Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 412–423

Nonlocal Constructions in the Geometry of PDE

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We give an overview of recent developments of nonlocal constructions in the geometry of partial differential equations and their applications to Bäcklund transformations, Hamiltonian and symplectic structures, recursion operators, etc.

This is an overview of recent results obtained by the authors and related to using of nonlocal constructions in geometry of partial differential equations. For general references concerning geometry of PDE see [1, 15].

Let $\mathscr{E} \subset J^\infty(\pi) \xrightarrow{\pi_\infty} M$ be an infinitely prolonged differential equation considered as a submanifold in an appropriate manifold of infinite jets. Then $\mathscr{E}$ is endowed with a natural finite-dimensional integrable distribution[1] (the *Cartan distribution* denoted by $\mathscr{C}$) locally spanned by the total derivatives. Invariantly, at any point $\theta \in \mathscr{E}$ the space $\mathscr{C}_\theta$ is the span of all tangent planes to the graphs of jets passing through θ. A fiber bundle $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is called a *covering* over $\mathscr{E}$ if (a) $\tilde{\mathscr{E}}$ is endowed with an integrable distribution $\tilde{\mathscr{C}}$, $\dim \tilde{\mathscr{C}} = \dim \mathscr{C}$ and (b) $\tau_* \tilde{\mathscr{C}}_y = \mathscr{C}_{\tau(y)}$ for any $y \in \tilde{\mathscr{E}}$. A $\pi_\infty \circ \tau$-vertical vector field X is called a *nonlocal symmetry* of $\mathscr{E}$ if it preserves $\tilde{\mathscr{C}}$. Note that since the distributions $\mathscr{C}$ and $\tilde{\mathscr{C}}$ are *horizontal*, they determine flat connections in the bundles π_∞ and $\pi_\infty \circ \tau$ respectively. We call these connections the *Cartan connections* and use the same notation, $\mathscr{C}$ and $\tilde{\mathscr{C}}$, for them.

A $\tau \circ \pi_\infty$-vertical vector field X on $\tilde{\mathscr{E}}$ is called a *nonlocal τ-symmetry* if $[X, \tilde{\mathscr{C}}] \subset \tilde{\mathscr{C}}$. They form a Lie algebra denoted by $\mathrm{sym}_\tau \mathscr{E}$. Nonlocal symmetries can be expressed in finite terms rather rarely. A good (and rather useful) substitute is the notion of a *shadow* (that is often mixed up with nonlocal symmetries themselves!). Naively, in local coordinates a shadow can be introduced as follows. Note that any vector field X lying in the Cartan distribution can be uniquely lifted to a field $\tilde{X}$ lying in $\tilde{\mathscr{C}}$ such that $\tau_* \tilde{X} = X$. Moreover, one has $\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]$. Consequently, any linear differential operator Δ on $\mathscr{E}$ expressed in total derivatives (a so-called *$\mathscr{C}$-differential operator*) is lifted to an operator $\tilde{\Delta}$ on $\tilde{\mathscr{E}}$. In particular, assume that $\mathscr{E}$ is given by a differential operator F and $\ell_{\mathscr{E}}$ is the restriction of its linearization to $\mathscr{E}$. Then $\ell_{\mathscr{E}}$ can be lifted to an operator $\tilde{\ell}_{\mathscr{E}}$ on $\tilde{\mathscr{E}}$. Consider a solution φ of the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$. Then to such a φ there corresponds a derivation $S_\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. This derivation (or φ itself) is called a shadow in the covering τ, or a *τ-shadow*. The *Reconstruction Theorem* (see [10, 14] and below) states that for any shadow S_φ in τ there exist a covering $\mathscr{E}_\varphi \xrightarrow{\tau_\varphi} \tilde{\mathscr{E}} \xrightarrow{\tau} \mathscr{E}$ and a nonlocal symmetry S in this covering such that $S|_{C^\infty(\mathscr{E})} = S_\varphi$. Vice versa, any derivation $S\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ that takes the Cartan distribution on $\mathscr{E}$ to that on $\tilde{\mathscr{E}}$ is locally identified with a τ-shadow. Thus, this property of S can be taken for the global definition of shadows.

In a similar way, one can consider the adjoint operator $\ell_{\mathscr{E}}^*$ and its lifting $\tilde{\ell}_{\mathscr{E}}^*$ to $\tilde{\mathscr{E}}$. In what follows, we assume that the equation under consideration satisfies the conditions of *Vinogradov's Two-Line Theorem* [18–20]. Then solutions of the equation $\tilde{\ell}_{\mathscr{E}}^*(\psi) = 0$ are called *nonlocal gen-*

[1] Integrability in this case means that for any two vector fields $X, Y \in \mathscr{C}$ their commutator $[X, Y]$ lies in $\mathscr{C}$ as well.

erating functions. Nonlocal symmetries (and their shadows) and generating functions naturally arise in various problems related to geometry of PDE. We shall consider some of them below.

