Ricci-Hamilton flow on surfaces:

lectures on works of R.Hamilton and G.Perelman

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Abstract

In this book, we give a full expository lecture about the Ricci-Hamilton flow on surfaces. Hence we derive the Li-Yau-Hamilton's Harnack inequality, the entropy estimate of R.Hamilton, the isoperimetric estimate , and the improved gradient estimates of W.X.Shi, B.Chow, Bartz-Struwe-Ye, for the Ricci-Hamilton flow. We discuss the Maximum principle for tensors of R.Hamilton and the moving plane method. We also derive in full detail some estimates for curve shortening flow in a Riemannian manifold. We give a detailed computation for our estimates involved. We also introduce G.Perelman's W-functional and give some geometric application. In an appendix, we consider the Nirenberg problem on S^2 by using the perturbation method.

This book is based on my lectures given at Tsinghua University at Beijing, National University of Singapore, and Nankai University at Tianjin given in the past three years. My aim for these lectures is to introduce graduate students into this exciting area of mathematics. I am sure, with a nice understanding of the material presented in this book, people can read works of R.Hamilton and G.Perelman well.

Preface

I made up my mind to write this book in December 2002 just after the appearance of G.Perelman's first paper on Ricci-Hamilton flow. It is clear that earlier papers of R.Hamilton on Ricci-Hamilton flow are beautiful and key point oriented, however, many details and further explanations have to be given for graduate students and non-experts to understand. In my lectures in this direction, I often found difficulty to explain the detail for the Maximum principle for tensors. So I add some description in an appendix. As us know that for the scalar linear partial differential equations, the positivity of the first eigenvalue of the corresponding operator implies the maximum principle, so one can expect such an result true for tensors.

Once we have a local existence of the flow, we try to extend the flow. That is equivalent to finding the long-term behavior of the system. To study the global behavior of the flow, one often likes to do a priori estimates for the flow, then one can use the Arzela-Ascoli theorem to get a convergence result. Actually, this principle is often used in the study of the Ricci-Hamilton flow. However, the estimates are not easy. The guiding line is to use the maximum principle. The first difficulty is to choose nice geometric quantities (often scaling-invariant quantities) and find nice equations (like the Bochner-type formula). In geometric problems, the Bianchi identities are always the conservation laws. Then the symmetry or invariant property preserved by the flow helps to find the key point to do analysis for convergence. So one has to study the self-similar or soliton problem. The common feature is that the soliton equation is a fixed point equation. Again one has to face a priori estimate. Occasionally, with the help of symmetry, one can reduce the soliton equation to an nice ordinary differential equation and luckily find solitons by solving the dynamical system. An important trick in finding the a priori estimate is the blow up analysis for the flow. Then one has to meet the limiting flows (sometimes it is a soliton problem). One may call such problem as singularity analysis. This is like the tangent cone problem in minimal surface theory. Classifying the limits and studying which singularities are real ones in the flow are the essential objects. In this program, geometric and analytical considerations play important roles. The key part is the Harnack type estimate. A beautiful work done by P.Li and S.T.Yau in 1986 serves as a model for such goal. R.Hamilton further developed this estimate. By now, such an estimate is called Li-Yau-Hamilton Harnack estimate.

In some cases, one can expect global convergence, so one tries to do pinching estimates. Pinching set may be defined by convex or concave functions on matrix space. R.Hamilton found that the associated ordinary differential equations (i.e. ODE's for short) help this goal a lot. Usually, the ODE's are monotone systems. They are rare in the dynamical system, but often occur in geometric evolution equations. Finding nice flows in geometry with nice pinching set are important.

All these ideas may be explained in the Ricci-Hamilton flow on closed surfaces. So this is a good model to introduce non-experts or graduate students to this fascinating mathematical area. This is our aim for writing this book. We have to point out that the geometric computations are easy to follow in this case. The other feature is that in dimension two, the Ricci-Hamilton flow and the Yamabe flow are the same, so one can easily get the local existence of the flow in this case.

