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On the Geometry of Liouville Equation: Symmetries, Conservation Laws, and Bäcklund Transformations

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Abstract. Various geometrical structures related to the Liouville equation are considered. Properties of the symmetry algebra are discussed and local conserved currents are constructed. Bäcklund transformations for the Liouville equation are integrated and nontrivial generalizations of the latter are studied as well as the structures on them.

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Introduction

The two-dimensional elliptic Liouville equation

$$\mathcal{E} = \{u_{xx} + u_{yy} - \exp(2u) = 0\}, \quad x \equiv x^1, \ y \equiv x^2, \tag{1}$$

appears in numerous models of mathematical physics. In Riemannian geometry, it represents the Gauss equation expressed in isothermic coordinates, for the Lobatchevsky plane ([9]). Equation (1) may be treated as an example of the continuous elliptic Toda system ([12, 15]), see also ([18]) associated with the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$; the coefficient 2 in the exponent stands for the Cartan 1×1 matrix of the algebra. Finding N-instanton solutions, minimizing the free Yang–Mills equations' action, for self-duality equations $F_{\mu\nu} = *F_{\mu\nu}$, where $F_{\mu\nu}$ is the stress tensor, also leads ([20]) to the Liouville equation.

Equation (1) was considered by Poincaré in [16], where uniformization of algebraic curves was studied. It is known [1, 22] that for surfaces of genus 0, the specially regularized action, computed on a classical solution of the Liouville equation, is a generating function for accessor parameters that characterize uniformization of a Riemannian surface; moreover, the action for Equation (1) is a potential for the Weil–Peterson metric on the Teichmüller space of marked Riemannian surfaces of $(0, N)$ type.

The elliptic Liouville equation plays an important role in modern field theory, namely, in the theory of strings, where the quantum Liouville field appears as a conformal anomaly ([1, 17]). Also Equation (1) relates to self-similar solutions of the Kadomtsev–Pogutse equations ([6]).

Equation (1) may be obtained formally from the hyperbolic equation ([13])

$$\mathcal{E}' = \{F \equiv u_{\xi\eta} - \exp(\lambda u) = 0\}, \quad \xi \equiv y^1,\ \eta \equiv y^2, \tag{2}$$

with $\lambda = 2$ by the substitution $\xi = (x + iy)/2$, $\eta = (x - iy)/2$. In many cases investigating Equation (1) in coordinates $\xi \in \mathbb{C}$, $\eta \in \mathbb{C}$, $\bar{\xi} = \eta$ is convenient, because the results of computations in the internal coordinates $\xi, \eta, u, u_k \equiv \partial^k u/\partial\xi^k$, ${}_l u \equiv \partial^l u/\partial\eta^l$ are simpler than similar expressions in real internal coordinates considered on (1).

In Section 1 of this paper, a nontrivial n-dimensional generalization of (1) is proposed. In Section 2 generating sections of symmetries and conservation laws are studied and the Lie algebra of classical symmetries of the Liouville equation is built. Invariant solutions are considered and a special property of the energy-momentum tensor is discussed. Classical and higher conservation laws for the hyperbolic Liouville equation are constructed in Section 3. For a special case of higher generating sections, an infinite series of rational (nonpolynomial) higher conserved currents is obtained. In Section 4, shadows of classical nonlocal symmetries corresponding to Abelian coverings, based on an arbitrary conserved current for the Liouville equation, are described. Bäcklund autotransformation for the Liouville equation $u_{\xi\eta} = \exp(2u)$ and transformations between the latter, the wave equation $v_{\xi\eta} = 0$, and the scal$^+$-Liouville equation $w_{\xi\eta} = \exp(-2w)$ are integrated. Classical symmetries and conserved currents for the elliptic Toda systems associated with the semisimple Lie algebras of rank 2 are obtained in Section 5.

1. The n-Dimensional Elliptic Liouville Equation

1.1. CONFORMAL EQUIVALENCE OF TWO-DIMENSIONAL METRICS

Consider two pointwise conformally equivalent Riemannian metrics $\mathrm{d}s_j^2 = f_j(x, y)$ $(\mathrm{d}x^2 + \mathrm{d}y^2)$, $f_j > 0$, $j = 1, 2$, on an open two-dimensional Riemannian manifold of constant Gauss curvature

$$K_j = -(2f_j)^{-1}\Delta \ln f_j = \mathrm{const}_j.$$

Let $f_2 = f_1 \exp(2u)$, then $u(x, y)$ satisfies the equation ([9])

$$\Delta u = -K_2 f_1 \exp(2u) - K_1 f_1.$$

It is easy to see that the two-dimensional elliptic Liouville equation (1) corresponds to the pair consisting of the flat metric with $f_1 \equiv 1$, $K_1 \equiv 0$ and the metric on the Lobatchevsky plane ($K_2 \equiv -1$).

COROLLARY 1. *Every solution of Equation* (1) *implies existence of the Lobatchevsky plane model conformally equivalent to Euclidean space with the diagonal metric* $g_{ij} = \delta_{ij}$.

DEFINITION 1. We call the equation corresponding to $f_1 \equiv 1$, $K_1 \equiv 0$, $K_2 \equiv +1$, *the* scal$^+$*-Liouville equation.*

1.2. GENERAL CASE: $n \in \mathbb{N}$

Let the number n of independent variables $x^1, \dots, x^n$ be greater than 2. Then constancy of the open n-dimensional Riemannian manifold's scalar curvature

$$\text{scal} \equiv R = \text{const} \tag{3}$$

is a natural condition appearing in a general case of pointwise conformal equivalence between the Euclidean metric and the metric on the n-dimensional Riemannian manifold. In order to adjust this construction to the two-dimensional case (1) let us fix $R = -2$ (i.e., the Gauss curvature $K = -1$ for $n = 2$) and let

$$\mathrm{d}s^2 = \exp(2u)\, \mathrm{d}x^k \mathrm{d}x^k. \tag{4}$$

Condition (3) is a nonlinear PDE for $u(x^1, \dots, x^n)$.

THEOREM 1. *Condition* (3) *is*

$$(n-1)\Delta u + \frac{(n-1)(n-2)}{2}(\operatorname{grad} u)^2 = \exp(2u), \tag{5}$$

where Δ *is the Laplacian in the Euclidean space* $\mathbb{E}^n$.