1 The long exact sequence of a covering

Let $\mathscr{E} \subset J^\infty(\pi)$, as before, be an infinitely prolonged differential equation understood as a submanifold in the manifold of infinite jets, $\pi_\infty\colon \mathscr{E} \to M$ be the natural projection. Consider a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and the *structure element*[2] $U_\tau \in \mathrm{D^v}(\Lambda^1(\tilde{\mathscr{E}})) \subset \mathrm{D}(\Lambda^1(\tilde{\mathscr{E}}))$, where (and below) $\mathrm{D}(\Lambda^i(\tilde{\mathscr{E}}))$ denotes the module of $\Lambda^i(\tilde{\mathscr{E}})$-valued derivations $C^\infty(\tilde{\mathscr{E}}) \to \Lambda^i(\tilde{\mathscr{E}})$ while $\mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}}))$ consists of π_∞-vertical derivation. Then (see [12]) the $\mathscr{C}$*-complex*

$$0 \to \mathrm{D^v}(\tilde{\mathscr{E}}) \xrightarrow{\partial_\tau} \mathrm{D^v}(\Lambda^1(\tilde{\mathscr{E}})) \to \cdots \to \mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}})) \xrightarrow{\partial_\tau} \mathrm{D^v}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \to \cdots \tag{1}$$

arises, where the differential $\partial_\tau = [\![U_\tau, \cdot]\!]^{\mathrm{n}}$ is defined by the *Frölicher–Nijenhuis bracket*. The corresponding cohomology is denoted by $H^i_{\mathscr{C}}(\tilde{\mathscr{E}})$ and called the $\mathscr{C}$*-cohomology* of the covering τ.

Consider the module $\mathrm{D^g}(\Lambda^i(\tilde{\mathscr{E}})) = \{X \in \mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}})) \mid X(C^\infty(\mathscr{E})) = 0\}$, where $C^\infty(\mathscr{E})$ is understood as a subalgebra in $C^\infty(\tilde{\mathscr{E}})$. Let also $\mathrm{D^s}(\Lambda^i(\tilde{\mathscr{E}})) = \mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}}))/\mathrm{D^g}(\Lambda^i(\tilde{\mathscr{E}}))$ be the quotient module. By basic properties of the Frölicher–Nijenhuis bracket, $\partial_\tau(\mathrm{D^g}(\Lambda^i(\tilde{\mathscr{E}}))) \subset \mathrm{D^g}(\Lambda^{i+1}(\tilde{\mathscr{E}}))$ and thus the short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D^g}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D^s}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial_\tau} & & \downarrow{\scriptstyle\partial_\tau} & & \downarrow{\scriptstyle\partial_\tau} & & \\
0 & \longrightarrow & \mathrm{D^g}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D^v}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D^s}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array} \tag{2}$$

arises (we preserve the notation ∂_τ for the differential in both quotient and subcomplexes). Denote by $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ the cohomology groups of the quotient and subcomplexes, respectively.

Definition 1. The groups $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ are called *shadow* and *gauge* $\mathscr{C}$-cohomologies of the covering τ, respectively. The cohomological sequence

$$\begin{aligned}
&0 \to H^0_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots \\
&\quad \to H^i_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^i_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^i_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{i+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots
\end{aligned} \tag{3}$$

corresponding to (2) is called the *long exact sequence* of the covering τ.

Remark 1. Recall (see [12]) that the modules $\Lambda^i(\tilde{\mathscr{E}})$ split into the direct sum

$$\Lambda^i(\tilde{\mathscr{E}}) = \oplus_{p+q=i}\Lambda^{p,q}(\tilde{\mathscr{E}}), \qquad \Lambda^{p,q}(\tilde{\mathscr{E}}) = \mathscr{C}\Lambda^p(\tilde{\mathscr{E}}) \otimes \Lambda^q_h(\tilde{\mathscr{E}}), \tag{4}$$

where $\mathscr{C}\Lambda^p(\tilde{\mathscr{E}})$ and $\Lambda^q_h(\tilde{\mathscr{E}})$ are modules of *Cartan* and *horizontal* forms, respectively. To (4) there corresponds the splitting $\mathrm{D^v}(\Lambda^i(\tilde{\mathscr{E}})) = \oplus_{p+q=i}\mathrm{D^v}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$ and the differential ∂_τ takes

[2]The structure element U_τ may be understood as the *connection form* of the Cartan connection $\tilde{\mathscr{E}}$. In a similar way, we call the connection form of $\mathscr{C}$ the structure element of $\mathscr{E}$.

elements $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$ to $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q+1}(\tilde{\mathscr{E}}))$. This leads to the cohomology groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$, $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and (3) actually splits into the series of the following sequences

$$0 \to H^{p,0}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,0}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,0}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$
$$\cdots \to H^{p,q}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,q+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$

The groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$ are identified with the *horizontal cohomology* of $\tilde{\mathscr{E}}$ with coefficients in $\mathrm{D}^{\mathrm{v}}(\mathscr{C}\Lambda^p(\tilde{\mathscr{E}}))$ [17].

1.1 Interpretations

Before discussing particular applications of the above constructions, let us describe the geometrical meaning of some of the groups in (3) that we shall need below.

$H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$: elements of this group are identified with *nonlocal τ-symmetries*;

$H^0_{\mathrm{g}}(\tilde{\mathscr{E}})$: these are *gauge symmetries* in τ, i.e., τ-vertical symmetries (the corresponding diffeomorphisms, if they exist, are automorphisms of τ);

$H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$: this cohomology group consists of *τ-shadows* of nonlocal symmetries, i.e., π_∞-vertical derivations $X\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ preserving the Cartan distributions. As it was already mentioned, in local coordinates, shadows are described by vector-functions $\varphi = (\varphi^1, \dots, \varphi^m)$, $\varphi^l \in C^\infty(\tilde{\mathscr{E}})$, $m = \dim \pi$, satisfying the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$, where $\tilde{\ell}_{\mathscr{E}}$ is the linearization of $\mathscr{E}$ naturally lifted to $\tilde{\mathscr{E}}$;

