The general symmetry algebra structure of the underdetermined equation $u_x=(v_{xx})^2$

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In a recent paper, Anderson, Kamran, and Olver ["Interior, exterior, and generalized symmetries," preprint (1990)] obtained the first- and second-order generalized symmetry algebra for the system $u_x = (v_{xx})^2$, leading to the noncompact real form of the exceptional Lie algebra G_2. Here, the structure of the general higher-order symmetry algebra is obtained. Moreover, the Lie algebra G_2 is obtained as ordinary symmetry algebra of the associated first-order system. The general symmetry algebra for $u_x = f(u,v,v_x,...,)$ is established also.

I. INTRODUCTION AND GENERAL

In a recent paper Anderson, Kamran and Olver[1] discussed symmetries and first- and second-order generalized symmetries of a peculiar kind of equation, merely an "underdetermined system" of equations

$$u_x = v_{xx}^2, \tag{1.1}$$

where u, v are functions of x.

Due to the fact that the system (1.1) is underdetermined, one can raise the question of existence of generalized symmetries. As a result the authors obtained as a Lie algebra of second-order symmetries the noncompact real form of the exceptional Lie algebra G_2.

Symmetries can be described in a differential geometric way as vector fields V satisfying the conditions

$$\mathscr{L}_V I \subset D'I, \tag{1.2}$$

where I is a closed ideal of differential forms associated to the differential equation, $\mathscr{L}_V$ the Lie derivative by the vector field, and finally $D'I$ the prolongation of the ideal I upto some finite order.[2,3]

In Sec. II we discuss the results obtained by software developed to construct symmetries.[2] In Sec. III we give a derivation of the general Lie algebra structure of (1.1). In Sec. IV we discuss some special cases. Finally, in Sec. V, we give a beautiful and short derivation of the Lie algebra of generalized symmetries for a completely general "system"

$$u_x = f(u,v,v_x,v_{xx},...). \tag{1.3}$$

II. SYMMETRIES, 2nd- AND 4th-ORDER GENERALIZED SYMMETRIES

We shall present here the results obtained using the computer algebra package[2] to construct (generalized) symmetries of differential equations, applied to

$$u_x = (v_{xx})^2. \tag{2.1}$$

In Sec. II A we obtain point symmetries of (2.1), while in Sec. II B we derive 2nd-order generalized symmetries. In Sec. II C we prove that the generalized symmetries obtained in B are equivalent to ordinary point symmetries of an associated first-order system of differential equations. Finally, in Sec. II D we compute 63 4th-order generalized symmetries.

A. Ordinary infinitesimal symmetries

In order to compute ordinary symmetries for (2.1) we describe the differential equation in terms of an exterior differential system I of differential forms on $\mathbb{R}^5 = \{(x,u,v,v_1,v_2)\}$, $v_1 = v_x$, $v_2 = v_{xx}$,... . The ideal I is generated by the differential one-forms

$$\begin{aligned} \alpha_1 &= du - v_2^2\,dx, \\ \alpha_2 &= dv - v_1\,dx, \\ \alpha_3 &= dv_1 - v_2\,dx, \end{aligned} \tag{2.2a}$$

and the exterior derivatives

$$d\alpha_1, \quad d\alpha_2, \quad d\alpha_3. \tag{2.2b}$$

An infinitesimal symmetry (point symmetry) of the ideal I (2.2) or the differential equation (2.1) is a vector field V defined on $\mathbb{R}^5$, i.e.,

$$V = V^x\partial_x + V^u\partial_u + V^v\partial_v + V^{v_1}\partial_{v_1} + V^{v_2}\partial_{v_2} \tag{2.3}$$

where V^x, V^u, V^v are functions of x, u, v, such that

$$\mathscr{L}_V I \subset I. \tag{2.4}$$

In (2.4) $\mathscr{L}_V$ denotes the Lie derivative with respect to V. Condition (2.4) leads to an overdetermined system of partial differential equations for $V^x,...,V^{v_2}$ which can be solved in a straightforward way, leading to a six-dimensional Lie algebra generated by the vector fields

$$\begin{aligned} &V_1 = \partial_x, \quad V_4 = x\partial_x + \tfrac{3}{2}v\partial_v + \tfrac{1}{2}v_1\partial_{v_1} - \tfrac{1}{2}v_2\partial_{v_2}, \\ &V_2 = \partial_u, \quad V_5 = u\partial_u + \tfrac{1}{2}v\partial_v + \tfrac{1}{2}v_1\partial_{v_1} - \tfrac{1}{2}v_2\partial_{v_2}, \\ &V_3 = \partial_v, \quad V_6 = x\partial_v + \partial_{v_1}, \end{aligned} \tag{2.5}$$

whereas the Lie algebra structure is given by

$$\begin{array}{c|cccccc} i\downarrow \; [V_i,V_j] \; \overset{\to j}{} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & * & 0 & 0 & V_1 & 0 & V_3 \\ 2 & & * & 0 & 0 & V_2 & 0 \\ 3 & & & * & \tfrac{3}{2}V_3 & \tfrac{1}{2}V_3 & 0 \\ 4 & & & & * & 0 & -\tfrac{1}{2}V_6 \\ 5 & & & & & * & -\tfrac{1}{2}V_6 \\ 6 & & & & & & * \end{array} \quad . \tag{2.5a}$$

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B. Second-order generalized symmetries

Due to the choice we made in (2.1), (2.2) to consider (2.1) as a first-order differential equation expressing u in terms of v and not taking the description

$$v_{xx} = \pm\sqrt{u_x}, \tag{2.6}$$

a fact that is reflected in the choice of the set of algebraic variables, we are now searching for second-order (in v) generalized symmetries.