Proof. Scalar curvature R of metric (4) is defined by the formula $R = \exp(-2u)$ R^i_{qqi} (summation on repeated indices is assumed). One has

$$R^i_{qqi} = \partial_i \Gamma^i_{qq} - \partial_q \Gamma^i_{qi} + \Gamma^i_{pi}\Gamma^p_{qq} - \Gamma^i_{pq}\Gamma^p_{qi} \quad \text{(no summation)},$$

where the Christoffel symbols are

$$\Gamma^k_{ij} = \partial_i u\, \delta^k_j + \partial_j u\, \delta^k_i - \partial_l u\, \delta_{ij}\delta^{kl}.$$

Computing sums on $q, i, p = 1, \dots, n$, one obtains (5).

COROLLARY 2. *On a Riemannian manifold with constant negative scalar curvature* -2*, there exists a class of conformal metrics parametrized by solutions of Equation* (5).

2. Classical Symmetries of the Liouville Equation

The determining equation on generating sections φ of symmetries and conservation laws ([2]) for the elliptic Liouville equation (1) is

$$D_1^2\varphi + D_2^2\varphi - 2\exp(2u)\varphi = 0.$$

Let $z, \bar{z}$ be the complex coordinates on the plane (x, y):

$$z = x + iy, \qquad \bar{z} = x - iy, \tag{6}$$

and let $w(z)$ be an arbitrary analytic function.

Generating sections of classical symmetries and conservation laws depend on z and $\bar{z}$, function u and its first derivatives $u_z, u_{\bar{z}}$: $\varphi = \varphi(z, \bar{z}, u, u_z, u_{\bar{z}})$.

PROPOSITION 1. *Generating sections of classical symmetries and conservation laws over* $\mathbb{R}$ *for Equation* (1) *are*:

$$\varphi = w(z)u_z + \overline{w(z)}u_{\bar{z}} + \tfrac{1}{2}(w'(z) + \overline{w'(z)}). \tag{7}$$

The real Lie algebra of classical infinitesimal symmetries of the two-dimensional elliptic Liouville equation consists of vector fields

$$X_w = -w(z)\partial - \overline{w(z)}\,\overline{\partial} + \frac{1}{2}(\partial w + \overline{\partial w})\frac{\partial}{\partial u} + \cdots, \tag{8}$$

where ∂ and $\overline{\partial}$ correspond to differentiating with respect to z and $\bar{z}$.

Proposition 1 and formula (8) imply

THEOREM 2. *The algebra* $\operatorname{sym}\mathcal{E}$ *of classical infinitesimal symmetries of the elliptic Liouville equation is isomorphic to the Lie algebra* $\operatorname{Vect} S^1$ *of analytic vector fields on circumference* S^1 ([8]).

The isomorphism is a projection $-\mathrm{d}z(X_w)$ for one turn and for another turn it is a reconstruction of the Lie field (8) by generating section (7) corresponding to the analytic $w(z)$.

Remark 1. The Lie fields X_w possess an obvious complex structure $J\colon X_w \mapsto X_{iw}$ such that $J(JX_w) = -X_w$.

Remark 2. The Liouville equation has a discrete symmetry $x^1 \to x^2$ invoking change of orientation of the plane (x^1, x^2); under its action analytic functions of complex argument z are mapped to conjugate ones, i.e., to antianalytic ones.

2.1. INVARIANT SOLUTIONS

Obtaining solutions of Equation (1), invariant under the action of a certain point symmetry X_w of the form (8), is described in [6]. Denote by $W(z) = \alpha \int \mathrm{d}z/w(z) + i\alpha\beta$, $\alpha, \beta \in \mathbb{R}$; then the answer is

$$u_1(z, \bar{z}) = \tfrac{1}{2} \ln \frac{W'\overline{W'}}{\sinh^2(\operatorname{Im} W)}, \tag{9a}$$

$$u_2(z, \bar{z}) = \tfrac{1}{2} \ln \frac{W'\overline{W'}}{(\operatorname{Im} W)^2}, \tag{9b}$$

$$u_3(z, \bar{z}) = \tfrac{1}{2} \ln \frac{W'\overline{W'}}{\sin^2(\operatorname{Im} W)}. \tag{9c}$$

Remark 3. Functions (9a)–(9c) map successively one into another:

$$W_1(z) = i \ln H(z), \qquad W_2(z) = \tan \frac{W_1(4z)}{2}, \qquad W_3(z) = \ln W_2(z),$$

where $H(z)$ is an arbitrary analytic function ([13]), and finally they provide general solution of the Liouville:

$$u[H] = \frac{1}{2} \ln \frac{4H'\overline{H'}}{(1 - H\overline{H})^2}. \tag{10}$$

Remark 4. One easily checks that the analytic function $w(z)$ is expressed via harmonic one $v(x, y) \equiv \operatorname{Im} W(x + iy)$ by

$$w(z) = \frac{\text{const}}{v_y + i v_x}, \quad \text{const} \in \mathbb{R} \setminus \{0\}.$$

Remark 5. General solution (10) may be also obtained by integrating the elliptic Toda system associated with the simple Lie algebra A_1 [12, 15].

Remark 6. According to Bianchi, 1879 (cf. [7]), one can construct a solution of Equation (1) using an arbitrary harmonic function $v(x, y)$ in a way similar to (9a)–(9c):

$$u_4(z, \bar{z}) = \frac{1}{2} \ln \frac{v_x^2 + v_y^2}{\cos^2 v}. \tag{9d}$$

2.1.1. *The Energy-Momentum Tensor on $u = \infty$ Curves*

Let *caustic* [5] be a connected component of the curve

$$\{ z \mid |H(z)| = 1,\ \partial^l H(z) \neq 0,\ l \geqslant 1 \},$$

where solution (10) becomes ∞. The traceless energy-tensor for Equation (1) may be considered as $T(z) = \partial^2 u - (\partial u)^2$, $\overline{\partial} T = 0$.

PROPOSITION 2. *The energy-momentum tensor $T(z)$ is continuous on the caustics.*

Proof. It is easy to check that for arbitrary solution (10)

$$T(z) = \frac{1}{2}\left(\frac{H'''}{H'} - -\frac{3}{2}\left(\frac{H''}{H'}\right)^2\right) \equiv \tfrac{1}{2}\{H, z\},$$

where $\{H(z), z\}$ is the Shwarzian derivative. The fact that $T(z)$ does not depend on values $H(z)$ with $|H| = 1$, implies the result.

2.1.2. *Nonlinear Superposition for the Liouville Equation*

The Cauchy–Riemann equation $\partial H/\partial\bar{z} = 0$ is linear, and superposition principle is valid for it. Thus one obtains nonlinear superposition principle for Equation (1) by means of formula (10): if H an everywhere nongenerating function and a linear combination of arbitrary analytic functions $H_1(z)$ and $H_2(z)$, then $u[H]$, $u'[H_1]$, and $u''[H_2]$ are independent solutions of the elliptic Liouville equation (see (10) for the definition of $u[H]$).

