(B)f(C)$ is δ-thin, it follows that the quasi-geodesic triangle $f(\Delta ABC)$ is $(2c + \delta)$-thin. Thus the triangle ΔABC is $L(2c + \delta) + A$-thin.

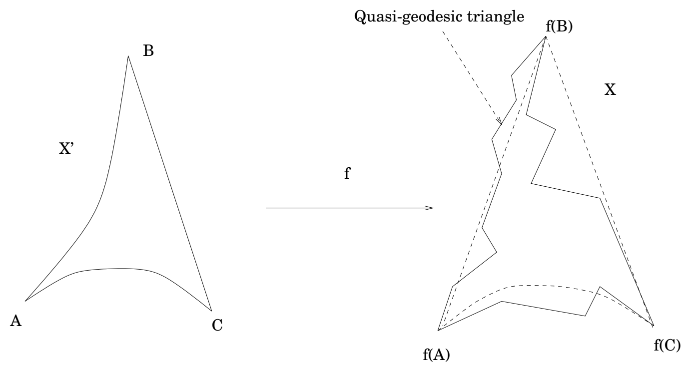

Figure 10: *Image of a geodesic triangle.*

□

Here is another example of application of asymptotic cones to study quasi-isometries.

Lemma 71. *Suppose that $X = \mathbb{R}^n$ or $\mathbb{R}_+$, $f : X \to X$ is an (L, A)-quasi-isometric embedding. Then $N_C(f(X)) = X$, where $C = C(L, A)$.*

Proof. I will give a proof in the case of $\mathbb{R}^n$ as the other case is analogous. Suppose that the assertion is false, i.e. there is a sequence of (L, A)-quasi-isometries $f_j : \mathbb{R}^n \to \mathbb{R}^n$, sequence of real numbers r_j diverging to infinity and points $y_j \in \mathbb{R}^n \setminus Image(f)$ such that $d(y_j, Image(f)) = r_j$. Let $x_j \in \mathbb{R}^n$ be a point such that $d(f(x_j), y_j) \leq r_j + 1$. Using x_j, y_j as basi-points on the domain and target to f_j rescale the metrics on the domain and the target by $1/r_j$ and take the corresponding ultralimits. In the limit we get a bi-Lipschitz embedding

$$f_\omega : \mathbb{R}^n \to \mathbb{R}^n,$$

whose image misses the point $y_\omega \in \mathbb{R}^n$. However each bilipshitz embedding is necessarily proper, therefore by the invariance of domain theorem the image of f_ω is both closed and open. Contradiction. □

Remark 72. Alternatively, one can prove the above lemma as follows: Approximate f by a continuous mapping g. Then, since g is proper, it has to be onto.

3.3 The asymptotic cone of a metric space

Let X be a metric space and ω be a non-principal ultrafilter on $I = \mathbb{N}$. Suppose that we are given a sequence λ_i so that $\omega\text{-lim}\, \lambda_i = 0$ and a sequence of base-points $x_i^0 \in X$. Given this data the *asymptotic cone* $\mathrm{Cone}_\omega(X)$ of X is defined as the based ultralimit of rescaled copies of X:

$$\mathrm{Cone}_\omega(X) := X_\omega^0, \qquad \text{where} \quad (X_\omega^0, x_\omega^0) = \omega\text{-}\lim_i (\lambda_i \cdot X, x_i^0).$$

The discussion in the previous section implies:

Proposition 73. *1.* $\mathrm{Cone}_\omega(X \times Y) = \mathrm{Cone}_\omega(X) \times \mathrm{Cone}_\omega(Y)$.

2. $Cone_\omega \mathbb{R}^n \cong \mathbb{R}^n$.

3. The asymptotic cone of a geodesic space is a geodesic space.

4. The asymptotic cone of a CAT(0)-space is CAT(0).

5. The asymptotic cone of a space with a negative upper curvature bound is a metric tree.

Remark 74. Suppose that X admits a cocompact discrete action by a group G of isometries. The problem of dependence of the topological type of $Cone_\omega X$ on the ultrafilter ω and the scaling sequence λ_i was open until recently counterexamples were constructed in [53], [15]. However in the both examples the group G is not finitely presentable. Moreover, if a finitely-repsentable group has *an asymptotic cone* which is a tree, then the group is hyperbolic and hence *each* asymptotic cone is a tree, see [33].

To get an idea of the size of the asymptotic cone, we will see below that in the most interesting cases it is homogeneous.

We call an isometric action $G \curvearrowright X$ *cobounded* if there exists $D < \infty$ such that for some point $x \in X$,

$$\bigcup_{g \in G} g(B_D(x)) = X,$$

i.e. $G \cdot x$ is a D-net in X. Equivalently, given any pair of points $x, y \in X$, there exists $g \in G$ such that $d(g(x), y) \leq 2D$. We call a metric space X *quasi-homogeneous* if the action $\mathrm{Isom}(X) \curvearrowright X$ is cobounded.

Suppose that X is a metric space and $G \subset \mathrm{Isom}(X)$ is a subgroup. Given a nonprincipal ultrafilter ω define the group G^* to be the ultraproduct

$$G^* = \prod_{i \in I} G/\omega.$$

By obusing notation we will refer to points in G^* as *sequences.* Given a sequence λ_i so that $\omega\text{-}\lim \lambda_i = 0$ and a sequence of base-points $x_i^0 \in X$, let $\mathrm{Cone}_\omega(X)$ be the corresponding asymptotic cone. It is clear that G^* acts isometrically on the ultralimit

$$U := \omega\text{-}\lim_i (\lambda_i \cdot X).$$

Let $G_\omega \subset G^*$ denote the stabilizer in G^* of the component $\mathrm{Cone}_\omega(X) \subset U$. In other words,

$$G_\omega = \{(g_i) \in G^* : \omega\text{-}\lim_i \lambda_i d(g_i(x_i^0), x_i^0) < \infty\}.$$

Thus $G_\omega \subset \mathrm{Isom}(\mathrm{Cone}_\omega(X))$. Observe that if (x_i^0) is a bounded sequence in X then the group G has a diagonal embedding in G_ω.

Proposition 75. *Suppose that $G \subset \mathrm{Isom}(X)$ and the action $G \curvearrowright X$ is cobounded. Then for every asymptotic cone $\mathrm{Cone}_\omega(X)$ the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is transitive. In particular, $\mathrm{Cone}_\omega(X)$ is a homogeneous metric space.*

Proof. Let $D < \infty$ be such that $G \cdot x$ is a D-net in X. Given two sequences $(x_i), (y_i)$ of points in X there exists a sequence (g_i) of elements of G such that

$$d(g_i(x_i), y_i) \leq 2D.$$

Therefore, if $g_\omega := (g_i) \in G^*$, then $g_\omega((x_i)) = (y_i)$. Hence the action

$$G^* \curvearrowright X_\omega = \omega\text{-}\lim_i (\lambda_i \cdot X)$$

is transitive. It follows that the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is transitive as well.

Example 76. Construct an example of a metric space X and an asymptotic cone $\mathrm{Cone}_\omega(X)$ so that for the isometry group $G = \mathrm{Isom}(X)$ the action $G_\omega \curvearrowright \mathrm{Cone}_\omega(X)$ is not effective (i.e. has nontrivial kernel). Construct an example when the kernel of $G_\omega \to \mathrm{Isom}(\mathrm{Cone}_\omega(X))$ contains the entire group G embedded diagonally in G_ω.

Lemma 77. *Let X be a quasi-homogeneous δ–hyperbolic space with uncountable number of ideal boundary points. Then for every nonprincipal ultrafilter ω the asymptotic cone $\mathrm{Cone}_\omega(X)$ is a tree with uncountable branching.*

Proof. Let $x^0 \in X$ be a base-point and $y, z \in \partial_\infty X$. Denote by γ the geodesic in X with the ideal endpoints z, y. Then $\mathrm{Cone}_\omega([x^0, y))$ and $\mathrm{Cone}_\omega([x^0, z))$ are geodesic rays in $\mathrm{Cone}_\omega(X)$ emanating from x^0_ω. Their union is equal to the geodesic $\mathrm{Cone}_\omega \gamma$. This produces uncountably many rays in $\mathrm{Cone}_\omega(X)$ so that any two of them have precisely the base-point in common. The homogeneity of $\mathrm{Cone}_\omega(X)$ implies the assertion. $\square$

3.4 Extension of quasi-isometries of hyperbolic spaces to the ideal boundary

Lemma 78. *Suppose that X is a proper δ-hyperbolic geodesic space. Let $Q \subset X$ be a (L, A)-quasigeodesic ray or a complete (L, A)-quasigeodesic. Then there is Q^* which is either a geodesic ray (or a complete geodesic) in X so that the Hausdorff distance between Q and Q^* is $\le C(L, A, \delta)$.*

Proof. I will consider only the case of quasigeodesic rays $\rho : [0, \infty) \to Q \subset X$ as the other case is similar. Consider the sequence of geodesic segments $\gamma_i = \overline{\rho(0)\rho(i)}$. By Morse lemma, each γ_i is contained within $N_c(Q)$, where $c = c(L, A, \delta)$. By local compactness, the geodesic segments γ_i subconverge to a complete geodesic ray $Q^* = \gamma(\mathbb{R}_+)$ which is contained in $N_c(Q)$.

It remains to show that Q is contained in $N_D(Q^*)$, where $D = D(L, A, \delta)$. Consider the nearest-point projection $p : Q^* \to Q$. This projection is clearly a quasi-isometric embedding with the constants depending only on L, A, δ. Lemma 71 shows that the image of p is ϵ-dense in Q with $\epsilon = \epsilon(L, A, \delta)$. Hence each point of Q is within distance $\le D = \epsilon + c$ from a point of Q^*. $\square$

Observe that this lemma implies that for any divergent sequence $t_j \in \mathbb{R}_+$, the sequence of points $\rho(t_j)$ on a quasi-geodesic ray in X, converges to a point $\eta \in \partial_\infty X$, $\eta = \gamma(\infty)$. Indeed, if γ, γ' are geodesic rays Hausdorff-close to Q then γ, γ' are Hausdorff-close to each other as well, therefore $\gamma(\infty) = \gamma'(\infty)$.

We will refer to the point η as $\rho(\infty)$. Note that if ρ' is another quasi-geodesic ray which is Hausdorff-close to ρ then $\rho(\infty) = \rho'(\infty)$.

Theorem 79. *Suppose that X and X' are Gromov-hyperbolic proper geodesic metric spaces. Let $f : X \to X'$ be a quasi-isometry. Then f admits a homeomorphic extension $f_\infty : \partial_\infty X \to \partial_\infty X'$. This extension is such that the map $f \cup f_\infty$ is continuous at each point $\eta \in \partial_\infty X$.*

Proof. First, we construct the extension f_∞. Let $\eta \in \partial_\infty X$, $\eta = \rho(\infty)$ where ρ is a geodesic ray in X. The image of this ray $\rho' := f \circ \rho : \mathbb{R}_+ \to X'$ is a quasi-geodesic ray, hence we set $f_\infty(\eta) := \rho'(\infty)$. Observe that $f_\infty(\eta)$ does not depend on the choice of a geodesic ray asymptotic to η. Let $\bar{f}$ be quasi-inverse of f. It is clear from the construction that $(\bar{f})_\infty$ is inverse to f_∞. It remains therefore to verify continuity.

Suppose that $x_n \in X$ is a sequence which converges to η in the cone topology, $d(x_n, \rho) \leq c$. Then $d(f(x_n), \rho') \leq Lc + A$ and $d(f(x_n), (\rho')*) \leq C(Lc + A)$, where $(\rho')*$ is a geodesic ray in X' asymptotic to $\rho'(\eta)$. Thus $f(x_n)$ converges to $f_\infty(\eta)$ in the cone topology.

Finally, let $\eta_n \in \partial_\infty X$ be a sequence which converges to η. Let ρ_n be a sequence of geodesic rays asymptotic to η_n with $\rho_n(0) = \rho(0) = x_0$. Then, for each $T \in \mathbb{R}_+$ there exists n_0 such that for all $n \geq n_0$ and $t \in [0, T]$ we have

$$d(\rho(t), \rho_n(t)) \leq 2\delta,$$

where δ is the hyperbolicity constant of X. Hence

$$d(f(\rho_n(t)), \rho(t)) \leq 2L\delta + A.$$

Set $\rho'_n := f \circ \rho_n$. Then

$$(\rho'_n)^*([0, L^{-1}T - A]) \subset N_C((\rho')^*([0, LT + A])),$$

for all $n \geq n_0$. Thus the geodesic rays $(\rho'_n)^*$ converge to a ray within finite distance from $(\rho')^*$. It follows that the sequence $f_\infty(\eta_n)$ converges to $f_\infty(\eta)$. $\square$

Lemma 80. *Let X and X' be proper geodesic δ-hyperbolic spaces. In addition we assume that X is quasi-homogeneous and that $\partial_\infty X$ consists of at least four points. Suppose that $f, g : X \to X'$ are (L, A)-quasi-isometries such that $f_\infty = g_\infty$. Then $d(f, g) \leq D$, where D depends only on L, A, δ and the geometry of X.*

Proof. Let γ_1, γ_2 be complete geodesics in X which are asymptotic to the points ξ_1, η_1, ξ_2, η_2 respectively, where all the points $\xi_1, \eta_1, \xi_2, \eta_2$ are distinct. There is a point $y \in X$ which is within distance $\leq r$ from both geodesics γ_1, γ_2. Let G be a group acting isometrically on X so that the $GB = X$ for an R-ball B in X. Pick a point $x \in X$: Our goal is to estimate $d(gf(x), g(x))$. By applying an element of G to x we can assume that $d(x, y) \leq R$, in particular, $d(x, \gamma_1) \leq R + r, d(x, \gamma_2) \leq R + r$. Thus the distance from $f(x)$ to the quasi-geodesics $f(\gamma_1), f(\gamma_2)$ is at most $L(R + r) + A$. We now apply the quasi-inverse $\bar{g}$ the to quasi-isometry g: $\bar{g}f(\gamma_i)$ is an $(L^2, LA + A)$-quasi-geodesic in X; since $f_\infty = g_\infty$, these quasi-geodesics are asymptotic to the points ξ_i, η_i, $i = 1, 2$. Since the Hausdorff distance from $\bar{g}f(\gamma_i)$ to γ_i is at most $C + 2\delta$ (where $C = C(L^2, LA + A, \delta)$ is the constant from Lemma 78) we conclude that

$$d(\bar{g}f(x), \gamma_i) \leq C' := C + 2\delta.$$

See Figure 11.

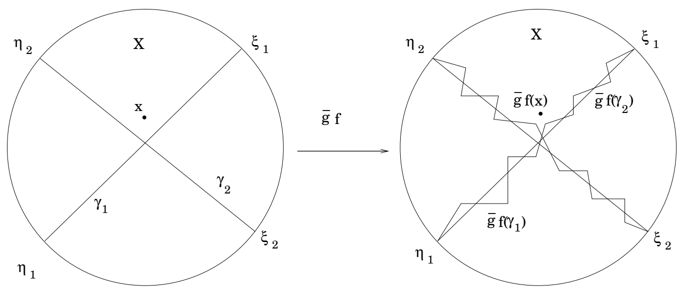

Figure 11:

Since the geodesics γ_1, γ_2 are asymptotic to distinct points in $\partial_\infty X$, it follows that the diameter of the set $\{z \in X : d(z, \gamma_i) \leq \max(C', r + R), i = 1, 2\}$ is at most C'', where C'' depends only on the geometry of X and the fixed pair of geodesics γ_1, γ_2. Hence $d(\bar{g}f(x), x) \leq C''$. By applying g to this formula we get:

$$d(g(x), g\bar{g}f(x)) \leq L(C'' + A) + A,$$

$$d(f(x), g\bar{g}f(x)) \leq A.$$

Therefore

$$d(f(x), g(x)) \leq 2A + L(C'' + A). \quad \square$$

Remark 81. The line $X = \mathbb{R}$ is 0-hyperbolic, its ideal boundary consists of 2 points. Take a translation $f : X \to X$, $f(x) = x + a$. Then f_∞ is the identity map of $\{-\infty, \infty\}$ but there is no bound on the distance from f to the identity.

4 Tits alternative

Theorem 82 (Tits alternative, [54]). *Let L be a Lie group with finitely many components and $\Gamma \subset L$ be a finitely generated subgroup. Then either Γ is virtually solvable or Γ contains a free nonabelian subgroup.*

I will give a detailed proof of this theorem in the case $L = SL(2,\mathbb{R})$ and will outline the proof in the general case. Our proof in the $SL(2,\mathbb{R})$ case does not require Γ to be finitely generated.

The projectivization $PSL(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ is the orientation-preserving subgroup of the isometry of group of the hyperbolic plane $\mathbb{H}^2$. If we use the upper half-plane model of $\mathbb{H}^2$ then $PSL(2,\mathbb{R})$ acts on $\mathbb{H}^2$ via linear-fractional transformations:

$$P(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) : z \mapsto \frac{az+b}{cz+d}.$$

It is clear that Tits alternative for $PSL(2,\mathbb{R})$ implies Tits alternative for $SL(2,\mathbb{R})$, since they differ by finite center.

Classification of isometries γ of $\mathbb{H}^2$:

Let $A \in SL(2,\mathbb{R})$. Then the fixed points for the action of $P(A)$ on $\bar{\mathbb{C}}$ correspond to the eigenvectors of the matrix A. Thus we get:

Case 1. $|tr(A)| > 2 \iff P(A)$ has 2 distinct fixed points on $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$. Then $\gamma = P(A)$ is called *hyperbolic*. It acts as a translation along a geodesic in $\mathbb{H}^2$ connecting the fixed points of γ.

Case 2. $|tr(A)| < 2 \iff P(A)$ has 2 distinct fixed points on $\bar{\mathbb{C}} \setminus \bar{\mathbb{R}}$, one in the upper and one in the lower half-plane. Then γ is called *elliptic*, in the unit disk model, if we send the fixed point to the origin, γ acts as a rotation around the origin.