$H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$: these are equivalence classes of nontrivial *infinitesimal deformations* of U_τ, i.e., of the element defining the basic geometrical structure on $\tilde{\mathscr{E}}$; a deformation is infinitesimally trivial if and only if its cohomological class in $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ vanishes. On the other hand, elements of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ act on $H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) = \mathrm{sym}_\tau\mathscr{E}$ by contraction: $R(X) = \mathrm{i}_X(R)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, $R \in H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$. Nontrivial actions may correspond to elements of $H^{1,0}_{\mathscr{C}}(\tilde{\mathscr{E}})$ only;

$H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$: those cohomological classes are *τ-vertical*, or *gauge*, deformations. They deform the covering structure itself only and do not change the structure of the underlying equation $\mathscr{E}$;

$H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$: Similar to $H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$, these are also shadows of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., classes of π_∞-vertical derivations $C^\infty(\mathscr{E}) \to \Lambda^1(\tilde{\mathscr{E}})$ that preserve the Cartan distributions. Elements of $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ have the following local description: $\mu = (\mu^1, \dots, \mu^m)$, $\mu^l \in \mathscr{C}\Lambda^1(\tilde{\mathscr{E}})$ lies in $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ if and only if $\tilde{\ell}_{\mathscr{E}}(\mu) = 0$.

1.2 Reconstruction of shadows

Consider again a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and a shadow $\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. We say that φ is *reconstructible* in τ if there exists a symmetry $X\colon C^\infty(\tilde{\mathscr{E}}) \to C^\infty(\tilde{\mathscr{E}})$ such that $X\big|_{C^\infty(\mathscr{E})} = \varphi$. Of course, given a covering τ, not any shadow can be reconstructed in this covering, but a weaker result was found by N. Khor'kova and proved in [14]:

Theorem 1 (Reconstruction Theorem). *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and φ be a τ-shadow. Then there exists a covering $\tau_\varphi\colon \mathscr{E}_\varphi \to \tilde{\mathscr{E}}$ such that φ is reconstructible in the composition covering $\tau \circ \tau_\varphi\colon \mathscr{E}_\varphi \to \mathscr{E}$.*

Remark 2. How the theorem works in practice may be seen in the classical example of the KdV equation

$$u_t = uu_x + u_{xxx}. \tag{5}$$

It is known that this equation possesses two infinite series of symmetries – a local one, independent of x and t and consisting of higher KdV equations, and another, (x,t)-dependent series whose first two terms are local (the Galilean boost and a scaling symmetry) and all others are nonlocal. Take the first nonlocal one. It has the form $\psi_5 = tu_{xxxxx} + \cdots$ and contains the nonlocal variable w^1 defined by the relations

$$w^1_x = u, \qquad w^1_t = \tfrac{1}{2}u^2 + u_{xx}. \tag{6}$$

Actually, ψ_5 is not a symmetry but only a shadow in the covering corresponding to (6). To reconstruct the symmetry, one needs to find the coefficient at $\partial/\partial w^1$. But when we do this, a new nonlocal variable w^2 arises that is to satisfy the relation $w^2_x = u^2$, etc. As it was shown in [10], this process 'stops' at infinity only: to reconstruct the initial shadow, one has to add all nonlocal variables of the form $w^j_x = c^j$, where c^j is the density of the jth conservation law. As we shall see, this situation is somewhat typical.

Proof of Theorem 1. Consider the following part of (3) $H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$. Let $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ be a shadow in the covering τ. It is reconstructible if there exists a symmetry $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$ such that $\beta(X) = \varphi$. Due to the exactness, this is equivalent to $\partial(\varphi) = 0$. Thus the element $\omega_\varphi = \partial(\varphi)$ is the obstruction to reconstructibility of the shadow φ. So, the intermediate result is

Proposition 1. *A shadow $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ is reconstructible in the covering τ if and only if the obstruction $\omega_\varphi \in H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}})$ vanishes.*

The obstruction ω_φ is represented by a vector-valued horizontal 1-form on $\tilde{\mathscr{E}}$, the number of its components ω^l_φ equals the dimension of the covering τ. Each form ω^l_φ is closed and thus determines a 1-dimensional covering τ^l over $\tilde{\mathscr{E}}$, see [14]. This covering is trivial if and only if this form is exact (with respect to the horizontal de Rham differential) and $\omega_\varphi = 0$ if and only if all ω^l_φ are exact. Thus, if we choose those ω^l_φ whose horizontal cohomology classes are nontrivial, take the corresponding 1-dimensional coverings and consider the Whitney product $\tilde{\tau}\colon \tilde{\tilde{\mathscr{E}}} \to \tilde{\mathscr{E}}$ then the obstruction ω_φ will vanish in this covering, but a new obstruction of the same nature may arise. If the latter vanishes the shadow is reconstructible in $\tilde{\tau} = \tau_\varphi$, otherwise we must repeat the construction, etc. Eventually, we shall either stop at some finite step, or shall arrive to an infinite covering τ_φ. In both cases, there will be no obstruction to reconstruct the shadow φ. ∎

Remark 3. From the proof of the theorem it can be seen that the shadow φ completely determines the 'minimal' covering, where it reconstructs to a symmetry. In the case of a two-dimensional base M (i.e., when the equation $\mathscr{E}$ is in two independent variables) the cohomology classes of the forms ω^l_φ are identified with *conservation laws* of $\mathscr{E}$, generally nonlocal. But in some cases (for the KdV and similar equations) the reconstruction procedure deals with local conservation laws only, see [10].

