Introduction to symplectic topology

Lecture notes

1. Linear symplectic geometry.

1.1. Let V be a vector space and ω a non-degenerate skew-symmetric bilinear form on V. Such ω is called a linear symplectic structure. We write $\omega(u,v)$ for $u,v \in V$. The only difference with (pseudo)Euclidean structure is that the latter is symmetric.

Fix a dot product in V. Then one can write:

$$\omega(u,v) = Ju \cdot v$$

where J is non-degenerate operator on V. Since ω is skew-symmetric, $J^* = -J$. Taking determinants,

$$\det J^* = \det J = (-1)^n \det J$$

where $n = \dim V$. Thus n is even.

Examples. 1. The plane with an area form (i.e., cross-product) is a symplectic space. All 2-dimensional symplectic spaces are symplectomorphic to this one. In formulas, $\omega = dp \wedge dq$ (using the language of linear differential forms).

2. One can take direct sum of the previous example to obtain symplectic $\mathbf{R}^{2n}$ with the symplectic structure $dp \wedge dq = dp_1 \wedge dq_1 + ... + dp_n \wedge dq_n$. One has: $\omega(q_i, q_j) = \omega(p_i, p_j) = 0$ and $\omega(p_i, q_j) = \delta_{ij}$. This is a *symplectic basis*; the respective coordinates are called *Darboux coordinates*.

3. More conceptually, let W be a vector space. Then $V = W \oplus W^*$ is a symplectic space. The structure is as follows:

$$\omega((u_1, l_1), (u_2, l_2)) = l_1(u_2) - l_2(u_1).$$

Check that this is non-degenerate.

Exercise. Let J be a skew-symmetric matrix: $J^* = -J$. Then $\det J$ is a polynomial in the entries of J,, and this polynomial is the square of another polynomial in the entries of J:

$$\det J = (\text{Pf } J)^2.$$

The latter is called Pfaffian. Show that

$$\text{Pf } (A^* J A) = \det A \ \text{Pf } J.$$

As in Euclidean geometry, one defines (skew)orthogonal complement of a space. Unlike Euclidean geometry, one may have: $U \subset U^\perp$. For example, this is the case when dim $U = 1$. If $U \subset U^\perp$ then U is called *isotropic*. One has: dim $U^\perp = 2n-$ dim U where $2n$ is the dimension of the ambient space. Thus if U is isotropic then dim $U \le n$. An isotropic subspace of dimension n is called *Lagrangian*.

Exercises. 1. Let $A : W \to W^*$ be a linear map. Then $A^* = A$ if and only if the graph $GrA \subset W \oplus W^*$ is a Lagrangian subspace (with respect to the structure of Example 3 above).

2. Given two symplectic spaces (V_1, ω_1) and (V_2, ω_2) of the same dimensions and a linear map $A : V_1 \to V_2$, the map A is a symplectomorphism if and only if $GrA \subset V_1 \oplus V_2$ is a Lagrangian subspace with respect to the symplectic structure $\omega_1 \ominus \omega_2$.

1.2. Similarly to Euclidean spaces, the dimension is the only linear symplectic invariant.

Linear Darboux Theorem. *Two symplectic spaces of the same dimension are linearly symplectomorphic.*

Proof. Given a symplectic space V^{2n}, pick a Lagrangian subspace $W \subset V$. To construct W, choose $v_1 \in V$, consider $v_1^\perp$, choose $v_2 \in v_1^\perp$, consider $v_1^\perp \cap v_2^\perp$, choose $v_3 \in v_1^\perp \cap v_2^\perp$, etc., until one has $v_1, ..., v_n$ such that $\omega(v_i, v_j) = 0$. These vectors span a Lagrangian space.

Claim: V is linearly symplectomorphic to the symplectic space $W \oplus W^*$ of Example 3 in 1.1. To see this, pick another Lagrangian subspace U, transverse to W. Then U is identified with W^*: the pairing between U and W is given by ω. Since $V = U \oplus W$, we have the desired symplectomorphism.

Thus, one may choose one's favorite model of a symplectic space. For example, one may identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$, and then J from 1.1 is the operator of multiplication by $\sqrt{-1}$.

Recall that a complex space can be given a Hermitian structure $< z, w >= z\bar{w}$. Then Re $< z, w >$ is the dot product and Im $< z, w >$ is the symplectic structure.

1.3. Consider the set of all Lagrangian subspaces in $2n$-dimensional symplectic space; it is called the *Lagrangian Grassmanian* and denoted by Λ_n.

"Recall" the construction of the Grassman manifold $G_{k,n}$ of k-dimensional subspaces in $k+n$-dimensional space. The orthogonal group $O(n+k)$ acts transitively on $G_{k,n}$ (why?), and the isotropy subgroup of a given k-subspace is generated by orthogonal transformations of this subspace and of its orthogonal complement. Thus

$$G_{k,n} = O(n + k)/O(k) \times O(n).$$

To see that $G_{k,n}$ is a smooth manifold, consider a k-subspace V and let U be its orthogonal complement. Then every k-subspace near V is the graph of a linear map $A : V \to U$. Thus a neighborhood of V in $G_{k,n}$ is identified with the space of $k \times n$ matrices. In particular, dim $G_{k,n} = kn$.

Let us do a similar thing with Λ_n. Consider the symplectic space as a complex one. Let W be a Lagrangian subspace. Choose an orthonormal basis in W. Then this is also a unitary basis. It follows that the group $U(n)$ acts transitively on Λ_n, and the subgroup preserving W is $O(n)$. Thus $\Lambda_n = U(n)/O(n)$. To describe a neighborhood of W, choose a transverse Lagrangian subspace U. As in the proof of Theorem 1.2, $U = W^*$. Then, by an Exercise in 1.1, all Lagrangian subspaces near W are the graphs of self-adjoint maps $A : W \to W^*$. Thus a neighborhood of W in Λ_n identifies with the space of symmetric $n \times n$ matrices, and dim $\Lambda_n = n(n + 1)/2$.

Example. Λ_1 is the space of lines through the origin in the plane, i.e., $\mathbf{RP}^1$, topologically, a circle.

Exercise*. What is the topology of Λ_2?

Describe a related classical construction realizing Λ_2 as a quadratic hypersurface of signature $(+++--)$ in $\mathbf{RP}^4$. Let the symplectic space be $\mathbf{R}^4$ with $\omega = p_1 \wedge q_1 + p_2 \wedge q_2$. Given a 2-plane U, choose $u_1, u_2 \in U$ and consider the bivector $\phi = u_1 \wedge u_2$. Thus we assign $\phi \in \Lambda^2 U$, and ϕ is defined up to a factor. We have constructed a map $G_{2,2} \to P(\Lambda^2 U) = \mathbf{RP}^5$. The bivectors in $\Lambda^2 U$ corresponding to 2-planes, satisfy $\phi \wedge \phi = 0$ (and this is sufficient too). Thus $G_{2,2}$ is realized a quadratic hypersurface in $\mathbf{RP}^5$ of signature $(+++---)$.

If U is a Lagrangian plane then $\phi \wedge \omega = 0$ (why?) and this is also sufficient. This is a linear condition that determines a hyperplane $\mathbf{RP}^4 \subset \mathbf{RP}^5$. This hyperplane is transverse to the image of $G_{2,2}$ (why?), and the intersection is the Lagrangian Grassmanian.

1.4. Given a symplectic space (V^{2n}, ω), the group of linear symplectomorphisms is called the *linear symplectic group* and denoted by $Sp(V)$ or $Sp(2n, \mathbf{R})$. A symplectic space has a volume element $\omega^{\wedge n}$, therefore $Sp(2n)$ is a subgroup of $SL(2n)$.

Let $A \in Sp(2n)$. Then $\omega(Au, Av) = \omega(u, v)$ for all u, v. Thus $A^* J A = J$. This is interesting to compare with the orthogonal group: $A^* A = E$. The relations between the classical groups are as follows.

Lemma. *One has:*

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbf{C}) = O(2n) \cap GL(n, \mathbf{C}) = U(n).$$

Proof. One has:

$$A \in GL(n, \mathbf{C}) \ \text{ iff } \ AJ = JA; \quad A \in Sp(2n) \ \text{ iff } \ A^* J A = J;$$

and

$$A \in O(2n) \ \text{ iff } \ A^* A = E.$$

Any two of these conditions imply the third. A linear map that preserves the Euclidean and the symplectic structures also preserves the Hermitian one, that is, belongs to $U(n)$.

Exercises. 1. Let $A \in Sp(2n)$ and λ be an eigenvalue of A. Prove that so are $\bar{\lambda}$ and $1/\lambda$.

2. Prove that if A is symplectic then A^* is antisymplectic, that is, $\omega(A^* u, A^* v) = -\omega(u, v)$ or, equivalently, $AJA^* = -J$.

In fact, $U(n)$ is the maximal compact subgroup of $Sp(2n)$, and the latter is homotopically equivalent to the former. As a consequence, $\pi_1(Sp(2n)) = \mathbf{Z}$. Indeed, one has a fibration det: $U(n) \to S^1$ with fiber $SU(n)$. The latter group is simply connected as follows inductively from the exact homotopy sequence of the fibration $SU(n) \to S^{2n-1}$ with fiber $SU(n-1)$.

1.5. Let us describe the Lie algebra $sp(2n)$ of the Lie group $Sp(2n)$. Let $A \in Sp(2n)$ be close to the identity: $A = E + tH + O(t^2)$. Then the condition $\omega(Au, Av) = \omega(u, v)$

for all u, v implies: $\omega(Hu, v) + \omega(u, Hv) = 0$; in other words, $JH + H^*J = 0$. Thus H is skew-symmetric with respect to ω. Such H is called a *Hamiltonian operator*. The commutator of Hamiltonian operators is again a Hamiltonian operator.

To a Hamiltonian operator there corresponds a quadratic form $h(u) = \omega(u, Hu)/2$ called the Hamiltonian (function) of H. One can recover H from h since $\omega(u, Hv) = h(u+v) - h(u) - h(v)$. This gives a one-one correspondence between $sp(2n)$ and quadratic forms on V^{2n}. Thus dim $sp(2n) = n(2n+1)$.

In terms of quadratic forms, the commutator writes as follows:

$$\{h_1, h_2\}(u) = \omega(u, (H_2H_1 - H_1H_2)u)/2 = \omega(H_1u, H_2u).$$

The operation on the LHS is called the *Poisson bracket*.

To write formulas, it is convenient to identify linear operators with linear vector fields: the operator H is understood as the linear differential equation $u' = Hu$. Let (p, q) be Darboux coordinates.

Lemma. *The next formulas hold:*

$$H = h_p\partial q - h_q\partial p; \quad \{h_1, h_2\} = (h_1)_p(h_2)q - (h_1)_q(h_2)_p.$$

Proof. To prove the first formula we need to show that

$$2h = \omega((p, q), (-h_q, h_p)).$$

The RHS is $ph_p + qh_q = 2h$, due to the Euler formula. Then the Poisson bracket is given by

$$\{h_1, h_2\} = \omega(((h_2)_p, -(h_2)_q), ((h_1)_p, -(h_1)q)) = (h_1)_p(h_2)q - (h_1)_q(h_2)_p,$$

as claimed.

More conceptually, given a quadratic form h, one considers its differential dh which is a linear differential 1-form. The symplectic structure determines a linear isomorphism $V \to V^*$ which makes dh into a linear vector field H, that is, $i_H\omega = -dh$.

1.6. One of the first, and most celebrated, results of symplectic topology was Gromov's nonsqueezing theorem (1985). Let us discuss its linear version (which is infinitely simpler).

Given a ball $B^{2n}(r)$ of radius r and a symplectic cylinder $C(R) = B^2(R) \times \mathbf{R}^{2n-2}$ (where the 2-disc is spanned by the Darboux coordinates p_1, q_1), assume that there is an affine symplectic map F that takes $B^{2n}(r)$ inside $C(R)$.

Proposition. *Then $r \le R$.*

Note that this is false for volume-preserving affine maps.

Proof. The map writes $F : v \to Av + b$ where $A \in Sp(2n)$ and $b \in \mathbf{R}^{2n}$. Assume $r = 1$. Consider A^* and its two columns ξ_1 and ξ_2 corresponding to p_1, q_1 Darboux coordinates. Since A^* is antisymplectic (Exercise in 1.4), $|\omega(\xi_1, \xi_2)| = 1$, and therefore $|\xi_1||\xi_2| \ge 1$. Assume that $|\xi_1| \ge 1$, and let $v = \xi_1/|\xi_1|$. Note that ξ_1 and ξ_2 are rows of A (corresponding to coordinates p_1, q_1). Since $F(v) \in C(R)$, one has:

$$(\xi_1 \cdot v + b_1)^2 + (\xi_2 \cdot v + b_2)^2 \le R^2,$$

and then $|(|\xi_1|+b_1)| \le R$. For $b_1 \ge 0$ this implies $R \ge 1$, and for $b_1 < 0$ one should replace v by $-v$.

One defines an affine symplectic invariant called *linear symplectic width* of a subset $A \subset \mathbf{R}^{2n}$:

$$w(A) = \max\ \{\pi r^2 | F(B^{2n}(r)) \subset A \text{ for some affine symplectic } F\}.$$

Symplectic width is monotonic: if $A \subset B$ then $w(B) \ge w(A)$, homogeneous of degree 2 with respect to dilations and nontrivial: $w(B^{2n}(r)) = w(C(r)) = \pi r^2$.

To get a better feel of linear symplectic space, classify the ellipsoids. In Euclidean space, every ellipsoid can be written as

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i^2}.$$

A corresponding symplectic result is as follows.