Case 3. $|tr(A)| = 2$ and $\gamma \neq Id$. Then γ has a unique fixed point in $\bar{\mathbb{C}}$, this fixed point belongs to $\bar{\mathbb{R}}$. Then γ is called *parabolic*. Conjugate γ in $PSL(2,\mathbb{R})$ so that the fixed point of γ is infinity. Then $\gamma(z) = z + c$, $c \in \mathbb{R}$, i.e. γ acts as a Euclidean translation.

This is a complete classification of orientation-preserving isometries of $\mathbb{H}^2$. If γ is an orientation-reversing isometry of $\mathbb{H}^2$ then either:

(a) γ is a reflection in a geodesic $L \subset \mathbb{H}^2$, or

(b) γ is a *glide-reflection*, i.e. it is the composition of a reflection in a geodesic $L \subset \mathbb{H}^2$ with a hyperbolic translation along L.

Dynamics: Suppose that γ is hyperbolic or parabolic. Then the sequence γ^n, $n \in \mathbb{N}$, converges uniformly on compacts in $\bar{\mathbb{C}} \setminus Fix(\gamma)$ to the constant map $z \mapsto \xi$, where ξ is one of the fixed points of γ. If γ is hyperbolic then ξ is the attractive fixed point of γ.

Lemma 83 (Ping-Pong lemma). *Suppose that $g, h \in PSL(2,\mathbb{R})$ are hyperbolic or parabolic with disjoint fixed point sets. Then there exists $n \in \mathbb{N}$ such that the group $\langle g^n, h^n\rangle$ is free of rank 2.*

Proof. Is will consider the case when g, h are hyperbolic since the other cases are similar. Let A_- be a neighborhood of the repulsive fixed point of g, bounded by a geodesic in $\mathbb{H}^2$ and disjoint from the axis of h. Similarly, define B_-, a neighborhood of the repulsive fixed point of h, bounded by a geodesic in $\mathbb{H}^2$ and disjoint from the axis of h and from A_-. By taking sufficiently large n we can assume that the complements to $g^n(A_-)$ and $h^n(B_-)$ in $\mathbb{H}^2$ are domains A_+, B_+, as in the Figure 12, so that all four domains A_-, A_+, B_-, B_+ are pairwise disjoint. Let Φ denote the domain in $\mathbb{H}^2$ which is the complement to $A_- \cup A_+ \cup B_- \cup B_+$. Set $g := g^n, h := h^n$. I claim that the group $G := \langle g, h\rangle$ is free of rank 2. To prove this consider a reduced nonempty word w in the generators g, h. I claim that $w(\Phi) \cap \Phi = \emptyset$. This would imply that w is a nontrivial element of G which in turn would imply that G is free of rank 2.

Moreover, suppose that the last letter in w is g (or g^{-1}, or h, or h^{-1} resp.), i.e. $w = w'g$. I claim that $w(\Phi) \subset A_+$ (resp. A_-, B_+, B_-). Let's prove this by induction on the length of w. I consider the case when $w = w'g$, where w' is a reduced word whose last letter is not g^{-1}. Hence by induction, $w'(\Phi)$ is in one of the regions A_+, B_+, B_-, but not in A_-. Then it is clear from the action of the isometry g that $g(A_+ \cup B_+ \cup B_-) \subset A_+$. Thus $w(\Phi) \subset A_+$. $\square$

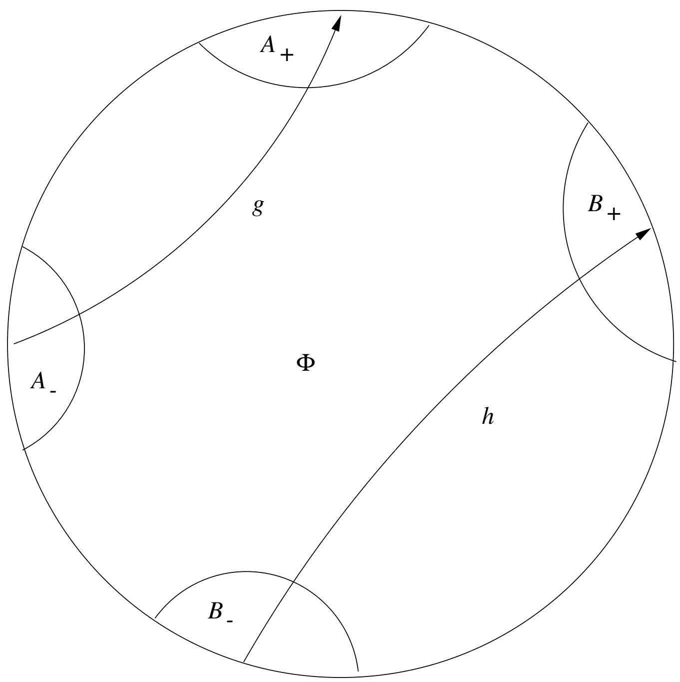

Figure 12:

This proves Tits alternative in the case when $G \subset PSL(2,\mathbb{R})$ contains two hyperbolic/parabolic elements which do not share a fixed point.

Definition 84. A subgroup of $PSL(2,\mathbb{R})$ is *elementary* if it either fixes a point in $\bar{\mathbb{C}}$ or preserves a 2-point subset of $\bar{\mathbb{C}}$.

Corollary 85. *Suppose that $\Gamma \subset PSL(2,\mathbb{R})$ is a nonelementary subgroup which contains a hyperbolic or parabolic element. Then Γ contains $\mathbb{F}_2$.*

Proof. Case 1. Suppose first that Γ contains a parabolic element γ whose fixed point is ξ; since Γ does not fix ξ, there exists $\alpha \in \Gamma$ such that $\eta = \alpha(\xi) \neq \xi$; then $\beta := \alpha\gamma\alpha^{-1}$ is a parabolic isometry with the fixed point $\eta \neq \xi$. Then Ping-Pong lemma implies that $\langle \gamma^n, \beta^n \rangle$ is isomorphic to $\mathbb{F}_2$ for large n.

Case 2. Now, suppose that $\gamma \in \Gamma$ is a hyperbolic isometry with the fixed points ξ, η. There exists $\alpha \in \Gamma$ such that $\alpha(\xi) \neq \xi$ and $\alpha(\eta) \neq \eta$ and $\alpha(\{\xi,\eta\}) \neq \{\xi,\eta\}$. If

$$\alpha(\{\xi,\eta\}) \cap \{\xi,\eta\} = \emptyset,$$

then we are done by the Ping-Pong lemma, analogously to the parabolic case above. Suppose that $\alpha(\eta) = \xi$. Define $\beta := \alpha\gamma\alpha^{-1}$: it is a hyperbolic isometry which fixes ξ and does not fix η. It is easy to see that the commutator $[\gamma, \beta]$ is a parabolic isometry which fixes ξ (just assume that $\xi = \infty$, $\eta = 0$ and then compute the commutator). Therefore Γ contains a parabolic isometry and we are done by Case 1. $\square$

The most difficult case is when Γ contains only elliptic elements.

Lemma 86. *If Γ contains only elliptic elements, then Γ fixes a point in $\mathbb{H}^2$.*

Proof. Suppose that there are elliptic elements α, β in Γ with distinct fixed points $a, b \in \mathbb{H}^2$. By assumption, their product $\gamma = \beta \circ \alpha$ is also an elliptic element; its fixed point c is necessarily distinct from a and b. Consider the geodesic triangle in $\mathbb{H}^2$ with the vertices a, b, c; let $J = R_1, R_2, R_3$ denote reflections in the sides $[ab], [bc]$ and $[ca]$ respectively. Then

$$\alpha = R_1 R_3, \beta = R_2 R_1, \gamma = R_2 R_3.$$

See Figure 13.

Then $[\beta^{-1}, \alpha^{-1}] = J\gamma J\gamma = (J\gamma)^2$. Note that $J\gamma$ is an orientation-reversing isometry. If $J\gamma$ is a reflection then $[\beta^{-1}, \alpha^{-1}] = Id$, which would imply that $a = b$. Thus $J\gamma$ is a glide-reflection; it follows that $(J\gamma)^2$ is a hyperbolic isometry (a translation along the axis of of $J\gamma$). Hence Γ contains a hyperbolic element. Contradiction. $\square$

Lemma 87. *If Γ is an elementary subgroup of $PSL(2,\mathbb{R})$, then Γ is virtually solvable.*

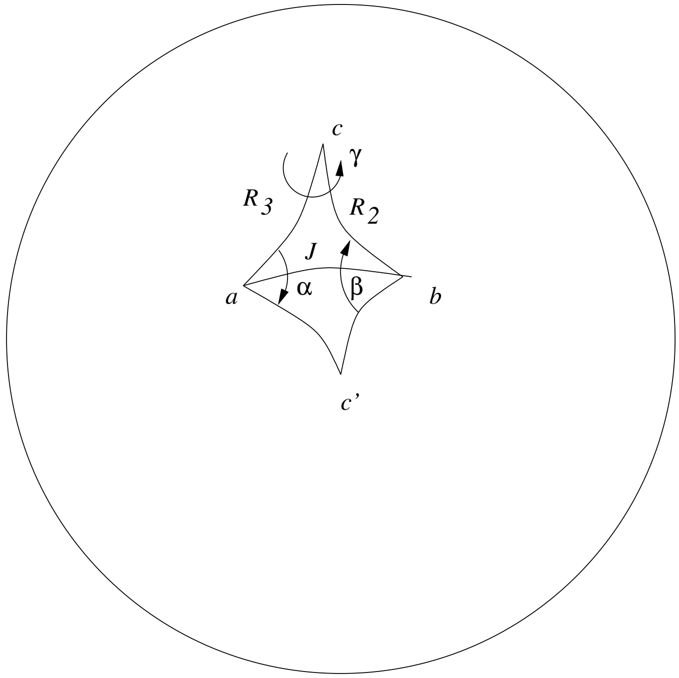

Figure 13:

Proof. If Γ preserves a 2-point set then its index 2 subgroup fixes a point. Therefore it suffices to consider the case when Γ fixes a point ξ in $\bar{\mathbb{C}}$.

Case 1. $\xi \notin \partial_\infty \mathbb{H}^2 = \bar{\mathbb{R}}$. Then Γ fixes a point in the hyperbolic plane $\mathbb{H}^2$ (either ξ or its complex conjugate). By using the unit disk model we can assume that Γ fixes the origin in the unit disk. Then $\Gamma \subset SO(2)$; since the latter is abelian it follows that Γ is abelian as well.

Case 2. $\xi \in \partial_\infty \mathbb{H}^2 = \bar{\mathbb{R}}$. We can assume that $\xi = \infty$; then Γ is contained in the group S of affine transformations $z \mapsto az + b$. The group S contains abelian subgroup A which consists of translations $z \mapsto z + b$. The group $A = [S, S]$ is the commutator subgroup of S. Therefore S is solvable. It follows that Γ is solvable as well. $\square$

Outline of the proof of Tits' alternative in the general case. By taking a homomorphism $L \to ad(L) \curvearrowright Lie(L)$, where $Lie(L)$ is the Lie algebra of L, it suffices to prove Tits alternative for subgroups $\Gamma \subset GL(n, \mathbb{R})$. Let G denote *Zariski closure* of Γ in $GL(n, \mathbb{R})$, i.e. the smallest algebraic subgroup (i.e. subgroup given by algebraic equations) of $GL(n, \mathbb{R})$ which contains Γ. If the identity component of G happens to be solvable then we are done. Otherwise the identity component of G has nontrivial semisimple part; by dividing G by its solvable radical we can assume that G is semisimple, i.e. its Lie algebra is a direct sum of simple Lie algebras. It suffices of course to treat the case when G is simple (by considering projections of Γ to the simple components of G). There are two cases which can occur:

(A) G is noncompact.

(B) G is compact.

(A) First, let's consider the noncompact case. There is a Riemannian manifold X, called *symmetric space*, associated with G on which G acts isometrically and transitively: $X = G/K$, where K is a maximal compact subgroup of G. The most important feature of X is that X has nonpositive sectional curvature and moreover, the sectional curvature is negative in certain directions. Thus one can use X as a replacement of the hyperbolic plane as we have done it in the case of $SL(2, \mathbb{R})$. There is a classification of isometries of X similar to the classification of isometries of $\mathbb{H}^2$: There are hyperbolic, parabolic and elliptic isometries. The elliptic ones fix points in X, hyperbolic isometries act as translations along certain geodesics in X. The fact that G is the Zariski closure of Γ then implies that Γ contains hyperbolic isometries. Then one can run a version of Ping-Pong lemma as we did in the case of $\mathbb{H}^2$ to show that Γ contains $\mathbb{F}_2$.

(B) The noncompact case is much more complicated. Let $\gamma_1, ..., \gamma_m$ denote generators of Γ and consider the field F in $\mathbb{R}$ generated by the matrix entries of the generators. If the field F happens to be a transcendental extension of $\mathbb{Q}$ one can show that there are homomorphisms $\phi_j : \Gamma \to G$ which converge (on each generator) to the identity embedding so that $\phi_j(\Gamma)$ have the property: The fields F_j associated with $\phi_j(\Gamma)$ as above are algebraic extensions of $\mathbb{Q}$. The reason for that is that we can assume that G is defined over $\mathbb{Q}$ (i.e. is given by equations with rational coefficients), thus the variety $Hom(\Gamma, G)$ is defined over $\mathbb{Q}$ as well; therefore algebraic points are dense in this variety. Because Γ was Zariski dense in G, there exists j such that $\phi_j(\Gamma)$ is Zariski dense as well and we are reduced to the case where the field F is contained in $\bar{\mathbb{Q}}$. Let $G(F)$ denote the group of F-points in G (i.e. points whose coordinates belong to F). Consider the action of the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the field F. Every such $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ will induce (a discontinuous!) automorphism σ of the complexification $G(\mathbb{C})$ of the group G, and therefore it will send the groups $\Gamma \subset G(F)$ to $\sigma(\Gamma) \subset G(\sigma(F)) \subset G(\mathbb{C})$. The homomorphism $\sigma : \Gamma \to \Gamma' := \sigma(\Gamma)$ is $1 - 1$ and therefore, if for some σ the group $G(\sigma(F))$ happens to be a non-relatively compact subgroup of $G(\mathbb{C})$ we are back to the noncompact case (A).

However it could happen that for each σ the group $G(\sigma(F))$ is relatively compact and thus we seemingly have gained nothing. There is a remarkable construction which saves the proof.

Adeles. (See [39, Chapter 6].) The ring of adeles was introduced by A. Weil in 1936. For the field F consider various norms $|\cdot| : F \to \mathbb{R}_+$. A norm is called *nonarchimedean* if instead of the usual triangle inequality one has:

$$|a+b| \leq \max(|a|, |b|).$$

For each norm ν we define F_ν to be the completion of F with respect to this norm. For each nonarchimedean norm ν the *ring of integers* $O_\nu := \{x : |x|_\nu \leq 1\}$ is an open subset of F_ν: If $|x|_\nu = 1$, $|y|_\nu < 1/2$, then for $z = x+y$ we have: $|z|_\nu \leq \max(1, |y|_\nu) = 1$. Therefore, if z belongs to a ball of radius $1/2$ centered at x, then $z \in O_\nu$.

Example 88. (A). **Archimedean norms.** Let $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, then the embedding $\sigma : F \to \sigma(F) \subset \mathbb{C}$ defines a norm ν on F by restriction of the norm (the usual absolute value) from $\mathbb{C}$ to $\sigma(F)$. Then the completion F_ν is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. Such norms (and completions) are *archimedean* and each archimedean norm of F appears in this way.

(B). **Nonarchimedean norms.** Let $F = \mathbb{Q}$, pick a prime number $p \in \mathbb{N}$. For each number $x = q/p^n \in \mathbb{Q}$ (where both numerator and denominator of q are not divisible by p) let $\nu_p(x) := p^n$. One can check that ν is a nonarchimedean norm and the completion of $\mathbb{Q}$ with respect to this norm is the field of p-adic numbers.

Let $Nor(F)$ denote the set of all norms on F which restrict to either standard or one of the p-adic norms on $\mathbb{Q} \subset F$. Note that for each $x \in \mathbb{Q}$, $x \in O_p$ (i.e. p-adic norm of x is ≤ 1) for all but finitely many p's, since x has only finitely many primes in its denominator. The same is true for elements of F: For all but finitely many $\nu \in Nor(F)$, $\nu(x) \leq 1$.

Product formula: For each $x \in \mathbb{Q} \setminus \{0\}$

$$\prod_{\nu \in Nor(\mathbb{Q})} \nu(x) = 1.$$

Indeed, if $x = p$ is prime then $|p| = p$ for the archimedean norm, $\nu(p) = 1$ if $\nu \neq \nu_p$ is a nonarchimedean norm and $\nu_p(p) = 1/p$. Thus the product formula holds for prime numbers x. Since norms are multiplicative functions from $\mathbb{Q}^*$ to $\mathbb{R}_+$, the product formula holds for arbitrary $x \neq 0$. A similar product formula is true for an arbitrary algebraic number field F:

$$\prod_{\nu \in Nor(F)} (\nu(x))^{N_\nu} = 1,$$

where $N_\nu = [F_\nu : \mathbb{Q}_\nu]$, see [39, Chapter 6].

Definition 89. The ring of *adeles* is the *restricted product*

$$\mathbb{A}(F) := \prod_{\nu \in Nor(F)} F_\nu,$$

i.e. the subset of the direct product which consists of points whose projection to F_ν belongs to O_ν for all but finitely many ν's.