Due to the occurrence of equivalence classes of generalized symmetries we restrict our attention to vertical vector fields[4,5]

$$V = F^u\partial_u + F^v\partial_v + \text{prolongation}, \tag{2.7}$$

where (2.7) F^u, F^v are functions of x, u, v, v_1, v_2, i.e.,

$$\begin{aligned} F^u &= F^u(x,u,v,v_1,v_2), \\ F^v &= F^v(x,u,v,v_1,v_2). \end{aligned} \tag{2.8}$$

Vertical vectors fields are vector fields with vanishing ∂_x component. The generalized symmetry condition is[2]

$$\mathscr{L}_V(I) \subset D^2I, \tag{2.9}$$

where D^2I is the second prolongation of I and D^2I is generated by

$$\begin{aligned} &\alpha_1,\alpha_2,\alpha_3 \ \text{ as in (2.2)}, \\ &\alpha_4 = dv_2 - v_3\,dx, \\ &\alpha_5 = dv_3 - v_4\,dx, \end{aligned} \tag{2.10}$$

together with $d\alpha_1,\ldots,d\alpha_5$.

The resulting overdetermined system of partial differential equations arising from condition (2.9) is given by

$$\begin{aligned} &F^u_x + F^u_u v_2^2 + F^u_v v_1 + F^4_{v_1} v_2 + F^4_{v_2} v_3 - 2v_2 F^{v_2} = 0, \\ &F^v_x + F^v_u v_2^2 + F^v_v v_1 + F^v_{v_1} v_2 + F^v_{v_2} v_3 - F^{v_1} = 0, \\ &F^{v_1}_x + F^{v_1}_u v_2^2 + F^{v_1}_v v_1 + F^{v_1}_{v_1} v_2 + F^{v_1}_{v_2} v_3 + F^{v_1}_{v_3} v_4 - F^{v_2} = 0. \end{aligned} \tag{2.11}$$

The resulting overdetermined is solved in a complete elementary way leading to a set of 14 vector fields that are given by

$$\begin{aligned}
\mathrm{VF}(1) &:= 3*D(U)*(3*U^2 + 4*V*V2^3 - 4*V1^2*V2^2) \\
&\quad + D(V)*(9*U*V - 4*V1^3), \\
\mathrm{VF}(2) &:= 4*D(U)*V2^2*(2*V1 - V2*X) \\
&\quad + D(V)*(-3*U*X + 4*V1^2), \\
\mathrm{VF}(3) &:= 4*D(U)*(3*U*V1 + 3*V*V2^2 \\
&\quad - 4*V1*V2^2*X + V2^3*X^2) \\
&\quad + D(V)*(3*U*X^2 + 12*V*V1 - 8*V1^2*X), \\
\mathrm{VF}(4) &:= 4*D(U)*(9*U*V - 9*U*V1*X \\
&\quad - 9*V*V2^2*X + 4*V1^3 + 6*V1*V2^2*X^2 \\
&\quad - V2^3*X^3) + 3*D(V)*(-U*X^3 \\
&\quad + 12*V^2 - 12*V*V1*X + 4*V1^2*X^2), \\
\mathrm{VF}(5) &:= 4*D(U)*V2^3 + 3*D(V)*U, \\
\mathrm{VF}(6) &:= D(U), \\
\mathrm{VF}(7) &:= D(V)*X, \\
\mathrm{VF}(8) &:= 4*D(U)*V1 + D(V)*X^2, \\
\mathrm{VF}(9) &:= 12*D(U)*(V - V1*X) - D(V)*X^3, \\
\mathrm{VF}(10) &:= D(V), \\
\mathrm{VF}(11) &:= 2*D(U)*U + D(V)*V, \\
\mathrm{VF}(12) &:= D(U)*(3*U + V2^2*X) + D(V)*V1*X, \\
\mathrm{VF}(13) &:= D(U)*(4*V1^2 - V2^2*X^2) \\
&\quad + D(V)*X*(3*V - V1*X), \\
\mathrm{VF}(14) &:= D(U)*V2^2 + D(V)*V1,
\end{aligned} \tag{2.12}$$

where $D(U) = \partial_u$, $V1 = v_1$,..

The result is in complete agreement with the result obtained by Anderson *et al.*[1]

C. Infinitesimal symmetries of the associated first-order system

Motivated by results obtained by several authors[6] on the equation

$$y_{xx} = 0, \tag{2.13a}$$

and the related system of equations

$$\begin{aligned} (y_1)_x &= y_2, \\ (y_2)_x &= 0, \end{aligned} \tag{2.13b}$$

we describe (2.1), i.e.,

$$u_x = (v_{xx})^2 \tag{2.14}$$

as an underdetermined system of three differential equations in four dependent variables (u,v,w,y), i.e.,

$$u_x = y^2, \quad v_x = w, \quad w_x = y. \tag{2.15}$$

The exterior differential system $\tilde{I}$ associated to (2.15) defined on $\mathbb{R}^6 = \{(x,u,v,w,y,y_1)\}$ is generated by

$$\begin{aligned} \beta_1 &= du - y^2\,dx, \\ \beta_2 &= dv - w\,dx, \\ \beta_3 &= dw - y\,dx, \\ \beta_4 &= dy - y_1\,dx, \end{aligned} \tag{2.16a}$$

and their exterior derivatives

$$d\beta_1, \quad d\beta_2, \quad d\beta_3, \quad \text{and } d\beta_4. \tag{2.16b}$$

A vector field V defined on $\mathbb{R}^6$, i.e.,

$$V = F^x\partial_x + F^u\partial_u + F^v\partial_v + F^w\partial_w + F^y\partial_y + F^{y_1}\partial_{y_1} \tag{2.17}$$

is a symmetry of $\tilde{I}$ if

$$\mathscr{L}_V(\tilde{I}) \subset \tilde{I}. \tag{2.18}$$

Condition (2.18) leads to the following overdetermined system of partial differential equations for the functions $V^x,\ldots,V^y$ functions which depend on x, u, v, w, y:

$$\begin{aligned}
&F^u_x + y^2F^u_u + wF^u_v + yF^u_w + y_1F^u_y - y^2F^x_x \\
&\quad - y^4F^x_u - wy^2F^x_v - y^3F^x_w - y^2y_1F^x_y - 2yF^y = 0, \\
&F^v_x + y^2F^v_u + wF^v_v + yF^v_w \\
&\quad + y_1F^v_y - wF^x_x - wy^2F^x_u - w^2F^x_v - wyF^x_w \\
&\quad - wy_1F^x_y - F^w = 0,
\end{aligned} \tag{2.19}$$

$$F_x^w + y^2F_u^w + wF_v^w + yF_w^w + y_1F_y^w - yF_x^x - y^3F_u^x - ywF_v^x - y^2F_w^x - yy_1F_y^x - F^y = 0,$$

and the defining relation for F^{y_1}.

