Symplectic or Contact Structures on Lie Groups

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Introduction

In the sequel G stands for a Lie group (supposed to be connected as a matter of simplicity) with Lie algebra $\mathfrak{g} := T_\varepsilon(G)$, where ε is the unit of G. If G is endowed with a left invariant differential 1-form α^+ such that

$$\alpha^+ \wedge \left(d\alpha^+\right)^p \neq 0$$

where $2p+1$ is the dimension of G, we will say that the pair (G, α^+) is a contact Lie group and that $(\mathfrak{g}, \alpha)$ is a contact Lie algebra; here $\alpha := \alpha^+_\varepsilon$.

Following Lichnerowicz-Medina [19] a pair (G, Ω^+) where Ω^+ is a left invariant symplectic form, is termed a symplectic Lie group and the corresponding infinitesimal object $(\mathfrak{g}, \omega)$, where $\omega := \Omega^+_\varepsilon$, is referred to as a symplectic Lie algebra.

In [21] (see also [5], [6]) a method of construction of symplectic Lie algebras, called "Symplectic Double Extension", is described. According to the theorem 2.5 in [21] every nilpotent symplectic Lie algebra is obtained from a sequence of "Symplectic Double Extension" starting from the trivial abelian Lie algebra consisting on only one element.

This result immediately implies that every nilpotent contact Lie algebra can be obtained by two operations, namely: the "Symplectic Double Extension" and the contactization.

Corresponding to those operations are inverse operations, well-known by geometers, the symplectic reduction and the symplectization.

Here are some few words about the main results and the organization of this work.

The section 1 gives a geometric description of the Contact Lie groups. In the theorem 1, they arise to be fibre bundles with connections the fibre being one dimensional, over a reductive homogeneous space

$$H \overset{i}{\hookrightarrow} G \xrightarrow{\pi} M = G/H$$

provided with a symplectic form, satisfying $\pi^*(\Omega) = \widehat{\Omega}$ where $\widehat{\Omega}$ is the curvature form of the connection α^+ (see [3]). Sometimes G/H is a symplectic Lie group.

The section 2 supplies a necessary and sufficient condition for a filiform Lie group to possess a left invariant contact form (see the Theorem 4). Such a contact form is unique up to a non zero scalar multiple and has a simple expression in terms of an adapted basis (see the Theorem 5).

Here it is convenient to recall some known facts. According to a result from Gromov, every Lie group of odd dimension admits a non necessary left invariant contact form. A symplectic Lie group (G, Ω^+) is endowed with a left invariant affine structure (see [4]) defined by the following formulas for a, b, c in $\mathfrak{g}$

$$\omega(ab, c) = -\omega(b, [a, c])$$

$$\bigtriangledown_{a^+} b^+ := (ab)^+$$

where a^+ is the left invariant vector field on G such that $a^+_\varepsilon = a$. Such connection $\bigtriangledown$ is fundamental in the description of the symplectic Lie groups and specially Kählérian Lie groups (see [5], [6]). Unlike the symplectic case there exists contact Lie groups with no left invariant affine structure. This is what happens for semi-simple contact Lie groups. More surprising, there even exists nilpotent contact Lie groups that never admit such an affine structure: a direct verification allows us to check that the example of Benoist (of dimension 11) supplied in [2] is among them.

Our work ends by supplying all nilpotent symplectic Lie algebras of dimension ≤ 6.

In this paper the following standard convention will be used without explicit mentioning: for a concrete basis $X_1, \ldots, X_n$ of a Lie algebra only those brackets $[X_i, X_j]$ which are nonzero and for which $i < j$ will be explicitly defined .

1 Contact Lie Groups as principal bundles with connection

The aim of this section is to prove the more or less known following results (see [3], [11], [12]).

Theorem 1.1 *Let (G, α^+) be a connected contact Lie group and H the isotropy subgroup of $\alpha := \alpha^+_\varepsilon$, for the coadjoint action. Then*

(a) The Lie group H is 1-dimensional and the homogeneous space $M := G/H$ is reductive in the sense of Nomizu.

(b) The form α^+ is a "connection form" on the canonical principal bundle

$$H \overset{i}{\hookrightarrow} G \overset{\pi}{\to} M = G/H \tag{1}$$

the curvature form $\widetilde{\Omega}$ of which satisfies the condition $\widetilde{\Omega} = d\alpha^+$.

(c) There exists a symplectic form Ω on M such that $\pi^*(\Omega) = \widetilde{\Omega}$.

(d) The canonical action of G on $(G/H_0, \Omega_0)$ is Hamiltonian, where H_0 is the connected component of the unit in H and $\Omega_0 = p^*(\Omega)$ with p being the natural projection of G/H_0 onto G/H.

Proof. It is clear that $H := \{\sigma \in G : \mathrm{Ad}^*(\sigma)(\alpha) = \alpha\}$ is a closed (hence embedded) subgroup of G, the Lie algebra $L(H) = \{x \in \mathfrak{g} : \mathrm{ad}^*(x)(\alpha) = 0\}$ of which coincides with the radical $\mathrm{Rad}(d\alpha)$ of the bilinear form $d\alpha$. Set $\dim G = 2p+1$ and let's prove that $\dim H = 1$. As α^+ is a contact form, $\mathrm{Ker}\alpha \cap \mathrm{Rad}(d\alpha) = \{0\}$ so that one has

$$\begin{aligned} 0 \leq \dim(\mathrm{Ker}\alpha + \mathrm{Rad}(d\alpha)) &= \dim \mathrm{Ker}\alpha + \dim \mathrm{Rad}(d\alpha) \\ &= 2p + \dim \mathrm{Rad}(d\alpha) \leq 2p+1 \end{aligned}$$

that is $0 \leq \dim H \leq 1$ [11].

If $\dim H = 0$, a fortiori G is of odd dimension, as the manifold G/H and $Orb(\alpha)$, orbit of α via Ad^*_G, are diffeomorphic. This is absurd. Thus $\dim H = 1$.

Let's prove that $M := G/H$ is reductive.. Let $z \in L(H)$ such that $\alpha(z) = 1$; one has $L(H) = \mathbb{R}z$. Set $\mathfrak{m} := \{x \in \mathfrak{g} : \alpha(x) = 0\}$, then we get $\mathfrak{g} = L(H) \oplus \mathfrak{m}$. Furthermore for $x \in \mathfrak{m}$ and $\tau \in H$ we have

$$\alpha\left(\mathrm{Ad}\left(\tau^{-1}\right)(x)\right) = \mathrm{Ad}^*(\tau)(\alpha)(x) = \alpha(x) = 0$$

i.e. $\mathrm{Ad}^*(H)(\mathfrak{m}) \subset \mathfrak{m}$.

Let z^+ be the left invariant vector field in G with $z^+_\varepsilon = z$. For every $X \in T_\sigma(G)$, let

$$\theta_\sigma(X) := \alpha^+_\sigma(X)\, z^+_\sigma$$

Let's check that θ is a connection form.