We topologize $\mathbb{A}(F)$ via the product topology. For instance, if $F = \mathbb{Q}$ then $\mathbb{A}(\mathbb{Q})$ is the restricted product

$$\mathbb{R} \times \prod_{p \text{ is prime}} \mathbb{Q}_p.$$

Now a miracle happens:

Theorem 90. *(See [39, Chapter 6, Theorem 1].) The image of the diagonal embedding $F \hookrightarrow \mathbb{A}(F)$ is a discrete subset in $\mathbb{A}(F)$.*

Proof. It suffices to verify that 0 is an isolated point. Take the archimedean norms $\nu_1, ..., \nu_m$ (there are only finitely many of them) and consider the open subset

$$U = \prod_{i=1}^{m} \{x \in F_{\nu_i} : \nu_i(x) < 1/2\} \times \prod_{\mu \in Nor(F) \setminus \{\nu_1, \dots, \nu_m\}} O_\mu$$

of $\mathbb{A}(F)$. Then for each $(x_\nu) \in U$,

$$\prod_{\nu \in Nor(F)} \nu(x_\nu) < 1/2 < 1.$$

Hence, by the product formula, the intersection of U with the image of F in $\mathbb{A}(F)$ consists only of $\{0\}$. ☐

Thus the embedding $F \hookrightarrow \mathbb{A}(F)$ induces a discrete embedding

$$\Gamma \subset G(F) \hookrightarrow G(\mathbb{A}(F)).$$

For each norm $\nu \in Nor(F)$ we have the projection $p_\nu : \Gamma \to G(F_\nu)$. If the image $p_\nu(\Gamma)$ is relatively compact for each ν then Γ is a discrete compact subset of $G(\mathbb{A}(F))$, which implies that Γ is finite, a contradiction! Thus there exists a norm $\nu \in Nor(F)$ such that the image of Γ in $G(F_\nu)$ is not relatively compact. If ν happens to be archimedean we are done as before. The more interesting case occurs if ν is nonarchimedean. Then one can define a metric space X_ν on which the group $G(F_\nu)$ acts isometrically, faithfully and cocompactly (although the quotient is not a point but a Euclidean simplex). The space X_ν is called a *Euclidean building*, it is a nonarchimedean analogue of the symmetric space. It has nonpositive curvature in the sense that the geodesic triangles in X_ν are "thinner" than geodesic triangles in the Euclidean plane. The space X_ν is covered by isometrically embedded copies of the Euclidean space E^r, called *apartments*, so that each pair of points in X_ν belongs to an apartment. The number r is called *rank* of the space X_ν.

Example 91. If $r = 1$ then X_ν is a simplicial metric tree where each edge has unit length.

We note that the homomorphism $\Gamma \to G(F_\nu) \to Aut(X_\nu)$ is an embedding. The isometries of X_ν admit a classification similar to the isometries of $\mathbb{H}^2$: Each isometry is either hyperbolic (i.e. a translation along a geodesic contained in one of the apartments) or elliptic, i.e. fixes a point in X_ν. The group Γ is Zariski dense in $G(F_\nu)$, therefore it contains hyperbolic isometries. This allows one to run an analogue of Ping-Pong type arguments in X_ν and show that Γ contains $\mathbb{F}_2$. $\square$

5 Growth of groups and Gromov's theorem

Let X be a metric space of bounded geometry and $x \in X$ is a base-point. We define the *growth function*

$$\beta_{X,x}(R) := |B(x, R)|,$$

the cardinality of R-ball centered at x. We introduce the following *asymptotic inequality* between functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\beta \prec \alpha,$$

if there exist constants C_1, C_2 such that $\beta(R) \le C_1\alpha(C_2R)$ for sufficiently large R. We say that two functions are equivalent, $\alpha \sim \beta$, if

$$\alpha \prec \beta \quad \text{and} \quad \beta \prec \alpha.$$

Lemma 92. *(Equivalence class of growth is QI invariant.) Suppose that $f : (X, x) \to (Y, y)$ is a quasi-isometry. Then $\beta_{X,x} \sim \beta_{Y,y}$.*

Proof. Let $\bar{f}$ be a coarse inverse to f, assume that $f, \bar{f}$ are L-Lipschitz. Then both f, $\bar{f}$ have multiplicity $\leq m$ (since X and Y have bounded geometry). Then

$$f(B(x,R)) \subset B(y, LR).$$

It follows that $|B(x,R)| \leq m|B(y,LR)|$ and $|B(y,R)| \leq m|B(x,LR)|$. $\square$

Corollary 93. *$\beta_{X,x} \sim \beta_{X,x'}$ for all $x, x' \in X$.*

Henceforth we will suppress the choice of the base-point in the notation for the growth function.

Definition 94. X has polynomial growth if $\beta_X(R) \prec R^d$ for some d. X has exponential growth if $e^R \prec \beta_X(R)$. X has subexponential growth if for each $c > 0$, $\beta_X(R) \leq e^{cR}$ for all sufficiently large R.

Example 95. Show that for each (bounded geometry) space X, $\beta_X(R) \prec e^R$.

For a group G with finite generating set S we sometimes will use the notation $\beta_S(R)$ for $\beta_G(R)$, where S is used to metrize the group G. Since G acts transitively on itself, this definition does not depend on the choice of a base-point.

Example 96. Suppose that $G = \mathbb{F}_r$ is a free nonabelian group. Show that G has exponential growth.

Suppose that H is a subgroup of G. It is then clear that

$$\beta_H \prec \beta_G.$$

Note that if $\phi : G \to \mathbb{F}_r$ is an epimorphism, then its admits a left inverse $\iota : \mathbb{F}_r \to G$. Hence G contains $\mathbb{F}_r$ and if $r \geq 2$ it follows that G has exponential growth.

The main objective of this chapter is to prove

Theorem 97. *(Gromov, [27]) If G is a finitely generated group of polynomial growth then G is virtually nilpotent.*

We will also verify that all virtually nilpotent groups have polynomial growth.

Corollary 98. *Suppose that G is a finitely generated group which is quasi-isometric to a nilpotent group. Then G is virtually nilpotent.*

Proof. Follows directly from Gromov's theorem since polynomial growth is a QI invariant. $\square$

Remark 99. An alternative proof of the above corollary (which does not use Gromov's theorem) was recently given by Y. Shalom [51].

5.1 Nilpotent and solvable groups

Given a group G and a subgroup $S \subset G$ define $[G, S]$ as the subgroup generated by the commutators $[g, s], g \in G, s \in S$. Define the *lower central series* of G:

$$G = G_0 \supset [G, G] = G_1 \supset [G, G_1] = G_2 \supset [G, G_2] = G_3 ...$$

and the *derived series* of G:

$$G = G^0 \supset [G, G] = G^1 \supset [G^1, G^1] = G^2 \supset [G^2, G^2] = G^3 ...$$

The group G is called *nilpotent*, resp. *solvable*, if the lower central, resp. derived, series of G terminates at the trivial group. The group G is called s-step nilpotent if its lower central series is

$$G_0 \supset G_1 \supset ...G_{s-1} \supset 1,$$

where $G_{s-1} \neq 1$.

Given a nilpotent group G we note that all the subgroups G_i are finitely generated, their generators are the iterated commutators of the generators of G. We also have finitely generated abelian groups $A_i := G_i/G_{i+1}$. After passing to a finite index subgroup in G we can assume that each A_i is torsion-free. Let ψ be an automorphism of G, then it preserves the lower central series and induces automorphisms of the free abelian groups A_i. Each such automorphism ψ_i is given by a matrix with integer coefficients. After taking sufficiently high power of ψ we can assume that none of these matrices have a root of unity (different from 1) as an eigenvalue. If each ψ_i has only 1 as an eigenvalue then after "refining" the lower central series we can assume that each ψ_i is trivial. Then the extension $\tilde{G}$ of G by ψ is again a nilpotent group.

Theorem 100. *Suppose that $1 \to G \to \tilde{G} \to \mathbb{Z} \to 1$ is an extension of G by ψ and at least one eigenvalue of one of the ψ_i's is different from 1. Then $\tilde{G}$ has exponential growth.*

Proof. We begin with

Lemma 101. *Let A be a finitely generated free abelian group and $\alpha \in Aut(A)$. Then:*

If α has an eigenvalue ρ such that $|\rho| \geq 2$ then there exists $a \in A$ such that

$$\epsilon_0 a + \epsilon_1 \alpha(a) + ... + \epsilon_m \alpha^m(a) + ... \in A$$

(where $\epsilon_i \in \{0, 1\}$ and $\epsilon_i = 0$ for all but finitely many i's) are distinct for different choices of the sequences (ϵ_i).

Proof. The transpose matrix α^T also has ρ as its eigenvalue. Hence there exists a nonzero linear function $\beta : A \to \mathbb{C}$ such that $\beta \circ \alpha = \rho\beta$. Pick any $a \in A \setminus Ker(\beta)$. Then

$$\beta(\sum_{i=0}^{\infty} \epsilon_i \alpha^i(a)) = (\sum_{i=0}^{\infty} \epsilon_i \rho^i(a))\beta(a).$$

Suppose that

$$\sum_{i=0}^{\infty} \epsilon_i \alpha^i(a) = \sum_{i=0}^{\infty} \delta_i \alpha^i(a).$$

Then

$$\sum_{i=0}^{\infty} \eta_i \alpha^i(a) = 0,$$

where $|\eta_i| \leq 1$ for each i. Let N be the maximal value of i for which $\eta_i \neq 0$. Then

$$|\rho|^N \leq \sum_{i=0}^{N-1} |\rho|^i = \frac{|\rho|^N - 1}{|\rho| - 1} \leq |\rho|^N - 1.$$

Contradiction. $\square$

We now can prove theorem 100. Suppose there is i such that α_i has an eigenvalue ρ which is not a root of unity. After taking appropriate iteration of ψ and possibly replacing ψ with ψ^{-1} we can assume that such that $|\rho| \geq 2$. Let $x \in G_i$ be an element which projects to $a \in A_i$ under the homomorphism $G_i \to G_i/G_{i+1}$. Let $z \in \tilde{G}$ denote the generator corresponding to the automorphism ψ. Define elements

$$x^{\epsilon_0}(zx^{\epsilon_1}z^{-1})...(z^m x^{\epsilon_m} z^{-m}) \in G_i, \epsilon_i \in \{0, 1\}.$$

After canceling out z's we get:

$$x^{\epsilon_0} z x^{\epsilon_1} z x^{\epsilon_2} z ... z x^{\epsilon_m} z^{-m}$$

The norm of each of these elements in $\tilde{G}$ is at most $3(m + 1)$. These elements are distinct for different choices of (ϵ_i)'s, since their projections to A_i are distinct according to the above lemma. Thus we get 2^m distinct elements of $\tilde{G}$ whose word norm is at most $3(m + 1)$. This implies that $\tilde{G}$ has exponential growth. $\square$

Proposition 102. *Suppose that G is a group of subexponential growth, which fits into a short exact sequence*

$$1 \to K \to G \xrightarrow{\psi} \mathbb{Z} \to 1.$$

Then K is finitely generated. Moreover, if $\beta_G(R) \prec R^d$ then $\beta_K(R) \prec R^{d-1}$.

Proof. Let $\gamma \in G$ be an element which projects to the generator 1 of $\mathbb{Z}$. Let $\{f_1, ..., f_k\}$ denote a set of generators of G. Then for each i there exists $s_i \in \mathbb{Z}$ such that $\psi(f_i\gamma^{s_i}) = 0 \in \mathbb{Z}$. Define elements $g_i := f_i\gamma^{s_i}$, $i = 1, ..., k$. Clearly, the set $\{g_1, ..., g_k, \gamma\}$ generates G. Without loss of generality we may assume that each generator g_i is nontrivial. Define

$$S := \{\gamma_{m,i} := \gamma^m g_i \gamma^{-m}, m \in \mathbb{Z}, i = 1, ..., k\}.$$

Then the (infinite) set S generates K. Given i consider products of the form:

$$\gamma_{0,i}^{\epsilon_1} ... \gamma_{m,i}^{\epsilon_m}, \quad \epsilon_i \in \{0, 1\}, m \geq 0.$$

We have 2^{m+1} words like this, each of length $\leq 2m$. Hence subexponential growth of G implies that for a certain $m = m(i)$, two of these words are equal:

$$\gamma_{0,i}^{\epsilon_1} ... \gamma_{m,i}^{\epsilon_m} = \gamma_{0,i}^{\delta_1} ... \gamma_{m,i}^{\delta_m},$$

$\epsilon_m \neq \delta_m$. It follows that

$$\gamma_{m,i} = w(\gamma_{0,i}, ..., \gamma_{m-1,i}) \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle,$$

where w is a certain word in the generators $\gamma_{0,i}, ..., \gamma_{m-1,i}$. Consider

$$\gamma_{m+1,i} = \gamma\gamma_{m,i}\gamma^{-1} = \gamma w(\gamma_{0,i}, ..., \gamma_{m-1,i})\gamma^{-1} = w'(\gamma_{1,i}, ..., \gamma_{m,i}).$$

Here w' is the word in the generators $\gamma_{1,i}, ..., \gamma_{m,i}$ which is obtained from w by inserting the products $\gamma^{-1} \cdot \gamma$ between each pair of letters in the word w and then using the fact that

$$\gamma_{j+1,i} = \gamma\gamma_{j,i}\gamma^{-1}, j = 0, ..., m-1.$$

However $w' \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle$, since

$$\gamma_{m,i} \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle.$$

Thus $\gamma_{m+1,i} \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle$ as well. We continue by induction: It follows that $\gamma_{n,i} \in \langle \gamma_{0,i}, ..., \gamma_{m-1,i} \rangle$ for each $n \geq 0$. The same argument works for the negative values of m and therefore there exists $M(i)$ so that each $\gamma_{j,i}$ is contained in the subgroup of K generated by

$$\{\gamma_{l,i}, |l| \leq M(i)\}.$$

Hence the subgroup K is generated by the finite set

$$\{\gamma_{l,i}, |l| \leq M(i), i = 1, ..., k\}.$$

This proves the first assertion of the Proposition.

Now let us prove the second assertion which estimates the growth function of K. Take a finite generating set Y of the subgroup K and set $X := Y \cup \{\gamma\}$, where γ is as above. Then X is a generating set of G. Given $n \in \mathbb{N}$ let $N := \beta_Y(n)$, where β_Y is the growth function of K with respect to the generating set Y. Thus there exists a subset

$$H := \{h_1, ..., h_N\} \subset K$$

where $\|h_i\|_Y \leq n$ and $h_i \neq h_j$ for all $i \neq j$. Then we get a set T of $(2n+1) \cdot N$ pairwise distinct elements

$$h_i\gamma^j, \quad -n \leq j \leq n, \quad i = 1, ..., N.$$

It is clear that $\|h_i\gamma^j\|_X \leq 2n$ for each $h_j\gamma^j \in T$. Therefore

$$n\beta_Y(n) \leq (2n+1)\beta_Y(n) = (2n+1)N \leq \beta_X(2n) \leq C(2n)^d = 2^d C \cdot n^d$$

It follows that

$$\beta_Y(n) \leq 2^d C \cdot n^{d-1} \prec n^{d-1}. \quad \square$$

5.2 Growth of nilpotent groups

Consider an s-step nilpotent group G with the lower central series

$$G_0 \supset G_1 \supset ...G_{s-1} \supset 1,$$

and the abelian quotients $A_i = G_i/G_{i+1}$. Let d_i denote the rank of A_i (or, rather, the rank of its free part). Define

$$d(G) := \sum_{i=0}^{s-1} (i+1)d_i.$$

Theorem 103. *(Bass, [2])* $\beta_G(R) \sim R^{d(G)}$.

Example 104. Prove Bass' theorem for abelian groups.

Our goal is to prove only that G has polynomial growth without getting a sharp estimate.

For the proof we introduce the notion of *distortion* for subgroups which is another useful concept of the geometric group theory. Let H be a finitely generated subgroup of a finitely generated group G, let d_H, d_G denote the respective word metrics on H and G, let $B_G(e, r)$ denote r-ball centered at the origin in the group G.

Definition 105. Define the *distortion function* $\delta(R) = \delta(H : G, R)$ as

$$\delta(R) := \max\{d_H(e, h) : h \in B_G(e, R)\}.$$

The subgroup H is called *undistorted* (in G) if $\delta(R) \sim R$.

Example 106. Show that H is undistorted iff the embedding $\iota : H \to G$ is a quasi-isometric embedding.

In general, distortion functions for subgroups can be as bad as one can imagine, for instance, nonrecursive.

Example 107. Let $G := \langle a, b : aba^{-1} = b^p \rangle$, $p \geq 2$. Then the subgroup $H = \langle b \rangle$ is exponentially distorted in G.

Proof. To establish the lower exponential bound note that:

$$g_n := a^n b a^{-n} = b^{p^n},$$

hence $d_G(1, g_n) = 2n + 1$, $d_H(1, g_n) = p^n$, hence

$$\delta(R) \geq p^{[(R-1)/2]}.$$

It will leave the upper exponential bound as a exercise (compare the proof of Theorem 109). ☐

Recall that each subgroup of a finitely generated nilpotent group is finitely generated itself.

The following theorem was originally proven by M. Gromov in [29] (see also [59]); later on, an explicit computation of the degrees of distortion was established by D. Osin in [45]:

Theorem 108. *Let G be a finitely generated nilpotent group, then every subgroup $H \subset G$ has polynomial distortion.*

I will prove only a special case of this result which will suffice for our purposes:

Theorem 109. *Let G be a finitely generated nilpotent group, then it commutator subgroup $G_1 := [G, G] \subset G_0 := G$ has at most polynomial distortion.*

Proof. As the equivalence class of a distortion function is a commensurability invariant, it suffices to consider the case when $A = G/G_1$ is free abelian. Suppose that G is n-step nilpotent. We choose a generating set T of G as follows. Set $T := T_0 \sqcup T_1 \sqcup ... \sqcup T_n$, where T_0 projects to the set of free generators of A, $T_i \subset G_i$. Let $x_i, i = 1, ..., p$, denote the elements of T_0. We assume that each T_{i+1} contains all the commutators $[y_k^{\pm 1}, x_j^{\pm 1}]$, where $y_j \in T_{i-1}$, $i = 1, ..., n$.

For each word w in the generating set T define its i-length $\ell_i(w)$ to be the total number of the letters $y_j^{\pm 1} \in T_i$ which appear in w. Clearly,

$$\|w\| = \sum_{i=0}^{n} \ell_i(w).$$

Given an appearance of the letter $a = x_k^{\pm 1}$ in the word w let's "move" this letter through w so that the resulting word w' equals to w as an element of G and that the letter a appears as the first letter in the new word w'.