Remark 7. The constants $H = z_0 \in \mathbb{C}$ may be used for constructing new solutions though none is defined by formula 10 for them solely; the trivial function $H = 0$ (or $v = 0$) must be regarded as the neutral element (matching no solution (10)!).

COROLLARY 3. *Solution $u[H]$ and 'antisolution' $u[-H]$ vanish in presence of an arbitrary solution $u[H_1]$.*

Really, $u[(H_1 + H) + (-H)] = u[H']$.

Remark 8. The Lie algebra structure $[H_1, H_2] = \text{Wronsk}(H_1, H_2)$ in the ring $\mathcal{A}$ of analytic functions provides another mapping $\mathcal{A} \times \mathcal{A} \to \text{Sol}\,\mathcal{E}$ into the set of solutions to the Liouville equation, i.e., $H_1 \otimes H_2 \mapsto u[[H_1, H_2]]$.

3. Conservation Laws and Higher Symmetries for the Liouville Equation

3.1. CLASSICAL CONSERVATION LAWS FOR THE HYPERBOLIC LIOUVILLE EQUATION

The results obtained in [10] may be carried over literally to the hyperbolic case (2), which is a simplification in some sense. Namely, the following statement is valid:

PROPOSITION 3. *For the hyperbolic Liouville equation one has*

(1) *Generating sections $\gamma(\xi, \eta, u, u_\xi, u_\eta)$ of classical symmetries and conservation laws are*

$$\gamma^{(\xi)}(\xi, u_\xi) = \left(\frac{\partial}{\partial\xi} + \lambda u_\xi\right)\Phi(\xi) \quad or \quad \gamma^{(\eta)}(\eta, u_\eta) = \left(\frac{\partial}{\partial\eta} + \lambda u_\eta\right)\Psi(\eta), \tag{11}$$

where $\Phi(\xi)$, $\Psi(\eta)$ are arbitrary functions.

(2) *Each of the generating sections corresponds a nontrivial conservation law.*

(3) *The symmetries above are Noetherian symmetries of the action with the Lagrangian density $L = u_\xi u_\eta/2 + \exp(\lambda u)/\lambda$.*

(4) *The generating section $\gamma^{(\xi)}$ corresponds to the conserved current*

$$\begin{aligned} h \;=\; & \left(u\Phi'' + \frac{\lambda}{2}uu_\xi\Phi' + \frac{\lambda}{2}uu_{\xi\xi}\Phi\right)\mathrm{d}\xi\, + \\ & + \left(\frac{\lambda}{2}u\Phi\exp(\lambda u) - \Phi\exp(\lambda u) + u_\eta\Phi' + \frac{\lambda}{2}u_\xi u_\eta\Phi\right)\mathrm{d}\eta, \end{aligned} \tag{12}$$

the current for $\gamma^{(\eta)}$ may be obtained by the interchange $\xi \leftrightarrow \eta$, $\Phi \leftrightarrow \Psi$.

Proof of the proposition is quite similar to the one given in [10] and is omitted.

Remark 9. Conserved currents h for both Equations (1) and (2) form a Lie algebra with the bracket induced by the Lie algebra of analytic functions $[a, b] = \mathrm{Wronsk}(a, b)$.

3.2. GENERATING SECTIONS OF HIGHER SYMMETRIES AND CONSERVATION LAWS

Generating sections γ of higher symmetries and conservation for Equation (2) are [7, 21]:

$$\gamma_n = (D_\xi + \lambda u_1)\Phi(\xi, w, w_1, \ldots, w_{n-3}), \tag{13}$$

where

$$u_k \equiv \partial^k u/\partial\xi^k, \qquad w \equiv u_2 - \frac{\lambda}{2}u_1^2, \qquad w_k \equiv D_\xi^k w,$$

D_ξ is the total derivative with respect to ξ, Φ is an arbitrary function (there exists a similar class of higher generating sections with ξ and η interchanged; second-order sections are absent).

Remark 10. The noncommutative Lie algebra of higher symmetries of Equation (2) contains an infinite commutative subalgebra defined by the higher Korteweg–de Vries equations [21].

The complex substitution $u(\xi,\eta)\big|_{(z/2,\bar z/2)} = u(z,\bar z)$ with $\lambda = 2$ transforms (2) to Equation (1) in coordinates (6): $4u_{z\bar z} = \exp(2u)$, and also gives generating sections of symmetries and conservation laws, defined over $\mathbb{C}$, for the latter. In order to define these structures over $\mathbb{R}$ and apply them to Equation (1), one has to compensate imaginary part with adding the conjugate structure:

$$\varphi = (D_z + 2u_z)\Phi(z, w, w_1, \ldots) + (D_{\bar z} + 2u_{\bar z})\bar\Phi(\bar z, \bar w, \bar w_1, \ldots).$$

EXAMPLE 1. The function $\varphi = 2u_{xxx} - 3\exp(2u)u_x - u_x^3 + 3u_x u_y^2$ is a generating section of higher symmetry and higher conservation law for Equation (1). The corresponding conserved current $h = S_1\,\mathrm{d}y - S_2\,\mathrm{d}x$ may be reconstructed with the same method [19] as in the case of classical conserved currents [10]; the components of h are

$$\begin{aligned}
4S_1 \;=\;& 3u_x^2u_y^2 - u_x^4 - 6uu_xu_yu_{xy} - 6\exp(2u)u_x^2 + 2\exp(2u)u_{xx} + \\
&+ 4u_xu_{xxx} - 3\exp(4u) + 3u_y^2u\exp(2u) - 3u_y^2uu_{xx} + 8u\exp(2u)u_{xx} + \\
&+ 3u_x^2uu_{xx} - 6u_y^2\exp(2u) - 4uu_{xxxx} + \frac{6u_y^2}{u}\exp(2u) + \\
&+ \frac{3}{u}\exp(4u) - \frac{3u_{xx}}{u}\exp(2u) - \frac{3u_y^2}{u^2}\exp(2u) + 13u\exp(2u)u_x^2,\\
4S_2 \;=\;& 6u_xu_yuu_{xx} + 4u_yu_{xxx} - u_yu_x^3 + 3u_xu_y^3 - 4uu_{xxxy} - 3uu_{xy}u_y^2 + \\
&+ 3uu_x^2u_{xy} + 6u_{xy}\exp(2u) - \frac{6u_xu_y}{u}\exp(2u) - \\
&- \frac{3u_{xy}}{u}\exp(2u) + \frac{3u_xu_y}{u^2}\exp(2u).
\end{aligned}$$

It is interesting that in the simplest case the conserved current is not polynomial, but a rational function.

