The Geometry of Infinite Dimensional Groups

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Contents

Chapter I

Preliminaries

In this introductory chapter, we collect some basic notions from the theory of Lie groups and algebras. Since its content should be familiar to most readers, the chapter mainly serves to set up notations.

1 Lie Groups and Lie Algebras

1.1 Lie groups

We start directly with the main definition of this chapter:

Definition 1.1 A *Lie group* is a smooth manifold G with a group structure such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps.

Since the Lie groups we will be considering throughout this book will be infinite-dimensional, we have to say what we mean by an infinite-dimensional manifold. Roughly speaking, an infinite-dimensional manifold is a manifold modeled on an infinite-dimensional locally compact vector space just as a finite-dimensional manifold is modeled on $\mathbb{R}^n$.

Definition 1.2 Let V, W be *Fréchet spaces*, i.e. complete Hausdorff metrizable vector spaces. A map $f : U \subset V \to W$ is said to be *differentiable* at a point $u \in U$ in direction $v \in V$ if the limit

$$Df(u; v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} \tag{1.1}$$

exists. The function is said to be continuously differentiable on U, if the limit exists for all $u \in U$ and all $v \in V$, and if the function $Df : U \times V \to W$ is continuous as a function on $U \times V$. In the same way, we can build the second derivative $D^2 f$, which (if it exists) will be a function $D^2 f : U \times V \times V \to W$ and so on. A function $f : U \to V$ is called *smooth* or C^∞, if all its derivatives exist and are continuous.

Definition 1.3 A *Fréchet manifold* is a Hausdorff space with a coordinate atlas taking values in a Fréchet space such that all transition functions are smooth maps.

Now one can start defining differential forms, vector fields, tangent spaces, principal bundles and the like on a Fréchet manifold exactly in the same way as for finite-dimensional manifolds (see e.g. [37] for more on this). From now on, whenever we speak of an infinite-dimensional manifold, we implicitly mean a Fréchet manifold. In particular, our infinite-dimensional Lie groups are *Fréchet Lie groups.*

Remark 1.4 Instead of Fréchet manifolds, one could consider manifolds modeled on Banach spaces. This would lead to the category of Banach manifolds. The main advantage of Banach manifolds is, that strong theorems from finite-dimensional analysis, such as the inverse function theorem, hold in Banach spaces but not necessarily in Fréchet spaces. Some of the Lie groups we will be considering such as the diffeomorphism groups are not Banach manifolds. For this reason we stay within the more general framework of Fréchet manifolds.

1.2 The Lie algebra of a Lie group

Definition 1.5 Let G be a Lie group with the identity element $e \in G$. The *Lie algebra* $\mathfrak{g}$ of G is the tangent space of G at the identity element.

Example 1.6 Here are several examples of finite-dimensional Lie groups:

- $Gl(n,\mathbb{R})$, the set of nondegenerate $n \times n$ matrices, is a Lie group with Lie algebra $\mathfrak{gl}(n,\mathbb{R}) = Mat(n \times n, \mathbb{R})$
- $Sl(n,\mathbb{R}) = \{A \in Gl(n,\mathbb{R}) \mid \det(A) \neq 0\}$ is a Lie group with Lie algebra $\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$
- $SO(n,\mathbb{R}) = \{A \in Gl(n,\mathbb{R}) \mid AA^\dagger = Id,\ \det A > 0\}$ is a Lie group of transformations of $\mathbb{R}^n$ preserving the Euclidean inner product of vectors (and orientation) in $\mathbb{R}^n$, i.e. $< Au,\ Av >=< u, v >$ for all vectors $u, v \in \mathbb{R}^n$. Its Lie algebra is the space of skew-symmetric matrices $\mathfrak{so}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A + A^\dagger = 0\}$.
- $Sp(2n,\mathbb{R})$ is the group of transformations of $\mathbb{R}^{2n}$ preserving the non-degenerate skew-product of vectors.

Exercise 1.2.1 Give an alternative definition of $Sp(2n,\mathbb{R})$ with the help of the equation the corresponding matrices have to satisfy. Find the corresponding Lie algebra.

The following examples are the first infinite-dimensional Lie groups we shall meet.

Example 1.7 Let M be a compact finite-dimensional manifold. Then the set $Diff(M)$ of diffeomorphisms of M is an open subspace of the Fréchet space of all smooth maps from M to M. One can check that the composition and inversion are smooth maps so that $Diff(M)$ is a Lie group. Its Lie algebra is given by $Vect(M)$, the Lie algebra of smooth vector fields on M (see [37]).

Given a volume form μ on M, one can define the group of volume preserving diffeomorphisms

$$SDiff(M) := \{\phi \in Diff(M) \mid \phi^*\mu = \mu\}\,.$$

Since $SDiff(M)$ is a closed subgroup of $Diff(M)$, it is a Lie group. Its Lie algebra is the Lie algebra of divergence-free vector fields $SVect(M) := \{v \in Vect(M) \mid Div(v) = 0\}$.

Example 1.8 Let M be a finite-dimensional compact manifold and let G be a finite-dimensional Lie group. Set the group of currents on M to be $G^M = C^\infty(M, G)$. We can define a multiplication on G^M pointwise, i.e., we set $(\varphi \cdot \psi)(g) = \varphi(g)\psi(g)$ for all $\varphi, \psi \in G^M$. This multiplication gives G^M the structure of a (Fréchet) Lie group, as we discuss below.

Example 1.9 A slight but important generalization of the example above is the following: Let M and G be as above, and let P be a principal G-bundle on M. Let $Gau(P)$ be the gauge group of P, i.e. the group of bundle automorphisms containing the identity automorphism. Every element $\varphi \in Gau(P)$ defines a map $s : P \to G$ such that $\varphi(p) = s(p)p$ for all $p \in P$. The map s has the property that $s(gp) = gs(p)g^{-1}$ for all $p \in P$ and $g \in G$. Hence $Gau(P)$ can be identified with the set $C^\infty(P, G)^G$ of all smooth G-equivariant maps from P to G, where G acts on itself by conjugation. Therefore, $Gau(P)$ is a Lie group with Lie algebra $gau(P) = C^\infty(P, \mathfrak{g})^G$ of G-equivariant maps from P to $\mathfrak{g}$, where the G-action on $\mathfrak{g}$ is the adjoint action (see below). Since $Gau(P)$ is a closed subgroup of G^P, the exponential map of $Gau(P)$ is given by restriction of the exponential map of G^P.

Definition 1.10 Define the *exponential map* from a Lie algebra to the corresponding Lie group $\exp : \mathfrak{g} \to G$. Given a vector $X \in \mathfrak{g}$ one can associate to it a vector field $\widetilde{X}$ on G by left translation: $\widetilde{X}(g) = l_{g*}X$, where $l_g : G \to G$ denotes the left translation by a group element g. Obviously, the vector field $\widetilde{X}$ is invariant under left translations by elements of G. That is, $l_{g*}\widetilde{X} = \widetilde{X}$ for all $g \in G$. On the other hand, any left-invariant vector field $\widetilde{X}$ on the group G defines an element $\widetilde{X}(e) \in \mathfrak{g}$. This shows that the Lie algebra $\mathfrak{g}$ can be identified with the set of left-invariant vector fields on the group G.

Now let us fix some $X \in \mathfrak{g}$ and let $\widetilde{X}$ denote the corresponding left-invariant vector field. The flow of the field $\widetilde{X}$ is a map $\phi_X : G \times \mathbb{R} \to G$ such that $\frac{d}{dt}|_{t=0}\phi_X(g,t) = \widetilde{X}(g)$ and $\phi_X(g, 0) = g$. The flow ϕ_X of this differential equation is unique (when it exists) and one can show that $\phi_X(e, t+s) = \phi_X(e,t)\phi_X(e,s)$. In particular, $\phi_X(e, .) : \mathbb{R} \to G$ is a subgroup of G, the so called one-parameter subgroup corresponding to X. When the one-parameter subgroup $\phi_X(e, .)$ exists for all $X \in \mathfrak{g}$, we define the exponential map $\exp : \mathfrak{g} \to G$ via the time-one map $X \mapsto \phi_X(1)$.

For each of the finite-dimensional Lie groups considered above, the exponential map is given by the usual exponential map for matrices: $\exp : A \mapsto \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$.

Remark 1.11 Note that the existence of the exponential map depends on the existence of solutions of certain first order differential equations. By the theory of differential equations, the exponential map exists for all finite-dimensional Lie groups and more generally for Lie groups modeled on Banach spaces. However, there are Fréchet Lie groups which do not admit an exponential map. Moreover, in the cases when the exponential map of an infinite-dimensional group exists, it can exhibit rather peculiar properties.

Let us first look at the case of the current group G^M, where the exponential map exists and endows this group with the structure of a Fréchet Lie group. Namely, the space $\mathfrak{g}^M = C^\infty(M, \mathfrak{g})$ endowed with the topology of uniform convergence is a Fréchet space.

Moreover, the map $\exp : \mathfrak{g} \to G$ can be used to define a map $\widetilde{\exp} : \mathfrak{g}^M \to G^M$ pointwise. In a sufficiently small neighborhood of $0 \in \mathfrak{g}^M$, the map $\widetilde{\exp}$ is bijective. Thus it can be used to define a local system of open neighborhoods of the identity in G^M. We can use left translation to transfer this system to any point in G^M and thus define a topology on the group G^M. Again using the exponential map, we can define coordinate charts on G^M. This definition implies that multiplication and inversion are smooth maps. So G^M is an infinite-dimensional Lie groups (see e.g. [37] for more details).

From the construction of the Lie groups structure on G^M, it is clear that its Lie algebra is $\mathfrak{g}^M$, and that the exponential map $\mathfrak{g}^M \to G^M$ is the map $\widetilde{\exp}$ we described above. Note however that $\widetilde{\exp}$ is not, in general, surjective, even if $\exp : \mathfrak{g} \to G$ is surjective. As an example, take both M and G to be group $U(1)$. As a manifold we can identify $U(1)$ with S^1. Then the identity map $\mathrm{id} : U(1) \to U(1)$ does not lie in the image of $\widetilde{\exp}$.

In the example of a diffeomorphism group $Diff(M)$ the exponential map $\exp : Vect(M) \to Diff(M)$ exists and assigns to each vector field on M the time-one map for its flow. In contrast to the exponential map in Example 1.8, this exponential map is not, in general, even locally surjective. This is already true for the group $Diff(S^1)$ as we shall see in the following Proposition.

Proposition 1.12 *The exponential map* $\exp : Vect(S^1) \to Diff(S^1)$ *is not locally surjective.*

PROOF. Observe first that any nowhere vanishing vector field on S^1 is conjugate under $Diff(S^1)$ to a constant vector field. Indeed, if $\xi(\theta) = v(\theta)\frac{\partial}{\partial\theta}$ is such a vector field, we can define a diffeomorphism $\psi : S^1 \to S^1$ via $\psi(\theta) = a\int_0^\theta \frac{dt}{v(t)}$. Here, a is chosen such that $\psi(2\pi) = 2\pi$. Then $\psi_*(\xi \circ \psi^{-1})$ is a constant vector field on S^1.

From this observation we can conclude that any diffeomorphism of S^1 which lies in the image of the exponential map and which does not have any fixed points is conjugate to a rigid rotation of S^1. Hence to see that exp is not locally surjective, it is enough to construct diffeomorphisms arbitrarily close to the identity which do not have any fixed points and which are not conjugate to a rigid rotation. For this one can take diffeomorphisms which have isolated periodic points, i.e., fixed points for a certain nth iteration of this diffeomorphism. Indeed, if such a diffeomorphism ψ belonged to the image of the exponential map, so would its nth power ψ^n. Then the corresponding vector field defining the ψ^n as the time-one map would either have zeros or be nonvanishing everywhere. In the former case, the periodic points of ψ must be actually fixed points, while in the latter case, ψ^n, as well as ψ, would be conjugate to a rigid rotation and hence all ponts of ψ would be periodic. Both cases give us a contradiction.

Explicitly, a family of such diffeomorphisms can be constructed as follows: Let us identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Then consider the map $\psi_{n,\epsilon} : x \mapsto x + \frac{1}{n} + \epsilon \sin(nx)$. For ϵ small enough, this is indeed a diffeomorphism of S^1. Furthermore, choosing n big and ϵ small, $\psi_{n,\epsilon}$ can be moved arbitrarily close to the identity. Finally, for $\epsilon \neq 0$, $\psi_{n,\epsilon}$ can not be conjugate to a rigid rotation. If it was conjugate to a rotation, it would have to be the rotation $\psi_{n,0}$ since $\psi^n_{n,\epsilon}(0) = 0$. But in this case, we would have $\psi^n_{n,\epsilon} = id$, which is not true for $\epsilon \neq 0$. $\square$

2 Adjoint and coadjoint orbits

2.1 The adjoint representation

A representation of a topological group G on a topological vector space V is a linear action of the group G on V which is continuous in the sense that the map $G \times V \to V$, $(g, v) \mapsto gv$ is continuous. If V is real, (V, φ) is called a real representation, otherwise it is a complex representation. Often, V is assumed to be a Hilbert space. In that case, the representation is said to be unitary if the inner product on V is invariant under the action of G. In the case when G is a Lie group, we require V to be a Fréchet space and the action of G on V to be smooth.

Every Lie group has two distinguished representations, the adjoint and the coadjoint representation. Since they will play a prominent role in this book we describe them in some detail.

Any element $g \in G$ defines an automorphism c_g of G by conjugation $c_g : h \in G \mapsto ghg^{-1}$. The differential of c_g at the identity maps the Lie algebra of G to itself and thus defines an element $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$.

Definition 2.1 The map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ defines a representation of the group G which is called the *group adjoint representation.* The orbits of the group G in its Lie algebra $\mathfrak{g}$ are called the *adjoint orbits* of G.

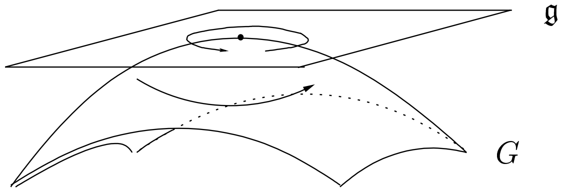

Figure 2.1: Conjugation c_g on the group G generates the adjoint representation Ad_g on the Lie algebra $\mathfrak{g}$.

The differential of $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at the identity defines a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, the *adjoint representation of the Lie algebra* $\mathfrak{g}$. Using the map ad, we can define a bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

via

$$[X, Y] = \mathrm{ad}_X(Y).$$

The map $[\ ,\ ]$ is called the *Lie bracket* of $\mathfrak{g}$.

Using the identification of $\mathfrak{g}$ with the set of left-invariant vector fields on G one can show that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ coincides with the bracket (or, commutator) of two left invariant vector fields. (Recall that the commutator $[\widetilde{X}, \widetilde{Y}]$ of two left-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ is defined as $[\widetilde{X}, \widetilde{Y}] = L_{\widetilde{X}}\widetilde{Y} - L_{\widetilde{Y}}\widetilde{X}$, where $L_{\widetilde{X}}$ denotes the Lie derivative of $\widetilde{Y}$ with respect to $\widetilde{X}$.) From this it follows that the bracket [,] is skew symmetric and satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. These two properties can be taken as an axiomatic definition of an abstract Lie algebra bracket on a vector space $\mathfrak{g}$. Thus, the bilinear operation on $\mathfrak{g}$ defined by the bracket [,] gives $\mathfrak{g}$ the structure of an abstact Lie algebra and $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ can be viewed as a representation of $\mathfrak{g}$.

Example 2.2

- Let $g \in Gl(n, \mathbb{R})$ and $A \in \mathfrak{gl}(n, \mathbb{R})$. Then $\mathrm{Ad}_g A = gAg^{-1}$. Hence the adjoint orbits are given by sets of similar matrices in $\mathfrak{gl}(n, \mathbb{R})$. The adjoint representation of $\mathfrak{gl}(n, \mathbb{R})$ is given by $\mathrm{ad}_A(B) = [A, B] = AB - BA$.

- The adjoint orbits of $SO(3, \mathbb{R})$ are spheres centered at the origin and the origin itself.

- The adjoint orbits of $SL(2, \mathbb{R})$ are again given by matrices of equal Jordan normal form. Hence, writing $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2, \mathbb{R})$, one sees that the adjoint orbits lie in the level sets of $a^2 + bc = const$: similar matrices have the same determinant.

- Let G be the set of orientation preserving affine transformations of the real line. That is $G = \{(a,b) \mid a, b \in \mathbb{R},\ a > 0\}$, and $(a, b) \in G$ acts on $x \in \mathbb{R}$ via $x \mapsto ax + b$. The Lie algebra of G is $\mathbb{R}^2$ and the adjoint orbits on G are the affine lines

  $$\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha = const \neq 0, \quad \beta \text{ arbitrary}\},$$

  the two rays $\{(\alpha, \beta) \in \mathbb{R}^2, \alpha = 0, \beta \neq 0\}$, and the origin $\{(0,0)\}$ (see figure 2.5).

- Let M be a compact manifold. The adjoint orbits of $Gl(n, \mathbb{C})^M$ in $\mathfrak{gl}(n, \mathbb{C})^M$ are given by fixing the Jordan normal form of $\psi(x)$ for each $x \in M$.

- Let M be a compact manifold. The adjoint representation of $Diff(M)$ on $Vect(M)$ is given by coordinate changes of the vector field: $\phi \in Diff(M) : v \mapsto \phi_* v \circ \phi^{-1}$. The adjoint representation of $Vect(M)$ on itself is given by the negative of the usual Lie bracket of vector fields: $\mathrm{ad}_v w = -L_v w = \frac{\partial v}{\partial x} w(x) - \frac{\partial w}{\partial x} v(x)$ in any local coordinate x.

  Exercise 2.1.1 verify the latter formula from the definition of the group adjoint action. Hint: expess the diffeomorphisms corresponding to the vector fields $v(x)$ and $w(x)$ in the form

  $$g(t): \quad x \mapsto x + tv(x) + o(t), \qquad h(s): \quad x \mapsto x + sw(x) + o(s), \qquad t, s \to 0,$$

  and find the first several terms of $g(t)h(s)g^{-1}(t)$.

2.2 The coadjoint representation

Dually to the adjoint representation of G on $\mathfrak{g}$, one has the coadjoint representation of G on $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$:

Definition 2.3 The coadjoint representation Ad^* of the group G on the space $\mathfrak{g}^*$ is the dual of the adjoint representation. Let $\langle\ ,\ \rangle$ denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Then the *coadjoint action of the group* G on the dual space $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{Ad}_g^*(\xi), X\rangle := \langle \xi, \mathrm{Ad}_{g^{-1}}(X)\rangle$$

for all ξ in $\mathfrak{g}^*$ and $X \in \mathfrak{g}$. The differential $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ of Ad^* at e is called the *coadjoint representation of the Lie algebra* $\mathfrak{g}$. Explicitly it is given by

$$\langle \mathrm{ad}_Z^*(\xi), X\rangle = -\langle \xi, \mathrm{ad}_Z(X)\rangle.$$

Remark 2.4 The dual space of a Fréchet space is not necessarily again a Fréchet space. In this case, instead of considering the full dual space to a Lie algebra $\mathfrak{g}$, we will usually confine ourselves to considering only the "smooth duals," the functionals from a certain G-invariant Fréchet subspace $\mathfrak{g}_s^* \subset \mathfrak{g}^*$. The pair $(\mathfrak{g}_s^*, \mathrm{Ad}^*|_{\mathfrak{g}_s^*})$ is called the regular (or smooth) part of the coadjoint representation of G, and, abusing notations, we will usually skip the index $_s$.

Example 2.5

- In the first three examples of 2.2, there exists a G-invariant inner product on $\mathfrak{g}$ which induces a G-equivariant isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$. Hence the adjoint and coadjoint representations of the groups G are isomorphic, and the coadjoint orbits coincide with the adjoint ones.

- The group of affine transformations of the real line in Example 2.2 has two two-dimensional coadjoint orbits, the upper and lower half-planes in $\mathbb{R}^2$, and a set of zero-dimensional orbits, namely, the points $(\alpha, 0)$ for each $\alpha \in \mathbb{R}$ (see Figure 2.5).

- We can define a non-degenerate G^M-invariant product on $\mathfrak{gl}(n, \mathbb{R})^M$ by setting

$$\langle X, Y\rangle = \int_M \mathrm{tr}(X(\theta) \cdot Y(\theta))\, dvol$$

  for $X, Y \in \mathfrak{gl}(n, \mathbb{R})^M$ and some fixed volume form $dvol$ on M. This inner product can be used to identify $\mathfrak{g}^M$ with a subspace $\mathfrak{g}_s^* \subset \mathfrak{g}^*$. The space $\mathfrak{g}_s^*$ is called the smooth (or regular) part of $\mathfrak{g}^*$. Thanks to the nondegenerate pairing, the smooth part of the coadjoint representation of G^M is isomorphic to the adjoint representation.

Note that each of the finite-dimensional coadjoint orbits above is even-dimensional. This is a consequence of the general fact that coadjoint orbits are symplectic manifolds, which we discuss later.

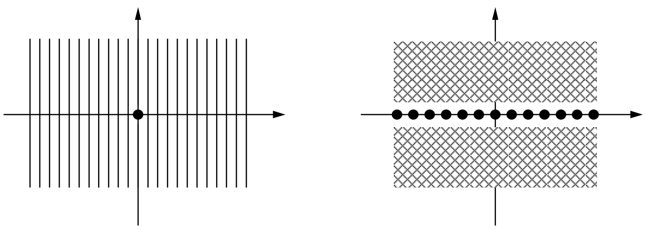

Figure 2.2: Adjoint and coadjoint orbits of the group of affine transformations on the line

Remark 2.6 In what follows we will pay a particular attention to the structure and description of coadjoint orbits of infinite-dimensional Lie groups. According to the orbit method, a general principle due to A. Kirillov, the information on the set of unitary representations of a Lie group G is contained in the group coadjoint orbits. This method has become a powerfull tool in the study of Lie groups and it has been worked out in detail for large classes of finite-dimensional Lie groups such as nilpotent and compact Lie groups (see [59]). In infinite dimensions, the correspondence between coadjoint orbits and representations has only been fully understood in a few isolated cases such as affine Lie groups (cf. [27],[82], [91]), but there are indications that it works for other groups as well.

3 Central extensions

3.1 Lie algebra central extensions

Definition 3.1 A *central extension of a Lie algebra* $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is a Lie algebra $\widetilde{\mathfrak{g}}$, whose the vector space $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ is equipped with the following Lie bracket:

$$[(X,u),(Y,v)]^{\sim} = ([X,Y],\omega(X,Y)).$$

(Note that ω depends only on X and Y, but not on u and v, which means that the extension is *central*: the space $\mathfrak{n}$ belongs to the center of the new Lie algebra.) The skew symmetry and the Jacobi identity for the new Lie bracket on $\widetilde{\mathfrak{g}}$ are equivalent to the following conditions on the map ω. Such a map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ has to be a *two-cocycle on the Lie algebra* $\mathfrak{g}$, i.e. has to be bilinear, antisymmetric, and to satisfy the *cocycle identity*

$$\omega([X,Y],Z) + \omega([Z,X],Y) + \omega([Y,Z],X) = 0$$

for any triple of elements $X, Y, Z \in \mathfrak{g}$.

A two-cocycle ω on $\mathfrak{g}$ with values in $\mathfrak{n}$ is called a two-coboundary if there exists a linear map $\alpha : \mathfrak{g} \to \mathfrak{n}$ such that $\omega(X,Y) = \alpha([X,Y])$ for all $X, Y \in \mathfrak{g}$. One can easily see that the central extension defined by such a two-coboundary becomes the trivial extension by the zero cocycle after the change of coordinates $(X,u) \mapsto (X, u - \alpha(X))$.

Hence in describing different central extensions we are interested only in the two-cocycles modulo two-coboundaries, i.e., in the *second cohomology* $H^2(\mathfrak{g};\mathfrak{n})$ of the Lie algebra $\mathfrak{g}$ with values in $\mathfrak{n}$: $H^2(\mathfrak{g};\mathfrak{n}) = \mathcal{Z}(\mathfrak{g};\mathfrak{n})/\mathcal{B}(\mathfrak{g};\mathfrak{n})$, where $\mathcal{Z}(\mathfrak{g};\mathfrak{n})$ is the vector space of all two-cocycles on $\mathfrak{g}$ with values in $\mathfrak{n}$, and $\mathcal{B}(\mathfrak{g};\mathfrak{n})$ is the subspace of two-coboundaries.

Remark 3.2 A central extension of a Lie algebra $\mathfrak{g}$ by an Abelian Lie algebra $\mathfrak{n}$ can be defined by the following exact sequence

$$\{0\} \longrightarrow \mathfrak{n} \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \{0\}$$

of Lie algebras such that $\mathfrak{n}$ lies in the center of $\widetilde{\mathfrak{g}}$. A morphism of two central extensions is a pair (φ, ν) of Lie algebra homomorphisms $\varphi : \mathfrak{n} \to \mathfrak{n}'$ and $\mu : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \widetilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mu} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{n}' & \longrightarrow & \widetilde{\mathfrak{g}}' & \xrightarrow{\pi'} & \mathfrak{g} & \longrightarrow & 0, \end{array} \tag{3.2}$$

Two extensions are said to be equivalent if the map μ is an isomorphism and $\varphi = id$.

Exercise 3.1.1 Prove the following equivalence:

Proposition 3.3 *There is a one-to-one correspondence between the set of equivalence classes of central extensions of* $\mathfrak{g}$ *by* $\mathfrak{n}$ *and the set* $H^2(\mathfrak{g}; \mathfrak{n})$.

Example 3.4 Consider the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^2$, and let $\omega \in \Lambda^2(\mathbb{R}^2)$ be an arbitrary skew symmetric bilinear form on $\mathbb{R}^2$. Then ω defines a two-cocycle on $\mathbb{R}^2$ with values in $\mathbb{R}$ (in this case, the cocycle condition is trivial, since $\mathfrak{g}$ is abelian). The resulting central extension is $\mathfrak{g}' = \mathbb{R}^2 \oplus \mathbb{R}$ with Lie bracket $[(v_1, h_1), (v_2, h_2)] = (0, \omega(v_1, v_2))$. Moreover, since $\mathfrak{g}$ is abelian, $\mathcal{B}(\mathfrak{g}; \mathbb{R}) = \{0\}$ whence $H^2(\mathfrak{g}; \mathbb{R}) = \Lambda^2(\mathbb{R}^2) \cong \mathbb{R}$. Note that all $\omega \neq 0 \in \Lambda^2(\mathbb{R}^2)$ lead to isomorphic Lie algebras. A representative of this isomorphism class is called the three-dimensional Heisenberg algebra.

By taking a nondegenerate skew-symmetric form ω in R^{2n}, we can define in the same way the Heisenberg algebra in R^{2n}.

An infinite-dimensional analog of the Heisenberg algebra is as follows. Consider the space of smooth functions on the circle $\mathfrak{g} = C^\infty(S^1)$ and regard it as an abelian Lie algebra. Define the two-cocycle by $\omega(f, g) = \int_{S^1} f'g\, dx$. (One can view these algebra and cocycle as the "limit" $n \to \infty$ of the example above by considering the functions in Fourier components.)

Exercise 3.1.2 Check the cocycle identity for $\omega(f, g)$.

Definition 3.5 A central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ is called *universal*, if for any other central extension $\widetilde{\mathfrak{g}}'$, there is a morphism $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}'$ of central extensions.

Obviously, if it exists, the universal central extension of a Lie algebra $\mathfrak{g}$ is unique up to isomorphism.

Remark 3.6 There is the following simple criterion to determine whether a Lie algebra possesses a universal central extension. For a Lie algebra $\mathfrak{g}$ to have a universal central extension, it is necessary and sufficient for it to be perfect, i.e., to coincide with its own derived algebra: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (see e.g. [76]).

Any finite-dimensional semisimple Lie algebra is perfect. An abelian Lie algebra is not perfect.

Example 3.7 Let M be a finite-dimensional manifold. One can show that the Lie algebra $Vect(M)$ of vector fields on M is perfect. We will determine the universal central extension of $Vect(M)$ explicitly for the case $M = S^1$ in section 2 of chapter II.

Example 3.8 For any n-dimensional compact manifold M and perfect finite-dimensional Lie algebra $\mathfrak{g}$, the Lie algebra $\mathfrak{g}^M$ is perfect. If moreover $\mathfrak{g}$ is simple, the universal central extension $\widetilde{\mathfrak{g}}^M$ of $\mathfrak{g}^M$ can be constructed as follows: Let $\Omega^1(M)$ denote the set of one-forms on M and $d\Omega^0(M)$ the subset of exact one-forms. Furthermore, let $\langle\ ,\ \rangle$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. Then we can define a two-cocycle ω on $\mathfrak{g}^M$ with values in $\Omega^1(M)/d\Omega^0(M)$ via $\omega(X, Y) = \langle X, dY \rangle$. The antisymmetry of ω is immediate and the cocycle identity follows from the Jacobi identity in $\mathfrak{g}^M$. So ω defines a central extension of $\mathfrak{g}^M$. For a proof of the fact that this is indeed the universal central extension of $\mathfrak{g}^M$, see e.g. [82].

Example 3.9 More generally, let G be a compact or complex semisimple Lie group and $P \to M$ be a principal G-bundle on a compact manifold M. Then $gau(P)$ is perfect. Its universal central extension is given by $\Omega^1(P)^G / d\Omega^0(P)^G$, where $\Omega^1(P)^G$ is the space of G-invariant one-forms on P, and $d\Omega^0(P)^G$ denotes the subspace of exact G-invariant one-forms.

3.2 Central extensions of Lie groups

Central extensions of Lie groups can be defined similarly to those of Lie algebras. However, as we will see below, unlike the case of Lie algebras, not all group extensions can be described explicitly by cocycles. This is why we start with the alternative definition of the extensions via exact sequences.

Definition 3.10 A central extension $\widetilde{G}$ of a Lie group G by an abelian Lie group H is an exact sequence of Lie groups

$$\{e\} \to H \to \widetilde{G} \to G \to \{e\}$$

such that the image of H lies in the center of $\widetilde{G}$. Morphisms and equivalence of two central extensions are defined analogously to the case of Lie algebras.

If the central extension $\widetilde{G}$ is topologically a direct product of G and H: $\widetilde{G} = G \times H$ (or, equivalently, if there is a smooth section in the principle H-bundle $\widetilde{G} \to G$) one can define the multiplication in $\widetilde{G}$ as follows: $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, c(g_1, g_2) h_1 h_2)$, similar to the case of Lie algebra central extensions. The associativity of this multiplication corresponds to the so called cocycle identity on the map c.

Definition 3.11 A smooth map $c : G \times G \to H$ which satisfies $c(g_1 g_2, g_3)c(g_1, g_2) = c(g_1, g_2 g_3)c(g_2, g_3)$ is called a smooth group two-cocycle on G with values in H.

A smooth two-cocycle on G with values in H is called a two-coboundary if there exists a smooth map $\lambda : G \to H$ such that $c(g_1, g_2) = \lambda(g_1)\lambda(g_2)\lambda(g_1 g_2)^{-1}$. As before, the group two-coboundaries correspond to the trivial group extensions, after a possible change of coordinates. Similarly, two group two-cocycles define isomorphic extensions if they differ by a two-coboundary. This explains the following

Proposition 3.12 *There is a one-to-one correspondence between the set of central extensions of G by H which admit a smooth section and the elements in the second cohomology group $H^2(G, H) := Z(G, H)/B(G, H)$. Here $Z(G, H)$ and $B(G, H)$ denote respectively the sets of smooth two-cocycles and two-coboundaries on G, with the natural abelian group structure.*

However, in contrast to the case of Lie algebras, there exist central extensions of Lie groups which do not admit a smooth section, and hence cannot be defined by explicit two-cocycles. We will encounter examples for such groups in Chapter II.

A central extension of a Lie group G always defines a central extension of the corresponding Lie algebra. The converse need not be true: the existence of a Lie group for a given Lie algebra is not guaranteed in infinite dimensions. Instead, one says that a central extension $\widetilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ lifts to the group level if there exists a central extension $\widetilde{G}$ of the group G whose Lie algebra is given by $\widetilde{\mathfrak{g}}$. If the group central extension $\widetilde{G}$ by H is defined by a group two-cocycle c, one can recover the Lie algebra two-cocycle defining the corresponding central extension $\widetilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ directly from the group cocycle c by appropriate differentiation.

Proposition 3.13 *Let H be an abelian Lie group with a Lie algebra $\mathfrak{h}$, and let c be an H-valued two-cocycle on G defining a central extension $\widetilde{G}$. Then the $\mathfrak{h}$-valued two cocycle ω defining the central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ is given by*

$$\omega(X, Y) = \frac{d^2}{dtds}|_{t=0,s=0} c(g_t, h_s) - \frac{d^2}{dtds}|_{t=0,s=0} c(h_s, g_t)\,,$$

where g_t is a smooth curve in G such that $\frac{d}{dt}|_{t=0} g_t = X$, and h_s is a smooth curve in G such that $\frac{d}{ds}|_{s=0} h_s = Y$.

Exercise 3.2.1 prove this Proposition.

Example 3.14 Let $G = \mathbb{R}^2$ with the natural abelian group structure. Then

$$\widetilde{G} = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \}$$

is a central extension of G, called the three-dimensional Heisenberg group. One verifies directly that the central extension is defined via the $\mathbb{R}$-valued group two-cocycle c given by $c((a,c),(a',c')) = ac'$. Using Proposition 3.13, we see that the infinitesimal form of the cocycle c is given by

$$\omega((A,C),(A',C')) = AC' - A'C,$$

so that the Lie algebra of $\widetilde{G}$ is the Heisenberg algebra discussed in Example 3.4.

4 The Euler equations for Lie groups

4.1 Poisson structures on manifolds

Definition 4.1 A *Poisson structure* on a manifold M is a bilinear operation on functions

$$\{\ ,\ \}: \quad C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the following properties:

i) antisymmetry:

$$\{f,g\} = -\{g,f\},$$

ii) the Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\} = 0, \quad \text{and}$$

iii) the Leibniz identity:

$$\{f,gh\} = \{f,g\}h + \{f,h\}g$$

for any functions $f,g,h \in C^\infty(M)$.

The first two properties mean that a Poisson structure is a Lie algebra structure $\{\ ,\ \}$ on the space $C^\infty(M)$ of functions, while the third property gives that $\{f,\cdot\} : C^\infty(M) \to C^\infty(M)$ is a derivation for all $f \in C^\infty(M)$. Since any differentiation of the space of function is the Lie derivative along an appropriate vector field, Poisson structures can be thought of as the maps from functions to the corresponding vector fields on the manifold:

Definition 4.2 Let $H : M \to \mathbb{R}$ be any smooth function on a Poisson manifold M. Such a function H defines a vector field on M by $L_{\xi_H} g = \{H, g\}$ for any test function $g \in C^\infty(M)$. The corresponding vector field ξ_H is called the *Hamiltonian field* corresponding to the *Hamiltonian function* H with respect to the Poisson bracket $\{\ ,\ \}$.

Remark 4.3 Let M be a manifold with a Poisson structure $\{\ ,\ \}$. The Hamiltonian vector fields on M evaluated at a fixed point $m \in M$ span a subspace of the tangent space $T_m M$. Thus, varying the point m defines a distribution on the manifold M (i.e. a subbundle of the tangent bundle TM). Obviously, the commutator of two Hamiltonian vector field is again Hamiltonian. Therefore, by the Frobenius theorem, this distribution is integrable and gives rise to a *foliation* of the Poisson manifold M. Furthermore, since the tangent space of the leaf through the point m is generated by Hamiltonian vector fields, we can define a two-form ω on each leaf by fixing it on Hamiltonian vector fields ξ_f and ξ_g:

$$\omega(\xi_f, \xi_g) = \{f,g\}\,.$$

The leaves of this foliation are called *symplectic leaves.* [1]

Exercise 4.1.1 Show that the two-form ω defined on the leaf through a point $m \in M$ is closed and non-degenerate.

Definition 4.4 A pair (N, ω) consisting of a manifold N and a two-form ω on N is called a *symplectic manifold* if ω is closed ($d\omega = 0$) and non-degenerate. In the case of infinite dimensions, the form is non-degenerate in the sense that for each point $p \in N$ and any non-zero vector $X \in T_pN$, there exists another vector $Y \in T_pM$ such that $\omega_p(X, Y) \neq 0$.

4.2 Hamiltonian equations on the dual of a Lie algebra

Let G be a Lie group (finite of infinite dimensional) with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ denote (the smooth part of) its dual.

Definition 4.5 The natural *Lie-Poisson* (also, *Kirillov-Kostant* Poisson) structure $\{\ ,\ \}_{LP}$ on the dual Lie algebra $\mathfrak{g}^*$

$$\{\ ,\ \}_{LP} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$$

is defined via

$$\{f, g\}_{LP}(m) = \langle [df_m,\ dg_m],\ m\rangle$$

for any $m \in \mathfrak{g}^*$ and any two smooth functions f, g on $\mathfrak{g}^*$. (Here df_m is the differential of the smooth function f taken at the point m, understood as an element of the space $\mathfrak{g}$ itself; and $\langle .\ ,\ .\rangle$ is the natural pairing between the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.)

Figure: $df, dg \in \mathfrak{g}$, while $m \in \mathfrak{g}^*$.

Proposition 4.6 *The Hamiltonian equation corresponding to the function H and the natural Lie-Poisson structure $\{\ ,\ \}_{LP}$ on $\mathfrak{g}^*$ is given by*

$$\frac{d}{dt} m(t) = -\mathrm{ad}^*_{dH_{m(t)}} m(t).$$

This equation is called the Euler-Poisson equation *on $\mathfrak{g}$.*

PROOF. Let $g \in C^\infty(\mathfrak{g}^*)$ be an arbitrary function. Then

$$L_{\xi_H} g(m) = \{H, g\}(m) = \langle [dH_m, dg_m], m\rangle = \langle \mathrm{ad}_{dH_m}(dg_m), m\rangle = -\langle dg_m, \mathrm{ad}^*_{dH_m}(m)\rangle$$

implies $\xi_H(m) = -\mathrm{ad}^*_{dH_m}(m)$, which is the assertion. $\square$

[1] Alternatively, a Poisson structure can be defined by specifying a *bivector field* Π on the manifold M, i.e. a section of $TM^{\wedge 2}$:

$$\{f, g\} = \Pi(df, dg).$$

Such a bivector field defines a distribution on M as the images of the map $\Pi : T^*M \to TM$. Symplectic leaves are, by definition, integral submanifolds of this distribution.

Corollary 4.7 *The symplectic leaves of* $\{\ ,\ \}_{LP}$ *on* $\mathfrak{g}^*$ *are the coadjoint orbits of* G. *In particular, all (finite-dimensional) coadjoint orbits have even dimension.*

PROOF. Let $m \in \mathfrak{g}^*$ be a point in the dual space, $\mathcal{O}_m$ the coadjoint orbit through m, and $H \in C^\infty(\mathfrak{g}^*)$ a function on the dual. For any vector $v \in \mathfrak{g}$ of the Lie algebra, one can represent it as dH_m by taking an appropriate function H. Therefore, one can obtain as Hamiltonian vectors $\mathrm{ad}^*_{dH_m}(m)$ at the point m all vectors in the image of $\mathrm{ad}^*_{\mathfrak{g}}(m)$, i.e., all vectors in the tangent space to the orbit $T_m\mathcal{O}_m := T_m(Ad^*_G(m))$. By definition, all Hamiltonian vectors span the tangent space to symplectic leaves, which proves that coadjoint orbits are exactly the symplectic leaves of the Lie-Poisson bracket. □

Definition 4.8 Consider a quadratic Hamiltonian function $H(m) := \frac{1}{2}\langle A^{-1}m, m\rangle$ defined with the help of an invertible operator $A : \mathfrak{g} \to \mathfrak{g}^*$, called the *inertia operator.*

Corollary 4.9 *For the quadratic Hamiltonian function* $H : \mathfrak{g}^* \to \mathbb{R}$ *defined by* $H(m) := \frac{1}{2}\langle A^{-1}m, m\rangle$ *the corresponding Hamiltonian equation is*

$$\frac{d}{dt}m(t) = -\mathrm{ad}^*_{A^{-1}m(t)}m(t)\,. \tag{4.3}$$

Indeed, $dH_m(m) = A^{-1}m$ *for any* $m \in \mathfrak{g}^*$.

4.3 A Riemannian approach to the Euler equations

It turns out that the Euler-Poisson equations with quadratic Hamiltonians have a beautiful Riemannian reformulation.

V. Arnold in [9] suggested the following general set-up for the Euler equation describing a geodesic flow on an arbitrary Lie group. Consider a (possibly infinite-dimensional) Lie group G, which can be thought of as the configuration space of some physical system. (Examples from [9]: $SO(3)$ for a rigid body or the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms for an ideal fluid filling a domain M.) The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, the "energy," on $\mathfrak{g}$. We consider left (or right) translations of this quadratic form to the tangent space at any point of the group (the "translational symmetry" of the energy). This way the energy defines a left- (respectively, right-) invariant Riemannian metric on the group G. The geodesic flow on G with respect to this energy metric represents the extremals of the least action principle, i.e. the actual motions of our physical system. [2] To describe a geodesic on the Lie group G with an initial velocity $v(0)$, we transport its velocity vector at any moment t to the identity of the group by using the left (respectively, right) translation. This way we obtain the evolution law for $v(t)$ on the Lie algebra $\mathfrak{g}$.

To fix the notations, let $(\ ,\ )$ be some left-invariant metric on the group G. The geodesic flow with respect to this metric is a dynamical system on the tangent bundle TG

[2] Usually, the finite-dimensional examples below are related to the left invariance, while the infinite-dimensional ones with the right invariance of the metric. In particular, for a rigid body one has to consider left translations on $SO(3)$, while for fluids the right ones on $\mathrm{SDiff}(M)$.

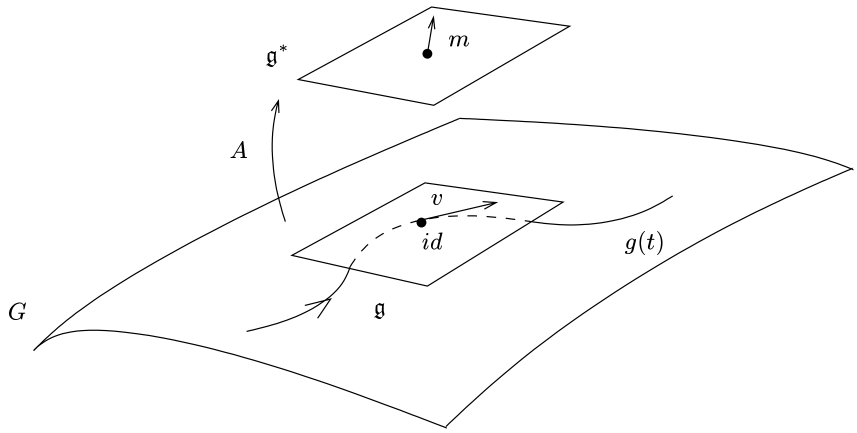

Figure 4.3: The vector v in the Lie algebra $\mathfrak{g}$ traces the evolution of the velocity vector of a geodesic $g(t)$ on the group. The inertia operator A sends v to a vector m in the dual space $\mathfrak{g}^*$.

of the group G. We can pull back this system to the Lie algebra $\mathfrak{g}$ of the group G by left translation. That is, if $g(t)$ is a geodesic in the group G with tangent vector $g'(t)$, then the pull back $v(t) = l^*_{g(t)^{-1}} g'(t)$ is an element of the Lie algebra $\mathfrak{g}$. (In the case of a right invariant metric, we set $v(t) = r^*_{g(t)^{-1}} g'(t)$). Hence, the geodesic equations for $g(t)$ give us a dynamical system

$$\frac{d}{dt} v(t) = B(v(t)) \,, \tag{4.4}$$

on the Lie algebra $\mathfrak{g}$ of the group G, where $B : \mathfrak{g} \to \mathfrak{g}$ is a (non-linear) operator.

Definition 4.10 The dynamical system (4.4) on the Lie algebra $\mathfrak{g}$ describing the evolution of the velocity vector of a geodesic in a right invariant metric on the Lie group G is called the *Euler (or, Euler-Arnold) equation* corresponding to this metric on G.

It turns out that the Euler equation for a Lie group G can be viewed as a Hamiltonian equation on the dual of the Lie algebra $\mathfrak{g}$ in the following way. Observe that the metric $(\ ,\ )_e$ at the identity $e \in G$ defines a non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$ and, therefore, it also determines an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ such that $(v, w)_e = \langle A(v), w \rangle$ for all $v, w \in \mathfrak{g}$. This identification $A : \mathfrak{g} \to \mathfrak{g}^*$ allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$, see figure 4.3. Now, setting $m = A(v)$ allows us to relate the geodesic equation (4.4) on the Lie algebra $\mathfrak{g}$ to the Hamiltonian equation on the dual $\mathfrak{g}^*$ with respect to the Hamiltonian function $H(m) = \frac{1}{2}\langle m, A^{-1}m \rangle$:

Theorem 4.11 (Arnold) *Fix some left-invariant metric on G and let $A : \mathfrak{g} \to \mathfrak{g}^*$ denote the corresponding inertia operator. The geodesic equation (4.4) takes the form*

$$\frac{d}{dt} m(t) = -\mathrm{ad}^*_{A^{-1}m(t)} m(t) \,.$$

Remark 4.12 The underlying reason for the Riemannian reformulation is the fact that any geodesic problem in *Riemannian* geometry can be described in terms of *symplectic* geometry. Geodesics on M are extremals of a quadratic Lagrangian (metric) on TM. They can also be described by the Hamiltonian flow on T^*M for the quadratic Hamiltonian function obtained from the Lagrangian via the Legendre transform.

If the manifold is a group G with a left-invariant metric then there exists the group action on the tangent bundle TG, as well as on the cotangent bundle T^*G. By taking the quotient with respect to the group action, we obtain from the (symplectic) cotangent bundle T^*G the Lie-Poisson structure on the cotangent space $T^*G|_e = \mathfrak{g}^*$, i.e., on the dual to the Lie algebra. The Hamiltonian function on T^*G is dual to the Riemannian metric (viewed as a form on TG), and its restriction to $\mathfrak{g}^*$ is the quadratic form $H(m) = \frac{1}{2}\langle A^{-1}m, m\rangle, \quad m \in \mathfrak{g}^*$.

The geodesics of a left-invariant metric on G correspond to the Hamiltonian function $-H(m)$, while those of a right-invariant metric correspond to $H(m)$.

PROOF. Let us fix the function $L : TG \to \mathbb{R}$ by

$$L(g, v) = \frac{1}{2}(v, v)_g \,.$$

Suppose that $g(t)$ is a geodesic path for the left-invariant metric $(\ ,\ )$ on the group G. Then, by definition, the path $g(t)$ satisfies the variational principle

$$\delta \int L(g(t), g'(t))dt = 0 \tag{4.5}$$

with fixed endpoints. (Here and later, δ denotes the variational derivative.)

