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The Cartan covering and complete integrability of the KdV–mKdV system

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ABSTRACT

The coupled KdV–mKdV system arises as the classical part of one of superextensions of the KdV equation. For this system, we prove its complete integrability, i.e., existence of a recursion operator and of infinite series of symmetries. After giving a short introduction into the theory of symmetries, coverings, and the notion of Cartan-covering, the recursion operator will be constructed as a symmetry in the Cartan covering of the KdV–mKdV system.

INTRODUCTION

There are several supersymmetric extensions of the classical Korteweg–de Vries equation (KdV) [6,9,10]. One of them is of the form (the so-called $N = 2$, $A = 1$ extension [2])

$$\begin{aligned} u_t &= -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1 \\ &\quad + 6\psi\varphi_1 w - 6\varphi\psi_1 w - 6\varphi\psi w_1, \\ \varphi_t &= -\varphi_3 + 3\varphi u_1 + 3\varphi_1 u - 3\psi_2 w - 3\psi_1 w_1 + 3\varphi_1 w^2 + 6\varphi w w_1, \\ \psi_t &= -\psi_3 + 3\psi u_1 + 3\psi_1 u + 3\varphi_2 w + 3\varphi_1 w_1 + 3\psi_1 w^2 + 6\psi w w_1, \\ w_t &= -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w, \end{aligned}$$

where u and w are classical (even) independent variables while φ and ψ are odd ones (here and below the numerical subscript at an unknown variable denotes its

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derivative over x of the corresponding order). Being completely integrable itself, this system gives rise to an interesting system of even equations

$$(1)\qquad \begin{aligned} u_t &= -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1, \\ w_t &= -w_3 + 3w^2w_1 + 3uw_1 + 3u_1w, \end{aligned}$$

which can be considered as a sort of coupling between the KdV (with respect to u) and the modified KdV (with respect to w) equations. In fact, setting $w = 0$, we obtain

$$u_t = -u_3 + 6uu_1,$$

while for $u = 0$ we have

$$w_t = -w_3 + 3w^2w_1.$$

The above indicates why we call (1) the KdV–mKdV system.

In what follows, we prove complete integrability, cf. [1], of system (1) by establishing existence of infinite series of symmetries and/or conservation laws. Toward this end we construct a recursion operator using the techniques of deformation theory introduced in [5] and extensively described and exemplified in [6].

In practical situations the construction of a deformation of the equation structure boils down to the construction of symmetries in an augmented setting of the equation or system at hand.

In Section 1 of this lecture we shall set out the nonlocal setting for differential equations and describe the notion of nonlocal symmetries in this setting.

Section 2 deals with a particular type of nonlocality, the Cartan covering of an equation.

Section 3 combines the two previous types of coverings, and it is this covering where the recursion operator for symmetries is obtained as a symmetry in this covering.

In these first three sections the classical KdV equations acts as the main example. Finally in Section 4 we shall present symmetries, conservation laws, nonlocalities and the recursion operator for symmetries for the coupled KdV–mKdV system (1).

1. Nonlocal setting for differential equations

As standard example, to illustrate the notions, we take KdV-equation

$$(2)\qquad u_t = uu_x + u_{xxx}.$$

We consider $Y \subset J^\infty(x,t;u)$ the infinite prolongation of (2), cf. [7,8], where coordinates in the infinite jet bundle $J^\infty(x,t;u)$ are given by $(x, t, u, u_x, u_t, \ldots)$ and Y is formally described as the submanifold of $J^\infty(x,t;u)$ defined by

$$
\begin{aligned}
u_t &= uu_x + u_{xxx},\\
u_{xt} &= uu_{xx} + u_x^2 + u_{xxxx},\\
&\vdots
\end{aligned}
\tag{3}
$$

As internal coordinates in Y one chooses $(x, t, u, u_x, u_{xx}, \ldots)$ while $u_t, u_{xt}, \ldots$ are obtained from (3).

The Cartan distribution on Y is given by the total partial derivative vector fields

$$
\begin{aligned}
\widetilde{D}_x &= \partial_x + \sum_{n\geqslant 0} u_{n+1}\partial_{u_n},\\
\widetilde{D}_t &= \partial_t + \sum_{n\geqslant 0} u_{nt}\partial_{u_n},
\end{aligned}
\tag{4}
$$

where $u_1 = u_x, u_2 = u_{xx}, \ldots; u_{1t} = u_{xt}; u_{2t} = u_{xxt}, \ldots.$

Classically the notion of a *generalized* or *higher symmetry* of a differential equation $F = 0$ is defined as a vertical vector field V

$$
V = \ni_f = f\partial_u + \widetilde{D}_x(f)\partial_{u_1} + \widetilde{D}_x^2(f)\partial_{u_2} + \cdots,
\tag{5}
$$

where $f \in C^\infty(Y)$ such that,

$$
\ell_F(f) = 0,
\tag{6}
$$

where in (6) ℓ_F is the *universal linearisation operator* [11,7] which reads in the case of KdV-equation (3)

$$
\widetilde{D}_t(f) - \widetilde{D}_x(f) - u_1\cdot f - (\widetilde{D}_x)^3(f) = 0.
\tag{7}
$$

Let now $W \subset \mathbb{R}^m$ with coordinates $(w_1, \ldots, w_m)$.