This involves at most $\|w\|$ "crossings" of the letters in w. Each "crossing" results in introducing a commutator of the corresponding generators:

$$y_j a \to a y_j [y_j^{-1}, a^{-1}].$$

Therefore,

$$\ell_{i+1}(w') \le \ell_{i+1}(w) + \ell_i(w). \tag{110}$$

We will apply this procedure inductively to each letter $a = x_k^{\pm 1}$ in the word w, so that the new word w_* starts with a power of x_1, then comes the power of x_2, etc, by moving first all appearances of x_1 to the left, then of x_2 to the left, etc. In other words

$$w_* = x_1^{\alpha_1} x_2^{\alpha_2} ... x_p^{\alpha_p} \cdot u,$$

where u is a word in the generators $S = T_1 \cup .. \cup T_n$ of the group G_1. We have to estimate the length of the word u. We have a sequence of words

$$w_0 = w, w_1, ..., w_m = w_*,$$

where each w_i is the result of moving a letter in w_{i-1} to the left and $m \le \ell_0(w)$. Clearly, $\ell_0(w_j) = \ell_0(w)$ for each j. By applying the inequality (110) inductively we obtain

$$\ell_{i+1}(w_j) \le \ell_{i+1}(w) + j\ell_i(w_{j-1})$$

and hence:

$$\ell_{i+1}(w_m) \leq \ell_{i+1}(w) + m\ell_i(w) + m(m-1)\ell_{i-1}(w) + ... + \frac{m!}{(m-i-1)!}\ell_0(w)$$

$$\leq m^{i+1}\sum_{j=0}^{i+1}\ell_j(w) \leq m^{i+1}\|w\|.$$

Therefore, $\ell_i(w_*) \leq \|w\|^{i+1}$ for each i. By adding up the results we get:

$$\|u\| = \sum_{i=1}^{n}\ell_i(w_*) \leq \sum_{i=1}^{n}\|w\|^{i+1} \leq n\|w\|^{n+1}.$$

Suppose now that w represents an element g of $H = G_1$. Then, since $x_1, ..., x_p$ project to a free generating set of A, it follows that $\ell_0(w_*) = 0$ and therefore $w_* = u$ is a word in the generators of the group $H = G_1$. Thus for each $g \in H$ we obtain:

$$d_H(1, g) \leq n d_G(1, g)^{n+1}.$$

Hence the distortion function δ of H in G satisfies $\delta(R) \prec R^{n+1}$. $\square$

Theorem 111. *Each nilpotent group has at most polynomial growth.*

Proof. The proof is by induction on the number of steps in the nilpotent group. The assertion is clear is G is 1-step nilpotent (i.e. abelian). Suppose that each $s-1$-step nilpotent group has at most polynomial growth. Consider s-step nilpotent group G:

$$G = G_0 \supset G_1 \supset ...G_{s-1} \supset 1.$$

By the induction hypothesis, G_1 has growth $\prec R^d$ and, according to Theorem 109, the distortion of G_1 in G is at most R^D. Let r denote the rank of the abelinization of G.

Consider an element $\gamma \in B_G(e, R)$, then γ can be written down as a product $w_0 w_1$ where w_0 is a word on T_0 of the form:

$$x_1^{k_1}...x_n^{k_n},$$

and w_1 is a word on T_1. Then $\|w_0\| \leq R$, and $\|w_1\| \leq \|w_0\| + \|\gamma\| \leq 2R$. The number of the words w_0 of length $\leq R$ is $\prec R^r$. Since G_1 has distortion $\prec R^D$ in G, the length of the word w_1 on the generators T_1 is $\prec (2R)^D$. Since $\beta_{G_1} \prec R^d$ we conclude that

$$\beta_G(R) \prec R^r \cdot (2R)^{dD} \sim R^{dD+r}. \quad \square$$

Corollary 112. *A solvable group G has polynomial growth iff G is virtually nilpotent.*

Proof. It remains to show that if G is a solvable and has polynomial growth then G is virtually nilpotent group. By considering the derived series of G we get the short exact sequence

$$1 \to K \to G \to \mathbb{Z}.$$

Suppose that G has polynomial growth $\prec R^d$, then K is finitely generated, solvable and has growth $\prec R^{d-1}$. By induction, we can assume that K is virtually nilpotent. Then Theorem 100 implies that G is also virtually nilpotent. □

Corollary 113. *Suppose that G is a finitely generated linear group. Then G either has polynomial or exponential growth.*

Proof. By Tits alternative either G contains a nonabelian free subgroup (and hence G has exponential growth) or G is virtually solvable. For virtually solvable groups the assertion follows from Corollary 112. □

R. Grigorchuk [26] constructed finitely generated groups of *intermediate growth*, i.e. their growth is superpolynomial but subexponential. Existence of finitely-presented groups of this type is unknown.

5.3 Elements of the nonstandard analysis

Our discussion here follows [25], [58].

Let I be a countable set. Recall that an *ultrafilter* on I is a finitely additive measure with values in the set $\{0, 1\}$ defined on the power set 2^I. We will assume that ω is nonprincipal. Given a set S we have its *ultrapower*

$$S^* := S^I/\omega,$$

which is a special case of the ultraproduct. Note that if G is a group (ring, field, etc.) then G^* has a natural group (ring, field, etc.) structure. If S is totally ordered then S^* is totally ordered as well: $[f] \le [g]$ (for $f, g \in S^I$) iff $f(i) \le g(i)$ for ω-all $i \in I$.

For subsets $P \subset S$ we have the canonical embedding $P \hookrightarrow \hat{P} \subset S^*$ given by sending $x \in P$ to the constant function $f(i) = x$.

Thus we define the ordered semigroup $\mathbb{N}^*$ (the nonstandard natural numbers) and the ordered field $\mathbb{R}^*$ (the nonstandard real numbers). An element $R \in \mathbb{R}^*$ is called *infinitely large* if given any $r \in \mathbb{R} \subset \mathbb{R}^*$, one has $R \ge r$. Note that given any $R \in \mathbb{R}^*$ there exists $n \in \mathbb{N}^*$ such that $n > R$.

Definition 114. A subset $W \subset S_1^* \times ... \times S_n^*$ is called *internal* if "membership in W can be determined by coordinate-wise computation", i.e. if for each $i \in I$ there is a subset $W_i \subset S_1 \times ... \times S_n$ such that for $f_1 \in S_1^I, ..., f_n \in S_n^I$

$$([f_1], ..., [f_n]) \in W \iff (f_1(i), ..., f_n(i)) \in W_i \text{ for } \omega - \text{all } i \in I.$$

The sets W_i are called *coordinates* of W.

Using this definition we can also define *internal functions* $S_1^* \to S_2^*$ as functions whose graphs are internal subsets of $S_1^* \times S_2^*$. Clearly the image of an internal function is an internal subset of S_2^*.

Lemma 115. *Suppose that $A \subset S$ is infinite subset. Then $\hat{A} \subset S^*$ is not internal.*

Proof. Suppose that $A_i, i \in I$, are coordinates of $\hat{A}$. Let $a_1, a_2, ...$ be an infinite sequence of distinct elements of A. Define the following function $f \in S^I$:

Case 1. $f(n) = a_j$, where $j = \max\{j' : a_{j'} \in A_n\}$ if the maximum exists,

Case 2. $f(n) = a_{n+j}$, where $j = \min\{j' : a_{n+j'} \in A_n\}$ if the maximum above does not exist.

Note that for each $n \in I$, $f(n) \in A_n$, therefore $[f] \in \hat{A}$. Since $\hat{A}$ consists of (almost) constant functions, there exists $m \in \mathbb{N}$ such that $f(n) = a_m$ for ω-all $n \in I$.

It follows that the Case 2 of the definition of f cannot occur for ω-all $n \in I$. Thus for almost all $n \in I$ the function f is defined as in Case 1. It follows that for almost all $n \in I$, $a_{m+1} \notin A_n$. Thus $a_{m+1} \notin A$, which is a contradiction. ☐

Corollary 116. *$\mathbb{N}$ is not an internal subset of $\mathbb{N}^*$.*

Suppose that (X, d) is a metric space. Then X^* has a natural structure of $\mathbb{R}^*$-metric space where the "distance function" d takes values in $\mathbb{R}_+^*$:

$$d([f], [g]) := [i \mapsto d(f(i), g(i))].$$

We will regard d^* as a generalized metric, so we will talk about metric balls, etc. Note that the "metric balls" in X^* are internal subsets.

A bit of logic. Let Φ be a statement about elements and subsets of S. The *non-standard interpretation* Φ^* of Φ is a statement obtained from Φ by replacing:

1. Each entry of the form "$x \in S$" with "$x \in S^*$".
2. Each entry of the form "$A \subset S$" with "A an internal subset of S^*".

Theorem 117. *(Los) A statement Φ about S is true iff its nonstandard interpretation Φ^* about S^* is true.*

As a corollary we get:

Corollary 118. *1. (Completeness axiom) Each nonempty bounded from about internal subset $A \subset \mathbb{R}^*$ has supremum. (Note that $\mathbb{R} \subset \mathbb{R}^*$ does not have supremum.)*

2. (Nonstandard induction principle.) Suppose that $S \subset \mathbb{N}^*$ is an internal subset such that $1 \in S$ and for each $n \in S$, one has $n+1 \in S$. Then $S = \mathbb{N}^*$. (Note that this fails for $S = \mathbb{N} \subset \mathbb{N}^*$.)

Example 119. 1. Give a direct proof of the completeness axiom for $\mathbb{R}^*$.

2. Use the completeness axiom to derive the nonstandard induction principle.

Suppose we are given $a_n \in \mathbb{R}^*$, where $n \in \mathbb{N}^*$. Using the nonstandard induction principle on can define the nonstandard products:

$$a_1 ... a_n, n \in \mathbb{N}^*,$$

as an internal function $f : \mathbb{N}^* \to \mathbb{R}^*$ given by $f(1) = a_1$, $f(n+1) = f(n)a_{n+1}$.

5.4 Regular growth theorem

A metric space X is called *doubling* if there exists a number N such that each R-ball in X is covered by N balls of radius $R/2$.

Exercise 120. Show that doubling implies polynomial growth for spaces of bounded geometry.

Although there are spaces of polynomial growth which are not doubling, the Regular Growth Theorem below shows that groups of polynomial growth exhibit doubling-like behaviour.

Our discussion here follows [58].

Theorem 121 (Regular growth theorem). *Suppose that G is a finitely generated group such that $\beta_G(R) \prec R^d$. Then there exists an infinitely large $\rho \in \mathbb{R}^*$ such that for all $i \in \mathbb{N} \setminus \{1\}$ the following assertion $P(\rho, i)$ holds:*

If $x_1, ..., x_t \in B(e, \rho/2) \subset G^*$ and the balls $B(x_j, \rho/i)$ are pairwise disjoint ($j = 1, ..., t$) then $t \leq i^{d+1}$.

Here e is the identity in G^*.

Proof. Start with an arbitrary infinitely large $R \in \mathbb{R}^*$ (for instance, represented by the sequence n, $n \in \mathbb{N} = I$). I claim that the number ρ can be found within the interval $[\log R, R]$ (here logarithm is taken with the base 2). Suppose to the contrary, that for each $\rho \in [\log R, R]$ there exists $i \in \mathbb{N} \setminus \{1\}$ such that $P(\rho, i)$ fails. Observe that the assertion $P(\rho, i)$ also makes sense when $i \in \mathbb{N}^*$. Then we define the function

$$\iota : [\log R, R] \to \mathbb{N}^*, \quad \iota(\rho) \text{ is the smallest } i \text{ for which } P(\rho, i) \text{ fails.}$$

Since i is less than any nonstandard natural number, it follows that the image of ι is contained in $\mathbb{N}$ (embedded in $\mathbb{N}^*$ diagonally). Since the nonstandard distance function is an internal function, the function ι is internal as well. Therefore, according to Lemma 115, the image of ι has to be finite. Thus there exists $K \in \mathbb{N}$ such that

$$\iota(\rho) \in [2, K], \quad \forall \rho \in [\log R, R].$$

We now define (using the nonstandard induction) the following elements of G^*:

1. $x_1(1), ..., x_{t_1}(1) \in B(e, R/2)$ such that $t_1 = i_1^{d+1}$, (for $i_1 = \iota(R)$) and the balls $B(x_j(1), R/i_1)$, contained in $B(e, R)$ are pairwise disjoint.

2. Each nonstandard ball $B(x_j(1), R/i_1)$ is isometric to $B(e, R/i_1)$. Therefore failure of $P(R/i_1, i_2)$ (where $i_2 = \iota(R/i_1)$) implies that in each ball $B(x_j(1), R/(2i_1))$ we can find points

$$x_1(2), ..., x_{t_2}(2), t_2 = i_2^{d+1},$$

so that the balls $B(x_j(2), R/(i_1 i_2)) \subset B(x_j(1), R/i_1)$ are pairwise disjoint.

We continue via the nonstandard induction. Given $u \in \mathbb{N}^*$ such that the points $x_1(u), ..., x_{t_u}(u)$ are constructed, we construct the next generation of points

$$x_1(u+1), ..., x_{t_{u+1}}(u+1)$$

within each ball $B(x_j(u), R/(2i_1...i_u))$ so that the balls

$$B(x_j(u+1), R/(i_1...i_{u+1}))$$

are pairwise disjoint and $t_{u+1} = i_u^{d+1}$. Here and below the product $i_1...i_{u+1}$ is understood via the nonstandard induction as in the end of the previous section.

Note that

$$B(x_j(u+1), R/(i_1...i_{u+1})) \subset B(x_j(u), R/(i_1...i_u)).$$

In particular,

$$B(x_j(u+1), R/(i_1...i_{u+1})) \cap B(x_k(u+1), R/(i_1...i_{u+1})) = \emptyset$$

when $j \neq k$.

Remark 122. Thus in the formulation of the Assertion $P(\rho, i)$ it is important to consider points in the ball $B(e, \rho/2)$ rather than in $B(e, \rho)$.

This induction process continues as long as $R/(i_1...i_{u+1}) \geq \log R$. Recall that $i_j \geq 2$, hence

$$R/(i_1...i_u) \leq 2^{-u}R.$$

Therefore, if $u > \log R - \log\log R$ then

$$R/(i_1...i_u) < \log R.$$

Thus there exists $u \in \mathbb{N}^*$ such that

$$R/(i_1...i_u) \geq \log R, \text{ but } R/(i_1...i_{u+1}) < \log R.$$

Let's count the "number" (nonstandard of course!) of points $x_i(k)$ we have constructed between the step 1 of induction and the u-th step of induction:

We get $i_1^{d+1} i_2^{d+1} ... i_u^{d+1}$ points; since

$$R/(K \log R) \leq R/(i_{u+1} \log R) < i_1...i_u,$$

we get:

$$(R/(K \log R))^{d+1} \leq (i_1 i_2 ... i_u)^{d+1}.$$

What does this inequality actually mean? Recall that R and u are represented by sequences of real and natural numbers R_n, u_n respectively. The above inequality thus implies that for ω-all $n \in \mathbb{N}$, one has:

$$\left(\frac{R_n}{K \log R_n}\right)^{d+1} \leq |B(e, R_n)|.$$

Since $|B(e, R)| \leq CR^d$, we get:

$$R_n \leq Const(\log(R_n))^{d+1},$$

for ω-all $n \in \mathbb{N}$. If $R_n = 2^{\lambda_n}$, we obtain

$$2^{\lambda_n/(d+1)} \leq Const\lambda_n,$$

where ω-$\lim \lambda_n = \infty$. Contradiction. $\square$

5.5 Topological group actions

The proof of Gromov's polynomial growth theorem relies heavily upon the work of Montgomery and Zippin on Hilbert's 5-th problem (characterization of Lie group as topological groups). Therefore in this section we collect several *elementary* facts in point-set topology and review, *highly nontrivial* results of Montgomery and Zippin.

Recall that a *topological group* is a group G which is given topology so that the groop operations (multiplication and inversion) are continuous. A continuous group action of a topological group G on a topological space X is a continuous map

$$\mu : G \times X \to X$$

such that $\mu(e, x) = x$ for each $x \in X$ and for each $g, h \in G$

$$\mu(gh, x) = \mu(g) \circ \mu(h)(x).$$

In particular, for each $g \in G$ the map $x \mapsto \mu(g)(x)$ is a homeomorphism $X \to X$. Thus each action μ defines a homomorphism $G \to Homeo(X)$. The action μ is called *effective* if this homomorphism is injective.

Throughout this section we will consider only *metrizable* topological spaces X. We will topologize the group of homeomorphisms $Homeo(X)$ via the compact-open topology, so that we obtain a continuous action $Homeo(X) \times X \to X$.

Lemma 123. *Fix some $r > 0$ and let Y be a geodesic metric space where each metric r-ball is compact. Then Y is a proper metric space.*

Proof. Pick a point $o \in Y$ and a number ϵ in the open interval $(0, r)$. We will prove inductively that for each $n \in \mathbb{N}$ the ball

$$B(o, n(r - \epsilon))$$

is compact. The assertion is clear for $n = 1$ since $B(o, r - \epsilon)$ is a closed subset of the compact $B(o, r)$. Suppose the assertion holds for some $n \in \mathbb{N}$. Then the metric sphere $S(o, n(r - \epsilon)) = \{y \in Y : d(y, o) = n(r - \epsilon)\}$ is compact. Let $\{x_j\}$ be a finite $\epsilon/2$-net in $S(o, n(r - \epsilon))$. Since Y is a geodesic metric space, for each point $y \in Y$ such that $d(y, o) = R > n(r - \epsilon)$, there exists a point $y' \in S_{n(r-\epsilon)}(o)$, which lies on a geodesic connecting o and y, such that $d(y', y) = R - n(r - \epsilon)$. Therefore, given each point $y \in B(o, (n + 1)(r - \epsilon))$, there exists a point x_j as above such that

$$d(y, x_j) \leq r - \epsilon + \frac{\epsilon}{2} \leq r.$$

Therefore $y \in B(x_j, r)$. Therefore the finite union of compact metric balls $B(x_j, r)$ cover $B(o, (n+1)(r-\epsilon))$. Thus $B(o, (n+1)(r-\epsilon))$ is compact. $\square$

Example 124. Construct an example of a *non-geodesic* metric space where the assertion of the above lemma fails.

