Symmetry algebras of Lagrangian Liouville-type systems

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Abstract

The generators and commutation relations are calculated explicitly for higher symmetry algebras of a class of hyperbolic Euler–Lagrange systems of Liouville type (in particular, for 2D Toda chains associated with semi-simple complex Lie algebras).

Key words: Symmetries, 2D Toda chains, Liouville-type systems, Hamiltonian hierarchies, brackets.

Introduction. We give a description of the generators and relations in higher symmetry algebras for a class of Darboux-integrable hyperbolic Euler–Lagrange systems of Liouville type [1, 2, 3]. There exist many non-equivalent definitions of this type of PDEs [1, 3, 4]; we investigate the systems $\mathcal{E}_{\mathrm{L}}$ that admit as many first integrals of the characteristic equations $D_y(w) \doteq 0$ and $D_x(\bar{w}) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$ as there are unknown functions. The 2D Toda chains $\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u})$ associated with semi-simple complex Lie algebras are the most well studied example of such equations [1, 2, 5, 6, 7]. The systems of this class are known to possess higher symmetries $\varphi = \square(\boldsymbol{\phi})$ that depend on free functional parameters $\boldsymbol{\phi} = {}^t\bigl(\phi_1(x,[w]),\ \ldots,\ \phi_r(x,[w])\bigr)$ and belong to the image of matrix total differential operators $\square$ (linear operators in total derivatives) [6, 8, 9, 10]. The existence of such operators $\square$ for Liouville-type systems was observed in [1, 6] and [3, 10], where the importance of the linearizations $\ell_w^{(u)}$ of the first integrals w in the construction of $\square$ was revealed. In the paper [9] we proved that the additional assumption for $\mathcal{E}_{\mathrm{L}}$ be Euler–Lagrange strengthens known results and even makes the description of $\square$ explicit, see formula (3) below.

In this paper we establish the transformation rules for the operators $\square$ under unrelated reparametrizations of the coordinates in their domains and images. We show that, under natural assumptions on the geometry of $\mathcal{E}_{\mathrm{L}}$, the images of these operators are closed with respect to the commutation, whence the Lie algebra structure on their domains appears. We calculate the brackets on the domains explicitly, which yields, by the push forward of the Lie algebra structure, the commutation relations in the symmetry algebras $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$. To do this, we introduce auxiliary Hamiltonian operators which have the same domain as $\square$.

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Remark. We do not assume the presence of a symmetry $x \leftrightarrow y$ in $\mathcal{E}_{\mathrm{L}}$. We work with 'the x-half' of the algebra $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$ related to the first integrals $w^i \in \ker D_y\big|_{\mathcal{E}_{\mathrm{L}}}$; the reasonings hold for the respective 'y-half' of $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$, and the two subalgebras commute between each other. For the Euler–Lagrange systems $\mathcal{E}_{\mathrm{L}}$ at hand, the integrals $\bar{w}^{\bar{\imath}} \in \ker D_x\big|_{\mathcal{E}_{\mathrm{L}}}$ are not used in the proofs, unlike in [10] for arbitrary Liouville-type systems.

The full list of assumptions on the systems $\mathcal{E}_{\mathrm{L}}$ and their integrals is given in our main Theorem 6, see also Remark 2 on p. 9. However, the reasonings in section 1 hold under less restrictive conditions. In particular, the number of first integrals w^1, ..., w^r for the characteristic equation on $\mathcal{E}_{\mathrm{L}}$ can be less than the number of the unknowns u^1, ..., u^m in $\mathcal{E}_{\mathrm{L}}$. In that case, the auxiliary $(r \times r)$-matrix operators $\hat{A}_k$ defined in (7) become smaller in size but remain Hamiltonian (see [9] for the second Poisson structure for KdV provided by the 2D Toda chains with a unique integral).

The paper is organized as follows. First we define the operators $\square$ that determine symmetry generators for the systems $\mathcal{E}_{\mathrm{L}}$ and introduce auxiliary Hamiltonian operators. Here we re-derive the higher Poisson structures for the Drinfel'd–Sokolov hierarchies [11] on 2D Toda chains related to semi-simple complex Lie algebras; an example is given for the A_2-Toda chain. Then in section 2 we establish the commutation closure for images of the operators $\square$ and calculate the structural relations in the algebras $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$; an illustration is given for the Kaup–Boussinesq equation. Finally, in section 3 we discuss some properties of the operators that yield symmetries of non-Lagrangian Liouville-type systems.

All notions and constructions from geometry of PDE are standard [12, 13, 14]. We follow the notation of [9, 15, 16]. This paper develops further the concept of [9].

1. Symmetry generators for $\mathcal{E}_{\mathrm{L}}$.

Definition. A *Liouville-type system* $\mathcal{E}$ is a system $\{\boldsymbol{u}_{xy} = f(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y; x, y)\}$ of m hyperbolic equations upon $\boldsymbol{u} = (u^1, \ldots, u^m)$ which admits nontrivial *first integrals*

$$w^1,\ \ldots,\ w^r \in \ker D_y\big|_{\mathcal{E}};\quad \bar{w}^1,\ \ldots,\ \bar{w}^{\bar{r}} \in \ker D_x\big|_{\mathcal{E}},\qquad 0 < r, \bar{r} \leq m,$$

for the linear first order characteristic equations $D_y\big|_{\mathcal{E}}(w^i) \doteq 0$ and $D_x\big|_{\mathcal{E}}(\bar{w}^{\bar{\jmath}}) \doteq 0$ that hold by virtue ($\doteq$) of $\mathcal{E}$.

Example 1. In [2] it was proved that the 2D Toda chains [5] $u^i_{xy} = \exp(K^i_j u^j)$ related to semi-simple complex Lie algebras with the Cartan matrices K admit maximal ($r = \bar{r} = m$) sets of the integrals. Various methods for reconstruction of w^i, $\bar{w}^{\bar{\jmath}}$ for these exponential-nonlinear Toda chains were proposed in [3, 4, 7]. The differential orders (after a shift by -1) of the integrals w^1, ..., w^r w.r.t. $\boldsymbol{u}$ are equal to the exponents of the corresponding semi-simple complex Lie algebras of rank r, which follows from [2, p. 21].

For instance, in the sequel we consider the Euler–Lagrange 2D Toda system $\mathcal{E}_{\mathrm{Toda}}$ associated with the simple Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, see [1, 5, 7],

$$\mathcal{E}_{\mathrm{Toda}} = \bigl\{u_{xy} = \exp(2u - v),\ v_{xy} = \exp(-u + 2v),\qquad K = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)\bigr\}. \quad (1)$$

The integrals of respective orders 2 and 3 for system (1) are (e.g., see [17]) $w^1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2$ and $w^2 = u_{xxx} - 2u_x u_{xx} + u_x v_{xx} + u_x^2 v_x - u_x v_x^2$.

The generators $\varphi = \square\bigl(\boldsymbol{\phi}(x,[w])\bigr)$ of higher symmetry algebras for Liouville-type equations are given by matrix total differential operators $\square$, see [3, 6]. For Euler–Lagrange Liouville-type systems $\mathcal{E}_{\mathrm{L}} = \{F \equiv \mathbf{E}(\mathcal{L}) = 0\}$, see [8, 9, 18], the existence of *certain* factorizations for at least a *part* of symmetries is rigorous and can be readily seen as follows. For integrals w such that $D_y(w) = \nabla(F)$ vanishes on the differential ideal $\{F = 0\}^\infty$ by virtue of an operator ∇, and for any $I(x,[w])$, the generating section $\psi_I = \Bigl[\nabla^* \circ \bigl(\ell_w^{(u)}\bigr)^* \circ \bigl(\ell_I^{(w)}\bigr)^*\Bigr](1)$ for a conservation law $\int I\,\mathrm{d}x$ solves the equations $\ell^*_{\mathbf{E}(\mathcal{L})}(\psi_I) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$, see [12, 13, 14]. The Helmholtz condition $\ell_{\mathbf{E}(\mathcal{L})} = \ell^*_{\mathbf{E}(\mathcal{L})}$ for the linearization (the Frechét derivative) implies that the vector

$$\varphi[\boldsymbol{\phi}] = \Bigl[\nabla^* \circ \bigl(\ell_w^{(u)}\bigr)^*\Bigr]\bigl(\boldsymbol{\phi}(x,[w])\bigr) \in \ker \ell_{\mathbf{E}(\mathcal{L})}\bigr|_{\mathcal{E}_{\mathrm{L}}} \tag{2}$$

is a symmetry of $\mathcal{E}_{\mathrm{L}}$ for any $\boldsymbol{\phi} = \bigl(\ell_I^{(w)}\bigr)^*(1) = \mathbf{E}_w(I\,\mathrm{d}x)$. A standard homological reasoning (see [13, Ch. 5] or [14, §7.8]) shows that formula (2) yields symmetries of the system $\mathcal{E}_{\mathrm{L}}$ even if sections $\boldsymbol{\phi}$ do not belong to the image of the variational derivative $\mathbf{E}_w$ w.r.t. w.

