What is new with Connes' approach to the Riemann hypothesis?

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In one of his lectures during the IPM conference on noncommutative geometry in September 2005 in Tehran, Alain Connes unveiled a new program (joint with Katia Consani and Matilde Marcolli) to attack the Riemann hypothesis. This was also later sketched in the Connes-Marcolli article *A walk in the noncommutative garden*. In a nutshell this program charters a new route towards a proof of the Riemann Hypothesis (RH), and its generalizations, via noncommutative geometry and is based on Connes' original work on RH in his 1999 paper *"Trace formula in noncommutative geometry and the zeros of the Riemann zeta function"* (Sel. Math. New ser. 5 (1999), 29-106). To try to reach to that magnificent summit of number theory, one must, of course, conquer new territories and ascend several vertical walls, which is a strong motivation for further developing noncommutative geometry along a path sketched in this program.

Recall that Riemann's zeta function is defined, for $Re(s) > 1$, by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which has a holomorphic extension to $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$. There is a very deep relationship between zeros of zeta and distribution of prime numbers. Euler observed that the so called *Euler product formula* for the zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

immediately implies that there are infinite number of primes (and in fact $\sum \frac{1}{p} = \infty$). But much more is known. Let $\pi(x)$ denote the number of primes

$\leq x$. Gauss, based on numerical evidence, already knew that $\pi(x)$ is very well approximated by the *Logarithmic Integral function* $Li(x) = \int_2^x (logt)^{-1}dt$. The celebrated *prime number theorem* proved, independently, in 1896 by Hadamard and de la Vallee Poussin, states that $\pi(x)$ and $Li(x)$ are asymptotically equal. In effect they showed that $\pi(x) - Li(x) = O(xe^{-a\sqrt{logx}})$ for some positive number a. To obtain, however, more precise information on $\pi(x)$, better estimates for the error term $\pi(x) - Li(x)$ are desirable. Riemann in his epoch-making paper of 1859 entitled "On the Number of Primes Less Than a Given Magnitude" wrote down a formula, now called the *Riemann explicit formula*, that gives a relationship between zeros of zeta and this error term

$$\pi'(x) = Li\,(x) - \sum_{\rho} Li\,(x^{\rho}) + \int_x^{\infty} \frac{du}{u(u^2-1)logu} - log\,\xi(0),$$

where the summation, which is only conditionally convergent, is over the set of zeros ρ of zeta with positive real part and $\pi'(x)$ is the Moebius transform of $\pi(x)$ defined by

$$\pi'(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots.$$

Now the Euler product formula implies that zeta has no zeros to the right of the line $s = 1$ and the functional equation, which was established by Riemann but won't be reproduced here, shows that there are no zeros in the region $Re(s) < 0$ except for trivial zeros at $s = -2, -4, \cdots$. Riemann conjectured, based on numerical evidence, it seems, that all nontrivial zeros of zeta should lie on the *critical line* $Re(s) = \frac{1}{2}$. He showed that from this hypothesis, now called the Riemann Hypothesis, Gauss's conjectural estimate for $\pi(x)$ (the prime number theorem) would follow. It is however known that the RH is much stronger than the prime number theorem and in fact is equivalent to the estimate $\pi(x) - Li(x) = O(x^{\frac{1}{2}}logx)$ for the error term. Despite huge progress, RH has remained unsolved so far.