Denote x^* the vertical (relative to the fibration (1)) vector field on G associated to $x \in L(H)$. For $\sigma \in G$, one has

$$x^*_\sigma := \frac{d}{dt}\,|_{t=0}\,(\sigma \exp tx) = x^+_\sigma$$

As $x = \lambda z$ for some $\lambda \in \mathbb{R}$, it follows

$$x^+_\sigma := (L_\sigma)_{*,\varepsilon}(x) = (L_\sigma)_{*,\varepsilon}(\lambda z) = \lambda (L_\sigma)_{x,\varepsilon}(z) = \lambda z^+_\sigma = x^*_\sigma$$

Thus

$$\theta_\sigma(x^*_\sigma) = \theta_\sigma\left(\lambda z^+_\sigma\right) = \lambda\theta_\sigma\left(z^+_\sigma\right) = \lambda\alpha^+_\sigma\left(z^+_\sigma\right) z^+_\sigma = \lambda z^+_\sigma = x^*_\sigma$$

Now let's prove that for every $\tau \in H$ we have $(R_\tau)^*\theta = \mathrm{Ad}\left(\tau^{-1}\right)\theta$. From the equalities $\mathrm{Ad}^*(\tau)\alpha = \alpha$ and $L^*_\tau\alpha^+ = \alpha^+$, its follows that $R^*_\tau\alpha^+ = \alpha^+$ for every $\tau \in H$.

For $X_\sigma \in T_\sigma(G)$ and for every $\tau \in H$, $\sigma \in G$ we have:

$$(R^*_\tau\theta)(X_\sigma) = \theta_{\sigma\tau}\left((R_\tau)_{*,\alpha}\,X_\sigma\right) = \alpha^+_{\sigma\tau}\left((R_\tau)_{*,\alpha}\,X_\sigma\right) z^+_{\sigma r}$$

$$\theta_\sigma(X_\sigma) = \alpha^+_\sigma(X_\sigma)\, z^+_\sigma$$

As $\alpha^{+}_{\sigma\tau}\left((R_\tau)_{*,\alpha}\, X_\sigma\right) = \alpha^{+}_{\sigma}\left(X_\sigma\right)$ it follows that $R^{*}_{\tau}\theta = \theta$. Hence we must check that $\theta = \mathrm{Ad}\left(\tau^{-1}\right) \cdot \theta = R^{*}_{\tau}\theta$ for $\tau \in H$, that is

$$\theta\left(\left(R_\tau\right)_{*} X\right) = \mathrm{Ad}\left(\tau^{-1}\right)\theta\left(X\right)$$

for every $\tau \in H$. But this arises from the fact that H is commutative. Thus θ is a connection form.

Let $\widetilde{\Omega}$ be the curvative form of θ. From the fact that $\dim H = 1$, the relation

$$d\theta\left(X,Y\right) = -\frac{1}{2}\left[\theta\left(X\right), \theta\left(Y\right)\right] + \widetilde{\Omega}\left(X,Y\right)$$

for all $X, Y \in T_\sigma\left(G\right)$ then reads

$$d\theta\left(X,Y\right) = \widetilde{\Omega}\left(X,Y\right) = d\alpha^{+}\left(X,Y\right)$$

Let's prove (c). As $\widetilde{\Omega} = d\alpha^{+}$ we will have $L^{*}_{\sigma}\widetilde{\Omega} = \widetilde{\Omega}$ for all $\sigma \in G$. Furthermore, for every τ in H,

$$R^{*}_{\tau}\widetilde{\Omega} = R^{*}_{\tau}\left(d\alpha^{+}\right) = d\left(R^{*}_{\tau}\alpha^{+}\right) = d\alpha^{+} = \widetilde{\Omega}$$

Let $[\sigma] \in M$ and u, v in $T_{[\sigma]}\left(M\right)$. Set

$$\Omega_{[\sigma]}\left(u,v\right) := \widetilde{\Omega}_{\sigma}\left(u_\sigma, v_\sigma\right)$$

where u_σ (respectively v_σ) is the horizontal lifts of u (respectively of v) at σ. Let's see first that Ω is well defined. Let $u_{\sigma\tau}$, $v_{\sigma\tau}$ be the horizontal lifts of u and v at $\sigma\tau$ with $\tau \in H$. One has

$$\widetilde{\Omega}_{\sigma\tau}\left(u_{\sigma\tau}, v_{\sigma\tau}\right) = \widetilde{\Omega}_{\sigma\tau}\left(\left(R_\tau\right)_{*,\alpha} u_\sigma, \left(R_\tau\right)_{*,\alpha} v_\sigma\right) = \widetilde{\Omega}_{\sigma}\left(u_\sigma, v_\sigma\right)$$

as $\widetilde{\Omega}$ is R_τ invariant, $\tau \in H$.

In addition the equalities

$$\pi^{*}\left(d\Omega\right) = d\left(\pi^{*}\Omega\right) = d\widetilde{\Omega} = d\left(d\alpha^{+}\right) = 0$$

imply that $d\Omega = 0$ and taking into account the following

$$\pi^{*}\left(\Omega^{p}\right) = \left(\pi^{*}\Omega\right)^{p} = (\widetilde{\Omega})^{p} = \left(d\alpha^{+}\right)^{p} \neq 0$$

we then deduce that Ω is symplectic and invariant by the canonical action of G on M.

The canonical map $p : G/H_0 \to G/H, \quad \sigma H_0 \to \sigma H$, is obviously a covering map. Let $\Omega_0 := p^{*}\left(\Omega\right)$. It is clear that Ω_0 is symplectic and invariant by the canonical action of G on G/H_0. Furthermore, one has $R^{*}_{\tau}\widetilde{\Omega} = \widetilde{\Omega}$ for all τ in H_0. We have $\pi^{*}_{0}\Omega_0 = \widetilde{\Omega}$ where $\pi_0 : G \to G/H_0 =: M_0$ is the canonical injection.

We are going to prove now that the canonical action

$$\phi : G \times M_0 \to M_0 \qquad \left(\sigma, [\rho]\right) \mapsto [\sigma\rho] =: \phi_\sigma\left([\rho]\right)$$

is a Hamiltonian action. The action ϕ is symplectic. For $x \in \mathfrak{g}$, let $[\widetilde{x}]$ be the fundamental vector field on M_0 associated to x. It is clear that the following diagram is commutative:

$$\begin{tikzcd}
G \arrow[r, "L_\sigma"] \arrow[d, "\pi_0"'] & G \arrow[d, "\pi_0"] \\
M_0 \arrow[r, "\phi_\sigma"] & M_0
\end{tikzcd}$$

for every σ in G. Let's denote by $\widetilde{x}$ the horizontal lift (relative to θ_0) of $[\widetilde{x}]$ on the total space of the fiber with connection $H_0 \hookrightarrow G \to M_0$. This vector field is invariant under the R_τ for $\tau \in H_0$. Moreover, as the flows of $\widetilde{x}$ and $[\widetilde{x}]$ are the same via π_0 and

$$[\widetilde{x}]_{[\sigma]} := \frac{d}{dt}|_{t=0} \exp tx \cdot [\sigma] = \frac{d}{dt}|_{t=0} [\exp tx \cdot \sigma]$$

it follows that the flow of $\widetilde{x}$ consists on left translations on G. Thus $\widetilde{x}$ is a right invariant vector field on G. Let's emphasize on the fact that we are not pretending that $\widetilde{x}$ is the right invariant vector field x^- associated to x. We only have $\widetilde{x} = y^-$ for some $y \in \mathfrak{g}$ satisfying $\pi_{*,\varepsilon}(y) = [x]_{\pi_0(\varepsilon)}$.

