Change of Variables in Multiple Integrals: Euler to Cartan

From formalism to analysis and back; methods of proof come full circle.

VICTOR J. KATZ
University of the District of Columbia
Washington, DC 20008

Leonhard Euler first developed the notion of a double integral in 1769 [**7**]. As part of his discussion of the meaning of a double integral and his calculations of such an integral, he posed the obvious question: what happens to a double integral if we change variables? In other words, what happens to $\int\int_{\mathbf{A}} f(x,y)\,dx\,dy$ if we let $x=x(t,v)$ and $y=y(t,v)$ and attempt to integrate with respect to t and v? The answer is provided by the change-of-variable theorem, which states that

$$\int\int_{\mathbf{A}} f(x,y)\,dx\,dy=\int\int_{\mathbf{B}} f(x(t,v),y(t,v))\left|\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial t}\right|dt\,dv \tag{1}$$

where the regions **A** and **B** are related by the given functional relationship between (x,y) and (t,v). This result, and its generalization to n variables, are extremely important in allowing one to transform complicated integrals expressed in one set of coordinates to much simpler ones expressed in a different set of coordinates. Every modern text in advanced calculus contains a discussion and proof of the theorem. (For example, see [**5**], [**1**], [**18**].)

Euler interpreted this result formally; namely, he considered $dx\,dy$ as an "area element" of the plane. So his aim was to show that his area element transformed into a new "area element" $\left|\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}\right|dt\,dv$ under the given change of variables. Obviously, if we merely change coordinates by a translation, rotation, and/or reflection, the area element is transformed into a congruent one. So Euler noted that if t and v are new orthogonal coordinates related to x and y by

Leonhard Euler 1707–1783

a translation through constants a and b, a clockwise rotation through the angle θ whose cosine is m, and a reflection through the x-axis, i.e.,

$$x = a + mt + v\sqrt{1-m^2}$$

$$y = b + t\sqrt{1-m^2} - mv,$$

then $dx\,dy$ should be equal to $dt\,dv$. Unfortunately, when he performed the obvious formal calculation

$$dx = m\,dt + dv\sqrt{1-m^2},$$

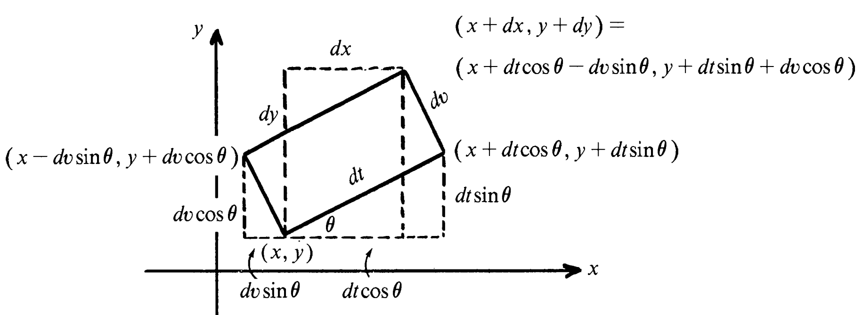
$$dy = dt\sqrt{1-m^2} - m\,dv$$

and multiplied the two equations, he arrived at

$$dx\,dy = m\sqrt{1-m^2}\,dt^2 + (1-2m^2)\,dt\,dv - m\sqrt{1-m^2}\,dv^2,$$

which, he noted, was obviously wrong and even meaningless (see FIGURE 1). Even more so, then, would a similar calculation be wrong if t and v were related to x and y by more complicated transformations. It was thus necessary for Euler to develop a workable method; i.e., one that in the above situation gives $dx\,dy = dt\,dv$ and, in general, gives $dx\,dy = Z\,dt\,dv$, where Z is a function of t and v.