The Cartan distribution on $Y \otimes W$ is given by

$$
\begin{aligned}
\overline{D}_x &= \widetilde{D}_x + \sum_{j=1}^{m} X^j \frac{\partial}{\partial w_j},\\
\overline{D}_t &= \widetilde{D}_t + \sum_{j=1}^{m} T^j \partial_{w_j},
\end{aligned}
\tag{8}
$$

where $X^j, T^j \in C^\infty(Y \otimes W)$ such that

$$
[\overline{D}_x, \overline{D}_t] = 0
\tag{9}
$$

which yields the so-called *covering condition*

$$
D_x(T) - D_t(X) + [X, T] = 0
$$

whereas in (9) [*,*] is the Lie bracket for vector fields $X = \sum_{j=1}^{m} X^j \partial_{w_j}$, $T = \sum_{j=1}^{m} T^j \partial_{w_j}$ defined on W.

A *nonlocal symmetry* is a vertical vector field on $Y \otimes W$, i.e., of the form (5), which satisfies ($f \in C^\infty(Y \otimes W)$)

$$\bar{\ell}_F(f) = 0 \tag{10}$$

which for KdV results in

$$\overline{D}_t(f) - u\overline{D}_x(f) - u_1 f - (\overline{D}_x)^3(f) = 0. \tag{11}$$

Formally this is just what is called the *shadow* of the symmetry, i.e., not bothering about the ∂_{w^j}, $j = 1, \dots, m$, components.

The construction of the associated ∂_{w^j}, $j = 1, \dots, m$, components is called the *reconstruction problem* [4]. For reasons of simplicity, we omit this reconstruction problem, i.e., reconstructing the vector field from its shadow.

The classical Lenard recursion operator $\mathcal{R}$ for KdV equation,

$$\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1} \tag{12}$$

which is just such, that

$$\begin{aligned} &f_0 = u_1, \\ &\mathcal{R}f_0 = f_1 = uu_1 + u_3, \\ &\mathcal{R}f_1 = f_2 = u_5 + \frac{5}{3}u_3 u + \frac{10}{3}u_2 u_1 + \frac{5}{6}u_1 u^2, \end{aligned} \tag{13}$$

i.e., creating the (x, t)-independent hierarchy of higher symmetries, has an action on vertical symmetry $\ni_{\bar{f}_{-1}}$ (Gallilei-boost)

$$\begin{aligned} &\bar{f}_{-1} = (1 + tu_1)/3, \\ &\mathcal{R}\bar{f}_{-1} = \bar{f}_0 = 2u + xu_1 + 3t(u_3 + uu_1), \\ &\bar{f}_1 = \mathcal{R}\bar{f}_0 = 3t(f_2) + x(f_0) + 4u_2 + \frac{4}{3}u^2 + \frac{1}{3}u_1 D_x^{-1}(u). \end{aligned} \tag{14}$$

If we introduce the variable $p(= w_1)$ through

$$\begin{aligned} &p_x = u, \\ &p_t = u_2 + \frac{1}{2}u^2, \\ &\text{i.e.,} \quad D_t(u) = D_x\left(u_2 + \frac{1}{2}u^2\right) \end{aligned} \tag{15}$$

then $\ni_{\bar{f}_1}$ is the shadow of a nonlocal symmetry in the one-dimensional covering of KdV-equation by

$$p = w_1, \qquad X_1 = u, \qquad T_1 = u_2 + \frac{1}{2}u^2.$$

So, by its action the Lenard recursion operator creates nonlocal symmetries in a natural way.

More applications of nonlocal symmetries can be found in, e.g., [6].

2. A special type of covering: The Cartan-covering

We discuss a special type of the nonlocal setting indicated in the previous section, the so-called Cartan-covering. As mentioned before we shall illustrate this by the KdV-equation.

Let $Y \subset J^\infty(x, t; u)$ be the infinite prolongation of KdV-equation (3). Contact one forms on $TJ^\infty(x, t; u)$ are given by

$$\begin{aligned} \alpha_0 &= du - u_1\, dx - u_t\, dt, \\ \alpha_1 &= du_1 - u_2\, dx - u_{1t}\, dt, \\ \alpha_2 &= du_2 - u_3 dx - u_{2t} dt. \end{aligned} \tag{16}$$

From the total partial derivative operators of the previous section we have

$$\begin{aligned} &\widetilde{D}_x(\alpha_1) = \alpha_1, \qquad \widetilde{D}_x(\alpha_1) = \alpha_2,\ \ldots, \\ &\widetilde{D}_t(\alpha_0) = \alpha_0 u_x + \alpha_1 u + \alpha_3 = \alpha_t, \\ &\widetilde{D}_t(\alpha_i) = (\widetilde{D}_x)^i(\alpha_t). \end{aligned} \tag{17}$$