Theorem. *There exist Darboux coordinates in which an ellipsoid writes:*

$$\sum_{i=1}^{n} \frac{p_i^2+q_i^2}{r_i^2},$$

and the radii $0 \le r_1 \le ... \le r_n$ *are uniquely defined.*

Proof. Recall how to prove the Euclidean fact. We have two Euclidean structures: $u \cdot v$ and $Au \cdot v$. Here A is self-adjoint, and we assume, it is in general position. Consider a relative extremum problem of $Au \cdot u$ relative $u \cdot u$. The extremum condition (Lagrange multipliers!) is that $Audu = \lambda udu$, that is, $Au = \lambda u$. The function $Au \cdot u$ is an even function on the unit sphere, that is, a function on $\mathbf{RP}^{n-1}$, and it has n critical points. Thus A has n real eigenvalues $a_1, ..., a_n$, and the respective eigenspaces are orthogonal. We obtain the desired expression.

A symplectic analog is as follows. We have a dot product and a symplectic structure $\omega(u,v) = Ju \cdot v$. Consider a relative extremum problem of $\omega(u,v)$ relative $u \cdot u$ and $v \cdot v$. The extremum condition is:

$$Ju\ dv - Jv\ du = \lambda u\ du + \mu v\ dv, \text{ that is, } Ju = \mu v, Jv = -\lambda u.$$

Thus u, v are eigenvectors of J^2 (with eigenvalue $-\lambda\mu$), a self-adjoint operator. In general position, these eigenspaces are 2-dimensional and pairwise orthogonal.

Thus the space is the orthogonal sum of 2-dimensional subspaces. Claim: they are also symplectically orthogonal. Indeed,

$$\omega(u_1, u_2) = Ju_1 \cdot u_2 = \mu_1 v_1 \cdot u_2 = 0$$

and, likewise, with $\omega(u_1, v_2)$ and $\omega(v_1, v_2)$. It remains to choose an orthogonal basis p_i, q_i in each 2-space so that $p_i \cdot p_i = q_i \cdot q_i$ and $\omega(p_i, q_i) = 1$.

The last thing to check is that the radii r_i are uniquely defined. Let $D(r)$ be the diagonal matrix with the entries $1/r_i^2$. Assume that, for a symplectic matrix A, one has: $A^*D(r)A = D(r')$. Since A is symplectic, $A^*JA = J$, or $A^* = JA^{-1}J^{-1}$. Thus $A^{-1}J^{-1}D(r)A = J^{-1}D(r')$, that is, the eigenvalues of the matrices $J^{-1}D(r)$ and $J^{-1}D(r')$ coincide. It follows that $r = r'$.

2. Symplectic manifolds.

2.1. Let M be a smooth manifold. A *symplectic structure* on M is a non-degenerate closed 2-form ω. Since it is non-degenerate, dim $M = 2n$. In other words, $d\omega = 0$ and $\omega \wedge ... \wedge \omega$ (n times) is a volume form. In particular, M is oriented. Also, $H^2(M, \mathbf{R}) \neq 0$. Hence $S^{2n},\ n \geq 2$ is not symplectic.

A *symplectomorphism* is a diffeomorphism $f : M \to M$ such that $f^*(\omega) = \omega$. Symplectomorphisms form an infinite-dimensional group.

2.2. Examples.

(a) Linear symplectic space $\mathbf{R}^{2n}$ with $\omega = dp \wedge dq$.

(b) Any oriented surface with an area form. For example, S^2, with the (standard) area form $\omega_x(u, v) = \det(x, u, v)$.

(c) (Archimedes) Consider the unit sphere and the circumscribed cylinder with its standard area form. Consider the radial projection π from the sphere to the cylinder.

Exercise. Prove that π is a symplectomorphism.

(d) The product of symplectic manifolds is a symplectic manifold.

(e) Cotangent bundle (important for mechanics!). On T^*M one has a canonical 1-form λ called the Liouville (or action) form. Let $\pi : T^*M \to M$ be the projection and ξ be a tangent vector to T^*M at point (x, p). Define: $\lambda(\xi) = p(\pi_*(\xi))$. In coordinates, $\lambda = pdq$ where q are local coordinates on M and p are the corresponding covectors (momenta). The canonical symplectic structure on T^*M is $\omega = d\lambda$, locally, $dp \wedge dq$.

Exercise. Let α be a 1-form on M. Then α determines a section γ of the cotangent bundle. Prove that $\gamma^*(\lambda) = \alpha$.

(f) $\mathbf{CP}^n$. First, consider $\mathbf{C}^{n+1} = \mathbf{R}^{2n+2}$ with its linear symplectic structure Ω. Consider the unit sphere S^{2n+1}. The restriction of Ω on S^{2n+1} has a 1-dimensional kernel. Claim: at point x, this kernel is generated by the vector Jx. Indeed, if $u \perp x$ then $\Omega(Jx, u) = J(Jx) \cdot u = 0$. The vector field Jx generated a foliation on circles, and the space of leaves is $\mathbf{CP}^n$. The symplectic structure Ω induces a new symplectic structure ω on $\mathbf{CP}^n$. The construction is called *symplectic reduction*.

Complex projective varieties are subvarieties of $\mathbf{CP}^n$; they have induced symplectic structures, and this is a common source of examples.

(g) Another example of symplectic reduction: the space of oriented lines in $\mathbf{R}^{n+1}$. Start with $T^*\mathbf{R}^{n+1}$ with its canonical symplectic structure $\Omega = dp \wedge dq$. Consider the hypersurface $|p| = 1$. Claim: the kernel of the restriction of Ω on this hypersurface at point (p, q) is generated by the vector $p\partial q$. Indeed, $(dp \wedge dq)(u, p\partial q) = (pdp)(u) = 0$. We get a foliation whose leaves are oriented lines (geodesics). We obtain a symplectic structure on the space of oriented lines $\omega = dp \wedge dq$ where p is a unit (co)vector and $q \cdot p = 0$.

Exercise. Prove that the above space is symplectomorphic to T^*S^n.

(h) Orbits of the coadjoint representation of a Lie group. Let G be a Lie group and g its Lie algebra. The action of G on itself by conjugation has e as a fixed point. Since $g = T_eG$, one obtains a representation Ad of G in g called adjoint. Likewise, one has the coadjoint representation Ad^* in g^*. One also has the respective representations of g denoted by ad and ad^*. In formulas,

$$ad_x y = [x, y], \quad (ad_x^* \xi)(y) = \xi([x, y]), \quad x, y \in g, \ \xi \in g^*.$$

Theorem (Lie, Kirillov, Kostant, Souriau). *An orbit of the coadjoint representation of G has a symplectic structure.*

Proof. Let $\xi \in g^*$. Then the tangent space to the orbit of the coadjoint representation at ξ identifies with g/g_ξ where

$$g_\xi = \{x \in g | \ ad_x^* \xi = 0\}.$$

On the space g/g_ξ one has a skew-symmetric bilinear form $\omega(x, y) = \xi([x, y])$ (why is it well defined?). This 2-form is closed as follows from the Jacobi identity for the Lie algebra g.

Exercise. Prove the last statement.

Example. Let $G = SO(3)$, then $g = so(3)$, skew-symmetric 3×3 matrices. Identify them with $\mathbf{R}^3$. Given $A \in so(3)$, consider $Tr(A^2)$. This gives a Euclidean structure on $so(3)$ that agrees with that in $\mathbf{R}^3$. We identify g and g^*. Clearly, $Tr(A^2)$ is invariant under the (co)adjoint action. The orbits are level surfaces of $Tr(A^2)$, that is, concentric spheres and the origin.

Exercise. Similarly study another 3-dimensional Lie group, $SL(2)$.

2.3. Being non-degenerate, a symplectic form defines an isomorphism between vector fields and 1-forms: $X \mapsto i(X)\omega$. A field is called *symplectic* if $i(X)\omega$ is closed, in other words, if $L_X\omega = 0$; the latter follows from the Cartan formula: $L_X = di(X) + i(X)d$. Symplectic fields form the Lie algebra of the group of symplectomorphisms. Let M be a closed symplectic manifold.

Lemma. *Given a 1-parameter family of vector fields X_t, consider the respective family of diffeomorphisms ϕ_t:*

$$d\phi_t(x)/dt = X_t(\phi_t(x)), \quad \phi_0 = \mathrm{Id}.$$

Then ϕ_t are symplectomorphisms for all t iff X_t are symplectic for all t. Given symplectic fields X and Y, the field $[X, Y]$ is symplectic with $i([X, Y])\omega = d\omega(X, Y)$.

Proof. One has: $d\phi_t^*(\omega)/dt = \phi_t^*(L_{X_t}\omega)$, and this implies the first claim. As to the second, we use the (somewhat non-traditional) definition:

$$[X, Y] = L_Y X = d\psi_t^* X/dt|_{t=0},$$

and then

$$i([X, Y])\omega = di(\psi_t^* X)\omega/dt|_{t=0} = L_Y i(X)\omega = di(Y)i(X)\omega = d\omega(X, Y),$$

as claimed.

Note that with the definition of the Lie bracket above one has $L_{[X,Y]} = -[L_X, L_Y]$ (cf. McDuff-Salamon, p. 82).

For linear symplectomorphisms, we had a relation with quadratic functions. Likewise, given a (Hamiltonian) function H on M, define its Hamiltonian vector field X_H by $i(X_H)\omega = dH$. In Darboux coordinates, $X_H = H_p\partial q - H_q\partial p$. On closed M, this gives a 1-parameter group of symplectomorphisms called the Hamiltonian flow of H.

Lemma. *The vector X_H is tangent to a level hypersurface $H = const$.*

Proof. One has: $dH(X_H) = \omega(X_H, X_H) = 0$.

This lemma is related to the symplectic reduction construction. If S is given by $H = const$ then X_H generates the *characteristic foliation* on S, tangent to the field of kernels of the restriction of ω to S. Indeed, $\omega(X_H, v) = dH(v) = 0$ once v is tangent to S.

Define the *Poisson bracket* $\{F, G\} = \omega(X_F, X_G) = dF(X_G)$. In Darboux coordinates, the formulas are as in 1.5. This bracket satisfies the Jacobi identity; we will deduce it from Darboux theorem.

Lemma. *The correspondence $H \mapsto X_H$ is a Lie algebra homomorphism.*

Proof. We want to show that $[X_F, X_G] = X_{\{F,G\}}$. One has

$$[X_F, X_G] = -d\phi_t^*(X_G)/dt|_{t=0} = -dX_{G(\phi_F^t)}/dt|_{t=0}.$$

Then

$$i([X_F, X_G])\omega = -d(dG(\phi_F^t))/dt|_{t=0} = -d(dG(X_F)) = d(\{F, G\}),$$

as claimed.

To summarize, here are the main formulas, in Darboux coordinates:

$$\omega = dq \wedge dp; \quad X_H = H_p\partial q - H_q\partial p; \quad \{H, F\} = H_pF_q - H_qF_p.$$

Note that the Hamiltonian vector field X_H is also often called the *symplectic gradient* of the function H.

2.4. Unlike Riemannian manifolds, symplectic manifolds do not have local invariants.

Darboux Theorem. *Symplectic manifolds of the same dimension are locally symplectomorphic.*

Proof. Consider two symplectic manifolds with fixed points (M_1, O_1, ω_1) and (M_2, O_2, ω_2). We want to construct a local symplectomorphism $(M_1, O_1, \omega_1) \to (M_2, O_2, \omega_2)$. First consider a local diffeomorphism $(M_1, O_1) \to (M_2, O_2)$; now we have two (germs of) symplectic structures ω_0, ω_1 on the same manifold (M, O), and since there is only one symplectic vector space of a given dimension (the linear Darboux theorem), we assume that ω_0 and ω_1 coincide at point O.

Claim: There is a local diffeomorphism $f : M \to M$, fixing O and such that $f^*(\omega_0) = \omega_1$.

Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1$. This is a symplectic structure for all $t \in [0,1]$ in a small neighborhood of the origin. We need to find a family of diffeomorphisms ϕ_t, fixing O, such that $\phi_t^*\omega_t = \omega_0$. This is equivalent to finding a time-dependent symplectic vector field X_t, related to ϕ_t in the usual way, $d\phi_t(x)/dt = X_t(\phi_t(x))$, and vanishing at O, such that

$$L_{X_t}\omega_t + \omega_1 - \omega_0 = 0.$$

Choose a 1-form α such that $d\alpha = \omega_1 - \omega_0$; this α is defined up to summation with df. Then we have the equation

$$i(X_t)\omega_t + \alpha = 0.$$

This is solvable for all α since ω_t is non-degenerate. It remains to show that X_t may be taken trivial at O. For this, we need to replace α by a 1-form that vanishes at O.

Every 1-form can be locally written as

$$\alpha = \sum x_i\alpha_i + \sum c_i dx_i$$

where α_i are 1-forms and c_i are constants. Then we replace α by $\alpha - d(\sum c_i x_i)$, and this 1-form vanishes at O.

In fact, a similar *homotopy method*, due to Moser, applies to a more general situation in which the points O_1, O_2 are replaced by germs of submanifolds N_1, N_2 such that the pairs $(N_1, \omega_1|_{N_1})$ and $(N_2, \omega_2|_{N_2})$ are symplectomorphic.

2.5. A *Lagrangian submanifold* of a symplectic manifold (M^{2n}, ω) is a manifold L^n such that $\omega|_L = 0$. An informal principle is that every symplectically meaningful object is a Lagrangian manifold.

Examples.

(a) Every curve is Lagrangian in a symplectic surface.