The structure of this note is as follows. In section two, we prove the local existence of the flow. In section three, we derive the evolution equation for curvature and prove that the global flow exists. Section four studies the behavior of the solution at infinity in the case when the average of the scalar curvature $r \leq 0$. In the section five, we consider Li-Yau-Hamilton inequality of Harnack type for the curvature. In section six, we derive the decay entropy estimate. In sections seven and eight, we give the uniform estimates for curvature and its derivatives. In Chapter one, we give a full argument of Theorems 1.1.1 and 1.1.2.

In Chapters two and three, we present two other proofs of Theorem 1.1.2 on the sphere based on the works [4] and [19]

We introduce G.Perelman's entropy functionals in chapter four. We give some basic properties of these functionals and geometric applications.

We add four appendices, and we hope this will help readers' understanding of key ideas used in the Ricci-Hamilton flow.

The lesson we learned from the works of Li-Yau , R.Hamilton and G. Perelman is that a deep understanding of geometric parabolic partial differential equations can yield important new results in Riemannian Geometry and in Differential Topology.

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Chapter 1

Ricci-Hamilton flow on surfaces

1.1 Introduction

In 1988, R.Hamilton [17] studied the evolution of a Riemannian metric (g_{ij}) on a compact surface by its scalar curvature R under the flow

$$\partial_t g_{ij} = (r - R) g_{ij}, \tag{1.1.1}$$

where r is the average value of R. We will call this flow the normalized Ricci-Hamilton flow on M. In short, we just call it the Ricci-Hamilton flow in this chapter except expressed explicitly. It is clear that this flow make sense in higher dimensions and it is called the Yamabe flow, which has been studied by many people. We will call such a flow on surfaces the Ricci-Hamilton flow.

R.Hamilton [17] proved the following

Theorem 1.1.1 *For any initial data, the solution exists for all time. For the convergence at infinity, we have that*

(1) If $r \leq 0$, then the metric converges to one of constant curvature;

(2) If $R > 0$, the metric converges to one of constant curvature

The global existence of the flow was obtained early by H.D.Cao in [5]. The proof of Hamilton's Theorem 1.1 is based on two outstanding estimates. One is the so called Li-Yau-Hamilton inequality of Harnack type for the scalar curvature. The other is a decay estimate for the entropy which is defined as

$$\mathcal{S}(R) := \int_M R \log R dv_g. \tag{1.1.2}$$

We can use the remarkable work of G. Perelman [30] to study the Ricci Hamilton flow on surfaces, but we will not present it in this book.

Later, B.Chow [9] removed the condition in Theorem 1.1 (2). His result is

Theorem 1.1.2 *For any metric on S^2, then under the Ricci-Hamilton flow, the scalar curvature becomes positive in finite time.*

Combining the two theorems above yields

Corollary 1.1.3 *If (g_{ij}) is any metric on a closed surface, then under the Ricci-Hamilton flow, the metric converges to one of constant curvature.*

This last result implies what we called the (differential geometric) uniformazation theorem. It says that

Theorem 1.1.4 *any Riemannian metric on a two dimensional closed surface is point-wise conformal to a metric of constant curvature.*

The Ricci-Hamilton flow on surfaces is also studied by B.Osgood ,R.Phillips and P.Sarnack [29] and J.Bartz, M.Struwe, and R.Ye [4], and others. The work of [4] used a moving plane method introduced by A. D. Alexandrov [2].

We use the following notations for the Riemannian metric surface (M^2, g) except explicitly stated.
Δ_g is the Laplacian operator of the metric g, and
D is the covariant derivative of g.

For a metric family $(g(t))$, we let
R be the scalar curvature of $g = g(t)$

In following, we write
R_{max} as the maximum of R at time t.
R_{min} as the minimum of R at time t.