We now define the Cartan-covering of Y by $Y \otimes \mathbb{R}^\infty$, where local coordinates are given $(x, t, u, u_1, \ldots, \alpha_0, \alpha_1, \ldots)$ by

$$\begin{aligned} D_x^C &= \widetilde{D}_x + \sum_i (\alpha_{i+1}) \frac{\partial}{\partial \alpha_i}, \\ D_t^C &= \widetilde{D}_t + \sum_i (\widetilde{D}_x)^i \alpha_t \frac{\partial}{\partial \alpha_i}. \end{aligned} \tag{18}$$

It is a straightforward check, and obvious that

$$\left[D_x^C, D_t^C\right] = 0, \tag{19}$$

i.e., they form a Cartan distribution on $Y \otimes \mathbb{R}^\infty$.

Note 1. Since at first α_i $(i = 0, \ldots)$ are contact forms, they constitute a Grassmann algebra (graded commutative algebra) $\Lambda(\alpha)$, where

$$\alpha_i \wedge \alpha_j = -\alpha_j \wedge \alpha_i,$$

i.e.,

$$xy = (-1)^{|x||y|} yx,$$

where x, y are contact $(*)$-forms of degree $|x|$ and $|y|$ respectively. So in effect we are dealing with a *graded* covering.

Note 2. Once we have introduced the Cartan-covering by (18) we can forget about the specifics of α_i $(i = 0, \ldots)$ and just treat them as (odd) ordinary variables, associated with their differentiation rules.

One can discuss nonlocal symmetries in this type of covering just as in the previous section, the only difference being:

$$f \in V^{\infty}(Y) \otimes \bigwedge(\alpha).$$

In the next section we shall combine constructions of the previous section and this section, in order to construct the recursion operator for symmetries.

3. The recursion operator as symmetry in the Cartan-covering

We shall discuss the recursion operator for symmetries of KdV-equation as a geometrical object, i.e., a symmetry in the Cartan-covering.

Our starting point is the four-dimensional covering of the KdV-equation in $Y \otimes \mathbb{R}^4$ where

$$\begin{aligned} \overline{D}_x &= D_x + u\partial_{w_1} + \frac{1}{2}u^2\partial_{w_2} + \left(u^3 - 3u_1^2\right)\partial_{w_3} + w_1\partial_{w_4}, \\ \overline{D}_t &= D_t + \left(\frac{1}{2}u^2 + u_2\right)\partial_{w_1} + \left(\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2\right)\partial_{w_2} \\ &\quad + \left(\frac{3}{4}u^4 - 6u_1u_3 + 3u^2u_2 - 6uu_1^2 + 3u_2^2\right)\partial_{w_3} + (u_1 + w_2)\partial_{w_4}, \end{aligned} \tag{20}$$

$\overline{D}_x, \overline{D}_t$ satisfy the covering condition (9), and note that due to the fact that the coefficients of ∂_{w_i} $(i = 1, 2, 3)$ in (20) are independent of w_j $(j = 1, 2, 3)$. These coefficients constitute conservation laws for the KdV-equation. We have the following "formal" variables.

$$\begin{aligned} w_1 &= \int u\,dx, \\ w_2 &= \int \frac{1}{2}u^2\,dx, \\ w_3 &= \int \left(u^3 - 3u_1^2\right)dx, \\ w_4 &= \int w_1\,dx, \end{aligned} \tag{21}$$

where in (21) w_4 is of a higher nonlocality.

We now build the Cartan-covering of the previous section on the covering given by (20) by introduction of the contact forms $\alpha_0, \alpha_1, \alpha_2, \ldots$ (16) and

$$\begin{aligned}\alpha_{-1} &= dw_1 - u\,dx - \left(\frac{1}{2}u^2 + u_2\right)dt,\\ \alpha_{-2} &= dw_2 - \frac{1}{2}u^2\,dx - \left(\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2\right)dt\end{aligned} \tag{22}$$

and similarly for α_{-3}, α_{-4}. It is straightforward to prove the following relations

$$\begin{aligned}&\overline{D}_x(\alpha_{-1}) = \alpha_0, \qquad \overline{D}_t(\alpha_{-1}) = u\alpha_0 + \alpha_0,\\ &\overline{D}_x(\alpha_{-2}) = u\alpha_0, \qquad \overline{D}_t(\alpha_{-2}) = u^2\alpha_0 - u_1\alpha_1 + u\alpha_2 + u_2\alpha_0,\\ &\overline{D}_x(\alpha_{-3}) = 3u^2\alpha_0 - 6u_1\alpha_1, \ \ldots.\end{aligned} \tag{23}$$

We are now constructing symmetries in this Cartan-covering of KdV-equation which are linear w.r.t. α_i $(i = -4, \ldots, 0, 1, \ldots)$.