(b) Consider T^*M with its canonical symplectic structure, and let α be a 1-form on M. This form determines a section γ of T^*M whose image is Lagrangian iff α is closed. Indeed,

$$\omega|\gamma(M) = \gamma^*(\omega) = d\gamma^*(\lambda) = d\alpha.$$

(c) Let $N \subset M$ be a submanifold. Its conormal bundle $P \subset T^*M|_N$ consists of the covectors, equal to zero on N. Then P is Lagrangian. Indeed, choose local coordinates $q_1, ..., q_n$ in M so that $q_1 = ... = q_k = 0$ is N. Let p, q be the respective coordinates in T^*M. Then P is given by $q_1 = ... = q_k = 0, p_{k+1} = ... = p_n = 0$. If N is a point, one obtains a "delta-function".

(d) Let $f : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism. Then the graph $G(f)$ is a Lagrangian submanifold in $(M_1 \times M_2, \omega_1 \ominus \omega_2)$. Indeed, consider $u_1, v_1 \in TM_1$, and let $u_2 = Df(u_1), v_2 = Df(v_1)$. Then (u_1, u_2) and (v_1, v_2) are two tangent vectors to $G(f)$, and

$$(\omega_1 \ominus \omega_2)((u_1, u_2), (v_1, v_2)) = \omega_1(u_1, v_1) - \omega_2(u_2, v_2) = (\omega_1 - f^*(\omega_2))(u_1, v_1) = 0.$$

(e) Let $N^{n-1} \subset \mathbf{R}^n$ be a hypersurface. Consider the set L of oriented normal lines to N; this is a Lagrangian submanifold of the space of oriented lines in $\mathbf{R}^n$. To prove this, recall that the space of oriented lines in $\mathbf{R}^n$ is symplectomorphic to T^*S^{n-1} with its symplectic structure $dp \wedge dq$; $q \in S^{n-1}, p \in T_q^*S^{n-1}$. Let $n(x)$ be the unit normal vector to N at point $x \in N$. Then L is given parametrically by the equations

$$q = n(x),\ p = x - (x \cdot n(x))\ n(x), \quad x \in N.$$

Hence

$$pdq = xdn - (x \cdot n)\ ndn = xdn = d(x \cdot n) - ndx = d(x \cdot n),$$

where $ndn = 0$ since $n^2 = 1$ and $ndx = 0$ on N since n is a normal. Therefore $dp \wedge dq = 0$ on L.

The function $(x \cdot n(x))$ is called the *support function* of the hypersurface N; it plays an important role in convex geometry.

Let $f : S^{n-1} \to \mathbf{R}$ be a support function. How can one construct the corresponding hypersurface N? Claim: N is the locus of points

$$y = f(x)x + \text{grad} f(x).$$

Indeed, we need to show that x is a normal to N at point y, i.e., $xdy = 0$. Note that $x\text{grad}f(x) = 0$ hence $x\ d\text{grad}f(x) + \text{grad}f(x)\ dx = 0$. Note also that $x^2 = 1$ hence $xdx = 0$. Now one has:

$$xdy = fxdx + x^2 df + x\ d\text{grad}f(x) = df - \text{grad}f(x)\ dx = 0$$

as needed.

Exercise. Let L^{n-1} be a submanifold of the space of oriented lines in $\mathbf{R}^n$. When does L consist of lines, orthogonal to a hypersurface? If this is the case, how many such hypersurfaces are there?

Exercise. Let $f : S^1 \to \mathbf{R}$ be the support function of a closed convex plane curve. Express the following characteristics of the curve in terms of f: curvature, area, perimeter length.

Exercise*. Let L be a Lagrangian submanifold in a symplectic manifold M. Prove that a sufficiently small neighborhood of L in M is symplectomorphic to a neighborhood of the zero section in T^*L. This statement is a version of Darboux theorem, and it can be proved along similar lines.

2.6. A *Lagrangian foliation* is an n-dimensional foliation of a symplectic manifold M^{2n} whose leaves are Lagrangian. Similarly one defines a *Lagrangian fibration.* An example is given by the cotangent bundle whose fibers are Lagrangian.

An *affine structure* on an n-dimensional manifold is given by an atlas whose transition maps are affine transformations. An affine manifold is *complete* if every line can be extended indefinitely. Examples include $\mathbf{R}^n$ and n-torus.

Theorem. *The leaves of a Lagrangian foliation have a canonical affine structure.*

Proof. Let M^{2n} be a symplectic manifold, $\mathcal{F}^n$ a Lagrangian foliation and $p : M \to N^n = M/\mathcal{F}$ the (locally defined) projection. Consider a function F on N and extend it to M as $F \circ p$. Let u be a tangent vector to a leaf. Then $dF(u) = 0$. Therefore X_F is skew-orthogonal to the tangent space of the leaf, that is, is tangent to the leaf. Then $\{F, G\} = \omega(X_F, X_G) = 0$, so the functions, constant on the leaves, Poisson commute.

Fix a point $x \in N$. If F is a function on N such that $dF(x) = 0$ then $X_F = 0$ on the leaf $\mathcal{F}_x$. Choose a basis in $T_x^* N$, choose n functions F_i whose differentials at x form this basis and consider the vector fields X_{F_i}. These are commuting vector fields, and we obtain a locally effective action of $\mathbf{R}^n$ on $\mathcal{F}_x$.

This action is well defined if the quotient space N is defined, for example, if $\mathcal{F}$ is a Lagrangian fibration. In general, N is defined only locally, and going from one chart to another changes the respective commuting vector fields by affine transformations. Thus an affine structure on the leaf is well defined.

Corollary. *If a leaf of a Lagrangian foliation is a closed manifold then it is a torus.*

Proof. If a leaf is complete then it is a quotient of $\mathbf{R}^n$ by a discrete subgroup. If the leaf is compact, the subgroup is a lattice $\mathbf{Z}^n$.

Here is what it boils down to in dimension 2. A Lagrangian foliation is given by a function $f(x, y)$: the leaves are the level curves. The function f is defined up to composition with functions of 1 variable: $f \mapsto \bar{f} = \phi \circ f$. The vector field X_f is tangent to the leaves and, on a leaf, one can introduce a parameter t such that $X_f = \partial t$. Changing f to $\bar{f}$, the field X_f multiplies by a constant ϕ' (depending on the leaf), and the parameter t also changes to $\bar{t} = ct$. This parameter, defined up to a constant, give an affine structure.

2.7. A consequence is the so-called Arnold-Liouville theorem in integrable systems.

Theorem. *Let M^{2n} be a symplectic manifold with n functions $F_1, ..., F_n$ that Poisson commute: $\{F_i, F_j\} = 0$. Consider a non-singular level manifold $M_c = \{F_i = c_i, i = 1, ..., n\}$ and a Hamiltonian function $H = H(F_1, ..., F_n)$. Then M_c is a smooth manifold, invariant under the vector field X_H. There is an affine structure on M_c in which the field X_H is constant. If M_c is closed and connected then it is a torus.*

Proof. The mapping $(F_1, ..., F_n) : M \to \mathbf{R}$ is a fibration near the value c, and its leaves are Lagrangian. The fields X_{F_i} are constant in the respective affine coordinates, and X_H is a linear combination of these n fields with the coefficients, constant on M_c (why?)

There is a version of this theorem in which X_H is replaced by a symplectomorphism $\phi : M \to M$ such that $F_i \circ \phi = F_i$ for all i. Then ϕ preserves M_c and the affine structure therein. Moreover, ϕ preserves each vector field X_{F_i}, and therefore ϕ is a parallel translation $x \mapsto x + c$.

Corollary. *Let ϕ and ψ be two symplectomorphisms that preserve the same Lagrangian foliation leafwise. Then ϕ and ψ commute.*

Proof. Both maps are parallel translations in the same affine coordinate system, and parallel translations commute.

2.8. Billiards. An example of a symplectic map is provided by billiards. Consider a strictly convex domain $M \subset \mathbf{R}^n$ with a smooth boundary N^{n-1}. Let U be the space of oriented lines that intersect M; it has a symplectic structure discussed in 2.2. Consider the billiard map $T : U \to U$ given by the familiar law of geometrical optics: the incoming and outgoing rays lie in one 2-plane with the normal at the impact point and make equal angles with this normal.

Theorem. *The billiard transformation is a symplectic map.*

Proof. Consider T^*M with its canonical symplectic structure $\omega = dp \wedge dq$ where q, p are the usual coordinates. We identify tangent and cotangent vectors by Euclidean structure. Consider two hypersurfaces in T^*M:

$$Y = \{(q,p) | p^2 = 1\}, \quad Z = \{(q,p) | q \in N\}.$$

The characteristics of Y are oriented lines in $\mathbf{R}^n$ (section 2.2, example g), and the symplectic reduction yields U with its symplectic structure. What are characteristics of Z? Consider the projection $\pi : Z \to T^*N$ given by the restriction of a covector on TN.

Claim: the characteristics of Z are the fibers of the projection π. Indeed, let $n(q)$ be the unit normal vector to N at point $q \in N$. Then the fibers of π are integral curves of the vector field $n(q)\partial p$. One has: $i(n(q)\partial p)\omega = n(q)dq = 0$ since n is a normal vector. It follows that the symplectic reduction of Z is the space $V = T^*N$.

Let $W = Y \cap Z$, the set of unit vectors with foot point on N. Consider W with the symplectic structure $\omega|_W$. The projections of W on U and V along the leaves of the characteristic foliations of Y and Z are double coverings. These projections are symplectic mappings (why?) One obtains two symplectic involutions σ and τ on W that interchange the preimages of a point under each projection. The billiard map T can be considered as a transformation of W equal to $\sigma \circ \tau$. Therefore T is a symplectomorphism.

The proof shows that the billiard map can be also considered as a symplectic transformation of T^*N realized as the set of inward unit vectors with foot points on N.

Exercise. Let $n = 2$. Denote by t an arc length parameter along the billiard curve and by α the angle between this curve and and the inward unit vector. The phase space of the billiard map is an annulus with coordinates (t, α). Prove that the invariant symplectic form is $\sin\alpha \; d\alpha \wedge dt$.

An alternative proof proceeds as follows. Let q_1q_2 be an oriented line, $q_1, q_2 \in N$. Let p_1 be the unit vector from q_1 to q_2. The billiard map acts as follows: $(q_1, p_1) \mapsto (q_2, p_2)$ where the covectors (q_2, p_1) and (q_2, p_2) have equal projections on $T_{q_2}N$. Consider the *generating function* $L(q_1, q_2) = |q_1q_2|$. Then

$$\partial L/\partial q_1 = -p_1, \quad \partial L/\partial q_2 = p_1.$$

Consider the Liouville form $\lambda = pdq$ and restrict everything on T^*N. Then one has: $T^*\lambda - \lambda = dL$. Therefore $\omega = d\lambda$ is T-invariant.

Corollary. *Billiard trajectories are extrema of the perimeter length function on polygons inscribed into N.*

Example. It is classically known that the billiard inside an ellipse is integrable: the invariant curves consist of the lines tangent to a confocal conic. Consider two confocal ellipses and the respective billiard transformations T_1, T_2. It follows from Corollary 2.7

that $T_1 \circ T_2 = T_2 \circ T_1$, an interesting theorem of elementary geometry (especially its particular case, "The most elementary theorem of Euclidean geometry")!

Exercise*. Let N be a smooth hypersurface in $\mathbf{R}^n$, and let X be the set of oriented lines in $\mathbf{R}^n$ with its canonical symplectic structure. Consider the hypersurface $Y \subset X$ that consists of the lines tangent to N. Prove that the characteristics of Y consist of the lines, tangent to a geodesic curve on N.

3. Symplectic fixed points theorems and Morse theory.

3.1. The next result was published by Poincaré as a conjecture shortly before his death and proved by Birkhoff in 1917. Consider the annulus $A = S^1 \times I$ with the standard area form and its area preserving diffeomorphism T, preserving each boundary circle and rotating them in the opposite directions. This means the a lifted diffeomorphism $\bar{T}$ of the strip $S = \mathbf{R} \times [0, 1]$ satisfies:

$$\bar{T}(x, 0) = (X, 0) \text{ with } X > x \text{ and } \bar{T}(x, 1) = (X, 1) \text{ with } X < x.$$

Theorem (Poincaré-Birkhoff). *The mapping T has at least two distinct fixed points.*

Both conditions, that T is area preserving and that the boundary circles are rotated in the opposite sense, are necessary (why?).

Proof. We prove the existence of one fixed point, the hardest part of the argument. Assume there are no fixed points. Consider the vector field $v(x) = \bar{T}(x) - x,\ x \in S$. Let point x move from lower boundary to the upper one along a simple curve γ, and let r be the rotation of the vector $v(x)$. This rotation is of the form $\pi + 2\pi k,\ k \in \mathbf{Z}$. Note that r does not depend on the arc γ (why?). Note also that T^{-1} has the same rotation r since the vector $T^{-1}(y) - y$ is opposite to $T(x) - x$ for $y = T(x)$.

To compute r, let $\varepsilon > 0$ be smaller than $|T(x), x|$ for all $x \in A$; such ε exists because A is compact. Let F_ε be the vertical shift of the plane through ε and let $\bar{T}_\varepsilon = F_\varepsilon \circ \bar{T}$. Consider the strip $S_\varepsilon = \mathbf{R} \times [0, \varepsilon]$. Its images under $\bar{T}_\varepsilon$ are disjoint. Since $\bar{T}_\varepsilon$ preserves the area, the image of S_ε will intersect the upper boundary. Let k be the least number of needed iterations, and let P_k be the upper most point of the upper boundary of this k-th iteration. Let $P_0, P_1, ..., P_k$ the respective orbit, P_0 on the lower boundary of S. Join P_0 and P_1 by a segment and consider its consecutive images: this is a simple arc γ. For ε small enough, the rotation r almost equals the winding number of the arc γ. In the limit $\varepsilon \to 0$, one has: $r = \pi$.