To see how he arrived at his method, we must first consider his definition and calculation of double integrals. After noting that $\int\int Z\,dx\,dy$ means an "indefinite" double integral, i.e., a function of x and y which when differentiated first with respect to x and with respect to y gives $Z\,dx\,dy$, Euler proceeded to calculate "definite" integrals over specified planar regions **A** in the way familiar to calculus students. Thus, he wrote the integral as $\int dx \int Z\,dy$ and holding x constant, he integrated with respect to y between the functions $y = f_1(x)$ and $y = f_2(x)$ which bounded the region **A**; finally he integrated with respect to x between its minimum and maximum values in **A**. He interpreted this integral in the obvious way as a volume. In particular, he integrated



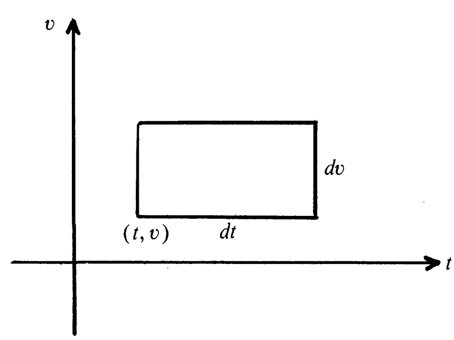


FIGURE 1

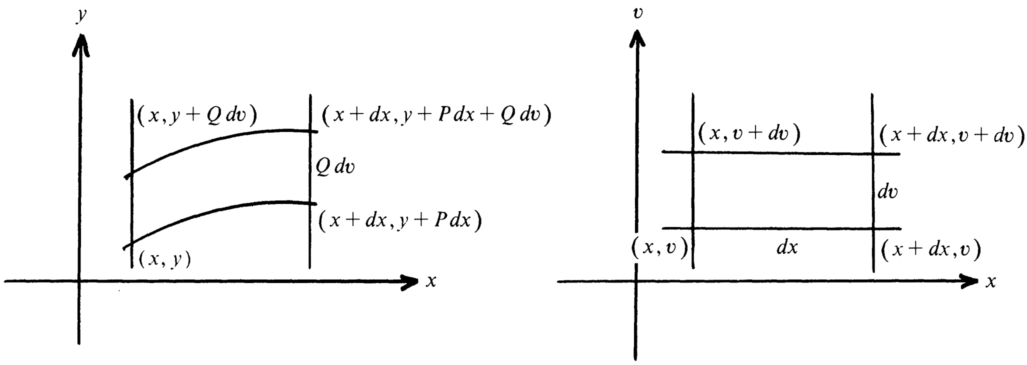


FIGURE 2

$\int\int\sqrt{c^2-x^2-y^2}\,dx\,dy$ over various regions to calculate volumes of portions of a sphere. Finally, he noted that $\int\int_{\mathbf{A}} dx\,dy$ is precisely the area of $\mathbf{A}$ and explicitly calculated the area of the circle given by $(x-a)^2+(y-b)^2=c^2$ to be πc^2.

Since the method of double integration involves leaving one variable fixed while dealing with the other, Euler proposed a similar method for the change-of-variable problem: change variables one at a time. First he introduced the new variable v and assumed that y could be represented as a function of x and v. So $dy = P\,dx + Q\,dv$ where P and Q are the appropriate partial derivatives. Now by assuming x fixed, he obtained $dy = Q\,dv$ and $\int\int dx\,dy = \int\int Q\,dx\,dv = \int dv \int Q\,dx$ (FIGURE 2). Next, he let x be a function of t and v and put $dx = R\,dt + S\,dv$. So by holding v constant, he calculated $\int dv \int Q\,dx = \int dv \int QR\,dt = \int\int QR\,dt\,dv$. This gave Euler the first solution to his problem: $dx\,dy = QR\,dt\,dv$.

Obviously, this was not completely satisfactory, since Q may well depend on x, and, in addition, the method was not symmetric. So Euler continued, now representing y as a function of t and v, hence $dy = T\,dt + V\,dv$. Then, formally, $dy = P\,dx + Q\,dv = P(R\,dt + S\,dv) + Q\,dv = PR\,dt + (PS+Q)\,dv$. So $PR = T$ and $PS + Q = V$, which gives $QR = VR - ST$. Euler's final answer was that $dx\,dy = (VR-ST)\,dt\,dv$. He noted again that simply multiplying the expressions for dx and dy together and rejecting the terms in dt^2 and dv^2 gives $(RV+ST)\,dt\,dv$, which differs by a sign from the correct answer. After a further note that one must always take the absolute value of the expression $VR-ST$ (since area is positive) he proceeded to confirm the correctness of his result through several increasingly complex examples.