The symmetry condition for $f \in C^\infty(Y \otimes \mathbb{R}^4) \otimes \Lambda^1(\alpha)$ is just given by (7)

$$\bar{\ell}_F^C(f) = 0, \tag{24}$$

which for the KdV equation results in

$$\overline{D}_t^C(f) - u\overline{D}_x^C(f) - u_x f - \left(\overline{D}_x^C\right)^3 f = 0.$$

As solutions of these equations we obtained

$$\begin{aligned}f^0 &= \alpha_0,\\ f^1 &= \left(\frac{2}{3}u\right)\alpha_0 + \alpha_2 + \left(\frac{1}{3}u_1\right)\alpha_{-1},\\ f^2 &= \left(\frac{4}{9}u^2 + \frac{4}{3}u_2\right)\alpha_0 + (2u_1)\alpha_1 + \left(\frac{4}{3}u\right)\alpha_2 + \alpha_4\\ &\quad + \frac{1}{3}(uu_1 + u_3)\alpha_{-1} + \frac{1}{9}(u_1)\alpha_{-2}.\end{aligned}$$

As we mentioned above we are working in effect with form-valued vector fields $\ni_{f^0}, \ni_{f^1}, \ni_{f^2}$. For these objects one can define Frölicher–Nijenhuis and (by contraction) Richardson–Nijenhuis brackets [5,6]. Without going into details, for which the reader is referred to [5], we can construct the contraction of a (generalized) symmetry and a form valued symmetry p.e.

$$R = \left(\frac{2}{3}u\alpha_0 + \alpha_2 + \frac{1}{3}u_1\alpha_{-1}\right)\frac{\partial}{\partial u} + \cdots. \tag{25}$$

The contraction being defined by

$$(V_1 \lrcorner R) = (V \lrcorner R_u)\partial_u + \overline{D}_x^C(V \lrcorner R_u)\partial_{u_1} + \cdots. \tag{26}$$

Start now with

$$V_1 = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \cdots \tag{27}$$

whose prolongation in the setting $Y \otimes \mathbb{R}^4$ is

$$\overline{V}_1 = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots + u \frac{\partial}{\partial w_1} + \frac{1}{2} u^2 \frac{\partial}{\partial w_2} + \left(u^3 - 3u_1^2\right) \frac{\partial}{\partial w_3} + w_1 \frac{\partial}{\partial w_4} \tag{28}$$

then

$$(\overline{V} \lrcorner R) = \left[\left(\frac{2}{3} u\right) u_1 + 1 \cdot u_3 + \frac{1}{3} u_1 \cdot u\right] \frac{\partial}{\partial u} + \cdots = (u_3 + u u_1) \frac{\partial}{\partial u} + \cdots = V_3 \tag{29}$$

and similarly

$$(\overline{V}_3 \lrcorner R) = \left(u_5 + \frac{5}{3} u_3 u + \frac{10}{3} u_2 u_1 + \frac{5}{6} u^2 u_1\right) \frac{\partial}{\partial u} + \cdots = V_5. \tag{30}$$

The result given above means that the well known Lenard recursion operator for symmetries of KdV-equation is represented as a *symmetry*, $\ni_{f_1}$, in the Cartan-covering of this equation and in effect is a geometrical object.

4. The coupled KdV–mKdV system

In this section we shall discuss the complete integrability of the KdV–mKdV system $\mathcal{E}$, given in (1), i.e.,

$$\begin{aligned} u_t &= -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1, \\ w_t &= -w_3 + 3w^2w_1 + 3uw_1 + 3u_1w. \end{aligned} \tag{31}$$

In order to demonstrate the complete integrability of this system, we shall construct the recursion operator for symmetries of this coupled system, leading to infinite hierarchies of symmetries and, most probably, of conservation laws.

Due to the very special form of the final results, it seems that integrability of this system, which looks at first glance quite ordinary, has not been discussed before. In order to discuss complete integrability, we shall start to discuss conservation laws in Subsection 4.1 leading to the necessary nonlocal variables.

In Subsection 4.2 we shall discuss local and nonlocal symmetries of the system, while in Subsection 4.3 we construct the recursion operator or deformation [5], by the construction of a symmetry in the Cartan covering of Eq. (31).

4.1. Conservation laws and nonlocal variables

Here we shall construct conservation laws for (31) in order to arrive at an Abelian covering of the coupled KdV–mKdV system as was shown KdV equation (2).

So we construct $X = X(x, t, u, \ldots, w \ldots)$, $T = T(x, t, u, \ldots, w \ldots)$ such that

(32) $$D_x(T) = D_t(X),$$

where D_x, D_t are defined as the total partial derivative operators on the infinite jetbundle associated to Eq. (31) and in a similar way we construct nonlocal conservation laws by the requirement

(33) $$\overline{D}_x(\overline{T}) = \overline{D}_t(\overline{X}),$$

where $\overline{D}_*$ is defined as the prolongation of D_* towards the covering of the equation by nonlocal variables arising from local conservation laws; moreover $\overline{X}$, $\overline{T}$ are dependent on local variables x, t, $u, \dots, w, \dots$ as well as the already determined nonlocal variables, denoted here by p_* or $p_{*,*}$, which are associated to the conservation laws (X, T) by the formal definition

$$\begin{aligned} D_x(p_*) &= (p_*)_x = X, \\ D_t(p_*) &= (p_*)_t = T. \end{aligned}$$