Alternatively, we have a vector field $v(x) = x_1 - x$ with $x_1 = T(x)$ along γ. One can homotop this field as follows: for 1/2 time freeze x at P_0 and let x_1 traverse γ to P_k, and for the other 1/2 time freeze x_1 at P_k and let x traverse γ.

Now consider the map T^{-1}. Unlike T, it moves the lower boundary of S right and the upper one left. By the same argument, its rotation equals $-\pi$. On the other hand, by a remark above, this rotation equals that of T, a contradiction.

A consequence is the existence of periodic billiard trajectories inside smooth strictly convex closed plane curves. The billiard transformation T is an area preserving map of the annulus $A = S^1 \times [0, \pi]$ (we assume that the length of the curve is 1). The map T

fixes the lower boundary and translates the upper through 1. Let R be the rotation of A in the negative direction through 1. Then n-periodic orbits of T with rotation number r are fixed points of the map $R^r T^n$. This map translated the boundaries in the opposite directions and the Poincaré-Birkhoff theorem yields at least 2 periodic trajectories of every rotation number. Even for 2-periodic trajectories, the number 2 is not quite trivial: one is the diameter, the longest chord inside the table, but the other is of minimax type.

Another symplectic fixed point result which is proved by an ad hoc method is as follows.

Theorem (Nikishin, Simon, 1974). *An area preserving diffeomorphism of the 2-sphere has at least 2 distinct fixed points.*

Proof (Sketch). By algebraic topology, the number of fixed points, counted with multiplicities, is 2. One needs to show that there could not be a single fixed point. This is because the index of an isolated fixed point of an area preserving diffeomorphism in dimension 2 cannot be greater than 1. This is illustrated by a similar statement for an isolated zero of a Hamiltonian vector field.

3.2. A far reaching generalization is due to V. Arnold (mid-1960-s). One has a closed symplectic manifold M and its *Hamiltonian symplectomorphism* $f : M \to M$. This means that f is the time-1 map of a time-dependent family of Hamiltonian vector fields (as in 2.3):

$$d\phi_t(x)/dt = X_{H_t}(\phi_t(x)), \quad \phi_0 = \mathrm{Id}$$

where H_t is a time-dependent Hamiltonian. The family of symplectomorphisms ϕ_t is called a *symplectic isotopy.*

Arnold's Conjecture. *A Hamiltonian symplectomorphism f has at least as many fixed points as the least number of critical points of a smooth function on M.*

There are two cases, both meaningful: f is in general position (the graph of f intersects the diagonal transversally) and f is arbitrary. The corresponding cases for smooth functions are: Morse functions (the Hessian is non-degenerate at critical points) and arbitrary functions.

For a standard symplectic torus T^{2n}, an equivalent assumption on f is that it preserves the center of mass. This means that a lifted mapping $\bar{f}$ of $\mathbf{R}^{2n}$ satisfies:

$$\int_{I^{2n}} (\bar{f}(x) - x)dx \in \mathbf{Z}^{2n}.$$

The relation of this condition to Hamiltonian symplectomorphism will be discussed later.

One case in which Arnold's Conjecture trivially holds is when f is a time-1 map of time-independent Hamiltonian vector field X_H: a critical point of H is a fixed point of f. Another case, known to Poincaré, is when f is C^1-close to the identity.

Proposition. *If a Hamiltonian symplectomorphism f of a closed symplectic manifold M is C^1-close to the identity then it has at least as many fixed points as the least number* *of critical points of a smooth function on M (this means two results: for all Hamiltonian symplectomorphisms/all smooth functions and for generic Hamiltonian symplectomorphisms/Morse functions).*

Proof. There is a function F on M whose critical points are fixed points of f. To construct F, let $\Delta \subset M \times M$ be the diagonal. Consider the graph $G \subset M \times M$ of f. Then G is C^1-close to Δ. A neighborhood of Δ is symplectomorphic to a neighborhood of the zero section in $T^*\Delta$. The graph of a symplectomorphism is Lagrangian, thus G is a the graph of a closed 1-form α on Δ whose zeroes correspond to the intersections of G and Δ, the fixed points of f. It remains to show that α is exact: $\alpha = dF$.

What we need to show is that $\int_\gamma \alpha = 0$ for every loop γ on Δ. Let λ be the Liouville 1-form on $T^*\Delta$. Then we need to prove that $\int_\Gamma \Lambda = 0$ where Γ is the image of γ under the section α. Make a simplifying assumption that M is an exact symplectic manifold: $\omega = d\lambda$. Then, under the identifications as in Example 2.5 (d), $\Lambda = \lambda_1 \ominus \lambda_2$. Our condition reads now: $\int_{f(\delta)} \lambda - \int_\delta \lambda = 0$ for every closed curve δ, or $f^*(\lambda) - \lambda$ is an exact 1-form.

Write X_t for X_{H_t}. One has:

$$d\phi_t^*(\lambda)/dt = \phi^*(L_{X_t}\lambda) = \phi^*(di(X_t)\lambda + i(X_t)d\lambda) = \phi^* d(\lambda(X_t) + H_t),$$

and therefore $f^*(\lambda) - \lambda$ is exact.

The first of modern symplectic fixed point results is due to Conley and Zehnder (1983).

Theorem. *A generic Hamiltonian symplectomorphism of the standard symplectic torus T^{2n} has at least 2^{2n} fixed points and, in the degenerate case, at least $2n+1$ fixed points.*

Proof. We will reduce the theorem to a statement of Morse theory. The approach is due to A. Givental. Lift ϕ to a Hamiltonian symplectomorphism ψ of $\mathbf{R}^{2n}$. One decomposes ψ into $\psi_k \circ ... \circ \psi_1$ where each ψ_i is sufficiently C^1 small. By the above Proposition and its proof, each ψ_i has a C^2-small generating function, say, f_i.

Assume that k is odd. Consider $Z = \mathbf{R}^{2nk} = \mathbf{R}_1^{2n} \times ... \times \mathbf{R}_k^{2n}$ with the symplectic structure $\Omega = \omega \oplus ... \oplus \omega$ (k times). Consider two maps of Z:

$$T(z_1, ..., z_k) = (z_2, ..., z_k, z_1) \;\text{ and }\; \Psi(z_1, ..., z_k) = (\psi_1(z_1), ..., \psi_k(z_k)).$$

We want to solve the equation $T(z) = \Psi(z)$. Consider the graphs $G(T), G(\Psi) \subset (Z \times Z, \Omega \ominus \Omega) = T^*Z$. The space $G(\Psi)$ is an exact Lagrangian submanifold, that is, the graph of the differential of a function $F : Z \to \mathbf{R}$ where $F(z_1, ..., z_k) = f_1(z_1) + ... + f_k(z_k)$. Note that each function f_i is lifted from T^{2n}, and therefore F is also lifted from T^{2nk}.

Claim: $G(T)$ is also the graph of the differential. Introduce the following notation. Let $(x_i, y_i), i = 1, ..., k$, be Darboux coordinates in $\mathbf{R}_i^{2n}$ in the first copy of Z, and let (x_i', y_i') be those in $\mathbf{R}_i^{2n}$ in the second copy of Z. Set:

$$q_i = (x_i + x_i')/2,\; Q_i = (y_i + y_i')/2,\; p_i = (y_i' - y_i)/2,\; P_i = (x_i - x_i')/2.$$

Then $dx \wedge dy - dx' \wedge dy' = 2(dP \wedge dQ + dp \wedge dq)$, that is, $(Z \times Z, \Omega \ominus \Omega) = T^*Z$ where Z is the (q, Q)-space. The equations $x_{i+1} = x_i',\; y_{i+1} = y_i'$ translate to the following equations defining $G(T) \subset T^*Z$:

$$p_i + p_{i+1} = Q_{i+1} - Q_i, \quad P_i + P_{i+1} = q_i - q_{i+1}$$

or

$$p_j = -\sum_{r=1}^{k-1} (-1)^r Q_{j+r},\ P_j = \sum_{r=1}^{k-1} (-1)^r q_{j+r}$$

where the indices are cyclic mod k. Here we use that k is odd. It follows that $G(T)$ is the graph of the differential of the quadratic function

$$H(q,Q) = \sum_{i<j} (-1)^{i+j} (Q_i q_j - q_i Q_j).$$

Note that F is invariant under the diagonal action of Z^{2n} on each summand $\mathbf{R}^{2n}$. The quadratic function $H(q,Q)$ is also invariant under parallel translations $(q,Q) \mapsto (q + c, Q + c)$; here we again use that k is odd. Both functions can be considered as functions on $T^{2n} \times \mathbf{R}^{2n(k-1)}$.

Thus we are interested in critical points of the function $G = H - F : T^{2n} \times \mathbf{R}^{2n(k-1)} \to \mathbf{R}$. Viewed from far away, G is a small perturbation of a function, equal zero on T^{2n} and equal to a quadratic form of signature $(n(k-1), n(k-1))$ on $\mathbf{R}^{2n(k-1)}$. We want to conclude that such G has not fewer critical points than a smooth function on T^{2n}. This is what Morse theory provides.

3.3. Morse theory relates the topology of a manifold with critical points of a smooth function on it. Let M^n be a closed manifold and f a smooth function on M. Let $x \in M$ be a critical point: $df(x) = 0$. Then one considers the Hessian matrix $H(f) = \partial^2 f/\partial x_i \partial x_j$ in some local coordinates. In fact, H is a quadratic form on $T_x M$ (whose definition does not need local coordinates or metric).

Exercise. Define $H(f)$ as a quadratic form. Is it also well defined at a non-critical point?

A critical point x is *non-degenerate* if $H(f)$ is non-degenerate at x. The *Morse index* $\mu(x)$ is the number of negative squares in $H(f)$, and the respective negative subspace is called *unstable*. According to the Morse Lemma, near a non-degenerate critical point of Morse index q, there exist coordinates in which the function writes as $c + x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ where $p = n - q$.

Exercise. Let A be a self adjoint linear map of $\mathbf{R}^n$ with a simple spectrum. Consider the function $f(x) = Ax \cdot x$ on the unit sphere S^{n-1}. Find the critical points and their Morse indices.

A function $f : M \to \mathbf{R}$ is called *Morse function* if all its critical points are non-degenerate. Associate the following Poincaré polynomial with a Morse function:

$$P_t(f) = \sum_{x \in Crf} t^{\mu(x)}.$$

A similar Poincaré polynomial is responsible for the topology of the manifold:

$$P_t(M) = \sum_k \mathrm{dim} H_k(M)\ t^k$$

where the coefficients are taken in a field. The *Morse inequalities* can be written as follows:

$$P_t(f) = P_t(M) + (1+t)Q_t$$

where Q_t has non-negative coefficients. In particular, the number of critical points of index k is bounded below by the k-th Betti number. It also follows that the alternating sum of critical points equals the Euler characteristic.

Examples. 1. The height function f on S^n has min and max with the indices 0 and n. Thus $P_t(f) = 1 + t^n = P_t(S^n)$. The height function f on M_g, the surface of genus g, has a min, a max and $2g$ saddle critical points. Thus $P_t(f) = 1 + 2gt + t^2 = P_t(M_g)$, and f is a *perfect* function.

2. In $\mathbf{C}^{n+1}$, consider the unit sphere $\sum |z_k|^2 = 1$ and a function $f(z) = \sum \lambda_k |z_k|^2$. Then f is a function on $\mathbf{CP}^n$ whose critical points correspond to coordinate axes. Consider a critical point, say, $z_1 = ... = z_n = 0$. One can use $x_1, y_1, ..., x_n, y_n$ as local coordinates, and

$$f = C + \sum_{k=1}^{n} (\lambda_k - \lambda_0)(x_k^2 + y_k^2).$$

Thus the Morse index is twice the number of k for which $\lambda_k < \lambda_0$. It follows that

$$P_t(f) = 1 + t^2 + t^4 + ... + t^{2n}.$$

Therefore $Q_t = 0$ in the Morse inequality, and

$$P_t(\mathbf{CP}^n) = 1 + t^2 + t^4 + ... + t^{2n}.$$

There is a number of approaches to proving Morse inequalities. First, I outline the dynamical system method, mostly due to Smale. Give M a generic Riemannian metric and consider grad f. Given a critical point x, consider the union of trajectories that start at x and that end at x. These are cells W_x and W_x^*, and they intersect transversally at x. One has: dim $W_x = \mu(x)$ and dim $W_x^* = n - \mu(x)$. Thus one obtains two stratifications of M by cells whose dimensions are Morse indices of f.

By a small perturbation of the field grad f one can put all W_x and W_y^* in general position so that they intersect transversally. It follows that if a trajectory goes from x to y then $\mu(x) > \mu(y)$. Indeed, the trajectory lies in $W_x \cap W_y^*$, hence dim $W_x \cap W_y^* \geq 1$. Then dim $W_x \geq \dim W_y + 1$. This condition fails in the usual picture of a torus.

To summarize, one has a filtration K_p of M by the unions of cells of dimension $\leq p$, and $K_p - K_{p-1} = \cup W_x,\ \dim W_x = p$. The standard topological arguments imply the Morse inequalities.

Let us explain the "standard topological argument". First, the homology of a cell space can be computed as the homology of a complex whose p-dimensional space is generated by p-dimensional cells of the space. Secondly, consider a chain complex

$$... \to C_{i+1} \to C_i \to C_{i-1} \to ...$$

and let $B_i \subset Z_i \subset C_i$ be the spaces of boundaries and cycles. Consider the generating function (Poincaré polynomials) B_t, Z_t, C_t and H_t. One has: $B_i = C_{i+1}/Z_{i+1}$, that is, $tB_t = C_t - Z_t$. On the other hand, $H_i = Z_i/B_i$, that is, $H_t = Z_t - B_t$. Combine to get:

$$C_t = tB_t + Z_t = H_t + (1+t)B_t.$$

This has the form of Morse inequalities.

