Caputo Fractional Derivatives and Its Applications

MCS 491

Graduation Project I

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Introduction

Introduction

The aim of this project is to understand and to apply the fractional calculus to solve differential equation with Caputo Derivatives. The plan of my project is as following in three section. In first section I give the definiton of Riemann-Liouville Fractional Derivatives and Caputo Fractional Derivatives. In second section I give equations with Caputo Derivatives and last section I give some examples with Caputo Derivatives.

Fractional Integrals and Fractional Derivatives

Definition

Fractional Integrals and Fractional Derivatives: Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and in engineering. In its turn, mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors: the iteration method, the series method, the Fourier transform technique, special methods for fractional differential equations of rational order or for equations of special type, the Laplace transform technique, the operational calculus method.

Fractional Integrals and Fractional Derivatives

Definition

Since both the Riemann-Liouville and the Caputo derivatives are defined through the Riemann-Liouville fractional integral and this operator plays a very important role in the development of the corresponding operational calculus, there are some coinciding elements in the operational calculus for both fractional derivatives.

Riemann-Liouville Fractional Integrals and Fractional Derivatives

Statement

In this section we give the definitions of the Riemann - Lioville fractional integrals and fractional derivatives on a finite interval of the real line and present some of their properties in spaces of summable and continuous functions.More detailed information may be found in the book Samko et al. Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\Re$. The Riemann-Liouville fractional integrals $(I^{\alpha}_{a+}f)$ and $(I^{\alpha}_{b-}f)$ of order $\alpha \in$ C $(\Re(\alpha) > 0)$ are defined by

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$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad (x > a; \Re(\alpha) > 0)$$

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$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad (x < b; \Re(\alpha) > 0)$$

Riemann-Liouville Fractional Integrals and Fractional Derivatives

Left Riemann-Liouville Fractional Derivative

In this part, we formulate the problem in terms of the left Riemann-Liouville fractional derivatives, which are defined as follows, the left Riemann-Liouville fractional derivative;

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Riemann-Liouville Fractional Integrals and Fractional Derivatives

Right Riemann-Liouville Fractional Derivative

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Riemann-Liouville Fractional Integrals and Fractional Derivatives

Right Riemann-Liouville Fractional Derivative

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$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau)d\tau, $$

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$${}_aD_t^{\alpha}C = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$$

Caputo Derivatives

Statement

In this section we present the definitions and some properties of the Caputo fractional derivatives. Let $[a, b]$ be a finite interval of the real line $\Re$, and let $D^{\alpha}_{a+}[y(t)](x) \equiv (D^{\alpha}_{a+}y)(x)$ and $D^{\alpha}_{b-}[y(t)](x) \equiv (D^{\alpha}_{b-}y)(x)$ be the Riemann-Liouville fractional derivatives of order $\alpha \in C(\Re(\alpha) \geq 0)$. The fractional derivatives $({}^{C}D^{\alpha}_{a+}y)(x)$ and $({}^{C}D^{\alpha}_{b-}y)(x)$ of order $\alpha \in C(\Re(\alpha) \geq 0)$ on $[a, b]$ are defined by via the above Riemann-Liouville fractional derivatives by

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$$({}^{C}D^{\alpha}_{a+}y)(x) := \left(D^{\alpha}_{a+} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{k}}{k!}(t-a)^{k} \right] \right)(x) \qquad (1)$$

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and

$$({}^{C}D^{\alpha}_{b-}y)(x) := \left(D^{\alpha}_{b-}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{k}}{k!}(b-t)^{k}\right]\right)(x) \qquad (2)$$

$$n = [\Re(\alpha)] + 1 \quad for \quad \alpha \notin N_0; \quad n = \alpha \quad \alpha \in N_0.$$