Proceeding in this way, we obtained the following set of nonlocal variables

(34) $$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1},\ p_{3,2},\ p_{4,1},\ p_5,$$

where their defining equations are given by

$$\begin{aligned}
(p_1)_x &= u, \\
(p_1)_t &= 3u^2 + 3uw^2 - u_2 - 3ww_2, \\
(p_{0,1})_x &= w, \\
(p_{0,1})_t &= 3uw + w^3 - w_2, \\
(p_{0,2})_x &= p_1, \\
(p_{0,2})_t &= -6p_3 - u_1, \\
(p_{1,1})_x &= \cos(2p_{0,1})p_1 w + \sin(2p_{0,1})w^2, \\
(p_{1,1})_t &= \cos(2p_{0,1})\big(3p_1uw + p_1w^3 - p_1w_2 + uw_1 - u_1w - w^2w_1\big) \\
&\quad + \sin(2p_{0,1})\big(4uw^2 + w^4 - 2ww_2 + w_1^2\big), \\
(p_{1,2})_x &= \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1 w, \\
(p_{1,2})_t &= \cos(2p_{0,1})\big(4uw^2 + w^4 - 2ww_2 + w_1^2\big) \\
&\quad + \sin(2p_{0,1})\big(-3p_1uw - p_1w^3 + p_1w_2 - uw_1 + u_1w + w^2w_1\big), \\
(p_{2,1})_x &= \big(4\cos(2p_{0,1})p_{1,1}w^2 - 4\sin(2p_{0,1})p_1p_{1,1}w + w\big(p_1^2 - 2u + w^2\big)\big)/2, \\
(p_{2,1})_t &= \big(4\cos(2p_{0,1})p_{1,1}\big(4uw^2 + w^4 - 2ww_2 + w_1^2\big) \\
&\quad + 4\sin(2p_{0,1})p_{1,1}\big(-3p_1uw - p_1w^3 + p_1w_2 - uw_1 + u_1w + w^2w_1\big) \\
&\quad + 3p_1^2uw + p_1^2w^3 - p_1^2w_2 + 2p_1uw_1 - 2p_1u_1w - 2p_1w^2w_1 - 8u^2w \\
&\quad - uw^3 + 2uw_2 - 2u_1w_1 + 2u_2w + w^5 + 3w^2w_2\big)/2, \\
(p_3)_x &= \big(-u^2 - uw^2 + ww_2\big)/2,
\end{aligned}$$

$$
\begin{aligned}
(p_3)_t &= \big(-4u^3 - 9u^2w^2 + 2uu_2 - 3uw^4 + 11uww_2 - uw_1^2 - u_1^2 + u_1ww_1 \\
&\quad + 4u_2w^2 + 6w^3w_2 + 3w^2w_1^2 - ww_4 + w_1w_3 - w_2^2\big)/2, \\
(p_{3,1})_x &= \big(\cos(2p_{0,1})w\big(p_1^3 - 6p_1u + 39p_1w^2 - 24p_{1,1}p_{1,2}w + 12p_3 + 6u_1\big) \\
&\quad + 2\sin(2p_{0,1})w\big(12p_1p_{1,1}p_{1,2} + 18p_1w_1 + 2w^3 + 3w_2\big) \\
&\quad + 6p_{1,2}w\big(-p_1^2 + 2u - w^2\big)\big)/12, \\
(p_{3,2})_x &= \big(2\cos(2p_{0,1})w\big(12p_1p_{1,1}p_{1,2} - 18p_1w_1 - 2w^3 - 3w_2\big) \\
&\quad + \sin(2p_{0,1})w\big(p_1^3 - 6p_1u + 39p_1w^2 + 24p_{1,1}p_{1,2}w + 12p_3 + 6u_1\big) \\
&\quad + 6p_{1,1}w\big(-p_1^2 + 2u - w^2\big)\big)/12, \\
(p_{4,1})_x &= \big(8\cos(2p_{0,1})w\big(p_1^3p_{1,2} + 12p_1p_{1,1}^2p_{1,2} - 6p_1p_{1,2}u + 3p_1p_{1,2}w^2 \\
&\quad - 12p_{1,1}p_{1,2}^2w + 18p_{1,1}uw - 4p_{1,1}w^3 - 6p_{1,1}w_2 + 12p_{1,2}p_3 + 6p_{1,2}u_1\big) \\
&\quad + 8\sin(2p_{0,1})w\big(p_1^3p_{1,1} + 12p_1p_{1,1}p_{1,2}^2 - 6p_1p_{1,1}u + 3p_1p_{1,1}w^2 \\
&\quad + 12p_{1,1}^2p_{1,2}w + 12p_{1,1}p_3 + 6p_{1,1}u_1 - 18p_{1,2}uw + 4p_{1,2}w^3 + 6p_{1,2}w_2\big) \\
&\quad + w\big(-p_1^4 - 24p_1^2p_{1,1}^2 - 24p_1^2p_{1,2}^2 + 12p_1^2u - 6p_1^2w^2 - 48p_1p_3 \\
&\quad - 24p_1u_1 + 48p_{1,1}^2u - 24p_{1,1}^2w^2 + 48p_{1,2}^2u - 24p_{1,2}^2w^2 \\
&\quad - 60u^2 + 44uw^2 + 24u_2 - 13w^4 + 6ww_2\big)\big)/48, \\
(p_5)_x &= \big(12u^3 + 24u^2w^2 - 6uu_2 + 6uw^4 - 30uww_2 - 3u_2w^2 - 8w^3w_2 \\
&\quad + 6ww_4\big)/6.
\end{aligned}
$$