For simplicity of notation, let us write $g^{-1}(t)g'(t)$ for $l^*_{g^{-1}(t)}g'(t)$. (Note, that if G is a matrix group, this notation agrees with the usual meaning of the expression $g^{-1}(t)g'(t)$ as a matrix product). Since the metric $(\ ,\ )$ on the group G is left-invariant, we can write

$$(g'(t), g'(t))_{g(t)} = (g^{-1}(t)g'(t), g^{-1}(t)g'(t))_e \,.$$

Then we can calculate

$$\delta \int \frac{1}{2}(g^{-1}g', g^{-1}g')_e dt = \int (\delta(g^{-1}g'), g^{-1}g')_e dt \tag{4.6}$$

Note that we have

$$\delta(g^{-1}g') = -g^{-1}\delta g g^{-1}g' + g^{-1}\delta g'$$

and

$$(g^{-1}\delta g)' = -g^{-1}g'g^{-1}\delta g + g^{-1}\delta g'$$

so that we can write

$$\delta(g^{-1}g') = (g^{-1}\delta g)' + [g^{-1}g', g^{-1}\delta g] \,.$$

Thus, the left hand side in equation (4.6) becomes

$$\int (\delta(g^{-1}g'), g^{-1}g')_e dt = \int ((g^{-1}\delta g)', g^{-1}g')_e dt + \int ([g^{-1}g', g^{-1}\delta g], g^{-1}g')_e dt \tag{4.7}$$

$$= -\int (g^{-1}\delta g, (g^{-1}g')')_e dt + \int ([g^{-1}g', g^{-1}\delta g], g^{-1}g')_e dt \,, \tag{4.8}$$

where we have used integration by parts in the last step. Now, let us write $v(t)$ for $g^{-1}(t)g'(t)$, and let $A : \mathfrak{g} \to \mathfrak{g}^*$ denote the inertia operator defined by the metric $(\ , \ )_e$. Then right-hand-side in the latter equation becomes

$$-\int (g^{-1}\delta g, (g^{-1}g')')_e dt + \int ([g^{-1}g', g^{-1}\delta g], g^{-1}g')_e dt \tag{4.9}$$

$$= -\int \langle g^{-1}\delta g, A(v)'\rangle dt + \int \langle \mathrm{ad}_v(g^{-1}\delta g), A(v)\rangle dt \tag{4.10}$$

$$= -\int \langle g^{-1}\delta g, A(v)'\rangle dt + -\int \langle g^{-1}\delta g, \mathrm{ad}_v^*(A(v))\rangle dt = 0 \tag{4.11}$$

This implies

$$A(v)' = -\mathrm{ad}_v^*(A(v)) .$$

Writing $m = A(v)$ finishes the proof of Theorem 4.11. □

Remark 4.13 Instead of using a left-invariant metric on G, we could have used a right-invariant one. This changes the signs in the Euler equation so that one obtains

$$\frac{d}{dt}m = ad^*_{A^{-1}m}(m)$$

Example 4.14 Let us consider the group $SO(3)$. The Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ can be identified with $\mathbb{R}^3$ such that the Lie bracket on $\mathfrak{g}$ is the cross product on $\mathbb{R}^3$: $[u, v] = u \times v$. Let A be some non-degenerate 3×3-matrix which we can view as an inertia operator. Then by Corollary 4.9, the Euler equation on $\mathfrak{so}(3)^*$ is given by

$$\frac{d}{dt}m = A^{-1}m \times m ,$$

which is the Euler equation for a rigid body in $\mathbb{R}^3$. Similarly, for $G = SO(n)$, one obtains the Euler equation for a higher dimensional rigid body.

Many other conservative dynamical systems in mathematical physics also describe geodesic flows on appropriate Lie groups. Here we list several examples of such systems to emphasize the range of applications of this approach. The choice of a group G (column 1) and a energy metric E (column 2) defines the corresponding Euler equations (column 3)

Group	*Metric*	*Equation*
$SO(3)$	$< \omega, A\omega >$	Euler top
$SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equations for a body in a fluid
$SO(n)$	Manakov′s metrics	$n-$dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa – Holm equation
Virasoro	$\dot{H}^1$	Hunter – Saxton (or Dym) equation
$\mathrm{SDiff}(M)$	L^2	Euler ideal fluid
$\mathrm{SDiff}(M)$	H^1	averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
$\mathrm{Maps}(S^1, SO(3))$	H^{-1}	Landau – Lifschits equation

Many of this examples we discuss below. This list is by no means complete. There are plenty of other interesting systems, e.g., super-equations or gas dynamics, and we refer to [10, 71, **?**] for more details.

4.4 Poisson pairs and Bihamiltonian structures

A first integral (or a conservation law) for a vector field ξ on a manifold M is a function on M invariant under the flow of this field. In this section we will show that if the vector field ξ is a Hamiltonian vector field with respect to two different Poisson structures on the manifold M which are compatible in a certain sense, there is a way of constructing first integrals for such a field.

Definition 4.15 Two Poisson structures $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$ on a manifold M are said to be *compatible* (or *form a Poisson pair*) if for any $\lambda \in \mathbb{R}$, the linear combination $\{\ ,\ \}_0 + \lambda\{\ ,\ \}_1$ is again a Poisson bracket on M.

A Hamiltonian system $\frac{d}{dt}m = \xi_m$ on M is called *bi-Hamiltonian* if the vector field ξ is Hamiltonian with respect to both Poisson structures.

Our main example of a manifold which admits a Poisson pair is the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. One Poisson structure on the space $\mathfrak{g}^*$ is given by the usual Lie-Poisson bracket $\{\ ,\ \}_{LP}$. We can define a second Poisson structure on $\mathfrak{g}^*$ by "freezing" the Lie-Poisson bracket at any point $m_0 \in \mathfrak{g}^*$:

Definition 4.16 The constant Poisson bracket on $\mathfrak{g}^*$ associated to a point $m_0 \in \mathfrak{g}^*$ is the bracket $\{\ ,\ \}_0$ defined on two smooth functions f, g on $\mathfrak{g}^*$ by

$$\{f, g\}_0(m) = \langle [df_m, dg_m], m_0 \rangle \,.$$

The Poisson bracket $\{\ ,\ \}_0$ depends on the freezing point $m_0 \in \mathfrak{g}^*$. Note that in the point m_0 itself the two Poisson brackets $\{\ ,\ \}_{LP}$ and $\{\ ,\ \}_0$ coincide. While the symplectic leaves of the Lie-Poisson bracket $\{\ ,\ \}_{LP}$ are the coadjoint orbits $\mathcal{O}_m$ of the Lie group G , the symplectic leaves of the constant bracket $\{\ ,\ \}_0$ are given by translations of the tangent space $T_{m_0}\mathcal{O}_{m_0}$ of the coadjoint orbit through the point m_0.

Lemma 4.17 *The Poisson brackets $\{\ ,\ \}_{LP}$ and $\{\ ,\ \}_0$ are compatible for every "freezing point" $m_0 \in \mathfrak{g}^*$.*

Proof. We have to check that $\{\ ,\ \}_\lambda = \{\ ,\ \}_{LP} + \lambda\{\ ,\ \}_0$ is a Poisson bracket on $\mathfrak{g}^*$ for all $\lambda \in \mathbb{R}$. But this is clear since $\{\ ,\ \}_\lambda$ is simply the bracket $\{\ ,\ \}_{LP}$ shifted by $-\lambda m_0$. □

Remark 4.18 Explicitly, the Hamiltonian equation on $\mathfrak{g}^*$ with the Hamiltonian function F and computed with respect to the constant Poisson structure frozen at a point $m_0 \in \mathfrak{g}^*$ has the following form:

$$\frac{dm}{dt} = -\mathrm{ad}^*_{dF_m} m_0 \,, \tag{4.12}$$

as a modification of Proposition 4.6 shows.

Going back to the general situation, let $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$ be a Poisson pair on a manifold M. It turns out that in this case one can generate a bihamiltonian dynamical system by producing a sequence of hamiltonians in involution, according to the following *Lenard-Magri scheme* [69]. Consider the Poisson bracket $\{\ ,\ \}_\lambda = \{\ ,\ \}_0 + \lambda\{\ ,\ \}_1$, and let h_λ be a *Casimir function* for this bracket, that is, a function on M parameterized by λ and satisfying $\{h_\lambda, f\}_\lambda = 0$ for all smooth functions $f \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. Furthermore, suppose that the function h_λ can be expanded into a power series in λ, i.e. that we can write

$$h_\lambda = \sum_{i=0}^{\infty} \lambda^i h_i \tag{4.13}$$

where the h_i are smooth functions on M. Each of the functions h_i defines a Hamiltonian vector field ξ_i on M with respect to the Poisson bracket $\{\ ,\ \}_1$. That is, $L_{\xi_i} f = \{h_i, f\}_1$ for all $f \in C^\infty(M)$.

Theorem 4.19 *Let ξ_i denote the Hamiltonian vector field on M corresponding to the function h_i and the Poisson bracket $\{\ ,\ \}_1$. Then the vector field ξ_i is a Hamiltonian with respect to the Poisson bracket $\{\ ,\ \}_0$ and the Hamiltonian function $-h_{i+1}$. Furthermore, for $i \neq j$, we have $\{h_i, h_j\}_k = 0$ for $k = 0, 1$.*

PROOF. Since h_λ is a Casimir function for the Poisson bracket $\{\ ,\ \}_\lambda$, we have $\{h_\lambda, f\}_\lambda = 0$ for all smooth functions f on M. Substituting for h_λ its power series expansion (4.13), we get

$$\begin{aligned} 0 = \{h_\lambda, f\}_\lambda =& \{\sum_{i=0}^{\infty} \lambda^i h_i, f\}_\lambda \\ =& \{\sum_{i=0}^{\infty} \lambda^i h_i, f\}_0 + \lambda\{\sum_{i=0}^{\infty} \lambda^i h_i, f\}_1. \end{aligned}$$

Comparing coefficients of the powers of λ we find $\{h_0, f\}_0 = 0$ and

$$\{h_i, f\}_0 = -\{h_{i-1}, f\}_1 .$$

The first identity expresses the fact that h_0 is a Casimir function for the bracket $\{.,.\}_0$. The next one says that the Hamiltonian field for h_1 with respect to $\{.,.\}_0$ coincides with the Hamiltonian field for $-h_0$ and the bracket $\{.,.\}_1$, and so on.

To see that every function h_i is a first integral for the equation generated by h_j with respect to each bracket, we have to show that $\{h_i, h_j\}_k = 0$ for $i \neq j$ and $k = 0, 1$. Assume, for instance, that $i < j$ and $k = 1$. Then

$$\{h_i, h_j\}_1 = -\{h_{i+1}, h_j\}_0 = \{h_{i+1}, h_{j-1}\}_1 = \cdots = \pm\{h_l, h_l\}_s = 0$$

for some $s \in \{0, 1\}$. The other cases are checked by similar calculations. $\square$

Exercise 4.4.1 Show that if the symplectic leaves of $\{\ ,\ \}_\lambda$ are of higher codimension, and there are several independent Casimirs $h_\lambda^{(1)}, h_\lambda^{(2)}, \ldots$, all the coefficients of their expansions in λ are in mutual involution with respect to both brackets: e.g., $\{h_i^{(1)}, h_j^{(2)}\}_k = 0$.

Remark 4.20 The fact that $\{h_i, h_j\}_k = 0$ for $k = 0, 1$ means that the functions h_j are first integrals of the Hamiltonian vector fields ξ_i. So if the functions h_j are independent, theorem 4.19 provides us with an infinite list of first integrals for each of the ξ_i. In this situation one says that the h_i are the Hamiltonians of a hierarchy of bi-Hamiltonian systems. We will encounter an example for such a situation in section II.2.4 when we study the so called KdV equation and its relation to the Virasoro algebra.

5 Symplectic Reduction

5.1 Hamiltonian group actions

Throughout this section let (M, ω) be a symplectic manifold, i.e. a manifold M together with a non-degenerate closed 2-form ω on M. Furthermore, let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let us suppose that the exponential map exists. If the group G acts smoothly on M, each element X of the Lie algebra $\mathfrak{g}$ defines a vector field ξ_X on M via

$$\xi_X(m) = \frac{d}{dt}|_{t=0} \exp(tX)m\,.$$

The group action of G on the manifold M is called symplectic, if it leaves the symplectic form ω invariant, i.e. if $g^*\omega = \omega$ for all $g \in G$.

Definition 5.1 A symplectic G-action on a symplectic Manifold M is called *Hamiltonian* if there exists a G-equivariant smooth map $\Phi : M \to \mathfrak{g}^*$ (the *moment map*) such that for all $X \in \mathfrak{g}$, we have

$$d\langle \Phi, X\rangle = \iota_{\xi_X}\omega\,.$$

Remark 5.2 The fact that a group action on a manifold M is Hamiltonian implies that all vector fields ξ_X on M which are generated by the group action are Hamiltonian vector fields with Hamiltonian function $H_X = \langle \Phi, X\rangle$.

Exercise 5.1.1 Consider $M = \mathbb{R}^2$ with the usual symplectic form $\omega = dp \wedge dq$. Let the group $U(1)$ act on $\mathbb{R}^2$ by rotations. Show that this action is Hamiltonian with moment map $\Psi(p, q) = \frac{1}{2}(p^2 + q^2)$.

Exercise 5.1.2 Consider the coadjoint action of a Lie group G on the dual of its Lie algebra. Show that this action restricted to any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is Hamiltonian with moment map the inclusion $\iota : \mathcal{O} \hookrightarrow \mathfrak{g}^*$.

Exercise 5.1.3 Generalize the definition of Hamiltonian group actions to Poisson manifolds and show that the coadjoint action of a Lie group G on the dual of its Lie algebra is Hamiltonian with moment map given by the identity.

5.2 The Marsden-Weinstein reduction

Consider a symplecic manifold (M, ω) with a Hamiltonian action of the group G. The equivariance of the moment map $\Phi : M \to \mathfrak{g}^*$ implies that the inverse image $\Phi^{-1}(\lambda)$ is a union of G_λ orbits. The Theorem on symplectic reduction states that if λ is a regular value of the moment map, and if the set $\Phi^{-1}(\lambda)/G_\lambda$ of G_λ-orbits in $\Phi^{-1}(\lambda)$ is a manifold, then it inherits a natural symplectic structure from the one on M.

Exercise 5.2.1 Suppose that G is a finite dimensional Lie group acting on on the symplectic manifold M in a Hamiltonian way with moment map Φ. For $m \in M$, let $G.m$ denote the G-orbit through m and let G_m denote the stabilizer of m with Lie algebra $\mathfrak{g}_m$. Show that for all $m \in M$, the kernel and the image of the differential of the moment map th the point $m \in M$ are given by

$$\ker(d_m \Phi) = (T_m G.m)^\omega = \{\xi \in T_m G.m \mid \iota_\xi \omega = 0\},$$

$$\mathrm{im}(d_m \Phi) = \mathrm{ann}(\mathfrak{g}_m) = \{\lambda \in \mathfrak{g}^* \mid \lambda(X) = 0 \text{ for all } X \in \mathfrak{g}_m\}$$

Conclude that an element $\lambda \in \mathfrak{g}^*$ is a regular value of moment map if and only if for all $m \in \Phi^{-1}(\lambda)$ the stabilizer G_m discrete.

The symplectic form ω does not restrict to a symplectic form on the level sets $\Phi^{-1}(\lambda)$ of the moment map, as it might aquire a kernel. But the exercise above shows that the foliation of the level set defined by this kernel coincides with the foliation into G^0_λ-orbits in $\Phi^{-1}(\lambda)$ (G^0_λ beeing the connected component of the identity of the stabilizer grou G_λ). Hence, if the quotient space is reasonably nice, the 2-form ω descends to a symplectic form on the quotient.

Theorem 5.3 (Marsden-Weinstein, Meyer) *Suppose, λ is a regular value of the moment map and suppose that $\Phi^{-1}(\lambda)/G_\lambda$ is a manifold (this condition is satisfied if e.g. G_λ is compact and acts freely on $\Phi^{-1}(\lambda)$). Then there exists a unique symplectic structure ω_λ on the reduced space $\Phi^{-1}(\lambda)/G_\lambda$ such that*

$$\iota^* \omega = \pi^* \omega_\lambda.$$

(Here, ι denotes the embedding $\Phi^{-1}(\lambda) \hookrightarrow M$.)

Finally, if $H : M \to \mathbb{R}$ is a Hamiltonian function which is invariant under the G-action, it descends to a function H_λ on the quotient space $\Phi^{-1}(\lambda)/G_\lambda$. Furthermore, if two G-invariant functions F and H on M Poisson commute with respect to the Poisson structure on M defined by the symplectic form ω, the corresponding functions on the quotient $\Phi^{-1}(\lambda)/G_\lambda$ stillPoisson commute with respect to the quotient strucure.

Example 5.4 Consider the manifold $M = \mathbb{C}^{n+1}$ with its standard symplectic structure $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$. The group $U(1) = \mathbb{R}/\mathbb{Z}$ acts on $\mathbb{C}^{n+1}$ by rotation: $z \mapsto e^{2\pi i t} z$. The Moment map for this action is given by $\Phi(z) = \pi \|z\|^2$. The reduced space $\Phi^{-1}(1)/U(1)$ is the complex projective space $\mathbb{CP}^n$ with symplectic form (a multiple of) the Fubini-Study form.

Let $f_i : \mathbb{C}^{n+1} \to \mathbb{R}$ denote the function $f_i(z) = \|z_i\|^2$. The functions f_i are invariant under the $U(1)$-action on $\mathbb{C}^{n+1}$, and the corresponding Hamiltonian functions on the symplectic quotient generate the rotation in the $\mathbb{C}_i$-hyperplanes in $\mathbb{CP}^n$.

6 Bibliographical notes

The material of this chapter is standard. There are many books on Lie groups and Lie algebras ([90], [20], [15] to name a few). Our treatment of infinite dimensional Lie groups follows [37]. A more general approach to infinite dimensional manifolds is developed in [64]. The symplectic structure on the coadjoint orbits of a Lie group goes back to Kirillov, Kostant and Sorieau and can be found in almost any book on symplectic geometry. In section 3, we follow [76], and for the material of section 4, see [71]. The idea of freezing the Lie Poisson bracket can be found in [70]. The Marsden Weinstein reduction in section 5 goes back to [72], and [74]. We have followed [36] and [73].

Chapter II

Infinite dimensional Lie groups and their coadjoint orbits

1 Loop groups and affine Lie algebras

Let G be a finite dimensional Lie group and let M be a finite dimensional compact manifold. The current group G^M of smooth maps from M to G endowed with pointwise multiplication is an infinite-dimensional (Fréchet) Lie group. In this section we will consider the particular case when the manifold M is the circle S^1. The resulting current group G^{S^1} is usually called the loop group of G and is denoted by LG.

We are interested in the coadjoint representation of the group LG or rather of a certain (and, actually, unique) central extension $\widehat{LG}$ of the group LG. It turns out that in contrast to the nonextended group LG, the coadjoint representation of the centrally extended group $\widehat{LG}$ has a much nicer structure. In particular, all orbits have finite codimension and can be classified in terms of conjugacy classes of the finite dimensional group G. This is not true for the nonextended loop group LG: all its non-zero orbits are of infinite dimension and codimension. From this point of view it is more attractive to study the extended affine group $\widehat{LG}$ than the nonextended loop group LG: in accordance with Kirillov's method of orbits, which roughly states that the group representations are in correspondence with the group coadjoint orbits. The simple structure of the $\widehat{LG}$ orbits explains, in particular, its similarity with finite-dimensional Lie groups.

The centrally extended loop group $\widehat{LG}$ has been studied from various points of view as it is connected with different fields of mathematics and mathematical physics. In this chapter we shall only describe the geometric features of the theory.

1.1 The central extension of the loop Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The corresponding loop algebra $L\mathfrak{g}$ is the Lie algebra of smooth maps from the circle S^1 to $\mathfrak{g}$. The Lie bracket on $L\mathfrak{g}$ is the pointwise Lie bracket of $\mathfrak{g}$. In this section we are interested in a certain central extensions of $L\mathfrak{g}$. It turns out that if the finite dimensional Lie algebra $\mathfrak{g}$ is simple, there is essentially only one non-trivial central extension of the loop algebra $L\mathfrak{g}$.

Recall first the definition of an invariant bilinear form on a Lie algebra $\mathfrak{g}$:

Definition 1.1 A bilinear form $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is *invariant*, if $\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle$ for all $X,Y,Z\in\mathfrak{g}$.

Any invariant bilinear form on a Lie algebra $\mathfrak{g}$ gives rise to a Lie algebra two-cocycle and hence a central extension of the corresponding loop algebra $L\mathfrak{g}$ in the following way:

Proposition / Definition 1.2 *The map $\omega : L\mathfrak{g}\times L\mathfrak{g}\to\mathbb{R}$, which is defined by*

$$\omega(X,Y)=\frac{1}{2\pi}\int_0^{2\pi}\langle X(t),\frac{d}{dt}Y(t)\rangle dt$$

is a two-cocycle on $L\mathfrak{g}$. The corresponding one-dimensional central extension of $L\mathfrak{g}$ is denoted by $\widehat{L\mathfrak{g}}$ and is called the affine Lie algebra corresponding to $\mathfrak{g}$ and $\langle\ ,\ \rangle$.

PROOF. Anti-symmetry of the map ω follows by integration by parts. It remains to check the cocycle identity of ω. This is a direct calculation again using integration by parts and the invariance of the inner product of $\mathfrak{g}$. □

Remark 1.3 In the algebraic literature, one is usually interested in the algebra $L\mathfrak{g}_{\mathrm{pol}}$ of polynomial loops in $\mathfrak{g}$ (i.e., trigonometric polynomials with values in $\mathfrak{g}$). Obviously, the construction of the cocycle ω on $L\mathfrak{g}$ restricts to $L\mathfrak{g}_{\mathrm{pol}}$ and we shall denote the corresponding central extension by $\widehat{L\mathfrak{g}_{\mathrm{pol}}}$. If $\mathfrak{g}$ is a complex simple Lie algebra, there exists a unique invariant bilinear form on $\mathfrak{g}$. The corresponding affine Lie algebra is also called (untwisted) affine Kac-Moody algebra.

Equivalently, the algebra $L\mathfrak{g}_{\mathrm{pol}}$ can be thought of as the space of $\mathfrak{g}$-valued Laurent polynomials $L\mathfrak{g}_{\mathrm{pol}} := \mathfrak{g}\otimes[z,z^{-1}]$, while the integral over the circle is replaced by taking the residue at $z=0$. The equivalence of polynomial loops with the latter description is delivered by the change of variable $t\to z=e^{it}$, which sends Fourier polynomials in t to Laurent polynomials in z and identifies d/dt with $iz\cdot d/dz$.

Remark 1.4 Sometimes it is convenient to further extend the affine algebra by adding the "co-central direction." Namely, the map

$$d : X\mapsto\frac{d}{dt}X$$

is a derivation of the Lie algebra $L\mathfrak{g}$ and it extends to a derivation of the centrally extended algebra $\widehat{L\mathfrak{g}}$ which acts trivially on the center. So we can define the semi-direct product $\widetilde{L\mathfrak{g}}=\widehat{L\mathfrak{g}}\rtimes\mathbb{R}d$ which is also sometimes called the affine Lie algebra corresponding to $\mathfrak{g}$. The advantage of the "doubly extended" Lie algebra $\widetilde{L\mathfrak{g}}$ is that, unlike $\widehat{L\mathfrak{g}}$, it admits a non-degenerate invariant bilinear form, which is given by

$$\langle X+\alpha c+\beta d, Y+\gamma c+\delta d\rangle=\frac{1}{2\pi}\int_0^{2\pi}\langle X(t),Y(t)\rangle dt+\alpha\delta+\beta\gamma\,.$$

It turns out that if the Lie algebra $\mathfrak{g}$ is semi-simple, the affine Lie algebras corresponding to $\mathfrak{g}$ for different invariant forms are the only nontrivial central extensions of the loop algebra $L\mathfrak{g}$:

Proposition 1.5 *If the Lie algebra $\mathfrak{g}$ is the Lie algebra of a semisimple compact Lie group G, then any central extension of loop algebra $L\mathfrak{g}$ is given by a cocycle of the form from Proposition 1.2 for some bilinear invariant form $\langle\ ,\ \rangle$ on $\mathfrak{g}$.*

PROOF. (See [82]): We first note that it suffices to consider cocycles ω which are constant under conjugation by constant loops $g \in G$. Indeed, if ω is not invariant under conjugation, the cocycle

$$\widetilde{\omega} = \int_G g^*\omega$$

is invariant and belongs to the same cohomology class as ω.

Let us consider the complexification $L\mathfrak{g}_{\mathbb{C}} = L\mathfrak{g} \otimes \mathbb{C}$ of $L\mathfrak{g}$. An element $X \in L\mathfrak{g}_{\mathbb{C}}$ can be expanded into a Fourier series $X = \sum a_r z^r$, where z denotes the function e^{it}. Any two-cocycle ω on $L\mathfrak{g}$ can be extended to a bilinear map $L\mathfrak{g}_{\mathbb{C}} \times L\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. By continuity, ω is completely determined by its values on polynomials (i.e. on finite Fourier series). Let us define maps $\omega_{p,q} : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ by

$$\omega_{p,q}(X, Y) = \omega(Xz^p, Yz^q).$$

These maps are G-invariant and hence symmetric. This follows from the general fact that for semisimple $\mathfrak{g}$, any G-invariant bilinear map $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ has to be symmetric, which is an easy consequence of Schur's lemma. By anti-symmetry of the map ω, we get $\omega_{p,q} = -\omega_{p,q}$. Moreover, the cocycle identity for ω implies

$$\omega([Xz^p, Yz^q], Zz^r) + \omega([Zz^r, Xz^p], Yz^q) + \omega([Yz^q, Zz^r], Xz^p) = 0.$$

Rewriting this in terms of the maps $\omega_{p,q}$ gives

$$\omega_{p+q,r}([X, Y], Z) + \omega_{r+p,q}([Z, X], Y) + \omega_{q+r,q}([Y, Z], X) = 0$$

Using the G-invariance and symmetry of $\omega_{p,q}$ we get

$$\omega_{r+p,q}([Z, X], Y) = \omega_{r+p,q}([X, Y], Z)$$

and similarly for the third term. Finally, using the fact that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ for semisimple $\mathfrak{g}$, we obtain

$$\omega_{p+q,r} + \omega_{r+p,q} + \omega_{q+r,q} = 0 \tag{1.1}$$

From now on, we only have to exploit Equation 1.1. Setting $p = q = 0$ yields

$$\omega_{0,r} = 0.$$

Setting $r = -p - q$ gives

$$\omega_{p+q,-p-q} + \omega_{-p,p} + \omega_{-q,q} = 0,$$

hence by induction

$$\omega_{n,-n} = n\omega_{1,-1}.$$

Finally, putting $r = n - p - q$, we get

$$\omega_{p+q,n-p-q} + \omega_{n-p,p} + \omega_{n-q,q} = 0,$$

which implies $\omega_{n-k,k} = k\omega_{n-1,1}$ for all n and k. Hence

$$0 = nk\omega_{0,n} = k\omega_{n-1,1} = \omega_{n-k,k},$$

from which we can deduce $\omega_{p,q} = 0$ for all p, q with $p + q \neq 0$.

Returning to the cocycle ω, we obtain

$$\begin{aligned}\omega(\sum_p X_p z^p, \sum_q Y_q z^q) &= \sum_{p,q} \omega_{p,q}(X_p, Y_q) \\ &= \sum_r r\omega_{1,-1}(X_r, Y_{-r}) \\ &= \frac{i}{2\pi} \int_0^{2\pi} \omega_{1,-1}(X(t), Y'(t))dt\end{aligned}$$

which is of the required form (remember that we have defined $z = e^{it}$). □

Remark 1.6 Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and M a compact oriented manifold. Consider the current algebra $\mathfrak{g}^M = C^\infty(M, \mathfrak{g})$, the Lie algebra of smooth maps from M to $\mathfrak{g}$. The non-degenerate bilinear form on $\mathfrak{g}$ is unique up to a multiple. The universal central extension of $\mathfrak{g}^M$ is given by the two-cocycle

$$\omega(X, Y) = \langle X, dY \rangle$$

with values in the quotient space $\Omega^1(M)/d\Omega^0(M)$ of all 1-forms in M modulo all exact 1-forms, cf [82].

Note that if dimension of the manifold M is at least 2, then the space $\Omega^1(M)/d\Omega^0(M)$ is infinite-dimensional. However, if M is one-dimensional, i.e. a circle $M = S^1$, any one-form on M is closed. Hence $\Omega^1(M)/d\Omega^0(M)$ is exactly the first de Rham cohomology group of M which is one-dimensional. So in this case, the universal central extension of $\mathfrak{g}^M = L\mathfrak{g}$ is exactly the central extension from proposition 1.5.

To visualize the $\Omega^1(M)/d\Omega^0(M)$-valued central extension of the algebra $\mathfrak{g}^M$ for higher-dimensional M, we fix an element in the dual space to $\Omega^1(M)/d\Omega^0(M)$ and associate to it an $\mathbb{R}$-valued central extension. First note that the (smooth) dual to the quotient space $\Omega^1(M)/d\Omega^0(M)$ is the space $Z^{n-1}(M) = \Omega^1(M)/d\Omega^0(M)$ of all closed $(n-1)$-forms on M. Indeed, for any 1-form $u \in \Omega^1(M)$ and an $(n-1)$-form $\gamma \in Z^{n-1}(M)$ there is the natural pairing on the compact manifold: $\langle u, \gamma \rangle := \int_M u \wedge \gamma$. This pairing gives zero if $u = df$ is exact and γ is closed.

Now the real-valued two-cocycle associated to the form γ is

$$\omega_\gamma(X, Y) = \int_M \langle X, dY \rangle \wedge \gamma.$$

Note that if the closed $(n-1)$-form γ is not smooth, but is a δ-type form (i.e., a Green current) supported on a closed curve $\Gamma \subset M$, then the cocycle $\omega_\gamma(X, Y)$ degenerates exactly to the affine algebra two-cocycle on the curve Γ: $\int_M \langle X, dY \rangle \wedge \gamma = \int_\Gamma \langle X, dY \rangle$ (cf. Proposition 1.5).

1.2 Coadjoint orbits of affine groups

In the last section, we have constructed the affine Lie algebra $\widehat{L\mathfrak{g}}$ corresponding to a finite dimensional Lie algebra $\mathfrak{g}$. Let us assume for a moment that there exists a Lie group $\widehat{LG}$ with Lie algebra $\widehat{L\mathfrak{g}}$. We will construct such a group in the following section. Here, we are interested in the adjoint and coadjoint representation of $\widehat{LG}$.

Note that in order to understand the coadjoint orbits of the group $\widehat{LG}$ we do not actually have to know how the extended group looks like, but we need just its non-extended vesion. Indeed, if the group $\widehat{LG}$ exists, it will be a central extension of the loop group LG of the Lie group G corresponding to the Lie algebra $\mathfrak{g}$. So there exists a map $p : \widehat{LG} \to LG$ whose kernel is the center of $\widehat{LG}$.

Furthermore, one can immediately see that for any Lie group H, the center of H acts trivially in the adjoint representation. Applying this observation to the group $\widehat{LG}$, we see that the adjoint action of $\widehat{LG}$ factors through the map p. So in order to study the adjoint action of the centrally extended group $\widehat{LG}$, we only have to understand action of the group LG on the Lie algebra $\widehat{L\mathfrak{g}}$ and its dual. This action will be described explicitly in the following Lemma.

Lemma 1.7 *In the adjoint representation of $\widehat{LG}$, an element $g \in LG$ acts on $(X,a) \in \widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$ via*

$$g : (X,a) \mapsto \left(\mathrm{Ad}_g(X), a - \frac{1}{2\pi}\int_0^{2\pi} \langle g^{-1}(t)\frac{d}{dt}g(t), X(t)\rangle dt \right) .$$

Proof. One checks directly that the map above defines a representation of LG on $L\mathfrak{g} \oplus \mathbb{R}$. To see that this is indeed the adjoint representation of $\widehat{LG}$, we have to check how the corresponding infinitesimal action looks like: Let g_s be a path in LG tangent to $Y \in L\mathfrak{g}$. That is, g_s is family of loops in G depending smoothly on a parameter $s \in \mathbb{R}$ such that $g_0 = id$ and $\frac{d}{ds}|_{s=0} g_s = Y$. Then we have

$$\frac{d}{ds}|_{s=0}\frac{1}{2\pi}\int_0^{2\pi} \langle g_s^{-1}(t)g_s'(t), X(t)\rangle dt = \frac{1}{2\pi}\int_0^{2\pi}\langle Y'(t), X(t)\rangle dt\,.$$

So we get

$$\begin{aligned}\frac{d}{ds}|_{s=0}\mathrm{Ad}_{g_s}(X,a) &= \left(\mathrm{Ad}_{g_s}(X), a - \frac{1}{2\pi}\int_0^{2\pi}\langle g_s^{-1}(t)g_s'(t), X(t)\rangle dt\right)\\ &= \left([Y,X], a - \frac{1}{2\pi}\int_0^{2\pi}\langle Y'(t)t, X(t)\rangle dt\right)\\ &= \mathrm{ad}_Y(X,a)\,,\end{aligned}$$

where we have used integration by parts in the last step. □

Let us turn to the coadjoint representation of $\widehat{LG}$ on $\widehat{L\mathfrak{g}}^*$. The space $\widehat{L\mathfrak{g}}$ can be identified with a subspace of the dual space of $\widehat{L\mathfrak{g}}$ via the non-degenerate pairing

$$\langle (X,a),(A,\lambda)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle X(t), A(t)\rangle dt + a\lambda\,.$$

The image of $\widehat{L\mathfrak{g}}$ is the smooth part of the dual of $\widehat{L}\mathfrak{g}$ and we will denote it by $\widehat{L\mathfrak{g}}_s^*$.

As in the case of the adjoint representation, the coadjoint representation of the group $\widehat{LG}$ factors through the map $p : \widehat{LG} \to LG$, so that it is enough to consider the action of the loop group LG on $\widehat{L\mathfrak{g}}^*$.

Lemma 1.8 *In the coadjoint representation of $\widehat{LG}$ on $\widehat{L\mathfrak{g}}_s^*$, an element $g \in LG$ acts via*

$$\mathrm{Ad}_g^*(A,\lambda) = \left(\mathrm{Ad}_g(A) + \lambda(\frac{d}{dt}g)g^{-1}, \lambda\right) .$$

PROOF. Recall that the coadjoint action of LG on $\widehat{L\mathfrak{g}}_s^*$ is defined via

$$\langle \mathrm{Ad}_g^*(A,\lambda), (X,a)\rangle = \langle (A,\lambda), \mathrm{Ad}_{g^{-1}}(X,a)\rangle .$$

The lemma follows from this formula by a direct calculation. □

Corollary 1.9 *In the coadjoint representation of $\widehat{L\mathfrak{g}}$ on $\widehat{L\mathfrak{g}}_s^*$, an element $X \in L\mathfrak{g}$ acts via*

$$\mathrm{ad}_X^*(A,\lambda) = ([X,A] - \lambda X', 0) .$$

Exercise 1.2.1 *i*) Verify this formula for $\mathrm{ad}_X^*(A,\lambda)$ by differentiating the group coadjoint action above, as in the proof of Lemma 1.7. *ii*) Verify the same formula directly from the definition of the Lie bracket in the affine algebra; *iii*) Check that the formula for $\mathrm{Ad}_g^*(A,\lambda)$ indeed defines a representation of the corresponding group, i.e., that $\mathrm{Ad}_g^*\mathrm{Ad}_h^* = \mathrm{Ad}_{gh}^*$.

This excercise gives an alternative way of verifying the formula for loop group coadjoint action on $\widehat{L\mathfrak{g}}_s^*$ without calculating the corresponding group adjoint action.

Remark 1.10 Suppose, the Lie algebra $\mathfrak{g}$ is $\mathfrak{gl}(n,\mathbb{R})$. Then we can identify the affine subspace $L\mathfrak{g} \times \{1\} \subset \widehat{L\mathfrak{g}}_s^*$ with the set of matrix-valued first order differential operators on the circle:

$$\widehat{L\mathfrak{g}}_s^* \times \{1\} = \{\frac{d}{dt} - A \mid A \in L\mathfrak{g}\} .$$

In this identification, the loop group LG acts via gauge transformations on differential operators.

We now want to study the orbits in the coadjoint representation of $\widehat{LG}$. Since λ is invariant under the action of LG we can study the orbits in a fixed hyperplane $\lambda = const$. For simplicity of notation, let us fix $\lambda = 1$. The other hyperplanes with $\lambda \neq 0$ differ from this one by an overall rescaling. We will comment on the case $\lambda = 0$ in Remark 1.16.

Theorem 1.11 *Let G be a compact and simply connected Lie group. There is a one-to-one correspondence between the set of LG-orbits in the $(\lambda = 1)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$ and the set of conjugacy classes in the group G. Moreover, the stabilizer of an element $(A,1) \in \widehat{L\mathfrak{g}}_s^*$ is isomorphic to the centralizer of the corresponding conjugacy class in G.*

Corollary 1.12 *Every coadjoint orbit in the $(\lambda = 1)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$ is isomorphic to LG/H for some subgroup $H \subset G$. In particlar, the codimension of such orbits is always finite.*

PROOF. Let us assign to each $(A, 1) \in \widehat{L\mathfrak{g}}_s^*$ the differential equation

$$\frac{d}{dt}z - Az = 0\,, \tag{1.2}$$

and let z be a solution of this differential equation. Since A is a loop in the Lie algebra $\mathfrak{g}$, any solution z of equation 1.2 with $z(0) \in G$ stays in the group G and hence defines a path $z : \mathbb{R} \to G$ (see e.g. [62]). Furthermore, since $A(t)$ is periodic in t, the map $z(t+2\pi)$ is another solution of equation 1.2. Differentiating with respect to t shows that the path $z(t+2\pi)^{-1}z(t)$ does not depend on t and hence defines an element $g_z \in G$. Now, if ν is another solution of equation 1.2 then ν differs from z by a constant $h \in G$. That is, $\nu(t) = hz(t)$ for all $t \in \mathbb{R}$. This gives

$$g_\nu = h^{-1} g_z h\ ,$$

so that g_ν and g_z lie in the same conjugacy class of G. This shows that we can assign to each element $(A, 1)$ a conjugacy class on the group G.

It remains to show that this construction assigns the same conjugacy class of the group G to each element in the coadjoint orbit of $\widehat{LG}$ containing $(A, 1)$. To this end, let z be a solution of equation 1.2 and fix a loop $g \in LG$. It is a direct calculation to check that the path $t \mapsto gz(t)g(0)^{-1}$ is a solution of the differential equation

$$\frac{d}{dt}z = (gAg^{-1} - g'g^{-1})z\,.$$

But $g \in LG$ is periodic in t so that we have

$$g(0)z(t+2\pi)^{-1}g(t+2\pi)^{-1}g(t)z(t)g(0) = g(0)z(t+2\pi)^{-1}z(t)g(0)\,.$$

Therefore, the conjugacy class of G corresponding to $\mathrm{Ad}_g^*(A, 1)$ is the same as the conjugacy class corresponding to $(A, 1)$.

The discussion above already shows that the stabilizer of an element $(A, 1) \in \widehat{L\mathfrak{g}}_s^*$ in the coadjoint representation of LG, is equal to the stabilizer of the corresponding conjugacy class in G.

Finally, we have to check that the map LG-orbits in the $(\lambda = 1)$–hyperplane $\to$ conjugacy classes in G is surjective. This is a consequence of the surjectivity of the exponential map. Indeed, fix some $g \in G$, and let $H \in \mathfrak{g}$ be an inverse image of H under the exponential map. Then we can view $(H, 1)$ as an element of the $(\lambda = 1)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$, and the conjugacy class of G corresponding to the coadjoint orbit through $(H, 1)$ contains g. $\square$

Remark 1.13 The proof of Theorem 1.11 shows that every coadjoint orbit of $\widehat{LG}$ in the $(\lambda = 1)$–hyperplane contains a constant element.

Example 1.14 The stabilizer of $(0,1) \in \widehat{L\mathfrak{g}}_s^*$ is the whole group G, so that the coadjoint orbit through $(0,1)$ is isomorphic to LG/G. In general, if G is compact, and $T \subset G$ is a maximal torus in G, then every element of G is conjugate to some element of T. (For $G = SU(n)$, this is the classical fact that every unitary matrix is diagonalizable). Furthermore, two elements of T are conjugate in G if and they are conjugate under $W = N_G(T)/T$. Here, $N_G(T)$ denotes the normalizer of T in G. The group W is a finite group, the so called Weyl group of G. It follows from our discussion above, that the set of conjugacy classes in G and hence the set of coadjoint orbits in the $(\lambda = 1)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$ can be identified with the set T/W. This set in turn is a convex polytope, the so called fundamental alcove of G (see Appendix A.1). The codimension of a conjugacy class in G is determined by its position in this polytope. For example the stabilizer of every conjugacy class in the interior of fundamental alcove is the maximal torus T itself. Hence the corresponding coadjoint orbits of LG are isomorphic to LG/T.

Remark 1.15 In the proof of theorem 1.11, we have considered arbitrary solutions of the differential equation

$$\frac{d}{dt}z - Az = 0 .$$

There is a distinguished solution of this equation, the so called fundamental solution z_f, which is given by the initial condition $z_f(0) = id$. The corresponding element $g_{z_f} = z_f(t+2\pi)^{-1} z_f(t) \in G$ is the monodromy of the equation. Mapping each element of a given coadjoint orbit to the fundamental solution of the corresponding differential equation allows us to identify the coadjoint orbit with a space of paths in G starting at the identity and ending in the corresponding conjugacy class in G. In this picture, the closure of the coadjoint orbit is given by the set of all continuous maps from the interval $[0, 2\pi]$ to G starting at the identity and ending in the corresponding conjugacy class. This identification allows to use powerful methods from analysis such as Wiener integration on the (closure) of the orbits (see [27],[91]).

Remark 1.16 The LG-action on the $(\lambda = 0)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$ is given by the pointwise conjugation action of LG on its Lie algebra. Thus, $(A, 0)$ and $(B, 0)$ belong to the same coadjoint orbit if and only if $A(t)$ and $B(t)$ are conjugate for all $t \in S^1$. Hence the classification of coadjoint orbits reduces to the classification of families of matrices up to conjugation. Every function $f : S^1 \to \mathbb{R}$ which only depends on the eigenvalues of the family $A(t)$ is an invariant under the action of the group LG on the $(\lambda = 0)$–hyperplane in $\widehat{L\mathfrak{g}}_s^*$. This shows that every coadjoint orbit in this hyperplane has infinite codimension.

1.3 Construction of the central extension of the loop groups

Recall from section 1.1 that in the case that $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra, every central extension of the loop algebra $L\mathfrak{g}$ is given by means of a cocycle

$$\omega(X, Y) = \frac{1}{2\pi} \int_0^{2\pi} \langle X(t), Y'(t) \rangle dt , \tag{1.3}$$

where $\langle\ ,\ \rangle \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is some invariant bilinear form on $\mathfrak{g}$.

Suppose that the Lie algebra $\mathfrak{g}$ is simple. Then up to multiplication by scalars, there exists a unique invariant bilinear form on $\mathfrak{g}$, the so called Killing form. Recall that if $\mathfrak{g}$ is a matrix Lie algebra, then we can write $\langle X, Y \rangle = \mathrm{tr}(XY)$ for $X, Y \in \mathfrak{g}$. Throughout this section, we shall normalize the Killing form such that the long roots in $\mathfrak{g}$ have square length 2 (see Appendix A.1 for some facts about compact Lie groups). Abusing notation slightly, we will often write $\mathrm{tr}(X, Y)$ for the normalized Killing form.

The choice of the bilinear form $\langle\ ,\ \rangle$ fixes a two-cocycle ω and hence a central extension $\widehat{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$. The goal of this section is to show that this central extension lifts to the group level. Thus, we are looking for a central extension

$$\{e\} \to S^1 \to \widehat{LG} \to LG \to \{e\}$$

of the loop group LG whose Lie algebra is given by $\widehat{L\mathfrak{g}}$. If such a central extension of LG exists, it defines an S^1 bundle $\widehat{LG} \to LG$. The problem in constructing the central extension $\widehat{LG}$ comes from the fact that the S^1 bundle $\widehat{LG} \to LG$ turns out to be non-trivial. So it is not possible to write down a continuous group two-cocycle which defines the extension. Otherwise it would mean the existence of a global section and hence the triviality of the bundle.

Theorem 1.17 *Let G be a compact, simple and simply connected Lie group and let ω be the two-cocycle on $L\mathfrak{g}$ defined in formula 1.3, where the bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is specialized to the Killing form normalized in such a way that the long roots on $\mathfrak{g}$ have squared length 2. Then the Lie algebra central extension of $L\mathfrak{g}$ defined by the cocycle ω lifts to a central extension $\widehat{LG}$ of LG.*

Remark 1.18 In fact, the proof of Theorem 1.17 will show that a central extension of the loop algebra $L\mathfrak{g}$ defined by a multiple $k\omega$ of the cocycle ω lifts to a central extension of the loop group LG whenever k is an integer. Although we will not show it, it turns out that these are all possible central extensions of the loop group LG.

The rest of this section is devoted to the proof of Theorem 1.17.

Before we start with the construction of the central extension of the loop group LG, let us give an alternative construction of the Lie algebra $\widehat{L\mathfrak{g}}$. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the closed unit disk. The loop algebra $L\mathfrak{g}$ can be regarded as a quotient $L\mathfrak{g} = \mathfrak{g}^D / \mathfrak{g}^D_{S^1}$, where $\mathfrak{g}^D = C^\infty(D, \mathfrak{g})$ is the Lie algebra of smooth maps from the disk D to $\mathfrak{g}$, and $\mathfrak{g}^D_{S^1}$ denotes the subalgebra of maps that vanish on the boundary $\partial D = S^1$ of D. That is, $\mathfrak{g}^D_{S^1} = \{X \in \mathfrak{g}^D \mid X_{S^1} = 0\}$. We can define a map $\omega^D : \mathfrak{g}^D \times \mathfrak{g}^D \to \mathbb{R}$ by

$$\omega^D(X, Y) = \frac{1}{2\pi} \int_D \mathrm{tr}(dX \wedge dY)\,.$$

Here dX and dY should be viewed as matrix-valued one forms on D, so that $\mathrm{tr}(dX \wedge dY)$ is a two form on D which can be integrated over the disk D. A short calculation shows that ω^D is a two-cocycle on $\mathfrak{g}^D$. Hence defines a central extension $\widehat{\mathfrak{g}^D}$ of $\mathfrak{g}^D$. Furthermore, by the Stokes formula, we have

$$\frac{1}{2\pi} \int_D \mathrm{tr}(dX \wedge dY) = \frac{1}{2\pi} \int_D d\mathrm{tr}(XdY) = \frac{1}{2\pi} \int_{\partial D = S^1} \mathrm{tr}(XdY)\,.$$

This shows that the cocycle ω^D depends only on the boundary values of X and Y. In particular, it vanishes on the subalgebra $\mathfrak{g}^D_{S^1} \subset \mathfrak{g}^D$ so that $\mathfrak{g}^D_{S^1}$ is an ideal in the Lie algebra $\widehat{\mathfrak{g}}^D$. Obviously, the quotient $\widehat{\mathfrak{g}}^D/\mathfrak{g}^D_{S^1}$ is isomorphic to $\widehat{L\mathfrak{g}}$.

The next goal is to construct the group extension $\widehat{LG}$ as a quotient similar to the construction of the Lie algebra extension $\widehat{L\mathfrak{g}}$ above. In particular, we are looking for a central extension $\widehat{G}^D$ of the Lie group G^D whose Lie algebra is given by $\widehat{\mathfrak{g}}^D$. We will then embed $G^D_{S^1}$ as a normal subgroup in $\widehat{G}^D$ such that the Lie algebra of $G^D_{S^1}$ is isomorphic to $\mathfrak{g}^D_{S^1}$. From this it is clear that the Lie group $\widehat{LG} = \widehat{G}^D/G^D_{S^1}$ is a central extension of LG whose Lie algebra is $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^D/\mathfrak{g}^D_{S^1}$.

We will proceed in several lemmas.