Remark 4. Though the covering, where the shadow at hand reconstructs, is well defined (but not uniquely!) algorithmically, the symmetry that corresponds to this shadow is definitely not unique: even when a covering τ_φ is chosen, a symmetry corresponding to φ is defined up to elements of $H^0_{\mathrm{g}}(\mathscr{E}_\varphi)$, i.e., up to gauge symmetries in τ_φ. This means that to deal with nonlocal symmetries is not the same as to deal with their shadows.

1.3 Toward Bäcklund transformations [3]

Geometrically, a *Bäcklund transformation* between two differential equations $\mathscr{E}$ and $\mathscr{E}'$ is a pair of coverings $\mathscr{E} \xleftarrow{\tau} \tilde{\mathscr{E}} \xrightarrow{\tau'} \mathscr{E}'$. This geometrical construction may be understood as a differential relation between solutions of $\mathscr{E}$ and $\mathscr{E}'$. If $s \subset \mathscr{E}$ is a solution of $\mathscr{E}$ then $\tau^{-1}(s)$ is fibered by a $\dim \tau$-dimensional family of solutions to $\tilde{\mathscr{E}}$ while $\tau'(\tau^{-1}(s))$ delivers solutions to $\mathscr{E}'$, and vice versa.

In applications, one of the most important cases is when $\mathscr{E} = \mathscr{E}'$ and τ, τ' belong to a smooth family of coverings τ_t, $t \in \mathbb{R}$ being a nonremovable[3] parameter. Such a family may be regarded as a *deformation* of the covering τ_{t_0}. These deformations may be treated as follows.

Essentially, we repeat here the main result given in [3]. Let us first recall the following construction, see [14]. Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations $A_t\colon \mathscr{E} \to \mathscr{E}$. Consider an arbitrary lift of A_t to $\tilde{\mathscr{E}}$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathscr{E}} & \xrightarrow{\tilde{A}_t} & \tilde{\mathscr{E}} \\ \tau\big\downarrow & & \big\downarrow\tau \\ \mathscr{E} & \xrightarrow{A_t} & \mathscr{E} \end{array} \tag{7}$$

is commutative. Let us define on $\tilde{\mathscr{E}}$ a t-parameter family of distributions $\tilde{\mathscr{C}_t}$ by setting

$$(\tilde{\mathscr{C}_t})_{\tilde{\theta}} = (\tilde{A}_t)_*^{-1}\tilde{\mathscr{C}}_{\tilde{A}_t\tilde{\theta}}, \qquad \tilde{\theta} \in \tilde{\mathscr{E}}.$$

All distributions $\tilde{\mathscr{C}_t}$ are integrable, i.e., $[\tilde{\mathscr{C}_t}, \tilde{\mathscr{C}_t}] \subset \tilde{\mathscr{C}_t}$, and thus $\tilde{\mathscr{E}_t} = (\tilde{\mathscr{E}}, \tilde{\mathscr{C}_t})$ covers $\mathscr{E}$ by means of τ. Denote this covering by $\tau_t\colon \tilde{\mathscr{E}_t} \to \mathscr{E}$ and notice that if $\tilde{A}'_t$ is another lift satisfying (7) then for any t the covering τ_t is either equivalent to τ'_t or not. It for all t (sufficiently small) τ_t and τ'_t are equivalent then this means that X can be lifted to a symmetry of $\tilde{\mathscr{E}}$. Thus we obtain

Proposition 2. *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations and such that it cannot be lifted to a symmetry of $\tilde{\mathscr{E}}$. Then X generates a one-parameter family of (equivalence classes of) coverings $\tau_t\colon \tilde{\mathscr{E}_t} \to \tilde{\mathscr{E}}$ such that*

1. *$\tilde{\mathscr{E}_t}$ and $\tilde{\mathscr{E}_{t'}}$ isomorphic as manifolds with distributions;*
2. *τ_t and $\tau_{t'}$ are pair-wise inequivalent for sufficiently small t and t'.*

We say that a family τ_t satisfying properties 1 and 2 above is *irreducible*.

Remark 5. Let $\mathfrak{g}$ be a finite-dimensional Lie subalgebra in the algebra of classical symmetries of $\mathscr{E}$. Then we obtain a G/H-irreducible family, where G is the Lie group corresponding to $\mathfrak{g}$ and H is the stabilizer of τ under the above described action.

Remark 6. Proposition 2 means that if we have a covering τ and a shadow that cannot be reconstructed to a symmetry in this covering[4] then, under certain conditions, this shadow generates an irreducible family of coverings. As we see below, this is, in a sense, a general way to obtain irreducible families.

Consider an irreducible family of coverings $\tau_t\colon \tilde{\mathscr{E}} \to \mathscr{E}$, $\tau_0 = \tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$, and the following part of (3)

$$H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{0,1}_{\mathscr{C}}(\tilde{\mathscr{E}}).$$

The family τ_t may be considered as a deformation of τ; hence its infinitesimal part μ lies in $H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}})$. By Property 1 from Proposition 2, the corresponding deformation of $\tilde{\mathscr{E}}$ is trivial and thus $\alpha(\mu) = 0$. By the exactness, $\mu = \partial(\varphi)$, $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$. The deformation τ_t is infinitesimally nontrivial if and only if $\mu \neq 0$ and, again by the exactness, if and only if $\varphi \neq \beta(X)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., if and only if φ is not reconstructible to a τ-symmetry. To state the final result, it remains to note that the action of ∂ is given by $[\![U_\tau, \cdot]\!]^{\mathrm{n}}$.