Definition 125 (Property A, section 6.2 of [41]). Suppose that H is a separable, locally compact topological group. Then H is said to satisfy Property A if for each neighborhood V of e in H there exists a compact subgroups $K \subset H$ so that $K \subset V$ and H/K is a Lie group.

In other words, the group H can be *approximated by* the Lie groups H/K.

According to [41, Chapter IV], each separable locally compact group H contains an open and closed subgroup $\hat{H} \subset H$ such that $\hat{H}$ satisfies Property A.

Theorem 126 (Montgomery-Zippin, [41], Corollary on page 243, section 6.3). *Suppose that X is a topological space which is connected, locally connected, finite-dimensional and locally compact. Suppose that H is a separable locally compact group satisfying Property A, $H \times X \to X$ is a topological action which is effective and transitive. Then H is a Lie group.*

Suppose now that X is a metric space which is complete, proper, connected, locally connected. We give $Homeo(X)$ the compact-open topology.

Let $H \subset Homeo(X)$ be a closed subgroup for which there exists $L \in \mathbb{R}$ such that each $h \in H$ is L-Lipschitz. (For instance, $H = \mathrm{Isom}(X)$.) We assume that $H \curvearrowright X$ is transitive. Pick a point $x \in X$. It is clear that $H \times X \to X$ is a continuous effective action. It follows from the Arcela-Ascoli theorem that H is locally compact.

Theorem 127. *Under the above assumptions, the group H is a Lie group with finitely many connected components.*

Proof.

Lemma 128. *The group H is separable.*

Proof. Given $r \in \mathbb{R}_+$ consider the subset $H_r = \{h \in H : d(x, h(x)) \le r\}$. By Arcela-Ascoli theorem, each H_r is a compact set. Therefore

$$H = \bigcup_{r \in \mathbb{N}} H_r$$

is a countable union of compact subsets. Thus it suffices to prove separability of each H_r. Given $R \in \mathbb{R}_+$ define the map

$$\phi_R : H \to C_L(B(x,R),X)$$

given by the restriction $h \mapsto h|B(x,R)$. Here $C_L(B(x,R),X)$ is the space of L-Lipschitz maps from $B(x,R)$ to X. Observe that $C_L(B(x,R),X)$ is metrizable via

$$d(f,g) = \max_{y \in B(x,R)} d(f(y),g(y)).$$

Thus the image of H_r in each $C_L(B(x,R),X)$ is a compact metrizable space. Therefore $\phi_R(H_r)$ is separable. Indeed, for each $i \in \mathbb{N}$ take $\mathcal{E}_i \subset \phi_R(H_r)$ to be an $\frac{1}{i}$-net. The union

$$\bigcup_{i \in \mathbb{N}} \mathcal{E}_i$$

is a dense countable subset of $\phi_R(H_r)$. On the other hand, the group H (as a topological space) is homeomorphic to the *inverse limit*

$$\lim_{R \in \mathbb{N}} \phi_R(H),$$

i.e. the subset of the product $\prod_i \phi_i(H)$ (given the product topology) which consists of sequences (g_i) such that

$$\phi_j(g_i) = g_j, j \le i.$$

Let $E \subset \phi_i(H_r)$ be a dense countable subset. For each element $e_i \in E_i$ consider a sequence $(g_j) = \tilde{e}_i$ in the above inverse limit such that $g_i = e_i$. Let $\tilde{e}_i \in H$ be the element corresponding to this sequence (g_j). It is clear now that

$$\bigcup_{i \in \mathbb{N}} \{\tilde{e}_i \in H, e_i \in E_i\}$$

is a dense countable subset of H_r. ☐

Corollary 129. *Separability implies that for each open subgroup $U \subset H$, the quotient H/U is a countable set.*

Lemma 130. *The orbit $Y := \hat{H}x \subset X$ is open.*

Proof. If Y is not open then it has empty interior (since $\hat{H}$ acts transitively on Y). Since $\hat{H} \subset H$ is closed, the Arcela-Ascoli theorem implies that Y is closed as well.

Since $\hat{H}$ is open, by the above corollary, the coset $S := H/\hat{H}$ is countable. Choose representatives g_i of S. Then

$$\bigcup_i g_i Y = X.$$

Therefore the space X is a countable union of closed subsets with empty interior. However, by Baire's theorem, each first category subset in the locally compact metric space X has empty interior. Contradiction. $\square$

We now can conclude the proof of Theorem 127. Let $Z \subset Y$ be the connected component of Y containing x. Its stabilizer $F \subset \hat{H}$ again has the Property A. It is clear that F is an open subgroup of $\hat{H}$. Then the assumptions of Theorem 126 are satisfied by the action $F \curvearrowright Z$. Therefore F is a Lie group. However $F \subset H$ is an open subgroup; therefore the group H is a Lie group as well. Let K be the stabilizer of x in H. The subgroup K is a compact Lie group and therefore has only finitely many connected components. Since the action $H \curvearrowright X$ is transitive, X is homeomorphic to H/K. Connectedness of X now implies that H has only finitely many connected components. $\square$

We now verify that the isometry groups of asymptotic cones corresponding to groups of polynomial growth satisfy the assumptions of Theorem 127.

Proposition 131. *Let G be a group of growth $\prec R^d$. Suppose that $\rho = (\rho_n)$ is a sequence satisfying the assertion of the Regular Growth Theorem. Then the asymptotic cone X_ω constructed from the Cayley graph of G by rescaling via ρ_n^{-1}, is (a) a proper homogeneous metric space, (b) has the covering dimension $\leq d+1$.*

Proof. (a) Recall that X_ω is complete, geodesic and G_ω acts isometrically and transitively on X_ω, see Proposition 75.

Therefore, according to Lemma 123 it suffices to show that the metric ball $B(e_\omega, 1/2)$ is totally bounded. Let $\epsilon > 0$. Then there exists $i \in \mathbb{N}$, $i \geq 2$, such that $1/(2i) < \epsilon$. For the ball $B(e, \rho) \subset G^*$ consider a maximal collection of points $x_i \in B(e, \rho/2)$ so that the balls $B(x_j, \rho/i)$ are pairwise disjoint. Then, according to the regular growth theorem, the number t of such points x_j does not exceed i^{d+1}. Then the points $x_1, ..., x_t$ form a $2\rho/i$-net in $B(e, \rho/2)$. By passing to X_ω we conclude that the corresponding points $x_{1\omega}, ..., x_{t\omega} \in B(e_\omega, 1/2)$ form an ϵ-net. Since t is finite we conclude that $B(e_\omega, 1/2)$ is totally bounded and therefore compact.

(b) Recall that the (covering) dimension of a metric space Y is the least number n such that for all sufficiently small $\epsilon > 0$ the space Y admits a covering by ϵ-balls so that the *multiplicity* of this covering is $\leq n+1$.

To prove the dimension bound we first review the concept of *Hausdorff dimension* for metric spaces. Let K be a metric space and $\alpha > 0$. The α-Hausdorff measure $\mu_\alpha(K)$ is defined as

$$\lim_{r\to 0} \inf \sum_{i=1}^{N} r_i^\alpha,$$

where the infimum is taken over all finite coverings of K by balls $B(x_i, r_i)$, $r_i \le r$ $(i = 1, ..., N)$. Then the *Hausdorff dimension* of K is defined as:

$$\dim_{Haus}(K) := \inf\{\alpha : \mu_\alpha(K) = 0\}.$$

Example 132. Verify that the Hausdorff dimension of the Euclidean space $\mathbb{R}^n$ is n.

We will need

Theorem 133. *(Hurewicz-Wallman, [31])* $\dim(K) \le \dim_{Haus}(K)$, *where* $\dim$ *stands for the covering dimension.*

Thus it suffices to show finiteness of the Hausdorff dimension of X_ω. We first verify that the Hausdorff dimension of $B(e_\omega, 1/2)$ is at most $d+1$. Pick $\alpha > d+1$; for each i consider the covering of $B(e_\omega, 1/2)$ by the balls

$$B(x_{j\omega}, 2/i), j = 1, ..., t \le i^{d+1}.$$

Therefore we get:

$$\sum_{j=1}^{t} (2/i)^\alpha \le 2^\alpha i^{d+1}/i^\alpha = 2^\alpha i^{d+1-\alpha}.$$

Since $\alpha > d+1$, $\lim_{i\to\infty} 2^\alpha i^{d+1-\alpha} = 0$. Hence $\mu_\alpha(B(e_\omega, 1/2)) = 0$.

Thus, by homogeneity of X_ω, $\dim_{Haus}(B(x, 1/2)) \le d+1$ for each $x \in X_\omega$. Since the Hausdorff measure is additive, we conclude that for each compact subset $K \subset X_\omega$, $\dim_{Haus}(K) \le d+1$.

We now consider the entire space X_ω. Let A_n denote the closed *annulus*

$$\overline{B(e_\omega, n+1) \setminus B(e_\omega, n)}.$$

Then A_n is compact and hence $\mu_\alpha(A_n) = 0$ for each $\alpha > d+1$. Additivity of μ_α implies that

$$\mu_\alpha(X_\omega) \le \sum_{n=1}^{\infty} \mu_\alpha(A_n) = 0.$$

Therefore $\dim(X_\omega) \le \dim_{Haus}(X_\omega) \le d+1$. $\square$

5.6 Proof of Gromov's theorem

The proof is by induction on the degree of polynomial growth. If $\beta_G(R) \prec R^0 = 1$ then G is finite and there is nothing to prove. Suppose that each group of growth at most R^{d-1} is virtually nilpotent. Let G be a a (finitely generated group) of growth $\prec R^d$. Find a sequence λ_n satisfying the conclusion of the regular growth theorem and construct the asymptotic cone X_ω of the Cayley graph of G via rescaling by the sequence λ_n. Then X_ω is connected, locally connected, finite-dimensional and proper. Recall that according to Proposition 75, we have a homomorphism

$$\alpha : G_\omega \to L := \mathrm{Isom}(X_\omega)$$

such that $\alpha(G_\omega)$ acts on X_ω transitively. We also get a homomorphism

$$\ell : G \to L, \ell = \iota \circ \alpha,$$

where $\iota : G \hookrightarrow G_\omega$ is the diagonal embedding. Since the isometric action $L \curvearrowright X_\omega$ is effective and transitive, according to Theorem 127, the group L is a Lie group with finitely many components.

Remark 134. Observe that the point-stabilizer L_y for $y \in X_\omega$ is a compact subgroup in L. Therefore $X_\omega = L/L_x$ can be given a left-invariant Riemannian metric ds^2. Hence, since X_ω is connected, by using the exponential map with respect to ds^2 we see that if $g \in L$ fixes an open ball in X_ω pointwise, then $g = id$.

We have the following cases:

(a) The image of ℓ is not virtually solvable. Then by Tits' alternative, $\ell(G)$ contains a free nonabelian subgroup; it follows that G contains a free nonabelian subgroup as well which contradicts the assumption that G) has polynomial growth.

(b) The image of ℓ is virtually solvable and infinite. Then, after passing to a finite index subgroup in G, we get a homomorphism ϕ from G onto $\mathbb{Z}$. According to Proposition 102, $K = Ker(\phi)$ is a finitely generated group of growth $\prec R^{d-1}$. Thus, by the induction hypothesis, K is a virtually nilpotent group. Since G has polynomial growth, Theorem 100 implies that the group G is virtually nilpotent as well.

(c) $\ell(G)$ is finite.

To see that the latter case can occur consider an abelian group G. Then the homomorphism ℓ is actually trivial. How to describe the kernel of ℓ? For each $g \in G$ define the *displacement function* $\delta(g, r) := \max\{d(gx, x) : x \in B(e, r)\}$. Then

$$K := \{g \in G : g|B(e_\omega, 1) = id\} = \{g \in G : \omega\text{-}\lim \delta(g, \lambda_n)/\lambda_n = 0\}.$$

Here e_ω is the point in X_ω corresponding to the constant sequence (e) in G. On the other hand, by the above remark,

$$Ker(\ell) = K.$$

Let $G' \subset G$ be a finitely-generated subgroup with a fixed set of generators $g_1, ..., g_m$. Define

$$D(G', r) := \max_{j=1,...,m} \delta(g_j, r).$$

(This is an abuse of notation, the above function of course depends not only on G' but also on the choice of the generating set.)

Given a point in the Cayley graph, $p \in \Gamma_G$, we define another function

$$D(G', p, r) := \max\{d(g_j x, x), x \in B(p, r), j = 1, ..., m, \}.$$

Clearly $D(G', e, r) = D(G', r)$ and for $p \in G \subset \Gamma_G$,

$$D(G', p, r) = D(p^{-1}G'p, r),$$

where we take the generators $p^{-1}g_j p, j = 1, ..., m$ for the group $p^{-1}G'p$. The function $D(G', p, r)$ is 2-Lipschitz as a function of p.

Lemma 135. *Suppose that $D(G', r)$ is bounded as a function of r. Then G' is virtually abelian.*

Proof. Suppose that $d(g_j x, x) \le C$ for all $x \in G$. Then

$$d(x^{-1}g_j x, 1) \le C,$$

and therefore the conjugacy class of g_j in G has cardinality $\le \beta_G(C) = N$. Hence the centralizer $Z_G(g_j)$ of g_j in G has finite index in G: Indeed, if $x_0, ..., x_N \in G$ then there are $0 \le i \ne k \le N$ such that

$$x_i^{-1} g_j x_i = x_k^{-1} g_j x_k \Rightarrow [x_k x_i^{-1}, g_j] = 1 \Rightarrow x_k x_i^{-1} \in Z_G(g_j).$$

Thus the intersection

$$A := \bigcap_{j=1}^{m} Z_G(g_j)$$

has finite index in G; it follows that $A \cap G'$ is an abelian subgroup of finite index in G'. ☐

We now assume that $\ell(G)$ is finite and consider the subgroup of finite index $G' := Ker(\ell) \subset G$. Let $g_1, ..., g_m$ be generators of G'. By the previous lemma it suffices to consider the case when the function $D(G', r)$ is unbounded; then, since $G' = Ker(\ell)$, the function $D(G', r)$ has "sublinear growth", i.e.

$$\omega\text{-lim}\, \delta(g_j, \lambda_n)/\lambda_n = 0, j = 1, ..., m.$$

If the subgroup G' is virtually abelian, we are done. Therefore we assume that that this is not the case. In particular, the function $D(G', p, r)$ is unbounded as the function of $p \in G$.

Lemma 136. *Let ϵ such that $0 < \epsilon \leq 1$. Then there exists $x_n \in G$ such that*

$$\omega\text{-}lim\, \frac{D(x_n^{-1} G' x_n, \lambda_n)}{\lambda_n} = \epsilon.$$

Proof. For ω-all $n \in \mathbb{N}$ we have $D(G', \lambda_n) \leq \epsilon\lambda_n/2$. Fix n. Since $D(G', p, \lambda_n)$ is unbounded, there exists $q_n \in G$ such that

$$D(G', q_n, \lambda_n) > 2\lambda_n.$$

Hence, because Γ_G is connected and the function $D(G', p, \lambda_n)$ is continuous, there exists $y_n \in \Gamma_G$ such that

$$D(G', y_n, \lambda_n) = \epsilon\lambda_n.$$

The point y_n is not necessarily in the vertex set of the Cayley graph Γ_G. Pick a point $x_n \in G$ within the distance $\frac{1}{2}$ from y_n. Then, since the function $D(G', \cdot, \lambda_n)$ is 2-Lipschitz,

$$|D(G', x_n, \lambda_n) - \epsilon\lambda_n| \leq 1.$$

It follows that $|D(x_n^{-1} G' x_n, \lambda_n) - \epsilon\lambda_n| \leq 1$ and therefore

$$\omega\text{-lim}\, \frac{D(x_n^{-1} G' x_n, \lambda_n)}{\lambda_n} = \epsilon. \quad \square$$

Now, given $0 < \epsilon \leq 1$ and $g \in G'$ we define a sequence

$$[g] := [x_n^{-1} g x_n] \in G^*.$$

Note that since $D(x_n^{-1} G x_n, \lambda_n) = O(\epsilon)$, the elements $\ell_\epsilon(g_j)$ belong to G_ω. Therefore we obtain a homomorphism $\ell_\epsilon : G' \to G_\omega, \ell_\epsilon : g \mapsto [g]$.

We topologize the group L via the compact-open topology with respect to its action on X_ω, thus ϵ-neighborhood of the identity in L contains all isometries $h \in L$ such that

$$\delta(h, 1) \leq \epsilon,$$

where δ is the displacement function of h on the unit ball $B(e_\omega, 1)$. By our choice of x_n, there exists a generator $h = g_j$ of G' such that $\delta(\ell_\epsilon(h), 1) = \epsilon$. If there is an $N \in \mathbb{N}$ such that the order $|\ell_\epsilon(h)|$ of $\ell_\epsilon(h)$ is at most N for all ϵ, then L contains arbitrarily small finite cyclic subgroups $\langle \ell_\epsilon(h) \rangle$, which is impossible since L is a Lie group. Therefore

$$\lim_{\epsilon \to 0} |\ell_\epsilon(h)| = \infty$$

If for some $\epsilon > 0$, $\ell_\epsilon(G')$ is infinite we are done as above. Hence we assume that $\ell_\epsilon(G')$ is finite for all $\epsilon > 0$. We then use

Theorem 137. *(Jordan) Let L be a Lie group with finitely many connected components. Then there exists a number $q = q(L)$ such that each finite subgroup F in L contains an abelian subgroup of index $\leq q$.*

We prove this theorem in section 5.7.