In this section we recall the construction of operators $\square$ that determine symmetries for a class of Euler–Lagrange Liouville-type systems. We suppose that the integrals w are *minimal*, meaning $I \in \ker D_y\bigr|_{\mathcal{E}_{\mathrm{L}}}$ implies $I = I(x,[w])$.

Proposition 1 ([9])**.** Let κ be an invertible symmetric constant real $(m \times m)$-matrix. Suppose that $\mathcal{L} = \int L\,\mathrm{d}x\mathrm{d}y$ with the density $L = -\frac{1}{2}\sum_{i,j}\kappa_{ij}u_x^i u_y^j - H_{\mathrm{L}}(u;x,y)$ is the Lagrangian of a Liouville-type equation $\mathcal{E}_{\mathrm{L}} = \{\mathbf{E}(\mathcal{L}) = 0\}$. Let $\mathfrak{m} = \partial L/\partial u_y$ be the momenta and $w(\mathfrak{m}) = (w^1,\ldots,w^r)$ be the minimal set of integrals for $\mathcal{E}_{\mathrm{L}}$ that belong to the kernel of $D_y\bigr|_{\mathcal{E}_{\mathrm{L}}}$. Then the adjoint linearization

$$\square = \bigl(\ell_w^{(\mathfrak{m})}\bigr)^* \tag{3}$$

of the integrals w.r.t. the momenta yields Noether symmetries $\varphi_{\mathcal{L}}$ of $\mathcal{E}_{\mathrm{L}}$:

$$\varphi_{\mathcal{L}} = \square\bigl(\delta\mathcal{H}/\delta w\bigr) \qquad \text{for any } \mathcal{H} = \int H(x,[w])\,\mathrm{d}x. \tag{4}$$

Corollary 2. Under the assumptions and notation of Proposition 1, the section

$$\varphi = \square\bigl(\boldsymbol{\phi}(x,[w])\bigr) \tag{5}$$

is a symmetry of the Liouville-type equation $\mathcal{E}_{\mathrm{L}}$ for any r-tuple $\boldsymbol{\phi} = {}^t(\phi_1,\ldots,\phi_r)$.

Proof. Consider the jet bundle $J^\infty(\xi)$ over the fibre bundle $\xi\colon \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}$ with the base $\mathbb{R} \ni x$ and the fibres $\mathbb{R}^r$ with coordinates $w^1,\ldots,w^r$. By Proposition 1, the statement is valid for any $\boldsymbol{\phi}$ in the image of the variational derivative $\mathbf{E}_w$. Obviously, its image contains all variational covectors $\boldsymbol{\phi}$ whose components $\phi_i(x) \in C^\infty(\mathbb{R})$ are functions on the base of the new bundle ξ. The prolongation of the substitution $w = w\bigl[\mathfrak{m}[u]\bigr]\colon J^\infty(\pi) \to \Gamma(\xi)$ converts the components of sections $\boldsymbol{\phi}$ to smooth differential functions in u (which denotes the set of fibre coordinates in the bundle π over the same base $\mathbb{R} \ni x$). Now recall that $\square$ is an operator in total derivatives D_x whose action on differential functions $f[u]$ is defined by $j_\infty(s)\bigl(D_x(f)\bigr) := \frac{\partial}{\partial x}\bigl(j_\infty(s)(f)\bigr)$ through the restrictions $j_\infty(s)(f)$ of f onto the jets $j_\infty(s)$ of sections $u = s(x)$. Hence we obtain $\phi_i(x) = \phi_i\bigl(x, \bigl[w[\mathfrak{m}[s(x)]]\bigr]\bigr)$, and the assertion follows. $\square$

Remark. This proof combines taking the variational derivatives with respect to w on one jet space with calculating the total derivatives of differential functions in u on the other jet space over the same base. Whenever the two bundles coincide, the reasoning amounts to the definition of D_x. Then it is called the *substitution principle* ([13], see a detailed discussion in [14]).

Theorem 3. *Under differential reparametrizations $\tilde{w} = \tilde{w}[w]$ and $\tilde{u} = \tilde{u}[u]$ of the coordinates $w^1, \ldots, w^r$ and $u^1, \ldots, u^m$ in the infinite jet bundles over ξ and π that specify its domain and image, respectively, the operator $\square$ is transformed according to the formula*

$$\square \mapsto \tilde{\square} = \ell_{\tilde{u}}^{(u)} \circ \square \circ \bigl(\ell_{\tilde{w}}^{(w)}\bigr)^* \Big|_{\substack{w=w[u]\\ u=u[\tilde{u}]}}. \tag{6}$$

Proof. The transformation $\tilde{\varphi} = \ell_{\tilde{u}}^{(u)}(\varphi)$ of the velocities is obvious. Under differential reparametrizations $w = w[\tilde{w}]$ of the integrals, the sections $\boldsymbol{\phi} = \delta\mathcal{H}/\delta w$ are transformed by $\boldsymbol{\phi} = \bigl(\ell_{\tilde{w}}^{(w)}\bigr)^*(\tilde{\boldsymbol{\phi}})$, thence $\square$ becomes well defined on $\operatorname{im}\mathbf{E}_w$. Namely, it maps variational covectors for the fibre bundle ξ to evolutionary derivations in the jet space over the other fibre bundle π. Repeating the reasoning used in the proof of Corollary 2, we establish the transformation rule (6) for $\square$ on the entire domain. $\square$

In Theorem 3 we showed that sections in the domain of the operator $\square$ are transformed by the same rule as the arguments of Hamiltonian operators. There is indeed a deep reason for that.

The integrals $w[\mathfrak{m}]$ of Euler–Lagrange Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ determine the Miura substitutions from commutative modified KdV-type Hamiltonian hierarchies $\mathfrak{B}$ of Noether symmetries for $\mathcal{E}_{\mathrm{L}}$ to completely integrable KdV-type hierarchies $\mathfrak{A}$ of higher symmetries of the multi-component wave equations $\mathcal{E}_\varnothing = \{s_{xy} = 0\}$, see below. A natural example is given by the potential modified KdV equation $u_t = -\frac{1}{2}u_{xxx} + u_x^3$, which is transformed to the KdV equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ by $w = u_x^2 - u_{xx}$. This example was analysed in detail in [9]. The method for generating relevant differential substitutions by the integrals w of Liouville-type systems was discovered in [19], see [3] for discussion. This fact was used in [20] for a classification of the first-order differential substitutions.