Lemma 1.19 *There exists a central extension $\widehat{G}^D = G^D \times U(1)$ of G^D with Lie algebra $\widehat{L\mathfrak{g}}^D$.*

PROOF. Let $g \in G^D$ be a map from the disk D to G. Then $g^{-1}dg$ and dgg^{-1} are $\mathfrak{g}$-valued one-forms on D. Let us define a map

$$\gamma : G^D \times G^D \to \mathbb{R},$$

by

$$\gamma(g_1, g_2) = \frac{1}{4\pi} \int_D \mathrm{tr}(g_1^{-1}dg_1 \wedge dg_2 g_2^{-1}) \, .$$

Then the map $e^{i\gamma}$ is a group two-cocycle in G^D with values in $U(1)$ defining the desired central extension of G^D. Indeed, we have

$$(g_1g_2)^{-1}d(g_1g_2) = g_2^{-1}g_1^{-1}dg_1g_2 + g_2^{-1}dg_2 \, ,$$

and

$$d(g_1g_2)(g_1g_2)^{-1} = dg_1g_1^{-1} + g_1^{-1}g_2^{-1}dg_2g_1$$

so that we get

$$\gamma(g_1g_2, g_3) + \gamma(g_1, g_2) = \gamma(g_1, g_2g_3) + \gamma(g_2, g_3) \, .$$

Hence, $e^{i\gamma}$ is a group 2-cocycle with values in $U(1)$ and hence defines a central extension $\widehat{G}^D$ of G^D by $U(1)$. Recall that the multiplication in $\widehat{G}^D$ is given by

$$(g_1, a_1)(g_2, a_2) = (g_1g_2(x, y), a_1a_2e^{i\gamma(g_1,g_2)}) \, .$$

Finally, we have to check that the Lie algebra of $\widehat{G}^D$ is given by $\widehat{\mathfrak{g}}^D$. To do this, we calculate the infinitesimal version of the cocycle $e^{i\gamma}$: Let g_s and g_t be two smooth curves in G^D such that $\frac{d}{ds}|_{s=0}g_s = X$ and $\frac{d}{dt}|_{t=0}g_t = X$. According to Proposition 3.13 of chapter I, the infinitesimal version of $e^{i\gamma}$ is given by

$$\frac{d^2}{dtds}|_{t=0,s=0} \exp(i\gamma(g_t, g_s)) - \frac{d^2}{dtds}|_{t=0,s=0} \exp(i\gamma(g_s, g_t))$$
$$= \frac{1}{2\pi} \int_D \mathrm{tr}(dX \wedge dY) = \omega^D(X, Y) \, .$$

□

Our next goal is to embed $G^D_{S^1}$ as a normal subgroup into $\widehat{G}^D$. Let B be a three-dimensional ball bounded by the two-sphere S^2 which is represented as the union of two disks D and D' glued together along the common boundary. Given a G-valued map $g \in G^D_{S^1}$ we can extend it trivially to S^2 and then choose an arbitrary extension $\widetilde{g}$ of g to B (see figure 1.1). This is possible due to the following topological fact.

Fact 1 *Let G be a compact and simply connected Lie group. Then $\pi_2(G) = \{e\}$, that is any continuous map $f : S^2 \to G$ can be extended to a continuous map $\tilde{f} : B \to G$.*

Then we can define a number

$$\lambda(g) = \frac{1}{12\pi}\int_B \operatorname{tr}(\tilde{g}^{-1}d\tilde{g})^{\wedge 3}.$$

Remark 1.20 The real-valued 3-form $\operatorname{tr}(\tilde{g}^{-1}d\tilde{g})\wedge^3$ has the following expression in coordinates: ... This is the left-invariant differential form on the group G, whole value at the identity of the group on a triple of vectors $A, B, C \in \mathfrak{g}$ from its Lie algebra is given by the following exterior 3-form: $\langle A, [B, C]\rangle$.

Obviously, $\lambda(g)$ depends on the extension $\tilde{g}$, but we shall see in the following the dependence of $\lambda(g)$ on the choice of an extension is easily controlled.

Lemma 1.21 *Given two maps $g, h : S^2 \to G$ and extensions $\tilde{g}, \tilde{h} : B \to G$ of g and h. Then*

$$\lambda(\tilde{g}\tilde{h}) = \lambda(\tilde{g}) + \lambda(\tilde{h}) + \gamma(g, h) \tag{1.4}$$

Moreover, for any $g : S^2 \to G$, the number $e^{i\lambda(g)}$ does not depend on the choice of the extension $\tilde{g}$ of g to B.

Proof. Let $\tilde{g}$ and $\tilde{h}$ be extensions of g and h to B. Then

$$\begin{aligned}
\lambda(\tilde{g}\tilde{h}) =& \frac{1}{12\pi}\int_B \operatorname{tr}\left((\tilde{g}\tilde{h})^{-1}d(\tilde{g}\tilde{h})\right)^{\wedge 3} = \frac{1}{12\pi}\int_B \operatorname{tr}\left(\tilde{h}^{-1}\tilde{g}^{-1}d\tilde{g}\tilde{h} + \tilde{h}^{-1}d\tilde{h}\right)^{\wedge 3}\\
=&\lambda(\tilde{g}) + \lambda(\tilde{h}) + \frac{1}{4\pi}\int_B \operatorname{tr}\left(\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\wedge d\tilde{h}\tilde{h}^{-1} + \tilde{g}^{-1}d\tilde{g}\wedge d\tilde{h}\tilde{h}^{-1}\wedge d\tilde{h}\tilde{h}^{-1}\right)\\
=&\lambda(\tilde{g}) + \lambda(\tilde{h}) + \frac{1}{4\pi}\int_B d\operatorname{tr}\left(\tilde{g}^{-1}d\tilde{g}\wedge d\tilde{h}\tilde{h}^{-1}\right)\\
=&\lambda(\tilde{g}) + \lambda(\tilde{h}) + \gamma(g, h),
\end{aligned}$$

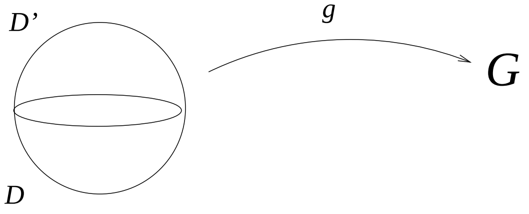

Figure 1.1: Extension of g to $\tilde{g} : B \to G$

where we have used the Stokes theorem in the last step.

Finally observe that any two extensions $\tilde{g}$ and $\tilde{\tilde{g}}$ of g to B differ by a map $\tilde{h} : B \to G$ such that $\tilde{h}|_{\partial B} = e$. That is, $\tilde{g} = \tilde{\tilde{g}}\tilde{h}$. Since $\tilde{h}|_{\partial B} = e$, we have $\gamma(g, \tilde{h}|_{\partial B}) = 0$. Hence by the first part of the lemma, $\lambda(g\tilde{h}|_{\partial B}) = \lambda(g) + \lambda(\tilde{h}|_{\partial B})$. Now we have to use another fact from the topology of compact Lie groups:

Fact 2 *Let G be a compact, simple and simply connected Lie group. Then the third cohomology group with values in $\mathbb{Z}$, $H^3(G, \mathbb{Z})$ is generated by a single element η. If $\tilde{g}$ is a map from a three-dimensional manifold M to G, then the pullback of η by $\tilde{g}$ is given by*

$$\tilde{g}^*\eta = \frac{1}{24\pi^2}\mathrm{tr}(\tilde{g}^{-1}d\tilde{g})^{\wedge 3} \ .$$

The map $\tilde{h}$ maps the boundary of B to the identity in G. So we can view it as a closed three-cycle in G. Using the fact that

$$\int_C \eta \in \mathbb{Z}$$

for any closed three-cycle C in G gives $\lambda(\tilde{h}) \in 2\pi\mathbb{Z}$ so that the number $e^{i\lambda(g)}$ indeed does not depend on the choice of extension of g. □

Since the number $e^{i\lambda(\tilde{g})}$ does not depend on the choice of the extension $\tilde{g}$ of the map g, we can define a map $\psi : G^D_{S^1} \to \widehat{G}^D$ via $g \mapsto (g, e^{i\lambda(\tilde{g})})$.

Lemma 1.22 *The map $\psi : G^D_{S^1} \to \widehat{G}^D$, defines an embedding of $G^G_{S^1}$ as a normal subgroup in $\widehat{G}^D$.*

PROOF. It follows from Lemma 1.21 that ψ is well defined and the injectivity of ψ is obvious. We only have to check that $\psi(G^D_{S^1})$ is a normal subgroup in $\widehat{G}^D$. Equation 1.4 shows that $\psi(G^D_{S^1})$ is indeed a subgroup of $\widehat{G}^D$. Normality of $G^D_{S^1}$ follows from the identity

$$\lambda(\tilde{g}\tilde{h}\tilde{g}^{-1}) = \lambda(\tilde{h}) + \gamma(gh, g^{-1}) + \gamma(h, g^{-1}),$$

which can be deduced directly from 1.4. □

Lemma 1.23 *Under the embedding $G^D_{S^1} \hookrightarrow \widehat{G}^D$, the Lie algebra of $G^D_{S^1}$ coincides with the Lie algebra $\mathfrak{g}^D_{S^1} \subset \widehat{\mathfrak{g}}^D$.*

PROOF. The form $\mathrm{tr}(\tilde{g}^{-1}d\tilde{g})^{\wedge 3}$ is a homogeneous polynomial of degree 3 in the derivatives of $\tilde{g}$ Hence all its derivatives at $\tilde{g} \equiv e$ vanish (see Figure 1.2). □

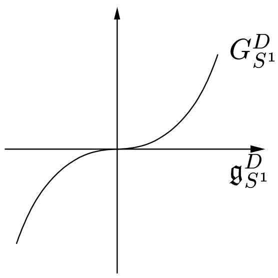

Figure 1.2: $G^D_{S^1}$ as a subset of $\widehat{G}^D$ in an neichborhood of the idntity of $\widehat{G}^D$.

Remark 1.24 Let us briefly remark on other central extensions of the loop group LG. Each multiple $k\omega^D$ of the cocycle ω^D defines a central extension of the Lie algebra $\mathfrak{g}^D$ and hence gives rise to an extension of the Lie algebra $L\mathfrak{g}$. Obviously, each of these extensions of $\mathfrak{g}^D$ lifts to a central extension $(\widehat{G}^D)^k$ of the group G^D. The problem, however, is that there exists a topological obstruction to the existence of an embedding of $G^D_{S^1}$ into the corresponding extension of G^D. Indeed, in order to embed the group $G^D_{S^1}$ into $(\widehat{G}^D)^k$, we have to use the map $g \mapsto (g, e^{ik\lambda(\tilde{g})})$. However, Lemma 1.21 shows that the number $e^{ik\lambda(\tilde{g})}$ depends on the choice of $\tilde{g}$ unless k is an integer. Hence, we can lift the central extension of the loop algebra $L\mathfrak{g}$ with cocycle $k\omega$ to the group level whenever $k \in \mathbb{Z}$. It can be shown [82], that these are in fact all possible central extensions of the loop group LG.

1.4 Bibliographical notes

There is a huge body on literature on the theory of loop groups and the corresponding Lie algebras. The standard reference for the material of this section is the book by Pressley and Segal [82]. The monograph [65] describes the theory of Kac-Moody groups from a more algebraic point of view, and the corresponding Lie algebras have been treated in the [47].

The central extension of the loop algebra goes back to Kac (cite ...) and Moody (cite ...). The classification of coadjoint orbits of the loop group has been obtained by Reiman and Semenof-Tian-Shansky (cite ...), Frenkel (cite ...) and Segal (cite ...). Frenkel has used this classification to give an analogue of Kirillov's character formula for compact Lie groups in the loop group setting. The construction of the central extension of the loop group as described in section 1.3 first appeared in the physics literature on the Wess-Zumino-Witten model [75]. Construction of this cocycle in different variations can be found in the papers of Faddeev-Shatashvili, Pressley-Segay, Mickelson, Witten, Moore-Nekrason-Losev-Shatashvili, etc.

2 Diff S^1 and the Virasoro Bott group

This section deals with the Lie group $Diff(S^1)$ of orientation preserving diffeomorphisms of the circle and its Lie algebra, the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle and their central extensions. In section 2.1 we we show that the Lie algebra of vector fields on the circle admits an essentially unique central extension, the so called Virasoro algebra. This central extension can be lifted to a central extension of the Lie group of orientation preserving diffeomorphisms of the circle. The corresponding group is called the Virasoro-Bott group. Similarly to the case of loop groups, the Virasoro-Bott group behaves much nicer in the coadjoint representation than the unextended group of orientation preserving diffeomorphisms of the circle, and its coadjoint orbits can be classified in terms of conjugacy classes of the finite dimensional group $SL_2(\mathbb{R})$. We will describe this classification in sections 2.2 and 2.3. Finally, in sections 2.4 and 2.5, we study the Euler equations with respect to some right-invariant metrics on the Virasoro-Bott group and show that they give rise to some interesting nonlinear partial differential equations.

2.1 Central extensions

Let us consider the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle. After fixing a coordinate θ of the circle, any vector field can be written as $f(\theta)\frac{\partial}{\partial\theta}$ where f is a smooth function on S^1. Under this identification, the commutator of two elements in $\mathrm{Vect}(S^1)$ is given by $[f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}] = (f'g - g'f)\frac{\partial}{\partial\theta}$, where f' denotes the derivative of the function f. Note that the Lie bracket is the negative of the usual Lie bracket of vector fields.

Proposition / Definition 2.1 *The map $\omega : \mathrm{Vect}(S^1) \times \mathrm{Vect}(S^1) \to \mathbb{R}$ given by*

$$\omega(f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}) = \int_{S^1} f'(\theta)g''(\theta)d\theta. \tag{2.5}$$

is a two-cocycle on $\mathrm{Vect}(S^1)$, called the Gelfand-Fuchs cocycle. The corresponding central extension of $\mathrm{Vect}(S^1)$ is called Virasoro algebra and is denoted by $\mathfrak{vir}$.

The proof that ω is indeed a two-cocycle is a straight forward calculation. The following proposition shows that up to isomorphism, the Virasoro algebra is the unique non-trivial central extension of the Lie algebra $\mathrm{Vect}(S^1)$.

Proposition 2.2 *We have $H^2(\mathrm{Vect}(S^1), \mathbb{R}) \cong \mathbb{R}$ where the generator of the cohomology is given by the Gelfand-Fuchs cocycle ω from Proposition 2.1.*

Proof. The proof of Proposition 2.2 is similar to the proof of proposition 1.5. Our goal is to show, that up to a coboundary, any two-cocycle ω on the Lie algebra $\mathrm{Vect}(S^1)$ is a multiple of the Gelfand-Fuchs cocycle. First, let us extend the cocycle ω on $\mathrm{Vect}(S^1)$ to a complex bilinear form on the complexification $\mathrm{Vect}(S^1)_{\mathbb{C}} = \mathrm{Vect}(S^1) \otimes \mathbb{C}$ of the Lie algebra $\mathrm{Vect}(S^1)$. An element $f\frac{\partial}{\partial\theta} \in \mathrm{Vect}(S^1)_{\mathbb{C}}$ can be expanded into a Fourier series

$$f(\theta) = \sum f_n e^{in\theta} .$$

By continuity, the cocycle ω is completely determined by its values on $L_n = e^{in\theta}\frac{\partial}{\partial\theta}$. Note that the commutator of L_n and L_m is given by

$$[L_n, L_m] = -i(m-n)L_{n+m}\,.$$

Let us set $\omega_{n,m} = \omega(L_n, L_m)$. Then, from the cocycle identity for ω, we get

$$(m-n)\omega_{n+m,k} + (k-m)\omega_{m+k,n} + (n-k)\omega_{k+n,m} \tag{2.6}$$

and

$$\omega_{n,m} = -\omega_{m,n}. \tag{2.7}$$

In particular, we have $\omega_{0,0} = 0$. Furthermore, we can define a one-cocycle α via $\alpha(L_n) = \frac{i}{n}\omega(L_n, L_0)$. Then we have $\omega(L_n, L_0) - d\alpha(L_n, L_0) = \omega(L_n, L_0) - \frac{i}{n}\omega([L_n, L_0], L_0) = 0$. So after changing ω by the exact cocycle $d\alpha$, we can assume $\omega_{n,0} = 0$ for all $n \neq 0$.

Setting $k = 0$ in equation 2.6 yields

$$-m\omega_{m,n} + n\omega_{n,m} = 0\,.$$

Whence, using the antisymmetry of ω, we get $\omega_{m,n} = 0$ whenever $n + m \neq 0$. Setting $k = -m-n$ in 2.6 gives

$$(m-n)\omega_{m+n,-n-m} - (2m+n)\omega_{-n,n} + (2n+m)\omega_{-m,m} = 0\,.$$

Finally, for $m = 1$ the equation above reads

$$(-n-1)\omega_{n+1,-n-1} - (n+2)\omega_{-n,n} + (2n+1)\omega_{-1,1} = 0\,.$$

Hence $\omega_{n,-n}$ is defined recursively by the knowledge of $\omega_{1,-1}$ and $\omega_{2,-2}$. This shows that the space of ω's which can satisfy condition 2.6 and 2.7 is at most two-dimensional. Two linear independent elements of this space are given by $\omega_{n,-n} = n^3$ and $\omega_{n,-n} = n$. But the cocycle defined by $\omega_{n,-n} = n$ is exact. In fact, it comes from the one-cocycle α defined above. So up to a coboundary, the two-cocycle ω we started with has the form

$$\omega(L_n, L_m) = c\delta_{n,-m}n^3$$

for some $c \in \mathbb{C}$.

It remains to show that ω is non-trivial. Suppose that $\omega = d\beta$ for some one cocycle β. This means that β is a linear map and $\omega(L_n, L_m) = \beta([L_n, L_m])$. In particular, we have $\beta([L_n, L_{-n}]) = 2ni\beta(L_0)$ which shows that in this case $\omega(L_n, L_{-n})$ would have to depend linearly on n which is a contradiction. □

Our next goal is to show that the central extension of the Lie algebra $\mathrm{Vect}(S^1)$ defined by the Gelfand-Fuchs cocycle ω can be lifted to a central extension of the group $Diff(S^1)$. Maybe surprisingly, the situation here is much simpler that in the case of loop groups. The central extension of the group $Diff(S^1)$ by S^1 corresponding to $\mathfrak{vir}$ is topologically trivial and hence can be defined by a continuous two-cocycle.

Proposition / Definition 2.3 *The map $c : Diff(S^1) \times Diff(S^1) \to S^1$ given by*

$$(\varphi, \psi) \mapsto \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi)' d \log \psi'$$

is a continuous two-cocycle on the group $Diff(S^1)$. The Lie algebra of the corresponding central extension $\widetilde{Diff}(S^1)$ is the Virasoro algebra $\mathfrak{vir}$. The two-cocycle c is called the Bott-cocycle, and the corresponding central extension of the group $Diff(S^1)$ is the Virasoro-Bott group.

PROOF. To show that the map c defines a cocycle, we have to check the identity

$$c(\varphi \circ \psi, \eta) + c(\varphi, \psi) = c(\varphi, \psi \circ \eta) + c(\psi, \eta)\,.$$

First, observe that we have

$$\int_{S^1} \log \psi' \, d \log \psi' = \int_{S^1} d(\log \psi')^2 = 0$$

so that we can calculate

$$c(\varphi \circ \psi, \eta) = \int_{S^1} \log(\varphi \circ \psi \circ \eta)' \, d \log \eta' = \int_{S^1} \log(\varphi' \circ \psi \circ \eta) \, d \log \eta' + c(\psi, \eta)\,.$$

On the other hand,we have

$$c(\varphi, \psi \circ \eta) = \int_{S^1} \log(\varphi \circ \psi \circ \eta)' \, d \log(\psi \circ \eta)' = c(\varphi, \psi) + \int_{S^1} \log(\varphi' \circ \psi \circ \eta) \, d \log \eta'\,.$$

This shows that the map c is indeed a group two-cocycle.

It remains to check that the infinitesimal version of the Bott cocycle c is the Gelfand-Fuchs cocycle ω. Let $f\frac{\partial}{\partial\theta}$ and $g\frac{\partial}{\partial\theta}$ be two smooth vector fields on S^1 and consider the corresponding flows φ_s and ψ_t on S^1. By Proposition 3.13 of chapter I we have to check that

$$\omega(f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}) = \frac{d^2}{dtds}|_{t=0,s=0} c(\varphi_t, \psi_s) - \frac{d^2}{dtds}|_{t=0,s=0} c(\psi_s, \varphi_t)$$

But

$$\frac{d}{d}|_{t=0} c(\varphi_t, \psi_s) = \frac{1}{2} \int_{S^1} \log'(\varphi_0' \circ \psi_s) f' \circ \psi_s d \log \psi_s' = \frac{1}{2} \int_{S^1} f' \circ \psi_s' d \log \psi_s'$$

and

$$\frac{d}{ds}|_{s=0} \frac{1}{2} \int_{S^1} f' \circ \psi_s' d \log \psi_s' = \frac{1}{2} \int_{S^1} f' dg'\,.$$

Similarly, we get

$$\frac{d^2}{dtds}|_{t=0,s=0} c(\psi_s, \varphi_t) = \frac{1}{2} \int_{S^1} g' df' = -\frac{1}{2} \int_{S^1} f' dg'\,.$$

Putting everything together yields the assertion. □

2.2 Coadjoint orbits of $Diff(S^1)$

Before we start classifying the coadjoint orbits of the Virasoro group, let us take a look at the coadjoint representation of the group $Diff(S^1)$ of orientation preserving diffeomorphisms of the circle. As we have seen in example 2.5 of chapter I, the smooth part of the dual of Vect(S^1) can be identified with the space

$$V = \left(\Omega^1(S^1)\right)^{\otimes 2}$$

of quadratic differentials on S^1. After choosing a coordinate θ on S^1, the elements of V can be written as $u(\theta)(d\theta)^2$ where u is a smooth function on S^1. Recall the following fact from example 2.5 of chapter I:

Proposition 2.4 *The coadjoint action of the group $Diff(S^1)$ on V is given by the formula*

$$\varphi : u(d\theta)^2 \mapsto u \circ \varphi^{-1} \cdot (\varphi^{-1'})^2 (d\theta)^2$$

which is just the change of variables formula.

It follows from this Proposition that $\sqrt{u(d\theta)^2}$ transforms under diffeomorphisms like a one-form on S^1. In particular, if the function u does not have any zeros, the number

$$\Phi(u(\theta)(d\theta)^2) = \int_{S^1} \sqrt{|u(\theta)|} d\theta$$

is invariant under the coadjoint action of $Diff(S^1)$. It turns out that this is the only invariant function since the coadjoint orbit through $u(d\theta)^2$ has codimension 1 in V. Indeed, the argument of the proof of proposition 1.12 of chapter I shows that whenever the function u does not have any zeros, the differential $u(d\theta)^2$ can be mapped under the action of $Diff(S^1)$ to a constant quadratic differential $u_0(d\theta)^2$, where the constant u_0 is the average of $\sqrt{u(\theta)}d\theta$ on the circle:

$$u_0 = \frac{1}{2\pi} \int_{S^1} (\sqrt{u(\theta)} d\theta \, .$$

The stabilizer of a constant quadratic differential is the group S^1 of rigid rotations so that the orbit through $u(d\theta)^2$ is diffeomorphic to $Diff(S^1)/S^1$ and hence has codimension 1 in V.

Now let us suppose that the function u does have zeroes on S^1 and denote its consecutive zeros by a_i. The numbers

$$c_i = \int_{a_i}^{a_{i+1}} \sqrt{|u|} d\theta \, ,$$

are preserved under the coadjoint action of the group $Diff(S^1)$. In particular, if the function u changes sign, the corresponding coadjoint orbit of $Diff(S^1)$ has necessarily codimension higher than 1, and there exist coadjoint orbits of the group $Diff(S^1)$ of arbitrary high codimension.

Remark 2.5 One can show that if the function u changes sign exactly twice, then the corresponding coadjoint of the group $Diff(S^1)$ has codimension 2. This is rather surprising since in finite dimension the existence of a symplectic structure on each coadjoint orbit forces the codimensions of the coadjoint orbits to vary in steps of multiples of 2. As we shall see in the next section, the coadjoint orbits of the Virasoro group behave much more like finite dimensional coadjoint orbits.

2.3 The Virasoro coadjoint action

Recall that elements of the Virasoro algebra are given by pairs $(f(\theta)\frac{\partial}{\partial\theta}, a)$, where $f(\theta)\frac{\partial}{\partial\theta}$ is a vector field on the circle, and a is a real number. Hence we can identify the smooth part $\mathfrak{vir}_s^*$ of the dual of the Virasoro algebra with

$$\mathfrak{vir}_s^* \cong V \oplus \mathbb{R},$$

where V is the space of quadratic differentials considered in the last section, and the pairing between $\mathfrak{vir}$ and $\mathfrak{vir}_s^*$ is given by

$$\langle (f(\theta)\frac{\partial}{\partial\theta}, a), (u(\theta)(d\theta)^2, \lambda)\rangle = \int_{S^1} f(\theta)u(\theta)d\theta + a\lambda\,.$$

Remark 2.6 From now on, we will simply write $\mathfrak{vir}^*$ for the smooth part of the dual of the Virasoro algebra since we will not talk about the full dual of $\mathfrak{vir}$.

Our goal in this section is to derive a classification of the coadjoint orbits of the Virasoro-Bott group $\widetilde{Diff}(S^1)$ which is similar to the classification of the coadjoint orbits of the centrally extended loop groups of section 1.2. Obviously, the center of the group $\widetilde{Diff}(S^1)$ has to act trivially in the coadjoint representation so that the representation factors through one of the unextended group $Diff(S^1)$. Using the pairing between $\mathfrak{vir} = \mathrm{Vect}(S^1)\oplus\mathbb{R}$ and $\mathfrak{vir}^*$ and the explicit construction of $\mathfrak{vir}$ as a central extension of $\mathrm{Vect}(S^1)$ one can find a formula for the resulting representation if $Diff(S^1)$.

Proposition 2.7 *In the coadjoint action of the Virasoro-Bott group, an element $\varphi \in Diff(S^1)$ acts via*

$$Ad^*_\phi(u(d\theta)^2, \lambda) = \left(u \circ \varphi^{-1} \cdot (\varphi^{-1\prime})^2(d\theta)^2 + \lambda S(\varphi^{-1})(d\theta)^2, \lambda\right)\,, \tag{2.8}$$

where

$$S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2$$

is the Schwarzian derivative of φ.

PROOF. The adjoint action of the Lie algebra $\mathfrak{vir}$ on itself is given by

$$\mathrm{ad}_{(v,a)}(f,b) = (v'f - f'v, -fv''')\,.$$

Hence, using the identity

$$\langle \mathrm{ad}_{(v,a)}(f,b),(u,\lambda)\rangle = -\langle (f,b),\mathrm{ad}^*_{(v,a)}(u,\lambda)\rangle$$

we find that the coadjoint action of the Lie algebra $\mathfrak{vir}$ on its dual $\mathfrak{vir}^*$ is given by

$$\mathrm{ad}^*_{(v,a)}(u,\lambda) = (-2v'u - u'v - \lambda v''', 0)\,. \tag{2.9}$$

It remains to check that equation (2.8) indeed defines an action of the group $Diff(S^1)$ on the space $\mathfrak{vir}^*$ and that the infinitesimal version of this action is given by equation (2.9). Both assertions can be checked by direct calculations which we leave to the reader. □

Remark 2.8 The Schwarzian derivative has the following important transformation law:

$$S(\varphi\circ\psi) = S(\varphi)\circ\psi\cdot(\psi')^2 + S(\psi)\,. \tag{2.10}$$

Equation (2.8) shows that the $(\lambda = const)$–hyperplanes in $\mathfrak{vir}^*$ are invariant under the coadjoint action of the Virasoro-Bott group. Our goal is to find a classification of the coadjoint orbits of the Virasoro-Bott group in a fixed $(\lambda \neq 0)$–hyperplane in $\mathfrak{vir}^*$ which is similar to the classification of the coadjoint orbits of the centrally extended loop groups in section 1. The basic idea in the classification is the following geometric interpretation of the group action (2.8). For a given element $(u(d\theta)^2,\lambda)\in\mathfrak{vir}^*$ with $\lambda\neq 0$ let us consider the differential equation

$$\lambda\frac{d^2}{d\theta^2}f + uf = 0 \tag{2.11}$$

on $\mathbb{R}$. This differential equation is called Hill's equation, and the differential operator $L_u = \lambda\frac{d^2}{d\theta^2}+u$ is called a Hill's operator. Equation (2.11) second order differential equation. So its space of solutions is two-dimensional.

Lemma 2.9 *Let f,g be two linearly independent solutions of equation (2.11). Then the Wronskian*

$$W(f,g) = \det\begin{pmatrix} f & g\\ f' & g'\end{pmatrix} = f'g - g'f$$

is constant.

PROOF. A short calculation using the fact that f and g satisfy equation (2.11) shows that $\frac{d}{d\theta}W(f,g) = 0$. Therefore $W(f,g)$ has to be constant. □

Since the solutions f and g of equation (2.11) are taken to be linearly independent, we know that the Wronskian $W(f,g)$ does not vanish. Hence, we can normalize f and g in such a way that we have $W(f,g)=1$. In particular, the functions f and g cannot vanish at the same time. So we can define a map $\psi = [f:g]:\mathbb{R}\to\mathbb{P}^1\mathbb{R}$. It turns out that the ratio ψ behaves nicely under the coadjoint action of the Virasoro group:

Proposition 2.10 *Under the Virasoro coadjoint action, a diffeomorphism φ of S^1 acts on the ratio ψ by reparametrization:*

$$\varphi : \psi \mapsto \psi \circ \varphi^{-1}$$

PROOF. Let us fix some $\mu \neq 0$, and let u be a smooth function on the circle. As before, L_u denotes the differential operator $L_u = \mu\frac{d^2}{d\theta^2} + u$. If f is a solution of the differential equation $L_u f = 0$, and if η is a diffeomorphism of S^1, then one can check, that the function

$$\widetilde{f} = f \circ \eta \cdot (\eta')^{-\frac{1}{2}} \tag{2.12}$$

is a solution of the differential equation $L_{\widetilde{u}}(\widetilde{f}) = 0$, where the function $\widetilde{u}$ is given by

$$\widetilde{u} = u \circ \eta (\eta')^2 + 2\mu S(\eta)\,.$$

Hence, substituting $\eta = \psi^{-1}$, we see that the action of the group $Diff(S^1)$ on solutions of equation (2.12) gives rise to the Virasoro adjoint action on the $(\lambda = 2\mu)$–hyperplane in equation (2.8). Now, if we change two linearly independent solutions f and g of equation $L_u f = 0$ according to equation (2.12), then the ratio of the two solutions simply changes by a reparametrization. □

On the other hand, given any map $\psi : \mathbb{R} \to \mathbb{P}^1\mathbb{R}$, we can reconstruct the potential u of the corresponding differential operator L_u:

Proposition 2.11 *Let $\psi : \mathbb{R} \to \mathbb{P}^1\mathbb{R}$ be a smooth map. Then the potential u of the corresponding differential operator L_u is given by $\frac{\lambda}{2}$ times the Schwarzian derivative of ψ:*

$$u = \frac{\lambda}{2} S(\psi)\,.$$

PROOF. Decomposing ψ into its components we can write $\psi = [f : g]$. Obviously, this does not define the functions f and g uniquely. But since we want both f and g to satisfy an equation of the form (2.11), the functions f and g should satisfy the relation

$$fg' - f'g \equiv const\,, \tag{2.13}$$

which can be seen by differentiating both sides of equation (2.13). Let us normalize f and g so that the constant equals 1. Writing $\psi = [1 : \widetilde{\psi}]$ with $\widetilde{\psi} = \frac{g}{f}$ whenever $f \neq 0$, we get

$$\widetilde{\psi}' = \frac{g'f - f'g}{f^2} = \frac{1}{f^2}\,,$$

which yields $f = \frac{1}{\sqrt{\widetilde{\psi}'}}$ and $g = \widetilde{\psi} f = \frac{\widetilde{\psi}}{\sqrt{\widetilde{\psi}'}}$.

Finally, given two functions $f, g : \mathbb{R} \to \mathbb{R}$ which satisfy $f'g - g'f = 0$, we want to find some $u : S^1 \to \mathbb{R}$ such that $L_u f = 0$ and $L_u g = 0$. One can check directly, that $u = \frac{\lambda}{2} S(\widetilde{\psi})$ has the desired properties. Indeed, since $\widetilde{\psi}' = (g/f)' = 1/f^2$, we get

$$S(\widetilde{\psi}) = -2\frac{f''}{f}$$

and hence

$$L_u f = \lambda f'' + fu = 0\,.$$

A similar calculation gives $L_u g = 0$. □

As a corollary of the considerations above, we obtain the following result:

Corollary 2.12 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Möbius transformation, i.e. $\varphi(x) = (ax+b)/(cx+d)$ for some a, b, c, d with $ad - bc = 1$. Then $S(\varphi) = 0$.*

PROOF. For a given Hill's operator L_u, the pair of linearly independent solutions f, g of the equation $L_u f = 0$ is not defined uniquely but up to a transformation of the pair (f, g) by a matrix from $SL_2(\mathbb{R})$. If we transform the pair (f, g) by a matrix from $SL_2(\mathbb{R})$, the ratio $\widetilde{\psi} = f/g$ changes by a Möbius transformation φ. But since the potential u of the Hill's operator L_u is defined uniquely by f and g, we can use the transformation law from equation (2.10) to conclude $S(\varphi) = 0$. □

All the considerations above were of local nature. To obtain a classification of the Virasoro coadjoint orbits we have to remember that the Hill's operators we are dealing with are periodic. In order to state the classification result, we have to introduce some notation. Let $\widetilde{SL}_2(\mathbb{R})$ denote the universal cover of the group $SL_2(\mathbb{R})$. The group $\widetilde{SL}_2(\mathbb{R})$ admits an outer automorphism of order 2. So we can consider the quotient $\widetilde{SL}_2(\mathbb{R})/\mathbb{Z}_2$. The main result of this section is the following theorem.

Theorem 2.13 *For fixed $\lambda \neq 0$, there is a one-to-one correspondence between the set of coadjoint orbits of the Virasoro-Bott group on the hyperplane $\{(u(\theta)(d\theta)^2, \lambda)\} \subset \mathfrak{vir}^*$ and the set of conjugacy classes in $\left(\widetilde{SL}_2(\mathbb{R}) \setminus \{id\}\right)/\mathbb{Z}_2$.*

PROOF. For a given $(u(d\theta)^2, \lambda) \in \mathfrak{vir}^*$, let us consider the corresponding Hill's operator $L_u = \frac{\lambda}{2}\frac{d^2}{d\theta^2} + u$ and consider a pair (f, g) of linearly independent solutions of the equation $L_u f = 0$. Since u is periodic, the values $(f(t), g(t))$ and $(f(t + 2\pi), g(t + 2\pi))$ are related by a monodromy matrix $M \in SL_2(\mathbb{R})$. That is,

$$(f(t + 2\pi), g(t + 2\pi)) = (f(t), g(t))M\,. \tag{2.14}$$

Equivalently, the values of the ratio $\psi(t) = [f(t) : g(t)]$ and $\psi(t+2\pi) = [f(t+2\pi) : g(t+2\pi)]$ are related by a Möbius transformation $\mathcal{M} \in PSL_2(\mathbb{R})$. The monodromy matrix M (respectively $\mathcal{M}$) is changed to a conjugate matrix if we change the pair of solutions (f, g).

On the other hand, in Proposition 2.11 we have shown that to each map $\psi : \mathbb{R} \to \mathbb{PR}^1$, we can assign a unique Hill's operator L_u. In particular, if ψ is quasiperiodic, i.e. if it satisfies an equation of type (2.14), it follows from the construction that the potential u is periodic. Furthermore, if we reparametrize ψ by a diffeomorphism of S^1, we see that the potential u changes according to the Virasoro coadjoint action. This shows that the Virasoro coadjoint orbits in the in a fixed ($\lambda = const$)-hyperplane are in one-to-one correspondence with equivalence classes of pairs $(\psi, \widetilde{M})$, where $\widetilde{M}$ is a conjugacy class in $PGL_2(\mathbb{R})$ and ψ is a path in PGL_2. The equivalence relation is given by the reparametrization of ψ with a diffeomorphism of S^1. The path ψ defines a lift of the conjugacy class $\widetilde{M}$ to the universal cover $\widetilde{SL}_2(\mathbb{R})$ of $PGL_2(\mathbb{R})$. Finally, the path ψ always goes in the positive direction. So we only reach "half" of the conjugacy classes in the universal cover $\widetilde{SL}_2(\mathbb{R})$, hence we have to quotient by $\mathbb{Z}_2$. Furthermore, the identity in $\widetilde{SL}_2(\mathbb{R})/\mathbb{Z}_2$ can never be assumed since $\psi'(0) \neq 0$. □

Corollary 2.14 *For fixed $\lambda \neq 0$, the Virasoro coadjoint orbits in the hyperplane $\{(u(d\theta)^2, \lambda)\} \subset \mathfrak{vir}^*$ are classified by the Jordan normal forms of matrices in $SL_2(\mathbb{R})$ together with a non-negative integer parameter, the winding number.*

Now, let us look at the stabilizers of the different coadjoint orbits. Since the codimension of a coadjoint orbit through $(u(d\theta)^2, \lambda)$ equals the codimension of in $SL_2(\mathbb{R})$ of the conjugacy class of the corresponding monodromy matrix we see that the codimension can be at most three. Matrices from the group $SL_2(\mathbb{R})$ fall into three classes, whose Jordan normal forms are the exponentials of the following three classes in the Lie algebra $sl_2(\mathbb{R})$.

$$i) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad ii) \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \text{ and } \quad iii) \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} \tag{2.15}$$

The codimension of the corresponding conjugacy classes in $SL_2, (\mathbb{R})$ is 3 in case $i)$ and it equals 1 in cases $ii)$ and $iii)$. Note that the class of real matrices which are exponentials of a matrix of type $ii)$ decomposes into rotation matrices (for $a \in i\mathbb{R}$) and hyperbolic rotation matrices (for $a \in \mathbb{R}$).

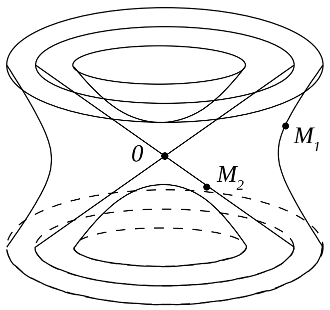

Figure 2.3: The points 0, M_1 and M_2 in $sl_2(\mathbb{R})$ correspond to Virasoro orbits of type $i)$, $ii)$, and $iii)$ respectively.

Finally, let us take a look at coadjoint orbits containing constant elements. We have to consider Hill's operators L_u with $u = p = const$. Obviously, the corresponding monodromy matrix $M_p \in SL_2(\mathbb{R})$ is given by

$$M_p = \begin{pmatrix} \cos(2\pi\sqrt{p}) & -\sin(2\pi\sqrt{p}) \\ \sin(2\pi\sqrt{p}) & \cos(2\pi\sqrt{p}) \end{pmatrix}$$

if $p \geq 0$ and

$$M_p = \begin{pmatrix} \cosh(2\pi\sqrt{-p}) & -\sinh(2\pi\sqrt{-p}) \\ \sinh(2\pi\sqrt{-p}) & \cosh(2\pi\sqrt{-p}) \end{pmatrix}$$

for $p < 0$. Now, the stabilizer of a differential operator L_p with $p > 0$ is the group S^1 of rigid rotations if $M_p \neq id \in PGL_2(\mathbb{R})$ and it is three dimensional if $M_p = id \in PGL_2(\mathbb{R})$, i.e. if M_p is of type i) in (2.15). This can only be the case if $p = \frac{1}{4}m^2$ for some $m \in \mathbb{Z}$. In this case, the stabilizer of L_p in $Diff(S^1)$ is the m-fold covering of $PGL_2(\mathbb{R})$ in $Diff(S^1)$.

2.4 The Virasoro group and the Korteweg-de Vries equation

The Korteweg-deVries (or KdV) non-linear differential equation which describes traveling waves in a shallow canal. It is given by

$$u_t = -3uu_\theta - cu_{\theta\theta\theta}\,, \tag{2.16}$$

where u is a function of t and θ, u_t and u_θ denote the partial derivatives $\frac{\partial}{\partial t}u$ and $\frac{\partial}{\partial\theta}u$, and $c \neq 0$ is a constant. A brief history of this equation can be found in [40].

Recall from section I4 that the Euler equation with respect to a right-invariant metric on a Lie group G is a dynamic system on the Lie algebra of G describing the evolution of the tangent vector of a geodesic on G pulled back to the Lie algebra of G by right translation. In this section we will show how the Korteweg-deVries equation appears as the Euler equation with respect to some right invariant metric on the Virasoro-Bott group.

Let us define an inner product on the Virasoro algebra $\mathfrak{vir} = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ by

$$\langle (v(\theta)\frac{\partial}{\partial\theta}, a), (w(\theta)\frac{\partial}{\partial\theta}, c)\rangle = \int_{S^1} v(\theta)w(\theta)d\theta + ac\,. \tag{2.17}$$

This inner product gives rise to the inertia operator $A : \mathfrak{vir} \to \mathfrak{vir}^*$, which is given by

$$(v\frac{\partial}{\partial\theta}, c) \mapsto (v(d\theta)^2, c)\,.$$

The main result of this section is the following theorem:

Theorem 2.15 *The Euler equations for the right invariant metric on the Virasoro group which is defined by right translating the inner product $\langle\ ,\ \rangle$ from equation (2.17) are given by*

$$u_t = -3uu_\theta - cu_{\theta\theta\theta} \tag{2.18}$$

$$c_t = 0\,. \tag{2.19}$$

Proof. Recall from equation (2.9) that in the coadjoint action, an element $(v\frac{\partial}{\partial\theta}, a) \in \mathfrak{vir}$ acts on an element $(u(d\theta)^2, c) \in \mathfrak{vir}^*$ via

$$\mathrm{ad}^*_{(v\frac{\partial}{\partial\theta}, a)}(u(d\theta)^2, c) = ((-2v'u - vu' - cv''')(d\theta)^2, 0)\,.$$

According to theorem 4.11, the Euler equation for the left invariant metric on the Virasoro-Bott group which comes from the inertia operator $A : \mathfrak{vir} \to \mathfrak{vir}^*$ is given by

$$\frac{d}{dt}m(t) = -\mathrm{ad}^*_{dH_{A^{-1}m(t)}}m(t)\,, \tag{2.20}$$

where m is a path in $\mathfrak{vir}^*$, and A is the inertia operator defined by the metric on the Virasoro algebra $\mathfrak{vir}$. Now, substituting $(u(d\theta), c)$ for m, we see that equation (2.20) becomes

$$u_t = 3uu_\theta + cu_{\theta\theta\theta} \tag{2.21}$$

$$c_t = 0 \tag{2.22}$$

Finally, by remark 4.13, switching from left to right invariant metrics changes the signs in the calculations above and thus yields the assertion. □

Remark 2.16 There is generalization of theorem 2.15: Consider the following two-parameter family of inner products on $\mathfrak{vir}$:

$$\langle (v\frac{\partial}{\partial\theta}, b), (w\frac{\partial}{\partial\theta}, c)\rangle_{H^1_{\alpha,\beta}} = \int_{S^1} (\alpha vw + \beta v_\theta w_\theta) d\theta + bc\,.$$

Note that for $\alpha = 1$ and $\beta = 0$, we recover the inner product from equation (2.17).

Theorem 2.17 *The Euler equations for the right invariant metric on the Virasoro group which is defined by right translating the inner product* $\langle\ ,\ \rangle_{H^1_{\alpha,\beta}}$ *with* $\alpha \neq 0$ *are given by*

$$\alpha(u_t + 3uu_\theta) - \beta(u_{\theta\theta t} + 2u_\theta u_{\theta\theta} + uu_{\theta\theta\theta}) + cu_{\theta\theta\theta} = 0\,, \tag{2.23}$$

$$c_t = 0\,. \tag{2.24}$$

Exercise 2.4.1 Give a proof of theorem 2.17 along the lines of the proof of theorem 2.15.

Remark 2.18 For $\alpha = 1$, $\beta = 0$, equation (2.23) recovers the KdV equation (2.18). For $\alpha = \beta = 1$, equation (2.23) is a generalized form of the Camassa-Holm equation, and for $\alpha = 0$, $\beta = 1$, equation (2.23) becomes the so called Hunter-Saxton equation.

The case $c = 0$ corresponds to the unextended Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle rather than the Virasoro algebra $\mathfrak{vir}$. In this case depending on the values of α and β one obtains the Hopf equation $u_t + 3uu_\theta = 0$ or the unextended Camassa-Holm equation $u_t + 3uu_\theta + 2u_\theta u_{\theta\theta} + uu_{\theta\theta\theta} + u_{\theta\theta t} = 0$.

2.5 Bi-Hamiltonian structure of the KdV equation

Recall from section 4.4 of chapter I that every point $m_0 \in \mathfrak{vir}^*$ gives rise to a "constant" Piosson bracket $\{\ ,\ \}_0$ on $\mathfrak{vir}^*$ by "freezing" the usual Lie-Poisson bracket $\{\ ,\ \}_{LP}$ in the point m_0. The constant Poisson bracket is defined for smooth functions f, g on $\mathfrak{vir}^*$ by

$$\{f, g\}_0(m) = \langle [df_m, dg_m], m_0 \rangle$$

Furthermore, we have seen in Lemma I.4.17 that the Poisson brackets $\{\ ,\ \}_{LP}$ and $\{\ ,\ \}_0$ are compatible for all choices of the point m_0. The main goal of this section is to show that for a certain choice of the "freezing point" m_0, the KdV equation is Hamiltonian with respect to the constant Poisson structure $\{\ ,\ \}_0$. Since the KdV equation is Hamiltonian with respect to the Lie-Poisson bracket as well, it is in fact bi-Hamiltonian.

Theorem 2.19 *The KdV equation (2.16) is Hamiltonian with respect to constant Poisson bracket on $\mathfrak{vir}^*$, "frozen" at the point $m_0 = (\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$.*

PROOF. Let $F(u(d\theta)^2, c)$ be a function on $\mathfrak{vir}^*$, and let

$$dF_{(u(d\theta)^2,c)} = (\delta F/\delta u, \delta F/\delta c) = (v\frac{\partial}{\partial\theta}, b) \in \mathfrak{vir}$$

be its derivative at the point $(u(d\theta)^2, c) \in \mathfrak{vir}^*$. A calculation which is similar to the proof of Proposition I.4.6 shows that the Hamiltonian equation with respect to the function F and the constant Poisson bracket frozen at a point $m_0 \in \mathfrak{vir}^*$ is given by

$$\frac{d}{dt}(u(d\theta)^2, c) = -\mathrm{ad}^*_{(v\frac{\partial}{\partial\theta}, b)} m_0 \,. \tag{2.25}$$

Specifying the "freezing" point to $m_0 = (\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$, equation (2.25) becomes

$$\frac{d}{dt}(u(d\theta)^2, c) = -\mathrm{ad}^*_{(v\frac{\partial}{\partial\theta}, b)}(\frac{1}{2}(d\theta)^2, 0) = (v', 0)\,. \tag{2.26}$$

The next step is to fix the function F via

$$F(u(d\theta)^2, c) = -\int_{S^1}\left(\frac{1}{3}u^3 - \frac{c}{2}(u_\theta)^2\right) d\theta\,.$$

In order to calculate the derivative dF of the function F at the point $(u(d\theta)^2, c)$, recall that the definition of the derivative of the function F in direction $(\xi(d\theta)^2, b) \in \mathfrak{vir}^*)$ gives

$$\langle(\xi(d\theta)^2, b), dF_{(u(d\theta)^2,c)}\rangle = \frac{d}{d\epsilon}|_{\epsilon=0} F(u(d\theta)^2 + \epsilon\xi(d\theta)^2, c + \epsilon b)\,.$$

Since in equation (2.26), only the partial derivative $\delta F/\delta u$ plays a role, it is enough to calculate

$$\frac{d}{d\epsilon}|_{\epsilon=0} F(u + \epsilon\xi, c) = -\int_{S^1}\left(\frac{1}{3}(u+\epsilon\xi)^3 - \frac{c}{2}(u_\theta + \epsilon\xi_\theta)^2\right) d\theta \tag{2.27}$$

$$= -\int_{S^1}\left(u^2\xi - cu_\theta\xi_\theta\right) d\theta \tag{2.28}$$

$$= -\int_{S^1}\left(u^2\xi + cu_{\theta\theta}\xi\right) d\theta \tag{2.29}$$

$$= -\langle((u^2 + cu_{\theta\theta})\frac{\partial}{\partial\theta}, c), (\xi(d\theta)^2, 0)\rangle\,. \tag{2.30}$$

Now plugging the derivative dF into equation (2.26) gives

$$u_t = -uu_\theta - cu_{\theta\theta\theta}$$

which is the KdV equation. $\square$

Corollary 2.20 *The KdV equation is bi-Hamiltonian with respect to the compatible Poisson structures* $\{\ ,\ \}_{LP}$ *and* $\{\ ,\ \}_0$ *on where* $\{\ ,\ \}_0$ *denotes the constant Poisson structure, frozen in the point* $(\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$.