Weil's 1941 proof as a blueprint

A pervasive and very fruitful idea in number theory is an analogy between *number fields* and *function fields*. Already in the 19th century, Kroencker, Dedekind, Weber, and others had noticed that facts about algebraic function fields (fields of meromorphic functions on a Riemann surface), once properly translated, have an analogue in the world of algebraic number fields. To make this analogy more precise one must work with function fields over finite fields and not over $\mathbb{C}$ (for a poetic exposition of this circle of ideas see Andre Weil's 1940 letter to his sister on the role of analogies in mathematics). As a rule, statements for function fields tend to be easier to establish. In 1941 Weil managed to establish the analogue of the Riemann hypothesis for the zeta function of a curve over a finite field (this zeta function was already defined by Artin in 1920's and was shown to satisfy RH in the genus 1 case by Hasse in 1930's). In his complete proof (which appeared only in 1945) he had to uncover a vast amount of new algebraic geometry: theory of correspondences and initial ideas about motives, and a suitable cohomology theory that would replace singular cohomology, with similar properties like Poincaré duality, for varieties over finite fields. But most and foremost it was the geometrization of the whole problem by showing that the *Riemann-Weil explicit formula* is equivalent to a *Lefschetz trace formula* for the action of the *Frobenius automorphism* on the cohomology of the curve, that made the proof possible. The Riemann hypothesis then follows from positivity of a certain convolution operator. The Frobenius map, sometimes dubbed as the 'king of number theory', is of course a purely finite characteristic phenomena and so far, i.e. before the recent work of Connes, Consani, and Marcolli, had found no analogue in the characteristic zero case.

The program outlined by Connes, Consani and Marcolli in their recent paper *"Noncommutative geometry and motives: the thermodynamics of endomotives"* (available as math.QA/0512138 in the Archive; cf. also the last section of *"A walk in the noncommutative garden"*, by Connes and Marcolli, available as math.QA/0601054, for a review) aims at creating an environment where something like Weil's 1941 proof can be repeated in the characteristic zero case. Among many other things, they produce an analogue of the Frobenius automorphism in characteristic zero in this paper. *Connes' trace formula* was shown to be equivalent to the Riemann hypothesis and its generalizations by Connes (cf. *"Trace formula in noncommutative geometry and the zeros of the Riemann zeta function"*, Sel. Math. New ser. 5 (1999), 29-106). Since Connes' trace formula is over the noncommutative space of *adèle classes*, the geometric setting is that of noncommutative geometry and they must go far beyond of what is done so far and import many ideas from modern algebraic geometry to noncommutative geometry. To achieve this, as a first step, good analogues of *étale cohomology*, the *category of motives*, and *correspondences* in noncommutative geometry must be introduced. Happily it turns out that some elements of these are already in place as cyclic (co)homology theory, and in the form of Connes' category of *cyclic modules*, and the *bivariant cyclic homology* and *KK-theory.*

Another important ingredient is quantum statistical mechanics and specially the fact, proved by the Japanese mathematician Tomita in the 1960's, that any state on a von Neumann algebra M is the equilibrium (in the sense of Kubo-Martin-Schwinger) state of a one-parameter group of automorphisms of M at inverse temperature $\beta = 1$. The construction of the Frobenius in characteristic zero follows a very general process that combines cyclic homology with quantum statistical mechanics in a novel way. Starting from a pair (A, φ) of an algebra and a state (a noncommutative space endowed with a 'measure'), they proceed by invoking the canonical one parameter group of automorphisms σ and consider the extremal equilibrium states Σ_β at inverse temperatures $\beta > 1$. Under suitable conditions there is an algebra map

$$\rho : A \rtimes_\sigma \mathbb{R} \to \mathcal{S}(\Sigma_\beta \times \mathbb{R}_+^*) \otimes \mathcal{L}.$$

The cyclic module $D(A, \varphi)$ is defined as the cokernel of the induced map by $Tr \circ \rho$ on the cyclic modules of these two algebras. The dual group $\mathbb{R}_+^*$ acts on $D(A, \varphi)$ and, in examples coming from number theory, replaces Frobenius in characteristic zero. The three steps involved in the construction of $D(A, \varphi)$ are called *cooling*, *distillation*, and *dual action* in the paper.

This is a very very brief (in fact dangerously brief, I am afraid) outline. For more on this fascinating new program we invite the readers to check the cited articles by Connes, Consani, and Marcolli. We also highly recommend a video taped lecture by Alain Connes available at KITP website http: online.itp.ucsb.edu/online/strings05/connes2/.