In the previous equations, we skipped explicit formulas for $(p_{3,1})_t$, $(p_{3,2})_t$, $(p_{4,1})_t$, and $(p_5)_t$, because they are too massive, though quite important for the setting to be well defined and in order to avoid ambiguities. The reader is referred to [3] for these explicit formulas.

It is quite a striking result that functions $\cos(2p_{0,1})$, $\sin(2p_{0,1})$ appear in the presentation of the conservation laws and their associated nonlocal variables.

We should note that p_1, $p_{0,1}$, p_3, p_5 arise from *local conservation laws* and we shall call p_1, $p_{0,1}$, p_3, p_5 *nonlocalities of first order*.

In a similar way we see that $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ arise from *nonlocal conservation laws*, where their x- and t-derivatives are dependent on the first-order nonlocalities. For this reason $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ are called *nonlocalities of second order*. Proceeding in this way $p_{2,1}$, $p_{3,1}$, $p_{3,2}$, $p_{4,1}$ constitute *nonlocalities of third order*.

4.2. Local and nonlocal symmetries

In this section we shall present results for the construction of local and nonlocal symmetries of system (31). In order to construct these symmetries, we consider the system of partial differential equations obtained by the infinite prolongation of (31) together with the covering by the nonlocal variables

$$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1},\ p_{3,2},\ p_{4,1},\ p_5.$$

So, in the augmented setting governed by (31), their total derivatives and the equations given in Subsection 4.1 we construct symmetries $Y = (Y^u, Y^w)$ which have to satisfy the symmetry condition

$$\bar{\ell}_{\mathcal{E}} Y = 0.$$

From this condition we obtained the following symmetries

$$Y_{0,1},\ Y_{1,1},\ Y_{1,2},\ Y_{1,3},\ Y_{2,1},\ Y_{3,1},\ Y_{3,2},\ Y_{3,3},$$

where generating functions $Y^u_{*,*}$, $Y^w_{*,*}$ are given as

$$\begin{aligned}
Y^u_{0,1} &= 3t\big(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2\big) + xu_1 + 2u,\\
Y^w_{0,1} &= 3t\big(3uw_1 + 3u_1w + 3w^2w_1 - w_3\big) + xw_1 + w,\\
Y^u_{1,1} &= u_1,\\
Y^w_{1,1} &= w_1,\\
Y^u_{1,2} &= \cos(2p_{0,1})(2uw - w_2) + \sin(2p_{0,1})(u_1 + 2ww_1),\\
Y^w_{1,2} &= -\cos(2p_{0,1})u - \sin(2p_{0,1})w_1,\\
Y^u_{1,3} &= \cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(-2uw + w_2),\\
Y^w_{1,3} &= -\cos(2p_{0,1})w_1 + \sin(2p_{0,1})u,\\
Y^u_{2,1} &= \big(2\cos(2p_{0,1})(p_{1,1}u_1 + 2p_{1,1}ww_1 - 2p_{1,2}uw + p_{1,2}w_2)\\
&\quad + 2\sin(2p_{0,1})(-2p_{1,1}uw + p_{1,1}w_2 - p_{1,2}u_1 - 2p_{1,2}ww_1)\\
&\quad + 2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3\big)/2,\\
Y^w_{2,1} &= \big(2\cos(2p_{0,1})(-p_{1,1}w_1 + p_{1,2}u) + 2\sin(2p_{0,1})(p_{1,1}u + p_{1,2}w_1)\\
&\quad - p_1u + u_1 + ww_1\big)/2,\\
Y^u_{3,1} &= \big(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2\big)/3,\\
Y^w_{3,1} &= \big(3uw_1 + 3u_1w + 3w^2w_1 - w_3\big)/3,\\
Y^u_{3,2} &= \big(\cos(2p_{0,1})\big(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w - 4p_1w^2w_1 + 2p_1w_3\\
&\quad + 8p_{1,1}p_{1,2}u_1 + 16p_{1,1}p_{1,2}ww_1 - 8p_{1,2}^2uw + 4p_{1,2}^2w_2 - 4p_{2,1}u_1\\
&\quad - 8p_{2,1}ww_1 + 10u^2w + 6uw^3 - 8uw_2 - 14u_1w_1 - 8u_2w - 11w^2w_2\\
&\quad - 14ww_1^2 + 2w_4\big) + 2\sin(2p_{0,1})\big(-8p_{1,1}p_{1,2}uw + 4p_{1,1}p_{1,2}w_2 - 2p_{1,2}^2u_1\\
&\quad - 4p_{1,2}^2ww_1 + 4p_{2,1}uw - 2p_{2,1}w_2 + 6uu_1 + 10uww_1 + 3u_1w^2 - u_3\\
&\quad + 2w^3w_1 - 3ww_3 - 5w_1w_2\big) + 4p_{1,2}\big(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w\\
&\quad + 2w^2w_1 - w_3\big)\big)/8,\\
Y^w_{3,2} &= \big(\cos(2p_{0,1})\big(p_1^2u - 2p_1u_1 - 2p_1ww_1 - 8p_{1,1}p_{1,2}w_1 + 4p_{1,2}^2u + 4p_{2,1}w_1\\
&\quad - 4u^2 - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2\big)\\
&\quad + 2\sin(2p_{0,1})\big(4p_{1,1}p_{1,2}u + 2p_{1,2}^2w_1 - 2p_{2,1}u - 3uw_1 - 3u_1w - 3w^2w_1\\
&\quad + w_3\big) + 4p_{1,2}(-p_1u + u_1 + ww_1)\big)/8,
\end{aligned}$$