Let $(\varphi_t)_t$ be the flow of $\widetilde{x} = y^-$. One has

$$0 = \mathcal{L}\left(y^-\right)\alpha^+ = \left(d \circ i\left(y^-\right) + i\left(y^-\right) \circ d\right)\alpha^+ = d\left(\alpha^+(y^-)\right) + \left(i\left(y^-\right) \circ d\right)\left(\alpha^+\right)$$

where $\mathcal{L}$ is the derivative. But one also has

$$i\left(y^-\right) d\alpha^+ = i\left(y^-\right)\pi_0^*\left(\Omega_0\right) = \pi_0^*\left(i\left[\widetilde{x}\right]\left(\Omega_0\right)\right)$$

so that

$$0 = d\left(\alpha^+\left(y^-\right)\right) + \pi_0^*\left(i\left[\widetilde{x}\right]\Omega_0\right) \tag{2}$$

Let's consider the function $f_y : G \to \mathbb{R}, \quad f_y(\sigma) := \alpha_\sigma^+\left(y_\sigma^-\right)$. Let's prove that f_y can be projected by π_0. Let $Y \in T_\sigma(G)$. From (2) we have,

$$d\left(\alpha^+\left(y^-\right)\right)\left(Y_\sigma\right) = Y_\sigma\left(\alpha^+\left(y^-\right)\right) = Y_\sigma\left(f_y\right) = -\Omega_0\left(\left[\widetilde{x}\right], \left(\pi_0\right)_{*,\sigma} Y_\sigma\right)$$

Consequently, if Y is tangent to the fiber (that is if $\left(\pi_0\right)_{*,\sigma} Y_\sigma = 0$), we'll have $Y_\sigma\left(f_y\right) = 0$ for every σ in G. Hence f is constant along H_0. This implies the existence of a smooth function $J_y : M_0 \to \mathbb{R}$ such that $J_y \circ \pi_0 = f_y$.

The following result is a complement of the theorem. It is directly proved by taking into account the ideas provided in the proof of the theorem.

Corollary 1.2 *[11]*

(a) If (G, α^+) is a contact Lie group of non discrete center $Z(G)$, then the quotient Lie group $G/Z(G)$ has left invariant symplectic form Ω^+ such that $\pi^*\Omega^+ = -d\alpha^+$, where $\pi : G \to G/Z(G)$ is the canonical projection.

(b) Conversely, if (K, Ω^+) is a symplectic Lie group, every Lie group G with Lie algebra $L(G) := \mathbb{R}_\omega \times L(K)$ (the central extension of $L(K)$ by $\mathbb{R}$ via ω), where $\omega := \Omega_\varepsilon^+$, admits a left invariant contact form α^+ satisfying $\pi^*\Omega^+ = -d\alpha^+$ (that is $\pi^*\omega = -d\alpha$).

Remark 1.1 (a) *Notice that the manifold G/H in the above theorem can be identified with the orbit $Orb\,(\alpha)$ of $\alpha \in \mathfrak{g}^*$ for the coadjoint representation.*

(b) *If two contact Lie algebras $(\mathfrak{g}_1, \alpha_1)$ and $(\mathfrak{g}_2, \alpha_2)$ with non trivial centers are contacto-isomorphic (that is there exists an isomorphism of Lie algebras $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$, such that, $\varphi^*(\alpha_2) = \alpha_1$), then the symplectic Lie algebras $(\mathfrak{g}_1/Z(\mathfrak{g}_1), \omega_1)$ et $(\mathfrak{g}_2/Z(\mathfrak{g}_2), \omega_2)$ are symplecto-isomorphic (that is there exists an isomorphism of Lie algebras $\varphi : \mathfrak{g}_1/Z(\mathfrak{g}_1) \to \mathfrak{g}_2/Z(\mathfrak{g}_2)$, such that, $\varphi^*(\omega_2) = \omega_1$) and $\pi_i^*\omega_i = -d\alpha_i, \quad i = 1, 2;$ where $\pi_i : \mathfrak{g}_i \to \mathfrak{g}_i/Z(\mathfrak{g}_i)$ are the canonical projections .*

(c) *Let η^+ be the left invariant $\mathfrak{g}$-valued invariant1-form on G defined by $\eta_\varepsilon(x) = x$ for every $x \in \mathfrak{g}$. As $M := G/H$ and $M_0 := G/H_0$ are reductive as stated in theorem, then the $L(H)$ is a component of η gives rise to a connection on the fiber bundles $H \overset{i}{\hookrightarrow} G \overset{\pi}{\to} M$ and $H_0 \overset{i}{\hookrightarrow} G \overset{\pi_0}{\to} M_0$ which are invariant under the action of G ([18], page 103). Such connections coincide with the ones described in the above theorem.*

2 Contact or Symplectic Filiform Lie algebras

2.1 Filiform Lie algebras (basic definitions and results)

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Let

$$C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset \ldots \supset C^{n-2}\mathfrak{g} \supset C^{n-1}\mathfrak{g} = \{0\}$$

be the central descending series of $\mathfrak{g}$, where $C^0\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = \left[\mathfrak{g}, C^{i-1}\mathfrak{g}\right]$, $1 \leq i \leq n-1$.

Definition 2.1 *A Lie algebra $\mathfrak{g}$ of dimension ≥ 3 is called filiform if* $\dim C^k\mathfrak{g} = n - k - 1$ *for $k = 1, ..., n-1$.*

We remark that the filiform Lie algebras have the maximal possible nilindex, that is $n - 1$. These algebras are the "least" nilpotent.

Examples of filiform Lie algebras

For each $n \in \mathbb{N}$ there exists several $(n+1)$-dimensional filiform Lie algebras which are specially remarkable. In the following description, the brackets are given relative to a basis $(X_0, X_1, ..., X_n)$.