This "proof" was typical of Euler's use of formal methods in many parts of his vast mathematical work. As a developer of algorithms to solve problems of various sorts, Euler has never been surpassed. (We can see that Euler's method, in modern notation, amounts to first factoring the transformation $x = x(t,v), y = y(t,v)$ into two transformations, the first being $x = x(t,v), v = v$ and the second $x = x, y = y(x,v)$. This can be done by "solving" $x = x(t,v)$ for t in the form $t = h(x,v)$ and then writing $y = y(h(x,v),v)$. Then $P = y_1\frac{\partial h}{\partial x}, Q = y_1\frac{\partial h}{\partial v} + y_2$, $R = x_1, S = x_2, T = y_1$ and $V = y_2$, where subscripts denote partial derivatives. Since $x(h(x,v),v) = x$ and $h(x(t,v),v) = t$, we calculate that $\frac{\partial h}{\partial x}x_1 = 1$ and $\frac{\partial h}{\partial x}x_2 + \frac{\partial h}{\partial v} = 0$, so $PR = T$ and $PS + Q = V$.)

In 1773 J. L. Lagrange also had need of a change-of-variable formula—this time for triple integrals [**12**]. He was interested in determining the attraction which an elliptical spheroid exercised on any point placed on its surface or in the interior. Since the general expression for attraction at any point was well known, the difficulty lay in integrating over the entire body. Even though the problem had already been solved geometrically, Lagrange, as part of his general philosophy of treating mathematics analytically, attempted a different solution.

To solve his problem, Lagrange had to calculate a triple integral. Since, following Euler's

Joseph-Louis Lagrange 1736–1813

method, this had to be done by first holding two variables constant, integrating with respect to the third from one surface of the body to another, then evaluating the ensuing double integrals, he was quickly led to very complicated integrands. He realized that new coordinates were needed to replace the rectangular ones in order to make the integration tractable. Thus he proceeded to develop a general formula for changing variables in a triple integral. Lagrange's method was similar to Euler's in that he let vary only one variable at a time, but the details differed.

Given, then, x, y, and z as functions of new variables p, q, r, Lagrange wrote

$$\begin{aligned} dx &= A\,dp + B\,dq + C\,dr \\ dy &= D\,dp + E\,dq + F\,dr \\ dz &= G\,dp + H\,dq + I\,dr \end{aligned} \tag{2}$$

where $A, B, \ldots, I$ are, of course, the appropriate partial derivatives. His aim was to calculate the volume of the infinitesimal parallelepiped $dx\,dy\,dz$ (the "volume element") in terms of $dp\,dq\,dr$. To do this, he calculated each "difference" (i.e., edge of the parallelepiped) separately, regarding the other two variables as constant. First x and y are held constant; thus $dx = 0$ and $dy = 0$; the first two equations in (2) become

$$\begin{aligned} A\,dp + B\,dq + C\,dr &= 0 \\ D\,dp + E\,dq + F\,dr &= 0. \end{aligned}$$

Lagrange solved these two equations for dp and dq in terms of dr and substituted in the expression for dz in (2) to get

$$dz = \frac{G(BF - CE) + H(CD - AF) + I(AE - BD)}{AE - BD}\,dr.$$

Next, x and z are assumed constant and only y varies; so $dx = 0$ and $dz = 0$. It follows immediately that $dr = 0$ and $A\,dp + B\,dq = 0$; therefore, $dp = -(B/A)\,dq$ and

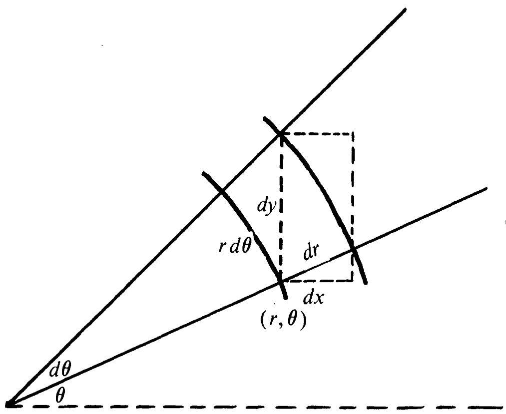
$$dy = \frac{AE - BD}{A}\,dq.$$

Finally, y and z are taken as constant, so $dy = 0$ and $dz = 0$. Thus $dr = 0$ and $dq = 0$, which implies that $dx = A\,dp$. By multiplying together the expressions obtained for dx, dy, and dz, Lagrange calculated his result:

$$dx\,dy\,dz = (AEI + BFG + CDH - AFH - BDI - CEG)\,dp\,dq\,dr. \tag{3}$$

This is, of course, our standard formula. The result for three-dimensional integrals is analogous to (1), and in modern notation, is written as

$$\iiint_{\mathbf{A}} f(x,y,z)\,dx\,dy\,dz = \iiint_{\mathbf{B}} f(x(p,q,r), y(p,q,r), z(p,q,r)) \left| \frac{\partial(x,y,z)}{\partial(p,q,r)} \right| dp\,dq\,dr$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$dx = 0 \Rightarrow dy = r\sec\theta\, d\theta$$
$$dy = 0 \Rightarrow dx = \cos\theta\, dr$$
$$dx\, dy = r\, dr\, d\theta$$

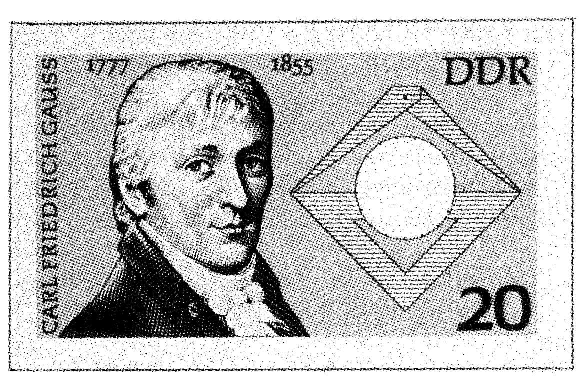
FIGURE 3

where $\dfrac{\partial(x,y,z)}{\partial(p,q,r)}$ is the functional determinant of x, y, z with respect to p, q, r. (FIGURE 3 illustrates Lagrange's idea for the case of two variables and polar coordinates.)

We note that Lagrange, like Euler, dealt with the differential forms formally; there is absolutely no infinitesimal approximation that we would require in a similar proof today. But this formalism is typical of some of Lagrange's other work, in particular, his attempt to develop the calculus without limits by the use of algebra and infinite series **[11]**, **[13]**. Also like Euler, Lagrange noted that the most obvious thing to do to try to obtain the change-of-variable formula would be to multiply together the original expressions (2) for dx, dy, and dz. However, he wrote, this product would contain squares and cubes of dp, dq, and dr and so would not be valid in an expression of a triple integral. Hence he had to use the step-by-step formal approach already outlined.

Lagrange applied his result to the case of spherical coordinates and was then able to perform the integrations he needed. Similarly, A. Legendre **[15]** and Pierre S. Laplace **[14]** soon after used essentially the same method to get similar results. These men were also interested in the change-of-variable formula in order to determine the attraction exercised by solids of various shapes, for which they needed to compute complicated integrals.

In 1813 Carl F. Gauss gave a geometric argument for a special case of the change-of-variable theorem for two variables, although in a somewhat different context **[8]**. Gauss' method of proof contrasts sharply with that of Euler. Gauss was developing the idea of a surface integral in connection with studying attractions. As part of this he gave a method for finding the element of surface in three-space so that he could integrate over such a surface. He started by parametrizing