Remark 2.21 In a similar fashion, one can extract the bi-Hamiltonian natures of the Hunter-Saxton and the Camassa-Holm equation mentioned in Remark 2.18. The respective "freezing" points of the constant Poisson bracket on $\mathfrak{vir}^*$ are $m_0 = ((d\theta)^2, 1)$ for the Camassa-Holm equation and $m_0 = (0, 1)$ for the Hunter-Saxton equation (see [49]).

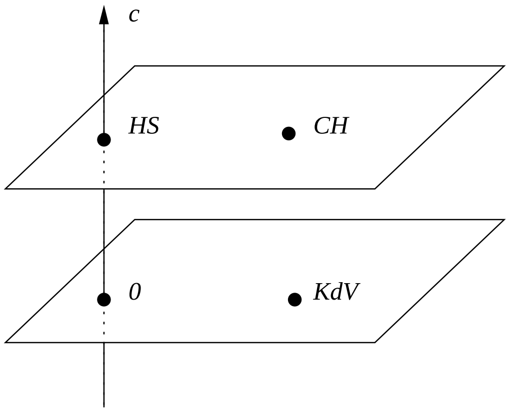

Figure 2.4: The freezing points of the constant Poisson bracket on $\mathfrak{vir}^*$ which give rise to the bi-Hamiltonian structures of the KdV, the Camassa-Holm and the Hunter-Saxton equation respectively.

Remark 2.22 Since the usual Lie-Poisson bracket and the constant Poisson bracket frozen in the point $m_0 = (0, 1)$ are compatible, the bracket $\{\ ,\ \}_{\lambda^2} = \{\ ,\ \}_{LP} + \lambda^2\{\ ,\ \}_0$ defines a Poisson bracket for all $\lambda \in \mathbb{R}$. which is the usual Lie Poisson bracket on $\mathfrak{vir}^*$ shifted by the point $m_0 = (0, 1) \in \mathfrak{vir}^*$.

We saw in Theorem 2.13 that the monodromy of the differential operator $c\frac{d^2}{d\theta^2} + u(\theta)$ which is a matrix in $SL_2(\mathbb{R})$ changes to a conjugate matrix under the Virasoro coadjoint action. Therefore, the trace of the monodromy is a Casimir function for the Poisson bracket $\{\ ,\ \}_{LP}$ on $\mathfrak{vir}^*$. Similarly, since $\{\ ,\ \}_{\lambda^2}$ is the usual Lie Poisson bracket on $\mathfrak{vir}^*$ shifted by the point $m_0 = (0, 1) \in \mathfrak{vir}^*$, we get

Lemma 2.23 *Let* M_λ *denote the monodromy of the differential operator* $c\frac{d^2}{d\theta^2} + u(\theta) - \lambda^2$. *Then the function*

$$h_\lambda = \log(\operatorname{tr}(M_\lambda))$$

is a Casimir function for the Poisson bracket $\{\ ,\ \}_{\lambda^2}$ *on* $\mathfrak{vir}^*$.

Finally, we can expand h_λ into a power series in λ and thereby produce first integrals of the KdV equation:

$$h_\lambda = 2\pi\lambda - \sum_{n=1}^{\infty} c_n h_{2n-1} \lambda^{1-2n},$$

where

$$h_1 = \int_{S^1} u(\theta) d\theta, \quad h_3 = \int_{S^1} u^2(\theta) d\theta, \quad h_5 = \int_{S^1} u^3(\theta) - \frac{1}{2}(u_\theta(\theta))^2 d\theta, \quad \ldots$$

and $c_1 = 1/2$, $c_n = (2n-3)!!/(2^n n!)$ for $n > 1$.

2.6 Bibliographical notes

The cohomology of the Lie algebra of vector fields on the circle was computed in [30], and the Virasoro algebra was introduced in [89]. The Bott cocycle on the group $Diff(S^1)$ first appeared in [14]. The classification of the coadjoint orbits of the Virasoro group goes back to [66], and in the present form to [85], [60, 61]. In section 2.4 and 2.5, we follow [80] and [49]. Much more details on the structure of the Virasoro-Bott group can be found in [35].

3 The group of volume preserving diffeomorphisms

In this section, we will study the coadjoint representation of the group of volume preserving diffeomorphisms of a compact manifold M. It turns out that the corresponding Euler equation on the group of volume preserving diffeomorphisms of a compact manifoldis the Euler equation for an incompressible fluid on the manifold M.

3.1 The group of volume preserving diffeomorphisms and its coadjoint representation

Recall the definition of the group of volume preserving diffeomorphisms of a compact manifold:

Proposition / Definition 3.1 *Let M be a n-dimensional compact manifold (possibly with boundary) and let μ be a volume form in M. The group*

$$SDiff(M) = \{\varphi \in Diff(M) \mid \varphi^* \mu = \mu\}$$

of volume preserving diffeomorphisms of M is an infinite-dimensional Lie group whose Lie algebra is the Lie algebra

$$S\mathrm{Vect}(M) = \{\xi \in \mathrm{Vect}(M) \mid L_\xi \mu = 0 \textit{ and } \xi \textit{ is tangent to } \partial M\}$$

of divergence free vector fields on M.

The following proposition identifies the smooth part of the dual of the Lie algebra $S\mathrm{Vect}(M)$ and the coadjoint representation of the group $SDiff(M)$ on it:

Proposition 3.2 *The smooth part of the dual of the Lie algebra $S\mathrm{Vect}(M)$ is identified with the space $\Omega^1(M)/d\Omega^0(M)$ of smooth one forms on M modulo exact one forms on M. In the coadjoint representation, an element $\varphi \in SDiff(M)$ maps an element $[u] \in \Omega^1(M)/d\Omega^0(M)$ to $[(\varphi^{-1})^* u]$. Accordingly, the coadjoint action of $S\mathrm{Vect}(M)$ on $\Omega^1(M)/d\Omega^0(M)$ is given by the negative of the Lie derivative:*

$$\xi : [u] \mapsto -[L_\xi u]$$

for any divergence free vector field ξ on M.

Proof. Let us suppose for a moment that the manifold M has no boundary. Each vector field ξ on M defines a $(n-1)$-form ν_ξ via interior multiplication with the volume form on M: $\nu_\xi = \iota_\xi \mu$. Thus, for a divergence free vector field $\xi \in S\mathrm{Vect}(M)$ we have $0 = L_\xi \mu = \iota_\xi d\mu + d\iota_\xi \mu = d\nu_\xi$, so that the $(n-1)$-form ν_ξ has to be closed. The pairing between $\Omega^1(M)/d\Omega^0(M)$ and $S\mathrm{Vect}(M)$ is given by

$$\langle [u], \xi \rangle = \int_M u \wedge \nu_\xi \,.$$

Since the ν_ξ is closed, we have $\langle df, \nu_\xi \rangle = 0$ for all functions f on M so that the pairing between $S\mathrm{Vect}(M)$ and $\Omega^1(M)/d\Omega^0(M)$ is indeed well defined.

The case of manifolds with boundaries can be dealt with along similar lines using the fact that the vector field ξ has to be tangent to the boundary of M.

Finally, recall that in the adjoint representation, the group $Diff(M)$ acts on its Lie algebra $Vect(M)$ by coordinate changes: $\mathrm{Ad}_\varphi(\xi) = \varphi_* \xi \circ \varphi^{-1}$. Hence we find

$$\langle \mathrm{Ad}_\varphi(\xi), [u] \rangle = \int_M u \wedge \iota_{\varphi_* \xi \circ \varphi^{-1}} \mu = \int_M (\varphi^{-1})^* u \wedge \iota_\xi (\varphi^{-1})^* \mu \,.$$

Since φ preserves the volume form μ, we have $\mathrm{Ad}^*_\varphi [u] = [(\varphi^{-1})^* u]$. The statement on infinitesimal coadjoint action of the Lie algebra $SVect(M)$ follows directly. $\square$

Exercise 3.1.1 Give a proof of Proposition 3.2 in the case that the manifold M has a non-empty boundary

3.2 The Euler equation of an ideal incompressible fluid

Let M be a n-dimensional Manifold or a subset of $\mathbb{R}^n$ and fix a volume form μ as well as a Riemannian metric $(\ ,\ )$ on M.

Definition 3.3 The Euler equation of an ideal, incompressible fluid on M is given by

$$\xi_t + (\xi, \nabla)\xi = -\nabla p \,, \qquad \mathrm{div}\xi = 0 \,. \tag{3.31}$$

Here, ξ denotes a time-dependent vector field on the manifold M (the velocity field), ξ_t is the partial derivative $\frac{\partial}{\partial t}\xi$, and p is a time-dependent function on M (the pressure). If the manifold M has a non-empty boundary, we additionally require that the vector field ξ is tangent to the boundary of M.

Remark 3.4 Let us make the following observation: For a vector field ξ on the manifold M, let $\phi : \mathbb{R} \times M \to M$ denote the flow of ξ. Then, by definition, we have $\frac{\partial}{\partial t}\phi(t,x) = \xi(t, \phi(t,x))$ and $\phi(0,x) = x$ for all $x \in M$. By the chain rule, we get

$$\frac{\partial^2}{\partial t^2}\phi = \xi_t + \xi_x \phi_t = \xi_t + \xi_x \xi = \xi_t + (\xi, \nabla)\xi \,.$$

Thus, rewritten as an equation for the flow ϕ, the Euler equation (3.31) is equivalent to the equation

$$\frac{\partial^2}{\partial t^2}\phi = -\nabla p(t, \phi(t,x)) \,. \tag{3.32}$$

Equation (3.32) has a natural interpretation as a geodesic equation: Indeed, the Riemannian metric $(\ ,\ )$ on the manifold M allows to define a quadratic form on $S\mathrm{Vect}(M)$ via

$$(\xi_1, \xi_2) = \int_M (\xi_1, \xi_2)\mu \,. \tag{3.33}$$

This quadratic form defines a right invariant metric on the group $SDiff(M)$ via right translation. Now, since the vector field ξ is divergence free, its flow $\phi(t,\,.\,)$ at time t takes values in the group $SDiff(M)$ of volume preserving diffeomorphisms of the manifold M.

Lemma 3.5 *Equation (3.32) implies that the path ϕ is geodesic with respect to the right-invariant metric on the group $SDiff(M)$ defined by equation (3.33).*

PROOF. Note that we have

$$\int_M (\nabla p, \xi)\mu = 0$$

for all divergence-free vector fields ξ and all smooth functions p on M. This implies that $\frac{\partial^2}{\partial t^2}\phi = \nabla p$ is a section of the normal bundle of $SDiff(M)$ viewed as a subset of the Fréchet space of all smooth maps from M to itself. Therefore the path ϕ is indeed geodesic. □

3.3 Hamiltonian structure and first integrals of the Euler equations for an incompressible fluid

Recall, that the smooth part of the dual of the Lie algebra of volume preserving vector fields on a compact manifold M is given by the space $\Omega^1(M)/d\Omega^0(M)$ of one-forms modulo exact one-forms. Furthermore, a metric $(\, , \,)$ on the manifold M determines a left-invariant metric on the group $SDiff(M)$, and hence an inertia operator

$$A : S\mathrm{Vect}(M) \to \Omega^1(M)/d\Omega^0(M)$$

by $A : \xi \mapsto [u]$ where u is the one form obtained from the vectorfield ξ by "raising the indices" with help of the metric. That is we set $u(\,.\,) = (\xi, \,.\,)$. Since the Riemannian metric is non-degenerate, the map A is surjective.

Lemma 3.6 *The Hamiltonian equation on $\Omega^1(M)/d\Omega^0(M)$ with respect to the Hamiltonian function $H([u]) = -\langle A^{-1}[u], [u]\rangle$ is given by*

$$\frac{d}{dt}[u] = -L_\xi[u]\,, \tag{3.34}$$

where $u = \langle \xi, \cdot \rangle$ is a fixed representative of $A^{-1}[u]$.

PROOF. Recall from corollary I.4.9 that the Hamiltonian equation with respect to the function H is given by

$$\frac{d}{dt}[u] = \mathrm{ad}^*_{A^{-1}[u]}[u]\,.$$

The coadjoint action of the Lie algebra $S\mathrm{Vect}(M)$ on its dual is given by the negative of the Lie derivative so that we get equation (3.34). □

Corollary 3.7 *The Hamiltonian equation (3.34) is equivalent to the Euler equation for an incompressible fluid on the Manifold M.*

PROOF. Rewriting equation (3.34) in terms of coset representatives gives

$$\frac{\partial}{\partial t}u = -L_\xi u - dp \tag{3.35}$$

for some time dependent function p. Passing back to vector fields, i.e. applying A on both sides of equation (3.35), we get our final equation:

$$\frac{\partial}{\partial t}\xi = -(\xi, \nabla)\xi - \nabla p\,,$$

This is the Euler equation for an incompressible ideal fluid on M where p is interpreted as the pressure function. □

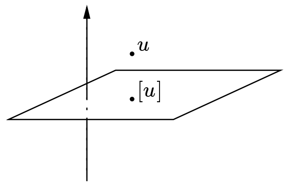

Figure 3.5: Projection of $u \in \Omega^1(M)$ to $\Omega^1(M)/d\Omega^0(M)$.

Our next goal is to determine invariants of the coadjoint action of $SDiff(M)$. To this end, we have the following proposition.

Proposition 3.8 *If the dimension of M is $n = 2m+1$, then the function*

$$I([u]) = \int_M u \wedge (du)^m$$

is invariant under the coadjoint action of the group $SDiff(M)$.

If $dim(M) = 2m$, then the function

$$I_k([u]) = \int_M \left(\frac{(du)^m}{\mu}\right)^k \mu$$

is invariant under the action of the group $SDiff(M)$ for each $k \in \mathbb{N}$.

Proof. We have to check that I and I_k are well defined functions on $\Omega^1(M)/d\Omega^0(M)$. This is clear, since for exact one-forms dv, we have $I([dv]) = 0$. Furthermore, the group $SDiff(M)$ acts on $\Omega^1(M)/d\Omega^0(M)$ by change of coordinates so that the integrals I and I_k are indeed invariant under this action. □

Corollary 3.9 *In odd, respectively even dimension, the functions I, respectively I_k on $S\mathrm{Vect}(D)$ are first integrals of the Euler equation for an incompressible ideal fluid.*

Proof. As we have seen in corollary 3.7, the Euler equation is Hamiltonian with respect to the usual Lie-Piosson bracket on the dual space of the Lie algebra $S\mathrm{Vect}(M)$. Therefore its flow-lines lie in fixed coadjoint orbits of the group $SDiff(M)$. But by Proposition 3.8, the functions I and I_k are constant on the coadjoint orbits and hence along the flow lines. □

Example 3.10 Consider a compact n-dimensional submanifold of $\mathbb{R}^n$ with the usual Euclidian metric. The inertia operator A maps a vector field $\xi = \sum \xi_i \frac{\partial}{\partial x_i}$ to the one form $u_\xi = \sum \xi_i dx_i$. Then the Euler equation has the form

$$\frac{\partial}{\partial t}\xi = -(\xi, \nabla)\xi - \nabla p\,.$$

Example 3.11 Specifying to the case of a 3-dimensional Manifold M, we see that the function

$$I(\xi) = \int_M u_\xi du_\xi$$

is a first integral of the Euler equation. A short direct calculation gives

$$I(\xi) = \int_M \langle \xi, curl(\xi)\rangle d^3x\,.$$

This last integral has a natural geometric meaning as the *helicity* of the vector field ξ:

Definition 3.12 Let M be a 3-dimensional compact manifold with a volume form μ and suppose that the second deRham cohomology group of M vanishes. Then, if ξ is a divergence-free vector field on M which is tangent to the boundary of M we know that the 2-form $\omega = \iota_\xi \mu$ is closed, hence exact. So we can find a 1-form αon M such that $\omega = d\alpha$. The *helicity* of the vector field ξ is defined by

$$H(\xi) = \int_M \iota_\xi \mu \wedge \alpha = \int_M d\alpha \wedge \alpha\,.$$

Exercise 3.3.1 Show that the helicity does not depend on the ambiguity in the definition of the 1-form α.

Remark 3.13 The helicity of a vector field ξ has a more topological interpretation as the "average linking number" of the trajectories of ξ (see [10] for details).

Example 3.14 Similarly, specifying to he case of a 2-dimensional submanifold $M \subset \mathbb{R}^2$, we find infinitely many first integrals of the Euler equation, namely

$$I_k(\xi) = \int_{R^2} (curl \xi)^k d^2x\,,$$

where $curl(\xi) = \frac{\partial \xi_1}{\partial x_2} - \frac{\partial \xi_2}{\partial x_1}$ is a function on $\mathbb{R}^2$.

3.4 More examples

Here we state (without going into any details) some other examples of groups of diffeomorphisms together with the Euler equation they give rise to.

Recall that for a compact manifold M, we can identify the smooth part of the dual of $\mathrm{Vect}(M)$ with the space $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^n(M)$ via the pairing

$$\langle \xi, \alpha \otimes \mu\rangle = \int_M \iota_\xi(\alpha)\mu\,.$$

Example 3.15 Consider the group $Diff(M) \ltimes C^\infty(M)$. Its Lie algebra is given as a vector space by $\mathrm{Vect}(M) \oplus C^\infty(M)$. The smooth part of the dual of $\mathrm{Vect}(M) \oplus C^\infty(M)$ can be identified with $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^n(M) \oplus \Omega^n(M)$ via the pairing

$$\langle (\xi, f), (\alpha \otimes \mu, \nu) \rangle = \int_M \iota_\xi(\alpha)\mu + \int_M f\nu \,.$$

Exercise 3.4.1 Show that the coadjoint action of the group $Diff(M) \ltimes C^\infty(M)$ on the space $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^n(M) \oplus \Omega^n(M)$ is given by

$$\mathrm{Ad}^*_{(\varphi,f)}(\alpha \otimes \eta, \nu) = (\varphi^*\alpha \otimes \varphi^*\eta) + \varphi^* df \otimes \varphi^*\nu, \varphi^*\nu) \ .$$

Exercise 3.4.2 Fix a map $h : C^\infty(M) \to C^\infty(M)$ which assigns to each function ρ (the density of the fluid) a function $h(\rho)$ (the pressure of the fluid). Define a Hamiltonian $H : \mathrm{Vect}(M) \oplus C^\infty(M) \to \mathbb{R}$ by

$$(\xi, \rho) \mapsto \int_M \left(\frac{1}{2}(\xi, \xi)\rho + \rho\Phi(\rho) \right) \mu \,,$$

where $\Phi(\rho)$ is chosen such that $\rho^2\Phi'(\rho) = h(\rho)$.

Show that the Euler equation corresponding to the Hamiltonian H is the Euler equation of a (baratropic) compressible fluid

$$\begin{cases} \rho \frac{d}{dt}\xi = -\rho(\xi, \nabla)\xi - \nabla h(\rho) \\ \frac{d}{dt}\rho + \mathrm{div}(\rho\xi) = 0 \,. \end{cases} \tag{3.36}$$

Exercise 3.4.3 Find Casimir functions similar to the ones we discussed in the case of the Euler equation of an ideal fluid in the preceeding section.

Example 3.16 As another example, we can consider the Lie group $SDiff(M) \ltimes S\mathrm{Vect}(M)$, where $S\mathrm{Vect}(M)$ is viewed as an Abelian Lie group. The underlying vector space of its Lie algebra is $S\mathrm{Vect}(M) \oplus S\mathrm{Vect}(M)$. Again choosing a Riemannian metric on M, we can define a Hamiltonian function $H : S\mathrm{Vect}(M) \oplus S\mathrm{Vect}(M) \to \mathbb{R}$ via

$$(\xi, B) \mapsto \int_M (\xi, \xi)\mu + \int_M (B, B)\mu \,.$$

In this case, the resulting Euler equation are the equations of magnetic hydrodynamics, where ξ is interpreted as the velocity field and B is the magnetic field.

Exercise 3.4.4 Work out the details for the example above.

3.5 Riemannian Geometry of the Group $SDiff$

In this section, we sidestep slightly from our theme of studying the geometry of the coadjoint orbits of the group of volume preserving diffeomorphisms $SDiff(M)$ and indicate without proofs some peculiarities of the geometry of the group itself.

Recall that a Riemannian metric $(\ ,\ )$ on the manifold M gives rise to a right-invariant metric on the group $SDiff(M)$ by fixing it at the identity via

$$(\xi_1, \xi_2)_e = \int_M (\xi_1, \xi_2)\mu \,.$$

Definition 3.17 Let $\phi_t : [0,1] \to SDiff(M)$ be a path in the group $SDiff(M)$, i.e. a family of volume preserving diffeomorphisms of M. The length of the path ϕ_t in the L^p-norm is defined to be

$$l_p(\phi_t) = \int_0^1 \|\frac{d\phi_t}{dt}\|_{L^p} dt = \int_0^1 \left(\int_M \|\frac{d\phi_t}{dt}\|^p \mu\right)^{\frac{1}{p}} dt \,.$$

Using the lengths of paths, one can define the L_p-distance between two diffeomorphisms φ and ψ by

$$\mathrm{dist}_p(\varphi, \psi) = \inf\{l_p(\phi_t) \mid \phi_0 = \varphi \text{ and } \phi_1 = \psi\} \,.$$

Finally, the L_p-diameter of a path-connected component of the group $SDiff(M)$ is defined by

$$\mathrm{diam}_p(SDiff(M)) = \sup_{\varphi,\psi \in SDiff(M)} \mathrm{dist}(\varphi, \psi) \,.$$

The following two theorems show that there is a fundamental difference between the groups of volume preserving diffeomorphisms of manifolds of different dimensions:

Theorem 3.18 (Eliashberg, Ratiu, 1991) *For all two-dimensional Riemannian manifolds M, the L_2-diameter of the group of volume-preserving diffeomorphisms is infinite:*

$$\mathrm{diam}_2(SDiff(M)) = \infty$$

Theorem 3.19 (Shnirelman 1985 and 1995) *Let M^n be the n-dimensional unit cube with $n \geq 3$ and the standard metric in $\mathbb{R}^n$. Then the L_2-diameter of the group of volume-preserving diffeomorphisms is finite and we have the following estimate:*

$$\mathrm{diam}_2(SDiff(M)) < 2\sqrt{\frac{n}{3}} \,.$$

We will not prove these theorems here. However, a related result will be proved in section 3.6. To give a rough idea, how the distinction between dimension $n = 2$ and dimension $n \geq 3$ comes about, consider a path in $SDiff(M)$ and associate to each point in the path the Jacobi matrix of the corresponding diffeomorphism which is a matrix in $SL_n(\mathbb{R})$. This defines a path in $SL_n(\mathbb{R})$. Now, the fundamental group of $SL_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}$, while for $n \geq 3$, the fundamental group of $SL_n(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence, in two dimensions, the shortest path from the identity to an element of the group $SDiff(M)$ can be very "twisted" which means that the element is far away from the identity in the L_p metric. In higher dimensions it is "easier" to "untwist" paths, so that any given element cannot be too far away from the identity.

3.6 Infinite Diameter of the group of Hamiltonian diffeomorphisms

In this section we will state and prove a result which is similar to Theorem 3.18 of Eliashberg and Ratiu on the diameter of the group of volume-preserving diffeomorphisms of a two-dimensional manifold.

To get started, let M^{2n} be a compact manifold with a symplectic form ω which is exact: $\omega = d\alpha$ for some one-form α on M. Note, that in this situation, the manifold M has to have a non-empty boundary since otherwise we would have

$$\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-1}) = 0\,,$$

which is a contradiction.

Definition 3.20 Let the group $Ham(M)$ be the group of Hamiltonian diffeomorphisms on M, i.e. the group of time-one flows of Hamiltonian vector fields on M. The Lie algebra of the group $Ham(M)$ is the Lie algebra of Hamiltonian vector fields on M.

The main goal of this section is to prove the following theorem of Eliashberg and Ratiu:

Theorem 3.21 *The L_p-diameter $\mathrm{diam}_p(Ham(M))$ of the group of Hamiltonian diffeomorphisms is infinite in any right-invariant L_p metric.*

Note that the strongest result is in the L^1-norm since

$$l_p(\,.\,) \geq C_{M,p} \cdot l_1(\,.\,)\,.$$

To give an example of a Hamiltonian diffeomorphism which is far away from the identity, let M be the unit disk in $B^2 \subset \mathbb{R}^2$ with the standard volume form. Consider the quadratic potential $H(x,y) = \frac{r}{2}(x^2+y^2-1)^2$. The time-one map of the corresponding Hamiltonian vector field is simply a rotation around the origin by the angle $2\pi r$. So for big r, the corresponding diffeomorphism is far away from the identity since two-dimensionality prevents us to "unwind" the rotation by a short path from the identity.

The rest of this section is devoted to the proof of theorem 3.21 in the special case of the $2n$-dimensional unit ball $B^{2n} \subset \mathbb{R}^{2n}$ with the standard symplectic form ω. Before we get started, we need some preparations:

Proposition / Definition 3.22 *Given a one-form α on B^{2n} and a Hamiltonian diffeomorphism $\varphi \in Ham(B^{2n})$ fixing the boundary ∂B^{2n}, there exists a unique function $h : B^{2n} \to \mathbb{R}$ vanishing with its gradient on ∂B^{2n} such that $dh = \alpha - \varphi^*\alpha$.*

Fix a one-form α such that $\omega = d\alpha$. The Calabi invariant *of a Hamiltonian diffeomorphism φ, defined by*

$$Cal(\varphi) = \frac{1}{n+1}\int_B h\omega^n$$

does not depend on the choice of α.

PROOF. First, we show the existence of the function h. To this end, observe that the one-form $\alpha - \varphi^*\alpha$ is closed. Indeed, we have

$$d(\alpha - \varphi^*\alpha) = d\alpha - \varphi^* d\alpha = \omega - \varphi^*\omega = 0$$

since φ fixes the symplectic form ω. Now, since α is closed, it has to be exact so that there exists a function h on B^{2n} such that $dh = \alpha$. The fact that φ restricted to the boundary of B^{2n} is the identity gives the condition $\nabla h|_{\partial B^{2n}} = 0$, which fixes h uniquely.

To see that the Calabi invariant is indeed invariant of the choice of α, suppose that we have another $\widetilde{\alpha} = \alpha + df$. Then we get $\widetilde{h} = h + (f - \varphi^* f)$ But we have

$$\int_{B^{2n}} (\varphi^* f)\omega^n = \int_{B^{2n}} f\omega^n$$

since φ fixes ω. □

Remark 3.23 The Calabi-invariant $Cal : Ham(B^{2n}) \to \mathbb{R}$ is a homomorphism. Indeed, if $\varphi = \varphi_2 \circ \varphi_1$, then we have

$$dh = \alpha - \varphi^*\alpha = \alpha - \varphi_2^*\alpha + \varphi_2^*\alpha - \varphi_2^* \circ \varphi_1^*\alpha = dh_2 - \varphi_1^* dh_1$$

so that we get

$$\int_{B^{2n}} h\omega^n = \int_{B^{2n}} h_1\omega^n + \int_{B^{2n}} h_2\omega^n .$$

There is an alternative definition of the Calabi invariant $Cal(\varphi)$: Let φ_t be a path in $Ham(B^{2n})$ such that $\varphi_0 = id$, and $\varphi_T = \varphi$. Then φ_t is the time-one flow of a Hamiltonian function $H_t : B^{2n} \times [0, T] \to \mathbb{R}$ (see figure 3.6).

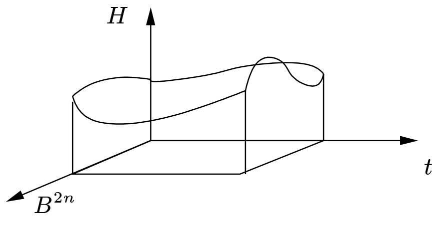

Figure 3.6: Graph of the function $H_t : B^{2n} \times [0, T] \to \mathbb{R}$

Theorem 3.24 *The Calabi-invariant of a Hamiltonian diffeomorphism φ of B^{2n} can be written as*

$$Cal(\varphi) = \int_0^T \left(\int_{B^{2n}} H_t\omega^n \right) dt .$$

In particular, the integral $\int_0^T \left(\int_{B^{2n}} H_t\omega^n \right) dt$ does not depend on the chice of the path φ_t.

PROOF. Let $\varphi_t \in Ham(B^{2n})$ be a path such that $\varphi_0 = id$ and $\varphi_T = \varphi$. We have to show that

$$\int_B h\omega^n = (n+1)\int_0^T \left(\int_{B^{2n}} H_t\omega^n\right) dt \tag{3.37}$$

Due to the homomorphism property of the Calabi invariant, it is enough to show this equality for infinitesimal short time. That is, it is enough differentiate both sides with respect to t at $t = 0$ and observe that the results coincide. That is, we need to prove the identity

$$\int_B \frac{d}{dt}|_{t=0} h\omega^n = (n+1)\int_{B^{2n}} H_t\omega^n\,.$$

Let ξ denote the Hamiltonian vector field $\xi = \frac{d}{dt}|_{t=0}\varphi_t$. Then, by the definition of the function h in Proposition 3.22, we get

$$d(\frac{d}{dt}h) = -L_\xi\alpha\,, \tag{3.38}$$

where $L_\xi\alpha$ denotes the Lie derivative of the one-form α with respect to the vector field ξ. We can apply Cartan's formula to find

$$-L_\xi\alpha = -d\iota_\xi\alpha - \iota_\xi d\alpha = d(H - \iota_\xi\alpha)\,, \tag{3.39}$$

since H is the Hamiltonian function corresponding to the vector field ξ, i.e. $-dH = \iota_\xi\omega = \iota_\xi d\alpha$. Equations (3.38) and (3.39) allows to reconstruct $\frac{d}{dt}h$:

$$\frac{d}{dt}h = H - \iota_\xi\alpha + const$$

for some constant $const \in \mathbb{R}$. The boundary conditions for h force the constant to be 0 so that we get

$$\frac{d}{dt}h = H - \iota_\xi\alpha\,.$$

Now, Theorem 3.24 follows from the next lemma: □

Lemma 3.25

$$-\int_{B^{2n}} \iota_\xi\alpha \wedge \omega^n = n\int_{B^{2n}} H\omega^n$$

PROOF. We can calculate

$$\begin{aligned}
-\int_{B^{2n}} \iota_\xi\alpha \wedge \omega^n =& -\int_{B^{2n}} \alpha \wedge \iota_\xi(\omega^n) = -n\int_{B^{2n}} \alpha \wedge \iota_\xi(\omega) \wedge \omega^{n-1} \\
=& n\int_{B^{2n}} \alpha \wedge dH \wedge \omega^{n-1} = n\int_{B^{2n}} d\alpha \wedge dH \wedge \omega^{n-1} - n\int_{\partial B^{2n}} \alpha \wedge H \wedge \omega^{n-1} \\
=& n\int_{B^{2n}} d\alpha \wedge dH \wedge \omega^{n-1}\,.
\end{aligned}$$

The boundary term vanishes due to the boundary conditions on the function H. This finishes the proof of Lemma 3.25. Theorem 3.24 follows from

$$\int_{B^{2n}} \left(\frac{d}{dt}|_{t=0}h\right)\omega^n = \int_{B^{2n}} (H - \iota_\xi\alpha) = (n+1)\int_{B^{2n}} H\omega^n\,.$$

□

Now we are ready to give a proof of Theorem 3.21 in the case $M = B^{2n}$, the unit ball in $\mathbb{R}^{2n}$.

PROOF. Let ω denote the standard symplectic structure on $\mathbb{R}^{2n}$, and let $\mu = \omega^n$ be the corresponding volume form. The L_2-length of a path φ_t joining the identity $\varphi_0 = id$ and an arbitrary Hamiltonian diffeomorphism $\varphi_1 = \varphi$ is given by

$$\begin{aligned} l_2(\varphi_t) &= \int_0^1 \left(\int_{B^{2n}} \|\frac{d\varphi_t}{dt}\|^2\mu\right)^{\frac{1}{2}} dt \\ &= \int_0^1 \left(\int_{B^{2n}} \|\nabla H_t(\varphi_t)\|^2\mu\right)^{\frac{1}{2}} dt \\ &= \int_0^1 \|\nabla H_t\|_{L_2} dt \end{aligned}$$

The estimate now follows from the Poincaré and Schwarz inequalities:

$$\begin{aligned} l_2(\varphi_t) = \int_0^1 \|\nabla H_t\|_{L_2}dt &\geq c_1 \int_0^1 \|H_t\|_{L_2} \geq c_2 dt \int_0^1 \|H_t\|_{L_1}dt \\ \geq& c_2 \int_0^1 \int_{B^{2n}} H_t\mu dt = c_2 \cdot Cal(\phi)\,. \end{aligned}$$

Now, since $Cal : Ham(B^{2n}) \to \mathbb{R}$ is a non-trivial homomorphism, one can find a Hamiltonian diffeomorphism of B^{2n} with arbitrarily large Calabi invariant. □

3.7 The symplectic structure on the space of curves

Let us fix the standard volume form $\mu = d^3x$ on $\mathbb{R}^3$. Consider an immersion of the circle S^1 into $\mathbb{R}^3$, i.e. a knot in $\mathbb{R}^3$. Throughout this section, we fix a parametrization of the circle S^1 so that the knot gives us a parametrized closed curve γ in $\mathbb{R}^3$. We can use this curve to define a functional on $S\mathrm{Vect}(\mathbb{R}^3)$, i.e. an element of the dual space $S\mathrm{Vect}(\mathbb{R}^3)^*$ as follows: Fix a compact oriented surface $S \subset \mathbb{R}^3$ such that the boundary of S coincides with γ: $\partial S = \gamma$. Such a surface is called a Seifert surface for γ. Then to any vector field $\xi \in \mathrm{Vect}(\mathbb{R}^3)$, we can associate its Flux through the surface S. If the vector field ξ is divergence free, the Flux does not depend on the choice of the surface S so that the curve γ defines a functional on the Lie algebra $S\mathrm{Vect}(\mathbb{R}^3)$:

$$\langle\gamma, \xi\rangle = Flux(\xi)|_S = \int_S \iota_\xi\mu\,. \tag{3.40}$$

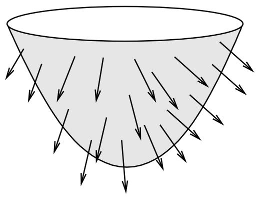

Figure 3.7: Seifert surface for the unknot

Remark 3.26 Obviously, the functional on the Lie algebra $S\mathrm{Vect}(\mathbb{R}^3)$ defined by the curve γ does not lie in the smooth part of the dual of $S\mathrm{Vect}(\mathbb{R}^3)$. But we can still associate a one form u on $\mathbb{R}^3$ to γ which however will not be a smooth one-form but a "δ-type" one-form which is supported on the surface S. Roughly speaking, the one form u integrated over a closed curve σ in $\mathbb{R}^3$ counts the intersections of σ with the surface S with appropriate multiplicities. As in the smooth case, the one-form u is not defined uniquely, but only up to an exact one form so that the curve γ determines an element $[u] \in \overline{\Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3)}$. Here $\overline{\Omega^1(\mathbb{R}^3)}$ denotes the space of δ-type one-forms which can be viewed as a completion of the space of usual one-forms and $\overline{\Omega^0(\mathbb{R}^3)}$ denotes the space of δ-functions on $\mathbb{R}^3$.

Exercise 3.7.1 Check the well-definedness of $[u]$.

Remark 3.27 As we have seen above, a knot γ in $\mathbb{R}^3$ can be seen as an element of the (full) dual space of the Lie algebra of divergence free vector fields on $\mathbb{R}^3$. From this viewpoint, the coadjoint orbit through γ is the equivalence class of γ. This leads to the somewhat rough statement that the knot-invariants are among the coadjoint invariants of the Lie group $SDiff(\mathbb{R}^3)$.

Definition 3.28 The *Marsden-Weinstein symplectic structure* ω on the space of unparametrized immersed curves in $\mathbb{R}^3$ is defined as follows: Let γ be a closed immersed curve in $\mathbb{R}^3$, and suppose that we have fixed a parametrization θ of the circle S^1. Let α and β two tangent vectors at γ. That is, α and β are vectorfields along γ which are not tangent to γ. Then we can define a two form ω on the space of immersed curves in $\mathbb{R}^3$, which, evaluated on two tangent vectors α and β at γ is defined to be the volume of the wedge spanned by $\alpha(\theta)$, $\beta(\theta)$, and $\gamma'(\theta)$ integrated over θ (see figure 3.7):

$$\omega(\alpha, \beta) = \int \mathrm{vol}(\alpha(\theta), \beta(\theta), \gamma'(\theta))d\theta = \int \iota_\alpha \iota_\beta \mu\,.$$

Proposition 3.29 *The Marsden-Weinstein symplectic structure at a curve γ coincides with the natural symplectic structure on the coadjoint orbit of the group $SDiff(\mathbb{R}^3)$ through the curve γ viewed as an element of the dual space of $S\mathrm{Vect}(\mathbb{R}^3)$.*

Exercise 3.7.2 Give a proof of this proposition.

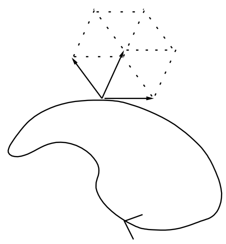

Figure 3.8: ω evaluated on two tangent vectors at the point γ

Remark 3.30 The same construction works for any 3-dimensional Riemannian manifold. In higher dimensions, a similar construction allows to define a symplectic structure on the space of immersed submanifolds of codimension 2.

In three dimensions, there exists a natural "energy" function H on the space of immersed paths which assigns to a path γ its length:

$$H(\gamma) = \int \|\gamma'(\theta)\| d\theta \,.$$

Proposition 3.31 *The Hamiltonian equation corresponding to the energy function H is given by the* binormal *or* filament equation

$$\gamma_t = \gamma' \times \gamma'' \,. \tag{3.41}$$

Here, γ' denotes the derivative $\frac{\partial}{\partial\theta}\gamma$.

Exercise 3.7.3 Give a proof of this proposition

Remark 3.32 Equation (3.41) is equivalent to the *Landau-Lifschitz* equation: Set $L = \gamma'$. Then equation (3.41) becomes

$$L_t = \gamma'_t = \gamma'' \times \gamma'' + \gamma' \times \gamma'' = \gamma' \times \gamma'' \,,$$

so that we get

$$L_t = L \times L''$$

which is the simplest form of the Landau-Lifschitz equation.

Remark 3.33 The Landau-Lifschitz equation has another interpretation as the Euler equation for a certain left-invariant metric on the loop group $L(SO(3)) = C^\infty(S^1, SO(3))$. To be more precise, let us identify the Lie algebra $L(\mathfrak{so}(3)) = C^\infty(S^1, \mathfrak{so}(3))$ with the smooth part of its dual via the pairing

$$\langle X, Y \rangle = \int_{S^1} \mathrm{tr}(X(\theta)Y(\theta)) d\theta \,.$$

Then we can define an inertia operator $A : L(\mathfrak{so}(3)) \to L(\mathfrak{so}(3))^*$ by prescribing its inverse:

$$A^{-1}(X) = X'' \,.$$

The corresponding left-invariant metric on the group $L(SO(3))$ is the so called H^{-1}-metric. Finally, the energy function on the dual space $L(\mathfrak{so}(3))^*$ is given by

$$E(X) = \int_{S^1} \mathrm{tr}(\partial^{-1}X, \partial^{-1}X)dt\,.$$

Exercise 3.7.4 Show that the Landau-Lifschitz equation is the Euler equation for the Energy E.

Remark 3.34 The Hasimoto transformation sends the binormal equation to the non-linear Schrödinger equation.

3.8 Bibliographical notes

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4 The group of pseudo differential operators

The Lie algebra of vector fields on the circle embeds naturally into the Lie algebra of differential operators on S^1, where the Lie bracket of two differential operators is given by their commutator: $[A, B] = A \circ B - B \circ A$. We have seen in section 2.1 that the Lie algebra of vector fields on the circle admits an essentially unique central extension, the Virasoro algebra. Hence it is natural to ask, if the Lie algebra of Differential operators on S^1 also admits a central extension which, restricted to the subalgebra of vector fields gives the Virasoro algebra. Such an extension has been constructed by Kac and Peterson in in [44].

In this section, we will study the Lie algebra ψDO of *pseudo differential operators* on S^1 which contains the algebra DO of differential operators as a subalgebra. It turns out that the Lie algebra ψDO admits a central extension which, restricted to subalgebra DO gives back the Kac Peterson extension $\widehat{DO}$. From this point of view, the Lie algebra of pseudo differential operators is the most general of the three Lie algebras mentioned above. Maybe surprisingly, the construction of the central extension of ψDO is rather easy and resembles the construction of the affine Lie algebras in section 1.

There does not seem to be a Lie group attached to the Lie algebra of pseudo differential operators or to its central extension $\widehat{\psi DO}$. However, after extending the Lie algebra $\widehat{\psi DO}$ further by adding an outer derivation, we can find a subalgebra of the big Lie algebra to which there is a Lie group attached, the group $\widetilde{G}_{int}$ of *α-pseudo differential operators*. We shall describe this group in section 4.4. It turns out, that the corresponding Lie algebra is a Lie *bi-algebra* which in turn implies that the group $\widetilde{G}_{int}$ is a *Poisson Lie group*, i.e. that it admits a Poisson structure which is compatible with the group structure. This Poisson structure is the so called Gelfand Dickey structure. In section 4.5 we describe a hierarchy of Hamiltonian attached to this Poisson structure.

4.1 The Lie algebra of pseudo differential operators

Definition 4.1 The associative algebra ψDO of *pseudo differential operators* consists of formal series

$$A(\theta, \partial) = \sum_{k=-\infty}^{n} a_k(\theta)\partial^k$$

with $a_k \in C^\infty(S^1)$. The multiplication in ψDO is given by the Leibnitz rule for multiplication of differential operators:

$$\partial^k \circ f\partial^n = f\partial^{n+k} + \sum_{i=1}^{\infty} \binom{k}{i} f^{(i)}\partial^{n+k-i}, \tag{4.42}$$

where

$$\binom{k}{i} = \frac{k(k-1)(k-2)\cdots(k-i+1)}{i!} \tag{4.43}$$

denotes the binomial coefficient. The Lie algebra structure on ψDO is given by the usual commutator $[A, B] = A \circ B - B \circ A$.

Remark 4.2 For $k > 0$, equation (4.42) should be interpreted as the Leibnitz rule for differential operators: For example, in the case $k = 1$, we have $\partial(fg) = f\partial g + f'g$. In particular, the Lie algebra

$$DO = \{\sum_{k=0}^{n} a_k(\theta)\partial^k \mid a_k \in C^\infty(S^1)\}$$

of differential operators on S^1 is a subalgebra of ψDO. Similarly, we can define the subalgebra $\mathfrak{g}_{int}$ if integral differential symbols to be

$$\mathfrak{g}_{int} = \{ \sum_{k=-\infty}^{-1} a_k(\theta)\partial^k \mid a_k \in C^\infty(S^1)\}\,.$$

As a vector space, the Lie algebra ψDO admits a direct sum decomposition $\psi DO = DO \oplus \mathfrak{g}_{int}$.

Remark 4.3 For $k < 0$, the binomial coefficients (4.43) are never zero so that the sum on the left hand side of equation (4.42) is infinite. For example, for $k = -1$, we get

$$\partial^{-1} \circ f = f\partial^{-1} - f'\partial^{-1} + f''\partial^{-3} - \ldots,$$

which makes sense, since we now have $\partial \circ (\partial^{-1} \circ f) = \partial^{-1} \circ (\partial \circ f) = f$ so that ∂ and ∂^{-1} are indeed inverse to each other.

One of the nice properties of the Lie algebra of pseudo differential operators is that admits an analogue of the trace of usual matrices:

Definition 4.4 The trace of a pseudo differential operator $A = \sum a_i(\theta)\partial^i$ is defined by

$$\mathrm{tr}\left(\sum_{i=-\infty}^{n} a_i(\theta)\partial^i\right) = \int_{S^1} a_{-1}(\theta)d\theta\,.$$

Exercise 4.1.1 Show that $tr[A, B] = 0$ for all a, $B \in \psi DO$. This implies that the bilinear form $(A, B) = tr(A \circ B)$ is invariant, i.e. that it satisfies $([A, B], C) = (A, [B, C])$.

Exercise 4.1.2 Show that the bilinear form $(A, B) = tr(A \circ B)$ is non-degenerate on ψDO and that the subalgebras DO and $\mathfrak{g}_{int}$ are isotropic subspaces of ψDO.

4.2 Outer derivations and central extensions of ψDO

Suppose, we want to find the logarithm of the differential operator ∂. To do this heuristically, note that in the definition of the product of two pseudo differential operators in formula (4.42) we can replace the integer k by an arbitrary real number α so that we get

$$[\partial^\alpha, f\partial^n] = \sum_{i=1}^{\infty} \binom{\alpha}{i} f^{(i)}\partial^{n+\alpha-i}\,, \tag{4.44}$$

where the binomial coefficients on the right hand side are defined exactly as in the integer case. Formally, we can write

$$\frac{d}{d\alpha}|_{\alpha=0}\partial^\alpha = (\log\partial)\partial^\alpha|_{\alpha=0} = \log\alpha\,.$$

Hence, differentiating both sides of equation (4.44) at $\alpha = 0$ gives

$$[\log\partial, f\partial^n] = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}f^{(i)}\partial^{n-i}. \tag{4.45}$$

We use this formula to define the map $[\log\partial, .] : \psi DO \to \psi DO$.

Proposition / Definition 4.5 *The linear map* $[\log\partial, .] : \psi DO \to \psi DO$ *which is given by formula (4.45) is an outer derivation of the Lie algebra* ψDO *of pseudo differential operators.*

It is a direct calculation to check that the map $[\log\partial, .]$ is indeed a derivation of the Lie algebra ψDO. The fact that it is outer will be clear after Proposition 4.6 (see Remark 4.9). The derivation $[\log\partial, .]$ can be used to define a central extension of the Lie algebra ψDO.

Exercise 4.2.1 Show that the map $c : \psi DO \times \psi DO \to \mathbb{R}$ which is given by

$$A, B \mapsto tr([\log\partial, A]\circ B)$$

is a Lie algebra two-cocycle on ψDO.

It turns out that the cocycle c on the Lie algebra ψDO generalizes some of the Lie algebra cocycles we have met before:

Proposition 4.6 *The restriction of the cocycle* c *to the subalgebra* $\mathrm{Vect}(S^1) \subset \psi DO$ *gives the Gelfand-Fuchs cocycle, i.e.*

$$c(f\partial, g\partial) = \frac{1}{6}\int_{S^1} g'f''$$

for all functions f *and* g *on* S^1.

PROOF. This is a direct calculation:

$$\begin{aligned}
c(f\partial, g\partial) =& tr([\log\partial, f\partial]\circ g\partial) = tr(g\partial\circ[\log\partial, f\partial]) \\
=& tr\left(g\partial\circ\sum_{i=0}^{\infty}\frac{(-1)^{i+1}}{i}f^{(i)}\partial^{1-i}\right) \\
=& tr\left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}g(f^{(i+1)}\partial^{1-i} + f^{(i)}\partial^{1-i+1})\right)
\end{aligned}$$

The coefficient of ∂^{-1} in the last sum is $-\frac{1}{2}gf''' + \frac{1}{3}gf''' = -\frac{1}{6}gf'''$. So by definition of the trace, we have

$$c(f\partial, g\partial) = -\int_{S^1} \frac{1}{6} g f''' = \frac{1}{6}\int_{S^1} g' f'',$$

which is the Gelfand-Fuchs cocycle. $\square$

Corollary 4.7 *The restriction of the cocycle c to the subalgebra of differential operators $DO \subset \psi DO$ is the Kac-Peterson cocycle.*

Exercise 4.2.2 Show that the restriction of the cocycle c to the Abelian subalgebra $C^\infty(S^1)$ of smooth functions on the circle gives the cocycle defining the infinite-dimensional Heisenberg algebra. That is

$$c(f, g) = \int_{S^1} f' g$$

for all smooth functions f and g on S^1.