1. The Lie algebra L_n :

   It is the simplest $(n+1)$-dimensional filiform Lie algebra. Its non trivial brackets are given by:

   $$[X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1.$$

2. The Lie algebra $Q_n \quad (n = 2k+1)$:

   $$\begin{aligned} &[X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1 \\ &[X_i, X_{n-i}] = (-1)^i X_n, \quad i = 1, ..., k \end{aligned}$$

In the basis $(Z_0, Z_1, ..., Z_n)$, where $Z_0 = X_0 + X_1, \quad Z_i = X_i, \quad i = 1, ..., n;$ this Lie algebra is defined by

$$\begin{array}{l} [Z_0, Z_i] = Z_{i+1}, \quad i = 1, ..., n-2 \\ [Z_i, Z_{n-i}] = (-1)^i Z_n, \quad i = 1, ..., k \end{array}$$

3. The Lie algebra R_n :

$$\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1 \\ [X_1, X_j] = X_{j+2}, \quad j = 2, ..., n-2 \end{array}$$

4. The Lie algebra W_n :

$$\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1 \\ [X_i, X_j] = \frac{6(i-1)!(j-1)!(j-i)}{(i+j)!} X_{i+j+1}, \quad 1 \leq i, j \leq n-2, \quad i+j+1 \leq n \end{array}$$

This Lie algebra can be defined also relative to a basis $(Y_1, Y_2, ..., Y_{n+1})$ by the brackets

$$[Y_i, Y_j] = (j-i) Y_{i+j}, \quad i+j \leq n+1$$

5. The Lie algebra $T_n \quad (n = 2k)$:

$$\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1 \\ [X_{k-i-1}, X_{k+i}] = (-1)^i X_n, \quad i = 0, 1, ..., k-2 \end{array}$$

6. The Lie algebra $T_n \quad (n = 2k+1)$:

$$\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1 \\ [X_{k-i-1}, X_{k+i+j}] = (-1)^i C_{i+j}^i X_{n+j-1}, \quad i = 0, 1, ..., k-2, \quad j = 0, 1 \end{array}$$

7. The Lie algebra $P_n \quad (n = 2k)$

$$\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, ..., n-1; \quad [X_{k-1}, X_k] = X_n; \\ [X_{k-i-1}, X_{k+i}] = (-1)^i \left(1 - \frac{2}{(k-1)(k-2)} C_{i+1}^{i-1}\right) X_n, \quad i = 1, ..., k-2; \\ [X_{k-i-2}, X_{k+i+j-1}] = (-1)^i \frac{2}{(k-1)(k-2)} C_{i+j}^i X_{n+j-2}, \ 0 \leq i \leq k-3; \ j = 0, 1. \end{array}$$

Let $\mathfrak{g}$ be a m-dimensional filiform Lie algebra. It is naturally filtered by descending central series and we can associate to $\mathfrak{g}$ a graded Lie algebra $gr\,\mathfrak{g}$ which is also filiform. This Lie algebra is defined on the vector space

$$gr\,\mathfrak{g} = \oplus_{i=1}^{m-1} \mathfrak{g}_i$$

where $\mathfrak{g}_i = C^{i-1}\mathfrak{g}/C^i\mathfrak{g}$, by the brackets $\left[x + C^i\mathfrak{g}, y + C^j\mathfrak{g}\right] = [x, y] + C^{i+j}\mathfrak{g}$, $x \in C^{i-1}\mathfrak{g}$, $y \in C^{j-1}\mathfrak{g}$.

Proposition 2.1 *[25] Let $\mathfrak{g}$ be a m-dimensional filiform Lie algebra. Then the graded Lie algebra $gr\,\mathfrak{g}$ is isomorphic to L_{m-1} , if m is odd, and isomorphic to L_{m-1} or Q_{m-1}, if m is even.*

Let Δ be the set of pairs of integers (k,r) such that $1 \leq k \leq n-1$, $2k+1 < r \leq n$, $r \geq 4$ (if n is odd we suppose that Δ contain also the pair $\left(\frac{n-1}{2}, n\right)$). For any element $(k,r) \in \Delta$, we can associate the 2-cocycle for the Chevalley cohomology of L_n with coefficients in the adjoint module denoted $\Psi_{k,r}$ and defined by

$$\Psi_{k,r}\left(X_i, X_j\right) = -\Psi_{k,r}\left(X_j, X_i\right) = (-1)^{k-i}\, C_{j-k-1}^{k-i} X_{i+j+r-2k-1}$$

if $1 \leq i \leq k < j \leq n$, $i+j+r-2k-1 \leq n$ and $\Psi_{k,r}\left(X_i, X_j\right) = 0$ otherwise. We remark that this formula for $\Psi_{k,r}$ is uniquely determined from the conditions :

$$\Psi_{k,r}\left(X_k, X_{k+1}\right) = X_r$$

$$\Psi_{k,r}\left(X_i, X_j\right) \in Z^2\left(L_n, L_n\right)$$

Proposition 2.2 *[25]Any $(n+1)$-dimensional filiform Lie algebra law $\mu \in F_m$ is isomorphic to $\mu_0 + \Psi$ where μ_0 is the law of L_n and Ψ is a 2-cocycle defined by*

$$\Psi = \sum_{(k,r)\in\Delta} a_{k,r} \Psi_{k,r}$$

and verifying the relation $\Psi \circ \Psi = 0$ with

$$\Psi \circ \Psi\,(x,y,z) = \Psi\left(\Psi\left(x,y\right), z\right) + \Psi\left(\Psi\left(y,z\right), x\right) + \Psi\left(\Psi\left(z,x\right), y\right)$$

Definition 2.2 *Let $\mathfrak{g}$ be a $(n+1)$-dimensional filiform Lie algebra with law μ. A basis $(X_0, X_1, ..., X_n)$ of $\mathfrak{g}$ is called adapted, if $[X_i, X_j] = \mu_0\left(X_i, X_j\right) + \Psi\left(X_i, X_j\right), \quad 0 \leq i, j \leq n$.*

Proposition 2.3 *Let $\mathfrak{g}$ be a filiform Lie algebra of dimension ≥ 4. Then $Der\mathfrak{g}$ is solvable.*

Proof. Consider an adapted basis $(X_0, X_1, ..., X_n)$ of $\mathfrak{g}$. As the central descending series of $\mathfrak{g}$ is an invariant flag under all derivations it is sufficient to show that the ideal $\langle X_1, ..., X_n\rangle$ is also an invariant. Let $d \in Der\mathfrak{g}$ and $d\left(X_1\right) = \sum_{i=0}^{n} a_i X_i$. For the 4-dimensional filiform Lie algebra $\overline{\mathfrak{g}} = \mathfrak{g}/C^3\mathfrak{g}$ (it is isomorphic to L_3) we have the derivation $\overline{d}$ with $\overline{d}\left(\overline{X}_1\right) = \sum_{i=0}^{3} a_i \overline{X}_i$. This is possible only if $a_0 = 0$.

Let $\mathfrak{g}$ be a Lie algebra. Consider in $Der\mathfrak{g}$ a maximally abelian subalgebra $\mathfrak{t}$ consisting of semisimple endomorphisms (a such subalgebra is called torus of $\mathfrak{g}$). According to a theorem by Mostow [22] two such subalgebras are conjugated by an inner automorphism. The common dimension of a tori on $\mathfrak{g}$ is called *rank* of $\mathfrak{g}$. Note that for nilpotent $\mathfrak{g}$ the rank cannot exceed the codimension of the derived ideal since $\mathfrak{g}$ is generated by any vector subspace of $\mathfrak{g}$ complementary to the derived ideal. For a filiform Lie algebra the only possible ranks are 0, 1 and 2.