The hierarchies $\mathfrak{A}$ and $\mathfrak{B}$ share the Hamiltonians $\mathcal{H}_i[\mathfrak{m}] = \mathcal{H}_i\bigl[w[\mathfrak{m}]\bigr]$ through the Miura substitution $w[\mathfrak{m}]$. The Hamiltonian structures for the Magri schemes of $\mathfrak{A}$ and $\mathfrak{B}$ are correlated by the operators $\square$, which map cosymmetries ϕ_i for the hierarchy $\mathfrak{A}$ to symmetries φ_{i+1} of the modified hierarchy $\mathfrak{B}$. We stress that the first Hamiltonian operator $\hat{B}_1 = \bigl(\ell_{\mathfrak{m}}^{(u)}\bigr)^*$ for $\mathfrak{B}$ originates from the differential constraint $\mathfrak{m} = \partial L/\partial u_y$ upon the coordinates u and the momenta $\mathfrak{m}$ for $\mathcal{E}_{\mathrm{L}}$. Using the explicit form of the 'junior' operator $\hat{B}_1$ and the differential-functional closure in $[w]$ for the velocities of the integrals, see [3], we realize the classical scheme for generating higher Poisson structures via Miura's substitutions [21].

Lemma 4. Introduce the linear differential operator

$$\hat{A}_k = \square^* \circ \hat{B}_1 \circ \square \tag{7}$$

that maps variational covectors for the jet bundle $J^\infty(\xi)$ over ξ to evolutionary vector fields on it, $\hat{A}_k\colon \Gamma\bigl(\widehat{\xi}\,\bigr) \otimes_{C^\infty(\mathbb{R})} C^\infty\bigl(J^\infty(\xi)\bigr) \to \Gamma(\xi) \otimes_{C^\infty(\mathbb{R})} C^\infty\bigl(J^\infty(\xi)\bigr)$. The operator (7) is Hamiltonian and determines[1] a Poisson structure for the KdV-type hierarchy $\mathfrak{A}$. The coefficients of $\hat{A}_k$ are differential functions in w.

Proof. By construction, the Poisson bracket $\{\mathcal{H}_1, \mathcal{H}_2\}_{\hat{A}_k} = \bigl\langle \mathbf{E}_w \mathcal{H}_1, \hat{A}_k\bigl(\mathbf{E}_w \mathcal{H}_2\bigr)\bigr\rangle$ satisfies the equality

$$\bigl\{\mathcal{H}_1[w], \mathcal{H}_2[w]\bigr\}_{\hat{A}_k} = \bigl\{\mathcal{H}_1[w[\mathfrak{m}]], \mathcal{H}_2[w[\mathfrak{m}]]\bigr\}_{\hat{B}_1}. \tag{8}$$

Therefore the left-hand side of (8) is bi-linear, skew-symmetric, and satisfies the Jacobi identity. Fourth, it measures the velocity of the integrals w along a Noether symmetry of $\mathcal{E}_{\mathrm{L}}$. Since evolutionary derivations are permutable with the total derivative D_y, the velocity $\{\mathcal{H}_1, \mathcal{H}_2\}_{\hat{A}_k}$ lies in $\ker D_y\bigr|_{\mathcal{E}_{\mathrm{L}}}$ and hence its density is a differential function of the minimal integrals w. $\square$

The multi-component wave equation $\mathcal{E}_\varnothing = \{s_{xy} = 0\}$, whose symmetries contain the hierarchy $\mathfrak{A}$ and such that $\hat{A}_1 = \bigl(\ell_w^{(s)}\bigr)^*$ encodes the differential constraint between the coordinates s and momenta w for $\mathcal{E}_\varnothing$, is chosen such that the first structure $A_1 = \hat{A}_1^{-1}$ for $\mathfrak{A}$ factors the higher Hamiltonian structure for $\mathfrak{B}$. Hence $B_{k'} = \square \circ A_1 \circ \square^*$, where $k' = k'\bigl(\square, \bigl(\ell_w^{(s)}\bigr)^*\bigr) \geq 2$.

Example 2. Consider the Euler–Lagrange 2D Toda system (1). The density L of its Lagrangian is

$$L = -\tfrac{1}{2}\bigl((2u_x - v_x)\cdot u_y + (2v_x - u_x)\cdot v_y\bigr) - \exp(2u - v) - \exp(2v - u).$$

Therefore we introduce the momenta $\mathfrak{m}^1 := 2u_x - v_x$ and $\mathfrak{m}^2 := 2v_x - u_x$, whence we express the integrals as $w = w[\mathfrak{m}]$. All symmetries (up to $x \leftrightarrow y$) of (1) are of the form $\varphi = \square\bigl(\boldsymbol{\phi}\bigl(x, [w^1], [w^2]\bigr)\bigr)$, where $\boldsymbol{\phi} = {}^t(\phi_1, \phi_2)$ is a pair of arbitrary functions and the (2×2)-matrix total differential operator is

$$\square = \ell_{w^1,w^2}^{(\mathfrak{m}^1,\mathfrak{m}^2)} = \begin{pmatrix} u_x + D_x & -\frac{2}{3}D_x^2 - u_x D_x - \frac{1}{3}u_x^2 - \frac{2}{3}u_x v_x + \frac{2}{3}v_x^2 + \frac{1}{3}u_{xx} - \frac{2}{3}v_{xx} \\ v_x + D_x & -\frac{1}{3}D_x^2 + \frac{2}{3}u_{xx} - \frac{1}{3}v_{xx} - \frac{2}{3}u_x^2 + \frac{2}{3}u_x v_x + \frac{1}{3}v_x^2 \end{pmatrix}.$$

The entries of the arising Hamiltonian operator $\hat{A}_2 = \|A_{ij}, 1 \leq i, j \leq 2\|$ are [13]

$$\begin{aligned}
A_{11} &= 2D_x^3 + 2w^1 D_x + w_x^1,\\
A_{12} &= -D_x^4 - w^1 D_x^2 + (3w^2 - 2w_x^1)\cdot D_x + (2w_x^2 - w_{xx}^1),\\
A_{21} &= D_x^4 + w^1 D_x^2 + 3w^2 D_x + w_x^2\\
A_{22} &= -\tfrac{2}{3}D_x^5 - \tfrac{4}{3}w^1 D_x^3 - 2w_x^1 D_x^2 + (2w_x^2 - 2w_{xx}^1 - \tfrac{2}{3}(w^1)^2)\cdot D_x\\
&\quad + \tfrac{1}{3}(3w_{xx}^2 - 2w_{xxx}^1 - 2w^1 w_x^1).
\end{aligned}$$

The shift $w^2 \mapsto w^2 + \lambda$ of the second integral yields the 'junior' Hamiltonian operator[2] $\hat{A}_1^{(2)} := \frac{d}{d\lambda}\Bigr|_{\lambda=0}\bigl(\hat{A}_2\bigr) = \bigl(\begin{smallmatrix} 0 & 3D_x \\ 3D_x & 0 \end{smallmatrix}\bigr)$, which is compatible with $\hat{A}_2$.

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[1] In most cases, this is one of the *higher* structures for $\mathfrak{A}$, which is indicated by the subscript $k = k(\square, \mathfrak{m}) \geq 2$. The choice of the 'junior' operator $\hat{A}_1$ for $\mathfrak{A}$ is discussed in what follows.

[2] The analogous operator $\hat{A}_1^{(1)} = \frac{d}{d\mu}\bigr|_{\mu=0}(\hat{A}_k)$, where $w^1 \mapsto w^1 + \mu$, is not Hamiltonian.

The pair $(\hat{A}_1^{(2)}, \hat{A}_2)$ is the bi-Hamiltonian structure for the Boussinesq equation $w^1_t = 2w^2_x - w^1_{xx}$, $w^2_t = -\frac{2}{3}w^1_{xxx} - \frac{2}{3}w^1 w^1_x + w^2_{xx}$. The symmetry $w_x = \bigl(\hat{A}_2 \circ \delta/\delta w\bigr)\bigl(\int w^1\,\mathrm{d}x\bigr)$ starts the second sequence of Hamiltonian flows in the Boussinesq hierarchy $\mathfrak{A}$. The modified Boussinesq hierarchy $\mathfrak{B}$ shares the two sequences of Hamiltonians with $\mathfrak{A}$ by the Miura substitution $w = w[\mathfrak{m}]$ with $\mathfrak{m} = \mathfrak{m}[u]$. Namely, for any Hamiltonian $\mathcal{H}[w]$, the flows $u_\tau = \delta\mathcal{H}[\mathfrak{m}]/\delta\mathfrak{m}$, $\mathfrak{m}_\tau = -\delta\mathcal{H}\bigl[\mathfrak{m}[u]\bigr]/\delta u$ belong to $\mathfrak{B}$. The velocities u_τ constitute the commutative subalgebra of Noether symmetries for the 2D Toda chain (1).