[3]I.e., for any $t_1 \neq t_2$ the coverings τ_{t_1} and τ_{t_2} are inequivalent.

[4]For the problem of reconstruction see Subsection 1.2.

Theorem 2. *Any irreducible family of coverings τ_t, $\tau_0 = \tau$, is infinitesimally generated by a τ-shadow that cannot be reconstructed to a nonlocal τ-symmetry.*

Remark 7. Of course, not to any shadow φ there corresponds an irreducible family of coverings. Anyway, we can construct a *formal deformation* $\exp(t\varphi)U_\tau$ and use it in some applications.

1.4 Action of recursion operators

We now pass to the last topic of this section. Let us first briefly recall the cohomological theory of recursion operators as it was exposed in [13]. Consider an equation $\mathscr{E}$ and the $\mathscr{C}$-complex (1) associated to it. Then, as it was already mentioned, the group $H^0_{\mathscr{C}}(\mathscr{E})$ is identified with the Lie algebra $\operatorname{sym}(\mathscr{E})$ of higher symmetries of $\mathscr{E}$. The contraction (or inner product) operation determines an action of elements of $H^1_{\mathscr{C}}(\mathscr{E})$ on $\operatorname{sym}(\mathscr{E})$ and nontrivial actions may correspond to the elements of $H^{1,0}_{\mathscr{C}}(\mathscr{E})$ only[5]. Locally, these elements are represented as Cartan vector-forms of degree 1, $\omega = (\omega_1, \dots, \omega_m)$, $\omega_l \in \mathscr{C}\Lambda^1(\mathscr{E})$, $m = \dim\pi$, satisfying the equation

$$\ell_{\mathscr{E}}(\omega) = 0. \tag{8}$$

Solving (8) and applying these solutions to the known symmetries we, in principle, can generate new symmetries. The same remains valid for nonlocal symmetries if we take $\tilde{\mathscr{E}}$ instead of $\mathscr{E}$. It is natural to anticipate that in this way we find recursion operators for symmetries of the equation at hand.

But in practice the picture is more complicated. If, for example, we realize the above procedure for the KdV equation (5) we shall obtain trivial solutions only. On the other hand, this equation possesses the recursion operator

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}D_x^{-1},$$

where D_x is *total derivative* with respect to x. Of course, this operator could not be found by the described above method because of the nonlocal term D_x^{-1}. Such a term may arise in the covering given by (6), but if we try the same in this nonlocal setting the result will be negative again.

What actually gives nontrivial results is the following (see all the examples in [13]). Let us show how it works for the KdV equation. Namely, we do the following:

1. We take the covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ given by (6). Note that $\tilde{\mathscr{E}}$ is isomorphic to the infinite prolongation of the equation

   $$w_t = w_{xxx} + \tfrac{1}{2}u_x^2.$$

2. Then we consider the module of Cartan 1-forms on $\tilde{\mathscr{E}}$. It is generated by the forms

   $$\omega_k = du_k - u_{k+1}\,dx - D_x^k(u_3 + u_1)\,dt, \qquad \omega_{-1} = dw^1 - u\,dx - \left(u_2 + \tfrac{1}{2}u^2\right)\,dt,$$

   where $k = 0, 1, \dots$ and $u_k = u\underbrace{x \dots x}_{k\ \text{times}}$, $u_0 = u$.

3. The next step is to solve the equation

   $$\tilde{\ell}_{\mathscr{E}}(\Omega) = 0, \tag{9}$$

   where $\tilde{\ell}_{\mathscr{E}}$ is the lift of the linearization to τ and $\Omega = \sum_{k\ge -1} f_k\omega_k$.

[5]Note also that the contraction determines a structure of an associative algebra with unit on $H^1_{\mathscr{C}}(\mathscr{E})$.

4. Solving (9), we obtain two independent solutions: $\Omega_1 = \omega_0$ and $\Omega_2 = \omega_2 + \frac{2}{3}u\omega_0 + \frac{1}{3}\omega_{-1}$. The first one corresponds to the identical action while the second one gives the classical recursion operator for the KdV equation (see above).

So, from this scheme we see that construction of recursion operators amounts to computation of the group $H_1^{1,0}(\tilde{\mathscr{E}})$. Indeed, we have the following

Theorem 3. *Let $\mathscr{E}$ be an equation and $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering. Then the contraction operation generates the action $R\colon H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \to H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$, where $R \in H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$, i.e., elements of $H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$ take τ-symmetries to τ-shadows*[6].

Remark 8. The result of this action, in general, is really a shadow. For example, applying the recursion operator of the KdV equation to the scaling symmetry $\psi_3 = tu_{xxx} + \cdots$, we obtain the element ψ_5 (see above) that reconstructs to a symmetry in the infinite-dimensional covering, where all conservation laws of the equation are 'killed'.

Proof of Theorem 3. The proof is very simple: consider the standard contraction

$$\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \lrcorner\, \mathrm{D}^{\mathrm{v}}(\Lambda^j(\tilde{\mathscr{E}})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}})).$$

It remains to note that if $X \in \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ and $Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^j(\tilde{\mathscr{E}}))$ then $X \lrcorner\, Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}}))$ and that the differential in (1) preserves the inner product and thus the latter is inherited in the cohomology groups. ∎

Remark 9. In some cases action of recursion operator on local symmetries leads to local symmetries again. For example, this is the case for classical hierarchies of integrable equations. Some remarks on how to establish locality may be found in [11].