Proposition 2.4 *[14]Let $\mathfrak{g}$ be a filiform Lie algebra of dimension $n+1$ and of rank 2. Then $\mathfrak{g}$ is isomorphic to L_n if n is even and isomorphic to L_n or Q_n if n is odd.*

The following theorem gives a description of the filiform Lie algebras of rank 1.

Theorem 2.5 *[14]Let $\mathfrak{g}$ be a filiform Lie algebra of dimension $n+1 \geq 7$ and of rank 1. There is a basis $(Y_0, Y_1, ..., Y_n)$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is one of the following families of Lie algebras:*

$$(i)\quad \begin{array}{l} \mathfrak{g} = A_{n+1}^r(\alpha_1, ..., \alpha_t), \quad 1 \leq r \leq n-3, \quad t = \left[\frac{n-r-1}{2}\right], \\ [Y_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n-1, \\ [Y_i, Y_j] = \left(\sum_{k=i}^t \alpha_k (-1)^{k-i} C_{j-k-1}^{k-i}\right) Y_{i+j+r}, \ 1 \leq i \leq j \leq n, \ i+j+r \leq n. \end{array}$$

$$(ii)\quad \begin{array}{l} \mathfrak{g} = B_{n+1}^r(\alpha_1, ..., \alpha_t), \ n = 2m+1, \ 1 \leq r \leq n-4, \ t = \left[\frac{n-r-2}{2}\right], \\ [Y_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n-2, \\ [Y_i, Y_{n-i}] = (-1)^i Y_n, \quad 1 \leq i \leq m, \\ [Y_i, Y_j] = \left(\sum_{k=i}^t \alpha_k (-1)^{k-i} C_{j-k-1}^{k-i}\right) Y_{i+j+r}, \\ 1 \leq i < j \leq n-1, \quad i+j+r \leq n-1. \end{array}$$

$$(iii)\quad \begin{array}{l} \mathfrak{g} = C_{n+1}(\alpha_1, ..., \alpha_t), \quad n = 2m+1, \quad t = m-1, \\ [Y_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n-2, \\ [Y_i, Y_{n-i}] = (-1)^i Y_n, \quad 1 \leq i \leq m, \\ [Y_i, Y_{n-i-2k}] = (-1)^i \alpha_k Y_n, \quad 1 \leq k \leq m-1, \quad 1 \leq i \leq n-2k-1 \end{array}$$

where C_q^s are the binomial coefficients (we suppose that $C_q^s = 0$ if $q < 0$ or $q < s$), $(\alpha_1, ..., \alpha_t)$ are the parameters satisfying the polynomial relations emanating from Jacobi's identity and at least one parameter $\alpha_i \neq 0$. A maximal torus of derivations is spanned by d, where :

If $\mathfrak{g} = A_{n+1}^r(\alpha_1, ..., \alpha_t)$:

$$d(Y_0) = Y_0, \quad d(Y_i) = (i+r)Y_i, \quad 1 \leq i \leq n.$$

If $\mathfrak{g} = B_{n+1}^r(\alpha_1, ..., \alpha_t)$:

$$d(Y_0) = Y_0, \quad d(Y_i) = (i+r)Y_i, \quad 1 \leq i \leq n-1, \quad d(Y_n) = (n+2r)Y_n.$$

If : $\mathfrak{g} = C_{n+1}(\alpha_1, ..., \alpha_t)$:

$$d(Y_0) = 0, \quad d(Y_i) = Y_i, \quad 1 \leq i \leq n-1, \quad d(Y_n) = 2Y_n.$$

Remark 2.1 (*a*) *Let $n \geq 13$ and $r = 1$. Then, up to isomorphism, there are only four Lie algebras $\mathfrak{g}$ of rank 1 if n is even and three Lie algebras of rank 1 if n is odd [16], [17]: If n is even and ≥ 14, then $\mathfrak{g}$ is isomorphic to one of the Lie algebras R_n, W_n, T_n, P_n. If n is odd and ≥ 13, $\mathfrak{g}$ is isomorphic to one of the Lie algebras R_n, W_n, T_n. If $n = 12$, then $\mathfrak{g}$ is isomorphic to one of the Lie algebras R_{12}, W_{12}, T_{12}.*

(*b*) *The laws $C_{n+1}(\alpha_1, ..., \alpha_t)$ satisfy the Jacobi's identity for all values of parameters $(\alpha_1, ..., \alpha_t)$.*

(*c*) *Let $\mathfrak{g}$ be a Lie algebra belonging to one of the families $(i), (ii), (iii)$ and at least one of parameters α_i be different to zero. Then we can transform one of these parameters to 1 using the automorphism ψ defined by $\psi(X_0) = aX_0$, $\psi(X_1) = bX_1$ (this is a unique type of automorphisms preserving the torus and the property of basis to be adapted). Modulo this transformation we have a classification up to isomorphism of filiform Lie algebras of rank 1.*

2.2 Symplectization and contactization of the Filiform Lie algebras

The following result shows that the class of Filiform Lie algebras is closed respect to the contactization and symplectization process described in the section 1.

Theorem 2.6 *Let (G, α^+) be a contact filiform Lie group. Then the quotient $G/Z(G)$ is a symplectic filiform Lie group. Conversely, if (K, ω) is a symplectic filiform Lie group, then every central extension*

$$0 \to \mathbb{R} \to G \to K \to 0$$

following ω is a contact filiform Lie group.

Proof. Let $\mathfrak{g} = L(G)$ be the Lie algebra of G. For an adapted basis $(X_0, X_1, \ldots, X_{2p})$ of $\mathfrak{g}$ we have $[X_0, X_i] = X_{i+1}, \quad i = 1, \ldots, 2p-1$ and $Z(\mathfrak{g}) = \mathbb{R} \cdot X_{2p}$. Following the corollary of theorem 1 the quotient $\mathfrak{g}/Z(\mathfrak{g})$ is a symplectic Lie algebra. Let $\pi : \mathfrak{g} \to \widetilde{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ be the canonical projection. Then we have

$$[\pi(X_0), \pi(X_i)] = \pi(X_{i+1}), \quad i = 1, \ldots, 2p-2;$$

and the Lie algebra $\widetilde{\mathfrak{g}}$ is also filiform.