Carl Friedrich Gauss 1777–1855

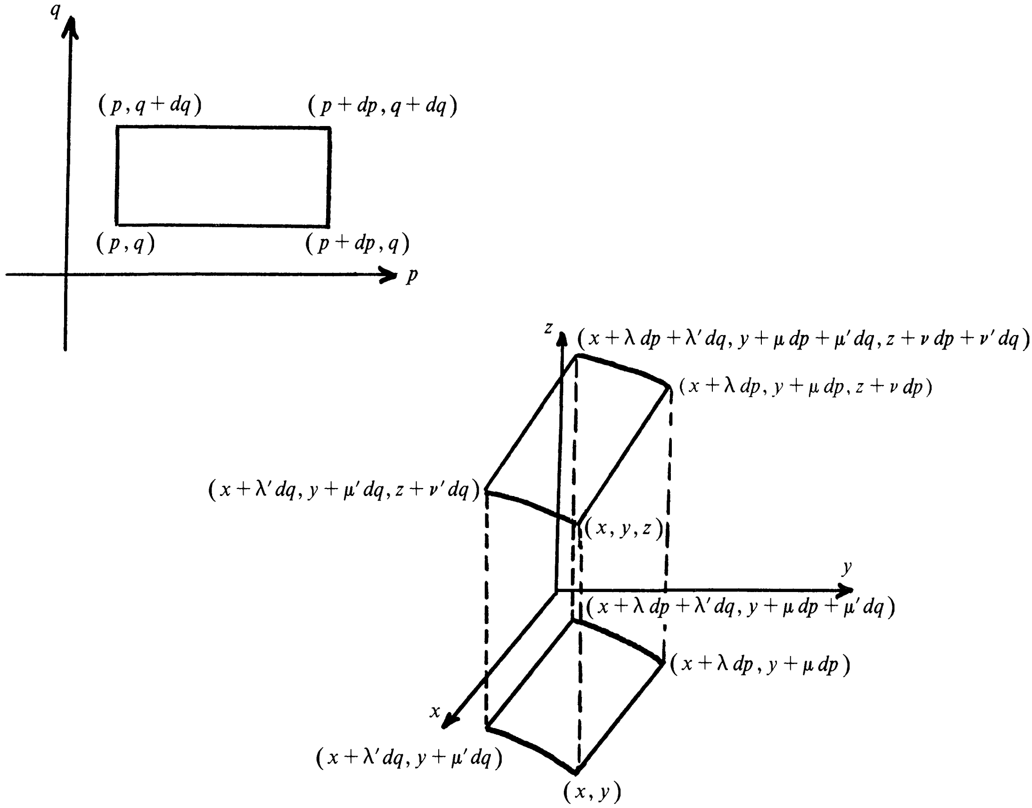


FIGURE 4

the surface using three functions x, y, z of the two variables p, q. He then noted that given an infinitesimal rectangle in the p-q plane whose vertices were (p, q), $(p+dp, q)$, $(p, q+dq)$, $(p+dp, q+dq)$, there was a corresponding "parallelogram" element in the surface whose vertices were (x, y, z), $(x+\lambda\, dp, y+\mu\, dp, z+\nu\, dp)$, $(x+\lambda' dq, y+\mu' dq, z+\nu' dq)$, and $(x+\lambda\, dp+\lambda' dq, y+\mu\, dp+\mu' dq, z+\nu\, dp+\nu' dq)$, where

$$\begin{aligned} dx &= \lambda\, dp + \lambda' dq \\ dy &= \mu\, dp + \mu' dq \\ dz &= \nu\, dp + \nu' dq. \end{aligned} \tag{4}$$

(One can easily calculate the above result from the definitions and properties of the relevant partial derivatives.) It follows that the projection of the infinitesimal parallelogram onto the x-y plane is the parallelogram whose vertices are (x, y), $(x+\lambda\, dp, y+\mu\, dp)$, $(x+\lambda' dq, y+\mu' dq)$, $(x+\lambda\, dp+\lambda' dq, y+\mu\, dp+\mu' dq)$ and whose area is clearly $\pm(\lambda\mu' - \mu\lambda')\, dp\, dq$. (See FIGURE 4.) Gauss was therefore able to compute the element of surface area as $dp\, dq((\mu\nu' - \nu\mu')^2(\nu\lambda' - \lambda\nu')^2(\lambda\mu' - \mu\lambda')^2)^{1/2}$ and thus to integrate this over the p-q region corresponding to his surface. (In this paper, Gauss used his special cases of the divergence theorem and his parametric method for calculating a surface element to evaluate certain "surface integrals" for the case of an ellipsoid given by $x = A\cos(p)$, $y = B\sin(p)\cos(q)$, $z = C\sin(p)\sin(q)$ for $0 \leqslant p \leqslant \pi$, $0 \leqslant q \leqslant 2\pi$.)