System (2.19) reduces to

$$F_y^u - y^2F_y^x = 0, \tag{2.20a}$$

$$F_y^v - wF_y^x = 0, \tag{2.20b}$$

$$F_y^w - yF_y^x = 0, \tag{2.20c}$$

together with

$$F_x^u + y^2F_u^u + wF_v^u + yF_w^u - y^2(F_x^x + y^2F_u^x + wF_v^x + yF_w^x) - 2yF^y = 0, \tag{2.21a}$$

$$F_x^v + y^2F_u^v + wF_v^v + yF_w^v - w \times (F_x^x + y^2F_u^x + wF_v^x + yF_w^x) - F^w = 0, \tag{2.21b}$$

$$F_x^w + y^2F_u^w + wF_v^w + yF_w^w - y \times (F_x^x + y^2F_u^x + wF_v^x + yF_w^x) - F^y = 0. \tag{2.21c}$$

We now differentiate (2.21b) with respect to y and use condition (2.20) that results in

$$2yF_u^v + F_w^v - w(2yF_u^x + F_w^x) = 0. \tag{2.22a}$$

Differentiation of (2.22a) with respect to y yields

$$2F_u^v + 2yF_{uy}^v + F_{wy}^v - w(2F_u^x + 2yF_{uy}^x + F_{wy}^x) = 0, \tag{2.22b}$$

i.e., $2F_u^v + F_y^x - 2wF_u^x = 0$, a result obtained by using (2.20b).

If we now differentiate (2.22b) with respect to y and use (2.20b) we arrive at

$$F_{yy}^x = 0. \tag{2.23a}$$

From this condition it is now a straightforward calculation to solve the overdetermined system (2.20), (2.21) leading to a 14-dimensional Lie algebra of ordinary infinitesimal symmetries of (2.16), (2.15).

The result is

$$\begin{aligned}
&\mathrm{VF}(1) := 6*D(X)*(-3*V*Y + 2*W^2) + 3*D(U)*(3*U^2 - 2*V*Y^3) + D(V)*(9*U*V - 18*V*W*Y + 8*W^3) \\
&\qquad + 9*D(W)*(U*W - V*Y^2) + 3*D(Y)*Y*(3*U - 2*W*Y), \\
&\mathrm{VF}(2) := 12*D(U)*(V - W*X) - D(V)*X^3 - 3*D(W)*X^2 - 6*D(Y)*X, \\
&\mathrm{VF}(3) := D(U), \\
&\mathrm{VF}(4) := 4*D(U)*W + D(V)*X^2 + 2*D(W)*X + 2*D(Y), \\
&\mathrm{VF}(5) := D(V)*X + D(W), \\
&\mathrm{VF}(6) := D(X)*X^2 + 4*D(U)*W^2 + 3*D(V)*V*X + D(W)*(3*V + W*X) + D(Y)*(4*W - X*Y), \\
&\mathrm{VF}(7) := 2*D(U)*U + D(V)*V + D(W)*W + D(Y)*Y, \\
&\mathrm{VF}(8) := 2*D(X)*(-4*W + 3*X*Y) + 2*D(U)*X*Y^3 + D(V)*(-3*U*X - 4*W^2 + 6*W*X*Y) \\
&\qquad + 3*D(W)*(-U + X*Y^2) + 2*D(Y)*Y^2, \\
&\mathrm{VF}(9) := 6*D(X)*Y + 2*D(U)*Y^3 + 3*D(V)*(-U + 2*W*Y) + 3*D(W)*Y^2, \\
&\mathrm{VF}(10) := 6*D(X)*X*(6*V - 4*W*X + X^2*Y) + 2*D(U)*(18*U*V - 18*U*W*X + 8*W^3 + X^3*Y^3) \\
&\qquad + 3*D(V)*(-U*X^3 + 12*V^2 - 4*W^2*X^2 + 2*W*X^3*Y) + 3*D(W)*(-3*U*X^2 + 12*V*W \\
&\qquad - 4*W^2*X + X^3*Y^2) + 6*D(Y)*(-3*U*X + 4*W^2 - 2*W*X*Y + X^2*Y^2), \\
&\mathrm{VF}(11) := 2*D(X)*(-6*V + 8*W*X - 3*X^2*Y) + 2*D(U)*(6*U*W - X^2*Y^3) + D(V)*X*(3*U*X + 8*W^2 \\
&\qquad - 6*W*X*Y) + D(W)*(6*U*X + 4*W^2 - 3*X^2*Y^2) + 2*D(Y)*(3*U + 2*W*Y - 2*X*Y^2), \\
&\mathrm{VF}(12) := -D(X)*X + D(U)*U - D(V)*V + D(Y)*Y, \\
&\mathrm{VF}(13) := D(X), \\
&\mathrm{VF}(14) := D(V).
\end{aligned} \tag{2.23b}$$

By taking the ∂_u, ∂_v components of the equivalent vertical vector field and, carrying through the transformation

$$x = x, \quad u = u, \quad v = v, \quad w = v_1, \quad y = v_2,$$

we arrive at the symmetry algebra G_2 as derived by Anderson *et al.*[1]

So the Lie algebra G_2 is nothing else but the ordinary symmetry algebra of the related system. At the moment we have no general theorem relating classes of higher-order symmetries to ordinary symmetries of associated systems of first-order equations. We hope to deal with this problem in the future.