3.2.1. *Higher Conservation Laws*

An infinite series of polynomial conserved currents for Equation (2) was described in [4], but the formulae given there do not describe *all* local conservation laws. Below higher conservation laws for Equation (2) are obtained. They are rational in $u(\xi,\eta)$. We obtain higher conserved current h corresponding to generating section γ_n defined in (13).

Let the divergence ω be

$$\omega = (u_{\xi\eta} - \exp(\lambda u))(D_\xi\Phi + \lambda u_1\Phi)\,\mathrm{d}\xi\wedge\mathrm{d}\eta, \qquad \omega\big|_{\mathcal{E}} = \bar{\mathrm{d}}h = 0.$$

Literally repeating the reasoning for classical case, we have

$$G(\ell_\omega \circ u) \;\equiv\; s^\xi\,\mathrm{d}\xi + s^\eta\,\mathrm{d}\eta = \left[uD_\xi\Big(\Phi_\xi + \sum_{r=0}^{n-3}\Phi_{w_r}w_{r+1} + \lambda u_1\Phi\Big)\right]\mathrm{d}\eta +$$

$$
\begin{aligned}
&+\Bigg[{}_1u\Big(\Phi_\xi+\sum_{r=0}^{n-3}\Phi_{w_r}w_{r+1}+\lambda u_1\Phi\Big)+\lambda(u_{\xi\eta}-\mathrm{e}^{\lambda u})\Phi+\\
&+\sum_{s=0}^{n-3}\sum_{t=1}^{s+2}\sum_{p=1}^{t}(-1)^{p+1}\binom{t}{p}D_\xi^{p-1}\bigg((u_{\xi\eta}-\mathrm{e}^{\lambda u})\Phi_{\xi w_s}\frac{\partial w_s}{\partial u_t}u_{t-p}\bigg)+\\
&+\sum_{s=0}^{n-3}\sum_{r=0}^{n-3}\sum_{t=1}^{s+2}\sum_{p=1}^{t}(-1)^{p+1}\binom{t}{p}\times\\
&\times D_\xi^{p-1}\bigg((u_{\xi\eta}-\mathrm{e}^{\lambda u})w_{r+1}\Phi_{w_s w_r}\frac{\partial w_s}{\partial u_t}u_{t-p}\bigg)+\\
&+\sum_{r=1}^{n-2}\sum_{t=1}^{r+2}\sum_{p=1}^{t}(-1)^{p+1}\binom{t}{p}D_\xi^{p-1}\bigg((u_{\xi\eta}-\mathrm{e}^{\lambda u})\Phi_{w_{r-1}}\frac{\partial w_r}{\partial u_t}u_{t-p}\bigg)+\\
&+\lambda\sum_{r=0}^{n-3}\sum_{t=1}^{r+2}\sum_{p=1}^{t}(-1)^{p+1}\binom{t}{p}\times\\
&\times D_\xi^{p-1}\bigg((u_{\xi\eta}-\mathrm{e}^{\lambda u})u_1\Phi_{w_r}\frac{\partial w_r}{\partial u_t}u_{t-p}\bigg)\Bigg]\,\mathrm{d}\xi,
\end{aligned}
$$

where G is the Green operator and $\ell_\omega = \sum_\sigma \partial\omega/\partial u_\sigma \cdot D_\sigma$ is the universal linearization operator ([2, 19]). The required conserved current $h = S^\xi\,\mathrm{d}\xi + S^\eta\,\mathrm{d}\eta$ corresponding to $G(\ell_\omega \circ u)$ is

$$
h=\int^0 \mathrm{d}\tau\, A_\tau^*(s^\xi\,\mathrm{d}\xi+s^\eta\,\mathrm{d}\eta),\quad \bar{\mathrm{d}}h=0,
$$

where $A_\tau\colon (y^i, u_\sigma)\mapsto (y^i, u_\sigma\exp(\tau))$.

3.2.2. *Infinite Series of Rational Higher Conserved Currents*

Let us obtain higher conserved currents h for Equation (2) in the case $\Phi = w_k$, $k \geq 0$. The function Φ supplies a higher generating section by means of 13.

Let $\begin{bmatrix}\bullet\\ \bullet,\dots,\bullet\end{bmatrix}$ be the coefficient in the expansion of the qth total derivative with respect to ξ for the exponent $\exp(\lambda u(\xi,\eta))$:

$$
D_\xi^q\exp(\lambda u)=\sum_{m=0}^{q}\sum_{j_1+\ldots+j_q=m}\exp(\lambda u)\lambda^m\begin{bmatrix}q\\ j_1,\dots,j_q\end{bmatrix}u_1^{j_1}\cdots u_q^{j_q},\quad q\geqslant 0.
$$

The coefficients $\begin{bmatrix}\bullet\\ \bullet,\dots,\bullet\end{bmatrix}$ satisfy the following recursion relation:

$$
\begin{bmatrix}q\\ j_1,\dots,j_q\end{bmatrix}=\begin{bmatrix}q\\ j_1-1,\dots,j_q\end{bmatrix}+\begin{bmatrix}q\\ j_1,\dots,j_{q-1}+1\end{bmatrix}\delta_{j_q,1}(j_{q-1}+1)+
$$

$$+\sum_{k=1}^{q-2}\begin{bmatrix} q \\ j_1,\ldots,j_k+1,j_{k+1}-1,\ldots,j_q\end{bmatrix}(j_k+1),$$

where

$$\sum_k j_k = m \in \mathbb{N}, \quad \text{and} \quad \begin{bmatrix} \cdots \\ \ldots, j_k-1, \ldots \end{bmatrix} \equiv 0,$$

if there exists k such that $j_k = 0$. The relation follows from the Leibniz rule.