2 Δ-coverings [7]

A covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is said to be *linear* if τ is a vector bundle and for any vector field X on M the field $\tilde{X}$ lifted to $\tilde{\mathscr{E}}$ by the Cartan connection $\tilde{\mathscr{C}}$ preserves the subspace of fiber-wise linear functions in $C^\infty(\tilde{\mathscr{E}})$. If Δ is a $\mathscr{C}$-differential operator over $\mathscr{E}$, then there exists a canonical way to construct a linear covering associated to Δ. This covering is called Δ-*covering*. From *technical* point of view, Δ-coverings are useful to solve the following *factorization problem*: take another $\mathscr{C}$-differential operator ∇ and find $\mathscr{C}$-differential operators V and V' such that the equation

$$\nabla \circ V = V' \circ \Delta \tag{10}$$

holds.

Theorem 4. *Solutions $V \mod (\Box \circ \Delta)$ of (10), where $\Box$ is a $\mathscr{C}$-differential operator, are in one-to-one correspondence with fiber-wise linear solutions s of the equation $\tilde{\nabla}(s) = 0$, where $\tilde{\nabla}$ is the lifting of ∇ to the Δ-covering.*

There are two canonical coverings associated to a given equation. Namely, taking the operator $\ell_{\mathscr{E}}$, we obtain the $\ell_{\mathscr{E}}$-*covering* and for $\ell^*_{\mathscr{E}}$ one gets the $\ell^*_{\mathscr{E}}$-*covering*. The first one plays the role of the tangent bundle to $\mathscr{E}$ while the second is the counterpart of the cotangent bundle in the category of differential equations. This determines *conceptual* importance of the introduced constructions.

[6] Actually, nontrivial actions may correspond to the elements of $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}}) \subset H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$ only.

In the simplest case of a scalar evolution equation $u_t = f(x,t,u,\dots,u_k)$ the ℓ-covering is described by additional coordinates $\omega = \omega_0,\ \omega_1,\ \dots$ that satisfy the relations

$$(\omega_i)_x = \omega_{i+1}, \qquad \omega_t = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}\omega_i,$$

while the ℓ^*-covering is described by the coordinates $p = p_0,\ p_1,\ \dots$ satisfying

$$(p_i)_x = p_{i+1}, \qquad p_t = \sum_{i=0}^{k} (-D_x)^i \left(\frac{\partial f}{\partial u_i} p\right).$$

Similar formulas can be obtained for equations of general form.

2.1 Recursion operators for symmetries [7, 13]

Let $\Delta = \nabla = \ell_{\mathscr{E}}$. Then equation (10) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}},$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to itself. In other words, V is a recursion operator for symmetries. From Theorem 4 we obtain an efficient method to compute such operators:

Corollary 1. *Recursion operators for symmetries are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}(s) = 0$, i.e., with linear shadows in the $\ell_{\mathscr{E}}$-covering.*

Remark 10. Evidently, this result is in full agreement with what we saw in Subsection 1.4.

2.2 Recursion operators for conservation laws [7]

Let $\Delta = \nabla = \ell_{\mathscr{E}}^*$. Then equation (10) takes the form $\ell_{\mathscr{E}}^* \circ V = V' \circ \ell_{\mathscr{E}}^*$, i.e., V takes $\ker \ell_{\mathscr{E}}^*$ (generating functions) to itself. In other words, V is a recursion operator for generating functions. From Theorem 4 we obtain an efficient method to compute such operators:

Corollary 2. *Recursion operators for generating functions are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}^*(s) = 0$, i.e., with nonlocal linear generating functions in the $\ell_{\mathscr{E}}^*$-covering.*

To obtain a recursion procedure for conservation laws, recall that for 'two-line equations' [18–20] their conservation laws (the term $E_1^{0,n-1}$ of the Vinogradov $\mathscr{C}$-spectral sequence) are related to generating functions (the term $E_1^{1,n-1}$) by the injective differential $d_1^{0,n-1}$, which, in the simplest case of evolution equations, coincides with the Euler operator $\mathcal{E}$, and a generating function ψ corresponds to a conservation law if and only if $\ell_\psi = \ell_\psi^*$.

2.3 Hamiltonian structures [7]

Let $\mathscr{E}$ be an evolution equation, $\Delta = \ell_{\mathscr{E}}^*$, and $\nabla = \ell_{\mathscr{E}}$. Then equation (10) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}}^*, \tag{11}$$

i.e., V takes $\ker \ell_{\mathscr{E}}^*$ (generating functions) to $\ker \ell_{\mathscr{E}}$ (symmetries). We call such operators *pre-Hamiltonian*. As above, Theorem 4 provides an efficient method to compute these operators:

Corollary 3. *Pre-Hamiltonian operators are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}(s) = 0$, i.e., with linear shadows in the $\ell_{\mathscr{E}}^*$-covering.*

A pre-Hamiltonian structure is *Hamiltonian* if (a) $V + V^* = 0$ and (b) $[\![s, s]\!]^{\mathrm{s}} = 0$, where $[\![\cdot, \cdot]\!]^{\mathrm{s}}$ is the *variational Schouten bracket* [4]. If $\mathscr{E}$ is of the form

$$u_t = f(x, t, u, \dots, u_k), \qquad u = (u^1, \dots, u^m), \qquad f = (f^1, \dots, f^m), \qquad u_l = \frac{\partial^l u}{\partial x^l}, \tag{12}$$

and $V = \| \sum_l a^l_{ij} D^l_x \|$, where D_x is the total derivative with respect to x, then conditions (a) and (b) can be reformulated in terms of the function[7] $W_V = \sum_{ijl} a^l_{ij} p^i_l p^j_0$ as follows

$$\sum_i \frac{\delta W_V}{\delta p^i} p^i_0 = -2W_V, \qquad \mathcal{E} \sum_i \left(\frac{\delta W_V}{\delta u^i} \frac{\delta W_V}{\delta p^i} \right) = 0$$

and are easily checked. Here $\delta/\delta u$, $\delta/\delta p$ are variational derivatives[8].