If we let $z = 0$ so that the "surface" is part of the x-y plane, then Gauss' argument shows that the new "area element" is $|\lambda\mu' - \mu\lambda'|\, dp\, dq$, hence that $\iint dx\, dy = \iint |\lambda\mu' - \mu\lambda'|\, dp\, dq$, a special case of the change-of-variable theorem from which the general case may easily be derived. Gauss' argument differs considerably from those of Euler and Lagrange. He essentially made use of analytic and geometric methods instead of using the formal approach of his predecessors. But as was typical of Gauss, he did not provide all the steps necessary to complete his analytic argument, especially since he was dealing with infinitesimals. The missing parts can, however, be readily supplied.

The next mathematician to break new ground in this field was Mikhail Ostrogradskii, in 1836. A Russian mathematician who studied in France in the 1820's, he later returned to St. Petersburg where he produced many works in applied mathematics. Unfortunately, some of his most important discoveries appear to have been totally ignored, at least in Western Europe. Not only did he give the first generalization of the change-of-variable theorem to n variables, but he also first proved and later generalized the divergence theorem [**10**], wrote integrals of n-forms over n-dimensional "hypersurfaces," and, as we shall see below, gave the first proof of the change-of-variable theorem for double integrals using infinitesimal concepts. All of these results were eventually repeated by other mathematicians with no credit to Ostrogradskii.

In his 1836 paper [**16**], Ostrogradskii generalized to n dimensions the change-of-variable theorem and Lagrange's proof of it. Given that $X, Y, Z, \ldots$ are all functions of $x, y, z \ldots$, Ostrogradskii first calculated $dX, dY, dZ, \ldots$ in terms of $dx, dy, dz, \ldots$. Then by holding all variables except X constant, he had $dY = dZ = \cdots = 0$, so he could solve for dX in terms of dx by using determinants; continuing with each variable in turn he calculated expressions for $dY, dZ, \ldots$ in terms of $dy, dz, \ldots$ and by multiplying showed that $dX\,dY\,dZ \ldots = \Delta\, dx\,dy\,dz \ldots$ where Δ is the functional determinant of $X, Y, Z, \ldots$ with respect to $x, y, z, \ldots$. Ostrogradskii did not state this result as a formula for transforming multiple integrals, but he did apply it to convert a hypersurface integral with $n+1$ terms of the form $dx\,dy \ldots$, to an ordinary n-dimensional integral in n new variables.

Both Carl Jacobi [**9**] and Eugene Catalan [**4**] published papers in 1841 giving clearly the general change-of-variable theorem for n-dimensional integrals. Catalan's proof was also similar to Lagrange's in its use of formal manipulations on one variable at a time. Jacobi's paper was the culmination of a series of articles concerning this theorem; it contained additional results such as the multiplication rule for the composition of several changes of variable. Jacobi's work was referred to shortly thereafter by Cauchy and soon his name became tied to the theorem. In fact, the functional determinant Δ is now known as the Jacobian rather than the "Ostrogradskian."

Two years after his 1836 paper, Ostrogradskii published in [**17**] a proof of the change-of-variable formula in two variables which used the same basic idea as had Gauss. He first criticized the proofs of Euler and Lagrange, and, by implication, his own earlier proof. He claimed that, assuming that x and y were functions of u and v, if one first used $dx = 0$ to solve for dy in terms of du (that is, to evaluate one side of the differential rectangle) one could not then assume that du would be 0 when one tried to evaluate dx by setting $dy = 0$ (to find the other side of the rectangle). In fact, he wrote, you would have to use a new set of differentials, δu and δv, in evaluating the other side, and, once you did that, you came up with an incorrect result.