Remark 11. The following two integrals appear in computation of the conserved current h with arbitrary $\sigma = (\sigma_1, \ldots, \sigma_k)$, $k > 0$:

$$\int^0 u_{\sigma_1}\cdots u_{\sigma_k}\exp(k\tau)\,\mathrm{d}\tau = \frac{1}{k}u_{\sigma_1}\cdots u_{\sigma_k};$$

if the integrand contains exponent of $u(\xi, \eta)$, one has to integrate by parts:

$$\int^0 u_{\sigma_1}\cdots u_{\sigma_k}\exp(k\tau)\exp(\lambda u\exp(\tau))\,\mathrm{d}\tau$$
$$= \frac{u_{\sigma_1}\cdots u_{\sigma_k}}{\lambda^k u^k}\sum_{j=1}^{k-1}(-1)^{j+1}\times$$
$$\times\left(\frac{(k-1)!}{(k-j)!}(\lambda u)^{k-j}\exp(\lambda u) - (-1)^k(k-1)!\,\exp(\lambda u)\right).$$

PROPOSITION 4. *The higher conserved current $h = S_1\,\mathrm{d}\eta - S_2\,\mathrm{d}\xi$ for Equation* (2) *reconstructed from generating section* (13) *with $\Phi = w_k$, $k \geq 0$, has the components*

$$S_1 = \frac{1}{2}\mathbb{1}uu_{k+3} - \frac{\lambda}{6}\mathbb{1}u\sum_{l=0}^{k+1}\binom{k+1}{l}u_{l+1}u_{k-l+2} + \frac{\lambda}{3}\mathbb{1}uu_1u_{k+2} - \frac{\lambda}{8}\mathbb{1}uu_1\times$$
$$\times\sum_{l=0}^{k}\binom{k}{l}u_{l+1}u_{k-l+1} + \sum_{p=0}^{k+2}\binom{k+2}{p}(-1)^{k-p}\sum_{m=0}^{k-p+2}\binom{k-p+2}{m}\times$$
$$\times u_{k-m+2}\sum_{n=0}^{m}\sum_{j_1+\ldots+j_m=n}\mathrm{e}^{\lambda u}\begin{bmatrix} m \\ j_1,\ldots,j_m\end{bmatrix}u_1^{j_1}\ldots u_m^{j_m}\times$$
$$\times\left(\frac{\lambda^n}{2} + \frac{1}{\lambda u^{n+1}}\left\{\sum_{j=1}^{n}(-1)^j(\lambda u)^{n-j+1}\frac{n!}{(n-j+1)!} - (-1)^n n!\right\}\right) -$$
$$-\frac{\lambda}{2}\sum_{l=0}^{k+1}\binom{k+1}{l}\sum_{p=0}^{l}\binom{l+1}{p}(-1)^{l-p}\sum_{i_1=0}^{l-p}\binom{l-p}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}\times$$

$$\times u_{k-p-i_1+2}u_{p+i_1-i_2}\sum_{m=0}^{i_2}\sum_{j_1+\ldots+j_{i_2}=m}\mathrm{e}^{\lambda u}\begin{bmatrix} i_2 \\ j_1,\ldots,j_{i_2}\end{bmatrix}u_1^{j_1}\ldots u_{i_2}^{j_{i_2}}\times$$

$$\times\left(\frac{\lambda^m}{3}+\frac{1}{\lambda^2u^{m+2}}\left\{\sum_{j=1}^{m+1}(-1)^j(\lambda u)^{m-j+2}\frac{(m+1)!}{(m-j+2)!}+\right.\right.$$

$$\left.\left.+(-1)^m(m+1)!\right\}\right)+\frac{\lambda}{2}\sum_{l=0}^{k+1}\binom{k+1}{l}\sum_{p=0}^{k-l+1}\binom{k-l+2}{p}(-1)^{k-l-p}\times$$

$$\times\sum_{i_1=0}^{k-l-p+1}\binom{k-l-p+1}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}u_{k-p+2}u_{p+i_1-i_2}\sum_{m=0}^{i_2}\times$$

$$\times\sum_{j_1+\ldots+j_{i_2}=m}\mathrm{e}^{\lambda u}\begin{bmatrix} i_2 \\ j_1,\ldots,j_{i_2}\end{bmatrix}u_1^{j_1}\ldots u_{i_2}^{j_{i_2}}\left(\frac{\lambda^m}{3}+\frac{1}{\lambda^2u^{m+2}}\times\right.$$

$$\times\left\{\sum_{j=1}^{m+1}(-1)^j(\lambda u)^{m-j+2}\frac{(m+1)!}{(m-j+2)!}+\right.$$

$$\left.\left.+(-1)^m(m+1)!\right\}\right)-\frac{\lambda}{3}uu_{\xi\eta}u_{k+2}+\frac{\lambda^2}{8}uu_{\xi\eta}\sum_{l=0}^{k}\binom{k}{l}u_{l+1}\times$$

$$\times u_{k-l+1}+u_{k+2}\mathrm{e}^{\lambda u}-\frac{u_{k+2}}{\lambda u}\mathrm{e}^{\lambda u}-\frac{1}{2}\sum_{l=0}^{k}\binom{k}{l}\times$$

$$\times\frac{u_{l+1}u_{k-l+1}}{\lambda u^2}\{\lambda^2u^2-2\lambda u+2\}\mathrm{e}^{\lambda u}+$$

$$+\lambda\sum_{p=0}^{k+1}\binom{k+2}{p}(-1)^{k-p}\sum_{i_1=0}^{k-p+1}\binom{k-p+1}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}\times$$

$$\times u_{k-p-i_1+2}u_{p+i_1-i_2}\sum_{m=0}^{i_2}\sum_{j_1+\ldots+j_{i_2}=m}\mathrm{e}^{\lambda u}\begin{bmatrix} i_2 \\ j_1,\ldots,j_{i_2}\end{bmatrix}\times$$

$$\times u_1^{j_1}\ldots u_{i_2}^{j_{i_2}}\left(\frac{\lambda^m}{3}+\frac{1}{\lambda^2u^{m+2}}\times\right.$$

$$\left.\times\left\{\sum_{j=1}^{m+1}(-1)^j(\lambda u)^{m-j+2}\frac{(m+1)!}{(m-j+2)!}+(-1)^m(m+1)!\right\}\right)-$$

$$-\frac{\lambda^2}{2}\sum_{l=0}^{k}\binom{k}{l}\sum_{p=0}^{l}\binom{l+1}{p}(-1)^{l-p}\sum_{i_1=0}^{l-p}\binom{l-p}{i_1}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}\times$$