2. Commutation relations in $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$**.** In this section we prove the commutation closure for the images of operators (3) by using the well-known analogous property of the auxiliary Hamiltonian operators (7). At once, we describe the relations in the symmetry algebra generated by (5) for $\mathcal{E}_{\mathrm{L}}$.

First, consider a linear total differential operator A whose arguments $\boldsymbol{\phi}(x,[w]) = {}^t\bigl(\phi_1,\ldots,\phi_r\bigr)$ are the variational covectors for the infinite jet bundle over ξ. Assume that the image of A in the Lie algebra of evolutionary vector fields ∂_φ is closed w.r.t. the commutation: $[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A$. By the Leibnitz rule, two sets of summands appear in the bracket of fields $A(\phi'), A(\phi'')$ that belong to the image of A:

$$\bigl[A(\phi'), A(\phi'')\bigr] = A\bigl(\partial_{A(\phi')}(\phi'') - \partial_{A(\phi'')}(\phi')\bigr) + \bigl(\partial_{A(\phi')}(A)(\phi'') - \partial_{A(\phi'')}(A)(\phi')\bigr).$$

In the first summand we have used the permutability of evolutionary derivations and total derivatives. The second summand hits the image of A by construction.

The commutator $[\,,\,]\bigr|_{\operatorname{im} A}$ induces a Lie algebra structure $[\,,\,]_A$ in the quotient $\Omega(\xi_\pi)$ of the domain of A by its kernel:

$$\bigl[A(\phi'), A(\phi'')\bigr] = A\bigl([\phi',\phi'']_A\bigr), \qquad \phi',\phi'' \in \Omega(\xi_\pi). \tag{9a}$$

This bracket, which is defined up to $\ker A$, equals

$$[\phi',\phi'']_A = \partial_{A(\phi')}(\phi'') - \partial_{A(\phi'')}(\phi') + \{\{\phi',\phi''\}\}_A. \tag{9b}$$

It contains the two standard summands and the skew-symmetric bilinear bracket $\{\{\,,\,\}\}_A$.

Lemma 5 ([13, 14])**.** The image of a Hamiltonian operator $\hat{A} = \bigl\|\sum_\tau A^{\alpha\beta}_\tau(x,[w])\cdot D_\tau\bigr\|$ is closed w.r.t. the commutation. The k-th component $(1 \le k \le r)$ of the arising bracket $\{\{\,,\,\}\}_{\hat{A}}$ on the domain of $\hat{A}$ is calculated by the formula

$$\{\{\phi',\phi''\}\}^k_{\hat{A}} = \sum_{|\sigma|\ge 0}\sum_{i=1}^{r}(-1)^\sigma\Bigl(D_\sigma \circ \Bigl[\sum_{|\tau|\ge 0}\sum_{j=1}^{r} D_\tau(\phi'_j)\cdot\frac{\partial A^{ij}_\tau}{\partial w^k_\sigma}\Bigr]\Bigr)(\phi''_i). \tag{10}$$

The coefficients of the bilinear terms in the bracket $\{\{\,,\,\}\}_{\hat{A}}$ are differential functions of the variables w.

Now we pass from the Hamiltonian operators (7) to the operators $\square$ that have the same domain as $\hat{A}_k$ but take values in a different Lie algebra. Here is our main result.

Theorem 6. *Let the following conditions*[3] *be satisfied on an open dense subset of the Euler–Lagrange system* $\mathcal{E}_{\mathrm{L}} = \bigl\{ \boldsymbol{u}_{xy} = f(\boldsymbol{u}; x, y) \bigr\}$ *of Liouville type (see Proposition* 1*):*

- *the constant symmetric real matrix* κ *in the kynetic term of the Lagrangian density* $\mathcal{L}$ *be invertible;*
- *the linearization* $\ell_f^{(u)} = \|\partial f^i/\partial u^j\| \equiv f'(\boldsymbol{u}; x, y)$ *of the right-hand side in* $\mathcal{E}_{\mathrm{L}}$ *be an invertible matrix;*
- *there be as many integrals* $w^i\bigl(x, [\mathfrak{m}]\bigr) \in \ker D_y\bigr|_{\mathcal{E}_{\mathrm{L}}}$ *as there are unknowns* u^j*;*
- *the integrals* w *be minimal:* $\Phi \in \ker D_y\bigr|_{\mathcal{E}_{\mathrm{L}}}$ *implies* $\Phi = \bigl(x, [w]\bigr)$*;*
- *the integrals* w *be differential-functional independent,*[4] *meaning that* $\Phi\bigl(x, \bigl[w[\mathfrak{m}]\bigr]\bigr) = 0$ *implies* $\Phi \equiv 0$*;*
- *the* $(r \times m)$*-matrix* $\Lambda = \bigl\|\partial w^i/\partial \mathfrak{m}^j_{d(i)}\bigr\|$*, where* $d(i) := \operatorname{ord}_x w^i$ *is the differential order of the* i-th *integral* $w^i[\mathfrak{m}]$*, be invertible.*

Then the following statements hold:

1. *the image of the operator* (3) *is closed w.r.t. the commutation of symmetries* $\varphi = \Box\bigl(\boldsymbol{\phi}(x, [w])\bigr) \in \operatorname{sym}\mathcal{E}_{\mathrm{L}}$*;*
2. *the bracket* $\{\{\,,\,\}\}_\Box$ *arising on the domain of the operator* $\Box$ *satisfies the equality*
$$\{\{\phi', \phi''\}\}_\Box = \{\{\phi', \phi''\}\}_{\hat{A}_k}, \qquad \phi', \phi'' \in \Omega(\xi_\pi); \tag{11}$$
3. *the coefficients of the bilinear terms in the bracket* $\{\{\,,\,\}\}_\Box$ *are differential functions of the integrals* w*.*

Remark 1. The first assumption of the theorem implies that the system $\mathcal{E}_{\mathrm{L}}$ is determined, normal, and ℓ-normal (see section 3). We emphasize that $\mathcal{E}_{\mathrm{L}}$ is the only system of equations imposed upon the sections $u = s(x, y) \in \Gamma(\pi)$.

Our second statement means that the ambiguity (up to $\ker \hat{A}_k$) in the choice of a representative from the equivalence class $\{\{\phi', \phi''\}\}_{\hat{A}_k}$ in the right-hand side of (11) amounts to the choice of an element from $\ker \Box \subseteq \ker \hat{A}_k$. We prove the equality of the kernels for all $\boldsymbol{\phi}([w]) \in \Omega(\xi_\pi)$. This implies that commutation relations in $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$, which are determined by the Lie algebra structure (9b) on $\Omega(\xi_\pi)$, are obtained explicitly via (10) for $\hat{A}_k$.

Being a corollary of Lemma 5 and the first two, our third statement is, at the same time, a special case of Proposition 7 (see below).

Proof of Theorem 6. We notice first that symmetries (5) are independent of u and of $u_y, u_{yy}, \ldots$. Hence this is also true for the commutator $\varphi = [\varphi', \varphi''] \in \operatorname{sym}\mathcal{E}_{\mathrm{L}}$ of two such symmetries $\varphi' = \Box\bigl(\boldsymbol{\phi}'(x, [w])\bigr)$ and $\varphi'' = \Box\bigl(\boldsymbol{\phi}''(x, [w])\bigr)$, because the Lie bracket is a local bi-differential operator.

[3] The list can be not minimal such that it is easier to verify each requirement.