For each ϵ consider the preimage G'_ϵ in G' of the abelian subgroup in $\ell_\epsilon(G')$ which is given by Jordan's theorem. The index of G'_ϵ in G' is at most q. Let G'' be the intersection of all the subgroups $G'_\epsilon, \epsilon > 0$. Then G'' has finite index in G and G'' admits homomorphisms onto finite abelian groups of arbitrarily large order. Since all such homomorphisms have to factor through the abelinization $(G'')^{ab}$, the group $(G'')^{ab}$ has to be infinite. Since $(G'')^{ab}$ is finitely generated it follows that it has nontrivial free part, hence G'' again admits an epimorphism to $\mathbb{Z}$. Thus we are done by the induction. $\square$

5.7 Proof of Jordan's theorem

In this section I outline a proof of Jordan's theorem, for the details see [48, Theorem 8.29]. Recall that each connected Lie group H acts on itself smoothly via the conjugation. This action fixes $e \in H$, therefore we can look at the derivatives $d_e(g) : T_eH \to T_eH$. We obtain a linear action of G on the vector space T_eH (the Lie algebra of H) called *adjoint representation.* The kernel of this representation is contained in the center $Z(H)$ of H. Therefore, each connected Lie group embeds, modulo its center to the group of real matrices. Therefore I will be assuming that $L = GL_n(\mathbb{R})$.

Given a subset $\Omega \subset L$ define inductively subsets $\Omega^{(i)}$ as $\Omega^{(i+1)} = [\Omega, \Omega^{(i)}]$, $\Omega^{(0)} := \Omega$.

Lemma 138. *There is a neighborhood Ω of $1 \in L$ such that*

$$\lim_{i\to\infty} \Omega^{(i)} = \{1\}.$$

Proof. Let $A, B \in L$ be near the identity; then $A = \exp(\alpha), B = \exp(\beta)$ for some α, β in the Lie algebra of L. Therefore

$$[A, B] = [1 + \alpha + \frac{1}{2}\alpha^2 + ..., 1 + \beta + \frac{1}{2}\beta^2 + ...] =$$

$$(1 + \alpha + \frac{1}{2}\alpha^2 + ...)(1 + \beta + \frac{1}{2}\beta^2 + ...)(1 - \alpha + \frac{1}{2}\alpha^2 - ...)(1 - \beta + \frac{1}{2}\beta^2 - ...)$$

By opening the brackets we see that the linear term in the commutator $[A, B]$ is zero and each term in the resulting infinite series involves both nonzero powers of α and of β. Therefore

$$\|1 - [A, B]\| \leq C\|1 - A\| \cdot \|1 - B\|.$$

Therefore, by induction, if $B_{i+1} := [A, B_i]$, $B_1 = B$, then

$$\|1 - [A, B_i]\| \leq C^i\|1 - A\|^i \cdot \|1 - B\|.$$

By taking Ω be such that $\|1 - A\| < C$ for all $A \in \Omega$, we conclude that

$$\lim_{i\to\infty} \|1 - B_i\| = 0. \quad \square$$

Lemma 139 (Zassenhaus lemma). *Let $\Gamma \subset L$ be a discrete subgroup. Then the set $\Gamma \cap \Omega$ generates a nilpotent subgroup.*

Proof. There exists a neighborhood V of 1 in L such that $V \cap \Gamma = \{1\}$; it follows from the above lemma that all the iterated commutators of the elements of $\Gamma \cap \Omega$ converge to 1. It thus follows that the iterated m-fold commutators of the elements in $\Gamma \cap \Omega$ are trivial for all sufficiently large m. Therefore the set $\Gamma \cap \Omega$ generates a nilpotent subgroup in Γ. $\square$

The finite subgroup $F \subset L$ is clearly discrete, therefore the subgroup $\langle F \cap \Omega\rangle$ is nilpotent. Then $\log(F \cap \Omega)$ generates a nilpotent subalgebra in the Lie algebra of L. Since F is finite, it is also compact, hence, up to conjugation, it is contained in the maximal compact subgroup $K = O(n) \subset GL(n, \mathbb{R}) = L$. The only nilpotent Lie subalgebras of K are abelian subalgebras, therefore the subgroup F' generated by

$F \cap \Omega$ is abelian. It remains to estimate the index. Let $U \subset \Omega$ be a neighborhood of 1 in K such that $U \cdot U^{-1} \subset \Omega$ (i.e. products of pairs of elements xy^{-1}, $x, y \in U$, belong to Ω). Let q denote $Vol(K)/Vol(U)$, where Vol is induced by the biinvariant Riemannian metric on K.

Lemma 140. *$|F : F'| \leq q$.*

Proof. Let $x_1, ..., x_{q+1} \in F$. Then

$$\sum_{i=1}^{q+1} Vol(x_i U) = (q+1)Vol(U) > Vol(K).$$

Hence there are $i \neq j$ such that $x_i U \cap x_j U \neq \emptyset$. Thus $x_j^{-1} x_i \in UU^{-1} \subset \Omega$. Hence $x_j^{-1} x_i \in F'$. □

This also proves Jordan's theorem. □

6 Quasiconformal mappings

Definition 141. Suppose that D, D' are domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a homeomorphism. The mapping f is called **quasiconformal** if the function

$$H_f(x) = \limsup_{r \to 0} \frac{\sup\{d(f(z), f(x)) : d(x, z) = r\}}{\inf\{d(f(z), f(x)) : d(x, z) = r\}}$$

is bounded from above in X. A quasiconformal mapping is called K-quasiconformal if the function H_f is bounded from above by K a.e. in X.

The notion of quasiconformality does not work well in the case when the domain and range are 1-dimensional. It is replaced by

Definition 142. Let $C \subset \mathbb{S}^1$ be a closed subset. A homeomorphism $f : C \to f(C) \subset \mathbb{S}^1$ is called **quasimoebius** if there exists a constant K so that for any quadruple of mutually distinct points $x, y, z, w \in \mathbb{S}^1$ their cross-ratio satisfies the inequality

$$K^{-1} \leq \frac{\lambda(|f(x) : f(y) : f(z) : f(w)|)}{\lambda(|x : y : z : w|)} \leq K \tag{143}$$

where $\lambda(t) = |\log(t)| + 1$.

Note that if f is K-quasimoebius then for any pair of Moebius transformations α, β the composition $\alpha \circ f \circ \beta$ is again K-quasimoebius.

Recall that a mapping $f : S^n \to S^n$ is *Moebius* if it is a composition of inversions. Equivalently, f is Moebius iff it is the extension of an isometry $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$. Yet another equivalent definition: Moebius mappings are the homeomorphisms of S^n which preserve the cross-ratio.

Here is another (analytical) description of quasiconformal mappings. A homeomorphism $f : D \to D'$ is called *quasiconformal* if it has distributional partial derivatives in $L^n_{loc}(D)$ and the ratio

$$R_f(x) := \|f'(x)\|/|J_f(x)|^{1/n}$$

is uniformly bounded from above a.e. in D. Here $\|f'(x)\|$ is the operator norm of the derivative $f'(x)$ of f at x. The essential supremum of $R_f(x)$ in D is denoted by $K_O(f)$ and is called the *outer dilatation* of f. Let us compare $H_f(x)$ and $R_f(x)$. Clearly it is enough to consider positive-definite diagonal matrices $f'(x)$. Let Λ be the maximal eigenvalue of $f'(x)$ and λ be the minimal eigenvalue. Then $\|f'(x)\| = \Lambda$, $H_f(x) = \Lambda/\lambda$ and

$$R_f(x) \le H_f(x) \le R_f(x)^n.$$

Two definitions of quasiconformality (using H_f and R_f) coincide (see for instance [49], [60], [57]) and we have:

$$K_O(f) \le K(f) \le K_O(f)^n.$$

In particular, quasiconformal mappings are differentiable a.e. and their derivative is a.e. invertible.

Note that quasiconformality of mappings and the coefficients of quasiconformality $K(f)$, $K_O(f)$ do not change if instead of the Euclidean metric we consider a conformally-Euclidean metric in D. This allows us to define quasiconformal mappings on domains in S^n, via the stereographic projection.

Examples:

1. If $f : D \to D'$ is a conformal homeomorphism the f is quasiconformal. Indeed, conformality of f means that $f'(x)$ is a similarity matrix for each x, hence $R_f(x) = 1$ for each x. In particular, Moebius transformations are quasiconformal.

2. Suppose that the homeomorphism f extends to a diffeomorphism $\overline{D} \to \overline{D}'$ and the closure $\overline{D}$ is compact. Then f is quasiconformal.

3. Compositions and inverses of quasiconformal mappings are quasiconformal. Moreover, $K_O(f \circ g) \le K_O(f)K_O(g)$, $K_O(f) = K_O(f^{-1})$.

Theorem 144. *(Liouville's theorem for quasiconformal mappings, see [49], [44].) Suppose that $f : S^n \to S^n$, $n \geq 2$, is a quasiconformal mapping which is conformal a.e., i.e. for a.e. $x \in S^n$, $R_f(x) = 1$. Then f is Moebius.*

Conformality of f at x means that the derivative $f'(x)$ exists and is a similarity matrix (i.e. is the product of a scalar and an orthogonal matrix).

Historical remark. Quasiconformal mappings for $n = 2$ were introduced in 1920-s by Groetch as a generalization of conformal mappings. Quasiconformal mappings in higher dimensions were introduced by Lavrentiev in 1930-s for the purposes of application to hydrodynamics. The discovery of relation between quasi-isometries of hyperbolic spaces and quasiconformal mappings was made by Efremovich and Tihomirova [16] and Mostow [44] in 1960-s.

Theorem 145. *Suppose that $f : \mathbb{H}^n \to \mathbb{H}^n$ is a (k, c)-quasi-isometry. Then the homeomorphic extension $h = f_\infty$ of f to $\partial_\infty \mathbb{H}^n$ constructed in Theorem 79 is a quasiconformal homeomorphism (if $n \geq 3$) and quasimoebius (if $n = 2$).*

Proof. I will verify quasiconformality of h for $n \geq 3$ and will leave the case $n = 2$ to the reader. According to the definition, it is enough to verify quasiconformality at each particular point x with uniform estimates on the function $H_h(x)$. Thus, after composing h with Moebius transformations, we can take $x = 0 = h(x)$, $h(\infty) = \infty$, where we consider the upper half-space model of $\mathbb{H}^n$.

Take a Euclidean sphere $S_r(0)$ in $\mathbb{R}^{n-1}$ with the center at the origin. This sphere is the ideal boundary of a hyperplane $P_r \subset \mathbb{H}^n$ which is orthogonal to the vertical geodesic $L \subset \mathbb{H}^n$, connecting 0 and ∞. Let $x_r = L \cap P_r$. Let $\pi_L : \mathbb{H}^n \to L$ be the nearest point projection. The hyperplane P_r can be characterized by the following equivalent properties:

$$P_r = \{w \in \mathbb{H}^n : \pi_L(w) = x_r\}$$

$$P_r = \{w \in \mathbb{H}^n : d(w, x_r) = d(w, L)\}.$$

Since quasi-isometric images of geodesics in $\mathbb{H}^n$ are uniformly close to geodesics, we conclude that

$$\operatorname{diam}[\pi_L(f(P_r))] \leq Const$$

where $Const$ depends only on the quasi-isometry constants of f. The projection π_L extends naturally to $\partial_\infty \mathbb{H}^n$. We conclude:

$$\operatorname{diam}[\pi_L(h(S_r(0)))] \leq Const.$$

Thus $h(S_r(0))$ is contained in a spherical shell

$$\{z \in \mathbb{R}^{n-1} : \rho_1 \le |z| \le \rho_2\}$$

where $\log[\rho_1/\rho_2] \le Const$. This implies that the function $H_h(0)$ is bounded from above by $K := \exp(Const)$. We conclude that the mapping h is K-quasiconformal. $\square$

7 Quasi-isometries of nonuniform lattices in $\mathbb{H}^n$.

Recall that a *lattice* in a Lie group G (with finitely many components) is a discrete subgroup Γ such that the quotient $\Gamma \setminus G$ has finite volume. Here the left-invariant volume form on G is defined by taking a Riemannian metric on G which is left-invariant under G and right-invariant under K, the maximal compact subgroup of G. Thus, if $X := G/K$, then this quotient manifold has a Riemannian metric which is (left) invariant under G. Hence, Γ is a lattice iff Γ acts on X properly discontinuously so that $vol(\Gamma \setminus X)$ is finite. Note that the action of Γ on X is a priori not free. A lattice γ is called *uniform* if $\Gamma \setminus X$ is compact and Γ is called *nonuniform* otherwise.

Note that each lattice is finitely-generated (this is not at all obvious), in the case of the hyperbolic spaces finite generation follows from the thick-thin decomposition above. Thus, if Γ is a lattice, then it contains a torsion-free subgroup of finite index (Selberg lemma). In particular, if Γ is a nonuniform lattice in $\mathbb{H}^2$ then Γ is virtually free of rank ≥ 2.

Example 146. Consider the subgroups $\Gamma_1 := SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$, $\Gamma_2 := SL(2,\mathbb{Z}[i]) \subset SL(2,\mathbb{C})$. Then Γ_1, Γ_2 are nonuniform lattices. Here $\mathbb{Z}[i]$ is the ring of Gaussian integers, i.e. elements of $\mathbb{Z} \oplus i\mathbb{Z}$. The discreteness of Γ_1, Γ_2 is clear, but finiteness of volume requires a proof.

Let's show that Γ_i, $i = 1, 2$, are not uniform. I will give the proof in the case of Γ_1, the case of Γ_2 is similar.

Note that the symmetric space $SL(2,\mathbb{R})/SO(2)$ is the hyperbolic plane. I will use the upper half-plane model of $\mathbb{H}^2$. The group Γ_1 contains the upper triangular matrix

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This matrix acts on $\mathbb{H}^2$ by the parabolic translation $\gamma : z \mapsto z + 1$ (of infinite order). Consider the points $z := (0, y) \in \mathbb{H}^2$ with $y \to \infty$. Then the length of the geodesic segment $\overline{z\gamma(z)}$ tends to zero as y diverges to infinity. Hence the quotient $S := \Gamma_1 \backslash \mathbb{H}^2$ has injectivity radius unbounded from below (from zero), hence S is not compact.

More generally, lattices in a Lie group can be constructed as follows: let $h : G \to GL(N, \mathbb{R})$ be a homomorphism with finite kernel. Let $\Gamma := h^{-1}(GL(N, \mathbb{Z}))$. Then Γ is an *arithmetic lattice* in G.

Recall that a *horoball* in $\mathbb{H}^n$ (in the unit ball model) is a domain bounded by a round Euclidean ball $B \subset \mathbb{H}^n$, whose boundary is tangent to the boundary of $\mathbb{H}^n$ in a single point (called the *center* or *footpoint* of the horoball). The boundary of a horoball in $\mathbb{H}^n$ is called a *horosphere*. In the upper half-space model, the horospheres with the footpoint ∞ are horizontal hyperplanes

$$\{(x_1, ..., x_{n-1}, t) : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\},$$

where t is a positive constant.

Theorem 147. *(Thick-thin decomposition) Suppose that Γ is a nonuniform lattice in $Isom(\mathbb{H}^n)$. Then there exists an (infinite) collection C of pairwise disjoint horoballs $C := \{B_j, j \in J\}$, which is invariant under Γ, so that $(\mathbb{H}^n \setminus \cup_j B_j)/\Gamma$ is compact.*

The quotient $(\mathbb{H}^n \setminus \cup_j B_j)/\Gamma$ is called the *thick part* of $M = \mathbb{H}^n/\Gamma$ and its (noncompact) complement in M is called *thin part* of M.

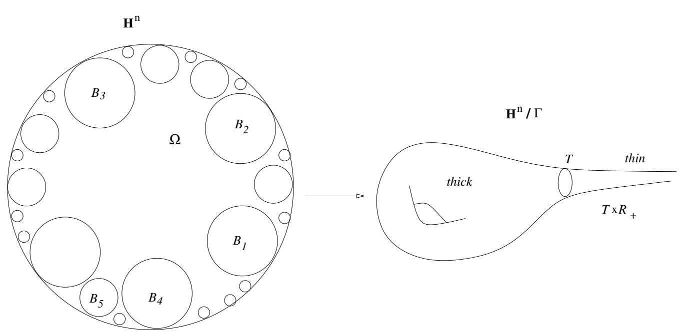

Figure 14: *Truncated hyperbolic space and thick-thin decomposition.*

The complement $\Omega := \mathbb{H}^n \setminus \cup_j B_j$ is called a *truncated hyperbolic space*. Note that the stabilizer Γ_j of each horosphere ∂B_j acts on this horosphere cocompactly with the quotient $T_j := \partial B_j/\Gamma_j$. The quotient B_j/Γ_j is naturally homeomorphic to $T_j \times \mathbb{R}_+$, this product decomposition is inherited from the foliation of B_j by the horospheres with the common footpoint ξ_j and the geodesic rays asymptotic to ξ_j. In the case Γ is torsion-free, orientation preserving and $n = 3$, the quotients T_j are 2-tori.

Definition 148. Let $\Gamma \subset G$ be a subgroup. The *commensurator* of Γ in G, denoted $Comm(\Gamma)$ consists of all $g \in G$ such that the groups $g\Gamma g^{-1}$ and Γ are *commensurable*, i.e. their intersection has finite index in the both groups.

Here is an example of the commensurator: let $\Gamma := SL(2, \mathbb{Z}[i]) \subset SL(2, \mathbb{C})$. Then the commensurator of Γ is the group $SL(2, \mathbb{Q}(i))$. In particular, the group $Comm(\Gamma)$ is nondiscrete in this case. There is a theorem of Margulis, which states that a lattice in G is *arithmetic* if and only if its commensurator is discrete. We note that each element $g \in Comm(\Gamma)$ determines a quasi-isometry $f : \Gamma \to \Gamma$. Indeed, the Hausdorff distance between Γ and $g\Gamma g^{-1}$ is finite. Hence the quasi-isometry f is given by composing $g : \Gamma \to g\Gamma g^{-1}$ with the nearest-point projection to Γ.

The main goal of the remainder of the course is to prove the following

Theorem 149. *(R. Schwartz [50].) Let $\Gamma \subset Isom(\mathbb{H}^n)$ is a nonuniform lattice, $n \geq 3$. Then:*

(a) For each quasi-isometry $f : \Gamma \to \Gamma$ there exists $\gamma \in Comm(\Gamma)$ which is within finite distance from f. The distance between these maps depends only on Γ and on the quasi-isometry constants of f.