$$
\begin{aligned}
&\times \sum_{i_3=0}^{i_2} u_{l-p-i_1+1} u_{k-l+i_1-i_2+1} u_{p+i_2-i_3} \sum_{m=0}^{i_3} \sum_{j_1+\ldots+j_{i_3}=m} \mathrm{e}^{\lambda u} \times \\
&\times \begin{bmatrix} i_3 \\ j_1, \ldots, j_{i_3} \end{bmatrix} u_1^{j_1} \ldots u_{i_3}^{j_{i_3}} \left(\frac{\lambda^m}{4} + \frac{1}{\lambda^3 u^{m+3}} \times \right. \\
&\times \left. \left\{ \sum_{j=1}^{m+2} (-1)^j (\lambda u)^{m-j+3} \frac{(m+2)!}{(m-j+3)!} - (-1)^m (m+2)! \right\} \right) - \\
&- \frac{\lambda^2}{2} \sum_{l=0}^{k} \binom{k}{l} \sum_{p=0}^{k-l} \binom{k-l+1}{p} (-1)^{k-l} \sum_{i_1=0}^{k-l} \binom{k-l}{i_1} \sum_{i_2=0}^{i_1} \binom{i_1}{i_2} \sum_{i_3=0}^{i_2} \times \\
&\times \binom{i_2}{i_3} u_{k-l-i_1+1} u_{l+i_1-i_2+1} u_{p+i_2-i_3} \sum_{m=0}^{i_3} \sum_{j_1+\ldots+j_{i_3}=m} \mathrm{e}^{\lambda u} \times \\
&\times \begin{bmatrix} i_3 \\ j_1, \ldots, j_{i_3} \end{bmatrix} u_1^{j_1} \ldots u_{i_3}^{j_{i_3}} \left(\frac{\lambda^m}{4} + \frac{1}{\lambda^3 u^{m+3}} \times \right. \\
&\times \left. \left\{ \sum_{j=1}^{m+2} (-1)^j (\lambda u)^{m-j+3} \frac{(m+2)!}{(m-j+3)!} - (-1)^m (m+2)! \right\} \right),
\end{aligned}
$$

$$
\begin{aligned}
S_2 \;=\; & -\frac{\lambda}{3} u u_1 u_{k+3} + \frac{\lambda^2}{8} u u_1 \sum_{l=0}^{k} \binom{k+1}{l} u_{l+1} u_{k-l+2} + \\
& + \frac{\lambda}{6} u \sum_{l=0}^{k+2} \binom{k+2}{l} u_{l+1} u_{k-l+3} - \frac{\lambda}{3} u u_2 u_{k+2} + \\
& + \frac{\lambda^2}{8} u u_2 \sum_{l=0}^{k} \binom{k}{l} u_{l+1} u_{k-l+1} - \frac{1}{2} u u_{k+4}.
\end{aligned}
$$

4. Coverings of the Liouville Equation, Nonlocal Symmetries, and Bäcklund Transformations

4.1. ON NONLOCAL SYMMETRIES

One can construct an Abelian covering $\tilde{c}^h$ structure [2] in the bundle $c^h\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ using a conserved current $h = S^\xi \,\mathrm{d}\xi + S^\eta \,\mathrm{d}\eta$ for an equation $\mathcal{E}$ with 2 independent variables. Then the fiber coordinate w is differentiated with respect to ξ and η in the following way: $w_\xi = S^\xi$, $w_\eta = S^\eta$; the total derivatives are extended to the form $\tilde{D}_i = \bar{D}_i + S^{y^i}\,\partial/\partial w$.

Consider the covering $\tilde{c}^h$, provided by classical conserved current (12) corresponding to generating section γ^ξ of the form (11). The search for shadows of nonlocal $\tilde{c}^h$-symmetries of Equation (2), i.e., solving the determining equation $\tilde{\ell}_F^{\mathcal{E}'}(\tilde{\gamma}) = 0$, where the universal linearization operator $\tilde{\ell}_F$ contains extended total derivatives $\tilde{D}_i$ leads to

PROPOSITION 5. *The shadows $\tilde{\gamma}(\xi, \eta, w, u, u_\xi, u_\eta)$ of the classical nonlocal $\tilde{c}^h$-symmetries are*

$$\tilde{\gamma}^{(\xi)}(\xi, u_\xi) = \left(\frac{\partial}{\partial\xi} + \lambda u_\xi\right)\tilde{\Phi}(\xi) \quad or \quad \tilde{\gamma}^{(\eta)}(\eta, u_\eta) = \left(\frac{\partial}{\partial\eta} + \lambda u_\eta\right)\tilde{\Psi}(\eta), \tag{11'}$$

where $\tilde{\Phi}(\xi)$, $\tilde{\Psi}(\eta)$ are arbitrary functions.

In other words, the set of shadows of classical nonlocal $\tilde{c}^h$-symmetries of Equation (2) is isomorphic to the Lie algebra of classical symmetries (11), and no new symmetries of the latter equation are obtained.

4.2. BÄCKLUND TRANSFORMATIONS

Bäcklund autotransformation for the Liouville equation and Bäcklund transformation between Equation (2), the wave equation $v_{\xi\eta} = 0$ and the equation $w_{\xi\eta} = \exp(-2w)$ corresponding to the Gaussian curvature $K_2 \equiv +1$ in the sense of Section 1.1, were found in [3]. Let us integrate all these cases.

Remark 12. These transformations depend on formal parameter k, but the latter can be eliminated with by a scale transformation.

4.2.1. *Bäcklund Autotransformation*

Equation (2) possesses the Bäcklund autotransformation

$$\begin{aligned}(u' - u)_\xi &= \frac{1}{k} \cdot \exp(u' + u),\\ (u' + u)_\eta &= k \cdot \left(\exp(u' - u) - \exp(u - u')\right),\end{aligned}$$

$k \neq 0$ being a formal parameter. Set $k = 1$ and denote $\tau^- = u' - u$, $\tau^+ = u' + u$. It is easy to see that the functions τ^- and τ^+ satisfy the system

$$\tau_\xi^- = \exp(\tau^+), \tag{14a}$$

$$\tau_\eta^+ = 2\sinh\tau^-. \tag{14b}$$

From (14a) it follows that $\tau^+ = \ln\tau_\xi^-$; then Equation (14b) implies $\tau_{\xi\eta}^- = 2\tau_\xi^-$ $\sinh\tau^-$, and

$$\tau_\eta^- = 2\cosh\tau^- + \delta(\eta), \tag{15}$$

where $\delta(\eta)$ is the integration constant. Resolving (15), one obtains the function $\tau^-(\xi,\eta)$ depending now on another integration constant $\epsilon(\xi)$, and thus one reconstructs $\tau^+(\xi,\eta)$. Knowing the functions τ^- and τ^+, one gets a pair of solutions u, u' of Equation (2), and these solutions are bound by Bäcklund autotransformation.

There exists a similar transformation with $\xi \leftrightarrow \eta$.