[4] The non-existence of a nontrivial Φ, and hence of its nonzero linearization $\ell_\Phi^{(\mathfrak{m})} = \ell_\Phi^{(w)} \circ \ell_w^{(\mathfrak{m})}$, is equivalent to $\bigl(\nabla \circ \ell_w^{(\mathfrak{m})}$ with $\nabla = \ell_\bullet^{(\mathfrak{m})}\bigr) \Rightarrow \nabla = 0$. This nondegeneracy requirement is dual to (12), see below.

The factorization (7) and Lemma 5 provide the diagram

$$
\begin{tikzcd}
 & \dot{w} = \Phi\bigl(x,[w]\bigr) & {[\phi',\phi'']}_{\hat{A}_k} \arrow[l] \arrow[d, equal] \\
\dot{\mathfrak{m}} = \psi\bigl(x,[\mathfrak{m}]\bigr) \arrow[ur, "\ell_w^{(\mathfrak{m})}"] & \circlearrowright & \boldsymbol{\phi} = \boldsymbol{\phi}\bigl(x,[w]\bigr). \arrow[ul, "\hat{A}_k"'] \arrow[dl, "\square"] \\
 & \dot{u} = \varphi\bigl(x,[u_x]\bigr) \arrow[ul, "\hat{B}_1 = \ell_{\mathfrak{m}}^{(u)}"] &
\end{tikzcd}
$$

The commutator $\varphi = [\varphi', \varphi'']$ determines the velocity $\Phi\bigl(x,[w]\bigr)$ of the integrals that equals[5] $\partial_{[\varphi',\varphi'']}(w) = \bigl[\hat{A}_k(\phi'), \hat{A}_k(\phi'')\bigr]$. Since the image of the Hamiltonian operator $\hat{A}_k$ is closed under commutation, we obtain the equivalence class $\boldsymbol{\phi}\bigl(x,[w]\bigr) = \bigl[\phi',\phi''\bigr]_{\hat{A}_k}$ of sections such that $\Phi = \hat{A}_k(\boldsymbol{\phi}) = \partial_{\square(\boldsymbol{\phi})}(w)$. By construction of $\hat{A}_k$, the commutator of φ' and φ'' belongs to the set of symmetries $\square(\boldsymbol{\phi})$. This proves the first statement of the theorem.

However, there may be many such $\varphi = \square(\boldsymbol{\phi})$ that induce the same velocity $\dot{w} = \Phi$. Since $\ker\square \subseteq \ker\hat{A}_k$, then, in principle, the equivalence class $[\phi',\phi'']_{\hat{A}_k}$ may contain elements that do not belong to $[\phi',\phi'']_{\square}$.

We claim that all representatives of the equivalence class $[\phi',\phi'']_{\hat{A}_k}$ determine a unique symmetry φ of $\mathcal{E}_{\mathrm{L}}$. Therefore this $\varphi = \square(\boldsymbol{\phi})$ is the commutator $\bigl[\square(\boldsymbol{\phi}'), \square(\boldsymbol{\phi}'')\bigr]$ of the two symmetries, because the image of $\boldsymbol{\phi}$ under $\square$ must contain it. It suffices to prove the uniqueness of the trivial solution φ for the linear homogeneous equation $\ell_w^{(u)}(\varphi) = \bigl(\ell_{w[\mathfrak{m}]}^{(\mathfrak{m})} \circ \ell_{\mathfrak{m}}^{(u)}\bigr)(\varphi) = 0$.

There exist three ways to obtain the zero velocity $\Phi\bigl(x,[w]\bigr) \equiv 0$ of the integrals $w\bigl[\mathfrak{m}[u]\bigr]$ along $\varphi \in \operatorname{sym}\mathcal{E}_{\mathrm{L}}$. The first possibility is that the integrals be differentially dependent, which is excluded by the assumption of the theorem. Second, the intermediate equation $\kappa\, D_x(\varphi) = 0$, $\det\kappa \neq 0$, may have nontrivial solutions only if some shifts $\varphi = \mathrm{const}$ are symmetries of $\mathcal{E}_{\mathrm{L}}$. However, the determining equation $\bigl(D_x D_y - f'(\boldsymbol{u}; x, y)\bigr)(\mathrm{const}) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$ then exprimes the linear dependence between the differentials of its right-hand sides. This contradicts another initial assumption.

Therefore the proof of the second statement is reduced to the uniqueness problem for the zero solution $\psi = 0$ of the linear homogeneous equation $\ell_{w[\mathfrak{m}]}^{(\mathfrak{m})}(\psi) = 0$. Since $\mathcal{E}_{\mathrm{L}}$ is the only system[6] of differential relations that are imposed upon the sections $u = s(x,y) \in \Gamma(\pi)$, the zero in the right-hand side of $\ell_w^{(\mathfrak{m})}(\psi) = 0$ is achieved identically w.r.t. $[\mathfrak{m}]$ (otherwise it would overdetermine $\mathcal{E}_{\mathrm{L}}$).

There remain two situations when a velocity ψ of the momenta $\mathfrak{m} = -\frac{1}{2}\kappa u_x$ makes $\dot{w} = \Phi$ zero. One reason is obvious: it is the use of the equation $\mathcal{E}_{\mathrm{L}} = \{\boldsymbol{u}_{xy} = f\}$. Indeed, if the 'time' along the flow $\dot{\mathfrak{m}} = \psi$ is the variable y, *i.e.*, $\mathfrak{m}_y = -\delta H_{\mathrm{L}}/\delta u$, then we have $\partial_{\mathfrak{m}_y}^{(\mathfrak{m})}(w) = D_y\bigl(w[\mathfrak{m}]\bigr) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$. But the presence of $\boldsymbol{u}$ in the list if arguments of f excludes such solutions ψ from further consideration.

[5] The ℓ-normality of $\mathcal{E}_{\mathrm{L}}$ implies that φ *is* its symmetry whenever the velocity $\partial_\varphi(w)$ of the minimal integrals lies in $\ker D_y\big|_{\mathcal{E}_{\mathrm{L}}}$, see (16) in section 3.

[6] The hyperbolic system $\mathcal{E}_{\mathrm{L}}$ is *formally integrable* [12]: its infinite prolongation $\mathcal{E}_{\mathrm{L}}^\infty$ exists and there is an epimorphism $\mathcal{E}_{\mathrm{L}}^\infty \to \mathcal{E}_{\mathrm{L}}$.

Without loss of generality we assume that the integral $w^r[\mathfrak{m}]$ has the highest differential order: $d(r) \geq d(i)$ for all $i < r$. Let us calculate the velocities of the non-minimal integrals $(w')^i := D_x^{d(r)-d(i)}(w^i)$. Using the permutability $\bigl[D_x, \partial_\psi^{(\mathfrak{m})}\bigr] = 0$ of evolutionary derivations with total derivatives, from the identities $\partial_\psi^{(\mathfrak{m})}(w^1) = \cdots = \partial_\psi^{(\mathfrak{m})}(w^r) = 0$ we deduce that $\partial_\psi^{(\mathfrak{m})}(w') = 0$. It is readily seen that the linearization of the new integrals is of the form $\ell_{w'}^{(\mathfrak{m})} = \Lambda \cdot D_x^{d(r)} + O(d(r)-1)$, where the matrix Λ is invertible by our initial assumption. Multiplying the new equation $\ell_{w'}^{(\mathfrak{m})}(\psi) = 0$ by Λ^{-1}, we obtain by induction that $\psi\bigl(x, [\mathfrak{m}]\bigr)$ does not depend on $[\mathfrak{m}]$, hence $\psi = \psi(x)$. Consequently, the admissible sections φ that solve the intermediate equation $\psi(x) = -\frac{1}{2}\kappa D_x(\varphi)$ also depend on x only: $\varphi = \varphi(x)$. However, such sections, whenever nonzero, can not be[7] symmetries of the hyperbolic system $u_{xy} - f(\boldsymbol{u}; x, y) = 0$ due to the nondegeneracy $\det f'(\boldsymbol{u}) \neq 0$. In this notation, the "symmetry" $\varphi(x)$ must satisfy the determining equation $\bigl(D_x \circ D_y - f'(\boldsymbol{u})\bigr)\,\varphi(x) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$. The first summand vanishes because $D_y(x) \equiv 0$. Thus we obtain $f'(u) \cdot \varphi(x) = 0$, where the linearization matrix $f'(\boldsymbol{u})$ is invertible. Hence $\varphi(x) \equiv 0$. This completes the proof. $\square$