D. Fourth-order generalized symmetries

In this subsection we present the computer results of the computation of 4th-order generalized symmetries of equation (2.1). In order to do so we prolong the ideal I of differential forms (2.2), (2.10), i.e., D^4I is generated by

$$\begin{aligned}\alpha_1&=du-v_2^2\,dx,\\ \alpha_2&=dv-v_1\,dx,\\ \alpha_3&=dv_1-v_2\,dx,\\ \alpha_4&=dv_2-v_3\,dx,\\ \alpha_5&=dv_3-v_4\,dx,\\ \alpha_6&=dv_4-v_5\,dx,\\ \alpha_7&=dv_5-v_6\,dx,\end{aligned} \tag{2.24a}$$

and their exterior derivatives

$$d\alpha_1,\ldots,d\alpha_7. \tag{2.24b}$$

The generalized symmetry condition is given as

$$\mathscr{L}_V(I)\subset D^4I, \tag{2.25}$$

which results in a system of three partial differential equations for the coefficients $F^u, F^v, F^{v_1}, F^{v_2}$ of the vertical vector field v. The conditions are of a form analogous to (2.11), and as an immediate result we arrive at

$$\frac{\partial F^v}{\partial v_4}=0, \tag{2.26}$$

so F^v is a function of x, u, v, v_1, v_2, v_3.

In order to obtain at least some solutions of the system satisfying (2.25) we put in the additional condition

$$\frac{\partial^2 F^v}{\partial v_3^2}=0, \tag{2.27}$$

i.e., F^v is linear with respect to v_3.

From the condition (2.27) the construction of the solutions was straightforward and did lead to a set of 63 4th-order generalized symmetries.

The general solution to condition (2.25) is given in the next section. Here we present some of the 63 generalized symmetries (x independent):

$$\begin{aligned}
\mathrm{VF}(1):=\;&D(U)*(81*U^4+648*U^2*V^2*V2*V4-324*U^2*V^2*V3^2\\
&+648*U^2*V*V1*V2*V3+216*U^2*V*V2^3-540*U^2*V1^2*V2^2+648*U*V^2*V2^3*V3\\
&-576*U*V*V1^3*V2*V4+288*U*V*V1^3*V3^2-864*U*V*V1^2*V2^2*V3-648*U*V*V1*V2^4\\
&-288*U*V1^4*V2*V3+864*U*V1^3*V2^3-180*V^2*V2^6-288*V*V1^3*V2^3*V3+576*V*V1^2*V2^5\\
&+128*V1^6*V2*V4-64*V1^6*V3^2+384*V1^5*V2^2*V3-432*V1^4*V2^4)+2*D(V)*(81*U^3*V\\
&+162*U^2*V^2*V3-162*U^2*V*V1*V2-36*U^2*V1^3-54*U*V^2*V2^3-144*U*V*V1^3*V3\\
&+108*U*V*V1^2*V2^2+72*U*V1^4*V2+24*V*V1^3*V2^3+32*V1^6*V3-48*V1^5*V2^2),\\
\mathrm{VF}(2):=\;&12*D(U)*(18*U^2*V*V2*V4-9*U^2*V*V3^2+9*U^2*V1*V2*V3+3*U^2*V2^3+18*U*V*V2^3*V3\\
&-8*U*V1^3*V2*V4+4*U*V1^3*V3^2-12*U*V1^2*V2^2*V3-9*U*V1*V2^4-5*V*V2^6\\
&-4*V1^3*V2^3*V3+8*V1^2*V2^5)+D(V)*(27*U^3+108*U^2*V*V3-54*U^2*V1*V2-36*U*V*V2^3\\
&-48*U*V1^3*V+36*U*V1^2*V2^2+8*V1^3*V2^3),\\
\mathrm{VF}(3):=\;&D(U)*(18*U^2*V2*V4-9*U^2*V3^2+18*U*V2^3*V3-5*V2^6)+3*D(V)*U*(3*U*V3-V2^3),\\
\mathrm{VF}(4):=\;&D(U)*(9*U^2*V2^2+72*U*V*V1*V2*V4-36*U*V*V1*V3^2+36*U*V*V2^2*V3\\
&+36*U*V1^2*V2*V3-36*U*V1*V2^3+36*V*V1*V2^3*V3-24*V*V2^5-32*V1^4*V2*V4\\
&+16*V1^4*V3^2-64*V1^3*V2^2*V3+36*V1^2*V2^4)+D(V)*(9*U^2*V1+36*U*V*V1*V3\\
&-9*U*V*V2^2-18*U*V1^2*V2-6*V*V1*V2^3-16*V1^4*V3+16*V1^3*V2^2),\\
\mathrm{VF}(5):=\;&2*D(U)*(9*U^3+36*U*V^2*V2*V4-18*U*V^2*V3^2+36*U*V*V1*V2*V3+12*U*V*V2^3\\
&-30*U*V1^2*V2^2+18*V^2*V2^3*V3-16*V*V1^3*V2*V4+8*V*V1^3*V3^2-24*V*V1^2*V2^2*V3\\
&-18*V*V1*V2^4-8*V1^4*V2*V3+24*V1^3*V2^3)+D(V)*(27*U^2*V+36*U*V^2*V3\\
&-36*U*V*V1*V2-8*U*V1^3-6*V^2*V2^3-16*V*V1^3*V3+12*V*V1^2*V2^2+8*V1^4*V2),\\
\mathrm{VF}(6):=\;&2*D(U)*(6*U*V*V2*V4-3*U*V*V3^2+3*U*V1*V2*V3+2*U*V2^3+3*V*V2^3*V3\\
&-3*V1*V2^4)+D(V)*(3*U^2+6*U*V*V3-3*U*V1*V2-V*V2^3),\\
\mathrm{VF}(7):=\;&4*D(U)*(6*U*V1*V2*V4-3*U*V1*V3^2+3*U*V2^2*V3+3*V1*V2^3*V3-2*V2^5)\\
&+D(V)*(12*U*V1*V3-3*U*V2^2-2*V1*V2^3),
\end{aligned}$$