Remark 4.8 One can easily generalize the cocycle c to the Lie algebra of $\mathfrak{gl}_n$-valued pseudo differential operators. Then the restriction to the subalgebra of Lie algebra valued functions gives back the cocycle defining the affine Lie algebra $\widetilde{\mathfrak{gl}}_n$.

Remark 4.9 Proposition 4.6 shows in particular that the cocycle c cannot be trivial. Indeed, otherwise it would have to be trivial restricted to all subalgebras of ψDO which, by Proposition 4.6, can not be true. This also implies that the derivation $[\log, .]$ is outer since otherwise the cocycle c would be trivial.

Remark 4.10 Note the analogy between the construction of the cocycle c on ψDO and the cocycle defining the central extension of the loop algebra $L\mathfrak{g}$ discussed in section 1: The cocycle on $L\mathfrak{g}$ was defined by

$$\omega(X, Y) = \int_{S^1} tr(X \frac{d}{d\theta} Y) = \langle [\frac{d}{d\theta}, X], Y \rangle \,.$$

So the outer derivation $\frac{d}{dt}$ of the loop algebra $L\mathfrak{g}$ is "replaced" by the outer derivation $\log \partial$ of the Lie algebra ψDO.

Remark 4.11 Obviously, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is not defined on the circle. Nevertheless, the commutator $[x, A]$ with any element $A \in \psi DO$ is a pseudo differential operator on S^1. This allows to define a 2-cocycle c' on the Lie algebra ψDO by

$$c'(A, B) = tr([x, A] \circ B) \,.$$

It has been shown by B. Feigin (cite...) that the cocycles c and c' span the second cohomology of the Lie algebra ψDO. Note however that the restriction of the cocycle c' to the subalgebra $DO \subset \psi DO$ vanishes: We have $c'(A, B) = 0$ for all differential operators A and B. Similarly, the cocycle c restricted to the subalgebra $\mathfrak{g}_{int}$ of integral pseudo differential operators vanishes.

This symmetry between the cocycles c and c' becomes even more apparent if we consider the Lie algebra of pseudo differential operators on $\mathbb{C}^*$. If we identify z with $e^{i\theta}$, the cocycle c' takes the form

$$c'(A,B) = tr([\log z, A] \circ B]\,.$$

4.3 The doubly extended Lie algebra $\widetilde{\psi DO}$ and Manin triples

The cocycle c discussed in the last section defines a central extension $\widehat{\psi DO}$ of the Lie algebra ψDO. In this section we are concerned with a slightly bigger Lie algebra which is obtained by extending $\widehat{\psi DO}$ by the derivation $\log \partial$.

Definition 4.12 Extend the derivation $[\log \partial, \,.\,]$ to a derivation on the Lie algebra $\widehat{\psi DO}$ by letting it act trivially on the center. The "double extension" of the Lie algebra ψDO is given as the semi-direct product

$$\widetilde{\psi DO} = \widehat{\psi DO} \rtimes \mathbb{R} \log \partial\,.$$

Remark 4.13 As a vector space, the Lie algebra $\widetilde{\psi DO}$ has a direct sum decomposition $\widetilde{\psi DO} = \widehat{DO} \oplus \widetilde{\mathfrak{g}}_{int}$, where $\widetilde{\mathfrak{g}}_{int}$ denotes the Lie algebra of integral differential operators $\mathfrak{g}_{int}$ extended by the "cocentral" direction $\log \partial$, and $\widehat{DO}$ denotes the centrally extended Lie algebra of differential operators on S^1 (see figure 4.13). $\widehat{DO}$ (see figure 4.13).

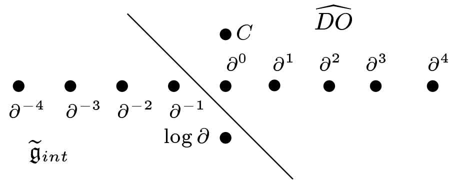


Figure 4.9: $\widehat{DO}$ and $\widetilde{\mathfrak{g}}_{int}$ as subalgebras of the Lie algebra $\widetilde{\psi DO}$.

The advantage of the doubly extended Lie algebra $\widetilde{\psi DO}$ is that it admits a non-degenerate invariant bilinear form (the "Killing form"):

Exercise 4.3.1 Show that the bilinear form $(\;,\;)$ on the Lie algebra $\widetilde{\psi DO}$ which is given by

$$(A + b_1 C + \alpha_1 \log \partial, B + b_2 C + \alpha_2 \log \partial) = tr(A \circ B) + b_1\alpha_1 + b_2\alpha_2\,. \tag{4.46}$$

is non-degenerate and invariant. Show that the subalgebras $\widehat{DO}$ and $\widetilde{\mathfrak{g}}_{int}$ are isotropic subspaces of $\widetilde{\psi DO}$.

Corollary 4.14 *The bilinear form $(\;,\;)$ on $\widetilde{\psi DO}$ identifies $\widetilde{\mathfrak{g}}_{int}$ with the (smooth part of) the dual of $\widehat{DO}$ and vice versa.*

A more elaborate form of stating these observations is to say that the Lie algebras $\widetilde{\psi DO}$, $\widehat{DO}$, and $\widetilde{\mathfrak{g}}_{int}$ form a Manin triple:

Definition 4.15 Three Lie algebras $\overline{\mathfrak{g}}$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ form a *Manin triple* if the following two conditions are satisfied:

1. The Lie algebras $\mathfrak{g}_\pm$ are nontrivial subalgebras of $\overline{\mathfrak{g}}$ such that $\overline{\mathfrak{g}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$.

2. There exists a non-degenerate invariant bilinear form on $\overline{\mathfrak{g}}$ such that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces.

The importance of Manin triples comes from the fact that there is a one to one correspondence between Manin triples and Lie bi-algebras:

Definition 4.16 A Lie algebra $\mathfrak{g}$ is called a Lie *bi-algebra* if its dual space $\mathfrak{g}^*$ comes equipped with a Lie algebra structure $[\ ,\ ]^*$ such that the map $\alpha : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ which is dual to the commutator map $[\ ,\ ]^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies

$$\mathrm{ad}_X \alpha(Y) = \alpha([X, Y])\,, \tag{4.47}$$

where ad denotes the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. [1]

Theorem 4.17 *Let $(\overline{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple. Then the Lie algebra $\mathfrak{g}_-$ is naturally identified with $\mathfrak{g}_+^*$ and gives $\mathfrak{g}_+$ the structure of a Lie bi-algebra. On the other hand, for any Lie bi-algebra $\mathfrak{g}$ there is a natural Lie algebra structure on $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ such that $(\overline{\mathfrak{g}}, \mathfrak{g}, \mathfrak{g}^*)$ forms a Manin triple.*

Given a Lie bi-algebra $\mathfrak{g}$, we can define a commutator on $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ by

$$[X + a, Y + b]_{\overline{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} - \mathrm{ad}_b^*(X) + \mathrm{ad}_a^*(Y) + [a, b]_{\mathfrak{g}^*} + \mathrm{ad}_X^*(b) - \mathrm{ad}_Y^*(a)$$

for all $X, Y \in \mathfrak{g}$ and $a, b \in \mathfrak{g}^*$. The condition that this commutator satisfies the Jacobi identity is equivalent to the condition (4.47) on the map α. For a full proof of Theorem 4.17 we refer the reader to e.g. [86].

Now we can restate our observations about the Lie algebra $\widetilde{\mathfrak{g}}_{int}$ in this more elaborate language.

Corollary 4.18 *The Lie algebras $\widehat{\psi DO}$, $\widehat{DO}$ and $\widetilde{\mathfrak{g}}_{int}$ form a Manin triple. In particular, $\widetilde{\mathfrak{g}}_{int}$ is a Lie bi-algebra.*

4.4 The Lie group of α-pseudo differential operators

The Lie algebra DO of differential operators on the circle and its central extension $\widehat{DO}$ do not seem to have Lie groups attached to them. However, the dual part $\widetilde{\mathfrak{g}}_{int}$ of the Manin triple $(\widehat{\psi DO}, \widehat{DO}, \widetilde{\mathfrak{g}}_{int})$, i.e. the Lie algebra of integral pseudo differential operators comes with a group attached to it, the so called group of α-differential operators $\widetilde{G}_{int}$. In this section we will describe this group and a natural Poisson structure on it which comes from the bi-algebra structure of its Lie algebra. In the final section of this chapter we will see how this Poisson structure gives rise to interesting Hamiltonian hierarchies.

[1] The condition on the map α states that α should be a Lie algebra 1-cocycle values in $\mathfrak{g}\wedge\mathfrak{g}$ corresponding to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

Definition 4.19 Define the group of *α-pseudo differential operators* to be the set of monic pseudo differential operators of degree $\alpha \in \mathbb{R}$:

$$\widetilde{G}_{int} = \{\partial^\alpha (1 + \sum_{i=-\infty}^{-1} a_i(\theta)\partial^i) \mid \alpha \in \mathbb{R},\ a_i \in C^\infty(S^1)\}\,.$$

The product on $\widetilde{G}_{int}$ is defined by

$$\partial^\alpha \circ f = f\partial^\alpha + \sum_{i=1}^{\infty} \binom{\alpha}{i} f^{(i)} \partial^{\alpha - i}\,.$$

Exercise 4.4.1 Check that $\widetilde{G}_{int}$ is indeed a group, i.e. that it is closed under taking inverses.

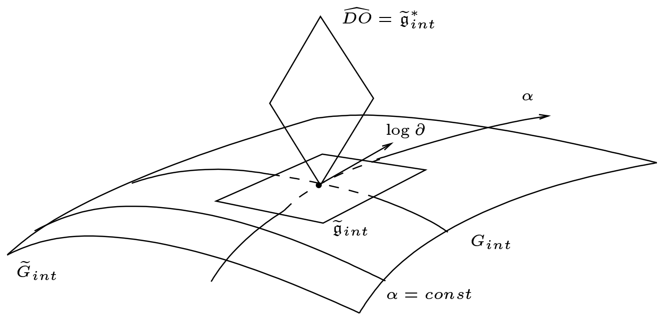


Figure 4.10: The group $\widetilde{G}_{int}$ of α-pseudo differential operators and its Lie algebra $\widetilde{\mathfrak{g}}_{int}$

Remark 4.20 To define a topology on $\widetilde{G}_{int}$, observe that the group $\widetilde{G}_{int}$ can be written as a semi-direct product

$$\widetilde{G}_{int} = G_{int} \rtimes \{\partial^\alpha \mid \alpha \in \mathbb{R}\}\,,$$

where G_{int} denotes the group of monic pseudo differential operators of degree 0, and $\{\partial^\alpha \mid \alpha \in \mathbb{R}\}$ is a one-parameter subgroup of $\widetilde{G}_{int}$. The group G_{int} is the inverse limit of Fréchet vector spaces (each one being the direct sum of finitely many $C^\infty(S^1)$'s with the usual C^∞ topology), hence it can be endowed with the inverse limit topology. In the α-direction we use the usual topology on $\mathbb{R}$. Finally we give $\widetilde{G}_{int}$ the direct product topology of G_{int} and the α-direction.

Remark 4.21 Suppose, there exists an exponential map $\exp : \widetilde{g}_{int} \to \widetilde{G}_{int}$. Let $A \in \mathfrak{g}_{int}$ be a pseudo differential operator. Then, if it existed, the element $P_t = \exp(t(a \log \partial + A))$ should satisfy the equation

$$\left(\frac{d}{dt}\right) \circ \left(P_t \circ P_t^{-1}\right) = a \log \partial + A\,. \tag{4.48}$$

It turns out that in the case of the Lie algebra $\widetilde{\mathfrak{g}}_{int}$, this equation can be used to define the one-parameter subgroup P_t and hence the exponential map. This is the content of the following theorem which we will prove in section 4.6.

Theorem 4.22 *The exponential map* $\exp : \widetilde{\mathfrak{g}}_{int} \to \widetilde{G}_{int}$ *which is defined by* $a \log \partial + A \mapsto P_1$*, where* P_1 *is defined by equation (4.48) is well defined and bijective.*

Remark 4.23 Since the exponential map $\exp : \widetilde{\mathfrak{g}}_{int} \to \widetilde{G}_{int}$ is bijective, the group G_{int} can be viewed as an infinite dimensional analogue of the finite dimensional unipotent group.

The fact that $\widetilde{\mathfrak{g}}_{int}$ is a Lie bi-algebra implies that the corresponding Lie group $\widetilde{G}_{int}$ is a Poisson Lie group:

Definition 4.24 A *Poisson Lie group* $(G, \{\ ,\ \})$ is a Lie group G together with with a Poisson structure $\{\ ,\ \}$ on the underlying manifold such that the multiplication is a Poisson map $G \times G \to G$ with respect to the product Poisson structure on $G \times G$ and the inverse $G \to G^-$ is a Poisson map $G \to G^-$ where G^- denotes the Lie group G equipped with the negative of the Poisson structure of G.

Theorem 4.25 *For any simply connected Lie group* G*, there is a one-to-one correspondence between the the set of Lie bi-algebra structures on its Lie algebra* $\mathfrak{g}$ *and Poisson Lie group structures on* G*.*

For a proof of this theorem, see e.g. [68]. Here we shall only recall the construction of the Poisson bracket on G from the Lie bi-algebra structure on $\mathfrak{g}$ (which is all we will need later). Since the Lie algebra $\mathfrak{g}$ is a bi-algebra, it is part of a Manin triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \mathfrak{g}^*)$. Suppose there is a Lie group $\bar{G}$ corresponding to the Lie algebra $\bar{\mathfrak{g}}$. We can define a Poisson structure on the group G as follows: Let $\xi,\, \eta \in T_g^*G$ be cotangent vectors at a point $g \in G$. Let $A = r_g^*\xi$ and $B = r_g^*\eta$ denote their pull-backs to $T_e^*G = \mathfrak{g}^*$, and let $\bar{A}$ and $\bar{B}$ be arbitrary extensions of A and B to cotangent vectors of the larger group, i.e. $\bar{A}, \bar{B} \in T_e^*\bar{G} = \bar{\mathfrak{g}}^*$. Furthermore, for $\bar{A} \in \bar{\mathfrak{g}}$, let $(\bar{A})_+$ denote the restriction of $\bar{A}$ to $\mathfrak{g}^* \subset \bar{\mathfrak{g}}$. Then the bivector field Π which defines the Poisson structure on the group G evaluated on the co-vectors ξ and η is gigen by

$$\Pi_g(\xi, \eta) = \langle (\bar{A})_+, \bar{B} \rangle - \langle (Ad_g \bar{A})_+, Ad_g \bar{B} \rangle\,.$$

Remark 4.26 Note that we don't need the existence of a Lie group $\bar{G}$ corresponding to the Lie algebra $\bar{\mathfrak{g}}$. For the construction above to work it is enough to know how the Lie group G acts on the Lie algebras $\bar{\mathfrak{g}}$.

Corollary 4.27 *The group* $\widetilde{G}_{int}$ *carries a natural Poisson structure which makes it into a Poisson Lie group.*

As it turns out, the Poisson structure on the group $\widetilde{G}_{int}$ has a very concrete description:

Definition 4.28 The quadratic (or second) generalized Gelfand-Dickey Poisson structure $\{\ ,\ \}_{GD}$ on the group $\widetilde{G}_{int}$ is defined as follows:

1. For two functions $f, g : \widetilde{G}_{int} \to \mathbb{R}$, the value of $\{f, g\}_{GD}$ at some point $\partial^{\alpha_0}(1 + \ldots) \in \widetilde{G}_{int}$ depends only on the restriction of f and g to the $\alpha = \alpha_0$ hyperplane.

2. We can identify the tangent space of the $\alpha = \alpha_0$-hyperplane with the set $\partial^{\alpha_0} \circ (1+\mathfrak{g}_{int})$ and the cotangent space with $DO \circ \partial^{-\alpha_0}$. The pairing between $\partial^{\alpha_0} \circ \mathfrak{g}_{int}$ and $DO \circ \partial^{-\alpha_0}$ is given by $\langle A, \delta L\rangle = tr(A \circ \delta L)$, where $A \in DO \circ \partial^{-\alpha_0}$ and $\delta L \in \partial^{\alpha_0} \circ \mathfrak{g}_{int}$ is a tangent vector at L.

3. Now it is sufficient to define the Poisson bracket on linear functionals $A, B \in DO \circ \partial^{-\alpha_0}$. We define
$$\{A, B\}_{GD}(L) = \langle B, V_A^2(L)\rangle ,$$
where $V_A^2(L)$ is defined by
$$V_A^2(L) = (L \circ A)_+ \circ L - L \circ (A \circ L)_+ .$$
Here X_+ denotes the positive (differential) part of a pseudo differential operator $X \in \psi DO$.

Remark 4.29 Note that if the degree of L is α and the degree of A is $-\alpha$, then the degree of $V_A^2(L)$ differs from α at most by an integer. This is important since otherwise the pairing $\langle B, (V_A^2(L))\rangle = tr(B \circ (V_A^2(L))$ would vanish.

Theorem 4.30 *The Poisson structure in $\widetilde{G}_{int}$ which comes from the Lie bi-algebra structure of its Lie algebra as described in Corollary 4.27 coincides with the negative of the quadratic generalized Gelfand-Dickey structure.*

Proof. We first show that the function $\deg : \widetilde{G}_{int} \to \mathbb{R}$ which assigns to a pseudo differential operator the degree of its leading coefficient is a Casimir function on $\widetilde{G}_{int}$. Indeed, the differential $d\deg_L$ is an element of the cotangent space $T_L^*\widetilde{G}_{int}$ and its pull-back to the identity is given by $r_L^*(d\deg_L) = (0,1) \in \widehat{DO} = DO \oplus \mathbb{R}$. Let $f : \widetilde{G}_{int} \to \mathbb{R}$ be any differentable function. Then $r_L^*(df_L) = (A, a)$ for some $(A, a) \in \widehat{DO}$. Let $(\bar{A}, a, \alpha)$ be any extension of (A, a) to $\widetilde{\psi DO} = \psi DO \oplus \mathbb{R} \oplus \mathbb{R}\log\partial$. Here, $\bar{A}$ denotes an extension of A to $\psi DO = DO \oplus \mathfrak{g}_{int}$. Similarly, let $(bar0, 1, \beta)$ be any extension of $(0, 1)$ to $\widetilde{\psi DO}$. Then, since $(0, 1)$ is central, we have $L \circ (O, 1) \circ L^{-1} = (0, 1)$ and $L \circ (\bar{A}, a, \alpha) = (\bar{B}, b, \alpha)$ for some $\bar{B} \in \psi DO$ and $n \in \mathbb{R}$. In particular, α is invariant under conjugation with L. This shows that
$$\{\deg(L), f(L)\} = \langle (0,1)_+, (\bar{A}, a, \alpha)\rangle - \langle (L \circ (0,1) \circ L^{-1})_+, L \circ (\bar{A}, a, \alpha) \circ L^{-1}\rangle = 0 .$$

Since the function deg is a Casimir function, all $\deg(L) = const$-hyperplanes in $\widetilde{G}_{int}$ are Poisson submanifolds and we can restrict our attention to their tangent spaces. Let X and Y be cotangent vectors in L at the $\alpha = \alpha_0$ hyperplane. Then we view X and Y as elements of $\partial^{-\alpha_0}DO$. Let $\bar{X}$ and $\bar{Y}$ denote an arbitrary extension of X and Y to $\partial^{-\alpha_0}\psi DO$. By the definition of the poisson bracket on $\widetilde{G}_{int}$, we can write
$$\begin{aligned}\Pi_L(X, Y) =& \langle (r_L * \bar{X})_+, r_L * \bar{Y}\rangle - \langle (\mathrm{Ad}_L(r_L * \bar{X}))_+, \mathrm{Ad}_L r_L * \bar{Y}\rangle \\ =& \langle (\bar{X} \circ L)_+, \bar{Y} \circ L\rangle - \langle (L \circ \bar{X})_+, L \circ \bar{Y}\rangle \\ =& \langle L \circ (\bar{X} \circ L)_+ - (L \circ \bar{X})_+ \circ L, \bar{Y}\rangle .\end{aligned}$$

which is the negative of the quadratic generalized Gelfand Dickey bracket. □

Remark 4.31 The so called first (or linear) generalized Gelfand Dickey structure can be defined on the $\alpha \in \mathbb{Z}$-hyperplanes in $\widetilde{G}_{int}$ by setting

$$V_A^1(L) = ([L, A])_+ - [L, (A)_+] \,. \tag{4.49}$$

It turns out that the first and the second generalized Gelfand Dickey structures are compatible: Indeed, we have

$$V_A^2(L + \lambda\partial^0) = V_A^2(L) + \lambda V_A^1(L)$$

so that $V_A^2(L) + \lambda V_A^1(L)$ is the second Gelfand Dickey structure shifted by $\lambda\partial^0$.

Usually, for example when studying the n-KdV equation, the operator L is taken to be a pure differential operator $L = \partial^n + u_1\partial^{n-1} + \ldots + u_1$ of degree $n \geq 0$. Linear functionals on such differential operators are given by pseudo differential operators A of degree $\deg(A) \leq -1$ via the paring $\langle A, L\rangle = tr(A \circ L)$. Since the degree of A is negative, we have $(A)_+ = 0$ so that the term $[L, (A)_+]$ does not appear in equation (4.49). Thus in this case, the first generalized Gelfand Dickey structure reduces to the usual first Gelfand Dickey structure which is defined by $V_A(L) = ([L, A])_+$. Note that this Poisson structure is nothing else but the usual Lie-Poisson structure on $\widetilde{\mathfrak{g}}_{int} \cong \widehat{DO}^*$.

4.5 Integrable hierarchies on $\widetilde{G}_{int}$

We begin by noting that for every element $L \in \widetilde{G}_{int}$, there exists a one-parameter subgroup L^t passing through L. This comes from the fact that the exponential map $\widetilde{\mathfrak{g}}_{int} \to \widetilde{G}_{int}$ is surjective. Let us call the exponent α of the leading term of L the degree $\deg(L)$ of the operator L. It follows from the explicit description of the exponential map that $\deg(L^t) = t\deg(L)$ for all $t \in \mathbb{R}$.

Define functions $H_m : G_{int} \to \mathbb{R}$ for $m \in \mathbb{N}$ by

$$H_m(L) = tr(L^{\frac{m}{\alpha}})$$

for any L of degree α. As remarked above, the degree of the operator $L^{\frac{m}{\alpha}}$ equals m, so that none of the functions H_m vanishes.

Theorem 4.32 *1. The functions H_m are in involution with respect to the Poisson structure on G_{int}.*

2. Each H_m defines a Hamiltonian equation on G_{int} which is given by

$$\frac{\partial L}{\partial t_m} = [L, (L^{\frac{m}{\alpha}})_+] \,,$$

where, as above, $(L^{\frac{m}{\alpha}})_+$ denotes the pure differential part of the operator $L^{\frac{m}{\alpha}}$.

3. The Hamiltonian equation corresponding to H_m is well defined on any $\deg(L) =$ const-hyperplane in G_{int}.

Before we can prove this theorem, we need the following general lemma.

Lemma 4.33 *Let L be a pseudo differential operator of order $\alpha \neq 0$ with leading coefficient 1. Then there exists a pseudo differential operator Q of the form*

$$Q = u_0(\theta) + u_1(\theta)\partial^{-1} + u_2(\theta)\partial^{-2} + \dots$$

such that u_0 is nowhere vanishing and $L_0 = QLQ^{-1}$ has constant coefficients.

Exercise 4.5.1 Give a proof of this Lemma by induction on the order of the (undetermined) coefficients of Q and L_0. Note in particular that up to a multiplicative constant, the coefficient u_0 is determined by a_1.

PROOF. OF THEOREM 4.32 We start by proving part 2. Let us fix a differential operator L of degree α. Since we can identify the tangent space of $\widetilde{G}_{int}$ at the point L with the space $\partial^\alpha \circ \mathfrak{g}_{int}$, the differential of the function H_m at $L \in \widetilde{G}_{int}$ which is an element of the cotangent space $T^*_L G_{int}$ can be identified with a pseudo differential operator $X = \bar{X} \circ \partial^{-\alpha}$. Note that the operator $\bar{X}$ is well defined only up to addition of an element of $\mathfrak{g}_{int}$.

Let δL be an infinitesimal variation of L. Then, we get

$$dH_m(L)(\delta L) = tr(L + \delta L)^{\frac{m}{\alpha}} - tr L^{\frac{m}{\alpha}} = \langle \bar{X} \circ \partial^{-\alpha}, \delta L \rangle = tr(\bar{X} \circ \partial^{-\alpha} \circ \delta L)\,, \qquad (4.50)$$

where $\langle\ ,\ \rangle$ denotes the pairing between dual spaces.

Let us show that we can take $X = \frac{m}{\alpha} L^{\frac{m}{\alpha}-1}$. Indeed, if L has constant coefficients, then the differential part of $\bar{X}$ has constant coefficients as well. So to find X it is enough to consider δL's which also have constant coefficients. But in this case, L and δL commute so that the left hand side of equation (4.50) equals $\frac{m}{\alpha} L^{\frac{m}{\alpha}-1}\delta L$ up to terms of order $o(\delta L)$. Hence, if the operator L has constant coefficients, we get $X = \frac{m}{\alpha} L^{\frac{m}{\alpha}-1}$ as desired. Now suppose that L has arbitrary coefficients. Then by lemma 4.33, we can write $L = QL_0Q^{-1}$, where L_0 has constant coefficients and Q is of degree 0. The functions H_m are constant under conjugation by such Q, so the differential of H_m at L is the image of the differential of H_m at L_0 under this conjugation. So we find $X = \frac{m}{\alpha} L^{\frac{m}{\alpha}-1}$ in general. Finally, we can use the third part of the definition of the generalized Gelfand Dickey structure to obtain

$$\frac{\partial L}{\partial t} = \frac{m}{\alpha}\left((L^{\frac{m}{\alpha}})_+ L - L(L^{\frac{m}{\alpha}})_+\right) = [L, (L^{\frac{m}{\alpha}})_+]\,.$$

This is part two of the Theorem. The first part follows directly, since the flow for H_m changes an operator into a conjugate one. Thus, the ad-invariant function H_n stays invariant on the flow lines so that the commutator $\{H_m, H_n\}$ vanishes for all m, n. Finally, the flow of H_m does not change the degree, which is the assertion of the third part of Theorem 4.32. □

Remark 4.34 It follows from the definition of the Gelfand Dickey structure on the group $\widetilde{G}_{int}$ that the function deg : $G_{int} \to \mathbb{R}$ is a Casimir function, i.e. $\{\deg, f\}_{GD} = 0$ for all $f \in C^\infty(G_{int})$. So we can restrict the Hamiltonian systems defined by the Hamiltonian functions H_m to the $\deg(L) = const$-hyperplanes in $\widetilde{G}_{int}$. This gives a hierarchy of Hamiltonian systems on each $\deg(L) = const$-hyperplane. The following cases are of particular interest:

1. On the $\deg(L) = 1$ hyperplane one gets the the so called Kadomtev-Petashvili (KP) hierarchy

2. On the $\deg(L) = n$-hyperplanes one recovers the n-KdV hierarchy
$$\frac{\partial L}{\partial t_m} = [L, (L^{m/n})_+]$$

3. Since products and inverses of Poisson submanifolds of a Poisson Lie group are also Poisson, we can consider the "quotient" of the $\deg(L) = n$ and the $\deg(L) = m$-hyperplanes, i.e. the set $\{L_1 L_2^{-1} \mid \deg(L_1) = m,\ \deg(L_2) = n\}$. This gives the fractional KP hierarchy.

4. We can also consider pseudo differential operators with complex coefficients. Fix Hamiltonians $H_m(L) = i^m tr(L^m)$. Then for $m = 2$ and $L = \partial + \psi^* \partial^{-1} \psi$ the coefficient of ∂^{-1} in the Hamiltonian give the non-linear Schroedinger equation:
$$i\frac{\partial \psi}{\partial t} = \psi_{\theta\theta} + |\psi|^2 \psi^* \, .$$

4.6 Surjectivity of the exponential map

In this section we will prove Theorem 4.22 on the surjectivity of the exponential map $\exp : \widetilde{\mathfrak{g}}_{int} \to \widetilde{G}_{int}$. Take an element $P \in \widetilde{\mathfrak{g}}_{int}$. If P was an element of a finite-dimensional matrix Lie algebra, the exponential $L_s = \exp(sP)$ should satisfy the differential equation
$$\left(\frac{d}{ds} L_s\right) = P \circ L_s. \tag{4.51}$$

We will use this equation to define the element $\exp(P) \in \widetilde{G}_{int}$. Before we deal with the general case, let us look at the subgroup G_{int} of pseudo differential operators of degree 0:

Proposition 4.35 *The map* $\exp : \mathfrak{g}_{int} \to G_{int}$ *is well defined and bijective.*

Proof. Fix some element
$$P = \sum_{i=-\infty}^{-1} a_i(\theta) \partial^i \ \in \mathfrak{g}_{int} \tag{4.52}$$
and let us write
$$L_s = 1 + \sum_{k=-\infty}^{-1} u_k(\theta; s) \partial^k \, .$$

Suppose that P and L_s satisfy the differential equation (4.51). Comparing the coefficients of ∂^k for $k \geq 1$ in 4.51 gives us an infinite system of differential equations of "triangular form":
$$\frac{d}{ds} u_k(\theta; s) = \Phi_k \left(u_1(\theta; s), \dots u_{k-1}(\theta; s) \right) \tag{4.53}$$

Here, Φ_k is a polynomial in the $u_1, \ldots, u_{k_1}$ and their derivatives with coefficients in $C^\infty(S^1)$. For example, we have $\frac{\partial}{\partial s} u_1(\theta; s) = a_1(\theta)$ and $\frac{\partial}{\partial s} u_2(\theta; s) = a_2(\theta) + a_1(\theta)u_1(\theta; s) + a_1(\theta)u_1'(\theta; s)$, where u_1' denotes the derivative with respect to θ. This system can be solved uniquely after fixing the initial condition $u(\theta; 0) = 0$. Furthermore, since the a_i's are periodic in θ, so will be the u_k's. Hence the exponential map is correctly defined for such P.

We can write down an explicit formula for $\exp(sP)$. Indeed, since $\deg(P^k) = k$ the coefficients of the ∂^i's in the sum $\sum_{k\geq 0} \frac{1}{k!} P^k$ are finite sums. So we can write

$$\exp(sP) = \sum_{k\geq 0} s^k \frac{1}{k!} P^k .$$

By the same argument, the series for $\log(1 + P) = \sum_{k\geq 1} \frac{(-1)^k}{k} P^k$ converges. So each $g \in G_{int}$ which is of the form $g = 1 + P$ for some $P \in \mathfrak{g}_{int}$ lies in the image of the exponential map. □

Before we can look at the exponential of a general element of $\widetilde{\mathfrak{g}}_{int}$, we need an observation which is also interesting by itself since it makes sense of the heuristic calculation $\frac{d}{ds}\partial^s = \log \partial$ we used in section 4.1.

Proposition 4.36 *The element* $\log \partial \in \mathfrak{g}_{int}$ *generates the Lie algebra of the one parameter subgroup* $\{\partial^s \mid s \in \mathbb{R}\} \subset G_{int}$.

Proof. Let $P \in \mathfrak{g}_{int}$ be of the form $P = a_i(\theta)\partial^i$. Then, for $\epsilon, \delta \in \mathbb{R}$ we get

$$\begin{aligned}
\partial^\epsilon \cdot \exp(\delta P) \cdot \partial^{-\epsilon} \exp(-\delta P) =& \partial^\epsilon \cdot (1 + \delta P) \cdot \partial^{-\epsilon} \cdot (1 + \delta P) \mod \delta^2 \\
=& 1 + \delta(\partial^\epsilon \cdot P\partial^{-\epsilon} - P) \mod \delta^2 \\
=& 1 + \delta[\partial^\epsilon, P]\partial^{-\epsilon} \\
=& 1 + \delta \sum_{k=1}^{\infty} \binom{\epsilon}{k} a_i^{(k)} \partial^{i-k} \\
=& 1 + \delta\epsilon \frac{\partial}{\partial\epsilon}|_{\epsilon=0} \sum_{k=1}^{\infty} \binom{\epsilon}{k} a_i^{(k)} \partial^{i-k} \mod \epsilon^2,
\end{aligned}$$

where we have used Taylor expansion in the last step. Since $\frac{\partial}{\partial\epsilon}|_{\epsilon=0} \binom{\epsilon}{k} = \frac{(-1)^{k+1}}{k}$, we get

$$\begin{aligned}
\partial^\epsilon \cdot \exp(\delta P) \cdot \partial^{-\epsilon} \exp(-\delta P) =& 1 + \delta\epsilon \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a_i^{(k)} \partial^{i-k} \mod \epsilon^2, \delta^2 \\
=& 1 + \delta\epsilon[\log \partial, P].
\end{aligned}$$

This proves proposition 4.36. □

Now we can continue the proof of theorem 4.22. Let us look at a general element of $P \in \widetilde{\mathfrak{g}}_{int}$. We can write $P = \lambda \log \partial + P_0$, where $P_0 = \sum_{i=-\infty}^{-1} a_i(\theta)\partial^i$ is an element of $\mathfrak{g}_{int}$. Equation (4.51) then reads

$$\frac{d}{ds} L_s = (\log \partial + P_0)\, L_s. \tag{4.54}$$

Let us write $L_s = (1 + Q_s)\partial^s$. Then the equation above is equivalent to the following one:

$$\frac{d}{ds} Q_s = [\log \partial, Q_s] + P \circ (1 + Q_s) \tag{4.55}$$

Indeed, replacing the differential equation for L_s by a finite difference, we get

$$\begin{aligned}
\epsilon(\frac{\partial}{\partial s} L_s)L_s^{-1} + \mathcal{O}(\epsilon^2) =& L_{s+\epsilon}L_s^{-1} = (1 + Q_{s+\epsilon})\partial^{s+\epsilon}\partial^{-s}(1 + Q_s)^{-1} \\
=& (1 + Q_{s+\epsilon})\partial^{\epsilon}(1 + Q_s)^{-1} \\
=& \left(\partial^{\epsilon}(1 + Q_{s+\epsilon}) - [\partial^{\epsilon}, Q_{s+\epsilon}]\right)(1 + Q_s)^{-1} \\
=& \left((\partial^{\epsilon} - 1)(1 + Q_{s+\epsilon}) + (1 + Q_{s+\epsilon}) - [\partial^{\epsilon}, Q_{s+\epsilon}]\right)(1 + Q_s)^{-1} \\
=& \epsilon \log \partial + \epsilon \left(\frac{d}{ds} Q_s\right)(1 + Q_s)^{-1} - \epsilon[\log \partial, Q_s](1 + Q_s)^{-1} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Hence equation 4.54 is equivalent to

$$\left(\frac{d}{ds} Q_s\right)(1 + Q_s)^{-1} - \epsilon[\log \partial, Q_s](1 + Q_s)^{-1} = P$$

which is the same as equation 4.55. Now, equation 4.55 gives rise to a triangular system of ordinary differential equations similarly to 4.53. Hence it can be solved so that we have defined the exponential map correctly on the whole Lie algebra $\widetilde{\mathfrak{g}}_{int}$.

Finally, we have to check the bijectivity of exp. We do this by constructing an inverse map to exp. Let us write $g = (1 + P_0)\partial^s$ for any $g \in G_{int}$. Here $P_0 = \sum_{i=-\infty}^{-1} a_i(\theta)\partial^i$ is an integral pseudo differential operator. By the Proposition 4.35, we can find some $Q \in \mathfrak{g}_{int}$ such that $(1 + P_0) = \exp(Q)$. Hence we get

$$g = (1 + P_0)\partial^s = \exp(Q) \circ \exp(s \log \partial)\,.$$

The Campbell-Hausdorff formula implies

$$g = \exp(Q + t \log \partial + t[Q, \log \partial] + \ldots)\,.$$

So we can define the inverse of the exponential map by

$$\log(g) = Q + t \log \partial + t[Q, \log \partial] + \ldots\,.$$

Note that the coefficients of each ∂^k in the expression for $\log(g)$ is a finite sum (since the terms have decreasing degree). Therefore, the map $\log : \widetilde{G}_{int} \to \widetilde{\mathfrak{g}}_{int}$ is well defined which implies that the exponential map $\exp : \widetilde{\mathfrak{g}}_{int} \to \widetilde{G}_{int}$ is indeed bijective.

4.7 Bibliographical notes

Our main references for this section are [56], [57], [47], and [21]. More facts on Poisson Lie groups can be found e.g. in [68] and [86].

5 Double loop groups and elliptic Lie algebras

Recall that for a compact manifold M and a simple finite dimensional Lie algebra $\mathfrak{g}$, the current algebra $\mathfrak{g}^M = C^\infty(M,\mathfrak{g})$ has a universal central extension whose center is the space $\Omega^1(M)/d\Omega^0(M)$ of one forms modulo exact one forms on M. The cocycle defining this central extension is given by $c(A,B) = (A, dB)$, where $(\ ,\ )$ denotes the Killing form on the Lie algebra $\mathfrak{g}$. Obviously, if the dimension of the manifold M is bigger that one, the space $\Omega^1(M)/d\Omega^0(M)$ is infinite-dimensional.

In this section we will consider the special case that $M = \Sigma$ is an elliptic curve, i.e. a 2-dimensional torus with a complex structure on it. The corresponding current algebra $\mathfrak{g}^\Sigma$ is usually called the *double loop algebra* corresponding to the finite dimensional Lie algebra $\mathfrak{g}$. The complex structure on the curve Σ can be used to define a 1-dimensional central extension of the Lie algebra $\mathfrak{g}^\Sigma$. The centrally extended Lie algebra $\hat{\mathfrak{g}}^\Sigma$ is sometimes called an *elliptic Lie algebra*. Maybe surprisingly, there is a classification of the coadjoint orbits of the corresponding Lie groups which is in complete analogy to the classification of the coadjoint orbits of the centrally extended loop groups.

The elliptic Lie algebras are "tied together" with the finite-dimensional simple Lie algebras and the affine Lie algebras by a class of integrable systems, the so called Calogero Moser systems. We will introduce these systems and describe their connection to the corresponding Lie algebras in section 6 of this chapter.

5.1 The central extension of G^Σ and its Lie algebra

Throughout this section fix an elliptic curve Σ over $\mathbb{C}$, i.e. a 2-dimensional torus with a complex structure. One can always write such a curve a a quotient $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau Z)$, where τ is a complex number with $Im(\tau) > 0$. Let dz denote the canonical $(1,0)$-form on Σ.

Definition 5.1 Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over the complex numbers. The *elliptic Lie algebra* corresponding to the Lie algebra $\mathfrak{g}$ and the elliptic curve Σ is the 1-dimensional central extension $\hat{\mathfrak{g}}^\Sigma = \mathfrak{g}^\Sigma \oplus \mathbb{C}$ of the current algebra $\mathfrak{g}^\Sigma$, which is defined by the following commutator:

$$[(X(z,\bar{z}),a)\,,(Y(z,\bar{z}),b)] = \left([X(z,\bar{z}),Y(z,\bar{z})]\,,\int_\Sigma \langle X, dY\rangle \wedge dz\right)\,.$$

Note that the complex structure of the elliptic curve Σ only appears in the cocycle ω^Σ, which defines the central extension. Furthermore, decomposing the operator $d = \partial + \bar{\partial}$ into its Dobleaut components ∂ and $\bar{\partial}$, one can write the cocycle ω^Σ as

$$\omega^\Sigma(X,Y) = \int_\Sigma (X, \bar{\partial} Y) \wedge dz\,. \tag{5.56}$$

Remark 5.2 The cocycle ω^Σ can be seen as a Lie algebra cocycle on $\mathfrak{g}^\Sigma$ with values in the dual of the space of holomorphic 1-forms on Σ. Its evaluation on the holomorphic 1-form dz is given by equation (5.56).

Proposition 5.3 *The elliptic Lie algebras $\hat{\mathfrak{g}}^{\Sigma_1}$ and $\hat{\mathfrak{g}}^{\Sigma_2}$ are isomorphic if and only if the elliptic curves Σ_1 and Σ_2 are isomorphic.*

PROOF. Let $\Phi : \hat{\mathfrak{g}}^{\Sigma_1} \to \hat{\mathfrak{g}}^{\Sigma_2}$ be an isomorphism of Lie algebras. We can regard the restriction of the map Φ to $\mathfrak{g}^{\Sigma_1}$ as an automorphism of the current algebra $\mathfrak{g}^{\Sigma_1}$. It is a general feature of current algebras that one can decompose such an automorphism Φ into $\Phi = h \circ \varphi_*$, where h is conjugation by an element of $Aut(\mathfrak{g})^{\Sigma_1}$, and φ is a diffeomorphism of Σ_1 to Σ_2 (see [82], Proposition 3.4.2). In order for the map Φ to extend to the central extension, it must take holomorphic differentials to holomorphic differentials which proves the assertion. $\square$

The Lie group corresponding to the current algebra $\mathfrak{g}^{\Sigma}$ is given by the current group G^{Σ}, where $G$is the simply connected complex Lie group corresponding to the Lie algebra $\mathfrak{g}$.

Theorem 5.4 *The central extension $\hat{\mathfrak{g}}^{\Sigma_\tau}$ of $\mathfrak{g}^{\Sigma_\tau}$ lifts to a central extension of the current group G^{Σ}, which sits inside an exact sequence*

$$e \to \Sigma \to \hat{G}^{\Sigma_\tau} \to G^{\Sigma_\tau} \to e$$

The central extension $\hat{G}^{\Sigma_\tau}$ can be constructed by the same methods which are used in [82] to construct the central extension of the loop group LG (see [22]) or by a generalization of the Mickelsson construction of the central extension of the loop group LG described in section 1.3. Since we will not need the explicit form of the central extension anywhere in this book, we refer the reader to the original literature for details.

Remark 5.5 Instead of restricting ourselves to elliptic curves, we could have considered the current algebra of any compact Riemann surface Σ of genus $g > 0$. Let H_Σ be the vector space of holomorphic differentials on Σ. This is complex vector space of dimension $\dim(H_\Sigma) = g$ Define a cocycle ω^{Σ} on $\mathfrak{g}^{\Sigma}$ with values in H^*_Σ via its evaluation on holomorphic 1-forms: Set

$$\omega^{\Sigma}(X,Y)(\alpha) := \int_\Sigma \langle X, \bar{\partial} Y\rangle \wedge \alpha$$

for all holomorphic one-forms α on Σ. This cocycle defines a g-dimensional central extension of the Lie algebra $\mathfrak{g}^{\Sigma}$. This central extension lifts to a central extension G^{Σ}, this time by the Jacobian J of Σ:

$$e \to J \to \hat{G}^{\Sigma} \to G^{\Sigma} \to e$$

If Σ is an elliptic curve, its Jacobian is given by Σ itself and one recovers the central extension of the double loop group which was described above.

5.2 Coadjoint orbits

In this section, we will study the coadjoint representation of the group $\hat{G}^{\Sigma_\tau}$.

Proposition 5.6 *The smooth part of the dual $(\hat{\mathfrak{g}}^\Sigma)^*$ of the Lie algebra $\hat{\mathfrak{g}}^\Sigma$ can be identified with the space $\mathfrak{g}^\Sigma \oplus \mathbb{C}$ via the pairing*

$$\langle (A,\lambda),(Y,\mu)\rangle = \int_\Sigma (A,Y)dz\wedge d\bar{z} + \lambda\mu\,,$$

where (,) is the Killing form on $\mathfrak{g}$.

In the coadjoint representation of $\hat{G}^\Sigma$ an element $g\in G^\Sigma$ acts on the space $(\hat{\mathfrak{g}}^\Sigma)^*$ via

$$g:(A,\lambda)\mapsto (\mathrm{Ad}_g(A)+\lambda\bar{\partial}gg^{-1},\lambda)\,.$$

This describes the coadjoint representation of the group $\hat{G}^\Sigma$ completely since the center acts trivially in the coadjoint representation.

Proof. The proof exactly same as the proof of the corresponding statement for affine Lie groups. □

The coadjoint orbits of the centrally extended group $\hat{G}^\Sigma$ have a nice classification in terms of holomorphic principal G-bundles over the elliptic curve Σ:

Theorem 5.7 *Fix some $\lambda\neq 0$. Then there is a one-to-one correspondence between the set of coadjoint orbits orbits of the group $\hat{G}^\Sigma$ in the hyperplane $(A,\lambda)\mid A\in\mathfrak{g}^\Sigma| \subset (\hat{\mathfrak{g}}^\Sigma)^*$ and the set of equivalence classes of holomorphic principal G-bundle on the elliptic curve Σ.*

Proof. Given an element $(A,\lambda)\in(\hat{\mathfrak{g}}^\Sigma)^*$, let us consider the partial differential equation

$$\lambda\bar{\partial}\psi = A\psi\,.$$

We can find an open covering U_i of the curve Σ such that there exist local solutions $\psi_i: U_i\to G$ of this equation. Then the functions $\varphi_{i,j}: U_i\cap U_j\to G$ defined by $\varphi_{i,j}=\psi_i^{-1}\psi_j$ are holomorphic and satisfy $\varphi_{i,j}=\varphi_{j,i}^{-1}$ and $\varphi_{i,j}\varphi_{j,k}\varphi_{k,i}=id$. Thus, they can be taken to be the gluing functions of a holomorphic principal G-bundle $P(A)$ on the curve Σ.

Conversely, every holomorphic principal bundle P on Σ_τ comes from some (A,λ) as above. Indeed, choose a local trivialization $\psi_i : U_i\to G$ of the bundle P such that the transition functions $\varphi_{i,j}=\psi_i^{-1}\psi_i$ are holomorphic. Then we have $\bar{\partial}\psi_i\psi_i^{-1} = \psi_j\bar{\partial}(\psi_j^{-1}\psi_i)\psi_i^{-1} - \psi_j\bar{\partial}\psi_j^{-1}\psi_i\psi_i^{-1} = \bar{\partial}\psi_j\psi_j^{-1}$. Hence, the $\bar{\partial}\psi_i\psi_i^{-1}$ fit together to define a global $\mathfrak{g}$-valued function $\frac{1}{\lambda}A$ on Σ_τ. Thus, the bundle P is isomorphic to the bundle $P(A)$ defined above.

Finally, two holomorphic principal G-bundles $P(A_1)$ and $P(A_2)$ are equivalent if and only if there exists some $g\in G^\Sigma$ such that $A_2 = gAg^{-1}-\lambda\bar{\partial}gg^{-1}$, i.e. if the differential operators $\lambda\bar{\partial}+A_1$ and $\lambda\bar{\partial}+A_2$ are related by a gauge transformation. □

Remark 5.8 Theorem 5.7 gives a classification of the coadjoint orbits of $\hat{G}^{\Sigma_\tau}$ in terms of holomorphic principal G-bundles on Σ_τ. A similar classification works for the centrally extended gauge groups on any Riemann surface. However, instead of the coadjoint representation of the group $\hat{G}^{\Sigma}$, one has to consider a representation of $\hat{G}^{\Sigma}$ on $\mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ which is defined by the cocycle $\omega(X,Y) = \int_{\Sigma}\langle X, dY\rangle \wedge \nu$, for some fixed where $(0,1)$-form ν on Σ. The proof of Theorem 5.7 adapts word by word to this more general case.

Exercise 5.2.1 Define an elliptic analogue of the Virasoro algebra, i.e. a central extension of the Lie algebra $\mathrm{Vect}_{(0,1)}(\Sigma)$ of $(0,1)$-vector fields on the elliptic curve Σ and describe its coadjoint orbits.

5.3 Holomorphic loop groups and monodromy

Recall that coadjoint orbits of a (centrally extended) loop group LG can be classified in terms of conjugacy classes of the corresponding finite dimensional Lie group G. The conjugacy class to a coadjoint orbit can be interpreted as the monodromy of a flat connection on the circle. As we shall see, coadjoint orbits of the centrally extended double loop group $\hat{G}^{\Sigma}$ admit a similar description.

Let $L_{Hol}G$ denote the group of holomorphic maps from the cylinder to the simply connected complex Lie group G with pointwise multiplication:

$$L_{Hol}G = \{g : \mathbb{C}/\mathbb{Z} \to G \mid g \text{holomorphic}\}$$

The group $L_{Hol}G$ is called the *holomorphic loop group* of G.