4.2.2. *Transformation Between the Liouville Equation and the Wave Equation*

Bäcklund transformation between Equation (2) and the equation $v_{\xi\eta} = 0$ is [3, 18]:

$$(v-u)_\xi = \frac{1}{k}\cdot\exp(v+u), \qquad (v+u)_\eta = -k\cdot\exp(u-v).$$

As before, denote $\tau^- = u - v$, $\tau^+ = v + u$; these functions satisfy the system

$$\tau^-_\xi = -\exp(\tau^+), \qquad \tau^+_\eta = -\exp(\tau^-),$$

i.e., each of the functions τ^-, τ^+ is a potential for the other, e.g., $\tau^+ = \ln(-\tau^-_\xi)$. Solving an easily obtained equation $\tau^-_{\xi\eta} = \exp(\tau^-)\tau^-_\xi$ for $\tau^-(\xi,\eta)$, one gets an equation with separating variables,

$$\frac{\mathrm{d}\theta}{\exp(\theta)} = \frac{\mathrm{d}\eta}{\exp(\delta(\eta))}, \tag{16}$$

where $\theta = (\tau^- + \delta)(\xi,\eta)$ and $\delta(\eta)$ is the integration constant. The variable ξ is a parameter in this equation, and thus another integration constant $\epsilon(\xi)$ appears. The function

$$\theta = -\ln\left(-\int_{\epsilon(\xi)}^{\eta}\exp(-\delta(\kappa))\,\mathrm{d}\kappa\right)$$

is a solution of Equation (16), and thus one successively reconstructs functions the τ^- and τ^+ and obtains a pair u, v, where $u(\xi,\eta)$ is a solution of and $v(\xi,\eta)$ is a solution of the wave equation.

Remark 13. Assuming $\xi = \bar{\eta} = z/2$, one has $\Delta_2 v(x,y) = 0$, i.e., the above-mentioned transformation bounds Equation (1) and the two-dimensional Laplace equation in complex coordinates (6).

4.2.3. *Transformation Between the Liouville Equation and the* scal$^+$*-Liouville Equation*

This transformation may be written down in the following way ([3]):

$$\begin{aligned}(w-u)_\xi &= \frac{1}{k}\cdot(\exp(w+u)+\exp(-w-u)),\\ (w+u)_\eta &= -k\cdot\exp(u-w),\end{aligned}$$

where $w_{\xi\eta} = \exp(-2w)$. Then $-w$ is a solution of the scal$^+$-Liouville equation (see Section 1.1). Denote $\tau^+ = u + w$, $\tau^- = u - w$, then

$$\tau^+_\eta = -\exp(\tau^-), \qquad \tau^-_\xi = -2\cosh\tau^+,$$

implying $\tau^- = \ln(-\tau^+_\eta)$ and $\tau^+_{\xi\eta} = 2\tau^+_\eta \cosh\tau^+$. Solving the last equation gives $\tau^+_\xi = 2\sinh\tau^+ + \delta(\xi)$, where $\delta(\xi)$ is the integration constant; further resolving the equation on $\tau^+(\xi,\eta)$ provides another integration constant $\epsilon(\eta)$. By $\tau^+(\xi,\eta)$ one reconstructs $\tau^-(\xi,\eta)$ and finally obtains a pair of solutions $u = (\tau^+ + \tau^-)/2$ and $w = (\tau^+ - \tau^-)/2$.

Remark 14. The function $w(\xi,\eta) = \frac{1}{2}\ln v_\xi v_\eta/\cosh^2 v$ is a solution of the equation $w_{\xi\eta} = \exp(-2w)$, where $v(\xi,\eta)$ is an arbitrary solution of the wave equation $v_{\xi\eta} = 0$.

Some more results concerning nonlocal structures (zero curvature representation and recursion operator for the Liouville equation) can be found in [7].

5. On the Toda Systems Associated with the Semisimple Lie Algebras of Rank 2

The two-dimensional elliptic Toda systems [12, 15] associated with the semisimple Lie algebras A_2, $B_2 \simeq C_2$, $D_2 \simeq A_1 \oplus A_1$, and G_2, of rank 2, represented in the coordinates z, $\bar z$, $u^k(z,\bar z)$ are:

$$u^k_{z\bar z} = \frac{1}{4}\exp\left(\sum_{l=1}^{2} K^k_l u^l\right), \quad k = 1, 2, \tag{17}$$

where K^k_l is the Cartan matrix of the corresponding Lie algebra. Denote $u^1 \equiv u$ and $u^2 \equiv v$, and let $w(z)$ be an arbitrary analytic function. The following results take place [11]:

PROPOSITION 6. *The following facts hold*:

(1) *Generating sections* $\varphi = (\varphi^1, \varphi^2)$ *for classical infinitesimal symmetries of Toda system* (17) *associated to the simple Lie algebra of rank* 2 *with the Cartan matrix*

$$K = \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}, \quad a = \text{const}, \tag{18}$$

are

$$\begin{aligned}\varphi^1 &= w(z)u^1_z + \overline{w(z)}u^1_{\bar z} + \frac{a+2}{4-a}\left(w_z + \bar w_{\bar z}\right),\\ \varphi^2 &= w(z)u^2_z + \overline{w(z)}u^2_{\bar z} + \frac{3}{4-a}\left(w_z + \bar w_{\bar z}\right).\end{aligned}$$

(2) *Classical conserved currents $H = S^z\,\mathrm{d}\bar{z} - S^{\bar{z}}\,\mathrm{d}z$ for this system have the components*

$$\begin{aligned}
S^z = {} & -av\bar{w}_{\bar{z}\bar{z}} - u\bar{w}_{\bar{z}\bar{z}} + \bar{w}u_{\bar{z}}^2 + w_z u_{\bar{z}} - \tfrac{1}{2}\exp(2u-av)w + awv_z v_{\bar{z}} - \\
& - u\bar{w}_{\bar{z}}u_{\bar{z}} + a\bar{w}_{\bar{z}}v_{\bar{z}} - \tfrac{1}{8}auw\exp(2v-u) + \tfrac{1}{2}au\bar{w}v_{\bar{z}\bar{z}} + \bar{w}_{\bar{z}}u_{\bar{z}} + \\
& + \tfrac{1}{2}av\bar{w}_{\bar{z}}u_{\bar{z}} - av\bar{w}_{\bar{z}}v_{\bar{z}} + \tfrac{1}{2}au\bar{w}_{\bar{z}}v_{\bar{z}} - \tfrac{1}{8}avw\exp(2u-av) - \\
& - \tfrac{1}{2}awv_z u_{\bar{z}} - \tfrac{1}{2}awv_{\bar{z}}u_z - a\bar{w}v_{\bar{z}}u_{\bar{z}} + aw_z v_{\bar{z}} + wu_z u_{\bar{z}} + \\
& + a\bar{w}v_{\bar{z}}^2 + \tfrac{1}{4}uw\exp(2u-av) + \tfrac{1}{4}avw\exp(2v-u) - \\
& - u\bar{w}u_{\bar{z}\bar{z}} + \tfrac{1}{2}av\bar{w}u_{\bar{z}\bar{z}} - av\bar{w}v_{\bar{z}\bar{z}} - \tfrac{1}{2}a\exp(2v-u)w, \\
S^{\bar{z}} = {} & -avw_{zz} - uw_{zz} + a\bar{w}v_z v_{\bar{z}} + w_z u_z + \bar{w}_{\bar{z}}u_z + wu_z^2 - \\
& - awu_z v_z - \tfrac{1}{2}a\bar{w}u_z v_{\bar{z}} - \tfrac{1}{2}a\bar{w}u_{\bar{z}}v_z - avw_z v_z + \tfrac{1}{2}avw_z u_z + \\
& + \tfrac{1}{2}auwv_{zz} + \tfrac{1}{4}av\bar{w}\exp(2v-u) - avwv_{zz} + \tfrac{1}{2}auw_z v_z - \\
& - \tfrac{1}{8}av\bar{w}\exp(2u-av) + \tfrac{1}{2}avwu_{zz} - \tfrac{1}{8}au\bar{w}\exp(2v-u) - \\
& - uwu_{zz} + \tfrac{1}{4}u\bar{w}\exp(2u-av) - uw_z u_z - \tfrac{1}{2}a\exp(2v-u)\bar{w} - \\
& - \tfrac{1}{2}\exp(2u-av)\bar{w} + a\bar{w}_{\bar{z}}v_z + aw_z v_z + \bar{w}u_z u_{\bar{z}} + awv_z^2.
\end{aligned}$$