Caputo Derivatives

Statement

These derivatives are called left-sided and right-sided Caputo fractional derivatives of order α.
In particular , when $0 < \Re(\alpha) < 1$, the relations (1.) and (2.) take the following forms:

$$({}^{C}D^{\alpha}_{a+}y)(x) := (D^{\alpha}_{a+}[y(t) - y(a)])(x)$$

and

$$({}^{C}D^{\alpha}_{b-}y)(x) := (D^{\alpha}_{b-}[y(t) - y(b)])(x)$$

If $\alpha \notin N_0$ and $y(x)$ is a function for which the Caputo fractional derivatives $({}^{C}D^{\alpha}_{a+}y)(x)$ and $({}^{C}D^{\alpha}_{b-}y)(x)$ of order $\alpha \in C(\Re(\alpha) \geq 0)$ exist together with the Riemann-Liouville fractional derivatives $({}^{C}D^{\alpha}_{a+}y)(x)$ and $({}^{C}D^{\alpha}_{b-}y)(x)$

Caputo Derivatives

Statement

Let $\Re(\alpha) \geq 0$ and let n be given. If $y(x) \in AC^n[a,b]$, then the Caputo fractional derivatives $({}^C D^{\alpha}_{a+} y)(x)$ and $({}^C D^{\alpha}_{b-} y)(x)$ exist almost everywhere on $[a,b]$.
(a) If $\alpha \notin N_0, ({}^C D^{\alpha}_{a+} y)(x)$ and $({}^C D^{\alpha}_{b-} y)(x)$ are represented by

$$({}^C D^{\alpha}_{a+} y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^n(t)dt}{(x-t)^{\alpha-n+1}} = (I^{n-\alpha}_{a+} D^n y)(x)$$

and

$$({}^C D^{\alpha}_{b-} y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^n(t)dt}{(t-x)^{\alpha-n+1}} = (-1)^n (I^{n-\alpha}_{b-} D^n y)(x)$$

respectively, where $D = d/dx$ and $n = [\Re(\alpha)] + 1$. In particular, when $0 < \Re(\alpha) < 1$ and $y(x) \in AC[a,b]$,

Caputo Derivatives

Statement

$$({}^C D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{y\prime(t)dt}{(x-t)^{\alpha}} = (I^{1-\alpha}_{a+}Dy)(x)$$

and

$$({}^C D^{\alpha}_{b-}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{y\prime(t)dt}{(t-x)^{\alpha}} = (I^{1-\alpha}_{b-}Dy)(x)$$

(b) If $\alpha = n \in N_0$, then $({}^C D^{n}_{a+}y)(x)$ and $({}^C D^{\alpha}_{b-}y)(x)$ are represented by (36.) and (37.). In particular ,

$$({}^C D^{0}_{a+}y)(x) = ({}^C D^{0}_{b-}y)(x) = y(x)$$

Equations with Caputo Derivatives

Statement

Here we consider problems for the nonlinear differential equations with the Caputo derivative $({}^{C}D^{\alpha}_{a+}y)(x)$.

The Cauchy Problem with the Initial Conditions at the Endpoint of the Interval Global Solution

Definition

In this section we consider the nonlinear differential equation of order $\alpha > 0$:

$$({}^C D^{\alpha}_{a+} y)(x) := f[x, y(x)] \quad (\alpha > 0; a \leq x \leq b),$$

involving the Caputo fractional derivative $({}^C D^{\alpha}_{a+} y)(x))$, on a finite interval $[[a, b]$ of the real axis $\Re$, with the initial conditions

$$y^k(a) = b_k, b_k \in \Re \quad (k = 1, 2, 3..., n-1); \quad n = -[-\alpha]$$

Volterra Integral Equation

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]dt}{(x-a)^{1-\alpha}} \quad (a \le x \le b).$$

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and then

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x ((x-t)^{n-1})(f[t, y(t)]dt)$$

$$(a \leq x \leq b).$$

Euler-Lagrange Equations

In this section we will obtain the necessary conditions for a continuoulsy differentiable function y which minimize the variational problem

$$I(y) = \int_a^b L(x, y(x), D_a^\alpha y(x))dx \quad y(a) = a_0, \quad y(b) = b_0$$