Remark 2. In the proof, we arrived at the linear ODE $\ell_w^{(\mathfrak{m})}\bigl(\psi(x)\bigr) = 0$ that holds simultaneously for all sections $s \in \Gamma(\pi)$, although nonzero solutions $\psi(x)$ do not contribute to the symmetry algebra. This is possible, first, if there is a total differential operator ∇ such that $\ell_w^{(\mathfrak{m})} \circ \nabla = 0$. (For instance, the identity $\left(\begin{smallmatrix} D_x & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -D_x \end{smallmatrix}\right)\bigl(h(x)\bigr) \equiv 0$ holds for all $h(x)$.) To avoid this, it is necessary

- to require the *nondegeneracy* of the linearization of the integrals:

$$\ell_{w[\mathfrak{m}]}^{(\mathfrak{m})} \circ \nabla = 0 \implies \nabla = 0. \tag{12}$$

  In the adjoint form, $\nabla^* \circ \square = 0 \Rightarrow \nabla^* = 0$, equation (12) exprimes the absence of linear differential relations[8] between components of symmetries $\varphi = \square\bigl(\boldsymbol{\phi}\bigl(x, [w]\bigr)\bigr)$.

This property is dual to the nondegeneracy $\nabla \circ \ell_w^{(\mathfrak{m})} = 0 \Rightarrow \nabla = 0$ that originates from the differential-functional independence $\Phi\bigl(x, [w]\bigr) = 0 \Rightarrow \Phi \equiv 0$ via $\nabla = \ell_\Phi^w$, see footnote 4 on p. 7.

Finally, let the section $s(x, y) \in \operatorname{Sol}\mathcal{E}_{\mathrm{L}}$ be a solution of the Darboux-integrable Liouville-type system $\mathcal{E}_{\mathrm{L}}$. Taking the restriction $\mathcal{L}^s = \ell_w^{(\mathfrak{m})}\big|_{j_\infty(s)}$ of the linearization operator onto the jet of s, we obtain the ordinary differential equation $\mathcal{L}^s\bigl(\psi(x)\bigr) = 0$. For each s, the linear space $\mathcal{O}(s)$ of its solutions is finite dimensional. (For example, its dimension is equal to the sum of the exponents of a semi-simple complex Lie algebra if $\mathcal{E}_{\mathrm{L}}$ is the associated 2D Toda chain.) Therefore

- the requirement

$$\bigcap_{s \in \operatorname{Sol}\mathcal{E}_{\mathrm{L}} \subset \Gamma(\pi)} \mathcal{O}(s) = \{0\},$$

[7] This is an immediate, point-by generalization of the fact (see above) that $\varphi = \mathrm{const} \neq 0$ is not a symmetry of $\mathcal{E}_{\mathrm{L}}$.

[8] Thence the nondegeneracy (12) is analogous to the notion of ℓ-normal differential equations in the analysis of their formal integrability, see section 3.

in combination with (12), permits to eliminate the excessive freedom in the choice of solutions $\psi(x)$ of the equation $\ell_w^{(\mathfrak{m})}(\psi) = 0$.

Theorem 6 is illustrated for semi-simple complex Lie algebras of rank two in [16], where the Hamiltonian operators $\hat{A}_1$ and $\hat{A}_k$ are constructed for the corresponding Drinfel'd–Sokolov hierarchies [11] and the commutation relations in $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$ are calculated for the 2D Toda chains $\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u})$.

Example 3 (The modified Kaup–Boussinesq equation)**.** Consider an Euler–Lagrange extension of the scalar Liouville equation [15],

$$A_{xy} = -\tfrac{1}{8}A\exp\bigl(-\tfrac{1}{4}B\bigr), \qquad B_{xy} = \tfrac{1}{2}\exp\bigl(-\tfrac{1}{4}B\bigr). \tag{13}$$

Denote the momenta by $a = \frac{1}{2}B_x$ and $b = \frac{1}{2}A_x$. The minimal integrals of system (13) are $w_1 = -\frac{1}{4}a^2 - a_x$ and $w_2 = ab + 2b_x$ such that $D_y(w_i) \doteq 0$ on (13), $i = 1, 2$. Hence the operator

$$\square = \Bigl(\ell_{w_1,w_2}^{(a,b)}\Bigr)^* = \begin{pmatrix} -\frac{1}{4}B_x + D_x & \frac{1}{2}A_x \\ 0 & \frac{1}{2}B_x - 2\,D_x \end{pmatrix}$$

determines (Noether, see (4)) symmetries of (13). The bracket $\{\{\,,\,\}\}_\square$ induced in the inverse image of $\square$ is

$$\{\{\boldsymbol{\psi}, \boldsymbol{\chi}\}\}_\square = \tfrac{1}{2}\cdot\begin{pmatrix} \psi_x^2\,\chi^1 - \psi^1\,\chi_x^2 + \psi_x^1\,\chi^2 - \psi^2\,\chi_x^1) \\ \psi_x^2\,\chi^2 - \psi^2\,\chi_x^2) \end{pmatrix},$$

where $\boldsymbol{\psi} = {}^t(\psi^1, \psi^2)$ and $\boldsymbol{\chi} = {}^t(\chi^1, \chi^2)$; we use upper indices for convenience.

Consider a symmetry of (13),

$$A_t = \tfrac{1}{2}A_xA_{xx} + \tfrac{1}{2}\left(\tfrac{1}{4}A_x^2 - 1\right)B_x, \qquad B_t = -2A_{xxx} + \tfrac{1}{8}A_xB_x^2 - \tfrac{1}{2}A_xB_{xx}. \tag{14}$$

Let us choose an equivalent pair of integrals $u = w_2$, $v = w_1 + \frac{1}{4}w_2^2$. The evolution of u and v along (14) equals

$$u_t = uu_x + v_x, \qquad v_t = (uv)_x + u_{xxx}. \tag{15}$$

This is the Kaup–Boussinesq system, and (14) is the potential twice-modified Kaup–Boussinesq equation, see [22]. The right hand side of the integrable system (14) belongs to the image of the adjoint linearization $\widetilde{\square} = \bigl(\ell_{(u,v)}^{(a,b)}\bigr)^*$. The operator $\widetilde{\square}$ factors the *third* Hamiltonian structure $\hat{A}_3^{\mathrm{KB}} = \widetilde{\square}^* \circ \bigl(\ell_{(a,b)}^{(A,B)}\bigr)^* \circ \widetilde{\square}$ for (15); we have $k = 3$ and

$$\hat{A}_3^{\mathrm{KB}} = \begin{pmatrix} u\,D_x + \frac{1}{2}u_x & D_x^3 + (\frac{1}{4}u^2 + v)\,D_x + \frac{1}{4}(u^2 + 2v)_x \\ D_x^3 + (\frac{1}{4}u^2 + v)\,D_x + \frac{1}{2}v_x & \frac{1}{2}(2u\,D_x^3 + 3u_x\,D_x^2 + (3u_{xx} + 2uv)D_x + u_{xxx} + (uv)_x) \end{pmatrix}.$$

By Theorem 6, the bracket $\{\{\,,\,\}\}_{\widetilde{\square}}$ is equal to $\{\{\,,\,\}\}_{\hat{A}_3^{\mathrm{KB}}}$, which is given by formula (10). We obtain

$$\{\{\boldsymbol{\psi}, \boldsymbol{\chi}\}\}_{\widetilde{\square}} = \{\{\boldsymbol{\psi}, \boldsymbol{\chi}\}\}_{\hat{A}_3^{\mathrm{KB}}} = \begin{pmatrix} \boldsymbol{\psi}\cdot\nabla_1(\boldsymbol{\chi}) - \nabla_1(\boldsymbol{\psi})\cdot\boldsymbol{\chi} \\ \boldsymbol{\psi}\cdot\nabla_2(\boldsymbol{\chi}) - \nabla_2(\boldsymbol{\psi})\cdot\boldsymbol{\chi} \end{pmatrix},$$

where $\nabla_1 = -\frac{1}{2}\left(\begin{smallmatrix} D_x & 0 \\ u\,D_x & D_x^3 + v\,D_x \end{smallmatrix}\right)$ and $\nabla_2 = -\frac{1}{2}\left(\begin{smallmatrix} 0 & D_x \\ D_x & u\,D_x \end{smallmatrix}\right)$. The operator $\hat{A}_1 = \left(\begin{smallmatrix} 0 & D_x \\ D_x & 0 \end{smallmatrix}\right)$ is the first Hamiltonian structure for (15); its inverse $A_1 = \hat{A}_1^{-1}$ factors the second Hamiltonian structure $B_2 = \widetilde{\square} \circ A_1 \circ \widetilde{\square}$ for (14).