An argument, similar to the one applied to Morse functions, applies to degenerate functions as well. Then one has a decomposition of M into pieces W_x which are not necessarily cells but are still contractible in M. We can enlarge them and obtain a open covering by sets, contractible in M. The least number of such sets is called *Lyusternik-Schnirelmann category* and denoted by $Cat(M)$. Thus the number of critical points of f is bounded below by $Cat(M)$, the Lyusternik-Schnirelmann inequality.

The category is hard to compute. However one has the following lower bound.

Proposition. *Assume M has k cohomology classes $u_1, ..., u_k$, dim $u_i > 0$, and $u_1 ... u_k \neq 0$. Then $Cat(M) \geq k+1$.*

Proof. Assume M can be covered by k contractible sets W_i. Then one can choose representatives $U_i \in H^*(M, W_i)$ (think of differential forms with support outside of W_i). Then $U_1 ... U_k \in H^*(M, \cup W_i) = 0$.

The maximal number k above is called the *cup-length* of M.

Example. The cup length of $\mathbf{RP}^n$ is n, therefore any smooth function has at least $n+1$ critical points on $\mathbf{RP}^n$.

Usually Lyusternik-Schnirelmann theory gives finer estimates than Morse theory but requires knowledge of the cohomology ring as opposed to Betti numbers only.

Another approach to Morse inequalities is by way of level surface method. One considers the set $M_c = \{x | f(x) \leq c\}$ and studies how its topology changes as c increases. As long as c remains non-critical, the homotopy equivalence class of M_c remains constant. Assume that there is only one critical point with Morse index μ on a critical level c. Choose sufficiently small ε and let $a = c - \varepsilon, b = c + \varepsilon$

Lemma. *The homotopy type of M_b is obtained from that of M_a by attaching a μ-cell:*

$$M_b \sim M_a \cup e_\mu.$$

Let us outline the proof. Consider a neighborhood of a critical point in which $f = -x^2 + y^2$ where x is μ-dimensional and y is $(n-\mu)$-dimensional. Consider the μ-dimensional disc $e = \{x^2 < \varepsilon,\ y = 0\}$. Then ∂e lies on the level surface $f^{-1}(-\varepsilon) = \partial M_a$ and M_b is obtained from M_a by attaching e and taking its neighborhood.

Consider now the two Poincaré polynomials, $P_t(f)$ and $P_t(M)$ restricted to M_c. How do they change from M_a to M_b? The former becomes greater by t^μ. With the latter, two scenarios are possible. The sphere ∂e is a cycle in M_a. If this cycle is a boundary, $\partial e = \partial \beta$ then $e \cup \beta$ gives a new homology class in M_b, and $P_t(M)$ increases by t^μ. If ∂e determines a homology class in M_a then the cell e kills it, and $P_t(M)$ decreases by $t^{\mu-1}$. In the first case, $P_t(f) - P_t(M)$ remains the same, and in the second increases by $t^\mu + t^{\mu-1} = (1+t)t^{\mu-1}$. Morse inequalities follow.

Exercises. 1. Let f be a harmonic function on $\mathbf{R}^n$. Prove that the Morse index of its critical point cannot equal 0 or n.

2. Let $F(z_1, ..., z_n)$ be an analytic function in $\mathbf{C}^n$ and $f = Re(F) : \mathbf{R}^{2n} \to \mathbf{R}$. Prove that the Morse index of every critical point of f equals n.

3.4. We discuss some applications of Morse theory. First, consider a submanifold $M^k \subset \mathbf{R}^n$. Let $p \in \mathbf{R}^n$ be a fixed point and $f : M \to \mathbf{R}$ is the distance to p. The critical points of f are those $q \in M$ for which $pq \perp M$. Thus the number of perpendiculars from p to M is bounded below by the sum of Betti numbers of M.

Let q be a critical point of f. We compute the Morse index. Assume that codim $M = 1$. Choose Cartesian coordinates near q so that M is given by

$$z = \frac{1}{2}(\sum a_i x_i^2 - \sum b_j y_j^2 + ...),\ a_i > 0, b_j > 0$$

(assuming that M is in general position and q is a generic point). Let $p = (0, r)$ with $r > 0$. Then, ignoring terms of order 3 and higher,

$$f(x, y) = (x^2 + y^2 + (z - r)^2)^{1/2} = r + \frac{1}{2r}(x^2 + y^2 - 2zr),$$

that is, the quadratic part is $x^2 + y^2 - r(\sum a_i x_i^2 - \sum b_j y_j^2)$. It follows that the Morse index is equal to the number of a_i satisfying $a_i > 1/r$.

Recall that the coefficients a_i are the principle curvatures of M at q, and that the points $q + (1/a_i)n$ are called focal points; here n is the unit normal vector at q. We proved that the Morse index of f equals the number of focal points on the segment pq.

Next, consider the case when codim $M^k > 1$. In this case there are many unit normals to M at $q \in M$, and each gives a quadratic form. Let $x(u)$ be a parameterization of M. For a unit vector vector n consider the quadratic form $(n \cdot \partial^2 x/\partial u_i \partial u_j)$. Its eigenvalues $\kappa_1, ..., \kappa_k$ are called the principal curvatures of M in the direction n. The points $q + \kappa_i^{-1} n$ are called focal points in the direction n.

Exercise. Prove that the Morse index of a non-degenerate critical point $q \in M$ of the function f is again equal to the number of focal points on the segment pq.

As an application, we prove the next result.

Theorem. *Let $M \subset \mathbf{C}^n$ be a smooth affine algebraic submanifold of real dimension $2k$. Then $H_i(M, \mathbf{Z}) = 0$ for $i > k$.*

Proof. Let $q \in M$. We need to consider the quadratic part of M at q. Let $z_1, ..., z_k$ be complex coordinates on M near q. The inclusion $M \subset \mathbf{C}^n$ gives a complex-analytic functions $w_1(z), ..., w_n(z)$. Let n be a unit normal vector to M at q. Then the Hermitian product $w \cdot \bar{n}$ decomposes in a complex series: Const + $Q(z_1, ..., z_k) + ...$ (why aren't there a linear term?)

To have real coordinates, let $z = x + iy$. Then Re $w \cdot \bar{n}$ decomposes in the series Const + $Q'(x_1, ..., x_k, y_1, ..., y_k) + ...$, and Q' is the second quadratic form of M in the direction n. We claim that the eigenvalues of Q' split into pair of opposite numbers. Indeed, $Q(iz_1, ..., iz_k) = -Q(z_1, ..., z_k)$. In other words, the operator J transforms Q' to

$-Q'$, or $J^*Q'J = -Q'$. Since J is an orthogonal transformation, $J^* = J^{-1} = -J$, and one has: $JQ'J = Q'$ or $JQ' = -Q'J$. Thus if $Q'(v) = \mu v$ then $Q'(Jv) = -\mu Jv$.

Returning to M, we see that the focal points on the line $q + tn$ are located symmetrically with respect to q. Now pick a generic point p. The sets $M_c = f^{-1}[0, c]$ are compact. If $q \in M$ is a critical point then its Morse index equals the number of focal points on pq. There are at most $2k$ focal points, and at most k lie on one side of q. Thus the index does not exceed k, and M has the homotopy type of a complex with cells of dimension not greater than k.

3.5. Morse theory was created in search of geodesics on Riemannian manifolds. Let M be a closed Riemannian manifold. Does is always carry a nontrivial closed geodesic? How many?

If we had a similar problem with fixed (distinct) ends then the minimum length string between the ends would be a solution. In the periodic problem this minimum may be a point. Let ΛM be the free loop space of maps $S^1 \to M$ (in the compact open topology). A classical result (Hadamard, Cartan,...) as follows.

Theorem. *Every component of ΛM other than the component of the trivial map contains a closed geodesic.*

Outline of proof. First, there is $\varepsilon > 0$ such that every two points at distance less than ε can be connected by a unique geodesic segment. Therefore every loop in ΛM can be subdivided into small pieces and homotoped to a geodesic n-gon with n sufficiently large. In particular, each component of ΛM contains a geodesic polygon.

Consider the space (finite dimensional!) $P_n M \subset M \times ... \times M$ consisting of $(x_1, ..., x_n)$ with
$$E(x) := \sum d(x_i, x_{i+1})^2 < \varepsilon.$$
Then every point in $P_n M$ determines a closed geodesic n-gon. Do you see why, for n large enough, this is not necessarily a short curve?

Now consider the energy function E. It is smooth near $P_n M \subset M^{(n)}$ and has a minimum. The minimum of E is attained inside $P_n M$ because it is defined as $E < \varepsilon$. This minimum is a smooth closed geodesics, i.e., has no corners.

Next consider the opposite case of a compact simply connected M. The next result was first proved by Lyusternik and Fet in the 1950-s.

Theorem. *M carries at least one closed geodesic.*

Outline of proof. The projection $\Lambda M \to M$ is a (Serre) fibration with fiber the space of based loops ΩM. This fibration admits a section, the constant loops. By algebraic topology,
$$\pi_i(\Lambda M) = \pi_i(M) \oplus \pi_i(\Omega M).$$
Another famous fibration, $PM \to M$ with fiber ΩM and contractible space gives: $\pi_i(M) = \pi_{i-1}(\Omega M)$. Thus
$$\pi_i(\Lambda M) = \pi_i(M) \oplus \pi_{i+1}(M).$$
Note that $\pi_i(M)$ cannot all be trivial. Choose a non-trivial element $x \in \pi_i(M)$ with minimal i. This gives $0 \neq y \in \pi_{i-1}(\Lambda M)$.

As before, replace ΛM by the space of polygons, $P_n M$, with n large enough. Then the homotopy groups will also coincide up to i. We want to find a critical point of E other than a constant path. If there are none then the gradient of E gives a homotopy equivalence of $P_n M$ to M. But there are no non-trivial elements in $\pi_{i-1}(M)$, a contradiction.

3.6. I will mention a number of further developments in Morse theory.

(i) *Morse theory on manifolds with boundary.* If $\partial M = N$ and f is a smooth function on M, then one should also count the critical points of $f|_N$. At such a point, grad f may have inward or outward direction, and we consider only the former points. Let $P_t(f)$ be the Poincaré polynomial of f in Int M and let $\bar{P}_t(f)$ be the Poincaré polynomial of $f|_N$, taking only the inward gradient points into account. Then

$$P_t(f) + \bar{P}_t(f) = P_t(M) + (1+t)Q_t$$

with non-negative Q_t.

Example. Let M^n be a disc and

$$f = \sum_{i=1}^{p} a_i x_i^2 - \sum_{j=p+1}^{n} b_j x_j^2.$$

Then $P_t(f) = t^q$ where $q = n - p$ and $\bar{P}_t(f) = 2(1 + t + ... + t^{q-1})$. Then

$$P_t(f) + \bar{P}_t(f) - P_t(M) = 1 + 2t(1 + t + ... + t^{q-1}) + t^q = (1+t)(1 + t + ... + t^{q-1}).$$

Exercises*. Prove these Morse inequalities. What happens if, instead of considering the inward gradient, one considers the outer one?

(ii) *Morse-Bott inequalities.* It often happens that instead of a critical point a function has a critical manifold. This is always the case when M is acted upon by a Lie group G and f is G-invariant. If $N \subset M$ is a critical submanifold then one looks at the Hessian in the normal direction (the normal bundle $\nu(N) = TM/TN$). Thus one defines Morse-Bott index $\mu(N)$. Assuming that the negative bundle of the Hessian over N is orientable, one has the Poincaré polynomial $P_t(f) = \sum_N t^{\mu(N)} P_t(N)$ and the usual inequalities.

Example. The function z on the standard T^2 in 3-space has two critical circles, and $P_t(z) = (1+t) + t(1+t)$. Thus this is a perfect function. The function z^2 on the standard S^2 has two critical points and a critical circle , and $P_t(z) = 2t^2 + (1+t)$.

(iii) *Morse-Smale-Witten complex.* Assume that the function f and the metric are generic so that the stable and unstable manifolds W_x and W_y^* of critical points x and y intersect transversally. Then $M(x,y) = W_x \cap W_y^*$ is the space of gradient lines from x to y. We saw that dim $M(x,y) = \mu(x) - \mu(y)$. One has also **R**-action on $M(x,y)$ by time translation in of the gradient flow.

One constructs a chain complex $\sum_{df(x)=0} \mathbf{Z}_2(x)$ with the boundary operator

$$\partial(x) = \sum_{y \in Crf, \mu(y) = \mu(x) - 1} (\#\text{ of gradient lines from } x \text{ to } y)\ (y).$$

One can prove that $\partial^2 = 0$, and the homology equal those of M. In fact, one can introduce orientations and do this over $\mathbf{Z}$.

A sketch of proof that $\partial^2 = 0$: We are given two critical points, x and z such that $\mu(x) - \mu(z) = 2$, and we want to show that there is an even number of gradient lines $x \to y \to z$. The space $M(x,z)/\mathbf{R}$ is 1-dimensional. It is a manifold with boundary, and therefore it consists of circles and intervals. Each interval contributes a pair of lines $x \to y \to z$, and their number is even.

Exercise. Compute the Morse-Smale-Witten complex for the function on T^2 given by $f(x,y) = \cos x + \cos y$.

In fact, one can reconstruct also the multiplication in the cohomology and other cohomological operations from Morse data (one may need more than one functions for that).