$$\mathrm{VF}(8):=4*D(U)*(6*U*V2^3+8*V1^3*V2*V4-4*V1^3*V3^2+12*V1^2*V2^2*V3-9*V1*V2^4)$$
$$+D(V)*(9*U^2+16*V1^3*V3-12*V1^2*V2^2),$$
$$\mathrm{VF}(9):=D(U)*(2*V2*V4-V3^2)+D(V)*V3,$$
$$\mathrm{VF}(10):=4*D(U)*(2*V1*V2*V4-V1*V3^2+V2^2*V3)+D(V)*(4*V1*V3-V2^2),$$
$$\mathrm{VF}(11):=2*D(U)*(6*V*V2*V4-3*V*V3^2+3*V1*V2*V3-2*V2^3)+3*D(V)*(2*V*V3-V1*V2),$$
$$\mathrm{VF}(12):=2*D(U)*(U*V2^2+4*V*V1*V2*V4-2*V*V1*V3^2+2*V*V2^2*V3+2*V1^2*V2*V3-2*V1*V2^3)$$
$$+D(V)*(2*U*V1+4*V*V1*V3-V*V2^2-2*V1^2*V2),$$
$$\mathrm{VF}(13):=3*D(U)*(6*V^2*V2*V4-3*V^2*V3^2+6*V*V1*V2*V3-4*V*V2^3+V1^2*V2^2)$$
$$+D(V)*(9*V^2*V3-9*V*V1*V2+4*V1^3),$$
$$\mathrm{VF}(14):=6*D(U)*(2*U*V2*V4-U*V3^2+V2^3*V3)+D(V)*(6*U*V3-V2^3),$$
$$\mathrm{VF}(15):=D(U)*(8*V1^2*V2*V4-4*V1^2*V3^2+8*V1*V2^2*V3-3*V2^4)+2*D(V)*V1*(2*V1*V3-V2^2). \tag{2.28}$$

III. DERIVATION OF THE GENERALIZED SYMMETRY ALGEBRA

We start the discussion at the ordinary differential equation

$$u_x=(v_{xx})^2, \tag{3.1}$$

where "x" is d/dx; u, v are functions of x.

So the equation at hand (3.1) is merely an underdetermined system of ordinary differential equations where u, v are dependent variables. The main aim of this section is the construction of the complete generalized symmetry algebra of (3.1). The final result is formulated in Theorem 3.1.

Generalized symmetries are formal vector fields defined on the infinite jet bundle $J(x;\,u,\,v)$ (cf. Refs. 4, 5) which leave invariant the differential equation together with its differential consequences $(u_x=u_1,\ v_x=v_1,\ v_{xx}=v_2,...)$ i.e.,

$$\begin{aligned} u_1&=v_2^2,\\ u_2&=2v_2v_3,\\ u_3&=2v_3^2+2v_2v_4,\\ &\vdots \end{aligned} \tag{3.2}$$

Local coordinates on $J(x;\,u,\,v)$ are given by

$$(x,u,v,u_1,v_1,u_2,v_2,u_3,v_3,...),$$

whereas local coordinates on the submanifold $\mathscr{Y}$ defined by (3.2) are

$$(x,u,v,v_1,v_2,v_3,...). \tag{3.3}$$

The formal total derivative vector field $\overline{D}$ on $J(x;\,u,\,v)$ is given by

$$\overline{D}=\partial_x+u_1\partial_u+v_1\partial_v+u_2\partial_{u_1}+v_2\partial_{v_1}+\cdots,$$

whereas the restriction of $\overline{D}$ to the submanifold $\mathscr{Y}$ defined by (3.2) is given by

$$D=\partial_x+v_2^2\partial_u+v_1\partial_v+v_2\partial_{v_1}+\cdots.$$

For later use we introduce the restriction $D^{(n)}$ of D to the $(n+1)$th-order jet bundle

$$D^{(n)}=\partial_x+v_2^2\partial_u+v_1\partial_v+\cdots+v_{n+1}\partial_{v_n}. \tag{3.4}$$

Suppose the vertical vector field V with characteristic functions $F^u(x,u,v_1,...,v_n)$, $F^v(x,u,v_1,...,v_n)$, $(F^x=0)$, is a generalized symmetry of (3.1), (3.2).

We then have the following symmetry conditions

$$D^{(n)}F^u[v_n]-2v_2F^{v_2}[v_{n+2}]=0, \tag{3.5a}$$

$$D^{(n)}F^v[v_n]-F^{v_1}[v_{n+1}]=0, \tag{3.5b}$$

$$D^{(n+1)}F^{v_1}[v_{n+1}]-F^{v_2}[v_{n+2}]=0. \tag{3.5c}$$

In (3.5) we introduced the notation

$$F[v_k]=F(x,u,v,v_1,...,v_k).$$

Equations (3.5b), (3.5c) are in effect the defining relations for the prolongation coefficients F^{v_1}, F^{v_2} of V:

$$V=F^u[v_n]\partial_u+F^v[v_n]\partial_v+F^{v_1}[v_{n+1}]\partial_{v_1}$$
$$+F^{v_2}[v_{n+2}]\partial_{v_2}+\cdots. \tag{3.6}$$

We now want to construct the general solution of (3.5). In order to do so we first solve (3.5c) for $F^{v_2}[v_{n+2}]$ i.e.,

$$F^{v_2}[v_{n+2}]=D^{(n+1)}F^{v_1}[v_{n+1}], \tag{3.7}$$

and the system (3.5) reduces to

$$D^{(n)}F^u[v_n]-2v_2D^{(n+1)}F^{v_1}[v_{n+1}]=0, \tag{3.8a}$$

$$D^{(n)}F^v[v_n]-F^{v_1}[v_{n+1}]=0. \tag{3.8b}$$

Remark: At this stage it is possible to solve (3.8b) but we decided not to do so! Now (3.8a) is a polynomial in v_{n+2} of degree 1 and (3.8) reduces to

$$v_{n+2}:-2v_2\frac{\partial F^{v_1}[v_{n+1}]}{\partial v_{n+1}}=0\Rightarrow F^{v_1}[v_{n+1}]=F^{v_1}[v_n], \tag{3.9a}$$

$$1:D^{(n)}F^u[v_n]-2v_2D^{(n)}F^{v_1}[v_n]=0, \tag{3.9b}$$

$$D^{(n)}F^v[v_n]-F^{v_1}[v_n]=0. \tag{3.9c}$$

In (3.9a) and further on, "v_{n+2}:" refers to the coefficient of v_{n+2} in a particular equation.