(b) Suppose that Γ, Γ' are non-uniform lattices which are quasi-isometric to each other. Then there exists an isometry $g \in Isom(\mathbb{H}^n)$ such that the groups Γ' and $g\Gamma g^{-1}$ are commensurable.

(c) Suppose that Γ' is a finitely-generated group which is quasi-isometric to a nonuniform lattice Γ above. Then the groups Γ, Γ' are weakly commensurable, i.e. there exists a finite normal subgroup $F \subset \Gamma'$ such that the groups $\Gamma, \Gamma'/F$ contain isomorphic subgroups of finite index.

The above theorem fails in the case of the hyperbolic plane (except for the last part).

7.1 Coarse topology of truncated hyperbolic spaces

On each truncated hyperbolic space Ω we put the path-metric which is induced by the restriction of the Riemannian metric of $\mathbb{H}^n$ to Ω. This metric is invariant under Γ and since the quotient Ω/Γ is compact, Ω is quasi-isometric to the group Γ. Note that the restriction of this metric to each peripheral horosphere Σ is a flat metric.

The following lemma is the key for distinguishing the case of the hyperbolic plane from the higher-dimensional hyperbolic spaces (of dimension ≥ 3):

Lemma 150. *Let Ω is a truncated hyperbolic space of dimension ≥ 3. Then each peripheral horosphere $\Sigma \subset \Omega$ does not coarsely separate Ω.*

Proof. Let $R < \infty$ and let B be the horoball bounded by Σ. Then the union of $N_R(\Sigma) \cup B$ is a horoball B' in $\mathbb{H}^n$ (where the metric neighborhood is taken in $\mathbb{H}^n$). The horoball B' does not separate $\mathbb{H}^n$. Therefore, for each pair of points $x, y \in \Omega \setminus B'$, there exists a PL path p connecting them within $\mathbb{H}^n \setminus B'$. If the path p is entirely contained in Ω, we are done. Otherwise, it can be subdivided into finitely many subpaths, each of which is either contained in Ω or connects a pair of points on the boundary of a complementary horoball $B_j \subset \mathbb{H}^n \setminus \Sigma$. The intersection of $N_R^{(\Omega)}(B')$ with $\Sigma_j = \partial B_j$ is a metric ball in the Euclidean space Σ_j (here $N_R^{(\Omega)}$ is the metric neighborhood taken within Ω). Note that a metric ball does not separate $\mathbb{R}^{n-1}$, provided that $n - 1 \geq 2$. Thus we can replace $p_j = p \cap B_j$ with a new path p_j' which connects the end-points of p_j within the complement $\Sigma_j \setminus N_R^{(\Omega)}(B')$. By making these replacements for each j we get a path connecting x to y within $\Omega \setminus N_R^{(\Omega)}(\Sigma)$. $\square$

Let now Ω, Ω' be truncated hyperbolic spaces (of the same dimension), $f : \Omega \to \Omega'$ be a quasi-isometry. Let Σ be a peripheral horosphere of Ω, consider its image $f(\Sigma)$ in Ω'.

Proposition 151. *There exists a peripheral horosphere $\Sigma' \subset \partial\Omega'$ which is within finite Hausdorff distance from $f(\Sigma)$.*

Proof. Note that Ω, being isometric to $\mathbb{R}^{n-1}$, has bounded geometry and is uniformly contractible. Therefore, according to Theorem 58, $f(\Sigma)$ coarsely separates $\mathbb{H}^n$; however it cannot coarsely separate Ω', since f is a quasi-isometry and Σ does not coarsely separate Ω. Let $R < \infty$ be such that $N_R(f(\Sigma))$ separates $\mathbb{H}^n$ into (two) deep components X_1, X_2. Suppose that for each complementary horoball B_j' of Ω' (bounded by the horosphere Σ_j'),

$$N_{-R}(B_j') := B_j' \setminus N_R(\Sigma_j') \subset X_1.$$

Then the entire Ω' is contained in $N_R(f(\Sigma))$. It follows that $f(\Sigma)$ does not coarsely separate $\mathbb{H}^n$, a contradiction. Thus there are complementary horoballs B_1', B_2' for Ω' such that $N_{-R}(B_1') \subset X_1, N_{-R}(B_2') \subset X_2$. If either Σ_1 or Σ_2 is not contained in $N_r(f(\Sigma))$ for some r then $f(\Sigma)$ coarsely separates Ω'. Thus we found a horosphere $\Sigma' := \Sigma_1'$ such that

$$\Sigma' \subset N_r(f(\Sigma)).$$

Our goal is to show that $f(\Sigma) \subset N_\rho(\Sigma')$ for some $\rho < \infty$. The nearest-point projection $\Sigma' \to f(\Sigma)$ defines a quasi-isometric embedding $h : \Sigma' \to \Sigma$. However Lemma 71 proves that a quasi-isometric embedding between two Euclidean spaces of the same dimension is a quasi-isometry. Thus there exists $\rho < \infty$ such that $f(\Sigma) \subset N_\rho(\Sigma')$. $\square$

Lemma 152. *$d_{Haus}(f(\Sigma), \Sigma')) \le r$, where r is independent of Σ.*

Proof. The proof is by inspection of the arguments in the proof of the previous proposition. First of all, the constant R depends only on the quasi-isometry constants of the mapping f and the uniform geometry/uniform contractibility bounds for $\mathbb{R}^{n-1}$ and $\mathbb{H}^n$. The inradii of the shallow complementary components of $N_R(f(\Sigma))$ again depend only on the above data. Therefore there exists a uniform constant r such that Σ_1 of Σ_2 is contained in $N_r(f(\Sigma))$. Finally, the upper bound on ρ such that $N_\rho(Image(h)) = \Sigma'$ (coming from Lemma 71) again depends only on the quasi-isometry constants of the projection $h : \Sigma' \to \Sigma$.

7.2 Hyperbolic extension

The main result of this section is

Theorem 153. *f admits a quasi-isometric extension $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$.*

Proof. We will construct the extension $\tilde{f}$ into each complementary horoball $B \subset \mathbb{H}^n \setminus \Omega$. Without loss of generality we can use the upper half-space model of $\mathbb{H}^n$ so that the horoballs B and B' are both given by

$$\{(x_1, ..., x_{n-1}, 1) : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

We will also assume that $f(\Sigma) \subset \Sigma'$. For each vertical geodesic ray $\rho(t), t \in \mathbb{R}_+$, in B_j we define the geodesic ray $\rho'(t)$ to be the vertical geodesic ray in B' with the initial point $f(\rho(0))$. This gives the extension of f into B:

$$\tilde{f}(\rho(t)) = \rho'(t).$$

Let's verify that this extension is coarsely Lipschitz. Let x and $y \in B$ be points within the (hyperbolic) distance ≤ 1. By the triangle inequality it suffices to consider the case when x, y belong to the same horosphere H_t (of the Euclidean height t) with the footpoint at ∞ (if x and y belong to the same vertical ray we clearly get $d(\tilde{f}(x), \tilde{f}(y)) = d(x, y)$). Note that the distance from x to y along the horosphere H does not exceed ϵ, which is independent of t. Let $\bar{x}, \bar{y}$ denote the points in Σ such that x, y belong to the vertical rays in B_j with the initial points $\bar{x}, \bar{y}$ respectively. Then

$$d_\Sigma(\bar{x}, \bar{y}) = t d_{H_t}(x, y) \le \epsilon t.$$

Hence, since f is (L, A)-coarse Lipshitz,

$$d_\Sigma(f(\bar{x}), f(\bar{y})) \leq L\epsilon t + A.$$

It follows that

$$d_\Sigma(\tilde{f}(x), \tilde{f}(y)) \leq L\epsilon + A/t \leq L\epsilon + A.$$

This proves that the extension $\tilde{f}$ is coarse Lipschitz in the horoball B. Since the coarse Lipschitz is a local property, the mapping $\tilde{f}$ is coarse Lipschitz on $\mathbb{H}^n$. The same argument applies to the hyperbolic extension $\tilde{f}'$ of the coarse inverse f' to the mapping f. It is clear that the mapping $\tilde{f} \circ \tilde{f}'$ and $\tilde{f}' \circ \tilde{f}$ have bounded displacement. Thus $\tilde{f}$ is a quasi-isometry. $\square$

Since $\tilde{f}$ is a quasi-isometry of $\mathbb{H}^n$, it admits a quasiconformal extension $h : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^n$. Let Λ, Λ' denote the sets of the footpoints of the peripheral horospheres of Ω, Ω' respectively. It is clear that $h(\Lambda) = \Lambda'$.

7.3 Zooming in

Our main goal is to show that the mapping h constructed in the previous section is Moebius. By the Liouville's theorem for quasiconformal mappings, h is Moebius iff for a.e. point $\xi \in S^{n-1}$, the derivative of h at ξ is a similarity. We will be working with the upper half-space of the hyperbolic space $\mathbb{H}^n$.

Proposition 154. *Suppose that h is not Moebius. Then there exists a quasi-isometry $F : \Omega \to \Omega'$ whose extension to the sphere at infinity is a linear map which is not a similarity.*

Proof. Since h is differentiable a.e. and is not Moebius, there exists a point $\xi \in S^{n-1} \setminus \Lambda$ such that $Dh(\xi)$ exists, is invertible but is not a similarity. By pre- and post-composing f with isometries of $\mathbb{H}^n$ we can assume that $\xi = 0 = h(\xi)$. Let $L \subset \mathbb{H}^n$ denote the vertical geodesic through ξ. Since ξ is not a footpoint of a complementary horoball to Ω, there exists a sequence of points $x_j \in L \cap \Omega$ which converges to ξ. For each $t \in \mathbb{R}_+$ define $\alpha_t : z \mapsto tz$, a hyperbolic translation along L. Let t_j be such that $\alpha_{t_j}(x_1) = x_j$. Set

$$\tilde{f}_j := \alpha_{t_j}^{-1} \circ \tilde{f} \circ \alpha_{t_j},$$

the quasiconformal extensions of these mappings to $\partial_\infty \mathbb{H}^n$ are given by

$$h_j(z) = \frac{h(t_j z)}{t_j}.$$

By the definition of differentiability,

$$\lim_{j\to\infty} h_j = A = Dh(0),$$

where the convergence is uniform on compacts in $\mathbb{R}^{n-1}$. Let's verify that the sequence of quasi-isometries $\tilde{f}_j$ subconverges to a quasi-isometry of $\mathbb{H}^n$. Indeed, since the quasi-isometry constants of all $\tilde{f}_j$ are the same, it suffices to show that $\{\tilde{f}_j(x_1)\}$ is a bounded sequence in $\mathbb{H}^n$. Let L_1, L_2 denote a pair of distinct geodesics in $\mathbb{H}^n$ through x_1, so that the point ∞ does not belong to $L_1 \cup L_2$. Then the quasi-geodesics $\tilde{f}_j(L_i)$ are within distance $\leq C$ from geodesics L^*_{1j}, L^*_{2j} in $\mathbb{H}^n$. Note that the geodesics L^*_{1j}, L^*_{2j} subconverge to geodesics in $\mathbb{H}^n$ with distinct end-ponts (since the mapping A is 1-1). The point $\tilde{f}_j(x_1)$ is within distance $\leq C$ from L^*_{1j}, L^*_{2j}. If the sequence $\tilde{f}_j(x_1)$ is unbounded, we get that L^*_{1j}, L^*_{2j} subconverge to geodesics with a common end-point at infinity. Contradiction.

We thus pass to a subsequence such that $\tilde{f}_j$ converges to a quasi-isometry $f_\infty : \mathbb{H}^n \to \mathbb{H}^n$. Note however that f_∞ in general does not send Ω to Ω'. Recall that $\Omega/\Gamma, \Omega'/\Gamma'$ are compact. Therefore there exist sequences $\gamma_j \in \Gamma, \gamma'_j \in \Gamma'$ such that $\gamma_j(x_j)$, $\gamma'_j(\tilde{f}(x_j))$ belong to a compact subset of $\mathbb{H}^n$. Hence the sequences $\beta_j := \alpha_{t_j}^{-1} \circ \gamma_j^{-1}, \beta'_j := \alpha_{t_j}^{-1} \circ \gamma_j'^{-1}$ is precompact in $Isom(\mathbb{H}^n)$ and therefore they subconverge to isometries $\beta_\infty, \beta'_\infty \in Isom(\mathbb{H}^n)$. Set

$$\Omega_j := \alpha_{t_j}^{-1}\Omega = \alpha_{t_j}^{-1} \circ \gamma_j^{-1}\Omega = \beta_j\Omega,$$

$$\Omega'_j := \alpha_{t_j}^{-1}\Omega' = \beta'_j\Omega',$$

then $\tilde{f}_j : \Omega_j \to \Omega'_j$. On the other hand, the sets Ω_j, Ω'_j subconverge to the sets $\beta_\infty\Omega, \beta'_\infty\Omega'$ and $\tilde{f}_\infty$ is a quasi-isometry between $\beta_\infty\Omega$ and $\beta'_\infty\Omega'$. Since $\beta_\infty\Omega$ and $\beta'_\infty\Omega'$ are isometric copies of Ω and Ω' the assertion follows. $\square$

The situation when we have a linear mapping (which is not a similarity) mapping Λ to Λ' seems at the first glance impossible. Here however is an example:

Example 155. Let $\Gamma := SL(2, \mathbb{Z}[i]), \Gamma' := SL(2, \mathbb{Z}[\sqrt{-2}])$. Then $\Lambda = \mathbb{Q}(i), \Lambda' = \mathbb{Q}(\sqrt{-2})$.

Define a real linear mapping $A : \mathbb{C} \to \mathbb{C}$ by sending 1 to 1 and i to $\sqrt{-2}$. Then A is not a similarity, however $A(\Lambda) = \Lambda')$.

Thus to get a contradiction we have to exploit the fact that the linear map in question is quasiconformal extension of an isometry between truncated hyperbolic spaces. This is done using a trick which replaces A with an *inverted linear map*, such maps are defined in the next section.

7.4 Inverted linear mappings

Let $A : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be an (invertible) linear mapping and I be the inversion in the unit sphere about the origin, i.e.

$$I(x) = \frac{x}{|x|^2}.$$

Definition 156. An inverted linear map is the conjugate of A by the inversion in the unit sphere centered at the origin, i.e. the composition

$$h := I \circ A \circ I,$$

which means that

$$h(x) = \frac{|x|^2}{|Ax|^2} A(x).$$

Lemma 157. *The function $\phi(x) = \frac{|x|^2}{|Ax|^2}$ is* asymptotically constant*, i.e. the gradient of ϕ converges to zero as $|x| \to \infty$.*

Proof. The function ϕ is a rational function of degree zero, hence its gradient is a rational vector-function of degree -1. $\square$

Note however that ϕ is not a constant mapping unless A is a similarity. Hence h is linear iff A is a similarity.

Corollary 158. *Let R be a fixed positive real number, $x_j \in \mathbb{R}^{n-1}$, $|x_j| \to \infty$. Then the function $h(x - x_j) - h(x_j)$ converges (uniformly on the R-ball $B(0, R)$) to a linear function, as $j \to \infty$.*

We would like to strengthen the assertion that ϕ is not constant (unless A is a similarity). Let G be a discrete group of Euclidean isometries acting cocompactly on $\mathbb{R}^{n-1}$. Fix a G-orbit Gx, for some $x \in \mathbb{R}^{n-1}$.

Lemma 159. *There exists a number R and a sequence of points $x_j \in Gx$ diverging to infinity such that the restrictions ϕ to $B(x_j, R) \cap Gx$ are not constant for all j.*

Proof. Let P be a compact fundamental domain for G, containing x. Let ρ denote $\mathrm{diam}(P)$. Pick any $R \geq 4\rho$. Then $B(x, R)$ contains all images of P under G which are adjacent to P. Suppose that the sequence x_j as above does not exist. This means that there exists $r < \infty$ such that the restriction of ϕ to $B(x_j, R)$ is constant for each

$x_j \in Gx \setminus B(0,r)$. It follows that the function ϕ is actually constant on $Gx \setminus B(0,r)$. Note that the set

$$\{y/|y|, y \in Gx \setminus B(0,r)\}$$

is dense in the unit sphere. Since $\phi(y/|y|) = \phi(y)$ it follows that ϕ is a constant function. $\square$

We now return to the discussion of quasi-isometries.

Let A be an invertible linear mapping (which is not a similarity) constructed in the previous section, by composing A with Euclidean translations we can assume that $0 = A(0)$ belongs to both Λ and Λ':

Indeed, let $p \in \Lambda \setminus \infty, q := A(p)$, define P, Q to be the translations by p, q. Consider $A_2 := Q^{-1} \circ A \circ P$, $\Lambda_1 := \Lambda - p$, $\Lambda'_1 := \Lambda' - q$, $\Omega_1 := \Omega - p$, $\Omega'_1 := \Omega' - q$. Then $A_2(\Lambda_1) = \Lambda'_1$, $A_2(0) = 0$, $0 \in \Lambda_1 \cap \Lambda'_1$.

We retain the notation $A, \Lambda, \Lambda', \Omega, \Omega'$ for the linear map and the new sets of foot-points of horoballs and truncated hyperbolic spaces.

Then $\infty = I(0)$ belongs to both $I(\Lambda)$ and $I(\Lambda')$. To simplify the notation we replace $\Lambda, \Omega, \Lambda', \Omega'$ with $I(\Lambda), I(\Omega), I(\Lambda'), I(\Omega')$ respectively. Then the truncated hyperbolic spaces Ω, Ω' have complementary horoballs B_∞, B'_∞.

Given $x \in \mathbb{R}^{n-1}$ define $h_*(x) := h(\Gamma_\infty x)$. Let Γ_∞, Γ'_∞ be the stabilizers of ∞ in Γ, Γ' respectively. Without loss of generality we can assume that $\infty \in \Lambda, \Lambda'$, hence Γ_∞, Γ'_∞ act cocompactly (by Euclidean isometries) on $\mathbb{R}^{n-1}$.