(iv) *Equivariant Morse theory.*

We already discussed the situation when M is acted upon by a compact Lie group G and the function f is G-equivariant. For example, the energy functional on loops is invariant under the S^1 action on the loops by shifting the parameter. If the action is free then M/G is a manifold, f is a function on it, and Morse inequalities apply to M/G. What does one do when the action is not free?

The recipe is known as the homotopy quotient or Borel construction. Note that if G acts on E freely then the diagonal action on $E \times M$ is also free. Note also that a contractible space is as good as a point. Define $M_G = (EG \times M)/G$ where EG is the universal G-space. For example, if M is a point then $M_G = BG$, the classifying space of G. One has a fibration $M_G \to BG$ with fiber M. Some common examples:

$$B\mathbf{Z} = S^1,\ B\mathbf{Z_2} = \mathbf{RP}^\infty,\ BS^1 = \mathbf{CP}^\infty,\ BU(n) = \mathbf{C}G_n(\infty).$$

In general, there is the Milnor construction for $EG = G * G * G * \ldots$.

The function f descends to M_G. It's Poincaré polynomial is as follows. Let O be a critical orbit on M. Then $O = G/H$, and

$$P_t^G(f) = \sum_{O \in Crf} t^{\mu(O)} P_t(BH).$$

Assume that the stabilizers H are connected. Then one has the usual Morse inequalities

$$P_t^G(f) = P_t^G(M) + (1+t)Q_t$$

where $P_t^G(M) = \sum t^i \dim H^i(M_G)$.

Example. Let $M = S^2$, $G = S^1$ and $f(x,y,z) = z$. Then both poles are critical with $H = S^1$. Thus $P_t^G(f) = (1+t^2)/(1-t^2)$. On the other hand, one finds from the fibration $M_G \to \mathbf{CP}^\infty$ that $P_t^G(M) = (1+t^2)/(1-t^2)$, so f is perfect.

Exercise. Find $P_t^G(f)$ for the function $f(x,y,z) = z^2$ on S^2.

(v) *Witten's approach.* Consider the de Rham complex $\Omega^0 \to \Omega^1 \to \ldots$ and perturb it by f as follows:

$$d_t\omega = d\omega + tdf \wedge \omega.$$

This is conjugated to the usual de Rham differential:

$$d_t = \exp(-tf) \circ d \circ \exp(tf),$$

so the homology remain the same for all finite t.

Very roughly speaking, one continues as follows. If one has a complex depending on a parameter, then the homology in general position is smallest, and for special values it may jump. This is because the kernel of a linear map can only jump while the image can only decrease for special values of t. Thus $H(d_t)$ is not greater than $H(d_\infty)$. The latter is the differential $\omega \mapsto df \wedge \omega$, and its homologies are concentrated on critical points of f. One arrives at Morse inequalities.

That the homology of d_∞ are related to the critical points of f is indicated by the following. If f had no critical points (say, replace df by a closed 1-form α) then one can find a vector field v such that $df(v) = 1$. Then

$$\omega = i_v(df \wedge \omega) + df \wedge i_v\omega = (i_v d_\infty + d_\infty i_v)\omega,$$

and d_∞ is acyclic.

The actual argument involves Hodge theory. Let $\Delta = d^*d + dd^*$ be the Laplacian acting on differential forms.

Lemma. $H(M) = \mathrm{Ker}\Delta$.

Proof. One has:

$$< (d^*d + dd^*)\omega, \omega > = < d^*\omega, d^*\omega > + < d\omega, d\omega > .$$

It follows that $\Delta\omega = 0$ if and only if $\omega \in \mathrm{Ker}d \cap \mathrm{Ker}d^*$. Next,

$$H = Z/B = Z \cap B^\perp = \mathrm{Ker}d \cap \mathrm{Ker}d^*$$

since $\mathrm{Ker}d^* = B^\perp$, and we are done.

Now we have Δ_t and Ω^i decomposes into the eigenspaces Ω^i_λ. Fix $c > 0$ and consider only $0 \le \lambda \le c$; this is a finite-dimensional subcomplex $\Omega^*_c(t) \subset \Omega^*(t)$ since the Laplace operator commutes with the differential. The cohomology of $\Omega^*_c(t)$ are still the de Rham cohomology of M. To obtain Morse inequalities one lets $t \to \infty$ and proves that dim $\Omega^i_c(\infty)$ equals the number of critical points of index i.

Example-Exercise. Consider $f(x) = -x^2/2$ near zero. Then

$$d_t(\phi) = \phi' dx - tx\phi dx,\ d_t^*(\phi dx) = -\phi' - tx\phi.$$

It follows that

$$\Delta^0_t(\phi) = -\phi'' + t^2x^2\phi + t\phi,\ \Delta^1_t(\phi dx) = -\phi'' dx + t^2x^2\phi dx - t\phi dx.$$

The second order differential operator $-d^2 + tx^2$ is known as quantum harmonic oscillator, and its spectrum is $|t|, 3|t|, 5|t|, ...$ (prove this!) It follows that, for $t > 0$,

$$\text{Spec}(\Delta_t^0) = 2t, 4t, 6t, ..., \quad \text{Spec}(\Delta_t^1) = 0, 2t, 4t, ...$$

A consequence is that if $f(x)dx$ is an eigenvector of Δ_t^1 for $t >> 1$ with a small eigenvalue then it concentrates near the maxima. For Δ_t^0, the situation is reversed, and the eigenfunctions $f(x)$ are concentrated near minima.

(vi) *Morse-Novikov theory.* It is clear that Witten's perturbation still works if one replaces df by a closed 1-form α. This already suggests a generalization of Morse theory from functions to closed 1-forms known as the Morse-Novikov theory. We consider the simplest case.

Let $[\alpha] \in H^1(M, \mathbf{Z})$. This 1-form determines a smooth map $g : M \to S^1$ given by integrating α over paths in M. There is a covering $p : N \to M$ with the group $\mathbf{Z}$ and a lifting $f : N \to \mathbf{R}$ such that $p^*(\alpha) = df$. One is interested in zeroes of α or critical points of f. A form is called Morse if so is the function f.

The usual constructions of Morse theory apply: each critical point of Morse index μ gives rise to a μ-dimensional cell going down but, unlike the case of compact manifold, it may go down to infinity, and the boundary of this cell may contain infinitely many cell of dimension $\mu - 1$. The action of $\mathbf{Z}$ adds constants to f and sends critical points to critical points of the same index.

To deal with this complication consider the ring K of Laurent series $\sum_{\text{const}<i} a_i t^i$ where t is thought of as the generator of the group of deck transformations $\mathbf{Z}$. The cell complex of the function f is considered as a complex of K-modules with finitely many generators (because α has finitely many zeroes on M):

$$0 \to C_n \to C_{n-1} \to ... \to C_1 \to C_0 \to 0.$$

Unlike the case of a Morse function, one can easily have $C_0 = 0$, and even all $C_i = 0$ if M fibers over S^1. One proves that the homology of the this Morse-Novikov complex is a homotopy invariant of (M, α), and the respective Morse-Novikov inequalities relate m_i, the number of zeroes of α of Morse index i with the rank of the respective homology group (all considered as K-modules). These numbers $B_i(M, [\alpha])$ play the role of Betti numbers in this theory, and the (weak) Morse-Novikov inequalities read: $m_i \geq B_i(M, [\alpha])$.

All this is closely related to Witten's theory. What follows applies to all 1-forms, not necessarily the ones with integer periods. Consider the deformation of the de Rham differential:

$$d_t\omega = d\omega + t\alpha \wedge \omega.$$

One proves that the homology is finite-dimensional for all t and that, as $t \to \infty$, the ranks stabilize to $B_i(M, [\alpha])$.

In fact, the homology of the differential d_t can be equivalently described as follows. Consider the representation of the fundamental group $\pi_1(M) \to \mathbf{C}^*$ given by the formula

$$\rho_t(\gamma) = \exp(i\ t \int_\gamma \alpha).$$

One obtains a local system of coefficients ρ_t on M, and the homology of d_t equals $H^*(M, \rho_t)$.

Digression. Recall the construction of the homology with local coefficients. We have a group $G = C^*$, and to a homotopy class of a path γ there corresponds a homomorphism $\rho(\gamma)$. Let c_q be the center of the standard q-simplex and $s_{q,i}$ be the straight segment connecting c_q with the center of the i-th face. A q-dimensional singular chain is $\sum c_j f_j$ where f_j is a singular q-simplex and $c_j \in G$. The boundary is defined as follows:

$$d(cf) = \sum_{i=0}^{q} (-1)^i \rho(f(s_{q,i}))(c)\ \Gamma_i(f),$$

where Γ_i is the i-th face operator.

The problem of estimating the numbers $B_i(M, [\alpha])$ is quite interesting on its own right. We have the differential d_t; denote by $H_{\text{g.p.}}$ the homology in general position with respect to t. Consider a more abstract set-up: let $d_t = d + t\delta$ be a differential in a finite dimensional space V.

Lemma. *The differential ∂ acts on $H(V, d)$ and*

$$\text{rk} H_{\text{g.p.}}(V, d_t) \leq \text{rk} H(H(V, d), \partial).$$

Proof. One has:

$$d_t^2 = d^2 + t(d\partial + \partial d) + t^2 \partial^2.$$

It follows that d and ∂ are (skew) commuting differentials, and therefore ∂ acts on $H(V, d)$.

Consider t as a formal parameter and work with infinite series in t. If $\omega_t = \omega_0 + t\omega_1 + t^2\omega_2 + ... \in \text{Ker} d_t$ then $d\omega_0 = 0,\ \partial\omega_0 = d\omega_1$, etc. Likewise, $\omega_t \in \text{Im} d_t$ if $\omega_0 = d\eta_0, \omega_1 = \partial\eta_0 + d\partial\eta_1$, etc. Now, given $[\omega_t] \in H_{\text{g.p.}}(V, d_t)$, assign to it the class $[\omega_0] \in H(H(V, d), \partial)$. One checks that this is a well defined and injective homomorphism.

In out case, the upper bound of the Lemma is the (graded) dimension of the kernel of the operator of multiplication by $[\alpha] \in H^1(M)$. More refined estimates are available along these lines involving Massey products.

Example-Exercise. In symplectic topology, one often needs to consider the following 1-form on the loop space ΛM of a symplectic manifold (M, ω). If $\omega = d\lambda$ then one has a function $\Lambda M \to \mathbf{R}$ given by

$$\gamma \mapsto \int_\gamma \lambda.$$

If ω is not exact one may take a surface $N^2 \subset M$ with $\partial N = \gamma$ and consider the function $\gamma \mapsto \int_N \omega$. This depends on the choice of N and is a multivalued function or a 1-form. Equivalently, one defines a 1-form on ΛM : to a vector field v along γ one assigns the number

$$\int_\gamma i_v \omega.$$

Prove that this 1-form is closed.

3.7. Return to the Conley-Zehnder theorem (sect. 3.2). We constructed a function G on $T^{2n} \times \mathbf{R}^{2n(k-1)}$ whose critical points correspond to the fixed points of the Hamiltonian symplectomorphism of the torus. Let $x \in T^{2n}$ and $y \in \mathbf{R}^{2n(k-1)}$, and let $Q(y)$ be a quadratic form of signature $(n(k-1), n(k-1))$. The function G had the following property: if $|y| >> 1$ then $dG(x,y) = dQ(y) + O(1)$. One can perturb G without creating new critical points so that, for $|y| >> 1$, one has: $G(x,y) = Q(y)$.

One wants to prove that, in general position, G has no fewer critical points than 2^{2n}. Indeed, let M_c be the sublevel set $\{G \leq c\}$. Then, for $c >> 1$, we have:

$$H^*(M_c, M_{-c}) = H^*(T^{2n} \times (D^{n(k-1)}, S^{n(k-1)-1})),$$

and the Poincaré polynomial of the latter space is $t^{n(k-1)}\ (1+t)^{2n}$. By Morse theory, one has no fewer than 2^{2n} critical points between levels $-c$ and c.

The main difference of this consideration from the closed geodesic problems is that the function is not bounded either below or above (unlike the energy or length that are bounded below).

To see the variational aspect, consider the space $\Lambda_0 T^{2n}$ of contractible loops on the torus. We will define a functional on this space whose critical points correspond to the fixed points of the Hamiltonian torus symplectomorphism. Given a contractible loop γ, lift it to $\mathbf{R}^{2n}$ as a loop $\tilde{\gamma}$. Define the action functional

$$F(\gamma) = \int_{\tilde{\gamma}} pdq - qdp - H(p,q,t)dt$$

where H is the time-dependent Hamiltonian defining the Hamiltonian symplectomorphism, lifted to $\mathbf{R}^{2n}$.

Lemma. *The extrema of F correspond to fixed points of the Hamiltonian symplectomorphism.*

Proof. Consider an infinitesimal perturbation of $\tilde{\gamma}$ given by a periodic vector field $a\partial p + b\partial q$. Then the increment of F is

$$dF(a\partial p + b\partial q) = \int_0^1 (p\dot{b} - q\dot{a} - H_p a - H_b b)dt = \int_0^1 (-\dot{p}b + \dot{q}a - H_p a - H_b b)dt.$$

This vanishes for all a, b if and only if $\dot{q} = H_p,\ \dot{p} = -H_q$, that is, the loop is a trajectory of the time-dependent Hamiltonian vector field with the Hamiltonian H.

The functional F can be considered as a perturbation of the quadratic function $\int pdq - qdp$ which is bounded neither below nor above.

Exercise. Let (M, ω) be a symplectic manifold and H a time-dependent Hamiltonian 1-periodic in t. Define a (multi-valued) functional on $\Lambda_0 M$, similar to F, and interpret its extremals as fixed points of the respective Hamiltonian symplectomorphism (cf. Example in 3.6).