1.2 preliminary material for Ricci-Hamilton flow on surfaces

Let

$$g = g_{ij}dx^i dx^j$$

be the Riemannian metric on the closed surface M, written in local coordinates (x^i). Let

$$(g^{ij}) = (g_{ij})^{-1}$$

be the inverse of the matrix (g_{ij}). Set

$$\mu = \sqrt{det g_{ij}}$$

be the volume density. Then along the Ricci-Hamilton flow, we compute

$$\partial_t \mu = \frac{1}{2} tr_g(g_t)\mu = \frac{1}{2} g^{ij}\partial_t g_{ij}\mu = (r - R)\mu$$

Hence for the total area $A = \int_M d\mu$, we have

$$A_t = \int_M (r - R)d\mu = 0.$$

That is, the total area is preserved along the flow.

By the Gauss-Bonnet formula we have

$$\int_M R d\mu = 4\pi\chi(M)$$

where $\chi(M)$ is the Euler number of the surface M. A well-known fact in Topology is that

$$\chi(M) = 2(1 - g),$$

where g is the genus of the closed surface M and it is the number of "holes" of M. For example, if $M = T^2$ is the torus, we have $g = 1$, and then $\chi(M) = 0$. We recall that the usual way to write the Gauss-Bonnet formula is the following

$$\int_M K d\mu = 2\pi\chi(M)$$

where $K = R/2$ is the Gauss curvature of M.

Since

$$r = \int_M R d\mu / A,$$

we get that

$$r = 4\pi\chi(M)/A$$

which is a constant along the flow, furthermore, if $\chi(M)$ is negative, we have $r < 0$.

1.3 Short-time existence of the flow

We point out that the flow exists at least in short time. We will prove this fact by using the conformal transformation. In fact, write $g = e^u g_0$, then we have

$$R = e^{-u}(-\Delta u + R_0)$$

(see appendix for derivation for this formula) and the Ricci-Hamilton flow is reduced into the following non-linear parabolic equation

$$u_t = e^{-u}(\Delta u - R_0 + re^u), \tag{1.3.1}$$

with initial data

$$u|_{t=0} = 1.$$

Many people just call this equation the Yamabe flow.

By the standard linearization and fixed point argument we can always find a short-time solution of (1.3.1) for any initial metric g_0. Hence we have

Theorem 1.3.1 *For any metric on a compact 2-manifold M^2, then there is a positive constant T such that the Ricci-Hamilton flow exists at the time interval $[0, T)$.*

We now consider the energy estimate for this Yamabe type flow.

Define

$$E(u) = \int_M (\frac{1}{2}|D_0 u|^2 + R_0 u) d\mu_0$$

Then its first variation is

$$\delta E(u) = (R - r)g,$$

Hence along the Ricci-Haimiton flow, we have

$$E_t = -\int_M (R - r)^2 d\mu_g.$$

Clearly from this we have the following estimate

$$\int_0^T dt \int_M (R_g - r)^2 d\mu \leq E(u_0).$$

where T is the maximal time for the flow to exist. The second variation of $E(\cdot)$ is

$$\delta^2(u)(v) = \int_M |D_0 v|^2 d\mu_0 \geq 0$$

Hence we can expect the global existence of the flow and its convergence at infinity.

The following Harnack type estimate is a key for the flow to convergence at infinity.

Assertion 1.3.2 *Assume $M = S^2$. Then along the Yamabe type flow, we have a uniform constant $C > 0$ such that*

$$\inf_t e^u \geq C \sup_t e^u.$$

We will call this assertion as Bartz-Struwe-Ye estimate since they obtained this estimate in [4]. This assertion will be proved in later sections. If we have this assertion, then we can use the volume restriction

$$\int_M d\mu_g = \int_M e^u d\mu_0$$

to get the uniform L^∞ bound on u. Using the standard linear theory of parabolic equations, we can get uniform bounds for the higher order derivatives of u. Then we can extend the solution in short time to a global one. In fact, if $T^* < +\infty$ is the maximal existence time for the Yamabe type flow, we shoud have

$$\lim_{t \to T^*} u_{max} = +\infty.$$

This is a contradiction to our uniform L^∞ bound on u. Here, we point out that all metrics $g(t) = e^{u(t)} g_0$ are uniformly equivalent to the metric g_0. Recall that our flow can also be written as $u_t = r - R$. From this, we can get the uniform bound of all derivatives of R.