For the Lie algebra A_2 the constant $a = 1$, for B_2, $a = 2$, and for G_2, $a = 3$. The Cartan matrix for C_2 is transpose one for B_2, so the change $u \leftrightarrow v$ is needed for this case. The Lie algebra D_2 is not simple, and the conserved current for this algebra is a sum of two independent conserved currents (12).

Computations of conserved currents H for system (17) were rather tedious and Jet [14] software was used.

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References

1. Belavin, A. A., Polyakov, A. N. and Zamolodchikov, A. B.: Infinite conformal symmetry group in two-dimensional quantum field theory, *Nuclear Phys.* **241**(2) (1984), 333–380.
2. Bocharov, A. V., Chetverikov, V. N., Duzhin, S. V., Khor'kova, N. G., Krasil'shchik, I. S., Samokhin, A. V., Torkhov, Yu. N., Verbovetsky, A. M. and Vinogradov, A. M.: *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, Amer. Math. Soc., Providence, RI, 1999. Edited and with a preface by Krasil'shchik and Vinogradov.
3. Dodd, R. K. and Bullough, R. K.: Bäcklund transformations for the sine-Gordon equations, *Proc. Roy. Soc. London A* **351** (1976), 499–523.

4. Dodd, R. K. and Bullough, R. K.: Polynomial conserved densities for the sine-Gordon equations, *Proc. Roy. Soc. London A* **352** (1977), 481–503.
5. Georgadze, G. P., Pogrebkov, A. K. and Polivanov, M. K.: On global solutions of the Cauchy problem for the Liouville equation $\varphi_{tt}(t,x) - \varphi_{xx}(t,x) = \frac{1}{2}m\exp(\varphi(t,x))$, *Dokl. AN SSSR* **243**(2) (1978), 318–320 (in Russian).
6. Gusyatnikova, V. N., Samokhin, A. V., Titov, V. S. *et al.*: Symmetries and conservation laws of Kadomtsev–Pogutse equations, *Acta Appl. Math.* **15**(1) (1989), 23–64.
7. Ibragimov, N. H. (ed): *CRC Handbook of Lie Group to Differential Equations*, *Vol. II. Applications in Engineering and Physical Sciences*, CRC Press, Boca Raton, FL, 1995.
8. Kac, V. G. and Raina, A. K.: *Bombai Lectures on Highest Weight Representation of Infinite Dimensional Lie Algebras*, World Scientific, Singapore, 1987.
9. Kazdan, J. L. and Warner, F. W.: Curvature functions for open 2-manifolds, *Ann. Math.* (*2*) **99**(2) (1974), 203–219.
10. Kiselev, A. V.: Classical conservation laws for the elliptic Liouville equation, *Moscow Univ. Phys. Bull.* (6) (2000), 11–13.
11. Kiselev, A. V.: Conservation laws for the elliptic Toda systems associated with the algebras A_2, B_2, C_2, D_2, G_2, *Proc. XXII Conf. of Young Sci. at MSU Mech.–Math. Dept.* (*2000*), MSU, Moscow, Russia.
12. Leznov, A. N. and Saveliev, M. V.: *Group-Theoretical Methods for Integration of Nonlinear Dynamical Systems*, Birkhäuser, Basel, 1992.
13. Liouville, J.: Sur l'équation aux différences partielles $\mathrm{d}^2\log\lambda/\mathrm{d}u\,\mathrm{d}v \pm \lambda/(2a^2) = 0$, *J. Math. Pure Appl.* **18**(1) (1853), 71–72.
14. Marvan, M.: *Jet. A software for differential calculus on jet spaces and diffieties, ver.* 3.9 (*August* 1997) *for Maple* V *Release* 4, Opava, Czhech Republic, 1997; http://diffiety.ac.ru/
15. Ovchnnikov, A. V.: Toda systems associated with the Lie algebras and the W-algebras in some problems of mathematical physics, PhD thesis, Physics Dept., MSU, Moscow, Russia, 1996 (in Russian).
16. Poincaré, H.: Les fonctions fuchsiennes et l'équation $\Delta u = \exp(u)$, *J. Math. Pures Appl. (5)* (1898), No 4, 157–230.
17. Polyakov, A. M.: *Gauge Fields and Strings*, Harwood Acad. Publs., New York, 1987.
18. Shadwick, W. F.: The Bäcklund problem for the equation $\partial^2 z/\partial x^1 \partial x^2 = f(z)$, *J. Math. Phys.* **19**(11) (1978), 2312–2317.
19. Vinogradov, A. M.: The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws, I. The linear theory. II. The nonlinear theory, *J. Math. Anal. Appl.* **100** (1984), 1–129.
20. Witten, E.: Some exact multipseudoparticle solutions of classical Yang–Mills theory, *Phys. Rev. Lett.* **38**(3) (1977), 121–124.
21. Zhiber, A. V. and Shabat, A. B.: The Klein–Gordon equations admitting nontrivial group, *Dokl. AN SSSR* **247**(5) (1979), 1103–1107 (in Russian).
22. Zograf, P. G. and Tachtajan, L. A.: On the Liouville equation, accessor parameters and geometry of the Teichmüller space for the Riemannian surfaces of genus 0, *Math. Sbornik* **132(174)**(2) (1987), 147–166 (in Russian).