Now we connect this variational approach with (pseudo)holomorphic curves. Recall that if $u(x, y)$ and $v(x, y)$ are the real and the imaginary parts of a holomorphic function then the Cauchy-Riemann equations hold: $u_x = v_y, u_y = -v_x$. This can be also written as

$$(u, v)_x + J(u, v)_y = 0$$

where $J(u, v) = (-v, u)$ is the operator of multiplication by $\sqrt{-1}$.

It turns out that the gradient lines of the functional F can be interpreted as solutions of a perturbed Cauchy-Riemann equation. The above formula read:

$$dF(a\partial p + b\partial q) = \int_0^1 (a\partial p + b\partial q) \cdot ((\dot{q} - H_p)\partial p - (\dot{p} + H_q)\partial q),$$

and therefore the field $(\dot{q} - H_p)\partial p - (\dot{p} + H_q)\partial q$ along γ is the gradient of F. Thus the equations for a gradient line is

$$p_s = q_t - H_p, \ q_s = -p_t - H_q$$

or

$$\frac{\partial w}{\partial s} + J\frac{\partial w}{\partial t} + \nabla H = 0$$

where s is the parameter in the gradient flow and $w = (p, q)$. This is the Cauchy-Riemann equation, perturbed by ∇H.

A. Floer used this approach to prove Arnold's conjecture for *monotone* symplectic manifolds that satisfy the following technical assumption: ω and c_1 (the first Chern class of the tangent bundle) are proportional over $\pi_2(M)$. Using J-holomorphic curves, Floer constructed a complex generated by fixed points of the Hamiltonian symplectomorphism whose homology are called Floer homology. For monotone symplectic manifolds they coincide with $H(M)$. Recently Arnold's conjecture has been finally proved for all closed symplectic manifolds.

4. Contact structures.

Contact geometry is an odd dimensional twin of symplectic one. The relation between contact and symplectic geometries is similar to the relation between projective and affine ones.

4.1. Consider a smooth field of hyperplanes (a distribution) ξ of a manifold M. Locally the field is given by a 1-form λ such that $\xi = \text{Ker}\lambda$. The distribution is called *integrable* if there is a codimension 1 foliation tangent to ξ at every point.

Lemma. *ξ is integrable if and only if $\lambda \wedge d\lambda = 0$.*

Proof. The distribution is integrable if and only if the commutator of every two sections of ξ is again its section. In other words, if $\lambda(u) = \lambda(v) = 0$ then $\lambda([u, v]) = 0$. It is also clear that $\lambda \wedge d\lambda = 0$ if and only if $d\lambda$ vanishes on ξ.

One has:

$$d\lambda(u, v) = \lambda([u, v]) - L_u\lambda(v) + L_v\lambda(u).$$

If ξ is integrable then $d\lambda(u,v) = 0$, and it follows that $\lambda([u,v]) = 0$. Conversely, if $d\lambda$ vanishes on ξ then $\lambda([u,v]) = 0$, and ξ is integrable.

Exercise. Let ξ be a codimension k distribution locally given by 1-forms $\lambda_1, ..., \lambda_k$. Then ξ is integrable (i.e., is a codimension k foliation) if and only if, for all i, the 2-form $d\lambda_i$ lies in the ideal $(\lambda_1, ..., \lambda_k)$ (Frobenius Theorem).

A $2n$-dimensional distribution ξ on M^{2n+1} is called a *contact structure* if it is maximally non-integrable: $\lambda \wedge d\lambda^n \neq 0$ for a 1-form λ that locally defines ξ. Clearly, this does not depend on the choice of this 1-form. The form may not exist globally; its existence is equivalent to coorientation of the distribution. A *contactomorphism* is a diffeomorphism that takes a contact structure to a contact structure.

Exercise. A contact 3-manifold is orientable.

An equivalent definition: $d\lambda$ is non-degenerate on ξ. Hence ξ must be even dimensional and M odd dimensional. If L^k is a submanifold, tangent to ξ, then λ and $d\lambda$ vanish on L, that is, T_xL is an isotropic subspace of $\xi(x), d\lambda$. Hence $k \leq n$. A submanifold is called *Legendrian* if it is n-dimensional and everywhere tangent to the contact distribution.

4.2. Examples.

(a) The form $dz - ydx$ gives a contact structure on $\mathbf{R}^3$. Likewise $dz - \sum y_i dx_i$ is a contact form on $\mathbf{R}^{2n+1}$. Such coordinates are called Darboux.

(b) Let V^{2n+2} be a symplectic vector space. Then $P(V)$ has a contact structure defines as follows. If $l \subset V$ is a line then E^{2n+1} is its skew-orthogonal complement. In projectivization one obtains a codimension 1 distribution $\xi = P(E)$. Let (p,q) be Darboux coordinates. Consider an affine chart $q_0 = 1$. In this chart, ξ is given by the 1-form

$$\sum pdq - qdp = -dp_0 + \sum_{i=1}^{n} p_i dq_i - q_i dp_i.$$

It follows that ξ is a contact structure.

Exercise. Consider the unit sphere in $\mathbf{C}^{n+1}$, and let $\xi(z)$ be the complex tangent space (n-dimensional complex) at $z \in S^{2n+1}$. Prove that this is a contact structure contactomorphic to the lift of the above described one in the projective space via the 2-fold covering $S^{2n+1} \to \mathbf{RP}^{2n+1}$. Show also that ξ consists of the orthogonal complements to the fibers of the Hopf bundle $S^{2n+1} \to \mathbf{CP}^n$.

(c) A *contact element* on a smooth manifold M^n is a hyperplane in a tangent space. The space of all contact elements is the projectivization of the cotangent bundle PT^*M. Likewise, the space of cooriented contact elements is ST^*M. The contact structure ξ on the space of contact elements is defined by the following skating condition: a velocity vector of a contact element belongs to ξ if the velocity vector of the base point belongs to the contact element. Let (q,p) be Darboux coordinates on T^*M. Consider a chart $p_0 = 1$. In this chart, ξ is given by the 1-form $dq_0 + \sum p_i dq_i$.

Let $N \subset M$ be a submanifold. Then $L(N)$, the set of contact elements, tangent to N, is a Legendrian submanifold in the space of contact elements. In particular, the fibration $PT^*M \to M$ is Legendrian, that is, has Legendrian fibers. For example, the space of contact element of the plane is the solid torus $S^1 \times \mathbf{R}^2$ with the contact form

$dy/dx = \tan\alpha$. A curve in the plane lifts to the solid torus as a Legendrian curve, and a Legendrian curve projects to the plane as a (wave) *front*. A front may have singularities but it has a tangent line at every point.

Exercise. Prove that the space of cooriented contact elements of S^2 is $SO(3) = \mathbf{RP}^3$.

(d) Projective duality is best understood via contact structures. Let P be a projective space. Then $PT^*P = PT^*P^*$. Indeed, both spaces consist of pairs (point of P, hyperplane in P). Thus one has two contact structures on this space.

Lemma. *The two contact structures coincide.*

Proof. Lift everything to the vector space V such that $P = P(V)$. A line is given by a vector v and hyperplane is characterized by its conormal ν. The space consists of pairs (v, ν) with $v\nu = 0$. The first contact form is given by $\nu dv = 0$ and the second by $v d\nu = 0$. Since $v d\nu + \nu dv = 0$, the two structures coincide.

Projective duality consists of lifting a submanifold in P to PT^*P as a Legendrian manifold and then projecting to P^* as a front. The dual of the dual of convex hypersurface is the original hypersurface.

Exercises. Prove that, for plane curves, cusps are dual to inflection points. What is dual to double tangents? Draw a curve, dual to $y = (x^2 - 1)^2$.

Duality takes graphs of convex functions to graphs of convex functions. The respective transformation of functions is called the *Legendre transform*.

Exercise. Find the Legendre transform of $f(x) = x^\alpha$.

(e) Contact structures are helpful in understanding the geodesic flow and, more generally, wave propagation. The geodesic flow in the space of cooriented contact elements is the motion of the contact element with unit speed along the geodesic, perpendicular to it. The time-t flow takes a hypersurface to the t-equidistant hypersurface (compare to the Huygens principle). It follows that, in the space of contact elements, the geodesic flow takes Legendrian manifolds to Legendrian ones. It follows that the geodesic flow is a 1-parameter group of contactomorphisms (why?)

Exercise. Describe the equidistant curves of an ellipse.

(f) Recall the notion of 1-jet of a function on a manifold. The space J^1M has a contact structure. Indeed, $J^1M = \mathbf{R} \times T^*M$, and in Darboux coordinates, the 1-form $dz - pdq$ is contact. If f is a function on M then its 1-jet extension (or 1-graph) is a Legendrian manifold.

Exercise. Prove that $ST^*\mathbf{R}^n$ is contactomorphic to J^1S^{n-1}. Hint: consider a contact element at point x given by a unit normal v. assign to it the 1-jet of the function $< x, . >$ at point $v \in S^{n-1}$.

4.3. Let (M^{2n-1}, ξ) be a contact manifold. Consider the manifold N^{2n} that consists of non-zero covectors on M, equal to zero on ξ. Then N is a fibration over M with fiber $\mathbf{R}^*$. For example, if M is the space of contact elements of manifold Q then $N = T^*Q - Q$. One has a canonical 1-form λ on N. Let $p : N \to M$ be the projection. The value of λ on tangent vector v at point α is $\alpha(dp(v))$. In other words, λ is the restriction of the Liouville form on $N \subset T^*M$.

Lemma. *The 2-form $d\lambda$ is a symplectic structure on N.*

Proof. Locally one can choose a contact form α. Then, also locally, $N = M \times \mathbf{R}^*$, and $\lambda = t\alpha$. Hence $d\lambda = td\alpha + dt \wedge \alpha$ and $d\lambda^n = t^{n-1} dt \wedge \alpha \wedge d\alpha^{n-1} \neq 0$.

The manifold N is called the *symplectization* of the contact manifold M. Symplectization translates contact questions to $\mathbf{R}^*$-homogeneous symplectic ones. For example, contactomorphisms of M are symplectomorphisms of N, commuting with the $\mathbf{R}^*$-action. Legendrian submanifolds in M are conical Lagrangian submanifolds in N.

We saw in 2.6 that the leaves of a Lagrangian foliation have an affine structure. Consider a Legendrian foliation. Symplectization transforms it into a Lagrangian foliation with a $\mathbf{R}^*$-action, and this gives a *projective structure* on the leaves. Here is an equivalent construction. Locally, a foliation is a fibration $p : M^{2n-1} \to B^n$ with Legendrian fibers. Let $x \in M$. Then $dp(\xi(x))$ is a contact element on B. We obtain a mapping $M \to PT^*B$ that sends the leaves to the fibers of $PT^*B \to B$, that is, to projective spaces. This gives a projective structure on the leaves.

A contact vector field v on M is an infinitesimal contactomorphism. If α is a contact 1-form then $L_v\alpha = f\alpha$ for a function f. A contact vector field on M lifts to a symplectic vector field on N, commuting with the $\mathbf{R}^*$-action. In particular, the Hamiltonian function can be taken homogeneous of degree 1: $H(tx) = tH(x)$.

Fix a contact 1-form on M. This gives a section of the fibration $N \to M$. Define the *contact Hamiltonian* of a contact vector field as the restriction on M of the homogeneous Hamiltonian of the symplectization of this field.

Lemma. *In Darboux coordinates, when $\alpha = dz + ydx$, the contact vector field v_f with the contact Hamiltonian f is given by the formula*

$$f_y \partial x + (yf_z - f_x)\partial y + (f - yf_y)\partial z.$$

Proof. One has coordinates x, y, z, t in N with $\lambda = tdz + tydx$. Change coordinates:

$$p = ty,\ q = x,\ p_0 = t,\ q_0 = z.$$

Then $\lambda = pdq$. Let $H(p, q, p_0, q_0)$ be the Hamiltonian; it is homogeneous of degree 1 in p, p_0. By Euler's formula, $(p_0\partial p_0 + p\partial p)H = H$ and therefore $p_0\partial p_0(H) = H - p\partial p(H)$. One has: $f(x, y, z) = H(y, x, 1, z)$. Now consider the Hamiltonian vector field:

$$\dot{q} = H_p,\ \dot{p} = t\dot{y} + y\dot{t} = -H_q,\ \dot{q}_0 = H_{p_0},\ \dot{p}_0 = -H_{q_0}.$$

It follows that

$$\dot{x} = f_y,\ t\dot{y} + y\dot{t} = -f_x,\ \dot{z} = f - yf_y,\ \dot{t} = -f_z,$$

and the result follows.

Corollary. *One has: $\alpha(v_f) = f$.*

Exercise. 1. The correspondence $f \to v_f$ makes it possible to define a structure of Lie algebra on the space of smooth functions on M (called the Lagrange bracket). Write down explicit formulas in Darboux coordinates. Does Leibnitz rule hold for this bracket?

Another operation is *contactization* of a symplectic manifold. Let (N, ω) be an exact symplectic manifold. Choose a 1-form α such that $d\alpha = \omega$. Consider the manifold $M = N \times \mathbf{R}$ with the contact 1-form $\alpha - dt$ (why is it contact?) If α' is a different potential for ω and $\alpha' - \alpha = df$ then the two contactizations are contactomorphic via $(x, t) \to (x, t + df(x))$.

Example. If $N = T^*Q$ then $M = J^1Q$.

The contactization of a Lagrangian submanifold $\Lambda \subset N$ is a Legendrian submanifold $L \subset M$ that projects diffeomorphically on Λ. This contactization exists if and only if $\alpha|_\Lambda$ is exact. Such a Lagrangian submanifold is called *exact*.