From (3.9) we arrive by (3.9b), (3.9c) being polynomials in v_{n+1} at

$$v_{n+1}: \frac{\partial F^u}{\partial v_n}[v_n] - 2v_2 \frac{\partial F^{v_1}}{\partial v_n}[v_n] = 0, \tag{3.10a}$$

$$1: D^{(n-1)} F^u[v_n] - 2v_2 D^{(n-1)} F^{v_1}[v_n] = 0, \tag{3.10b}$$

$$v_{n+1}: \frac{\partial F^v}{\partial v_n}[v_n] = 0, \tag{3.10c}$$

$$1: D^{(n-1)} F^v[v_n] - F^{v_1}[v_n] = 0. \tag{3.10d}$$

To solve system (3.10) first note that (3.10c)

$$F^v[v_n] = F^v[v_{n-1}], \tag{3.11}$$

and by differentiation of (3.10d) twice with respect to v_n

$$\frac{\partial^2 F^{v_1}}{\partial v_n^2}[v_n] = 0, \tag{3.12a}$$

and by consequence F^{v_1} is linear with respect to v_n

$$F^{v_1}[v_n] = H^1[v_{n-1}] + v_n H^2[v_{n-1}]. \tag{3.12b}$$

Now substitution of (3.11), (3.12b) into (3.10a)–(3.10d) yields

$$\frac{\partial F^u}{\partial v_n}[v_n] - 2v_2 H^2[v_{n-1}] = 0, \tag{3.13a}$$

$$D^{(n-1)} F^u[v_n] - 2v_2 D^{(n-1)} H^1[v_{n-1}] - 2v_2 v_n D^{(n-1)} H^2[v_{n-1}] = 0, \tag{3.13b}$$

$$D^{(n-1)} F^v[v_{n-1}] - H^1[v_{n-1}] - v_n H^2[v_{n-1}] = 0. \tag{3.13c}$$

We solve (3.13a) for $F^u[v_n]$, i.e.,

$$F^u[v_n] = 2v_2 v_n H^2[v_{n-1}] + H^3[v_{n-1}], \tag{3.14}$$

and from (3.13b), (3.13c) we arrive at

$$\begin{aligned} &2v_3 v_n H^2[v_{n-1}] \\ &\quad + 2v_2 v_n D^{(n-1)} H^2[v_{n-1}] + D^{(n-1)} H^3[v_{n-1}], \\ &- 2v_2 D^{(n-1)} H^1[v_{n-1}] - 2v_2 v_n D^{(n-1)} H^2[v_{n-1}] = 0, \end{aligned} \tag{3.15a}$$

$$D^{(n-1)} F^v[v_{n-1}] - H^1[v_{n-1}] - v_n H^2[v_{n-1}] = 0. \tag{3.15b}$$

Due to the cancellation of the second and fifth term in (3.15a) we obtain a resulting system of four equations

$$v_n: 2v_3 H^2[v_{n-1}] + \frac{\partial H^3}{\partial v_{n-1}}[v_{n-1}] - 2v_2 \frac{\partial H^1}{\partial v_{n-1}}[v_{n-1}] = 0, \tag{3.16a}$$

$$1: D^{(n-2)} H^3[v_{n-1}] - 2v_2 D^{(n-2)} H^1[v_{n-1}] = 0, \tag{3.16b}$$

$$v_n: \frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}] - H^2[v_{n-1}] = 0, \tag{3.16c}$$

$$1: D^{(n-2)} F^v[v_{n-1}] - H^1[v_{n-1}] = 0. \tag{3.16d}$$

From (3.16) we solve (3.16c) for $H^2[v_{n-1}]$,

$$H^2[v_{n-1}] = \frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}] \tag{3.17}$$

and then integrate (3.16a)

$$2v_3 \frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}] - 2v_2 \frac{\partial H^3}{\partial v_{n-1}}[v_{n-1}] - 2v_2 \frac{\partial H^1}{\partial v_{n-1}}[v_{n-1}] = 0, \tag{3.18}$$

which leads to

$$H^3[v_{n-1}] = 2v_2 H^1[v_{n-1}] - 2v_3 F^v[v_{n-1}] + H^4[v_{n-2}]. \tag{3.19}$$

By obtaining (3.19) we have to put in the requirement $(n-1>3)$, i.e.,

$$n > 4, \tag{3.20}$$

and we shall return to this case in Sec. IV.

Substitution of the results (3.17), (3.19) into (3.16) yields

$$\begin{aligned} &2v_3 H^1[v_{n-1}] + 2v_2 D^{(n-2)} H^1[v_{n-1}] - 2v_4 F^v[v_{n-1}] \\ &\quad - 2v_3 D^{n-2} F^v[v_{n-1}] + D^{(n-2)} H^4[v_{n-2}] \\ &\quad - 2v_2 D^{(n-2)} H^1[v_{n-1}] = 0, \end{aligned} \tag{3.21a}$$

$$D^{(n-2)} F^v[v_{n-1}] - H^1[v_{n-1}] = 0. \tag{3.21b}$$

By the cancellation of the second and sixth term in (3.21a) we finally arrive at

$$D^{(n-2)} F^v[v_{n-1}] - H^1[v_{n-1}] = 0, \tag{3.22a}$$

$$D^{(n-2)} H^4[v_{n-2}] - 2v_4 F^v[v_{n-1}] = 0, \tag{3.22b}$$

where (3.22a), (3.22b) can be considered as defining relation for $H^1[v_{n-1}]$, $F^v[v_{n-1}]$ in terms of an *arbitrary function* $H^4[v_{n-2}]$.