Definition 5.9 Fix some element $\tau \in \mathbb{C}$. Then the "τ-twisted" conjugacy classes of the group $L_{Hol}G$ are the orbits of the $L_{Hol}G$-action on itself which is given by

$$g(z) : h(z) \mapsto h^g = g(z-\tau)h(z)g(z)^{-1}.$$

Remark 5.10 The τ-twisted conjugacy classes of the group $L_{Hol}G$ can be seen as "restricted conjugacy classes" of the bigger group $\check{G} = L_{Hol}G \rtimes \mathbb{C}/\mathbb{Z}$, where the cylinder $\mathbb{C}/\mathbb{Z}$ acts on the group $L_{Hol}G$ by "rotating" the loops. The restricted conjugacy classes of $\check{G}$ are the $L_{Hol}G$-orbits in $\check{G}$, where $L_{Hol}G$ is viewed as a subgroup of $\check{G}$.

The next theorem is in complete analogy with the corresponding theorem for ordinary loop groups.

Proposition 5.11 *Let Σ be an elliptic curve which can be written as $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Fix some $\lambda \neq 0$. Then there is a one-to-one correspondence between the set of coadjoint orbits of the group $\hat{G}^{\Sigma}$ in the hyperplane $\{(X,\lambda) \mid X \in \mathfrak{g}^{\Sigma}\} \subset (\hat{\mathfrak{g}}^{\Sigma})^*$ and the set of τ-twisted conjugacy classes in the holomorphic loop group $L_{Hol}G$.*

PROOF. As before, we associate to an element $(A,\lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ the differential equation

$$\lambda\bar{\partial}\psi = A\psi. \tag{5.57}$$

Let ψ be a solution of equation 5.57) which is defined on the cylinder $\mathbb{C}/\mathbb{Z}$. Since A is periodic in τ, the function $\psi(z-\tau)$ is also a solution of (5.57). Furthermore, the function

$\nu(z) = \psi(z-\tau)^{-1}\psi(z)$ is holomorphic and periodic in $\mathbb{Z}$. Therefore it defines an element in $L_{Hol}(G)$. Now if $\psi_1(z)$ is any other solution of (5.57), it will have the form $\psi_1(z) = \psi(z)\mu(z)$, where μ is a G-valued function on $\mathbb{C}/\mathbb{Z}$. Also, we have $\bar{\partial}\mu = \bar{\partial}(\psi^{-1}\psi_1) = 0$ which shows that $\mu \in L_{Hol}(G)$. Finally, we have

$$\nu(z) = \psi(z-\tau)^{-1}\psi(z) = \mu(z-\tau)^{-1}\psi_1(z-\tau)^{-1}\psi_1(z)\mu(z)\,,$$

which shows that the τ-twisted conjugacy class in $L_{Hol}G$ corresponding to the element (A,λ) does not depend on the choice of a solution of equation (5.57).

Furthermore, let $g \in G^{\Sigma}$ and, as before, let ψ denote a solution of (5.57). Then the map $g\psi$ is a solution of the equation

$$\bar{\partial}\varphi = (\mathrm{Ad}(g)A + \lambda\bar{\partial}gg^{-1})\varphi\,.$$

Since g is periodic in τ, we get

$$\psi(z-\tau)^{-1}g(z-\tau)^{-1}g(g)\psi(z) = \psi(z-\tau)^{-1}\psi(z)\,,$$

which shows that the restricted conjugacy class associated to the element $(A,\lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ is well defined on the level of coadjoint orbits of the group $\hat{G}^{\Sigma}$. Thus, we can define a map

$$\{\hat{G}^{\Sigma}\text{–orbits in the hyperplane}\{(A,\lambda\} \subset \mathfrak{g}^{\Sigma}\oplus\mathbb{C}\} \to \{\tau\text{-twisted conjugacy classes in } L_{Hol}(G)\}.$$

Injectivity of this map is clear from the discussion above. Finally, every twisted conjugacy class in $L_{Hol}G$ comes from this construction so that the map is surjective as well. Indeed, given some element $g \in L_{Hol}G$, we can find a smooth map $\psi : \mathbb{C}/\mathbb{Z} \to G$ such that $g(z) = \psi(z-\tau)^{-1}\psi(z)$. Then we can set $A = \frac{1}{\lambda}\bar{\partial}\psi\psi^{-1}$. Now, the fact that $\psi(z-\tau)^{-1}\psi(z)$ is holomorphic implies that A is periodic in τ. Therefore, A is a smooth map from the elliptic curve Σ to the Lie algebra $\mathfrak{g}$. Obviously, the conjugacy class corresponding to the coadjoint orbit through $(A,\lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ is the τ-twisted conjugacy class through the element $g \in L_{Hol}G$. □

Remark 5.12 In the last section we have seen that there is a bijection between the set of coadjoint orbits of the group G^{Σ} in a fixed $\lambda = const$-hyperplane in $(\hat{\mathfrak{g}}^{\Sigma})^*$ and the set of equivalence classes of holomorphic G-bundles on the elliptic curve Σ. Hence, together with Proposition 5.11, we get a bijection between the set of τ-twisted conjugacy classes in the holomorphic loop group $L_{Hol}G$ and the set equivalence classes of holomorphic G-bundles on the elliptic curve Σ. This bijection can be easily constructed more directly: To any holomorphic map $g : \mathbb{C}/\mathbb{Z} \to G$ we can associate a holomorphic G-bundle P_g on the elliptic curve $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau Z)$ as the quotient

$$P_g = ((\mathbb{C}/\mathbb{Z}) \times G)/\tau\mathbb{Z}.$$

Here, $\tau\mathbb{Z}$ acts on $(\mathbb{C}/\mathbb{Z}) \times G$ via $\tau : (z,h) \mapsto (z+\tau, g(z)h)$. One checks directly that two bundles P_{g_1} and P_{g_2} are isomorphic if and only if g_1 and g_2 lie in the same τ-twisted conjugacy class of the group $L_{Hol}G$. On the other hand, every holomorphic G-bundle on the elliptic curve comes from this construction. Indeed, let P be a holomorphic G bundle on Σ. Consider the pull back π^*P of P to the cylinder $\mathbb{C}/\mathbb{Z}$ by the natural projection map $\pi : \mathbb{C}/\mathbb{Z} \to \Sigma = \mathbb{C}/(\mathbb{Z} + \tau Z)$. The bundle π^*P is holomorphically trivial, so we can find a global holomorphic section $g : \mathbb{C}/\mathbb{Z} \to G$. Obviously, the bundle P is isomorphic to the bundle P_g.

Remark 5.13 We can put all the considerations from this chapter into one picture: Let $\mathfrak{S}$ denote the set of all smooth maps ψ from the cylinder $\mathbb{C}/\mathbb{Z}$ to the group G such that $\psi(z-\tau)^{-1}\psi(z)$ is holomorphic and let $\mathfrak{B}$ denote the set of equivalence classes of holomorphic G-bundles on the elliptic curve Σ. The double loop group G^Σ acts on the set $\mathfrak{S}$ by multiplication from the left and the holomorphic loop group $L_{Hol}G$ acts on $\mathfrak{S}$ by multiplication on the right. Obviously, these two actions commute. The map $\psi(z) \mapsto \psi(z-\tau)^{-1}\psi(z)$ gives the set $\mathfrak{S}$ the structure of a G^Σ-bundle over $L_{Hol}G$ and it is $L_{Hol}G$-equivariant with respect to τ-twisted conjugation on $L_{Hol}G$. On the other hand, the map $\psi \mapsto \frac{1}{\lambda}\bar{\partial}\psi\psi^{-1}$ gives the set $\mathfrak{S}$ the structure of a $L_{Hol}G$-bundle over the affine space $\{(A,\lambda) \mid A \in \mathfrak{g}^\Sigma\}$ and it is G^σ-equivariant with respect to the coadjoint action of G^Σ on $\{(A,\lambda) \mid A \in \mathfrak{g}^\Sigma\}$. Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \mathfrak{S} \arrow[dl] \arrow[dr] & \\
L_{Hol}G \arrow[dr] & & \{(A,\lambda) \mid A \in \mathfrak{g}^\Sigma\} \arrow[dl] \\
 & \mathfrak{B} &
\end{tikzcd}$$

5.4 Bibliographical notes

The elliptic Lie algebras were introduced in [22], where most of the material of this section is taken from. Other references for this section are [28] and [24]. The observation in Remark 5.12 is due to E. Looijenga (see [22] and [13]). The commutative diagram in Remark 5.13 was shown to us by P. Slodowy.

It is also possible to define elliptic Lie algebras more abstractly, using the notion of elliptic Root systems ([84], [7], [81]). The coadjoint orbits of the corresponding Lie groups can be classified in terms of holomorphic bundles whose structure groups are non-simply connected (and possibly even non-connected), and in terms of twisted conjugacy classes of non-connected loop groups [92].

6 Calogero Moser Systems

In this section, we describe a family of integrable systems, the so-called Calogero-Moser systems and show how they can be recovered by Hamiltonian reduction from cotangent bundle of finite-dimensional simple, affine and elliptic Lie algebras respectively. This construction shows how these three classes of Lie algebras and the corresponding Lie groups are "tied together."

6.1 Definition of the systems

Consider a system of n interacting particles moving on the real line $\mathbb{R}$ governed by the Hamiltonian

$$H(q_1,\ldots,q_n,p_1,\ldots,p_n) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i>j} V(q_i - q_j)\,, \tag{6.58}$$

where the q_i denote the positions of the particles and the p_i denote their momenta. The potential V is supposed to depend only on the distance $q_{i,j} = q_i - q_j$ between the particles.

Definition 6.1 The Calogero-Moser systems are Hamiltonian systems of type (6.58), where the potential is given by

1. $V(\xi) = \frac{1}{\xi^2}$ (rational case),
2. $V(\xi) = \begin{cases} \frac{a^2}{\sin^2(2\xi)} \\ \frac{a^2}{\sinh^2(2\xi)} \end{cases}$ (trigonometric cases)
3. $V(\xi) = a^2\wp(\xi;\tau)$ (elliptic case).

Here, $\tau \in \mathbb{C}$ with $im(\tau) > 0$ should be viewed as the modular parameter of the elliptic curve $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and , $\wp(\,\cdot\,;\tau)$ is the Weierstrass $\wp$-function with periods 1 and τ which is defined by

$$\wp(\xi;\tau) = \frac{1}{\xi^2} + \sum_{\substack{m_1,m_2\in\mathbb{Z}^2 \\ (m_1,m_2)\neq(0,0)}} \left(\frac{1}{(\xi - m_1 - m_2\tau)^2} - \frac{1}{(m_1+m_2\tau)^2}\right)$$

It is known that the Calogero Moser systems are integrable. In the following sections, we will show, how they can be obtained by Hamiltonian reduction from the coadjoint representations of the finite dimensional compact Lie group $SU(n)$, the affine Lie group $\widehat{LSU(n)}$ and the elliptic Lie group $\widehat{SL(n,\mathbb{C})}^{\Sigma}$.

Remark 6.2 Note that in the limit $a \to \infty$, the Hamiltonian (6.63) degenerates to the Hamiltonian of the rational Calogero Moser system. Similarly, in the limit $\tau \to i\infty$ the function $\wp(z;\tau)$ reduces to $\frac{\pi^2}{\sinh^2(\pi z)} - \frac{\pi^2}{3}$ so that the Hamiltonian of the elliptic Calogero Moser degenerates to the trigonometric case.

Remark 6.3 The Calogero Moser systems can be defined for any simple root system. In this case, the Lie group $SU(n)$ which corresponds to a root system of type A_{n-1} has to be replaced by the corresponding simply connected compact Lie group G.

Remark 6.4 A relativistic analogue of the elliptic Calogero Moser system is the Ruijsenaars-Schneider model whose Hamiltonian is given by

$$H(p,q) = \sum_{i=1}^{n} (\cosh \beta p_i) \prod_{j \neq k} \sqrt{1 - g^2 \wp(q_j - q_k)}\,.$$

Analogues of the rational and trigonometric Calogero Moser systems can be obtained from this Hamiltonian by degenerations.

6.2 The rational Calogero Moser system and $SU(n)$

Let us look at the simple Lie group $G = SU(n)$. Its Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ is the Lie algebra of traceless skew-adjoint $n \times n$ matrices. Using the non-degenerate bilinear form $\langle A, B\rangle = -tr(AB)$, we can identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ itself. This identification gives an isomorphism $T^*\mathfrak{g} \cong \mathfrak{g} \oplus \mathfrak{g}$. The cotangent bundle of any manifold is carries a natural symplectic structure. Using the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$, the symplectic form ω on $T^*\mathfrak{g}$ is given by

$$\omega\left((Q_1, P_1), (Q_2, P_2)\right) = -tr(Q_1 P_2 - Q_2 P_1) \tag{6.59}$$

for $(Q_1, P_1), (Q_2, P_2) \in \mathfrak{g} \oplus \mathfrak{g}$.

The group G acts on $\mathfrak{g}$ by conjugation. The induced action of G on the cotangent bundle $T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$ is given by conjugation on each summand. In particular, the symplectic form ω is invariant under the G-action.

Exercise 6.2.1 Show that the map $\Phi : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ which is given by

$$\Phi\left((Q, P)\right) = [Q, P]$$

satisfies the moment map condition. That is, show that the map Φ is G-equivariant and $d\langle \Phi, X\rangle = \omega(\cdot, \xi_X)$ for all $X \in \mathfrak{g}$, where ξ_X denotes the vector field on $T^*\mathfrak{g}$ generated by an element $X \in \mathfrak{g}$.

We now want to perform Hamiltonian reduction. That is, given some element $\mu \subset \mathfrak{g}^*$, we consider the manifold $\Phi^{-1}(\mu)/G_\mu$, where G_μ denotes the stabilizer group of μ under the coadjoint representation.

Let us fix $\mu = i(-I + v^* \otimes v)$ where $v \in \mathbb{C}^n$ is the vector all whose entries are equal to 1. Then μ is a matrix with 0's on the diagonal and i's in all other entries. The inverse image of the moment map $\Phi^{-1}(\mu)$ is the set of all $(Q, P) \in \mathfrak{g} \oplus \mathfrak{g}$ such that

$$[Q, P] = \mu\,.$$

Lemma 6.5 *Let (Q, P) be an element of $\Phi^{-1}(\mu)$. Then we can simultaneously conjugate Q and P by an element $A \in G_\mu$ such that Q is a diagonal matrix. Furthermore, let $(iq_1, \ldots iq_n)$ denote the diagonal elements of the matrix Q. Then we can assume the q_j to be non-increasing.*

PROOF. Let is choose some unitary matrix C such that $D = CQC^{-1}$ is diagonal, and set $E = CPC^{-1}$ and $w = Cv$. Then $[E, D] = i(I - w^* \otimes w)$ is a matrix with 0's on the diagonal (since D is diagonal). This implies that $w_j^* w_j = 1$, whence $w_j = e^{it_j}$ for some $t_j \in \mathbb{R}$. Hence multiplying C by a diagonal matrix with entries e^{-it_j} shows that we can choose the matrix C to be an element of the stabilizer of μ. Finally, multiplying v by a permutation matrix does not change v, so we can arrange the diagonal entries of Q to be non-increasing. □

Lemma 6.5 shows that the reduced space $\Phi^{-1}(\mu)/G_\mu$ is given by the solutions of the equation

$$[Q, P] = \mu\,, \tag{6.60}$$

where $Q \in \mathfrak{su}(n)$ is a diagonal matrix, $P \in \mathfrak{su}(n)$ is arbitrary, and μ is the matrix with 0's on the diagonal and i's in all off-diagonal entries. Let $ip_{j,k}$ denote the entries of the matrix P and $(iq_1, \dots iq_n)$ be the entries of the diagonal matrix Q. Then equation (6.60) translates to $p_{j,k} = \frac{-1}{(q_j - q_k)}$ for $j \neq k$ and $p_{j,j}$ arbitrary.

Finally, let us consider a system of n free moving particles on the Lie algebra $\mathfrak{g}$. The Hamiltonian this system is given by its potential energy:

$$H(Q, P) = -\frac{1}{2} tr(P^2)\,.$$

The function H is is invariant under conjugation by elements of $SU(n)$, so it descends to a function $\widetilde{H}$ on the quotient $\Phi^{-1}(\mu)/G_\mu$ which is given by

$$\widetilde{H}(Q, P) = -\frac{1}{2} tr(P^2) = \frac{1}{2} \sum_j p_{j,j}^2 + \sum_{j>k} \frac{1}{(q_j - q_k)^2}\,.$$

This is the Hamiltonian of the rational Calogero Moser system. First integrals of this system are given by G-invariant functions on the Lie algebra $\mathfrak{g}$.

6.3 The trigonometric Calogero Moser system and $\widehat{LSU(n)}$

Here, we use similar ideas as in the last section to obtain the trigonometric Calogero-Moser system by Hamiltonian reduction. Let G be the group $SU(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ and let LG denote the loop group of G, i.e. the group of smooth maps from S^1 to G. Furthermore, let us denote by $\mathfrak{a}^*$ the vector space of smooth maps from the circle to the Lie algebra $\mathfrak{g}$ enlarged by a "co-central" direction:

$$\mathfrak{a}^* = C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{R}.$$

The space $\mathfrak{a}^*$ should be viewed as the smooth part of the dual of the affine Lie algebra $\widehat{L\mathfrak{g}} = C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{R}$ which was studied in section 1. As such, it carries a natural action of the loop group LG which comes from the coadjoint action of the centrally extended loop group $\widehat{LG}$. That is, an element $g \in LG$ acts on the space $\mathfrak{a}$ via

$$g : (Q, \lambda) \mapsto (gQg^{-1} + \lambda g' g^{-1}, \lambda)\,.$$

Let $\mathfrak{a}^{**}$ denote the full dual of the space $\mathfrak{a}$. We can write $\mathfrak{a}^{**}$ as a direct sum

$$\mathfrak{a}^{**} = \mathcal{F}(S^1, \mathfrak{g}) \oplus \mathbb{R},$$

where $\mathcal{F}(S^1, \mathfrak{g})$ denotes the space of $\mathfrak{g}^*$-valued distributions on S^1, and $\mathbb{R}$ denotes the "central" direction. If we use the Killing form to identify the Lie algebra $\mathfrak{g}$ with its dual space, we see that the space $\mathfrak{a}^{**}$ contains the affine Lie algebra $\widehat{L\mathfrak{g}}$ as a subspace. However, the Lie algebra structure of $\widehat{L\mathfrak{g}}$ does not extend to a Lie algebra structure on the space $\mathfrak{a}^{**}$. Nevertheless, the loop group LG acts naturally on the space $\mathfrak{a}^{**}$ via the dual to the coadjoint representation, and this representation contains the usual adjoint representation of the loop group LG on its Lie algebra as a sub-representation. Explicitly, the action of LG on $\mathfrak{a}^{**}$ is given by

$$g : (P, a) \mapsto \left(gPg^{-1}, a - \int_0^1 tr(Pg^{-1}g') \right).$$

Viewing the direct sum $\mathfrak{a}^* \oplus \mathfrak{a}^{**}$ as the cotangent bundle of the vector space $\mathfrak{a}$ allows to endow $\mathfrak{a}^* \oplus \mathfrak{a}^{**}$ with an LG-action and a natural symplectic structure ω which is easily seen to be invariant under the LG-action. (The symplectic structure can be written down explicitly by adapting equation (6.59) to the current situation.)

It turns out that the action of the loop group LG on the space $\mathfrak{a}^* \oplus \mathfrak{a}^{**} a$ is Hamiltonian, i.e. it admits an LG-equivariant moment map $\Phi : \mathfrak{a}^* \oplus \mathfrak{a}^{**} \to (L\mathfrak{g})^*$. However, the moment map does not take values in the smooth part of the dual of the Lie algebra $L\mathfrak{g}$ but in the full dual, i.e. in the space $\mathcal{F}(S^1, \mathfrak{g}^*)$ of $\mathfrak{g}^*$-valued distributions on S^1. The analogue of Exercise 6.2.1 in this situation is the following

Proposition 6.6 *The action of the loop group LG on the space $\mathfrak{a}^* \oplus \mathfrak{a}^{**}$ is Hamiltonian with moment map given by*

$$\Phi\left((Q, \lambda), (P, a)\right) = [P, Q] + \lambda \partial P$$

Here, $[P, Q]$ denotes the point-wise commutator of P and Q, and the derivative ∂P should be viewed in the distributional sense (i.e. if P has a finite jump at 0, its derivative ∂P acquires a Dirac-δ-type singularity at 0).

We now want to perform Hamiltonian reduction. Let us consider the element $\mu \otimes \delta_0 \in (L\mathfrak{g})^*$, where $\mu = i(-I + v^* \otimes v)$ is the matrix with 0's on the diagonal and i's on all off-diagonal entries, and δ_0 is the Dirac δ-function centered at $0 \in S^1$. Under the coadjoint action of the group LG on $(L\mathfrak{g})^*$, an element $g \in LG$ maps $\mu \otimes \delta_0$ to $g(0)\mu g(0)^{-1} \otimes \delta_0$.

Lemma 6.7 *Let $((Q, \lambda), (P, a))$ be an element in the inverse image $\Phi^{-1}(\mu \otimes \delta)$. Then, if $\lambda \neq 0$, we can find some element g in the stabilizer of $\mu \otimes \delta_0$ such that $gQg^{-1} + \lambda g^{-1}g'$ is a diagonal matrix whose entries are constants iq_j with $q_j \in \mathbb{R}$ such that the q_j are non-increasing*

PROOF. The assertion that Q is conjugate to a diagonal matrix with non-increasing diagonal entries follows from the classification of the coadjoint orbits of the smooth loop group LG in section 1.2. The argument that g can be chosen in the stabilizer group $(LG)_{\mu\otimes\delta}$ is the same as in the proof of Lemma 6.5 □

The moment map equation is given by

$$[P, Q] + \lambda\partial P = \mu \otimes \delta_0 \,,$$

where $Q \in \mathfrak{su}(n)$ is a diagonal matrix and $P \in (L\mathfrak{su}(n))^*$ is arbitrary. Let iq_j denote the entries of Q and $ip_{j,k}$ denote the entries of P. In these coordinates, the moment map equation reads

$$p_{j,k}(iq_k - iq_j) + \lambda\partial p_{j,k} = \delta_0 \,, \tag{6.61}$$

where the equation should be viewed as equations for $\mathbb{R}$-valued distributions on S^1.

Lemma 6.8 *For fixed $\lambda \neq 0$, the solutions of equation (6.61) are given by*

$$p_{j,k}(t) = \frac{e^{a_{j,k}t}}{\lambda(e^{a_{j,k}} - 1)} \tag{6.62}$$

with $a_{j,k} = \frac{i(q_j - q_k)}{\lambda}$.

Proof. Note that away from $t = 0$, equation (6.61) is an ordinary linear differential equation with constant coefficients so that the solution cerainly has to take the form (6.62). Viewed as a function on the circle $p_{j,k}$ has a finite jump at $t = 0$ which makes sense since its derivative should aquire a Dirac-δ type singularity. To prive the statement, we have to evaluate equation (6.61) on a test function $f : S^1 \to \mathbb{R}$. This gives

$$\begin{aligned}
\langle (iq_j - iq_k)p_{j,k} + \lambda\partial p_{j,k}, f\rangle &= (iq_j - iq_k)\int_{S^1} p_{j,k}(t)f(t)dt + \lambda\int_{S^1} p_{j,k}(t)f'(t)dt \\
&= (iq_j - iq_k)\int_{S^1} p_{j,k}(t)f(t)dt - \lambda\int_{S^1} p'_{j,k}(t)f(t)dt + \lambda f(0)(p_{j,k}(1) - p_{j,k}(0)) \\
&= f(0) = \langle \delta_0, f\rangle
\end{aligned}$$

In the last step we have used the fact that $p_{j,k}$ is of the form (6.62). In particular, we have $p_{j,k}(1) - p_{j,k}(0) = \frac{1}{\lambda}$. □

Finally, the function $H : \mathfrak{a}^* \oplus \mathfrak{a}^{**} \to \mathbb{R}$ which is given by

$$H\left((Q, \lambda), (P, a)\right) = -\frac{1}{2}\int_{S^1} tr(P^2)$$

is invariant under the action of the group LG on the space $\mathfrak{a} \oplus \mathfrak{a}^*$. Fixing λ and restricting the function H to the solutions of the moment map equation, we get

$$H\left((Q, \lambda), (P, a)\right) = \frac{1}{2}\sum_j p_{j,j}^2 + \sum_{j>k} \frac{1}{4\lambda^2 \sin^2(\frac{1}{2\lambda}(q_j - q_k))} \,. \tag{6.63}$$

For $\lambda = 1$ this is exactly the Hamiltonian of the trigonometric Calogero-Moser system. First integrals of this system are given by G-invariant functions on the Lie algebra $\mathfrak{g}$, integrated over the circle.

6.4 The elliptic Calogero Moser system and $\widehat{SL(n)}^{\Sigma}$

Here we use the same ideas as in the last section to obtain the elliptic Calogero Moser system by symplectic reduction from a Hamiltonian action of the elliptic Lie group corresponding to $SL_n(\mathbb{C})$. To this end, let $\Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ be an elliptic curve and let $G = SL_n(\mathbb{C})$ denote the special linear group with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. We denote by $\mathfrak{b}^*$ the space of smooth maps from the curve Σ to the Lie algebra $\mathfrak{g}$ enlarged by one "co-central" dimension:

$$\mathfrak{b}^* = C^\infty(\Sigma, \mathfrak{g}) \oplus \mathbb{C}\,.$$

The space $\mathfrak{b}^*$ should be thought of as the smooth part dual space of the elliptic Lie algebra $\widehat{\mathfrak{g}}^{\Sigma}$ which was introduced in section 5. Hence, it carries the coadjoint action of the current group G^{Σ} which is given by

$$g : (Q, \lambda) \mapsto gQg^{-1} + \lambda\bar{\partial}gg^{-1}$$

for an element $g \in G^{\Sigma}$.

Let $\mathfrak{b}^{**}$ denote the full dual of the space $\mathfrak{b}^*$, i.e. the space of $\mathfrak{g}^*$-valued distributions on the elliptic curve Σ enlarged by a "central" direction:

$$\mathfrak{b}^{**} = \mathcal{F}(\Sigma, \mathfrak{g}^*) \oplus \mathbb{C}, .$$

The vector space $\mathfrak{b}^{**}$ carries the dual of the coadjoint representation of G^{Σ}, which contains the usual adjoint representation of the group G^{Σ} as a subspace.

As in the last section, the action of the group G^{Σ} on the cotangent bundle $T^*\mathfrak{b}^* = \mathfrak{b}^* \oplus \mathfrak{b}^{**}$ turns out to be Hamiltonian, with moment map Φ taking values in the full dual of the current algebra $\mathfrak{g}^{\Sigma}$, i.e. in the space of $\mathfrak{g}^*$-valued distributions on the curve Σ:

Proposition 6.9 *The action of the current group G^{Σ} on the space $\mathfrak{b} \oplus \mathfrak{b}^*$ is Hamiltonian with moment map given by*

$$\Phi\left((Q, \lambda), (P, a)\right) = [P, Q] + \lambda\bar{\partial}X$$

Here, $[P, Q]$ denotes the point-wise commutator of P and Q, and the derivative $\bar{\partial}P$ should be viewed in the distributional sense.

Remark 6.10 In contrast to the cases of finite dimensional and affine Lie algebras, it is not true that in the coadjoint representation of the elliptic Lie algebra $\widehat{\mathfrak{g}}^{\Sigma}$, every element $(Q, \lambda) \in \mathfrak{b}^*$ with $\lambda \neq 0$ can be mapped to a constant diagonal matrix. However, recall that for fixed $\lambda_0 \neq 0$, the set of coadjoint orbits in the $(\lambda = \lambda_0)$-hyperplane in $\mathfrak{b}^*$ is in one-to-one correspondence with the set of equivalence classes of holomorphic $SL_n(\mathbb{C})$-bundles on the elliptic curve Σ. We call a holomorphic $SL_n(\mathbb{C})$-bundle *flat and unitary* if it admits a flat connection whose holonomy takes values in the maximal compact subgroup $SU(n) \subset SL_n(\mathbb{C})$. It is general fact that almost all holomorphic $SL_n(\mathbb{C})$-bundles on the elliptic curve Σ are flat and unitary. To be more precise, if $\{B_t\}_{t\in\mathcal{T}}$ is a holomorphic family of holomorphic $SL_n(\mathbb{C})$-bundles on Σ paramterized by a complex space $\mathcal{T}$ then the subset $\mathcal{T}_0$ of t such that B_t is flat and unitary is Zariski open in $\mathcal{T}$. Now, if $(Q, \lambda) \in \mathfrak{b}^*$ corresponds to a flat and unitary bundle, there exists a gauge transformation $g \in G^{\Sigma}$ such that $gQg^{-1} + \lambda g^{-1}\bar{\partial}g$ is a diagonal matrix (see Lemma 6.11).

Lemma 6.11 *If the element $(Q,\lambda)\in\mathfrak{b}^*$ corresponds to a flat and unitary bundle on the elliptic curve Σ, then there exists a gauge transformation $g\in G^\Sigma$ such that $gQg^{-1}+\lambda g^{-1}\bar\partial g$ is a diagonal matrix.*

PROOF. An $SL_n(\mathbb{C})$ bundle ξ on the elliptic curve $\Sigma=\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ which is flat and unitary can be defined as a quotient $\xi=(\mathbb{C}\times SL_n(\mathbb{C}))/(\mathbb{Z}\oplus\tau\mathbb{Z})$ where $\mathbb{Z}\oplus\tau\mathbb{Z}$ acts on $\mathbb{C}\times SL_n(\mathbb{C})$ via $(1,0):(z,h)\mapsto(z+1,t_1\,h)$, and $(0,\tau):(z,h)\mapsto(z+\tau,t_2\,h)$. Here $t_1,\,t_2\in SU(n)\subset SL_n(\mathbb{C})$ are the monodromies of the flat connection on the bundle ξ. Since the group $\mathbb{Z}\oplus\tau\mathbb{Z}$ is Abelian, the monodromies $t_1=\exp(H_1)$ and $t_2=\exp(H_2)$ have to commute. So we can diagonalize them simultaniously

Now, the bundle ξ is isomorphic to the bundle $\widetilde{\xi}=(\mathbb{C}/\mathbb{Z}\times SL_n(\mathbb{C}))/\tau\mathbb{Z}$, where $\tau\mathbb{Z}$ acts via $\tau:(z,h)\mapsto(z+\tau,\exp(H_2-\tau H_1)\,h)$. The isomorphism between the two bundles is performed by the map $(z,h)\mapsto(z,\exp(-zH_1)\,h)$. Now, $\exp(H_2-\tau H_1)$ can be viewed as an element of the holomorphic loop group $L_{Hol}SL_n(\mathbb{C})$, and $\widetilde{\xi}$ is the holomorphic $SL_n(\mathbb{C})$-bundle on the elliptic curve Σ which corresponds to the τ-twisted conjugacy class in $L_{Hol}SL_n(\mathbb{C})$ (see Remark 5.12). So the element (Q,λ) is gauge-equivalent to the constant diagonal matrix $(H_2-\tau H_1,\lambda)\in\mathfrak{b}^*$. □

Since the set of (Q,λ) which correspond to flat and unitary bundles is open and invariant under the action of the group G^Σ, we can restrict our attention to this set. To perform Hamiltonian reduction, we fix the element $\alpha\otimes\delta_0\in(\mathfrak{g}^\Sigma)^*$, where α is the matrix with 0's on the diagonal and 1's everywhere else, and δ_0 is the Dirac δ-function centered at $0\in\Sigma$.

Lemma 6.12 *Let $((Q,\lambda),(P,a))$ be an element in the inverse image $\Phi^{-1}(\mu\otimes\delta_0)$. Then, if $\lambda\neq 0$ and the element $(Q,\lambda)\in\mathfrak{b}$ corresponds to a flat and unitary $SL_n(\mathbb{C})$-bundle on the curve Σ, we can find some element g in the stabilizer of $\alpha\otimes\delta_0$ such that $gQg^{-1}+\lambda g^{-1}g'$ is a diagonal matrix whose entries are constants $q_j\in\mathbb{C}$.*

PROOF. Using Remark 6.10 and Lemma 6.11, the proof of Lemma 6.12 is the same as the proof of Lemma 6.7. □

Now, the moment map equation reads

$$[P,Q]+\lambda\bar\partial P=\alpha\otimes\delta_0\,,$$

where $Q\in\mathfrak{sl}_n(\mathbb{C})$ is a diagonal matrix and $P\in(\mathfrak{sl}_n(\mathbb{C})^\Sigma)^*$ is arbitrary. Let q_j denote the entries of Q and $p_{j,k}$ denote the entries of P. Then the moment map equation reads

$$p_{j,k}(q_k-q_j)+\lambda\bar\partial p_{j,k}=\delta_0\,, \tag{6.64}$$

where the equation should be viewed as equations for $\mathbb{C}$-valued distributions on the elliptic curve Σ. To solve this equation, we have to recall some facts about theta-functions:

Proposition / Definition 6.13 *The theta-function*

$$\theta_{1,1}(z;\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i n(z+\frac{1}{2})+\pi i\tau n^2}$$

is a holomorphic function in z with a simple zero at $z = 0$ on the complex plane $\mathbb{C}$. Furthermore, it satisfies the identities

$$\begin{aligned}\theta_{1,1}(z+1;\tau) &= -\theta_{1,1}(z;\tau)\\ \theta_{1,1}(z+\tau;\tau) &= -e^{-2\pi i z-\pi i\tau}\theta_{1,1}(z;\tau)\end{aligned}$$

Using the theta-function $\theta_{1,1}$, one can write down solutions for the moment map equation (6.64):

Lemma 6.14 *Write $\tau = \tau_1 + i\tau_2$ and $a_{j,k} = \frac{\tau_2(q_j-q_k)}{\pi\lambda}$. Then, for fixed $\lambda \neq 0$, the solutions of equation (6.64) are given by*

$$p_{j,k}(z) = e^{\pi a_{j,k}\frac{z-\bar{z}}{\tau_2}} \frac{\theta_{1,1}(z+a_{j,k};\tau)}{\theta_{1,1}(z;\tau)\theta_{1,1}(a_{j,k};\tau)}\,. \tag{6.65}$$

PROOF. Using the transformation properties of the theta-function $\theta_{1,1}$, one checks that the functions $p_{j,k}$ are periodic in 1 and τ. Furthermore, the function

$$\psi_{j,k}(z) = \frac{\theta_{1,1}(z+a_{j,k};\tau)}{\theta_{1,1}(z;\tau)\theta_{1,1}(a_{j,k};\tau)}$$

is meromorphic and has a first order pole in $z = 0$. Therefore, its derivative aquires a Dirac-δ type distribution at $z = 0$. Hence, evaluating equation (6.64) on a test function $f : \Sigma \to \mathbb{C}$ gives the assertion. □

Finally, we can write down G^{Σ}-invariant Hamiltonians on the space $\mathfrak{b}^* \oplus \mathfrak{b}^{**}$. The simplest one is given by

$$((Q,\lambda),(P,a)) = \frac{1}{2}\int_{\Sigma} tr(P^2)dz\wedge d\bar{z}\,. \tag{6.66}$$

Note that the functions $p_{j,k}p_{k,j} = \psi_{j,k}(z)\psi_{k,j}(z)$ are meromorphic periodic in z with periods 1 and τ and have a second order pole at $z = 0$ and a zeros at $z = \pm a_{j,k}$. This shows that we can write

$$p_{j,k}p_{k,j} = c(\wp(a_{j,k}) - \wp(z))$$

for some constant c. So after rescaling, the Hamiltonian H descends to the Hamiltonian of the elliptic Calogero-Moser system on the symplectic quotient. As in the cases of rational and trigonometric Calogero-Moser Systems, first integrals of this systems can be obtained from G-invariant functions on the Lie algebra $\mathfrak{g}$.

Remark 6.15 Instead of just looking at δ-functions centered at $z = 0$, one can consider δ-functions with finitely many poles. The corresponding integrable systems and their trigonometric analogues have been studied in [79].

6.5 Bibliographical notes

In this section we follow [46], [32] and [79]. [RW1]

RW1: Some historic references about Calogero Moser (SUTHERLAND) systems???

Chapter III

Real and complex gauge groups

1 Around holomorphic bundles

1.1 Holomorphic bundles

In this section, we will recall some basic notions and facts from the theory of holomorphic vector bundles. For a more comprehensive introduction, see e.g. [18] and [63].

Let M be a complex manifold let $E \to M$ a complex vector bundle, and denote the set of C^∞ sections of a vector bundle E by $\Gamma(E)$.

Definition 1.1 A *complex structure* on the complex vector bundle $E \to M$ is a covering of open sets $\{U_i\}_{i\in I}$ of the manifold M M together with local trivializations $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^n$ such that the transition functions

$$\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \times \mathbb{C}^n \to U_i \cap U_j \times \mathbb{C}^n$$

are holomorphic.

Remark 1.2 Since the transition functions $\varphi_i \circ \varphi_j^{-1}$ induce the identity on the first factor, they give rise to holomorphic maps $g_{i,j} : U_i \cap U_j \to Gl_n(\mathbb{C})$. It is easy to check that the $g_{i,j}$ satisfy the cocycle identity: $g_{i,j} = g_{j,i}^{-1}$ and $g_{i,j}g_{j,k}g_{k,i} = id$ on $U_i \cap U_j \cap U_k$.

On the other hand, given an open covering $\{U_i\}_{i\in I}$ of M together with a collection of holomorphic maps $g_{i,j}$ which satisfy the cocycle identity above, one can define a holomorphic vector bundle on M as follows. The total space E of the bundle can taken to be the union $\cup_{i\in I} U_i \times \mathbb{C}^n$ subject to the relation $(x, v) \sim (x, g_{i,j}(x)v)$ whenever $x \in U_i \cap U_j$. This is a well defined equivalence relation since the $g_{i,j}$ satisfy the cocycle condition. The transition functions of this bundle are given by $(id|_{U_i\cap U_j}, g_{i,j})$ which are holomorphic. Hence the bundle is indeed a holomorphic vector bundle on M.

Having defined the notion of a holomorphic vector bundle on a complex manifold, we want to know what its holomorphic sections are. To this end, remember that on a complex manifold M the complexified deRham complex (Ω^*_M, d) splits into a double complex $(\Omega^{*,*}_M, \partial, \bar\partial)$, where $\partial : \Omega^{p,q}_M \to \Omega^{p+1,q}_M$ and $\bar\partial : \Omega^{p,q}_M \to \Omega^{p,q+1}_M$ are the holomorphic and anti holomorphic differentials such that $\partial + \bar{\partial} = d$. In a local coordinate chart $(z_1, \ldots, z_n)$, the forms in $\Omega^{p,q}$ can be written as

$$\sum f_{i_1,\ldots i_p, j_1,\ldots j_q}(z_1, \ldots z_n, \bar{z}_1, \ldots, \bar{z}_n) dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

Definition 1.3 A function $f : U \to \mathbb{C}$ is holomorphic in if $\bar{\partial} f = 0$.

Remark 1.4 For any complex vector bundle E on M we can define the set of E-valued (p,q)-forms as $\Omega^{p,q}(E) = \Gamma(E) \otimes \Omega^{p,q}$. Given a holomorphic structure on the bundle E, we can define the operator $\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ uniquely by the properties

1. $\bar{\partial}_E(fs) = (\bar{\partial} f)s + f(\bar{\partial}_E s)$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$
2. $\bar{\partial}_E$ vanishes on open $U \subset M$ iff s is holomorphic in U.

Indeed, in any local trivialization (U, φ_U) of E, the operator $\bar{\partial}_E$ can be taken to be the standard differential $\bar{\partial}$. To see that $\bar{\partial}_E$ is independent of the choice of local trivialization, consider a different trivialization (V, φ_V). On $U \cap V$ the trivializations are related by a holomorphic map $g_{U,V} : U \cap V \to Gl_n(\mathbb{C})$. The differential of a section s is given by $\bar{\partial}_E(g_{U,V} s) = (\bar{\partial} g_{U,V}) + g_{U,V} \bar{\partial}_E(s) = g \bar{\partial}_E(s)$ since $g_{U,V}$ is holomorphic. This shows that the differential $\bar{\partial}_E$ is indeed well defined.

We have seen how a holomorphic structure on a complex vector bundle $E \to M$ gives rise to a differential operator $\bar{\partial}_E$ on the set of sections of E vanishing on the set of holomorphic sections. On the other hand, any connection d_A on a complex vector bundle E (i.e. any map $d_A : \Omega^0_M(E) \to \Omega^1_M(E)$, which satisfies the Leibnitz rule $d_A(fs) = (df)s + d_A s$ for all sections s and functions f) splits into partial connections $d_A = \partial_\beta + \bar{\partial}_\alpha$ with $\partial_\beta : \Omega^0_M(E) \to \Omega^{1,0}_M(E)$ and $\bar{\partial}_\alpha : \Omega^0_M(E) \to \Omega^{0,1}_M(E)$. Given a connection d_A on the bundle E, it is a natural question to ask, whether $\bar{\partial}_\alpha$ is integrable, i.e. whether it comes from a holomorphic structure on E. To find a criterion for the integrability of a partial connection $\bar{\partial}_\alpha$, let us extend $\bar{\partial}_\alpha$ to a map $\Omega^{0,p}_M(E) \to \Omega^{0,p+1}_M(E)$. Then, in a local C^∞-trivialization (U, φ), we can write $\bar{\partial}_\alpha = \bar{\partial} + \alpha^\varphi$, where α^φ is a matrix of 1-forms on U. Set $\Phi_\alpha = \bar{\partial}^2_\alpha \in \Omega^{(0,2)}(End E)$. In the local trivialization, we have $\Phi^\varphi_\alpha = \bar{\partial}\alpha^\varphi + \alpha^\varphi \wedge \alpha^\varphi$. Now we can state the integrability theorem for partial connections:

Theorem 1.5 *A partial connection $\bar{\partial}_\alpha$ on a complex vector bundle $E \to M$ is integrable if and only if $\bar{\partial}^2_\alpha = 0$.*

Theorem 1.5 is essentially a bundle-valued version of the Nirenberg-Newlander Theorem on the integrability of almost complex structures. The integrability condition $\bar{\partial}^2_\alpha = 0$ assures the maximal number of linearly independent solutions of the equation $\bar{\partial}_\alpha s = 0$. For a proof of theorem 1.5, see e.g. [63].

Remark 1.6 For any connection d_A on a complex vector bundle, its curvature $F_A = d_A^2 \in \Omega^2(End E)$ splits into $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$ Theorem 1.5 states that the $(0,1)$ part of a connection d_A defines a holomorphic structure on the corresponding bundle if and only if the $(0,2)$-part of its curvature vanishes.

Definition 1.7 Assume that there exists a Hermitian structure on the vector bundle E, i.e. a Hermitian metric (,) on each fiber which varies smoothly on the manifold M. A connection d_A on the bundle E is called *unitary* if it is compatible with the metric, i.e. if

$$d(s,t) = (d_A s, t) + (s, d_A t)$$

for any two sections s, t of E.

Lemma 1.8 *Let $\bar{\partial}_\alpha$ be a partial connection on a complex hermitian vector bundle E. Then there exists a unique unitary connection d_A on the bundle E such that the (0,1)-part of d_A is given by $\bar{\partial}_\alpha$.*

PROOF. Let A^φ be the matrix of 1-forms representing the connection d_A in a unitary trivialization (U, φ). Then the unitarity-condition on d_A reads $A^\varphi = -(A^\varphi)^*$. Now, given a partial connection $\bar{\partial}_\alpha$ with local connection matrix α^φ, we can define $A^\varphi = \alpha^\varphi - (\alpha^\varphi)^*$. This defines the connection d_A globally. □

Suppose that a complex vector bundle on the complex manifold M admits both a holomorphic structure and a Hermitian metric. Then there exists a unique connection d_A on the vector bundle E which is compatible with both structures. Since the curvature F_A of a unitary connection is skew adjoint we find $F_A^{0,2} = (F_A^{2,0})^*$. On the other hand, compatibility with the holomorphic structure forces $F_A^{0,2} = 0$ so that the curvature matrix F_A has to be a matrix of (1,1)-forms. The findings are summarized in the following Theorem.

Theorem 1.9 *A unitary connection on a hermitian complex vector bundle E over M is compatible with a holomorphic structure in E if and only if it has curvature of type (1,1). In this case the connection is uniquely determined by the metric and the holomorphic structure.*

Finally, let us recall the notion of a stable bundle. Let E be a vector bundle on a Riemann surface Σ. The rank $rk(E)$ is the dimension of its fibers, and the degree $deg(E)$ of the bundle E is the degree of its determinant bundle, i.e. the highest exterior power of the bundle E.

Definition 1.10 A holomorphic vector bundle E on a Riemann surface is stable if for any proper subbundle $F \subset E$ we have

$$\frac{deg(F)}{rk(F)} < \frac{deg(E)}{rk(E)} .$$

The bundle E is called semistable if the inequality above is replaced by $\leq$.

Example 1.11 To illustrate the notion of stability, we show that all endomorphisms of a stable vector bundle over a Riemann surface Σ, are isomorphisms: Indeed, let E be a stable vector bundle and let $f : E \to E$ be an endomorphism. Denote by I the image of E under the map f. This is a subbundle of E. Hence by stability, $\mu(I) \leq \mu(E)$. Since on the other hand, I is isomorphic to $E/\ker(f)$, we find $\mu(E) \leq \mu(I)$. By stability of E, the bundle I cannot be a proper subbundle of E. Thus, the bundles I and E have the same rank. Therefore the map f is surjective and hence an isomorphism.

Exercise 1.1.1 Let E be a stable vector bundle over a Riemann surface. Show that the only endomorphisms of E are homotheties.

Remark 1.12 The condition of stability of a vector bundle E on higher dimensional manifolds M^n is defined similarly. The degree of the bundle is now defined by $deg(E) = \int_M \omega^{n-1} \wedge c_1(E)$, where ω is a Kähler class on M and $c_1(E)$ is the first Chen class of the bundle E. So the degree of the bundle E depends on the choice of the Kähler class ω on M. Furthermore, instead of considering just subbundles of the given vector bundle E, one has to consider torsion free subsheaves of E.

1.2 Hitchin systems

In this section we will give a short introduction to the so called Hitchin systems. These are integrable systems on the cotangent bundle of the moduli space of stable vector bundle on a Riemann surface of genus $g > 1$. Similarly to the Calogero-Moser which we have introduced in section 6 of chapter II, systems they can be defined via symplectic reduction from the action of an infinite-dimensional Lie group on an affine space.

To get started, let us recall some facts about the moduli space of stable vector bundles on a Riemann surface: Throughout this section let us fix some Riemann surface Σ of genus g and let us and let $\mathcal{N}$ denote the space of equivalence classes of stable holomorphic vector bundles on Σ of a given topological type. That is, we fix a complex vector bundle E on the surface Σ and denote by $\mathcal{N}$ the moduli space of equivalent semistable holomorphic structures on E. It is a classical theorem due to Narashiman and Seshadri that $\mathcal{N}$ is a complex variety.

Remark 1.13 The dimension of the space $\mathcal{N}$ can be computed as follows. Let E_t be a family of holomorphic bundles depending on a parameter $t \in \mathbb{R}$ with $E_0 = E$ and let $g_{i,j}(t)$ denote the corresponding transition functions. Then the functions

$$g_{i,j}^{-1}(0)\frac{\partial}{\partial t}|_{t=0}g_{i,j}(t)$$

define a class in the sheaf cohomology group $H^1(\Sigma, EndE)$. Looking at all possible curves through a given holomorphic structure $g_{i,j}(0)$, we see that the tangent space of $\mathcal{N}$ at $g_{i,j}(0)$ can be identified with the cohomology group $H^1(\Sigma, EndE)$. The dimension of $H^1(\Sigma, EndE)$ is given by the Riemann-Roch theorem:

$$\dim H^0(\Sigma, EndE) - \dim H^1(\Sigma, EndE) = deg(EndE) + rank(EndE)(1-g)\,.$$

Now, $deg(EndE) = 0$ and stability of E implies that $\dim H^0(\Sigma, EndE) = 1$ (see Example 1.11 and Exercise 1.1.1). So we get

$$\dim \mathcal{N} = n^2(g-1) + 1\,.$$

The cotangent bundle $T^*\mathcal{N}$ admits a natural symplectic structure. We will give a second description of $T^*\mathcal{N}$ as a symplectic quotient much in the same way as we described the cotangent space of a coadjoint orbit as a symplectic quotient in section II.6. From this description it will be apparent that the cotangent bundle $T^*\mathcal{N}$ is the phase space of an integrable system.