$$
\begin{aligned}
Y^u_{3,3} = {} & \big(2\cos(2p_{0,1})\big(2p_{1,1}^2u_1 + 4p_{1,1}^2ww_1 - 4p_{2,1}uw + 2p_{2,1}w_2 - 6uu_1 \\
& - 10uww_1 - 3u_1w^2 + u_3 - 2w^3w_1 + 3ww_3 + 5w_1w_2\big) \\
& + \sin(2p_{0,1})\big(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w - 4p_1w^2w_1 + 2p_1w_3 \\
& - 8p_{1,1}^2uw + 4p_{1,1}^2w_2 - 4p_{2,1}u_1 - 8p_{2,1}ww_1 + 10u^2w + 6uw^3 \\
& - 8uw_2 - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14ww_1^2 + 2w_4\big) \\
& + 4p_{1,1}\big(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3\big)\big)/8, \\
Y^w_{3,3} = {} & \big(2\cos(2p_{0,1})\big(-2p_{1,1}^2w_1 + 2p_{2,1}u + 3uw_1 + 3u_1w + 3w^2w_1 - w_3\big) \\
& + \sin(2p_{0,1})\big(p_1^2u - 2p_1u_1 - 2p_1ww_1 + 4p_{1,1}^2u + 4p_{2,1}w_1 - 4u^2 \\
& - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2\big) + 4p_{1,1}(-p_1u + u_1 + ww_1)\big)/8.
\end{aligned}
$$

4.3. Recursion operator

Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of system of Eqs. (1) augmented by equations governing the nonlocal variables (34).

As demonstrated there, this symmetry is a form-valued vector field (or a vectorfield-valued one-form) and has to satisfy

$$\bar{\ell}^C_{\mathcal{E}}\mathcal{R} = 0. \tag{35}$$

In order to arrive at a nontrivial result as was explained for classical KdV equation (3), (25), we have to introduce nonlocal variables

$$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1},\ p_{3,2},\ p_{4,1},\ p_5$$

and their associated Cartan contact forms

$$\omega_{p_{0,1}},\ \omega_{p_{0,2}},\ \omega_{p_1},\ \omega_{p_{1,1}},\ \omega_{p_{1,2}},\ \omega_{p_{2,1}},\ \omega_{p_3},\ \omega_{p_{3,1}},\ \omega_{p_{3,2}},\ \omega_{p_{4,1}},\ \omega_{p_5}.$$

The final result, which is dependent on the nonlocal Cartan forms

$$\omega_{p_{0,1}},\ \omega_{p_1},\ \omega_{p_{1,1}},\ \omega_{p_{1,2}},$$

is given by

$$\mathcal{R} = R^u\frac{\partial}{\partial u} + R^w\frac{\partial}{\partial w} + \cdots, \tag{36}$$

where the components R^u, R^w are given by

(37)

$$
\begin{aligned}
R_u = {} & \omega_{u_2}(-1) + \omega_u\big(4u + w^2\big) + \omega_{w_2}(-2w) + \omega_{w_1}(-w_1) + \omega_w(3uw - 2w_2) \\
& + \omega_{p_{1,2}}\big(-\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2)\big) \\
& + \omega_{p_{1,1}}\big(\cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \omega_{p_1}(2u_1 + ww_1) + \omega_{p_{0,1}}\big(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3\big),\\
R_w &= \omega_{w_2}(-1) + \omega_w\big(2u + w^2\big) + \omega_u(2w)\\
&+ \omega_{p_{1,2}}\big(\cos(2p_{0,1})w_1 - \sin(2p_{0,1})u\big)\\
&+ \omega_{p_{1,1}}\big(\cos(2p_{0,1})u + \sin(2p_{0,1})w_1\big)\\
&+ \omega_{p_1}(w_1) + \omega_{p_{0,1}}(-p_1u + u_1 + ww_1).
\end{aligned}
$$