The final result can now be obtained by (3.17), (3.19):

$$H^2[v_{n-1}] = \frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}],$$

$$H^3[v_{n-1}] = 2v_2 H^1[v_{n-1}] - 2v_3 F^v[v_{n-1}] + H^4[v_{n-2}],$$

together with (3.22a), (3.22b), (3.14),

$$F^u[v_n] = 2v_2 v_n \frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}] + 2v_2 H^1[v_{n-1}] - 2v_3 F^v[v_{n-1}] + H^4[v_{n-2}], \tag{3.23a}$$

$$F^v[v_n] = F^v[v_{n-1}], \tag{3.23b}$$

whereas in (3.23a), (3.23b) $F^v[v_{n-1}]$, $H^1[v_{n-1}]$ are defined by (3.22a), (3.22b) in terms of the *arbitrary function* $H^4[v_{n-2}]$.

The general result of this section is now formulated in the following.

Theorem 3.1: Let H be an arbitrary function of $x,u,v,v_1,\ldots,v_{n-2}$, i.e.,

$$H = H(x,u,v,v_1,\ldots,v_{n-2}) = H[v_{n-2}]$$

and define

$$\begin{aligned} F^v[v_{n-1}] &= (1/2v_4) D^{(n-2)} H[v_{n-2}], \\ F^u[v_n] &= 2v_2 D^{(n-1)} F^v[v_{n-1}] - 2v_3 F^v[v_{n-1}] + H[v_{n-2}], \end{aligned} \tag{3.24}$$

then the vector field

$$V = F^u[v_n]\partial_u + F^v[v_{n-1}]\partial_v$$

is a generalized symmetry of (3.1). Conversely, given a generalized symmetry V of (3.1) then there exist a function H such that the components F^u, F^v of V are defined by (3.24) (cf. Sec. IV). □

IV. SPECIAL CASES

Due to the restriction (3.16) the result (3.19), (3.18) hold for

$$n = 5,...,$$

meaning $H^4[v_{n-2}]$ is a free function of $x,u,v,...,v_{n-2}$ and F^v is obtained by (3.18b), i.e.,

$$F^v[v_{n-1}] = (1/2v_4)D^{(n-2)}H^4[v_{n-2}]. \tag{4.1}$$

From (4.1), (3.18b) it is clear that $F^v[v_{n-1}]$ is linear with respect to v_{n-1} and

$$F^v[v_{n-1}] = \frac{v_{n-1}}{2v_4}\frac{\partial H^4}{\partial v_{n-2}} + \widetilde{F}^v[v_{n-2}]. \tag{4.2}$$

Moreover, the requirement $F^v[v_{n-1}]$ is independent of v_{n-1} reduces to $H^4[v_{n-2}]$ is independent of v_{n-2}, i.e.,

$$\frac{\partial F^v}{\partial v_{n-1}}[v_{n-1}] = 0 \Rightarrow H^4[v_{n-2}] = H^4[v_{n-3}]. \tag{4.3}$$

The result (4.3) holds for $n > 5$.

The results for generalized ($n \leqslant 5$) are obtained by imposing additional conditions on the coefficient F^v of the vertical vector field.

A. The case $n=5$

Here,

$$F^v[v_4] = \frac{1}{2v_4}\left\{\frac{\partial H^4}{\partial x}[v_3] + v_2^2\frac{\partial H^4}{\partial u}[v_3] + v_1\frac{\partial H^4}{\partial v}[v_3] + v_2\frac{\partial H^4}{\partial v_1}[v_3] + v_3\frac{\partial H^4}{\partial v_2}[v_3] + v_4\frac{\partial H^4}{\partial v_3}[v_3]\right\}. \tag{4.4}$$

The requirement $F^v[v_4]$ is independent of v_4 now leads to a genuine first-order partial differential equation

$$\frac{\partial H^4}{\partial x} + v_2^2\frac{\partial H^4}{\partial u} + v_1\frac{\partial H^4}{\partial v} + v_2\frac{\partial H^4}{\partial v_1} + v_3\frac{\partial H^4}{\partial v_2} = 0 \tag{4.5}$$

and the general solution is given in terms of the invariants of the corresponding vector field

$$V = \partial_x + v_2^2\partial_u + v_1\partial_v + v_2\partial_{v_1} + v_3\partial_{v_2}, \tag{4.6}$$

where the set of invariants is given by

$$\begin{aligned} z_1 &= v_3,\\ z_2 &= v_2 - v_3x,\\ z_3 &= 2v_1 - 2v_2x + v_3x^2,\\ z_4 &= 6v - 6v_1 + 3v_2x^2 - v_3x^3,\\ z_5 &= 3u - 3v_2^2x + 3v_2v_3x^2 - v_3^2x^3. \end{aligned} \tag{4.7}$$

So H^4 is given by

$$H^4 = H^4(z_1,...,z_5), \tag{4.8}$$

whereas the formulas for F^v, F^u reduce to

$$F^u = H^4 - v_2\frac{\partial H^4}{\partial v_2} - v_3\frac{\partial H^4}{\partial v_3} + v_2v_4\frac{\partial^2 H^4}{\partial v_3^2},$$

$$F^v = \tfrac{1}{2}\partial_{v_3}H^4.$$

B. The case $n=4$

The requirement F^v is independent of v_3 reduces to

$$\partial^2_{v_3}H^4 = 0 \tag{4.9a}$$

and (4.5)

$$\frac{\partial H^4}{\partial x} + v_2^2\frac{\partial H^4}{\partial u} + v_1\frac{\partial H^4}{\partial v} + v_2\frac{\partial H^4}{\partial v_1} + v_3\frac{\partial H^4}{\partial v_2} = 0. \tag{4.9b}$$

Substitution of (4.9a) into (4.9b) immediately leads to the condition

$$\partial_{v_2}\partial_{v_3}H^4 = 0,$$

i.e.,

$$F^v = F^v(x,u,v,v_1),$$

and the result completely reduces to the second-order symmetries obtained by Anderson *et al.*,[1] leading to the 14-dimensional Lie algebra G_2.

V. HIGHER ORDER SYMMETRIES OF A GENERALIZATION OF THE UNDETERMINED EQUATION

We derive the general formulas for higher-order symmetries of the differential equation

$$u_1 = f(u,v,v_1,v_2,...,v_k), \tag{5.1}$$

where in (5.1)

$$u_1 = u_x, \quad v_1 = v_x, \quad v_2 = v_{xx},... .$$

We shall restrict the derivation to the case

$$k = 3, \tag{5.2}$$

the most general case can be handled easily by an induction argument and the associated results are given at the end of this section.