Mikhail Ostrogradskii 1801–1861

So Ostrogradskii returned to the meaning of $\iint V dx\,dy$ as a sum of differential elements. Using a method similar to that of Gauss, although staying strictly in two dimensions, he proceeded to recalculate the area of these elements. He carefully chose each element to be bounded by two curves where u was constant and two curves where v was constant. If ω denotes the area of such an element, he noted that by the definition of the definite integral, $\iint V dx\,dy = \iint V\omega$. It is easy to calculate ω (see FIGURE 5) since the four vertices have coordinates (x, y), $\left(x + \frac{\partial x}{\partial u}du, y + \frac{\partial y}{\partial u}du\right)$, $\left(x + \frac{\partial x}{\partial v}dv, y + \frac{\partial y}{\partial v}dv\right)$, and $\left(x + \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, y + \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$. By elementary geometry, the area of this parallelogram is $\pm\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\,dv$ and so the integral formula becomes

$$\iint V dx\,dy = \pm\iint V\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\,dv.$$

Ostrogradskii further noted that this method could be easily extended to three dimensions but not more, since there is not a corresponding geometrical result in four dimensions. We must note, of course, that Ostrogradskii had not explicitly justified using the standard formula for the area of a parallelogram when, in fact, the area is actually that of a "curvilinear" parallelogram. However, it was common practice in that time (as we noted also about Gauss' proof), to ignore explicit arguments about infinitesimal approximation.

Only four years later, a proof similar to that of Ostrogradskii appeared in Augustus DeMorgan's text *Differential and Integral Calculus* [**6**], one of the first "analytic" textbooks to appear in English. It is doubtful that DeMorgan had read Ostrogradskii's work, for his approach is somewhat different; he was considering how to calculate a double integral over a plane region bounded by four curves, where the standard method of integrating, first with respect to one variable between two functions of the other and then with respect to the second between constant limits, will not work. But his method of attack, via the definition of the double integral as a limit, the division of the given region into subregions bounded by curves where u was constant and where v was constant, and the calculation of areas of curvilinear quadrilaterals, is very close to that of Ostrogradskii. DeMorgan went even further, however, to provide detailed reasoning as to why the errors of approximation—third order infinitesimals—may be safely ignored.

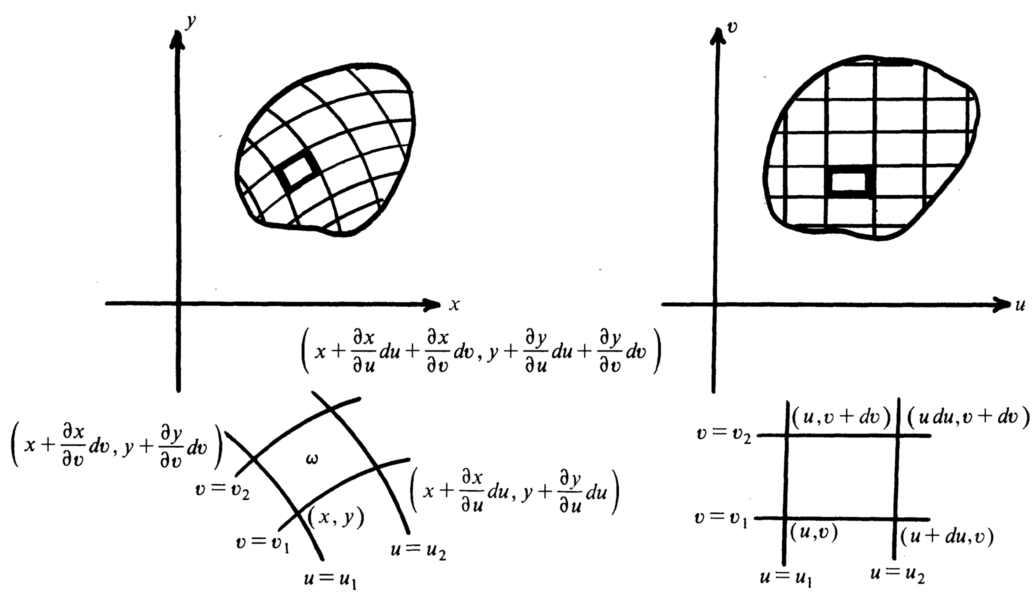


FIGURE 5

It is also interesting that DeMorgan prefaced his results by stating that Legendre's proof (which was identical to that of Lagrange) was "so obscure in its logic as to be nearly unintelligible, if not dubious."

Ostrogradskii and DeMorgan, then, had moved away from the formal symbolic approach of Euler and Lagrange. But we should emphasize that the former had not justified equating the "elements of area" $dx\,dy$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ themselves, as the latter had attempted to do. They had only showed the equality of the integrals over the appropriate regions. A new justification for the formal symbolic approach only came with Elie Cartan and his theory of differential forms.