Conversely suppose that $(\widetilde{\mathfrak{g}}, \omega)$ is a symplectic filiform Lie algebra of dimension $2p$. Consider an adapted basis $(Y_0, Y_1, \ldots, Y_{2p-1})$ of $\widetilde{\mathfrak{g}}$. As ω is a non degenerated form, there exists $0 \leq k \leq 2p-2$, such that $\omega(Y_k, Y_{2p-1}) \neq 0$. Let $\mathfrak{g} = \widetilde{\mathfrak{g}} \oplus_\omega \mathbb{R}$ be the central extension following ω. The central descending sequence $\left\{C^i\mathfrak{g}\right\}$ satisfies the condition $\dim C^{i-1}\mathfrak{g}/C^i\mathfrak{g} = 1$ for all $2 \leq i \leq 2p-2$ because this property holds in $\widetilde{\mathfrak{g}}$. As $\omega(Y_k, Y_{2p-1}) \neq 0$ we have also $\dim C^{2p-2}\mathfrak{g}/C^{2p-1}\mathfrak{g} = 1$ and $\dim C^{2p-1}\mathfrak{g} = 1$. Thus the nilindex of $\mathfrak{g}$ is equal to $2p$ and $\mathfrak{g}$ is filiform.

2.3 Existence of a left invariant contact form

Theorem 2.7 *Let G be a $(2p+1)$-dimensional filiform Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that the law μ of $\mathfrak{g}$ is written in an adapted basis by the formula*

$$\mu = \mu_0 + \sum_{(k,r)\in\Delta} a_{k,r}\Psi_{k,r}$$

Then G admits a left invariant contact form if and only if

$$A_j := \sum_{s=0}^{j-1} (-1)^s a_{p-j+s,2p-2(j-s-1)} C^s_{2j-s-2} \neq 0, \quad j = 1, 2, \ldots, p-1;$$

if this property holds the linear form $\alpha = b_0\alpha_0 + b_1\alpha_1 + \ldots + b_{2p}\alpha_{2p}$ is a contact form on $\mathfrak{g}$ if and only if $b_{2p} \neq 0$.

Proof. Let $A_j \neq 0, \quad j = 1, 2, \ldots, p-1$. We have

$$\begin{aligned}
[X_0, X_{2p-1}] &= X_{2p} \\
[X_1, X_{2p-2}] &= \textstyle\sum_{k=1}^{p-1} a_{k,2k+2}\Psi_{k,2k+2}(X_1, X_{2p-2}) = A_{p-1}X_{2p} \\
[X_2, X_{2p-3}] &= \textstyle\sum_{k=2}^{p-1} a_{k,2k+2}\Psi_{k,2k+2}(X_2, X_{2p-3}) = A_{p-2}X_{2p} \\
&\ldots \\
[X_{p-1}, X_p] &= a_{p-1,2p}\Psi_{p-1,2p}(X_{p-1}, X_p) = A_1 X_{2p}
\end{aligned}$$

and $[X_j, X_m] = 0$, if $m \geq 2p - j$. In the dual basis $(\alpha_0, \alpha_1, \ldots, \alpha_{2p})$ of $\mathfrak{g}^*$ the previous brackets give

$$\begin{aligned}
d(\alpha) \;\; = \;\; & -\alpha_0 \wedge \alpha_{2p-1} - A_{p-1}\alpha_1 \wedge \alpha_{2p-2} - A_{p-2}\alpha_2 \wedge \alpha_{2p-3} - \ldots \\
& -A_1\alpha_{p-1} \wedge \alpha_p + \sum_{m<2p-j-1} b_{j,m}\alpha_j \wedge \alpha_m
\end{aligned}$$

where $\alpha = a_0\alpha_0 + a_1\alpha_1 + \ldots + a_{2p-1}\alpha_{2p-1} + \alpha_{2p}$.
Thus

$$(d\alpha)^p = (-1)^p p! A_1 A_2 \cdots A_{p-1} \alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{2p-1}$$

and $\alpha \wedge (d\alpha)^p \neq 0$. This means that α is a contact form.

Conversely, we suppose now that the Lie algebra $\mathfrak{g}$ admits a contact form α. We put

$$\alpha = b_0\alpha_0 + b_1\alpha_1 + \ldots + b_{2p}\alpha_{2p}.$$

As $Z(\mathfrak{g})$ is not included in Kerα, then $b_{2p} \neq 0$. We have

$$\begin{aligned}
d\alpha = \;& b_0(d\alpha_0) + b_1(d\alpha_1) + \ldots + b_{2p}(d\alpha_{2p}) = \\
& = b_2(-\alpha_0 \wedge \alpha_1) + b_3(-\alpha_0 \wedge \alpha_2) + b_4(-\alpha_0 \wedge \alpha_3 - a_{1,4}\alpha_1 \wedge \alpha_2) + \ldots + \\
& + b_j(\textstyle\sum_{l+s<j} t_{l,s}\alpha_l \wedge \alpha_s) + \ldots + \qquad\qquad (*) \\
& + b_{2p}(-\alpha_0 \wedge \alpha_{2p-1} - A_{p-1}\alpha_1 \wedge \alpha_{2p-2} - A_{p-2}\alpha_2 \wedge \alpha_{2p-3} - \ldots - \\
& - A_1\alpha_{p-1} \wedge \alpha_p + \textstyle\sum_{m<2p-j-1} b_{j,m}\alpha_j \wedge \alpha_m).
\end{aligned}$$

As the basis $(X_0, X_1, ..., X_{2p})$ is adapted we have $X_{2p} \perp d\alpha_i, \quad 0 \leq i \leq 2p$, and thus $X_{2p} \perp d\alpha$. But $(d\alpha_{2p})^p \neq 0$ and

$$(d\alpha)^p = \lambda \alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{2p-1}$$

with $\lambda \neq 0$. Let us examine the terms of the expression $(*)$ which appear in the non null product $(d\alpha_{2p})^p$. In this expression, only there is one term containing the form α_{2p-1}; it is the term $-b_{2p}\alpha_0 \wedge \alpha_{2p-1}$. We deduce that

$$(d\alpha)^p = -b_{2p}\alpha_0 \wedge \alpha_{2p-1} \wedge \theta$$

where $\theta \in \wedge^{2p-2}\mathfrak{g}$, $\theta \neq 0$ and α_0, α_{2p-1} does not appear in θ. Let as examine now α_{2p-2}. As this containing in $(d\alpha)^p$, then $\theta = -b_{2p}A_{p-1}\alpha_1 \wedge \alpha_{2p-2} \wedge \theta_1$ with $\theta_1 \in \wedge^{2p-4}\mathfrak{g}$, $\theta_1 \neq 0$. In fact in the expression $t_{l,s}\alpha_l \wedge \alpha_s$ with $l+s < j, \quad j < 2p$, the index s cannot be equal to $2p-2$ except the case $l = 0$. Likewise in the term $b_{j,m}\alpha_j \wedge \alpha_m$, we have $m \neq 2p-2$ if $j \neq 0$.

Let us suppose now that the terms

$$-b_{2p}\alpha_0 \wedge \alpha_{2p-1}, \ -b_{2p}A_{p-1}\alpha_1 \wedge \alpha_{2p-2}, \ \ldots \ , \ -b_{2p}A_{p-k}\alpha_k \wedge \alpha_{2p-k-1}$$

of the expression $(*)$ are the factors of $(d\alpha)^p$. Then $A_{p-1}, \ldots, A_{p-k} \neq 0$ for $1 \leq k < p-1$. In the same way we show that the term $-b_{2p}A_{p-k}\alpha_{k+1} \wedge \alpha_{2p-k-2}$ is also a factor of $(d\alpha)^p$. By induction we have

$$(d\alpha)^p = (-1)^p p! b_{2p}^p A_1 A_2 \cdots A_{p-1} \alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{2p-1} \neq 0$$

and $A_1, A_2, \ldots, A_{p-1} \neq 0$.