2.4 Symplectic structures [7]

Let again $\mathscr{E}$ be an evolution equation, $\Delta = \ell_{\mathscr{E}}$, and $\nabla = \ell^*_{\mathscr{E}}$. Then equation (10) takes the form

$$\ell^*_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}}, \tag{13}$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to $\ker \ell^*_{\mathscr{E}}$ (generating functions). We call such operators *presymplectic*. Again, Theorem 4 gives an efficient method to compute these operators:

Corollary 4. *Presymplectic operators are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}^*_{\mathscr{E}}(s) = 0$, i.e., with linear generating functions in the $\ell_{\mathscr{E}}$-covering.*

A presymplectic structure is *symplectic* if condition (a) holds and s is 'variationally closed', that for equation (12) means that

$$\mathcal{E} \sum_i \frac{\delta W_V}{\delta u^i} p^i_0 = 0.$$

Remark 11. As it is shown in [8], any skew-adjoint operator V that satisfies the condition (11) *is automatically Hamiltonian*, provided that the symbol of the right-hand side of the corresponding evolution equation is nondegenerate. A similar statement is valid for symplectic structures if we take skew-adjoint solutions of (13).

2.5 Canonical representation for nonlocalities

As it was mentioned above, efficient construction of recursion operators and similar structures needs introduction of nonlocal variables. In all applications we used the following *canonical* choice of nonlocal variables:

for the ℓ-covering: ω_ψ defined by $(\omega_\psi)_x = \psi\omega_0$, where ψ is a generating function, i.e., satisfies the equation $\ell^*_{\mathscr{E}}(\psi) = 0$;

for the ℓ^*-covering: p_φ defined by $(p_\varphi)_x = \varphi p_0$, where φ is a symmetry, i.e., satisfies the equation $\ell_{\mathscr{E}}(\varphi) = 0$.

[7] The variables p^i_l, $l = 0, \dots, k$, $i = 1, \dots, m$, should be *odd*.

[8] In the formula above, as well as in the condition for symplectic structures below, the Euler operator contains variational derivatives both in u and in p.

These nonlocal variables lead to the operators of the form

$$\sum_{\alpha\geq 0} c_{ij}^{\alpha} D_y^{\alpha} + \sum_{\beta} d_j^{\beta} D_y^{-1} \circ e_i^{\beta},$$

where $\|c_{ij}^{\alpha}\|$ is an $m\times m$-matrix, $\|d_j^{\beta}\|$ is an $m\times l$-matrix, and $\|e_i^{\beta}\|$ is an $l\times m$-matrix for some $l>0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices d and e look for different types of operators.

Type of operator	Lines of matrix d	Columns of matrix e
Recursions for symmetries	Symmetry	Generating function
Recursions for generating functions	Generating function	Symmetry
Hamiltonian structures	Symmetry	Symmetry
Symplectic structures	Generating function	Generating function

2.6 Example: the coupled KdV-mKdV system

We shall now describe a Hamiltonian structure for the coupled KdV-mKdV system of the form

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3vv_{xxx} - 3v_x v_{xx} + 3u_x v^2 + 6uvv_x, \\ v_t &= -v_{xxx} + 3v^2 v_x + 3uv_x + 3u_x v. \end{aligned} \tag{14}$$

This system arises as the so-called *bosonic limit* of the $N=2$, $a=1$ supersymmetric extension of the KdV equation [6]; integrability properties of this system (existence of a recursion operator) were studied in [9]. In [5], by means of the prolongations structure techniques, a Lax pair for (14) was constructed.

2.6.1 Hamiltonian structures

Using the methods described above, we found the following Hamiltonian operator for system (14):

$$V = \begin{pmatrix} -D_x^3 + 4uD_x + 2u_1 & 2vD_x \\ 2vD_x + 2v_1 & D_x \end{pmatrix}. \tag{15}$$

Moreover, the following result is valid:

Theorem 5. *The coupled KdV-mKdV system* (14) *is Hamiltonian with respect to the the Hamiltonian operator* (15) *and possesses the Hamiltonian* $H=\frac{1}{2}(u^2+uv^2-vv_2)$. *The corresponding energy is given by the form*

$$\begin{aligned} \eta = \tfrac{1}{2}\big(u^2 + uv^2 - vv_2\big)\,dx + \tfrac{1}{2}\big(&4u^3 + 9u^2v^2 - 2uu_2 + 3uv^4 - 11uvv_2 + uv_1^2 \\ &+ u_1^2 - u_1vv_1 - 4u_2v^2 - 6v^3v_2 - 3v^2v_1^2 + vv_4 - v_1v_3 + v_2^2\big)\,dt. \end{aligned}$$

This structure is unique in the class of Hamiltonian structures independent of x and t and polynomial in u_k, v_k.