Beginning in the mid 1890's, Cartan wrote a series of papers in which he formalized the subject of differential forms, namely the expressions which appear under the integral sign in line and surface integrals. As part of this formalization, he used the Grassmann rules of exterior algebra for calculations with such forms. In a paper of 1896 [**2**], as an example of such a calculation, he was able to do what Euler could not; namely, if $x=x(t,v)$ and $y=y(t,v)$, he could multiply $dx=\frac{\partial x}{\partial t}dt+\frac{\partial x}{\partial v}dv$ and $dy=\frac{\partial y}{\partial t}dt+\frac{\partial y}{\partial v}dv$ using the rules $dt\,dt=dv\,dv=0$ and $dt\,dv=-dv\,dt$ to show that

$$dx\,dy=\left(\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial t}\right)dt\,dv.$$

In 1899 [**3**], Cartan went into much more detail on the rules for operating with these differential forms. And again, one of his first examples was the change-of-variable formula.

As a final point, we note that proofs using the methods of Euler, Lagrange, and Ostrogradskii all appeared in textbooks through the first third of the twentieth century. There were, naturally, attempts to make all three methods more rigorous. A readily available example of this (for the proofs of Euler and Ostrogradskii) occurs in Courant's *Differential and Integral Calculus* [**5**]. Most current textbooks, on the other hand, use an entirely different proof based on Green's theorem.

References

[1] R. C. Buck, Advanced Calculus, 2nd ed., McGraw-Hill, New York, 1965, p. 296 ff.

[2] E. Cartan, Le principe de dualité et certaines intégrales multiples de l'espace tangentiel et de l'espace réglé, Bull. Soc. Math. France, 24 (1896) 140–177; Oeuvres (2) 1, 265–302.

[3] ______, Sur certaines expressions différentielles et sur le problème de Pfaff, Annales École Normale, 16 (1899) 239–332; Oeuvres (2) 1, 303–397.

[4] E. Catalan, Sur la transformation des variables dans les intégrales multiples, Mémoires couronnés par l'Académie Royale des Sciences et Belles-Lettres de Bruxelles, 14 (1841) 1–48.

[5] R. Courant, Differential and Integral Calculus, vol. 2, Interscience, New York, 1959, p. 247 ff.

[6] A. DeMorgan, The Differential and Integral Calculus, Robert Baldwin, London, 1842.

[7] L. Euler, De formulis integralibus duplicatis, Novi comm. acad. scient. Petropolitanae, 14 (1769) 72–103; Opera (1) 17, 289–315.

[8] C. Gauss, Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractata, Comm. soc. reg. scient. Gottingensis, 2 (1813): Werke 5, 1–22.

[9] C. Jacobi, De determinantibus functionalibus, Crelle, 22 (1841) 319–359; Werke 3, 393–438.

[10] V. Katz, The history of Stokes' theorem, this MAGAZINE, 52 (1979) 146–156.

[11] J. Lagrange, Théorie des Fonctions Analytiques, Paris, 1797; Oeuvres 9.

[12] ______, Sur l'attraction des sphéroïdes elliptiques, Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin, (1773); Oeuvres 3, 619–658.

[13] ______, Leçons sur le Calcul des Fonctions, Paris, 1801.

[14] P. Laplace, Traité de Mécanique Céleste, Paris, 1799; Oeuvres 2.

[15] A. Legendre, Mémoire sur les intégrales doubles, Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique, (1788) 454–486.

[16] M. Ostrogradskii, Mémoire sur le calcul des variations des intégrales multiples, Mémoires de l'Académie Impériale des Sciences de St. Petersbourg, (6) 3 (part 1) (1836), 36–58; also in Crelle, 15 (1836) 332–354.

[17] ______, Sur la transformation des variables dans les intégrales multiples, Mémoires de l'Académie Impériale des Sciences de St. Petersbourg, cl. math., (1838) 401–407.

[18] A. Taylor and W. R. Mann, Advanced Calculus, 2nd ed., Xerox, Lexington, 1972, p. 490 ff.