Proposition / Definition 1.14 *Let $\mathcal{A}^s$ denote the set of stable complex structures on the complex vector bundle E. This set can be identified with the set of partial connections $\bar{\partial}_\alpha$ which are integrable and which give rise to a stable complex structure on the bundle E. The gauge group $\mathcal{G}(E)$ of the bundle E is the group of smooth automorphisms of the bundle E which fix the base Σ. This group acts on the set $\mathcal{A}^s$ by gauge transformations and the moduli space of stable holomorphic structures on the bundle E can be identified with the set*

$$\mathcal{N} = \mathcal{A}^s/\mathcal{G}_{\mathbb{C}}^{\Sigma}\,.$$

Remark 1.15 The cotangent bundle $T^*\mathcal{A}^s$ can be identified with the set of pairs

$$T^*\mathcal{A}^s = \{(\bar{\partial}_\alpha, \varphi)\}\,,$$

where $\bar{\partial}_\alpha$ is a point in $\mathcal{A}^s$ and φ is a $M(n,\mathbb{C})$-valued (1,0)-form on Σ. In the physics literature, the form φ is usually called *Higg's field* and the bundle $T^*\mathcal{A}^s$ is the *Higg's bundle.*

There exists a natural symplectic structure on the cotangent bundle $T^*\mathcal{A}^s$ which is given by

$$\omega_{\bar{\partial}_\alpha,\varphi}(\delta\bar{\partial}_\alpha, \delta\varphi) = \int_\Sigma \mathrm{tr}(\delta\bar{\partial}_\alpha \wedge \delta\varphi)\,.$$

Here, the pair $(\delta\bar{\partial}_\alpha, \delta\varphi)$ is a tangent vector at the affine space $T^*\mathcal{A}$ in the point $(\bar{\partial}_\alpha, \varphi)$. That is, $\delta\bar{\partial}_\alpha$ and $\delta\varphi$ are infinitesimal variations of $\bar{\partial}_\alpha$ and φ, i.e. tangent vectors at $\bar{\partial}_\alpha$ and φ.

The gauge group $\mathcal{G}(E)$ of the complex vector bundle E acts on the space $T^*\mathcal{A}^s$ by conjugation: $g : \varphi \mapsto g\varphi g^{-1}$ and $\bar{\partial}_\alpha \mapsto g\bar{\partial}_\alpha g^{-1}$. Obviously, the symplectic form ω is invariant under this action.

Exercise 1.2.1 The $\mathcal{G}(E)$-action on the space $T^*\mathcal{A}^s$ is Hamiltonian. The moment map

$$\mu : T^*\mathcal{A}^s \to Lie(\mathcal{G}(E))^*$$

is given by

$$\mu(\bar{\partial}_\alpha, \varphi) = [\bar{\partial}_\alpha, \varphi]\,.$$

Lemma 1.16 *The symplectic quotient $\mu^{-1}(0)/\mathcal{G}^\Sigma_\mathbb{C}$ is naturally identified with the cotangent bundle $T^*\mathcal{N}$.*

Lemma 1.16 shows that $T^*\mathcal{N}$ can be described as a symplectic quotient. Our next goal is to find a set of $\dim\mathcal{N}$ Poisson commuting Hamiltonians on $T^*\mathcal{N}$ which are functionally independent and thus form an integrable system. To this end, let K denote the canonical bundle on the Riemann surface Σ, and let T be the holomorphic tangent bundle of Σ. Furthermore, fix some $j \in \mathbb{N}$ and a basis $\nu_{j,k}$ of $H^1(\Sigma, K \otimes T^j)$.

Exercise 1.2.2 Show that the dimension of $H^1(\Sigma, K \otimes T^j)$ is $(2j-1)(g-1)$ for $j > 1$ and g for $j = 1$.

Let us define functions $H_{j,k} : T^*\mathcal{A}^s \to \mathbb{C}$ by

$$H_{j,k}(\bar{\partial}_\alpha, \phi) = \int_\Sigma \nu_{k,j}\, tr(\varphi^j)\,.$$

Exercise 1.2.3 Show that the functions $H_{j,k}$ Poisson-commute on $T^*\mathcal{A}^s$. Also show that they are gauge invariant, and therefore they Poisson commute after the symplectic reduction.

Note that the total number of the functions $H_{j,k}$ is

$$g + \sum_{j=2}^{n} (2j-1)(g-1) = n^2(g-1) + 1$$

which is exactly half the dimension of $T^*\mathcal{N}$. Finally, the functions $H_{j,k}$ are functionally independent. Hence they form an integrable system on $T^*\mathcal{N}$. These integrable systems are the so called *Hitchin systems*.

1.3 Poisson Structures on moduli spaces

In the following three sections section we will take a closer look at the moduli spaces of holomorphic vector bundles and show that they carry a symplectic structure. More generally, we will replace the moduli space of holomorphic vector bundles by the moduli space of holomorphic G-bundles, where G is a complex reductive Lie group. A classical theorem due to Ramanathan [83] states that this moduli space is isomorphic to the moduli space of flat connections on the corresponding K-bundle, where K is the maximal compact subgroup of G. We will describe this moduli space and its symplectic structure in section 1.4. If the Riemann surface Σ has a boundary, the moduli space of flat connections does not carry a symplectic structure but a Poisson structure. In this case, one can use the coadjoint orbits of the centrally extended loop groups described in section II.1 to classify the symplectic leafs of this structure.

In section 1.6 we will turn to the situation of holomorphic G-bundles on a complex surface S. If S admits a nowhere vanishing holomorphic n-form, it is a theorem of Mukai that the moduli space of holomorphic G-bundles on S carries a holomorphic symplectic structure. We shall give description of this symplectic structure which is analogous to our description of the moduli space of flat connections on a Riemann surface Σ. More generally, we will deal with the case that the complex surface S merely admits a meromorphic n-form with poles of at most first order. In this case the moduli space of holomorphic G-bundles on S inherits a Poisson structure and the symplectic leafs of this structure can be classified in terms of coadjoint orbits of the elliptic Lie groups we have been studying in section II.5. To make the constructions precise, we need a complex analogue of the Stokes' formula which we will describe in section 1.5.

1.4 Moduli spaces of flat connections

In this section, G will always denote a compact simply connected Lie group with Lie algebra $\mathfrak{g}$. As usual, $tr(\,.\,,\,.\,)$ denotes (a multiple of) the Killing form on $\mathfrak{g}$. Furthermore, let Σ be a compact Riemann surface possibly with boundary $\partial\Sigma$ consisting of the disjoint union of several copies of S^1 and let E be a principal G-bundle over Σ.

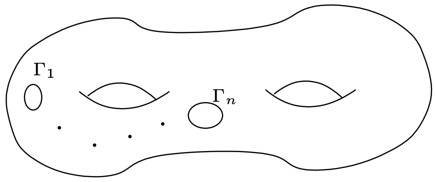


Figure 1.1: Riemann surface with boundary components $\Gamma_1 \dots \Gamma_n$

We denote by $\mathcal{A}^\Sigma$ the affine space of all smooth connections in E. Since the group G is simply connected, the bundle E is trivial. Hence we can write

$$\mathcal{A}^\Sigma = \{d + A \mid A \in \Omega^1(\Sigma, \mathfrak{g})\}\,.$$

This shows that $\mathcal{A}^\Sigma$ is an affine space with underlying vector space $\Omega^1(\Sigma, \mathfrak{g})$. In particular, we can identify the tangent space at any point of $\mathcal{A}^\Sigma$ with $\Omega^1(\Sigma, \mathfrak{g})$. This allows to define a symplectic structure ω on $\mathcal{A}^\Sigma$ by setting

$$\omega_A(\delta_1 A, \delta_2 A) = \int_\Sigma tr(\delta_1 A \wedge \delta_2 A)\,, \tag{1.1}$$

where $\delta_1 A$ and $\delta_2 A$ denote tangent vectors of $\mathcal{A}^\Sigma$ at the point $d + A$, i.e. elements of $\Omega^1(\Sigma, \mathfrak{g})$.

Since the principal G-bundle E on the surface Σ is trivial, the gauge group $\mathcal{G}$ of the bundle E is isomorphic to the current group with $G^\Sigma = C^\infty(\Sigma, G)$ and the Lie algebra of $\mathcal{G}$ is given by the current algebra $\mathfrak{g}^\Sigma = C^\infty(\Sigma, \mathfrak{g})$. The action of the group $\mathcal{G}$ on the affine space $\mathcal{A}^\Sigma$ is the usual gauge action

$$g : A \mapsto gAg^{-1} - dgg^{-1}.$$

This action preserves the symplectic structure ω and we shall see in a moment that it is Hamiltonian.

Proposition 1.17 *The Hamiltonian function H_ϵ generating the action of an infinitesimal gauge transformation $\epsilon \in \mathfrak{g}^\Sigma$ is given by*

$$H_\epsilon(A) = \int_\Sigma tr\left(\epsilon(dA + A \wedge A)\right) - \int_{\partial\Sigma} tr\epsilon A)\,.$$

Proof. The Hamiltonian vector field X corresponding to any function H is defined by its action on functions f by

$$L_X f = \{H, f\} = \int_\Sigma \langle \delta H \wedge \delta f\rangle, \tag{1.2}$$

where δf denotes the derivative of the function f $\mathcal{A}^\Sigma$. The latter expression in formula 1.2 is the Poisson bracket corresponding to the symplectic structure on $\mathcal{A}^\Sigma$.

In order to determine the Hamiltonian function H_ϵ, it suffices to consider the coordinate function $f(A) = A$. We then get

$$L_X A = \{H, A\} = \frac{\delta H}{\delta A}\,.$$

Thus, for the above Hamiltonian H_ϵ we have

$$L_{X_\epsilon} A = \frac{\delta H_\epsilon}{\delta A} = [\epsilon, A] - d\epsilon\,,$$

i.e. $L_{X_\epsilon} A$ is the infinitesimal gauge transformation of A by ϵ. Indeed, for $F(A) = dA + A \wedge A$ we have $\delta F(\delta A) = \delta(dA + A \wedge A) = d\delta A + \delta A \wedge A + A \wedge \delta A = \delta dA + [A, \delta A]$. This gives

$$\begin{aligned}\delta H_\epsilon(\delta A) &= \int_\Sigma tr(\epsilon \delta F)\rangle - \int_{\partial\Sigma} tr(\epsilon\delta A)\\ &= \int_\Sigma tr\left(\epsilon(\delta dA + [A, \delta A])\right) - \int_{\partial\Sigma} tr(\epsilon\delta A) = \int_\Sigma tr\left(\delta A \wedge ([\epsilon, A] - d\epsilon)\right)\,.\end{aligned}$$

In the last equality we have used Stokes' formula. □

Remark 1.18 In general, a Hamiltonian function H_ϵ corresponding to a vector field X is defined only up to a constant. Thus, the Poisson bracket of two Hamiltonian functions H_{ϵ_i} reproduces the Lie algebra $\mathfrak{g}^\Sigma$ only up to a two-cocycle c:

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1,\epsilon_2]} + c(\epsilon_1, \epsilon_2) \ ,$$

The cocycle c is an obstruction to the existence of a moment map. If the cocycle c is non-trivial, one has to consider the action of a central extension $\widehat{\mathfrak{g}}^\Sigma$ of the Lie algebra $\mathfrak{g}^\Sigma$ on the space $\mathcal{A}^\Sigma$ in order to have a moment map. Here, the central extension $\widehat{\mathfrak{g}}^\Sigma$ is the one coming from the cocycle c.

Proposition 1.19 *For the Hamiltonian functions H_ϵ considered above, the cocycle c from Remark 1.18 is given by*

$$c(\epsilon_1, \epsilon_2) = \int_{\partial\Sigma} \langle \epsilon_1, d\epsilon_2 \rangle$$

PROOF. Let us calculate $\{H_{\epsilon_1}, H_{\epsilon_2}\}$. Using formula 1.2 and the expression $\frac{\delta H_\epsilon}{\delta A} = [\epsilon, A] - d\epsilon$, we get

$$\begin{aligned}
\{H_{\epsilon_1}, H_{\epsilon_2}\} &= \int_\Sigma tr\left((-d\epsilon_1 + [\epsilon_1, A]) \wedge (-d\epsilon_2 + [\epsilon_2, A])\right) \\
&= \int_\Sigma tr(d\epsilon_1 \wedge d\epsilon_2) - \int_\Sigma tr(d\epsilon_1 \wedge [\epsilon_2], A) - \int_\Sigma tr([\epsilon_1, A] \wedge d\epsilon_2) \\
&\qquad + \int_\Sigma tr([\epsilon_1, A] \wedge [\epsilon_1], A) \\
&= \int_{\partial\Sigma} tr(\epsilon_1 d\epsilon_2) + \int_\Sigma tr\left([\epsilon_1, \epsilon_2](dA + A \wedge A)\right) - \int_{\partial\Sigma} tr([\epsilon_1, \epsilon_2], A) \\
&= \int_{\partial\Sigma} tr \epsilon_1 d\epsilon_2) + H_{[\epsilon_1,\epsilon_2]}.
\end{aligned}$$

We have used Stokes' formula several times. □

Remark 1.20 The cocycle c from Proposition 1.19 is nontrivial. Indeed, when restricted to one boundary component, it is exactly the cocycle defining the universal central extension of the loop algebra $L\mathfrak{g}$ (see section II.1.1). So c defines a non-trivial central extension $\widehat{\mathfrak{g}}^\Sigma$ of $\mathfrak{g}^\Sigma$.

Corollary 1.21 *The gauge action of G^Σ on $\mathcal{A}^\Sigma$ extends to a Hamiltonian action of the centrally extended group $\widehat{G}^\Sigma$.*

In order to describe the moment map of the action of the group $\widehat{G}^\Sigma$ on the space $\mathcal{A}^\Sigma$ more explicitly, we have to get our hands on the dual space of the Lie algebra

$$\widehat{\mathfrak{g}}^\Sigma = \{(\epsilon, z) \mid \epsilon \in \mathfrak{g}^\Sigma, \ z \in \mathbb{R}\}.$$

As usual, we will only be interested in the smooth part of the dual.

Lemma 1.22 *The smooth part* $(\widehat{\mathfrak{g}}^\Sigma)^*$ *of the dual of the Lie algebra* $\widehat{\mathfrak{g}}^\Sigma$ *is given by the space of triples*

$$(\widehat{\mathfrak{g}}^\Sigma)^* = \{(F, C, x) \mid F \in \Omega^2(\Sigma, \mathfrak{g}),\ C \in \Omega^1(\Sigma, \mathfrak{g}),\ x \in \mathbb{R}\}.$$

The non-degenerate pairing between $\widehat{\mathfrak{g}}^\Sigma$ *and* $(\widehat{\mathfrak{g}}^\Sigma)^*$ *is given by*

$$\langle (F, C, x), (\epsilon, z)\rangle = \int_\Sigma tr(\epsilon F) - \int_{\partial\Sigma} tr(\epsilon, C) + zx\,.$$

The group G^Σ acts on the Lie algebra $\widehat{\mathfrak{g}}^\Sigma$ by is adjoint action and dually on $(\widehat{\mathfrak{g}}^\Sigma)^*$ by its coadjoint action. To be more precise, we should consider the coadjoint action of the centrally extended group $\widehat{G}^\Sigma$ corresponding to $\widehat{\mathfrak{g}}^\Sigma$. But the center of $\widehat{G}^\Sigma$ acts trivially in the adjoint representation, it is enough to consider the group G^Σ itself. By Corollary 1.21, the group G^Σ acts on the space $\mathcal{A}^\Sigma$ in a Hamiltonian way. So there exists a moment map from $\mathcal{A}^\Sigma$ to $(\widehat{\mathfrak{g}}^\Sigma)^*$.

Proposition 1.23 *The moment map* $\mu : \mathcal{A}^\Sigma \to (\widehat{\mathfrak{g}}^\Sigma)^*$ *of the* G^Σ*-action on* $\mathcal{A}^\Sigma$ *is given by*

$$A \mapsto (F(A), A|_{\partial\Sigma}, 1)\ ,$$

where $F(A) = dA + A \wedge A$ *is the curvature of the connection* A.

Proof. Obviously, the map μ is equivariant since the curvature is equivariant. Let X_ϵ denote the vector field on $\mathcal{A}^\Sigma$ corresponding to an element $\epsilon \in \widehat{\mathfrak{g}}^\Sigma$. We have to check the moment map condition

$$\omega(X_\epsilon, \cdot) = d\langle \epsilon, \mu\rangle\,.$$

But this is exactly what we have shown in the proof of Proposition 1.17. □

Definition 1.24 For any submanifold $\Gamma \subset \Sigma$ let us denote by G^Σ_Γ the group of gauge transformations on Σ "based on Γ". That is, we set

$$G^\Sigma_\Gamma = \{g \in C^\infty(\Sigma, G) \mid g|_\Gamma = \mathrm{id}\},$$

and let $\mathfrak{g}^\Sigma_\Gamma$ be the corresponding Lie algebra.

A slight modification of Proposition 1.23 gives

Corollary 1.25 *The group* G^Σ_Γ *acts on* $\mathcal{A}^\Sigma$ *in a Hamiltonian way. The moment map for the action of the corresponding Lie algebra* $\mathfrak{g}^\Sigma_\Gamma$ *is the map* $\mathcal{A}^\Sigma \to (\mathfrak{g}^\Sigma_\Gamma)^*$ *given by the curvature:*

$$A \mapsto dA + A \wedge A\ .$$

We can consider the symplectic quotient $\mathcal{M}_{\Sigma,\Gamma} = \mathcal{A}^\Sigma /\!/ G^\Sigma_\Gamma$ of flat connections modulo gauge transformations from G^Σ_Γ. That is, we set

$$\mathcal{M}_{\Sigma,\Gamma} = \{d + A \in \mathcal{A}^\Sigma \mid F(A) = 0\}/G^\Sigma_\Gamma\ .$$

Remark 1.26 Since $\mathcal{M}_{\Sigma,\Gamma}$ is defined via Hamiltonian reduction, it inherits a symplectic structure from the space $\mathcal{A}^{\Sigma}$. Note however that if the boundary $\partial\Sigma$ of the Riemann surface Σ is non-empty, the space $\mathcal{M}_{\Sigma,\Gamma}$ will certainly be infinite-dimensional. Furthermore, it can have singularities. We shall only be concerned with the non-singular part of the moduli space $\mathcal{M}_{\Sigma,\Gamma}$.

For each boundary component Γ_i of the surface Σ, We can consider the restriction of a connection $A \in \mathcal{A}^{\Sigma}$ to each boundary component Γ_i of the surface Σ. This is a connection on Γ_i and hence can be viewed as an element in the dual space $(\widehat{\mathfrak{g}}^{\Gamma_i})^*$ of the centrally extended loop algebra $\widehat{\mathfrak{g}}^{\Gamma_i}$ corresponding to the boundary component Γ_i. Hence we can define a restriction map

$$\mathcal{A}^{\Sigma} \to \cup_i (\widehat{\mathfrak{g}}^{\Gamma_i})^* \;, \tag{1.3}$$

where the union ranges over the set of boundary components of Σ. Since the group G^{Σ}_{Γ} acts trivially on the boundary of Σ, the restriction map for connections factors through the symplectic quotient $\mathcal{M}_{\Sigma,\Gamma}$.

Lemma 1.27 *The restriction map*

$$\mathcal{M}_{\Sigma,\Gamma} \to \cup_i (\widehat{\mathfrak{g}}^{\Gamma_i})^*$$

which maps a connection to its restriction on the boundary of Σ is a Poisson map.

PROOF. The restriction map $\mathcal{A}^{\Sigma} \to \cup_i (\widehat{\mathfrak{g}}^{\Gamma_i})^*$ is obviously Poisson since it is just a component of the moment map $\mathcal{A}^{\Sigma} \to (\mathfrak{g}^{\Sigma}_{\Gamma})^*$. Factoring through $\mathcal{M}_{\Sigma,\Gamma}$ does not change this. □

Finally, note that since the group G^{Σ}_{Γ} of based gauge transformations is a normal subgroup in the group G^{Σ} of all gauge transformations, the group G^{Σ} acts on the space $\mathcal{M}_{\Sigma,\Gamma}$. Hence we can consider the quotient

$$\mathcal{M}_{\Sigma} = \mathcal{M}_{\Sigma,\Gamma}/G^{\Sigma} \,.$$

Furthermore, since the action of the group G^{Σ} on the space $\mathcal{A}^{\Sigma}$ is Hamiltonian, its induced action on the quotient $\mathcal{M}_{\Sigma,\Gamma}$ is still Hamiltonian. It is a general fact that the quotient of a symplectic manifold by a Hamiltonian group action is a (possibly singular) Poisson manifold. Hence the space $\mathcal{M}_{\Sigma}$ is a Poisson manifold. Furthermore, it is finite dimensional and compact. As before, we shall only be interested in the non-singular part of $\mathcal{M}_{\Sigma}$.

Remark 1.28 Note that the cocycle c is trivial on the Lie algebra $\mathfrak{g}^{\Sigma}_{\Gamma}$, so that the corresponding group G^{Σ}_{Γ} can be embedded as a normal subgroup into $\widehat{G}^{\Sigma}$. The quotient $\widehat{G}^{\Sigma}/G^{\Sigma}_{\Gamma}$ is then a central extension of the group $G^{\partial\Sigma}$. If the surface Σ is the unit disc, this is exactly the Mickelsson construction of the central extension of the loop group $L(G)$ which we described in section II.1.3.

We can now use the restriction map $\mathcal{M}_{\Sigma,\Gamma} \to \cup_i (\widehat{\mathfrak{g}}^{\Gamma_i})^*$ to describe the symplectic leaves of the Poisson manifold $\mathcal{M}_{\Sigma}$.

Proposition 1.29 *The symplectic leaves of the Poisson structure on the manifold $\mathcal{M}_\Sigma$ inherited from the symplectic structure on $\mathcal{A}^\Sigma$ are parametrized by the conjugacy classes of holonomies around the holes (that is, a symplectic leaf is singled out by fixing the conjugacy class of the holonomy around each hole).*

PROOF. The symplectic leaves of $\mathcal{M}_\Sigma$ are in one-to-one correspondence with the coadjoint orbits of the (centrally extended) affine Lie algebra on a circle $\widehat{\mathfrak{g}}^{\Gamma_j}$ (or the direct sum of several copies of the affine algebras if the boundary Γ of the surface Σ consists of several components). As we have seen in section II1.2, these coadjoint orbits, are parametrized by the conjugacy classes of holonomies around the circle. □

Remark 1.30 The above Proposition should not be understood as that the conjugacy classes of holonomies around the holes can be taken arbitrary, since the holonomies of a flat connection on the surface obey certain relations coming from the fundamental group $\pi_1(\Sigma)$. For example, if Σ is a sphere with n holes then the product of all n holonomies has to be id $\in G$ (provided one has chosen the same base point and a convenient orientation for all n loops encircling the holes).

Remark 1.31 There exists a completely finite-dimensional description of the Poisson structure on the space $\mathcal{M}_\Sigma$. The idea is to replace the surface Σ by a graph L which can be taken to be the 1-skeleton of a triangulation of the surface. The space of flat connections $\mathcal{A}^\Sigma$ is replaced by the set $\mathcal{A}^L$ of connections on the graph L. By definition, By definition, the set $\mathcal{A}^L = \prod_{\text{edges of } L} G$ is taken to be the product of G over all edges

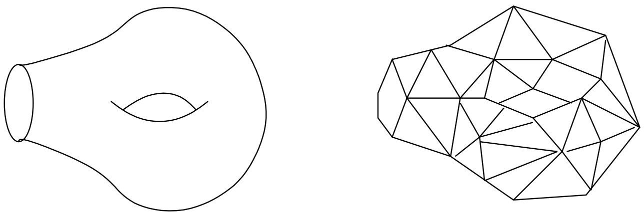


Figure 1.2: A Surface Σ is replaced by the 1-skeleton of a triangulation

of the graph L and should be viewed as the "holonomies along the edges." The gauge group $G^L = \prod_{\text{vertices of } L} G$ acts by conjugation at the vertices. The moduli space $\mathcal{M}_\Sigma$ of flat connections is diffeomorphic to the quotient $\mathcal{M}^L = \mathcal{A}^L/G^L$. It turns out that one can define Poisson strictures on both $\mathcal{A}^L$ and G^L such that the action of the group G^L on the space $\mathcal{A}^L$ is Poisson (i.e. the map $G^L \times \mathcal{A}^L \to \mathcal{A}^L$ is a Poisson map with respect to the sum of the two Poisson structures on $G^L \times \mathcal{A}^L$). In particular, the quotient $\mathcal{M}^L = \mathcal{A}^L/G^L$ is a Poisson manifold (see [25]).

There are several other finite dimensional constructions of the moduli space $\mathcal{M}_\Sigma$. (see e.g. [6, 5, 42, 45]).

1.5 Poincaré Residue

In the following section we want to treat the symplectic geometry of the moduli space of holomorphic vector bundles (or more generally, holomorphic G-bundles where $G \subset GL(n, \mathbb{C})$ is a complex reductive group) on a complex surface in a similar fashion as we treated the symplectic geometry of the moduli space of flat connections on a Riemann surface modulo gauge transformations. To do this, we need a complex analogue of Stokes' Theorem. This will be a multidimensional generalization of the Cauchy integral formula which we will now describe.

Let X be a compact complex n-dimensional manifold and let γ be a meromorphic n-form with singularities of at most first order on a smooth complex hypersurface $Y \subset X$. Let $z : U \to \mathbb{C}$ be a function defining Y locally in a neighborhood U of some point $p \in Y$. Then in U, the n-form γ can be decomposed into the sum

$$\gamma = \frac{dz}{z} \wedge \alpha + \beta \ ,$$

where α and β are holomorphic in U.

Lemma 1.32 *The restriction $\alpha|_Y$ is a well defined $(n-1)$-form on Y, that is, $\alpha|_Y$ is independent of the choice of f.*

This lemma can be easily proved using local coordinates. Since the 1-form $\alpha|_Y$ is independent of the choices made, the following definition makes sense:

Definition 1.33 The holomorphic $(n-1)$-form $\alpha|_Y$ on Y is called the Poincaré residue of the meromorphic form γ and is denoted by $\operatorname{res}_Y \gamma$.

Now we can state the main theorem of this section.

Theorem 1.34 (Cauchy-Stokes formula) *Let X, Y and γ be as above, and let u be a smooth $(n-1)$-form on X. Then the form $\gamma \wedge du$ is integrable on X, and*

$$\int_X \gamma \wedge du = (-1)^n 2\pi i \int_Y \operatorname{res}_Y \gamma \wedge u \ .$$

PROOF. Let z be a coordinate function locally defining Y. To see that $\gamma \wedge du$ is integrable, observe that $\gamma \wedge du$ grows like $\frac{1}{|z|}$ as $z \to 0$. Since the real codimension of Y in X equals 2, $\gamma \wedge du$ is indeed integrable.

To calculate the integral, choose some tubular neighborhood U_ϵ of radius ϵ of Y (see figure 1.5). Then we can write

$$\int_X \gamma \wedge du = \lim_{\epsilon \to 0} \int_{X \setminus U_\epsilon} \gamma \wedge du = (-1)^n \lim_{\epsilon \to 0} \int_{\partial U_\epsilon} \gamma \wedge u = (-1)^n \lim_{\epsilon \to 0} \int_Y \int_{|z| = \epsilon} \gamma \wedge u$$

where we have used Stokes' theorem to reduce the integral over $X \setminus U_\epsilon$ to an integral over its boundary. Finally, we can decompose $\gamma = \frac{dz}{z} \wedge \alpha + \beta$ and use Cauchy's theorem to write

$$\lim_{\epsilon \to 0} \int_Y \int_{|z| = \epsilon} \gamma \wedge u = 2\pi i \int_Y \operatorname{res}_Y \gamma \wedge u.$$

Note that the term involving $\beta \wedge u$ vanishes as $\epsilon \to 0$. □

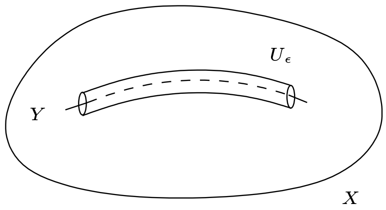


Figure 1.3: A tubular neighborhood U_ϵ of the polar divisor $Y \subset X$.

Remark 1.35 Note that in the Cauchy-Stokes' formula above, the polar divisor takes the part of the boundary in the usual Stokes' formula. In the same spirit, the meromorphic n-form can be seen as a "complex analogue" of an orientation. We will encounter similar analogies in subsequent sections.

Remark 1.36 Theorem 1.34 can be extended to the case when the meromorphic n-form γ has first order poles on a normal crossing divisor $Y = \cup_i Y_i \subset X$. (Normal crossing divisor means that the irreducible components Y_i of Y, each one appearing with multiplicity one, intersect generically.) Analogously to definition 1.33, one can define a residue $\text{res}_{Y_i} \gamma$ on each component of Y. The resulting $(n-1)$-forms $\text{res}_{Y_i} \gamma$ on Y_i are meromorphic and have first order poles at the intersections $Y_{i,j} = Y_i \cap Y_j$. One can now consider the repeated Poincaré residue of γ at $Y_{i,j}$. Let z_i and z_j be local coordinates of X representing Y_i and Y_j respectively. Then γ can be locally represented as $\gamma = \alpha \wedge \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j} + \beta$, where β is holomorphic. One then finds that

$$\text{res}_{i,j} \gamma := \text{res}_{Y_i}(\text{res}_{Y_j} \gamma) = \text{res}_{z_i=0} \left(\text{res}_{z_j=0} \, \alpha \wedge \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j} + \beta \right) = \alpha|_{Y_{i,j}} \ .$$

Note that

$$\text{res}_{i,j} \gamma = - \text{res}_{j,i} \gamma \ .$$

Otherwise $\text{res}_{i,j} \gamma$ does not depend on the choice of the local coordinates z defining Y_i and Y_j. In this situation we shall denote by $\text{res}\, \gamma$ the collection of $(n-1)$forms on the components Y_i of the normal crossing divisor $Y = \text{div}_\infty \gamma$.

1.6 Moduli spaces of holomorphic bundles

Let S be a compact complex surface ($\dim_{\mathbb{C}} S = 2$) and let $G \subset Gl(n, \mathbb{C})$ be a complex reductive group. In this section, we will describe a Poisson structure on the moduli space $\mathcal{M}_G$ of stable holomorphic vector bundles on S with the group G as structure group. As in section 1.4, when speaking about a moduli space we shall only mean its nonsingular part. The approach we take will be a complete analogy to our considerations in section 1.4, where we described the moduli space of flat connections on a Riemann surface. The basic idea is to replace the “real” notions of “orientation”and “boundary” of a Riemann surface by the notion of complex orientation (a non-vanishing meromorphic n-form on S) and its polar divisor which will play the role of the ”boundary” of the complex surface S (see Remark 1.35).

Let S be a compact complex surface as above, and fix some meromorphic two-form β on S which has only poles of first order and whose polar divisor $P \subset S$ is a smooth curve in S. Let us assume additionally that β has no zeroes. In this case, $\mathrm{res}_P\,\beta$ is a non-vanishing holomorphic one-form on P which shows that P is an elliptic curve or the union of several non-intersecting elliptic curves. According to Remark 1.35, these elliptic curves can be seen as the analogues of the circles constituting the boundary components of a real smooth surface. An example for such a situation is $S = \mathbb{CP}^2$ with the meromorphic 2-form defining the complex orientation on S given by $\beta = dxdy/xy$.

Now, let E be a smooth G-bundle over the surface S which admits a holomorphic structure and let $\mathrm{End}\,E$ be the corresponding bundle of endomorphisms with the fiber $\mathfrak{g} = \mathrm{Lie}(G)$. Furthermore, let $\mathcal{A}^S$ denote the infinite-dimensional affine space of smooth $(0,1)$-connections in E. Finally, let us choose a reference holomorphic structure, i.e. a $(0,1)$-connection $\bar\partial_0$ such that $\bar\partial_0^2 = 0$. The space $\mathcal{A}^S$ can be identified with the vector space $\Omega^{(0,1)}(S, \mathrm{End}\,E)$ of $(\mathrm{End}\,E)$-valued $(0,1)$-forms on S, i.e.

$$\mathcal{A}^S = \{\bar\partial_0 + A \mid A \in \Omega^{(0,1)}(S, \mathrm{End}\,E)\} \ .$$

The affine space $\mathcal{A}^S$ possesses a natural holomorphic symplectic structure which is given by

$$\omega_{\mathbb{C}}(\delta_1 A, \delta_2 A) = \int_S \beta \wedge tr(\delta_1 A \wedge \delta_2 A) \,,$$

where β is the meromorphic n-form on S (i.e. the complex ”orientation“ on S) and as usual, $tr(.,.)$ denotes the non-degenerate bilinear form on $\mathfrak{g}$ (i.e. the Killing form). Also, $\delta_1 A$ and $\delta_2 A$ are two tangent vectors at the affine space $\mathcal{A}^S$ in the point $\bar\partial + A$.

Following this observation, we can repeat the constructions from section 1.4 more or less word by word, always replacing the ”real” notions by their complex counterparts.

Let us denote the gauge group of the bundle E (i.e. the group of automorphisms of E which cover the identity on the base S) by G^S and its Lie algebra by $\mathfrak{g}^S = \Gamma(S, \mathrm{End}\,E)$ (where Γ denotes the space of C^∞-sections).

Proposition 1.37 *The symplectic structure $\omega_{\mathbb{C}}$ is invariant with respect to the gauge transformations*

$$A \mapsto gAg^{-1} - \bar\partial g g^{-1} \ ,$$

where $g \in G^S$ is an element of the group of gauge transformations, i.e., the group of automorphisms of the smooth bundle E.

As in section 1.4, we can find Hamiltonian functions generating the infinitesimal action of the gauge group G^S.

Proposition 1.38 *An infinitesimal gauge transformation ϵ is generated by the Hamiltonian function*

$$H_\epsilon(A) = \int_S \beta \wedge tr(\epsilon, (\bar{\partial}A + A \wedge A)) - 2\pi i \int_P \mathrm{res}_P\, \beta \wedge tr(\epsilon, A) \ .$$

PROOF. The proof is essentially the same as the proof of proposition 1.17. One just has to replace the use of Stokes' formula by Cauchy-Stokes. Indeed, writing $F = \bar{\partial}A + A \wedge A$ we have

$$\begin{aligned} \delta H_\epsilon &= \int_S \beta \wedge tr\,(\epsilon, \delta F(\delta A)) - 2\pi i \int_P \mathrm{res}_P\, \beta \wedge tr(\epsilon, \delta A) \\ &= \int_S \beta \wedge tr(\epsilon, \bar{\partial}\delta A + [A, \partial A]) - 2\pi i \int_P \langle \epsilon, \delta A\rangle = \int_S \beta \wedge tr(\delta A, [\epsilon, A] - \bar{\partial} A) \,. \end{aligned}$$

Here $[A, \epsilon] - \bar{\partial}\epsilon$ is the infinitesimal gauge transformation of a $(0,1)$-connection A by ϵ. □

We can also write down the commutation relations for the Hamiltonians with respect to the Holomorphic Poisson structure induces by ω. We have

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1, \epsilon_2]} + c(\epsilon_1, \epsilon_2)\,,$$

where c is a two cocycle on $\mathfrak{g}^S$.

Proposition 1.39 *For two Hamiltonian functions H_{ϵ_1}, H_{ϵ_2}, the 2-cocycle c is given by*

$$c(\epsilon_1, \epsilon_2) = 2\pi i \int_P \mathrm{res}_P\, \beta \wedge tr(\epsilon_1, \bar{\partial}\epsilon_2)\,.$$

Remark 1.40 If P is an elliptic curve, then the two cocycle c is

$$c(\epsilon_1, \epsilon_2) = 2\pi i \int_P \alpha \wedge tr(\epsilon_1, \bar{\partial}\epsilon_2)\,,$$

where α is some holomorphic one-form on P. In section II.5, we have used the same cocycle to construct a central extension of the double loop group $\mathfrak{g}^P$. This suggests that the centrally extended (elliptic) Lie algebra $\widehat{\mathfrak{g}}^P$ and the corresponding Lie group should play a similar role in the theory of holomorphic bundles as the centrally extended loop algebras (affine Lie algebras) played in the theory of flat connections on Riemann surfaces.

Lemma 1.41 *Let $\widehat{\mathfrak{g}}^{S,\beta}$ denote the gauge algebra $\mathfrak{g}^S$ centrally extended by means of the cocycle c from proposition 1.39, and let $\widehat{G}^{S,\beta}$ be the corresponding group. Identifying the Lie algebra $\widehat{\mathfrak{g}}^{S,\beta}$ with the set of pairs (ϵ, z) with $\epsilon \in \mathfrak{g}^S$ and $z \in \mathbb{C}$, we find that the smooth part of the dual $(\widehat{\mathfrak{g}}^{S,\beta})^*$ of $\widehat{\mathfrak{g}}^{S,\beta}$ is given by the space of triples (F, C, x), where F is a smooth $(0,2)$ form on the surface S, C in an (EndE)-valued $(0,1)$-form on P and x is a complex number. The non-degenerate pairing $<,>$ between the spaces $\widehat{\mathfrak{g}}^{(S,\beta)}$ and $(\widehat{\mathfrak{g}}^{(S,\beta)})^*$ is then given by:*

$$< (F, C, x), (\epsilon, z) >= \int_S \beta \wedge \langle \epsilon, F\rangle - 2\pi i \int_P \mathrm{res}_P\ \beta \wedge \langle \epsilon, C\rangle + zx \ .$$

The gauge group G^S acts on the affine space $\mathcal{A}^S$ by gauge transformations. This action gives rise to an action of the centrally extended group $\widehat{G}^{S,\beta}$ in which the center of $\widehat{G}^{S,\beta}$ acts trivially. The following result can be deduced from Proposition 1.38 exactly as Proposition 1.23 was deduced from 1.17 in section 1.4.

Proposition 1.42 *The centrally extended group of gauge transformations, $\widehat{G}^{(S,\sigma)}$, acts on the affine space $\mathcal{A}^S$ in a Hamiltonian way. The moment map for the action of the corresponding gauge algebra $\widehat{\mathfrak{g}}^{(S,\sigma)}$ is the mapping $\mathcal{A}^S \to (\widehat{\mathfrak{g}}^{(S,\sigma)})^*$ given by the $(0,2)$-curvature and by the restriction of the $\bar{\partial}$-connection form to the "boundary":*

$$A \mapsto (\bar{\partial}A + A \wedge A, \quad A|_P, \quad 1) \ .$$

As before, let us consider the group G_P^S of gauge transformations based at the polar divisor P:

$$G_P^S = \{g \in G^S \mid g|_P = \mathrm{id}\} ,$$

and denote by $\mathfrak{g}_P^S$ the corresponding Lie algebra. Since the cocycle c is trivial on $\mathfrak{g}_P^S$, we can view $\mathfrak{g}_P^S$ as a subalgebra of $\widehat{\mathfrak{g}}^S$. Accordingly, the group G_P^S can be viewed as a subgroup of $\widehat{G}^S$. Restricting the action of $\widehat{G}^S$ on $\mathcal{A}^S$, we get an action of G_P^S. A slight modification of the last proposition gives:

Corollary 1.43 *The group G_P^S acts on the space $\mathcal{A}^S$ in a Hamiltonian way. The moment map μ for the action of the corresponding Lie algebra $\mathfrak{g}_P^S$ is the mapping $\mathcal{A}^S \to (\mathfrak{g}_P^S)^*$ given by the $(0,2)$-curvature:*

$$A \mapsto \bar{\partial}A + A \wedge A \ .$$

Now observe, that the set $\mu^{-1}(0)$ is the set of holomorphic structures on the complex bundle E since it consists of $(0,1)$-connections whose $(0,2)$-curvature vanishes. So the (holomorphic) symplectic quotient

$$\mathcal{M}_{S,P} := \mathcal{A}^S // G_P^S = \mu^{-1}(0)/G_P^S$$

is the space of holomorphic structures on E modulo gauge transformations. [1] Coming from symplectic reduction, $\mathcal{M}_{S,P}$ acquires the structure of an (infinite-dimensional) symplectic manifold. In a second step, we will consider the quotient

$$\mathcal{M}_S := \mathcal{M}_{S,P}/\widehat{G}^{S,\beta} \ .$$

Being the quotient of a symplectic manifolds by a group of symplectic transformations, the quotient $\mathcal{M}_S$ will be a (possibly singular) Poisson manifold. We will now set out to describe the symplectic leaves of $\mathcal{M}_S$ outside the singularities. Before we do that, let is look at one particular situation.

Corollary 1.44 (Mukai) *Suppose that $P = \emptyset$. Then S admits a nowhere vanishing holomorphic n-form. Hence S has to be either an Abelian variety or a K3-surface. In this case, we also have $G_P^S = G^S$, so that $\mathcal{M}_S$ itself admits a holomorphic symplectic form.*

[1] Recall that we consider the moduli space only locally around smooth points

Going back to the general situation, recall that the polar divisor P of the meromorphic n-form β on S has to be the union of $\cup_{i\in I}\Sigma_i$ of elliptic curves. Using Remark 1.40 we see that the restriction of an element in $\mathcal{A}^S$ to P can be viewed as an element in $\cup_{i\in I}(\widehat{\mathfrak{g}}^{\Sigma_i})^*$, where $\widehat{\mathfrak{g}}^{\Sigma_i}$ is the centrally extended double loop algebra corresponding to the cocycle c. Similarly, the restriction map

$$\mathcal{A}^S \to \bigcup_{i\in I}(\widehat{\mathfrak{g}}^{\Sigma_i})^*$$

factors through the symplectic quotient $\mathcal{M}_{S,P}$ and we get:

Proposition 1.45 *The restriction map*

$$\mathcal{M}_{S,P} \to \bigcup_{i\in I}(\widehat{\mathfrak{g}}^{\Sigma_i})^* \ .$$

sending an integrable $(0,1)$*-connection on the surface* S *to its restriction to* P *is a Poisson mapping.*

The symplectic leaves of the Poisson manifolds $(\widehat{\mathfrak{g}}^{\Sigma_i})^*$ are the coadjoint orbits of the corresponding groups. These in turn are in one-to-one correspondence with the equivalence classes of holomorphic G-bundles on Σ_i (see section II.5.2). Finally, we can summarize the findings of this section:

Proposition 1.46 *The local moduli space* $\mathcal{M}_S$ *of holomorphic* G*-bundles on* S *possesses a (holomorphic) Poisson structure. The symplectic leaves of this structure are parametrized by the moduli of their restrictions to the anti-canonical divisor* $P \subset S$ *(that is a symplectic leaf is singled out by fixing the isomorphism class of the restriction to the elliptic curve* P*, or the isomorphism classes of restrictions to each curve if* P *consists of several such curves).*

Remark 1.47 The above proposition should not be understood as that the isomorphism classes of bundles on P can be taken arbitrary; rather they have to satisfy the condition that they arise as restrictions of bundles defined over S.

Remark 1.48 The discussion above applies with minor modifications to the case is the polar divisor P consists of several components intersecting transversally. In the latter case, the corresponding gauge group $\widehat{G}^{P,\beta}$ is the current group of a punctured Riemann surface.

1.7 Bibliographical notes

The material of section 1.1 is standard (see e.g. [18],[63]). The Hitchin systems in section 1.2 have been introduced in [39] (see also [40]). The moduli space of flat connections on a Riemann surface has been studied from various points of view. Our main references are the fundamental paper by Atiyah and Bott [12] and the papers [25] and [50]. The symplectic structure on the moduli space of stable sheaves on a K3 surface or an Abelian surface has been found by Mukai [78]. We follow [50].

2 Around the Chern-Simons Functional

In this somewhat informal section we describe the physics approach to the results of the last sections via the Chern-Simons functional. The Chern-Simons functional is a functional on the space of connections on a principal bundle over a three dimensional compact manifold N with boundary Σ. Its critical points are the flat connections. One can use this fact to recover the symplectic structure on the moduli space of flat connections on the boundary Σ of N. A complex version of the Chern-Simons functional is related to the holomorphic symplectic structure on the moduli space of stable bundles on an Abelian surface or a $K3$-surface in a similar way, as the usual Chern-Simons functional is related to the moduli spaces of flat connections. In section 2.3 we briefly mention how one can use path integration methods to to count linking numbers of closed curves embedded in three dimensional manifolds.

2.1 Reminder on the Lagrangian Formalism

A particle moving on a manifold M can be described using the Lagrangian formalism. That is, the physics on M is described by a Lagrangian function

$$L : TM \times \mathbb{R} \to \mathbb{R} \,,$$

where, as usual, TM denotes the tangent bundle of M. To each path $q : [t_0, t_1] \to M$ one can associate the action

$$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t) dt \ .$$

Here we have adopted the physics notation $\dot{q}(t) = \frac{d}{dt} q$. The movement of a particle on the manifold M is described by the condition that the action along the path $p(t)$ of its actual motion is minimal. To be more precise, let us suppose we know that the particle has position q_0 at time t_0 and q_1 at time t_1. Then we can pose the variational problem

$$S[q] = \min \tag{2.4}$$

subject to the conditions $q(t_0) = t_0$ and $q(t_1) = t_1$. A solution of this problem will be a local minimum of the functional S. Hence, if the path q is a solution of the variational problem and δq is a vector field along q with $\delta q(t_0) = \delta q(t_1) = 0$, we have the necessary condition

$$\delta S[\delta q] = 0 \,,$$

where $\delta S[\delta q]$ denotes the directional derivative of the functional S in direction δq. Writing out $\delta S[\delta q]$ explicitly, we get

$$\begin{aligned} \delta S[\delta q] &= \int_{t_0}^{t_1} \left(\frac{\partial L(q, \dot{q}, t)}{\partial q} \delta q + \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \frac{d}{dt} \delta q(t) \right) dt && (2.5) \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L(q, \dot{q}, t)}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \right) \delta q dt + \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \delta q|_{t_0}^{t_1} \ , && (2.6) \end{aligned}$$

where we have used integration by parts in the second step. The last term in equation (2.6) vanishes since we imposed the condition $\delta q(t_0) = \delta q(t_1) = 0$. So for q to be a solution of the variational problem from equation (2.4), we find the necessary condition

$$\frac{\partial L(q,\dot{q},t)}{\partial q} - \frac{d}{dt}\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} = 0\,. \tag{2.7}$$

Equation (2.7) is the famous Euler-Lagrange equation.

Example 2.1 The easiest example for the situation described here the case when $M = \mathbb{R}$, and the Lagrangian function L is given by $L(q,\dot{q},t) = \frac{1}{2}\dot{q}^2 - V(q)$. Here the function $V : \mathbb{R} \to \mathbb{R}$ denotes the potential energy of a particle moving on the real line. In this situation, the resulting Euler-Lagrange equation is given by

$$\frac{d}{dt}\dot{q} + V'(q) = 0\ .$$

Remark 2.2 In the considerations above, we have evaluated the derivative δS of the functional S on vector fields which vanish on the boundary. We could just as well insert more general vector fields into δS. That is, we can view δS as a one-form on the space $C^1([t_0,t_1],M)$ of differentiable paths in the manifold M. Thus, the functional S should be viewed a function on the space $C^1([t_0,t_1],M)$, and δS is the exterior differential of S.