2.4 Classes of contacto-isomorphisms

Two contact Lie algebras $(\mathfrak{g}_1, \alpha_1)$ and $(\mathfrak{g}_2, \alpha_2)$ called contacto-isomorphic if there exists an isomorphism of Lie algebras $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$, such that, $\varphi^*(\alpha_2) = \alpha_1$. The following result gives the classification up contacto-isomorphisms of the contact forms on a filiform Lie algebras.

Theorem 2.8 *Let $\mathfrak{g}$ be a filiform $(2p+1)$-dimensional Lie algebra. Let us consider an adapted basis $(X_0, X_1, \ldots, X_{2p})$ of $\mathfrak{g}$ and its dual basis $(\alpha_0, \alpha_1, \ldots, \alpha_{2p})$. If $\alpha = a_0\alpha_0 + a_1\alpha_1 + \ldots + a_{2p}\alpha_{2p}$ is a contact form on $\mathfrak{g}$, then the form $\beta = a_{2p}\alpha_{2p}$ is also a contact form on $\mathfrak{g}$ and $(\mathfrak{g}, \alpha)$ is contacto-isomorphic to $(\mathfrak{g}, \beta)$.*

Proof. The fact that β is a contact form on $\mathfrak{g}$ is a consequence of the proof of theorem 4. To prove the theorem it is sufficient to find an automorphism $\varphi \in$Aut $\mathfrak{g}$ such that $\varphi^*\alpha = \beta$.

Consider the derivation $d = -a_{2p-1}\text{ad } X_0$. As ad $X_0(X_i) = X_{i+1}$, $i = 1, \ldots, 2p-1$, the automorphism $A = \exp d$ satisfies the following property:

$$A(X_{2p-1}) = X_{2p-1} - a_{2p-1}X_{2p}, \quad A(X_{2p}) = X_{2p}.$$

Then

$$A^*\alpha_{2p} = a_{2p}\alpha_{2p} - a_{2p-1}\alpha_{2p-1} + \sum_{i<2p-1} c_i\alpha_i$$

and

$$A^*\alpha = a_{2p}\alpha_{2p} + \sum_{j\leq 2p-2} q_j\alpha_j$$

We suppose now that

$$\alpha = a_{2p}\alpha_{2p} + \sum_{j\leq 2p-k} b_j\alpha_j, \quad 2\leq k\leq 2p-1,$$

and we prove the existence of an automorphism $\varphi\in$Aut $\mathfrak{g}$ such that

$$\varphi^*\alpha = a_{2p}\alpha_{2p} + \sum_{j\leq 2p-k-1} b'_j\alpha_j.$$

Consider the derivation $d = -\lambda b_{2p-k}\text{ad } X_{k-1}$. From the proof of the theorem 4 we have

$$\text{ad}X_{k-1}(X_{2p-k}) = A_{p-k+1}X_{2p}, \quad 2\leq k\leq p,$$

and

$$\text{ad}X_{k-1}(X_{2p-k}) = -A_{k-p}X_{2p}, \text{ si } k>p.$$

We have also $\text{ad}X_{k-1}(X_i) = 0$ for all index $i > 2p-k$. Let us put

$$\lambda = \begin{cases} \frac{1}{A_{p-k+1}}, & \text{si} \quad 2\leq k\leq p \\ \frac{-1}{A_{k-p}}, & \text{si} \quad i>2p-2 \end{cases}$$

Then the automorphism $\varphi = \exp d$ satisfies the required condition. By induction we deduce the theorem.

Remark 2.2 (a) *The theorem 5 only concerns the filiform case. But, from a direct verification, we can affirm that this result remains true for every nilpotent Lie algebra of dimension less or equal to 7.*

(b) *Suppose that α_{2p} is a contact form on the filiform Lie algebra $\mathfrak{g}$. In general the contact Lie algebras $(\mathfrak{g},\ a\alpha_{2p})$ and $(\mathfrak{g},\ b\alpha_{2p})$ with $a\neq b$ are not contacto-isomorphic. But these contact Lie algebras are always contacto-isomorphic if $\mathfrak{g}$ admits a semisimple derivation (see [14] and the subsection 2.1 for their description).*

(c) *If $K=\mathbb{C}$, then the theorems 3, 4, 5 and the point (a) of this remark are valid. But the point (b) of this remark must be modified (see [9]).*

3 Symplectic Lie algebras of dimension ≤ 6

The studied relation between the classes of symplectic Lie algebras of dimension $2p$ and the contact Lie algebras of dimension $2p+1$ and the description of contact structures on a filiform Lie algebras and on a nilpotent Lie algebras of dimension ≤ 7 permits to obtain some classification results about symplectic Lie algebras. The following theorem gives a complete classification up to simplecto-isomorphism of the symplectic Lie algebras of dimension ≤ 6.

Theorem 3.1 *Every nilpotent symplectic Lie algebra of the dimension ≤ 6 is symplecto-isomorphic to one and only one of the following symplectic Lie algebras.*

Dimension 2

1. $\mathbb{R}^2$,

   $\omega = \alpha_1 \wedge \alpha_2$

Dimension 4

1. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4,$

   $\omega = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$

2. $[X_1, X_2] = X_3,$

   $\omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4$

3. $\mathbb{R}^4$,

   $\omega = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$

Dimension 6

1. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5,$
   $[X_1, X_5] = X_6, \qquad [X_2, X_3] = X_5, \qquad [X_2, X_4] = X_6,$

   $\omega = \alpha_1 \wedge \alpha_6 + (1-\lambda)\alpha_2 \wedge \alpha_5 + \lambda \alpha_3 \wedge \alpha_4, \qquad \lambda \in \mathbb{R} \setminus \{0, 1\}$

2. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5,$
   $[X_1, X_5] = X_6, \qquad [X_2, X_3] = X_6,$

   $\omega(\lambda) = \lambda(\alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_4 - \alpha_2 \wedge \alpha_5), \qquad \lambda \in \mathbb{R} \setminus \{0\}$

3. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5,$
   $[X_1, X_5] = X_6,$

   $\omega = \alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

4. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_6,$
$[X_2, X_3] = X_5, \qquad [X_2, X_5] = X_6,$

$$\omega(\lambda_1, \lambda_2) = \lambda_1 \alpha_1 \wedge \alpha_4 + \lambda_2(\alpha_1 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5),$$
$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 \in \mathbb{R} \setminus \{0\}$$

5. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = -X_6,$
$[X_2, X_3] = X_5, \qquad [X_2, X_5] = X_6,$

$$\begin{aligned}
\omega_1(\lambda_1, \lambda_2) &= \lambda_1 \alpha_1 \wedge \alpha_4 + \lambda_2(\alpha_1 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5), \\
&\hfill \lambda_1 \in \mathbb{R}, \quad \lambda_2 \in \mathbb{R} \setminus \{0\} \\
\omega_2(\lambda) &= \lambda(-\alpha_1 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 + \tfrac{1}{2}\alpha_1 \wedge \alpha_4 + \tfrac{1}{4}\alpha_1 \wedge \alpha_5 + \tfrac{1}{4}\alpha_2 \wedge \alpha_4 \\
&\hfill -\alpha_3 \wedge \alpha_5), \quad \lambda \in \mathbb{R} \setminus \{0\} \\
\omega_3(\lambda) &= \lambda(-\alpha_1 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 + \tfrac{3}{2}\alpha_1 \wedge \alpha_4 - \tfrac{3}{4}\alpha_1 \wedge \alpha_5 + \tfrac{1}{4}\alpha_2 \wedge \alpha_4 \\
&\hfill -\alpha_3 \wedge \alpha_5), \quad \lambda \in \mathbb{R} \setminus \{0\} \\
\omega_4(\lambda) &= \lambda(-\alpha_1 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 - \tfrac{1}{2}\alpha_1 \wedge \alpha_4 + \tfrac{5}{4}\alpha_1 \wedge \alpha_5 + \tfrac{1}{4}\alpha_2 \wedge \alpha_4 \\
&\hfill -\alpha_3 \wedge \alpha_5), \quad \lambda \in \mathbb{R} \setminus \{0\}
\end{aligned}$$

6. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5,$
$[X_2, X_3] = X_6,$

$$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4,$$
$$\omega_2 = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_4 - \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$$

7. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_5] = X_6,$
$[X_2, X_3] = X_6, \qquad [X_2, X_4] = X_6,$

$$\omega_1(\lambda) = \lambda(\alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5), \quad \lambda \in \mathbb{R} \setminus \{0\}$$
$$\omega_2(\lambda) = \lambda(\alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4), \quad \lambda \in \mathbb{R} \setminus \{0\}$$

8. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_5] = X_6,$
$[X_2, X_3] = X_6, \qquad [X_2, X_4] = X_6,$

$$\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4$$

9. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_5] = X_6,$
$[X_2, X_3] = X_6,$

$$\omega(\lambda) = \lambda(\alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5), \quad \lambda \in \mathbb{R}^+$$

10. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_3] = X_6,$
$[X_2, X_4] = X_6,$

$$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 - \alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_4,$$
$$\omega_2 = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 + \alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4$$

11. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5,$
$[X_2, X_3] = X_6, \qquad [X_2, X_4] = X_6,$

$$\omega_1(\lambda) = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \lambda \alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_4, \quad \lambda \in \mathbb{R}$$
$$\omega_2(\lambda) = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 + \lambda \alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4, \quad \lambda \in \mathbb{R}$$

12. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_3] = X_6,$
$[X_2, X_3] = -X_5, \qquad [X_2, X_4] = X_6,$

$$\omega(\lambda_1, \lambda_2) = \lambda_1 \alpha_1 \wedge \alpha_5 + \lambda_2 \alpha_2 \wedge \alpha_6 + (\lambda_1 + 1)\, \alpha_3 \wedge \alpha_4, \qquad \lambda_1 \in \mathbb{R} \setminus \{0, -1\}, \quad \lambda \in \mathbb{R}^+$$

13. $[X_1, X_2] = X_4, \qquad [X_1, X_3] = X_5, \qquad [X_1, X_4] = X_6,$
$[X_2, X_3] = X_6,$

$\omega_1(\lambda) = \alpha_1 \wedge \alpha_6 + \lambda \alpha_2 \wedge \alpha_5 + (\lambda - 1)\, \alpha_3 \wedge \alpha_4, \quad \lambda \in \mathbb{R} \setminus \{0, 1\}$
$\omega_2(\lambda) = \alpha_1 \wedge \alpha_6 + \lambda \alpha_2 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_5, \quad \lambda \in \mathbb{R} \setminus \{0\}$
$\omega_3 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \frac{1}{2} \alpha_2 \wedge \alpha_5 - \frac{1}{2} \alpha_3 \wedge \alpha_4$

14. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_6, \qquad [X_1, X_3] = X_5,$

$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5,$
$\omega_2 = \alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5,$
$\omega_3 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

15. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_6, \qquad [X_2, X_3] = X_5,$

$\omega_1 = -\alpha_1 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4,$
$\omega_2 = \alpha_1 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4,$
$\omega_3 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5,$
$\omega_4 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4,$
$\omega_5 = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

16. $[X_1, X_2] = X_5, \qquad [X_1, X_3] = X_6,$
$[X_2, X_4] = X_6, \qquad [X_3, X_4] = -X_5,$

$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_3 - \alpha_4 \wedge \alpha_5,$
$\omega_2 = \alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5$

17. $[X_1, X_3] = X_5, \qquad [X_1, X_4] = X_6, \qquad [X_2, X_3] = X_6,$

$\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

18. $[X_1, X_2] = X_4, \qquad [X_1, X_3] = X_5, \qquad [X_2, X_3] = X_6,$

$\omega_1(\lambda) = \alpha_1 \wedge \alpha_6 + \lambda \alpha_2 \wedge \alpha_5 + (\lambda - 1)\, \alpha_3 \wedge \alpha_4, \quad \lambda \in \mathbb{R} \setminus \{0, 1\},$
$\omega_2(\lambda) = \alpha_1 \wedge \alpha_5 + \lambda \alpha_1 \wedge \alpha_6 - \lambda \alpha_2 \wedge \alpha_5 + \alpha_2 \wedge \alpha_6 - 2\lambda \alpha_3 \wedge \alpha_4, \quad \lambda \in \mathbb{R} \setminus \{0\},$
$\omega_3 = -2\alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_4 - \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

19. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_5, \qquad [X_1, X_5] = X_6,$

$\omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5$

20. $[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4,$
$[X_1, X_4] = X_5, \qquad [X_2, X_3] = X_5,$

$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4,$
$\omega_2 = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

21. $[X_1, X_2] = X_4, \qquad [X_1, X_4] = X_6, \qquad [X_2, X_3] = X_6,$

$\omega = \alpha_1 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 - \alpha_3 \wedge \alpha_4 - \alpha_3 \wedge \alpha_5$

22. $[X_1, X_2] = X_5, \qquad [X_1, X_5] = X_6,$

$\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

23. $[X_1, X_2] = X_5, \qquad [X_1, X_3] = X_6,$

$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$
$\omega_2 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5$
$\omega_3 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5$

24. $[X_1, X_2] = X_6, \qquad [X_2, X_3] = X_5,$

$\omega_1 = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$
$\omega_1 = -\alpha_1 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4$

25. $[X_1, X_2] = X_6,$

$\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

26. $\mathbb{R}^6,$

$\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4$

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