We denote partial derivatives of f by

$$f_u = \frac{\partial f}{\partial u}, \quad f_0 = \frac{\partial f}{\partial v}, \quad f_1 = \frac{\partial f}{\partial v_1},... . \tag{5.3}$$

The generalized symmetry condition[4] for (5.1) results in

$$DF^u - f_uF^u - f_0F^v - f_1F^{v_1} - f_2F^{v_2} - f_3F^{v_3} = 0, \tag{5.4}$$

where

$$D = \partial_x + f\partial_u + v_1\partial_v + v_2\partial_{v_1} + \cdots, \tag{5.5}$$

and $F^u, F^v, F^{v_1},...$ are the components of the generalized symmetry V

$$V = F^u\partial_u + F^v\partial_v, \tag{5.6a}$$

while

$$F^{v_{i+1}} = DF^{v_i} \quad (i = 0,...). \tag{5.6b}$$

Condition (5.4) is easily rewritten as

$$DF^u - f_0 F^u - f_0 F^v - f_1 F^{v_1} - f_2 F^{v_2} - f_3 DF^{v_2} = 0. \quad (5.7)$$

First note that due to (5.6b) (5.7), if

$$F^u = F^u[v_{n+2}], \quad (5.8a)$$

i.e., $F^u = F^u(x,u,v,...,v_{n+2})$ then

$$F^v = F^v[v_n], \quad F^{v_1}[v_{n+1}], \quad F^{v_2} = F^{v_2}[v_{n+2}]. \quad (5.8b)$$

Now put

$$F^u[v_{n+2}] = f_3 F^{v_2}[v_{n+2}] + F_1^u[v_{n+2}], \quad (5.9)$$

where F_1^u is a function, yet to be determined, and substitute (5.9) into (5.7), then the highest-order terms imply

$$F_1^u[v_{n+2}] = F_1^u[v_{n+1}],$$

i.e., F_1^u does not depend on v_{n+2}, and (5.7) reduces to

$$DF_1^u - f_u F_1^u - f_0 F^v - f_1 F^{v_1} + \{(D - f_u) f_3 - f_2\} DF^{v_1} = 0. \quad (5.10)$$

Similar to (5.9) put

$$F_1^u = \{-(D - f_u) f_3 + f_2\} F^{v_1} + F_2^u[v_{n+1}] \quad (5.11)$$

and substitute into (5.10), then

$$F_2^u[v_{n+1}] = F_2^u[v_n]$$

and

$$DF_2^u - f_u F_2^u - f_0 F^v + \{-(D - f_u)^2 f_3 + (D - f_u) f_2 - f_1\} DF^v = 0. \quad (5.12)$$

Repeating the process once more, we put

$$F_2^u = \{(D - f_u)^2 f_3 - (D - f_u) f_2 + f_1\} F^v + F_3^u[v_n], \quad (5.13)$$

arriving at

$$F_3^u[v_n] = F_3^u[v_{n-1}] \quad (5.14a)$$

and

$$(D - f_u) F_3^u + \{(D - f_u)^3 f_3 - (D - f_u)^2 f_2 + (D - f_u) f_1 - f_0\} F^v = 0, \quad (5.14b)$$

from which we can solve for F^v in terms of an arbitrary function F_3^u (5.14a)

$$F^v = \{-(D - f_u)^3 f_3 + (D - f_u)^2 f_2 - (D - f_u) f_1 + f_0\}^{-1} (D - f_u) F_3^u,$$

and from (5.9), (5.11), (5.13) together with (5.6b), (5.14b) a formula for F^u

$$F^u = f_3 F^{v_2}[v_{n+2}] + \{-(D - f_u)(f_3) + f_2\} F^{v_1}[v_{n+1}] + \{(D - f_u)^2 (f_3) - (D - f_u)(f_2) + f_1\} F^v[v_n]. \quad (5.14c)$$

Of course it has to be noted that n must be chosen (sufficiently large) such that there is no contribution of the terms

$$f_3$$
$$(D - f_u) f_3 - f_2,$$
$$(D - f_u)^2 f_3 - (D - f_u) f_2 + f_1,$$
$$(D - f_u)^3 f_3 - (D - f_u)^2 f_2 + (D - f_u) f_1 - f_0,$$

in the highest-order terms of (5.7), (5.10), (5.12), (5.14).

We finish this section by formulating the following theorem.

Theorem 5.1: Consider the differential equation

$$u_1 = f(u,v,v_1,...,v_k)$$

and let H be an arbitrary function of $x,u,v,...,v_n$

$$H = H[v_n]. \quad (5.15a)$$

Define

$$F^v = \left\{ \sum_{l=0}^{k} (-1)^l (D - f_u)^l f_l \right\}^{-1} \cdot (D - f_u) H[v_n] \quad (5.15b)$$

and

$$F^u = \sum_{l=1}^{k} \left\{ \sum_{j=l}^{k} (-1)^{j-l} (D - f_u)^{j-l} (f_j) \right\} F^{v_{l-1}}[v_{n+l-1}] + H[v_n], \quad (5.15c)$$

where $F^{v_{l-1}}$ is obtained from F^v by prolongation (5.6b).

Then the vector field

$$V = F^u \partial_u + F^v \partial_v$$

is a generalized symmetry of (5.1).

Conversely any generalized symmetry of (5.1) arises from a specific choice of $H[v_n]$. □

As mentioned before the general result can be derived by an induction argument.

VI. CONCLUSION

Motivated by the result of computer "experiments" in the construction of generalized symmetries we derived the complete symmetry algebra for the ordinary underdetermined system

$$u_1 = f(u,v,v_1,...,v_k).$$

Moreover we showed that the generalized symmetry algebra of 2-*nd* order derived by Anderson *et al.* is equivalent to the ordinary symmetry algebra of the associated system.

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