Let us denote by $\mathrm{Sol}([t_0,t_1],M)$ the space of solutions of the Euler-Lagrange equations (2.7). Restricting the function S to the space $\mathrm{Sol}([t_0,t_1],M)$, we find

$$\delta S|_{\mathrm{Sol}([t_0,t_1],M)}(\delta q) = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta q|_{t_0}^{t_1} = \sigma_1(\delta q) - \sigma_0(\delta q)\ .$$

Here, σ_1 and σ_0 are one-forms on $\mathrm{Sol}([t_0,t_1],M)$. Applying the exterior derivative δ on both sides of the equation and using $\delta^2 = 0$, we get $\delta\sigma_1 = \delta\sigma_2$. Therefore, there exists a closed two-form

$$\omega = \delta\sigma_1 = \delta\sigma_2$$

on the space $\mathrm{Sol}([t_0,t_1],M)$. That is to say, the space $\mathrm{Sol}([t_0,t_1],M)$ is pre-symplectic. Note however that the two-form ω might be degenerate, so we can not expect $\mathrm{Sol}([t_0,t_1],M)$ to be symplectic.

Remark 2.3 Until now, we have been considering the situation of a particle moving on a manifold M. That is, we have been considering an action functional on the space of maps $C^1([t_0,t_1],M)$ of an interval $[t_0,t_1]$ to a manifold M. One can replace the interval $[t_0,t_1]$ by any higher dimensional manifold N with boundary: Let N be a n-dimensional manifold with boundary ∂N. After choosing a Lagrangian function

$$L : TM^{\otimes n} \to \mathbb{R},$$

we can associate to any differentiable map $\varphi : N \to M$ its action

$$S[\varphi] = \int_N L(\varphi(x), \partial_\nu\varphi(x))d\,\mathrm{vol}(x)\,.$$

As in the one-dimensional situation described above, one can pose a variational problem which leads to a generalization of the Euler-Lagrange equations. Again, one finds a closed (possibly degenerate) two-form on the space of solutions of these equations, which makes the space into a pre-symplectic manifold.

Another generalization of the situation described above is to replace maps $\varphi : N \to M$ by sections of a bundle on N. After defining an action functional, one can derive the corresponding Euler-Lagrange equations and see that the space of solutions of these equations is pre-symplectic. We will describe an example for such a situation in the next section.

2.2 Chern-Simons functionals

The real case

Let N be a compact three-dimensional manifold with boundary $\partial N = \Sigma$. Furthermore,

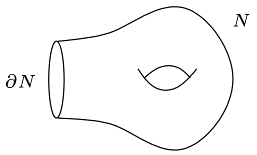


Figure 2.4: Three-dimensional manifold N with boundary $\partial N = \Sigma$.

let G be a compact Lie group with Lie-algebra $\mathfrak{g}$, and let $tr(. , .) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote the non-degenerate invariant bilinear form on $\mathfrak{g}$ (the Killing form). Fix a G-bundle $E \to N$ on N and let $\mathcal{A}$ denote the space of connections of the bundle E.

Definition 2.4 The Chern-Simons functional is a functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ which is defined by

$$\mathrm{CS}(A) = \int_N tr(A \wedge dA) + \frac{1}{3} \int_N tr(A \wedge [A \wedge A]) \, .$$

Proposition 2.5 *The set of solutions of the Euler-Lagrange equation for the Chern-Simons functional CS is the space of flat connections of the G-bundle E on the manifold N.*

Proof. Let $\delta A \in T_A\mathcal{A}$ be a tangent vector at the space $\mathcal{A}$ in the point A. That is, δA is a Lie algebra valued 1-form on the manifold N. Then we can calculate

$$\begin{aligned} \delta S(\delta A) &= \int_N tr(\delta A \wedge dA) + \int_N tr(A \wedge d\delta A) + \int_N tr(\delta A \wedge [A \wedge A]) \\ &= \int_N d\,tr(A \wedge \delta A)) + \int_N tr\left(\delta A \wedge (2dA + [A \wedge A])\right) \, . \end{aligned}$$

By Stokes theorem, the first term becomes $\int_{\partial N} \langle A \wedge \delta A \rangle$. Imposing the boundary condition $\delta A|_{\partial N} = 0$, we obtain the Euler-Lagrange equation

$$2dA + [A \wedge A] = 0 \, . \tag{2.8}$$

Using the fact that $[A \wedge A] = 2A \wedge A$, we see that the requirement that A is a solution of the Euler-Lagrange equation means that the curvature of A should vanish. Hence the space of solutions of equation (2.8) is exactly the space of flat connections on N. $\square$

The 1-form σ on the space of flat connections is given by

$$\sigma(\delta a) = \int_{\partial N} \langle a \wedge \delta a \rangle \,,$$

where a denotes the restriction of a connection A on the manifold N to the boundary $\Sigma = \partial N$. Obviously, the 1-form σ is invariant under the gauge group action. Therefore the closed 2-form $\partial\sigma$ is degenerate along the orbits of the action of the gauge group on the space of flat connections.

Exercise 2.2.1 Show that the 2-form $\partial\sigma$ is degenerate exactly along the orbits of the action of the gauge group G^{Σ}.

As a consequence of this exercise, we get another proof of the fact that the moduli space of flat connections carries a symplectic structure:

Corollary 2.6 *There exists a non-degenerate closed 2-form ω on the moduli space $\mathcal{M}^{\Sigma}$ of flat connections on $\Sigma = \partial N$.*

Remark 2.7 The Chern-Simons functional has the following invariance property: Assume that the manifold M is a compact three-dimensional manifold without boundary and suppose that A and $\widetilde{A}$ are connections of a G-bundle on M which are related by a gauge transformation g. That is

$$\widetilde{A} = gAg^{-1} - dgg^{-1} \,.$$

Then the Chern-Simons functional transforms via

$$CS(\widetilde{A}) = CS(A) + \frac{1}{12\pi} \int_M tr(g^{-1}dg)^{\wedge 3} \,.$$

The integral $\frac{1}{12\pi}\int_M tr(g^{-1}dg)^{\wedge 3}$ can be rewritten as the integral $\int_M g^*\mu$, where μ is a generator of the third cohomology group $H^3(G,\mathbb{Z})$ of the compact Lie group G. In particular, we have $\int_M g^*\mu \in \mathbb{Z}$ so that the exponential $e^{2\pi i CS(A)}$ is gauge invariant.

The complex case

According to the general philosophy we adopted in sections 1.5 and 1.6, the complex analogue of an oriented compact three dimensional manifold M should be a compact complex three dimensional variety X together with a meromorphic 3-form β which takes the part of the orientation. The polar divisor $\mathrm{div}_\infty \beta$ then plays the role of the boundary of the manifold M. From this point of view, it is rather obvious how to define an analogue of the Chern-Simons functional in this situation.

Let $G_{\mathbb{C}}$ be a complex Lie group and let $E_{\mathbb{C}}$ be a complex $G_{\mathbb{C}}$-bundle on the variety X. As before, let us denote by $\mathcal{A}^{(0,1)}$ the space of $(0,1)$-connections on X.

Definition 2.8 The complex Chern-Simons functional $\mathrm{CS}_{\mathbb{C}} : \mathcal{A}^{(0,1)} \to \mathbb{C}$ is defined via

$$\mathrm{CS}_{\mathbb{C}}(A) = \int_X \beta \wedge \left(\langle A \wedge \bar{\partial} A \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \right) .$$

Exactly the same discussion as in the real case applies to the complex case to give the Euler-Lagrange equation

$$\bar{\partial} A + \frac{1}{2}[A \wedge A] = 0 . \tag{2.9}$$

Remember that the condition on the boundary values of a "real" connection has to be replaced by a condition on the values of a complex connection on the polar divisor of the "orientation" β. The space of solutions of the Euler-Lagrange equation (2.9) is the space of connections with vanishing $(0,2)$-curvature. This in turn can be identified with the set of complex structures on the bundle $E_{\mathbb{C}}$.

Let us denote the polar divisor of the meromorphic 3-form β by Y. Then the residue $\mathrm{res}_Y \beta$ is a non-vanishing 2-form on the divisor Y. In particular, Y has to be a $K3$-surface or a complex torus. In any case, restricting solutions of the Euler-Lagrange equation (2.9) to Y gives a holomorphic pre-symplectic structure on the space of connections on Y with vanishing $(0,2)$-curvature. Explicitly, this structure is given as $\omega_{\mathbb{C}} = \delta\sigma_{\mathbb{C}}$, where σ is the holomorphic one-form

$$\sigma_{\mathbb{C}}(\delta a) = \int_Y \mathrm{res}_Y \beta \wedge \langle a \wedge \delta a \rangle .$$

Here, a is the restriction of some connection A on the 3-dimensional variety X to the polar divisor Y, and δa is a tangent vector at a. As in the real case, $\omega_{\mathbb{C}}$ is degenerate along the orbits of the complex gauge group so that we obtain a holomorphic symplectic structure on the moduli space of holomorphic bundles on the divisor Y:

Corollary 2.9 (Mukai) *There exists a non-degenerate holomorphic closed two-form $\omega_{\mathbb{C}}$ on the moduli space $\mathcal{M}^Y$ of holomorphic structures on $Y = \mathrm{div}_\infty \beta$.* [2]

Remark 2.10 The *Casson invartiant* for a homology 3-sphere X is defined as follows: Choose a smooth surface $\Sigma \subset X$ without boundary. Consider the moduli space $\mathcal{M}_\Sigma$ of flat connections on the trivial $SU(2)$-bundle on the surface Σ. The embedding of the surface Σ into the 3-manifold X allows to write X as a union $X = X_1 \cup_\Sigma X_2$. Accordingly, we can define two Lagrangian submanifolds L_1 and L_2 of the symplectic manifold $\mathcal{M}_\Sigma$ an the subspaces of flat connections on the surface Σ which extend to X_1 and X_2 respectively. Then the Casson invariant is defined as

$$\text{Casson Invariant} = c(\text{intersection } \# \text{ of } L_1 \text{ and } L_2)$$

for some constant c. **RW2**

This picture can be generalized to an analogue of the Casson invariant in the holomorphic setting (see [19], [87]).

[2] As in section 1.6 we are only concerned with the moduli space locally around a smooth point

RW2: Reference for this ???

2.3 Chern-Simons functionals and linking numbers

Linking numbers

Recall the definition of the linking number for two closed curves in a simply connected manifold.

Definition 2.11 Let M be a simply connected manifold and γ_1 and γ_2 two non-intersecting oriented closed curves in M. Since the manifold M is simply connected, we can pick an oriented surface $D_1 \subset M$ (a Seifert surface for the curve γ_1) such that the surface D_1 is bounded by the curve γ_1 and such that D_1 and γ_2 intersect transversally. We can now define the linking number $lk(\gamma_1, \gamma_2)$ to be the intersection number of the surface D_1 and the curve γ_2 (see figure 2.3):

$$lk(\gamma_1, \gamma_2) = \#(D_1, \gamma_2).$$

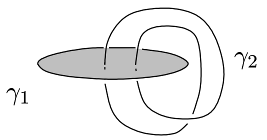


Figure 2.5: Linking of two oriented curves

Of course, one has to check that this number is well defined.

Proposition 2.12 *The linking number* $lk(\gamma_1, \gamma_2)$ *is*
(i) independent of the choice of D_1*,*
(ii) symmetric in γ_1 *and* γ_2*,*
(iii) invariant with respect to isotopy.
Furthermore, the linking number of two curves γ_1 *and* γ_2 *is well defined in any (not necessarily simply connected) 3 dimensional manifold* M *provided that both* γ_1 *and* γ_2 *are homologous to* 0*.*

Remark 2.13 The linking number is actually invariant with respect to changing the curve γ_1 to a curve $\widetilde{\gamma}_1$ which is homologous to γ_1 in $M \setminus \gamma_2$ (see figure 2.3).

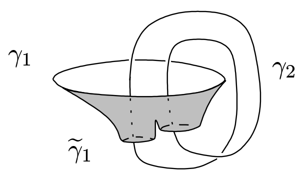


Figure 2.6: The homologous curves γ_1 and $\widetilde{\gamma}_1$ have the same linking number with the curve γ_2.

Remark 2.14 For the purpose of this section, we will need to write the linking number of two curves γ_1 and γ_2 in a slightly different way. Let $\Delta \subset M \times M$ denote the diagonal and let $\delta \in H_3(M \times M, \mathbb{R})$ denote its Poincaré dual. Then we can write

$$lk(\gamma_1, \gamma_2) = \#(\Delta, D_1 \times \gamma_2) = \int_{x \in D_1} \int_{y \in \gamma_2} \delta(x, y) \ . \tag{2.10}$$

One can split the three-form δ into

$$\delta = \delta^{3,0} + \delta^{2,1} + \delta^{1,2} + \delta^{0,3} \ ,$$

where $\delta^{i,j}$ denotes the component which is an i-form on the first factor of $M \times M$, and a j-form on the second factor. Note that in formula 2.10 we only had to integrate over the component $\delta^{2,1}$ since all the other integrals vanish.

Gaussian Integrals

Before we can describe the physics approach to linking numbers via the Chern-Simons functional we have to recall how infinite dimensional Gaussian integrals are dealt with in the physics literature. For a start, let us have a look at finite-dimensional Gaussian integrals: Let Q be a symmetric positive definite bilinear form on $\mathbb{R}^n$ and denote by $(. , .)$ the usual scalar product on $\mathbb{R}^n$. Then we can calculate

$$\int_{\mathbb{R}^n} e^{\frac{1}{2}(x,Qx)} dx^n = \left(\frac{(2\pi)}{\det Q} \right)^{\frac{1}{2}} =: Z(0) \, . \tag{2.11}$$

Now let $J \in \mathbb{R}^n$ be some vector and consider the integral

$$Z_Q(J) = \int_{\mathbb{R}^n} e^{\frac{1}{2}(x,Qx)+(J,x)} dx^n \, . \tag{2.12}$$

This integral can easily be solved by completing the square. Indeed, let x_0 be a solution of the equation $Qx_0 + J = 0$, i.e. $x_0 = -Q^{-1}J$. Then we can write $\widetilde{x} = x - x_0$, so that the integral becomes

$$\begin{aligned} Z_Q(J) =& \int_{\mathbb{R}^n} e^{\frac{1}{2}(\widetilde{x}+x_0, Q(\widetilde{x}+x_0))+(J,\widetilde{x}+x_0)} dx^n \\ =& \int_{\mathbb{R}^n} e^{\frac{1}{2}(\widetilde{x},Q\widetilde{x})+\frac{1}{2}(x_0,Qx_0)+(J,x_0)} d\widetilde{x}^n \\ =& e^{\frac{1}{2}(Q^{-1}J,J)-(J,Q^{-1}J)} \, Z_Q(0) = e^{-\frac{1}{2}(Q^{-1}J,J)} \, Z_Q(0) \, . \end{aligned}$$

Thus, we have

$$\frac{Z_Q(J)}{Z_Q(0)} = e^{-\frac{1}{2}(Q^{-1}J,J)} \, . \tag{2.13}$$

Remark 2.15 When the space $\mathbb{R}^n$ is replaced by some infinite dimensional vector space, the integrals defining $Z_Q(0)$ and $Z_Q(J)$ do usually not make sense. However, ignoring this fact one can "calculate" their ratio. That is, let Q be a symmetric positive definite operator on some inner product space $\mathcal{H}$, and let $J \in \mathcal{H}$ be some vector. Then the ratio $\frac{Z_Q(J)}{Z_Q(0)}$ is defined by equation (2.13).

The Chern-Simons functional and linking numbers

Let us apply these ideas in the case of the Chern-Simons functional. Let $\mathcal{A}$ denote the space of connections of a $U(1)$-bundle on a simply connected three-dimensional manifold M and let $CS : \mathcal{A} \to \mathbb{R}$ denote the Chern-Simons functional. Since the group $U(1)$ is Abelian, the term including $A \wedge [A \wedge A]$ vanishes. So in this case the Chern-Simons functional reads

$$\mathrm{CS}(A) = \frac{1}{2} \int_M A \wedge dA \, .$$

Let us choose some $J \in \mathcal{A}$ and define

$$S_J(A) = \frac{1}{2} \int_M A \wedge dA + \int_M A \wedge J \, .$$

The Gaussian integral $\int_{\mathcal{A}} e^{S_J}$ formally looks like the integral in equation (2.12). So we can apply the definition of the ratio $\frac{Z(J)}{Z(0)}$ from Remark 2.15 and calculate

$$\frac{Z(J)}{Z(0)} = e^{-\frac{1}{2} \int_M A_0 \wedge J} \, ,$$

where A_0 is a solution of the equation $dA + J = 0$.

To make a connection to the linking number, let γ_i, $i = 1, \dots, k$ be closed, non-intersecting curves in the manifold M and let δ_{γ_i} be Dirac-delta type 2-forms on M supported on the curve γ_i. To be more precise, remember that we have defined the 3-form $\delta \in H^3(M \times M, \mathbb{R})$ to be the Poincaré dual of the diagonal in $M \times M$. We can split the 3-form δ into its components $\delta = \delta^{3,0} + \delta^{2,1} + \delta^{1,2} + \delta^{0,3}$, where $\delta^{i,j}$ is the component of δ which is an i-form on the first factor of $M \times M$, and a j-form on the second factor (see Remark 2.10). Denoting the coordinates of the first factor of $M \times M$ by x and the components of the second factor by y, we can now write

$$\delta_{\gamma_i}(x) = \int_{y \in \gamma_i} \delta^{2,1}(x, y) \, .$$

Since the manifold M is simply connected, we can choose surfaces $D_i \subset M$ such that $\partial D_i = \gamma_i$. Let us define the delta-type 1-form supported on the surface D_i by

$$\delta_{D_i}(x) = \int_{y \in D_i} \delta^{1,2}(x, y) \, .$$

Then Stokes theorem gives

$$d_x \delta_{D_i}(x) = \delta_{\gamma_i}(x) \, ,$$

or more explicitly,

$$\delta_{\gamma_i}(x) = \int_{y \in D_i} d_x(\delta^{1,2})(x, y) \, ,$$

where d_x denotes the exterior derivative applied to the x-coordinates only.

Now let J be a linear combination of the delta-type 2-forms δ_{γ_i}. That is we write $J = \sum_i q_i \delta_{\gamma_i}$. Note that "the current" J is not an element of the space $\mathcal{A}$ but rather of its dual $\mathcal{A}^*$. Nevertheless, the definition of the Gaussian integral still applies to the function $e^{S_J(A)}$, and we can calculate

$$\begin{aligned}\frac{Z(J)}{Z(0)} =& e^{-\frac{1}{2}\int_M A_0\wedge J} = e^{-\frac{1}{2}\int_M A_0\wedge\sum_i q_i \int_{y\in\gamma_i}\delta^{2,1}(x,y)} \\ =& e^{-\frac{1}{2}\int_M A_0\wedge\sum_i q_i\int_{y\in D_i} d_x\delta^{1,2}(x,y)} = e^{\frac{1}{2}\int_M dA_0\wedge\sum_i q_i\int_{y\in D_i}\delta^{1,2}(x,y)} \\ =& e^{\frac{1}{2}\int_M \sum_j q_j\int_{z\in\gamma_j}\delta^{2,1}(x,z)\cdot\sum_i q_i\int_{y\in D_i}\delta^{1,2}(x,y)} \\ =& e^{\frac{1}{2}\sum_{i,j} q_i q_j \int_{z\in\gamma_j}\int_{y\in D_i}\delta(y,z)} = e^{\frac{1}{2}\sum_{i,j} q_i q_j lk(\gamma_i,\gamma_j)}\end{aligned}$$

Here we have used Stokes theorem several times as well as the definition of A_0 as a solution of the equation $dA + J = 0$.

Corollary 2.16 *The "path integral" $\frac{Z(J)}{Z(0)}$ counts the linking numbers of the γ_i.*

Remark 2.17 A similar discussion relates the holomorphic Chern-Simons functional with the polar linking number which will be defined in section 3.3.

Remark 2.18 Finally let us mention that replacing $U(1)$-bundles over the three dimensional manifold N by bundles based on non-Abelian groups, one can relate the Chern-Simons functional to knot invariants such as the Jones polynomial. Similar statements should be true in the case of the complex Chern-Simons functional but this is still an open problem ([29],[87]).

2.4 Bibliographical notes

An excellent account of the relation of the Chern-Simons functional to knot theory is the book by Atiyah [11]. The complex Chern-Simons functional has been introduced by several people, see [19], [29], [51], [88], [**?**] In this section, we follow [51], [19] and [94]. Mathematically rigorous approaches to the path integral in section 2.3 have been developed in [2],[3], and the non-Abelian case has been treated in [4].

3 Polar Homology

In sections 1 and 2 of this chapter, we have encountered several analogies between notions from differential topology and complex analysis which we list in the following table:

Real	Complex
real n-dimensional manifold	complex variety of dimension n
orientation	non-vanishing meromorphic n-form
boundary	polar divisor
Stokes Theorem	Cauchy-Stokes formula
flat connection	holomorphic structure
affine Lie algebra	elliptic Lie algebra
Chern-Simons functional	complex Chern-Simons functional
homology	polar homology

The last analogy of the table above is the subject of the current section. We begin by giving an informal introduction to polar homology before we state the precise definition on section 3.2. In section 3.3, we treat polar analogies of linking numbers and intersection numbers, and finally, in section 3.4 we briefly introduce polar homology of affine varieties.

3.1 Introduction to polar homology

In the analogy between real and complex manifolds described above, it seems natural to view the Dobleaut cohomology groups $H^{0,p}_{\bar\partial}(X)$ together with the $\bar\partial$-operator as a complex counterpart of the deRham cohomology $H^*(X)$ with the differential d. From this point of view, the natural question to ask is whether there exists a complex counterpart of the homology groups $H_i(X)$. That is, we would like to find "polar homology groups" $HP_p(X)$ such that there is a non-degenerate pairing

$$HP_p(X) \times H^{0,p}_{\bar\partial}(X) \to \mathbb{C}\,.$$

According to the table above, the part of the boundary operator in usual cohomology should be taken by an operator which assigns to a meromorphic form its polar divisor. Indeed, such groups can be constructed using the ideas of section 1.5. The Cauchy-Stokes formula (Theorem 1.34) gives an idea, how the corresponding chains should look like: The counterpart of a chain in singular cohomology should be a p-dimensional subvariety $W \subset X$ together with a meromorphic p-form β having poles of at most first order on the polar divisor $V \subset W$. According to the Cauchy-Stokes formula, the boundary operator ∂ sending p chains to $(p-1)$-chains should be given by $\partial(W,\beta) = 2\pi i(V, \mathrm{res}_V\,\beta)$. We will show later that one has $\partial^2 = 0$, so that the p-th polar cohomology group of X is given by $HP_p(X) = ker\partial/im\partial$.

Example 3.1 Consider the example $\alpha = \frac{dx\wedge dy}{xy}$ in $\mathbb{C}^2$. Then we have

$$\mathrm{res}\,|_{y=0}\,\mathrm{res}\,|_{x=0}\alpha = 1 = -\,\mathrm{res}\,|_{x=0}\,\mathrm{res}\,|_{y=0}\alpha\,.$$

Thus, the separated residue differs by a sign corresponding to the order of the divisor components. Hence, the total residue of the form α equals 0.

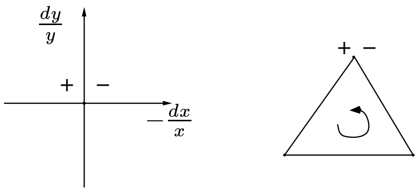


Figure 3.7: The relation $\partial^2 = 0$ in polar homology is a polar analogue of the relation $\partial^2 = 0$ in singular homology which comes from the fact that every vertex of the fundamental polygon appears twice with opposite signs.

Furthermore, we can define a pairing $\langle\, .\, ,\, .\, \rangle$ between the groups $HP_p(X)$ and $H^{(0,p)}_{\bar\partial}(X)$ via

$$\langle (W, \beta), u \rangle = \int_W \beta \wedge u \ .$$

By the Cauchy-Stokes formula, we have

$$\langle \partial(W, \beta), u \rangle = \langle (W, \beta), \partial u \rangle$$

so that the pairing is well defined.

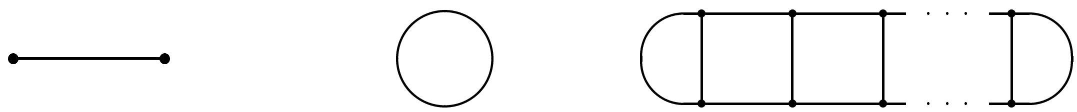
Example 3.2 Let us consider an example for the situation described above. Let Z be a complex projective curve and let us try to calculate the polar homology groups $HP_k(Z)$. In this (and in any) case, all the 0-chains are cycles. Let (P, a) and (Q, b) be two 0-cycles, where P, Q are points on Z and $a, b \in \mathbb{C}$. They are polar homologically equivalent iff $a = b$. Indeed, $a = b$ is necessary and sufficient for the existence of a meromorphic 1-form α on Z, such that $\mathrm{div}_\infty\, \alpha = P + Q$ and $\mathrm{res}_P\ \alpha = 2\pi i\, a$, $\mathrm{res}_Q\ \alpha = -2\pi i\, b$. (The sum of all residues of a meromorphic differential on a projective curve is zero by the Cauchy theorem.) Then, we can write in terms of polar chain complex (to be formally defined in the next section) that $(P, a) - (Q, a) = \partial\, (Z, \alpha)$. Thus, $HP_0(Z) = \mathbb{C}$.

Polar 1-cycles correspond to all possible holomorphic 1-forms on Z. On the other hand, there are no 1-boundaries, since there are no polar 2-chains in Z. Hence $HP_1(Z) \cong \mathbb{C}^g$, where g is the genus of the curve Z.

Remark 3.3 A real manifold whose homology groups have the same dimension over the reals as the polar homology groups of a Riemann surface Z_g of genus g is given by a wedge of g circles if $g > 1$, and by a closed interval if $g = 0$. These manifolds can be drawn as graphs as in figure 3.1.

Remark 3.4 One can define polar (k, l)-chains as pairs (A, α), where $A \subset X$ is a k-dimensional subvariety and α is an l-form on A. The relation $\partial^2 = 0$ still holds in this more general setup, and one obtains polar homology groups $HP_{k,l}$. Above, we have been considering the groups $HP_k = HP_{k,k}$. Another interesting case is the case $l = 0$ in which we consider pairs (A, a), where $a \in \mathbb{C}$ is a number. Two pairs (A_1, a_1) and (A_2, a_2) are equivalent if there exists a 1-form α on X which has opposite residues on A_1 and A_2.

$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g > 1$



Z is polar orientable (with polar boundary): there exists a 1-form with two simple poles and without zeros,

$$\dim HP_0(Z) = 1\,, \quad \dim HP_1(Z) = 0\,,$$

Z is polar orientable (without a polar boundary): there exists a 1-form without zeros or poles,

$$\dim HP_0(Z) = 1\,, \quad \dim HP_1(Z) = 1\,,$$

Z is not polar orientable: a generic holomorphic 1-form has $2g - 2$ zeros,

$$\dim HP_0(Z) = 1\,, \quad \dim HP_1(Z) = g\,.$$

Figure 3.8: The real analogue of a smooth projective curve Z of genus g.

Remark 3.5 One can view polar chains on the variety X as Green- or deRham currents, i.e. as functionals on the algebra of differential forms on X. Indeed, we can pair a polar chain (A, α) with a smooth differential form ω on X via

$$\langle (A, \alpha), \omega \rangle = \int_A \alpha \wedge \omega\,.$$

It follows immediately from the Cauchy-Stokes formula that this pairing is well-defined and gives a homomorphism

$$\rho : HP_k(X) \to H^{n,n-k}_{\bar\partial} \cong \left(H^{0,k}_{\bar\partial}\right)^*\,,$$

where $n = dim_{\mathbb{C}} X$ and more generally,

$$\rho : HP_{k,l}(X) \to H^{n-k+l,n-k}_{\bar\partial} \cong \left(H^{k-l,k}_{\bar\partial}\right)^*\,.$$

Remark 3.6 The intuitive definition of polar homology we have given in this section does not quite work in general. Rather than submanifolds (i.e. polar divisors) we should construct polar chains from images of varieties. This is in accordance with singular homology where one considers maps of abstract simplices into a manifold. In the following section, we will give a precise definition of the polar cohomology groups of a complex projective manifold.

3.2 Polar Homology of a projective variety

In this section we deal with complex projective varieties, i.e., subvarieties of a complex projective space. By a smooth projective variety we always understand a smooth and connected one. For a smooth variety X, we denote by Ω^p_X the sheaf of holomorphic p-forms on X. The sheaf $\Omega^{\dim X}_X$ of forms of the top degree on X will sometimes be denoted by K_X.

Before we can define the space of polar k-chains, we need to recall the notion of a push forward of a differential form.

Definition 3.7 Suppose that $f : X \to Y$ is a finite unramified covering and let φ be a function on X. Then the push-forward or trace $f_*\varphi$ of φ is a function on Y whose value at a point $y \in Y$ is calculated by summing the values of f over the set of pre-images of y taken with multiplicities.

Remark 3.8 The operation f_* can be generalized to holomorphic (resp. meromorphic) p-forms and to maps f which are only generically finite, i.e. if there exist analytic hyperplanes $D \subset Y$ such that f is an unramified finite covering outside D. Indeed, let ω be a meromorphic p-form on $U \subset (Y \setminus D)$ be an open set. Then $f^{-1}(U) = \cup_i U_i$ and $f|_{U_i}$ is an isomorphism. Let $s_j : U \to U_j$ be the inverse of $f^{-1}(U)$. Then on U, the p-form $f_*\omega$ is defined as

$$f_*\omega = \sum_j s_j^*\omega.$$

This defines a p-form on $Y \setminus D$. One can check that this form extends across the smooth points of the hyperplane D. Since the set of singular points has codimension greater that one in Y, the form $f_*\omega$ extends to a holomorphic (resp. meromorphic) p-form on Y provided the form ω itself was holomorphic (resp. meromorphic).

Example 3.9 Instead of giving a general proof of the existence of $f_*\omega$ which can be found in [33] let us consider the following example. Let $f : \mathbb{C} \to \mathbb{C}$ be defined as $x \mapsto f(x) = y = x^m$. Then any one-form $\omega = \psi(x)dx$ has a push-forward $f_*\omega = \varphi(y)dy$. Indeed, we have

$$\frac{dy}{y} = m\frac{dx}{x}$$

and hence, for $y \neq 0$, we can define

$$\varphi(y)dy = \frac{1}{m}\sum_j \frac{x_j\psi(x_j)}{y}dy \ ,$$

where x_j runs over all m^{th} roots of y. It remains to check that $\varphi(y)$ can be extended to $y = 0$. Expanding ψ into a power series

$$\psi(x) = \sum_{l\geq 0} a_l x^l \ ,$$

we find

$$\varphi(y) = \frac{1}{my}\sum_{l\geq 0} a_l \sum_j x_j^{l+1} = \frac{1}{my}\sum_{k\geq 1} a_{km-1}(my^k) = \sum_{k\geq 1} a_{km-1}y^{k-1} \ ,$$

where we have used that $\sum_j x_j^n = 0$ unless $n = km$. From this formula, we get

$$\lim_{y\to 0}\varphi(y) = \frac{1}{(m-1)!}\frac{d^{m-1}\psi(x)}{dx^{m-1}}|_{x=0}.$$

So ψ extends to a holomorphic function on $\mathbb{C}$.

The operations of push-forward and taking the residue are related in a nice way:

Proposition 3.10 *Let $f : X \to Y$ be a proper surjective holomorphic map between complex manifolds of the same dimension. Let ω be a meromorphic form on X with poles of at most first order on a smooth hypersurface $V \subset X$. Suppose that $f(V)$ is a smooth hypersurface in Y. Then $f_*\omega$ has first order poles on $f(V)$, and*

$$\operatorname{res} f_*\omega = \widetilde{f}_* \operatorname{res} \omega \ ,$$

where $\widetilde{f} : V \to f(V)$ is the restriction of f.

Finally, we can state the definition of polar k-chains. for a complex projective variety X with $\dim X = n$ which will be defined as a $\mathbb{C}$-vector space with certain generators and relations.

Definition 3.11 The space of *polar k-chains* $\mathcal{C}_k(X)$ is a vector space over $\mathbb{C}$ defined as the quotient $\mathcal{C}_k(X) = \hat{\mathcal{C}}_k(X)/\mathcal{R}_k$, where the vector space $\hat{\mathcal{C}}_k(X)$ is freely generated by the triples (A, f, α) described in (i),(ii),(iii) and $\mathcal{R}_k$ is a set of relations (R1),(R2),(R3) imposed on the triples.

(i) A is a smooth complex projective variety, $\dim A = k$;

(ii) $f : A \to X$ is a holomorphic map of projective varieties;

(iii) α is a rational k-form on A with first order poles on $V \subset A$, where V is a normal crossing divisor in A, i.e., $\alpha \in \Gamma(A, \Omega^k_A(V))$.

The relations are:

(R1) $\lambda(A, f, \alpha) = (A, f, \lambda\alpha)$

(R2) $\sum_i (A_i, f_i, \alpha_i) = 0$ provided that $\sum_i f_{i*}\alpha_i \equiv 0$, where $\dim f_i(A_i) = k$ for all i and the push-forwards $f_{i*}\alpha_i$ are considered on the smooth part of $\cup_i f_i(A_i)$.

(R3) $(A, f, \alpha) = 0$ if $\dim f(A) < k$.

Remark 3.12 Relation (R3) implies that $\mathcal{C}_k(X) = 0$ for for $k > dim_{\mathbb{C}}X$. Relation (R2) allows us, to deal with pairs instead of triples replacing a triple (A, f, α) by a pair $(\hat{A}, \hat{\alpha})$, where $\hat{A} = f(A) \subset X$, $\hat{\alpha}$ is defined only on the smooth part of $\hat{A}$ and $\hat{\alpha} = f_*\alpha$ there. Due to the relation (R2), such a pair $(\hat{A}, \hat{\alpha})$ carries precisely the same information as (A, f, α). (The only point to worry about is that such pairs cannot be arbitrary. In fact, by the Hironaka theorem on resolution of singularities, any subvariety $\hat{A} \subset X$ can be the image of some regular A, but the form $\hat{\alpha}$ on the smooth part of $\hat{A}$ cannot be arbitrary.)

One can also define a boundary operator.

Definition 3.13 The boundary operator $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ is defined by

$$\partial(A, f, \alpha) = 2\pi i \sum_i (V_i, f_i, \operatorname{res}_{V_i} \alpha)$$

(and by linearity), where V_i are the components of the polar divisor of α, $\operatorname{div}_\infty \alpha = \cup_i V_i$, and the maps $f_i = f|_{V_i}$ are restrictions of the map f to each component of the divisor.

Theorem 3.14 *The boundary operator ∂ is well defined, i.e. it is compatible with the relations* (R1), (R2), *and* (R3). *Furthermore, $\partial^2 = 0$.*

For a full proof of this theorem, we refer the reader to [52]. Note however that the relation $\partial^2 = 0$ is essentially a consequence of the equation $\operatorname{res}_{i,j}\gamma = -\operatorname{res}_{j,i}\gamma$ in Remark 1.36 (see example 3.1).

Definition 3.15 For a smooth complex projective variety X, with $\dim X = n$, the chain complex

$$0 \to \mathcal{C}_n(X) \xrightarrow{\partial} \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0(X) \to 0$$

is called the polar chain complex of X. Its homology groups, $HP_k(X), k = 0, \ldots, n$, are called the polar homology groups of X.

Example 3.16 The polar cohomology groups of a complex projective curve Z of genus g can be computed using the same arguments as in Example 3.2. We find that the polar homology groups are $HP_1(X) = \mathbb{C}$ and $HP_2(X) = \mathbb{C}^g$.

Finally, we can define a pairing between the Dobleaut cohomology groups $H^{(0,k)}(X)$ and the polar homology group $HP_{n-k}(X)$ as follows:

Definition 3.17 The pairing between an equivalence class of a triple $a = (A, f, \alpha)$ and a smooth $(0, k)$-form u on X, is given by the following integral:

$$\langle a \,, u \rangle \;:= \int_A \alpha \wedge f^* u \;.$$

The integral is well defined since the k-form α has only first order poles on a normal crossing divisor. It is now straightforward to show that the pairing $\langle \;,\; \rangle$ descends to the space of equivalence classes of triples $\mathcal{C}_k(X)$, i.e., that it is compatible with the relations (R1), (R2), (R3) of Definition 3.11. Indeed, (R1) is obvious, compatibility with (R3) follows from noticing that $f^*u = 0$ if $\dim f(A) < k$, and the compatibility with (R2) follows from the relation $\int_A \alpha \wedge f^*u = \int_{f(A)} f_*\alpha \wedge u$ if $\dim f(A) = k$, where the last integral is taken over the smooth part of $f(A)$.

Proposition 3.18 *The pairing defined above defines the following homomorphism in (co-)homology:*

$$\rho \colon HP_k(X) \to H_{\bar\partial}^{(n,n-k)},$$

where $n = \dim X$.

Proof. By Serre duality, ρ is the map $HP_k(X) \to (H_{\bar\partial}^{0,k}(X))^*$ and it is sufficient to verify that the pairing vanishes if $\partial a = 0$ and $u = \bar\partial v$, or if $\bar\partial u = 0$ and $a = \partial b$. This follows immediately from the Cauchy-Stokes formula (Theorem 1.34 of chapter 1.3):

$$\int_A \alpha \wedge f^*(\bar\partial u) = 2\pi i \int_{\operatorname{div}_\infty \alpha} (\operatorname{res} \alpha) \wedge f^*(u) \;,$$

that is $\langle a \,, \bar\partial u\rangle = \langle \partial a \,, u\rangle$. □

In fact, the homomorphism above is an isomorphism:

Theorem 3.19 (Polar deRham Theorem) *The map* $\rho : HP_k(X) \to H_{\bar{\partial}}^{(n,n-k)}$ *is an isomorphism.*

See [55] for a proof of this theorem.

3.3 Operations on polar homology

Polar intersection

In this section, we show how one can define a "polar analogue" for the intersection number in usual homology. Recall that in singular homology, one considers a smooth oriented closed manifold M and two oriented closed submanifolds $A, B \subset M$ of complementary dimensions, i.e., $\dim A + \dim B = \dim M$. Suppose, A and B intersect transversally at a finite number of points. Then to each intersection point P one assigns the local intersection index which is an element of $\{\pm 1\}$ by comparing the mutual orientations of the tangent vector spaces $T_P A, T_P B$, and $T_P M$.

Let X be a complex compact manifold of dimension n. In order to define a polar analogue of an intersection index, X needs to carry a "polar orientation". Recall that the analogue of an orientation on a real manifold is a holomorphic, nowhere vanishing n-form μ on X. Note that the existence of such a form requires the normal bundle of X to be trivial. Hence X has to be a Calabi-Yau manifold or a complex torus.

Definition 3.20 Given a polar oriented complex manifold X and two polar cycles (A, α) and (B, β) of complementary dimensions which intersect transversally in X. The forms α and β are to be viewed as "polar orientations" of the submanifolds A and B. The polar intersection number of the cycles (A, α) and (B, β) is defined by the formula

$$\langle (A, \alpha) \cdot (B, \beta) \rangle = \sum_{P \in A \cap B} \frac{\alpha(P) \wedge \beta(P)}{\mu(P)} .$$

Remark 3.21 At every intersection point P, the ratio in the right-hand-side is the "comparison" of the orientations of the polar cycles at that point with the orientation of the ambient manifold. This is a straightforward analogue of the use of mutual orientation of cycles in the definition of the topological intersection number. Note, that in the polar case the intersection number does not have to be an integer. Rather, it should be viewed as a function on the "parameters" $(A, \alpha), (B, \beta)$ and (X, μ)

Remark 3.22 Essentially the same formula can be used to define an intersection product of cycles when they intersect on a manifold of positive dimension. See e.g. [50] for more on this.

Polar linking number

A similar construction as in the previous section allows to define a polar analogue to the concept of a linking number. Recall that the linking number is defined for two non-intersecting oriented closed curves γ_1, γ_2 in a simply connected manifold M. Since M is simply connected, we can pick an oriented surface $D_1 \subset M$ such that D_1 is bounded by

γ_1 and such that D_1 and γ_2 intersect transversally. We can now define the linking number $lk(\gamma_1, \gamma_2)$ to be the intersection number of D_1 and γ_2:

$$lk(\gamma_1, \gamma_2) = \#(D_1, \gamma_2).$$

Using the polar intersection number, it is clear how to proceed and define a polar linking number.

Definition 3.23 Let (X, μ) be an oriented complex manifold and let (A_1, α_1) and (A_2, α_2) be two polar one cycles in X. Suppose that both cycles are boundaries (the analogue of a curve in M being contractible). Then we can pick a complex surface B_1 and a meromorphic 2-form β_1 on B_1 such that $\partial(B_1, \beta_1) = (A_1, \alpha_1)$. Suppose that the curves A_1 and A_2 do not have any common points. Then we can define the polar linking number of (A_1, α_1) and (A_2, α_2) as the polar intersection number of (B_1, β_1) and (A_2, α_2):

$$lk_{polar}\left((A_1, \alpha_1), (A_2, \alpha_2)\right) := \sum_{P \in B_1 \cap A_2} \frac{\beta_1(P) \wedge \alpha_2(P)}{\mu(P)} \ .$$

Theorem 3.24 *The polar linking number $lk_{polar}((A_1, \alpha_1), (A_2, \alpha_2))$ is*
(i) independent of the choice of (B_1, β_1),
(ii) symmetric in (A_1, α_1) and (A_2, α_2),
(iii) homolocically invariant when (A_1, α_1) is replaced by a cocycle (A_1', α_1') in the complement of A_2 which is polar homologous to (A_1, α_1).

Remark 3.25 The polar linking number is related to the complex Chern-Simons functional in the same way, the usual linking number is related to the real Chern-Simons functional (see section 2.3).

3.4 Polar homology for affine varieties

In the preceding sections, we have been studying polar homology of projective manifolds. Of course, it would be nice to have an analogue of homology of arbitrary varieties. In our analogy, projective varieties played the role of compact manifolds in usual topology. So it is natural to expect that the role of non-compact manifolds should be played by quasi projective varieties, i.e. Zariski open subsets in projective varieties. In this section we will indicate how a theory of polar homology for quasi projective varieties can be developed.

Let X be a quasi projective variety and let $\bar{X}$ be its projective closure. That is, X is Zariski open in $\bar{X}$. We take the space $\mathcal{C}_k(X)$ of k-chains in X to be the subset of k-chains in $\bar{X}$ such that the according k-forms vanish outside X.

$$\mathcal{C}_k(X) = \{(A, f, \alpha) \in \mathcal{C}_k(\bar{X}) \mid \alpha(x) = 0 \text{ for all } x \in \bar{X} \setminus X\} \ .$$

Proposition 3.26 *The spaces $\mathcal{C}_k(X)$, form a subcomplex in the polar chain complex $(\mathcal{C}_\bullet(\bar{X}), \partial)$ which depends only on the variety X and not on the choice of its compactification, the projective variety $\bar{X}$.*

The resulting homology groups of the chain complex $(\mathcal{C}_\bullet(X), \partial)$ are denoted as before by $PH_k(X)$ and are called polar homology groups of X. One of the good properties of the polar homology groups is that they behave like ordinary homology groups with respect to taking unions of Zariski open subsets of a projective variety $\bar{X}$.

Theorem 3.27 (Mayer–Vietoris sequence.) *Let a complex variety X (either quasi projective or projective) be the union of two Zariski open subsets U_1 and U_2 , $X = U_1 \cup U_2$. Then the following sequence of chains is exact:*

$$0 \to \mathcal{C}_k(U_1 \cap U_2) \xrightarrow{i} \mathcal{C}_k(U_1) \oplus \mathcal{C}_k(U_2) \xrightarrow{\sigma} \mathcal{C}_k(X) \to 0.$$

Here the map σ represents the sum of chains,

$$\sigma : a \oplus b \mapsto a + b,$$

and the map i is the embedding of the chain lying in the intersection $U_1 \cap U_2$ as a chain in each subset U_1 and U_2 :

$$i : c \mapsto (c) \oplus (-c).$$

This implies the following exact sequence in polar homology:

$$\cdots \to HP_k(U_1 \cap U_2) \xrightarrow{i} HP_k(U_1) \oplus HP_k(U_2) \xrightarrow{\sigma} HP_k(X) \to HP_{k-1}(U_1 \cap U_2) \to \dots$$

Example 3.28 As an example, let us consider the case when $X = \bar{X} \setminus \{P\}$ is a smooth projective curve without a point. Then $\dim HP_1(X) = g-1$ if $g > 0$ and $\dim HP_1(X) = 0$ if $g = 0$. Indeed, the space $HP_1(X)$ is the space of holomorphic 1-forms on $\bar{X}$ which vanish at P. To calculate $HP_0(X)$ in the case $g > 0$ it is sufficient to notice that for any two points $Q_1, Q_2 \in X$, the 0-cycle $(Q_1, q_1) + (Q_2, q_2)$ is homologically equivalent to zero if and only if $q_1 + q_2 = 0$ (the same condition as in the case of a non-punctured curve which we discussed in the introduction to this chapter). In the case of $g = 0$ an analogous statement requires three points to be involved (unlike the case of a non-punctured projective line): the corresponding 1-form on $\mathbb{CP}^1$ has to have at least one zero, and hence at least three poles. So we find $\dim HP_0(X) = 1$ if $g > 0$ and $\dim HP_0(X) = 2$ if $g = 0$.

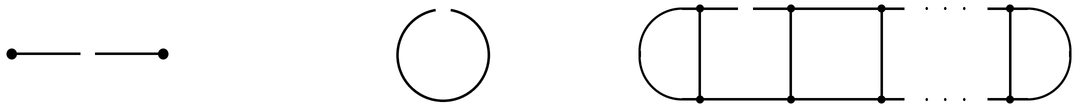
We collect the results about the curves in figure 3.28 (where we depict the complex curves by graphs, such that polar homology groups of the curves coincide with singular homology groups of the corresponding graphs).

Example 3.29 In a similar way, one can deal with the case of a smooth projective curve without two points, $X = \bar{X} \setminus \{P, Q\}$. We summarize the results in figure 3.4. Here, one has to distinguish the case of generic points P and Q, and the case when $P + Q$ is a special divisor in which case there are more 1-differentials with zeros at P, Q than generically.

3.5 Bibliographical notes

In this section, we followed the papers [50, 51, 52, 53, 55].

$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g > 1$

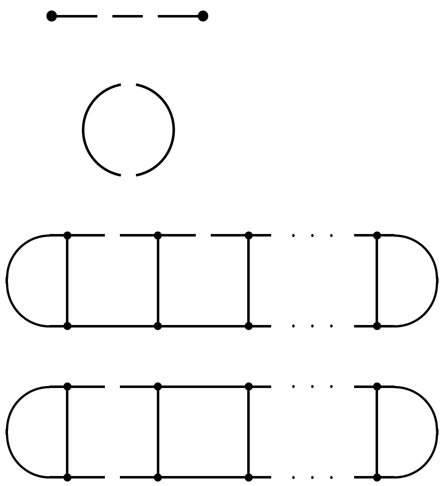


$\dim HP_0(X) = 2\,,$
$\dim HP_1(X) = 0\,,$

$\dim HP_0(X) = 1\,,$
$\dim HP_1(X) = 0\,,$

$\dim HP_0(X) = 1\,,$
$\dim HP_1(X) = g - 1\,.$

Figure 3.9: A smooth projective curve without a point, $X = \bar{X} \smallsetminus \{P\}$.



$g = 0$: $\dim HP_0(X) = 3\,,$ $\dim HP_1(X) = 0\,,$

$g = 1$: $\dim HP_0(X) = 2\,,$ $\dim HP_1(X) = 0\,,$

$g > 1\,,\ P, Q$ are generic: $\dim HP_0(X) = 1\,,$ $\dim HP_1(X) = g - 2\,,$

$g > 1\,,\ P + Q$ is special: $\dim HP_0(X) = 2\,,$ $\dim HP_1(X) = g - 1\,.$

Figure 3.10: A smooth projective curve without two points, $Z \smallsetminus \{P, Q\}$.

Appendices

A.1 Compact Lie groups

1.1 Maximal Tori and conjugacy

1.2 The fundamental group

1.3 Topology of semisimple simply connected compact Lie groups

1.4 Main structure Theorem

A.2 Root systems

2.1 Finite root systems

2.2 Affine root systems

2.3 Elliptic root systems

A.3 Twisted Calogero Moser Systems and elliptic root systems

This would be new...

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