2.6.2 Recursion operators

To describe the recursion operator for (14), let us first introduce the necessary nonlocal variables. They are denoted by w, w_1, w_{11}, w_{12} and defined by the equations

$$w_x = v, \qquad w_t = 3uv + v^3 - v_2;$$

$$
\begin{aligned}
&w_{1,x} = u, \qquad w_{1,t} = 3u^2 + 3uv - u_2 - 3vv_2;\\
&w_{11,x} = \cos(2w)w_1 v + \sin(2w)v^2,\\
&w_{11,t} = \cos(2w)(3wuv + wv^3 - wv_2 + uv_1 - u_1 v - v^2 v_1) + \sin(2w)(4uv^2 + v^4 - vv_2 - v_1^2);\\
&w_{12,x} = \cos(2w)v^2 - \sin(2w)wv,\\
&w_{12,t} = \cos(2w)(4uv^2 - v^4 - 2vv_2 + v_1^2) + \sin(2w)(-3wuv - wv^3 + wv_2 - uv_1 + v^2 v_1).
\end{aligned}
$$

Then the recursion operator is given by a 2×2-matrix operator R with the entries

$$
\begin{aligned}
R_{11} &= D_x^2 - 4u - v^2 - Y_{1,1}^u D_x^{-1} \circ \psi_{1,2}^u + Y_{1,2}^u D_x^{-1} \circ \psi_{1,1}^u - \tfrac{3}{2} Y_{1,0}^u D_x^{-1},\\
R_{12} &= 2vD_x^2 + v_1 D_x - 3uv + 2v_2 - Y_{1,1}^u D_x^{-1} \circ \psi_{1,2}^v + Y_{1,2}^u D_x^{-1} \circ \psi_{1,1}^v + Y_{2,1}^u D_x^{-1},\\
R_{21} &= -2v - Y_{1,1}^v D_x^{-1} \circ \psi_{1,2}^u + Y_{1,2}^v D_x^{-1} \circ \psi_{1,1}^u - \tfrac{3}{2} Y_{1,0}^v D_x^{-1},\\
R_{22} &= D_x^2 - 2u - v^2 - Y_{1,1}^v D_x^{-1} \circ \psi_{1,2}^v + Y_{1,2}^v D_x^{-1} \circ \psi_{1,1}^v + Y_{2,1}^v D_x^{-1},
\end{aligned}
$$

where $Y_{1,0}$, $Y_{1,1}$, $Y_{1,2}$, and $Y_{2,1}$ are symmetries of (14) with the components

$$
\begin{aligned}
&Y_{1,0}^u = u_1, \qquad Y_{1,0}^v = v_1;\\
&Y_{1,1}^u = -\cos(2w)\left(\tfrac{1}{2}u_1 + vv_1\right) + \sin(2w)\left(uv - \tfrac{1}{2}v_2\right), \qquad Y_{1,1}^v = \tfrac{1}{2}\cos(2w)v_1 - \tfrac{1}{2}\sin(2w)u;\\
&Y_{1,2}^u = \cos(2w)(uv - \tfrac{1}{2}v_2) + \sin(2w)\left(\tfrac{1}{2}u_1 + vv_1\right), \qquad Y_{1,2}^v = -\tfrac{1}{2}\cos(2w)u - \tfrac{1}{2}\sin(2w)v_1;\\
&Y_{2,1}^u = \cos(2w)(-w_{11}u_1 - 2w_{11}vv_1 + 2w_{12}uv - w_{12}v_2)\\
&\qquad + \sin(2w)(2w_{11}uv - w_{11}v_2 + w_{12}u_1 + 2w_{12}vv_1) - 2uv_1 - 3u_1 v - 2v^2 v_1 + v_3,\\
&Y_{2,1}^v = \cos(2w)(w_{11}v_1 - w_{12}u) - \sin(2w)(w_{11}u + w_{12}v_1) - (u_1 + vv_1),
\end{aligned}
$$

while

$$
\begin{aligned}
&\psi_{1,1}^u = -\sin(2w), \qquad \psi_{1,1}^v = 2(2\sin(2w)v - w_{12});\\
&\psi_{1,2}^u = -\cos(2w), \qquad \psi_{1,2}^v = 2(2\cos(2w)v + w_{11}).
\end{aligned}
$$

are the generating functions (cf. the remark above on the canonical representation of nonlocal objects).

2.7 Back to Bäcklund transformations

The above described scheme needs one generalization at least. Since recursion operators arising in practice usually contain terms of the D_x^{-1} type, these nonlocalities should be added in the initial setting[9]. This is being done by substituting equation $\mathscr{E}$ with its covering $\tilde{\mathscr{E}}$ in all constructions. Consequently, the operators obtained cease to be differential, but can be transformed to *$\mathscr{C}$-differential relations* between the objects where they act. Geometrically, these relations are realized as Bäcklund transformations relating:

- $\ell_{\mathscr{E}}$-covering with itself (recursion operators for symmetries);
- $\ell_{\mathscr{E}}^*$-covering with itself (recursion operators for generating functions);
- $\ell_{\mathscr{E}}$-covering with $\ell_{\mathscr{E}}^*$ covering (pre-Hamiltonian and presymplectic structures).

The first step in this direction was made in [16], where recursion operators for symmetries were treated as Bäcklund transformations. Clearly, generalized recursion operators for generating functions can be constructed in a similar way. As for the third type of Bäcklund transformations, one can see that they unify the concepts of pre-Hamiltonian and presymplectic structures. Following [2], these transformations may be called *pre-Dirac structures*. Their study is a subject for future research.

[9]Such terms arise also in *nonlocal* Hamiltonian and symplectic structures.

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