We shall now present this result in a more conventional form which appeals to expressions using operators of the form D_x and D_x^{-1}. In order to do this, we first split (37) into the so-called local part and nonlocal parts, consisting of terms associated to $\omega_{u_2}, \omega_u, \omega_{w_2}, \omega_{w_1}, \omega_w$ and those associated to $\omega_{p_{1,2}}, \omega_{p_{1,1}}, \omega_{p_1}, \omega_{p_{0,1}}$ respectively. The first part will account for D_x presentation, while the second one accounts for the D_x^{-1} part.

Due to the action of contraction $\ni_\varphi \lrcorner \mathcal{R}$, the local part is given by the following matrix operator:

$$
\begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix}.
$$

The nonlocal part will be split into parts associated to $\omega_{p_1}, \omega_{p_{0,1}}$ and $\omega_{p_{1,2}}, \omega_{p_{1,1}}$, respectively. The first one is given as

$$
\begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & (2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)D_x^{-1} \\ w_1D_x^{-1} & (-p_1u + u_1 + ww_1)D_x^{-1} \end{bmatrix}.
$$

To deal with the last part, let us introduce the notation:

$$
\begin{aligned}
A_1 &= \cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1),\\
A_2 &= \cos(2p_{0,1})u + \sin(2p_{0,1})w_1,\\
B_1 &= -\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2),\\
B_2 &= \cos(2p_{0,1})w_1 - \sin(2p_{0,1})u,
\end{aligned}
$$

being the coefficients at $\omega_{p_{1,1}}$ and $\omega_{p_{1,2}}$ in (37).

According to the presentations of $(p_{1,1})_x$ and $(p_{1,2})_x$, i.e.,

$$
\begin{aligned}
(p_{1,1})_x &= \cos(2p_{0,1})p_1w + \sin(2p_{0,1})w^2,\\
(p_{1,2})_x &= \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1w,
\end{aligned}
$$

we introduce their partial derivatives with respect to $p_{0,1}$, p_1, and w as

$$
\begin{aligned}
\alpha_1 &= -2p_1w\sin(2p_{0,1}) + 2w^2\cos(2p_{0,1}),\\
\alpha_2 &= w\cos(2p_{0,1}),\\
\alpha_3 &= p_1\cos(2p_{0,1}) + 2w\sin(2p_{0,1}),\\
\beta_1 &= -2w^2\sin(2p_{0,1}) - 2p_1w\cos(2p_{0,1}),
\end{aligned}
$$

$$\beta_2 = -w\sin(2p_{0,1}),$$
$$\beta_3 = 2w\cos(2p_{0,1}) - p_1\sin(2p_{0,1}).$$

From this we arrive in a straightforward way at the last nonlocal part of the recursion operator, i.e.,

$$\begin{bmatrix} A_1 D_x^{-1}\alpha_2 D_x^{-1} & A_1 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \\ A_2 D_x^{-1}\alpha_2 D_x^{-1} & A_2 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \end{bmatrix} + \begin{bmatrix} B_1 D_x^{-1}\beta_2 D_x^{-1} & B_1 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \\ B_2 D_x^{-1}\beta_2 D_x^{-1} & B_2 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \end{bmatrix}.$$

So, in the final form we obtain the recursion operator as

$$\begin{aligned} \mathcal{R} = {} & \begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1 D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix} \\ & + \begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & (2p_1 uw - p_1 w_2 + 2uw_1 + 3u_1 w + 2w^2 w_1 - w_3)D_x^{-1} \\ w_1 D_x^{-1} & (-p_1 u + u_1 + ww_1)D_x^{-1} \end{bmatrix} \\ & + \begin{bmatrix} A_1 D_x^{-1}\alpha_2 D_x^{-1} & A_1 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \\ A_2 D_x^{-1}\alpha_2 D_x^{-1} & A_2 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \end{bmatrix} \\ & + \begin{bmatrix} B_1 D_x^{-1}\beta_2 D_x^{-1} & B_1 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \\ B_2 D_x^{-1}\beta_2 D_x^{-1} & B_2 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \end{bmatrix}. \end{aligned}$$

5. Conclusion

We gave an outline of the theory of symmetries of differential equations, leading to the construction of recursion operators for symmetries of such equations. The extension of this theory to the nonlocal setting of differential equations is essential for getting nontrivial results. The theory has been applied to the construction of the recursion operator for symmetries for a coupled KdV–mKdV system, leading to a highly nonlocal result for this system. Moreover the appearance of nonpolynomial nonlocal terms in all results, e.g., conservation laws, symmetries and recursion operator is striking and reveals some unknown and intriguing underlying structure of the equations.Work on the construction of Bäcklund transformations for this system is in progress.

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