3. Non-Lagrangian Liouville-type systems. Let $\mathcal{E} = \{F = 0\}$ be a Liouville-type system; now it may not be Euler–Lagrange. Let a column $w \in \ker D_y\big|_{\mathcal{E}}$ be composed by minimal integrals for $\mathcal{E}$, thence $D_y(w) = \nabla(F)$ for some operator ∇. By construction of the Liouville-type systems $\mathcal{E}$, there are no differential relations (*syzygies*) between the hyperbolic equations $\{F^i = 0\}$ in them: $\Delta(F) = 0$ implies $\Delta = 0$. For the same reason, the systems $\mathcal{E}$ are ℓ-normal [12, 14]: $\Delta \circ \ell_F \doteq 0$ on $\mathcal{E}$ also requires $\Delta = 0$. Consequently, an evolutionary vector field ∂_φ is a symmetry of a Liouville-type system $\mathcal{E}$ if and only if it preserves the integrals,

$$D_y\bigl(\partial_\varphi(w)\bigr) = \partial_\varphi(\nabla)(F) + \nabla\bigl(\partial_\varphi(F)\bigr) \doteq \nabla\bigl(\ell_F(\varphi)\bigr) \text{ on } \mathcal{E}. \tag{16}$$

Consider the operator equation

$$D_y \circ \ell_w^{(u)} \doteq \nabla \circ \ell_F \text{ on } \mathcal{E}.$$

If, hypothetically, a total differential operator $\square$ such that

$$\ell_w^{(u)} \circ \square \in \mathcal{C}\mathrm{Diff}\left(\ker D_y\big|_{\mathcal{E}} \to \ker D_y\big|_{\mathcal{E}}\right) \tag{17}$$

were constructed, then it would assign symmetries $\varphi = \square(\boldsymbol{\phi})$ of the Liouville-type system $\mathcal{E}$ to arbitrary r-tuples $\boldsymbol{\phi}\bigl(x, [w]\bigr)$ of the integrals, see (5).

The recent paper [10] contains an algorithm for construction of operator solutions $\square$ for the equation in total derivatives

$$\ell_w^{(u)} \circ \square = \mathbf{1}_{m\times m} \cdot D_x^k \mod \mathcal{C}\mathrm{Diff}_{<k}\left(\ker D_y\big|_{\mathcal{E}} \to \ker D_y\big|_{\mathcal{E}}\right). \tag{18}$$

Most remarkably, the truncation from below for the sequence of coefficients of lower order derivatives in $\square$ is a consequence of the presence of a complete set of the integrals $\bar{w} \in \ker D_x\big|_{\mathcal{E}}$ for $\mathcal{E}$. However, the *minimal* integrals w must be 'spoilt' by differentiating w.r.t. x a suitable number of times. Consequently, instead of the *Hamiltonian* operator $\hat{A}_k = \ell_w^{(u)} \circ \square$, see (7), one obtains the r.h.s. of (18). Likewise, the images of operators constructed in [10] do not always span the entire x-components of the Lie algebras $\mathrm{sym}\,\mathcal{E}$, and the images are generally not closed under the commutation. Moreover, the transformation rules in the domains of $\square$ under reparametrizations $\tilde{w}[w]$ of the integrals remain unspecified for non-Lagrangian Liouville-type systems.

Proposition 7. If the image of a solution $\square$ of the operator equation (17) for a Liouville-type system $\mathcal{E}$ is closed under the commutation, then all coefficients of the bracket $\{\!\{\ ,\ \}\!\}_\square$ on its domain, see (9), belong to $\ker D_y\big|_{\mathcal{E}}$.

Proof. By assumption, we have that $\bigl(D_y \circ \ell_w^{(u)} \circ \square\bigr)\bigl([\phi', \phi'']_\square\bigr) \doteq 0$ for all $\phi', \phi''(x, [w])$. This equals

$$0 \doteq \bigl(D_y \circ \underline{\ell_w^{(u)} \circ \square}\bigr)\Bigl(\partial_{\square(\phi')}(\phi'') - \partial_{\square(\phi'')}(\phi') + \{\!\{\phi', \phi''\}\!\}_\square\Bigr) \doteq \bigl(\ell_w^{(u)} \circ \square\bigr)\bigl(D_y\bigl(\{\!\{\phi', \phi''\}\!\}_\square\bigr)\bigr),$$

because the underlined composition satisfies (17). Clearly, $D_y(\phi')$ and $D_y(\phi'')$ vanish on $\mathcal{E}$ for arbitrary ϕ', ϕ''. For the same reason, not only the whole bracket $\{\!\{\phi', \phi''\}\!\}_\square$, but each particular coefficient standing at the bilinear terms in it lies in $\ker D_y\big|_{\mathcal{E}}$. $\square$

Example 4. Consider the parametric extension of the scalar Liouville equation,

$$\mathcal{E}(\varepsilon) = \bigl\{u_{xy} = \exp(2u) \cdot \sqrt{1 + 4\varepsilon^2 u_x^2}\bigr\}, \qquad \varepsilon \in \mathbb{R}. \tag{19}$$

This equation is ambient w.r.t. the hierarchy of Gardner's deformation of the potential modified KdV equation, see [15]. The contraction $\mathcal{U} = \mathcal{U}(\varepsilon, [u(\varepsilon)])$ from (19) to the non-extended equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ is $\mathcal{U} = u + \frac{1}{2}\operatorname{arcsinh}(2\varepsilon u_x)$; it determines the third order integral for (19) using the integral $w = \mathcal{U}_x^2 - \mathcal{U}_{xx}$ at $\varepsilon = 0$. However, the regularized (at $\varepsilon = 0$) integral of order two for (19) is

$$w = \bigl(1 - \sqrt{1 + 4\varepsilon^2 u_x^2}\bigr)/2\varepsilon^2 + u_{xx}\big/\sqrt{1 + 4\varepsilon^2 u_x^2}. \tag{20}$$

The second integral for (19) is $\overline{w} = u_{yy} - u_y^2 - \varepsilon^2 \cdot \exp(4u) \in \ker D_x\big|_{\mathcal{E}(\varepsilon)}$. The operators $\bar{\square} = u_y + \frac{1}{2}D_y$ and

$$\square = \tfrac{1}{2}(1 + 4\varepsilon^2 u_x^2 - 2\varepsilon^2 u_{xx}) \cdot D_x + u_x + 4\varepsilon^2 u_x^3 - 2\varepsilon^2 u_{xxx} + \frac{12\varepsilon^4 u_x u_{xx}^2}{1 + 4\varepsilon^2 u_x^2} \tag{21}$$

assign symmetries $\varphi = \square\bigl(\phi(x, [w])\bigr)$ and $\overline{\varphi} = \overline{\square}\bigl(\overline{\phi}(y, [\overline{w}])\bigr)$ of (19) to its integrals.