We will see that these necessary conditions will be in form of Euler-Lagrange Equations. by parts

$$\int_a^b \frac{\partial L}{\partial y} y(x)dx + \int_a^b D_{b-}^\alpha \left(\frac{\partial L}{D_{a+}^\alpha y(x)} \right) y(x)dx$$

Fractional Lagrange Conditions :

$$\frac{\partial L}{\partial y} + D_{b-}^\alpha \left(\frac{\partial L}{D_{a+}^\alpha y(x)} = 0 \right)$$

Examples

If $\Re(\alpha) > 0$ and $\lambda > 0$, then

$$({}^C D_+^\alpha e^{\lambda t})(x) = \lambda^\alpha e^{\lambda x}$$

and

$$({}^C D_-^\alpha e^{-\lambda t})(x) = \lambda^\alpha e^{-\lambda x}$$

The Mittag-Leffler function $E_\alpha[\lambda(x-a)^\alpha]$ is invariant with respect to the Caputo derivative ${}^C D_{a+}^\alpha$,but it is not the case for the Caputo derivative ${}^C D_-^\alpha$.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha+1)}$$

$=$

$${}^C D_{a+}^\alpha \left(\sum^{\infty} \frac{z^k}{\Gamma(\alpha+1)} \right) = \sum^{\infty} \frac{z^k}{\Gamma(\alpha+1)} \left({}^C D_{a+}^\alpha z^k \right)$$

Example I

Example I

$$= ({}^C D^{\alpha}_{a+} E_{\alpha}[\lambda(x-a)^{\alpha}])(x)$$

$$= \sum_{k=0}^{\infty} \frac{{}^C D^{\alpha}_{a+}}{\Gamma(\alpha k+1)} (\lambda(x-a)^{\alpha})^{k}$$

$$= \frac{\lambda}{\Gamma(\alpha k+1)} E_{\alpha} \lambda (x-a)^{\alpha}$$

Example II

Example II

$$= ({}^C D_-^{\alpha} E_{\alpha}[\lambda(x-a)^{-\alpha}])(x)$$

$$={}^C\ D_-^{\alpha} - \left(x^{\alpha-1} \frac{(\lambda x^{-\alpha})^k}{\Gamma(\alpha k+1)} \right)$$

$$= \frac{1}{x} E_{\alpha,1-\alpha}(\lambda x^{-\alpha})$$

where

$$E_{\alpha,1-\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1 - \alpha)} (\alpha, 1-\alpha, \Re(\alpha) > 0)$$

Conclusion

Caputo Fractional Derivatives and Its Applications

By using the basic tool, Riemann-Liouville Fractional Integrals, it was possible to define Riemann-Liouville Fractional derivatives. These fractional derivatives generalize the standard derivative in usual calculus. The derivative of the constant function under Riemann-Liouville Fraction is not zero which is strange for me. However, the derivative of the constant function under Caputo Derivative is zero.

Both of the Riemann-Liouville and Caputo Derivatives are interesting to me. They appeared in many fields, mathematics, physics, biology and engineering.

I learned the Euler-Lagrange equation for Lagrangians containing fractional derivatives and realized that when $\alpha \to 1$ we obtain the Euler-Lagrange Equations that I learned in the calculus of variation course.

- A.A.KILBAS, H.M.VA, J.J.TRUJILLO *Theory and Applications of Fractional Differential Equations North-Holland*, 2006.
- Ixaru L. Gr. , *PODLUBNY, I., Fractional Differential Equations, Academic Press, San Diego*, 1999.
- Evans G. , *SAMKO, S.G., KILBAS, A.A., and MARICHEV, O. I., Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Switzerland*, 1993
- Davis P.J. and Rabinowitz P. , *WEST, B.J., BOLOGNA, M., and GRIGOLINI, P., Physical of Fractal Operators, Springer-Verlag, New York and Berlin*, 2003
- Ixaru L. Gr. , *ZASLAVSKY, G.M., Chaos and Fractional Dynamics, vol. 511 of Lect. Notes in Phys., Oxford University Press, Oxford*, 2005