Example. Let N be the plane with the standard area form. Every closed curve is a Lagrangian submanifold, and it is exact only if the curve bounds zero area. Note that such exact Lagrangian curve necessarily has self-intersections. A far-reaching generalization is the Gromov theorem: there are no exact Lagrangian embeddings of T^n into the standard symplectic $2n$-space.

One more relation between symplectic and contact manifolds is provided by the next construction. Let α be a contact 1-form on M. The *Reeb field* of α is the vector field such that $i_v d\alpha = 0$ and $\alpha(v) = 1$. It follows that $L_v \alpha = 0$. If the space of trajectories of the Reeb field is a manifold then this manifold has a symplectic structure coming from $d\alpha$.

Exercise. In the notation of the above lemma, prove that v_f is the Reeb field for the contact form $f\alpha$ (this somewhat explains the formulas for the vector field v_f from this lemma).

Exercises*. One has a version of Darboux theorem for contact manifolds (and contact 1-forms): locally they are all diffeomorphic. Moreover, if L is a Legendrian submanifold in a contact manifold M then a neighborhood of L in M is contactomorphic to a neighborhood of the zero section in the 1-jet space J^1L.

Another result of this kind is the Gray stability theorem: homotopic contact structures on closed manifolds are diffeomorphic. This is not true anymore for open manifolds. The proof uses the Moser homotopy method (cf. the proof of Darboux theorem).

4.4. This section is a very sketchy exposition of the theory of Legendrian knots. I do not include numerous figures here. Bennequin's discovery of non-standard contact structures in $\mathbf{R}^3$ in 1983 made use of knots.

First, a few words about Eliashberg's tight and overtwisted contact structures in dimension 3. If (M, ξ) is a contact 3-fold and S is a surface in M then the intersection with ξ defines a line field on S (in general, with singularities). of course, this foliation is called characteristic. An *overtwisted disc* in M is an embedded disc whose boundary is a characteristic curve. A contact manifold is called *overtwisted* if it contains an overtwisted disc; it is called *tight* otherwise.

Among other things, Eliashberg discovered the following results:

Theorem 1. *The isotopy classification of overtwisted contact structures on a closed 3-manifold is equivalent to the homotopy classification of 2-dimensional distributions.*

For example, the homotopy classes of 2-distributions on S^3 are in one-to-one correspondence with integers.

Theorem 2. *There is a unique tight contact structure on S^3, namely, the standard one.*

Bennequin's theorem can be formulated as stating that the standard contact structure in 3-space is tight (the total number of contact structures in $\mathbf{R}^3$ is countable – Eliashberg).

Consider Legendrian and transverse knots (and links) in the standard contact 3-space with the form $dz - ydx$. Legendrian curves are framed. The *Maslov index* μ is the rotation number of the curve inside the contact plane; the *Thurston-Bennequin index tb* is the self-linking of the curve. Transverse curves are also framed by the (homotopically unique) nonvanishing section of the contact distribution; the Bennequin index β is the respective self-linking number. Let γ be an oriented Legendrian curve. Slightly pushing the curve left or right inside the contact planes one obtains the curves $\gamma_\pm$. These curves are transverse, one ascending and another descending. One has:

$$\beta(\gamma_\pm) = tb(\gamma) \pm \mu(\gamma).$$

Exercise. Prove the last statements.

One can draw these curves in the (x, z) plane; for Legendrian ones, the missing y coordinate is the slope. A consequence of this representation is that every curve can be C^0-approximated by a Legendrian and by a transverse one. It also follows that every topological knot (or link) has a Legendrian, and a transverse, realization.

The elementary invariants, μ and tb, can be expressed in terms of the projection: μ is an algebraic sum of the cusps, and tb is the algebraic sum of the double points minus half the number of cusps.

Exercise. Prove the formulas.

Another projection is on the (x, y) plane. Then the missing coordinate is the area. This causes various problems: the lift is not unique; it is hard to tell whether a knot diagram agrees with the area restrictions. The numbers μ and tb can be easily expressed in term of this projections: μ is just the Whitney rotation number and tb is the writhe.

An analog of the Reidemeister theorem is as follows.

Lemma. *Two Legendrian knots are Legendrian isotopic if and only if their front projections are connected by a sequence of the three Legendre Reidemeister moves.*

What Bennequin proved was the next result.

Theorem. *Let γ be a transverse knot. Then $\beta(\gamma) \le -\chi(S)$ where S is a Seifert surface for γ.*

As a consequence, β(Legendrian unknot) < 0, and this fails for overtwisted contact structures in 3-space.

This result has shortcomings: insensitive to mirroring, hard to estimate in terms of a diagram, and very difficult to prove. Better results are given in terms of knot polynomials, the HOMFLY $\bar{F}(x, y)$ and the Kauffman $\bar{K}(x, y)$.

Theorem. *Let γ be a transverse link. Then*

$$\beta(\gamma) \le \min \deg_x \bar{F}_\gamma(x, y).$$

Let γ be a Legendrian link. Then

$$tb(\gamma) \leq \min \deg_x \bar{F}_\gamma(x,y) \text{ and } tb(\gamma) \leq \min \deg_x \bar{K}_\gamma(x,y).$$

For example, this gives 1 and -6 for right and left trefoils (why?) For some classes of knots, the result is sharp (for half of all mirror torus knots, for positive and for alternating links). Similar results hold for Legendrian and transverse knots in the space of cooriented contact elements in the plane.

One wants to know whether there are other invariants of Legendrian knots beyond topological ones and the Maslov and Bennequin invariants. The next result is due to Eliashberg and Eliashberg-Frazer.

Theorem. *For transverse and for Legendrian topological unknots there are no other invariants.*

A difficulty of the problem lies, in particular, in the zig-zag phenomenon: the stable Legendrian isotopy is the same as the topological one. This implies, in particular, that no Legendrian invariants of finite order (Vassiliev invariants) are capable of distinguishing Legendrian knots which are topologically isotopic and have equal Maslov and Bennequin invariants.

However Legendrian knot invariants, the *contact homology*, have been discovered by Eliashberg and by Chekanov. The proper framework in the symplectic field theory (ArXiv SG/0010059). The motivation comes from Morse theory for the following functional on the space of curves with end points on a Legendrian knot:

$$F(\gamma) = \int_\gamma dz - ydx.$$

Exercise. The extrema are the trajectories of the Reeb field (i.e., the vertical segments) with end points on the Legendrian knot, that is, the double points of the (x, y) projection.

We describe a purely combinatorial construction based on Chekanov's preprint.

One uses the (x, y) projection and considers the ring of non-commutative polynomials over $\mathbf{Z}_2$ whose generators correspond to double points. This ring is given a differential satisfying the Leibnitz rule. The differential on a generator is given by counting certain immersed disc in the knot diagram. More specifically, at every double point, there are two regions marked $+$ and two marked $-$. Let a be a generator, i.e., a double point. An immersed disc, contributing to da, has a unique positive vertex at a, its other vertices are negative.

The generators are graded, and the differential reduces the grading by 1. Assuming that the branches at a double point a intersect at the right angle, the grading is defines as follows. One leaves point a along an undercrossing branch and follows the diagram until one returns at a. Let α be the total winding of this path. Then $\deg a = (\alpha - \pi/2)/\pi$.

Exercise. Show that this grading is a well-defined integer if the Maslov number equals zero. Otherwise, it is an element of $\mathbf{Z}_{2\mu}$.

The main result is that the homology ring is a Legendrian knot invariant. There are examples of Legendrian knots that are topologically isotopic and have the same elementary invariants but are distinguished by the contact homology. It is not known whether this construction extends to transverse knots.

It is interesting to see how zig-zagging effects the contact homology. A zig or zag in the (x, z) projection is a kink in the (x, y) one.

Lemma. *If a Legendrian knot diagram has a small kink then the contact homology is trivial.*

Proof. Let a be a double point corresponding to a small kink. Then $da = 1 + A$, and we claim that $A = 0$. If this is proved then the contact homology is trivial. Indeed, if $dc = 0$ then $c = d(ac)$.

If a kink is small then one can deform the diagram can be deformed so that removing this kink yields a Legendrian knot. If $A \neq 0$ then this new diagram contains an immersed disc D with all negative vertices. Orient the disc clockwise and let C be the curve in space that projects to its boundary. Then

$$\int_C dz - ydx > 0$$

(why?) On the other hand, by Stokes' theorem

$$\int_C dz - ydx = \int_D dx \wedge dy < 0,$$

a contradiction.

Exercise. I draw a knot diagram in the (x, y) plane and claim that it is a diagram of a Legendrian knot. Describe an algorithm verifying or refuting this claim.

Among other results on Legendrian knots, let us mention a version of the Rolle theorem for Legendrian knots in J^1S^1 due to Eliashberg, and by a different method, to Chekanov.

Theorem. *Let γ be a Legendrian knot in J^1S^1, Legendrian isotopic to the zero section $z = y = 0$. Then there are at least two points on γ at which $y = 0$.*

This is not at all true for an arbitrary Legendrian knot in J^1S^1.

A more general result, implying the above theorem, is due to Eliashberg. Let A be a double point of a front at which both branches have the same direction. A positive resolution at A consists of a surgery removing the double point.

Theorem. *Let γ be the front of a Legendrian knot in J^1S^1, Legendrian isotopic to the zero section $z = y = 0$. Then, after a number of positive resolutions, γ becomes a union of the graph of a function and a number of "flying saucers".*

4.5. Since a semester is clearly not enough for a more-or-less comprehensive introduction to symplectic topology, this last section is a panorama of – by now – classic results in the field.

The first is the *Gromov nonsqueezing theorem*. As in 1.6, let $B^{2n}(r)$ be the ball of radius r in linear symplectic space and $C(R) = B^2(R) \times \mathbf{R}^{2n-2}$ be the symplectic cylinder. Assume that there is a symplectic embedding $B^{2n}(r) \to B^2(R) \times \mathbf{R}^{2n-2}$. Then, similarly to the linear case, $r \le R$.

Another result is the *symplectic "camel" theorem*. Consider the linear symplectic space $\mathbf{R}^{2n}$ with the "wall" $q_1 = 0$ and a "hole" of radius 1 in it. Consider the ball $B^{2n}(r)$ with $r > 1$ on one side of the wall. Can it be moved to the other side by a symplectic isotopy? The theorem asserts that this is impossible.

A related notion is that of *symplectic capacity*. This is a function C on symplectic manifolds of dimension $2n$ with values in $[0, \infty]$ satisfying the conditions:

(i) if (U, ω) symplectically embeds in (U', ω') then $c(U) \le c(U')$;

(ii) $c(U, \lambda\omega) = \lambda^2 c(U, \omega)$;

(iii) $0 < c(B^{2n}(1)) = c(B^2(1) \times \mathbf{R}^{2n-2}) < \infty$.

Do capacities exist? This is essentially equivalent to the nonsqueezing theorem. Indeed, one can define the Gromov width w by

$$w(U, \omega) = \sup\{\pi r^2 : B^{2n}(r) \text{ symplectically embeds in U}\}.$$

Then conditions (i) and (ii) hold, and $w(C(R)) = \pi R^2$ by the nonsqueezing theorem. There are other capacities, mostly based on the study of periodic trajectories of certain Hamiltonian vector fields associated with U. The original approach by Gromov was based on pseudo-holomorphic curves. The two approaches are related as we noted in 3.7.

One has the following result by Ekeland and Hofer.

Theorem. *An orientation preserving diffeomorphism f of the linear symplectic space $\mathbf{R}^{2n}$ is symplectic if and only if it preserves the capacity of all open subsets.*

Proof. We need to show that $df(x)$ is a linear symplectic map for every x. Without loss of generality, $x = f(x) = O$. Then $df(O)$ is the limit $t \to 0$ of the diffeomorphisms $f_t(x) = f(tx)/t$. Since f preserves capacity which behaves well under rescaling, f_t also preserves capacity. We want to conclude that the linear map $df(O)$ also preserves capacity of convex sets, in particular, ellipsoids. This follows from the following fact: the capacity of convex sets is continuous in the Hausdorff metric.

Recall that the Hausdorff distance between subsets in $\mathbf{R}^{2n}$ is defined as follows:

$$d(U, V) = \max_{x \in U} \min_{y \in V} |x - y| + \max_{y \in V} \min_{x \in U} |x - y|.$$

If U is a convex set containing the origin and $d(u, V)$ is small then $(1-\varepsilon)U \subset V \subset (1+\varepsilon)U$. This and property (i) imply continuity.

Now we have a linear map L of the linear symplectic space that preserves the capacity of ellipsoids. We claim that L is symplectic. Assume not; then there is a couple of vectors such that $\omega(u, v) \ne \omega(L^*u, L^*v)$. Without loss of generality,

$$0 < \lambda^2 = |\omega(L^*u, L^*v)| < \omega(u, v) = 1.$$

Construct two symplectic bases $u, v, ...$ and $L^*u/\lambda, L^*v/\lambda, ...$ and consider the linear symplectic maps A and B that take the standard basis $e_1, e_2, ...$ to the former and the latter. Let $C = B^{-1}L^*A$. Then $C(e_1) = \lambda e_1, C(e_2) = \lambda e_2$. It follows that the map C^* sends the unit ball into $C(\lambda)$.

On the other hand, C^* is the product of maps that preserve capacities of ellipsoids, so C^* preserves the capacity of the unit ball. This is a contradiction.

Corollary. *The group of symplectomorphisms is C^0-closed in the group of diffeomorphisms.*

Proof. Given a sequence of symplectomorphisms f_n, uniformly converging to a diffeomorphism f, we want to show that f is symplectic. But f preserves the capacities of ellipsoids because capacity is continuous in Hausdorff topology on convex sets.

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