The images of both operators $\square$ and $\overline{\square}$ are Lie subalgebras in $\operatorname{sym}\mathcal{E}(\varepsilon)$. The bracket $\{\{p, q\}\}_{\overline{\square}} = p_y q - p q_y$ for $\overline{\square}$ is familiar [3, 9]. The bracket induced in the domain of $\square$ has the following form: for any arguments p, q, we have

$$\begin{aligned}
\{\{p, q\}\}_{\square} &= \varepsilon^2 \cdot \bigl(p_{xx} q_x - p_x q_{xx}\bigr) - 2\varepsilon^2 \cdot \bigl(p_{xxx} q - p q_{xxx}\bigr) \\
&\quad - 12\varepsilon^4 \cdot \bigl(8\varepsilon^2 u_x^3 u_{xx} - 4\varepsilon^2 u_x^2 u_{xxx} + 4\varepsilon^2 u_x u_{xx}^2 + 2u_x u_{xx} - u_{xxx}\bigr) \\
&\qquad \times \bigl[1 + 8\varepsilon^2 u_x^2 + 16\varepsilon^4 u_x^4 - 2\varepsilon^2 u_{xx} - 8\varepsilon^4 u_x^2 u_{xx}\bigr]^{-1} \cdot \bigl(p_{xx} q - p q_{xx}\bigr) \\
&\quad + \bigl(\underline{1} + 288\varepsilon^4 u_x^4 - 288\varepsilon^4 u_x^2 u_{xx} + 28\varepsilon^2 u_x^2 - 16\varepsilon^2 u_{xx} - 288\varepsilon^6 u_x u_{xx} u_{xxx} \\
&\qquad - 96\varepsilon^6 u_{xx}^3 + 3072\varepsilon^{10} u_x^{10} + 24\varepsilon^6 u_{xxx}^2 + 24\varepsilon^4 u_{4x} + 1408\varepsilon^6 u_x^6 + 3328\varepsilon^8 u_x^8 \\
&\qquad - 768\varepsilon^{10} u_{4x} u_{xx} u_x^4 - 384\varepsilon^8 u_{4x} u_x^2 u_{xx} - 2304\varepsilon^8 u_x^3 u_{xx} u_{xxx} + 384\varepsilon^8 u_{xx}^2 u_x u_{xxx} \\
&\qquad - 4608\varepsilon^{10} u_x^5 u_{xx} u_{xxx} + 16\varepsilon^4 u_{xx}^2 - 5632\varepsilon^8 u_x^6 u_{xx} - 1920\varepsilon^6 u_{xx} u_x^4 + 3328\varepsilon^8 u_x^4 u_{xx}^2 \\
&\qquad + 512\varepsilon^6 u_{xx}^2 u_x^2 + 384\varepsilon^{10} u_x^4 u_{xxx}^2 - 960\varepsilon^{10} u_{xx}^4 u_x^2 - 48\varepsilon^4 u_x u_{xxx} - 3072\varepsilon^{10} u_x^7 u_{xxx} \\
&\qquad + 3072\varepsilon^{10} u_{xx}^3 u_x^4 - 2304\varepsilon^8 u_x^5 u_{xxx} - 576\varepsilon^6 u_x^3 u_{xxx} + 288\varepsilon^6 u_{4x} u_x^2 + 384\varepsilon^8 u_x^2 u_{xx}^3 \\
&\qquad + 6144\varepsilon^{10} u_{xx}^2 u_x^6 - 6144\varepsilon^{10} u_{xx} u_x^8 + 1152\varepsilon^8 u_{4x} u_x^4 + 1536\varepsilon^{10} u_{4x} u_x^6 + 192\varepsilon^8 u_{xxx}^2 u_x^2 \\
&\qquad + 240\varepsilon^8 u_{xx}^4 + 1536\varepsilon^{10} u_{xx}^2 u_x^3 u_{xxx} - 48\varepsilon^6 u_{4x} u_{xx}\bigr) \\
&\qquad \times \bigl[\underline{1} + 96\varepsilon^4 u_x^4 + 256\varepsilon^6 u_x^6 + 256\varepsilon^8 u_x^8 + 4\varepsilon^4 u_{xx}^2 - 48\varepsilon^4 u_x^2 u_{xx} + 32\varepsilon^6 u_{xx}^2 u_x^2 \\
&\qquad\quad - 4\varepsilon^2 u_{xx} - 256\varepsilon^8 u_x^6 u_{xx} + 64\varepsilon^8 u_x^4 u_{xx}^2 - 192\varepsilon^6 u_{xx} u_x^4 + 16\varepsilon^2 u_x^2\bigr]^{-1} \cdot \bigl(p_x q - p q_x\bigr).
\end{aligned}$$

The two underlined units correspond to the bracket $p_x q - p q_x$ on the domain of the operator $\square = \mathcal{U}_x + \frac{1}{2}D_x$ that provides symmetries of the Liouville equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ at $\varepsilon = 0$. In agreement with Lemma 7, the non-constant coefficients of bilinear terms $p_{xx}q - pq_{xx}$ and $p_x q - p q_x$ in $\{\{p, q\}\}_{\square}$ belong to $\ker D_y\big|_{\mathcal{E}(\varepsilon)}$. It is remarkable that, since the entire construction (19–21) contains formal power series $u(\varepsilon)$ in ε, so are these two rational functions: an attempt to express their dependence on $[w]$ leads to formal series with unbounded growth of the differential orders of its coefficients.

Discussion. The matrix operators $\square = \bigl(\square^{i,j},\ 1 \le i \le m,\ 1 \le j \le r\bigr)$ given by (3) are generalizations of tensors of type $(2,0)$ in the geometry of infinite jet bundles. We define the operators by using the two unrelated groups of differential reparametrizations for the coordinates in the domains and images, respectively. Furthermore, the operators $\square$ for the Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ generalize the theory of Hamiltonian structures as follows: they map variational covectors for one equation (we recall that $\operatorname{sym}\mathcal{E}_{\varnothing} \supset \mathfrak{A}$) to symmetries of the other system $\mathcal{E}_{\mathrm{L}}$ (such that $\operatorname{sym}\mathcal{E}_{\mathrm{L}} \supset \mathfrak{B}$).

Unlike in [8, 10], we do not attempt to solve equation (17) upon $\square$. On the contrary, we define the operators (3) by a geometric reasoning. Thence, first, we obtain the Hamiltonian operators $\hat{A}_k = \ell_w^{(u)} \circ \square$ for the KdV-type hierarchies on Euler–Lagrange systems of Liouville type [11, 9] and, second, we prove that the images of the operators $\square$ are involutive. In other words, we describe a direct algorithm aimed at constructing completely integrable equations.

Formulas (3) and (7) prescribe the differential order of $\hat{A}_k$. Estimates for the orders of the integrals w for the 2D Toda chains associated with semi-simple complex Lie algebras $\mathfrak{g}$ are known from [2], see Example 1, and were reformulated in [4, 16]. The *upper* bound, that the numbers $\operatorname{ord}_x w^i - 1$ are not greater than the exponents for $\mathfrak{g}$, is proved by verifying (via Schur polynomials) Serre's relations $(\operatorname{ad} Y_i)^{-K^i_j+1}(Y_j) = 0$, $i \ne j$, for the generators

$$Y_i = \sum\nolimits_{k\ge 0} \exp\Bigl(-\sum\nolimits_{j=1}^m K^i_j u^j\Bigr) \cdot D_x^k\Bigl(\exp\Bigl(\sum\nolimits_{j'=1}^m K^i_{j'} u^{j'}\Bigr)\Bigr) \cdot \partial/\partial u^i_{k+1}$$

of the characteristic Lie algebras (see [1, 2, 7] and also [16]), and by using Frobenius theorem. The fact that the vector fields Y_i coincide with the Chevalley generators $\mathfrak{f}_i$ of the semi-simple Lie algebra $\mathfrak{g}$ is important here. The same estimate from *below* follows from the absence of relations other than Serre's for the generators Y_i. This was established in [2, p.21] for the root systems A_n and D_n by listing the linear independent nonzero iterated commutators.

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