Lemma 160 (Scattering lemma). *Suppose that A is not a similarity. Then for each $x \in \mathbb{R}^{n-1}$, $h_*(x)$ is not contained in finitely many Γ'_∞-orbits.*

Proof. Let $x_j = \gamma_j x \in \Gamma_\infty x$ and $R < \infty$ be as in Lemma 159, where $G = \Gamma_\infty$. We have a sequence of maps $\gamma'_j \in \Gamma'_\infty$ such that $\gamma'_j h(x_j)$ is relatively compact in $\mathbb{R}^{n-1}$. Then the mapping $h|B(x_j, R) \cap \Gamma_\infty x$ is not linear for each j (Lemma 159). However the sequence of maps

$$\gamma'_j \circ h \circ \gamma_j := h_j$$

converges to an affine mapping h_∞ on $B(x, R)$ (since h is asymptotically linear). We conclude that the union

$$\bigcup_{j=1} h_j(\Gamma_\infty x \cap B(x, R))$$

is an infinite set. $\square$

Theorem 161. *Suppose that h is an inverted linear map which is not a similarity. Then h admits no quasi-isometric extension $\Omega \to \Omega'$.*

Proof. Let x be a footpoint of a complementary horoball B to Ω, $B \neq B_\infty$. Then, by the scattering lemma, $h_*(x)$ is not contained in a finite union of Γ'_∞-orbits. Let $\gamma_j \in \Gamma_\infty$ be a sequence such that the Γ'_∞-orbits of the points $x'_j := h\gamma_j(x)$ are all distinct. Let B'_j denote the complementary horoball to Ω' whose footpoint is x'_j. It follows that the Euclidean diameters of the complementary horoballs B'_j converge to zero. Let B_j be the complementary horoball to Ω whose footpoint is $\gamma_j x$. Then

$$dist(B_j, B_\infty) = dist(B_1, B_\infty) = -\log(\mathrm{diam}(B_1)) = D,$$

$$dist(B'_j, B'_\infty) = -\log(\mathrm{diam}(B'_j)) \to \infty.$$

If $f : \Omega \to \Omega'$ is an (L, A) quasi-isometry whose quasiconformal extension is h then

$$dist(B'_j, B'_\infty) \leq L(D + Const) + A.$$

Contradiction. $\square$

Therefore we have proven

Theorem 162. *Suppose that $f : \Omega \to \Omega'$ is a quasi-isometry of truncated hyperbolic spaces. Then f admits an (unique) extension to S^{n-1} which is Moebius.*

7.5 Proof of Theorem 149

(a) For each quasi-isometry $f : \Gamma \to \Gamma$ there exists $\gamma \in Comm(\Gamma)$ which is within finite distance from f.

Proof. The quasi-isometry f extends to a quasi-isometry of the hyperbolic space $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. The latter quasi-isometry extends to a quasiconformal mapping $h : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^n$. This quasiconformal mapping has to be Moebius according to Theorem 162. Therefore $\tilde{f}$ is within finite distance from an γ isometry of $\mathbb{H}^n$ (which is an isometric extension of h to $\mathbb{H}^n$). It remains to verify that γ belongs to $Comm(\Gamma)$. We note that γ sends the peripheral horospheres of Ω within (uniformly) bounded distance of peripheral horospheres of Ω. The same is of course true for all mappings of the group $\Gamma' := \gamma\Gamma\gamma^{-1}$. Thus, if $\gamma' \in \Gamma'$ fixes a point in Λ (a footpoint of a peripheral horosphere Σ), then it has to preserve Σ: Otherwise by iterating γ' we would get a contradiction. The same applies if $\gamma'(\xi_1) = \xi_2$, where $\xi_1, \xi_2 \in \Lambda$ are in the same Γ-orbit: $\gamma'(\Sigma_1) = \Sigma_2$, where ξ_i is the footpoint of the peripheral horosphere

Σ_i. Therefore we modify Ω as follows: Pick peripheral horospheres $\Sigma_1, ..., \Sigma_m$ with disjoint Γ-orbits and for each $\gamma' \in \Gamma'$ such that $\gamma'(\Sigma_i)$ is not contained in Ω, we replace the peripheral horosphere parallel to $\gamma'(\Sigma_i)$ with the horosphere $\gamma'(\Sigma_i)$. As the result we get a new truncated hyperbolic space Ω' which is invariant under both Γ and Γ'. Observe now that the group Γ'' generated by Γ, Γ' acts on Ω properly discontinuously and cocompactly: Otherwise the nontrivial connected component of the closure of Γ'' would preserve Ω' and hence the countable Λ, which is impossible.

Therefore the projections

$$\Omega/\Gamma \to \Omega/\Gamma'', \Omega/\Gamma' \to \Omega/\Gamma''$$

are finite-to-one maps. It follows that $|\Gamma'' : \Gamma|$ and $|\Gamma'' : \Gamma'|$ are both finite. Therefore the groups Γ, Γ' are commensurable and $\gamma \in Comm(\Gamma)$.

To prove a uniform bound on the distance $d(f, g|\Sigma)$ we notice that f and g have the same extension to the sphere at infinity. Therefore, by 80, the distance $d(f, g)$ is uniformly bounded in terms of the quasi-isometry constants of f.

(b) Suppose that Γ, Γ' are non-uniform lattices which are quasi-isometric to each other. Then there exists an isometry $g \in Isom(\mathbb{H}^n)$ such that the groups Γ' and $g\Gamma g^{-1}$ are commensurable.

Proof. The proof is analogous to (a): The quasi-isometry f is within finite distance from an isometry g. Then the elements of the group $g\Gamma g^{-1}$ have the property that they map the truncated hyperbolic space Ω' of Γ' within (uniformly) bounded distance from Ω. Therefore we can modify Ω' to get a truncated hyperbolic space Ω'' which is invariant under both Γ' and $g\Gamma g^{-1}$. The rest of the argument is the same as for (a).

(c) Suppose that Γ' is a finitely-generated group which is quasi-isometric to a nonuniform lattice Γ above. Then the groups Γ, Γ' are weakly commensurable, i.e. there exists a finite normal subgroup $K \subset \Gamma'$ such that the groups $\Gamma, \Gamma'/K$ contain isomorphic subgroups of finite index.

Proof. Let $f : \Gamma \to \Gamma'$ be a quasi-isometry and let $f' : \Gamma' \to \Gamma$ be its quasi-inverse. We define the set of uniform quasi-isometries

$$\Gamma'_f := f' \circ \Gamma' \circ f$$

of the truncated hyperbolic space Ω of the groups Γ. Each quasi-isometry $g \in \Gamma'_f$ is within a (uniformly) bounded distance from a quasi-isometry of Ω induced by an element g^* of $Comm(\Gamma)$. We get a map

$$\psi : \gamma' \mapsto f' \circ \gamma' \circ f \mapsto (f' \circ \gamma' \circ f)^* \in Comm(\Gamma).$$

I claim that this map is a homomorphism with finite kernel. Let's first check that this map is a homomorphism:

$$d(f' \circ \gamma_1' \gamma_2' \circ f, f' \circ \gamma_1' \circ f \circ f' \circ \gamma_2' \circ f) < \infty,$$

hence the above quasi-isometries have the same Moebius extension to the sphere at infinity. Suppose that $\gamma' \in Ker(\psi)$. Then the quasi-isometry $f' \circ \gamma' \circ f$ has a bounded displacement on Ω. Since the family of quasi-isometries

$$\{f' \circ \gamma' \circ f, \gamma' \in K\}$$

has uniformly bounded quasi-isometry constants, it follows that they have uniformly bounded displacement. Hence the elements $\gamma' \in K$ have uniformly bounded displacement as well. Therefore the normal subgroup K is finite. The rest of the argument is the same as for (a) and (b): The groups $\Gamma, \Gamma'' := \psi(\Gamma') \subset Comm(\Gamma)$ act on a truncated hyperbolic space Ω' which is within finite distance from Ω. Therefore the groups Γ'', Γ are commensurable. $\square$

8 A quasi-survey of QI rigidity

Given a group G one defines the *abstract commensurator* $Comm(G)$ as follows. The elements of $Comm(G)$ are equivalence classes of isomorphisms between finite index subgroups of G. Two such isomorphisms $\psi : G_1 \to G_2, \phi : G_1' \to G_2'$ are equivalent if their restrictions to further finite index subgroups $G_1'' \to G_2''$ are equal. The composition and the inverse are defined in the obvious way, making $Comm(G)$ a group.

Let X be a metric space or a group G. Call X *strongly QI rigid* if each (L, A)-quasi-isometry $f : X \to X$ is within finite distance from an isometry $\phi : X \to X$ or an element ϕ of $Comm(G)$ and moreover $d(f, \phi) \leq C(L, A)$.

Call a group G *QI rigid* if any group G' which is quasi-isometric to G is actually weakly commensurable to G.

Call a class of groups $\mathcal{G}$ *QI rigid* if each group G which is quasi-isometric to a member of $\mathcal{G}$ is actually weakly commensurable to a member of $\mathcal{G}$.

Theorem 163. *(Pansu, [46]) Let X be a quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$ $(n \geq 2)$ or the hyperbolic Cayley plane $\mathbb{H}^2_{Ca}$. Then X is strongly QI rigid.*

Theorem 164. *(Tukia, [55] for the real-hyperbolic spaces $\mathbb{H}^n, n \geq 3$ and Chow [11] for complex-hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}, n \geq 2$). Let X be a symmetric space of negative curvature which is not the hyperbolic plane $\mathbb{H}^2$. Then the class of uniform lattices in X is QI rigid.*

Theorem 165. *(Combination of the work by Gabai [21], Casson and Jungreis [10] and Tukia [56]) The fundamental groups of closed hyperbolic surfaces are QI rigid.*

Theorem 166. *(Stallings, [52]) Each nonabelian free group is QI rigid. Thus each nonuniform lattice in $\mathbb{H}^2$ is QI rigid.*

Theorem 167. *(Kleiner, Leeb, [37]) Let X be a symmetric space of nonpositive curvature such that each deRham factor of X is a symmetric space of rank ≥ 2. Then X is strongly QI rigid.*

Theorem 168. *(Kleiner, Leeb, [37]) Let X be a Euclidean building such that each deRham factor of X is a Euclidean building of rank ≥ 2. Then X is strongly QI rigid.*

Theorem 169. *(Kleiner, Leeb, [37]) Let X be a symmetric space of nonpositive curvature without Euclidean deRham factors. Then the class of uniform lattices in X is QI rigid.*

Theorem 170. *(Eskin, [17]) Let X be an irreducible symmetric space of nonpositive curvature of rank ≥ 2. Then each nonuniform lattice in X is strongly QI rigid and QI rigid.*

Theorem 171. *(Kleiner, Leeb, [38]) Suppose that Γ is a finitely-generated groups which is quasi-isometric to a Lie grooup G with the nilpotent radical N and semisimple quotient $G/N = H$. Then Γ fits into a short exact sequence*

$$1 \to K \to \Gamma \to Q \to 1,$$

where K is quasi-isometric to N and Q is weakly commensurable to a uniform lattice in H.

Problem 172. Prove an analogue of the above theorem for all Lie groups G (without assuming that the sol-radical of G is nilpotent).

Theorem 173. *(Bourdon, Pajot [4]) Let X be a thick hyperbolic building of rank 2 with right-angled fundamental polygon and whose links are complete bipartite graphs. Then X is strongly QI rigid.*

Problem 174. Construct an example of a hyperbolic group with Menger curve boundary, which is QI rigid.

Problem 175. Let G be a *random* k-generated group, $k \geq 2$. Is G QI rigid?

Randomness can be defined for instance as follows. Consider the set $B(n)$ of presentations

$$\langle x_1, ..., x_k | R_1, ..., R_l \rangle$$

where the total length of the words $R_1, ..., R_l$ is $\leq n$. Then a class C of k-generated groups is said to consist of random groups if

$$\lim_{n\to\infty} \frac{|B(n) \cap C|}{|B(n)|} = 1.$$

Here is another notion of randomness: fix the number l of relators, assume that all relators have the same length n; this defines a class of presentations $S(k, l, n)$. Then require

$$\lim_{n\to\infty} \frac{|S(k,l,n) \cap C|}{|S(k,l,n)|} = 1.$$

Theorem 176. *(Kapovich, Kleiner, [32]) There is a 3-dimensional hyperbolic group which is strongly QI rigid.*

Theorem 177. *Each finitely generated abelian group is QI rigid.*

Theorem 178. *(Farb, Mosher, [18]) Each solvable Baumslag-Solitar group*

$$BS(1, q) = \langle x, y : xyx^{-1} = y^q \rangle$$

is QI rigid.

Theorem 179. *(Whyte, [62]) All non-solvable Baumslag-Solitar groups*

$$BS(p, q) = \langle x, y : xy^p x^{-1} = y^q \rangle,$$

$|p| \neq 1, |q| \neq 1$ *are QI to each other.*

Theorem 180. *(Farb, Mosher, [19]) The class of non-polycyclic abelian-by-cyclic groups, i.e. groups* Γ *which fit into an exact sequence*

$$1 \to A \to \Gamma \to \mathbb{Z} \to 1$$

with A *is an abelian group, is QI rigid.*

Theorem 181. *(Dyubina, [14]) The class of finitely generated solvable groups is not QI rigid.*

Problem 182. Is the class of finitely generated polycyclic groups QI rigid?

Example 183. Let S be a closed hyperbolic surface, M is the unit tangent bundle over S. Then we have an exact sequence

$$1 \to \mathbb{Z} \to G = \pi_1(M) \to Q := \pi_1(S) \to 1.$$

This sequence does not split even after passage to a finite index subgroup in G, hence G is not weakly commensurable with $Q \times \mathbb{Z}$. However G is quasi-isometric to $Q \times \mathbb{Z}$. More generally, if Q is a hyperbolic group, then all groups G which fit into an exat sequence

$$1 \to \mathbb{Z} \to G \to Q \to 1,$$

are quasi-isometric.

Example 184. There are uniform lattices in $\mathbb{H}^n, n \geq 3$, which are not weakly commensurable.

Indeed, take an arithmetic and a nonarithmetic lattice in $\mathbb{H}^n$.

Example 185. The product of free groups $G = F_n \times F_m$, $(n, m \geq 2)$ is not QI rigid.

Proof. The group G acts discretely, cocompactly, isometrically on the product of simplicial trees $X := T \times T'$. However there are examples [63], [8], of groups G' acting discretely, cocompactly, isometrically on X so that G' contains no proper finite index subgroups. Then G is quasi-isometric to G' but these groups are clearly not weakly commensurable. □

Problem 186. Suppose that G is (a) a Mapping Class group, (b) $Out(F_n)$, (c) an Artin group, (d) a Coxeter group, (e) the fundamemtal group of one of the negatively curved manifolds constructed in [30], (f) $\pi_1(N)$, where N is a finite covering of the product of a hyperbolic surface by itself $S \times S$, ramified over the diagonal $\Delta(S \times S)$. Is G QI rigid?

One has to exclude, of course, Artin and Coxeter groups which are commensurable with the direct products of free groups.

Theorem 187. *(Kapovich, Leeb, [36]) The class of fundamental groups G of 3-dimensional Haken 3-manifolds, which are not Sol-manifolds*[1]*, is QI rigid.*

[1] I.e. excluding G which are polycyclic but not nilpotent.

Theorem 188. *(Papasoglu, [47]) The class of finitely-presented groups which split over $\mathbb{Z}$ is QI rigid. Moreover, quasi-isometries of 1-ended groups G preserve the JSJ decomposition of G*

Theorem 189. *(Kapovich, Kleiner, Leeb, [35]) Quasi-isometries preserve deRham decomposition of the universal covers of closed nonpositively curved Riemannian manifolds.*

Problem 190. Are there finitely generated (amenable) groups which are quasi-isometric but not bi-Lipschitz equivalent?

Problem 191. Suppose that G is a finitely-presented group. Does the topology of the asymptotic cone of G depend on the scaling sequence/ultrafilter?

Theorem 192. *(Gersten, [22] The cohomological dimension (over an arbitrary ring R) is a QI invariant within the class of finitely-presented groups of type FP (over R).*

I refer to [6] for the definitions of cohomological dimension and the type FP.

Theorem 193. *(Shalom, [51]) The cohomological dimension (over $\mathbb{Q}$) of amenable groups is a QI invariant.*

Problem 194. Is the cohomological dimension of a group (over $\mathbb{Q}$) a QI invariant?

Recall that a group G has *property (T)* if each isometric affine action of G on a Hilbert space has a global fixed point, see [13] for more thorough discussion. In particular, such groups cannot map onto $\mathbb{Z}$.

Theorem 195. *The property (T) is not a QI invariant.*

Proof. This theorem should be probably attributed to S. Gersten and M. Ramachandran; the example below is a variation on the *Raghunathan's example* discussed in [23].

Let Γ be a hyperbolic group which satisfies property (T) and such that $H^2(\Gamma, \mathbb{Z}) \neq 0$. To construct such a group, start for instance with an infinite hyperbolic group F satisfying Property (T) which has an aspherical presentation complex (see for instance [1] for the existence of such groups). Then $H^1(F, \mathbb{Z}) = 0$ (since F satisfies (T)), if $H^2(F, \mathbb{Z}) = 0$, add enough random relations to F, keeping the resulting groups F' hyperbolic, infinite, 2-dimensional. Then $H^1(F', \mathbb{Z}) = 0$ since F' also satisfies (T). For large number of relators we get a group $\Gamma = F'$ such that $\chi(\Gamma) > 0$ (the number of relator is larger than the number of generators), hence $H^2(\Gamma, \mathbb{Z}) \neq 0$. Now, pick a nontrivial element $\omega \in H^2(\Gamma, \mathbb{Z})$ and consider a central extension

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1$$

with the extension class ω. The cohomology class ω is *bounded* since Γ is hyperbolic; hence the groups G and $G' := \mathbb{Z} \times \Gamma$ are quasi-isometric, see [23]. The group G' does not satisfy (T), since it surjects to $\mathbb{Z}$. On the other hand, the group G satisfies (T), see [13, 2.c, Theorem 12]. $\square$

However the following question is still open:

Problem 196. A group G is said to be *a-T-menable* if it admits a proper isometric affine action on a Hilbert space. Is a-T-menability a QI invariant?

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