Using the energy bound

$$\int_0^{+\infty} dt \int_M |u_t|^2 d\mu = \int_0^{+\infty} dt \int_M (R_g - r)^2 d\mu \leq E(u_0).,$$

we can find a sub-sequence $t_j \to +\infty$ such that $u_t(t_j)$ converges to zero and $R_j = R(t_j)$ converges to r in $L^2(M)$. Using the Arzela-Ascoli theorem, we have that $u_t(t_j)$ converges to zero and $R_j = R(t_j)$ converges to r in $C^k(M)$ for any $k > 2$. In later sections, we can see that the whole $g(t)$ converges to a constant curvature metric.

We also point out that the Ricci-Hamilton flow is not strictly parabolic system, so one can not conclude the short-time existence of the flow from the standard argument. In fact, R.Hamilton [18] first used Nash-Moser implicit function theorem to obtain such a existence. Later De Turck [13] gave a short and beautiful proof based on a normalization argument. Their arguments work for any dimensions. However, in dimension two, we can avoid all these difficulties by using a conformal deformation trick as above.

1.4 Global existence of the flow

We now study long-time behavior of the normalized Ricci flow on the compact surface M:

$$\partial_t g_{ij} = (r - R)g_{ij}.$$

Then using the formula

$$R = e^{-u}(-\Delta u + R_0)$$

we have the flow for scalar curvature as follows.

$$\partial_t R = \Delta R + R(R - r). \tag{1.4.1}$$

Applying the maximum principle to the equation (1.4.1) we know that if the scalar curvature at $t = 0$ is non-positive, it remains so for $t > 0$, and if the scalar curvature at $t = 0$ is non-negative, it remains so for $t > 0$. Clearly, we have the following

Proposition 1.4.1 *There is a uniform constant C such that along the flow, we have*

$$R \geq -C.$$

In fact, we have the following two assertions:

(1). If $r \geq 0$ and the minimum of R at initial time is negative, the minimum of R is increases.

(2). If $r \leq 0$ and the minimum of R at initial time is less than r, the minimum of R is also increases.

1.4.1 negative scalar curvature

Using the maximum principle we have the following result

Theorem 1.4.2 *Assume that there are two positive constants $C_2 > C_1 > 0$ such that $-C_2 \leq R \leq -C_1$ at $t = 0$. Then the scalar curvature R remains so and there is a constant $C > 0$ such that*

$$re^{-C_2 t} \leq r - R \leq Ce^{rt}.$$

So the flow exists for all $t > 0$ and R approached r exponentially as $t \to +\infty$.

Proof: Note first that by the definition of r, we have $-C_2 \leq r \leq -C_1$. Then for fixed $t > 0$, at the maximum point of R we have $R - r \geq 0$ and the differential inequality

$$\frac{d}{dt}R \leq R(R-r) \leq 0.$$

Hence we have $R \leq -C_1$. Using this and the differential inequality again, we have

$$\frac{d}{dt}R \leq R(R-r) \leq -C_1(R-r).$$

So

$$(\log(R-r))' \leq -C_1.$$

and by integrating, we find that

$$R - r \leq (R(0) - r)e^{-C_1 t}$$

and

$$r - R \geq (r - R(0))e^{-C_1 t} \geq re^{-C_1 t}.$$

At the minimum point of R, we have

$$\frac{d}{dt}R \geq R(R-r) \geq r(R-r).$$

Then we immediately obtain that

$$r - R \leq C_2 e^{rt}.$$

Therefore, we have proven the theorem . □

1.4.2 positive scalar curvature

For the positive scalar curvature case, the flow is hard to treat. In this case, we have the following

Theorem 1.4.3 . *Assume that $r > 0$ and R is not a constant at $t = 0$.*

(1). If $R/r \geq c > 0$ at $t = 0$, then $c < 1$ and along the flow, we have the estimate

$$R \geq \frac{rc}{c + (1-c)e^{rt}}.$$

(2). If there is a positive constant C such that $R/r \leq C$ at $t = 0$, then we have that $C > 1$, and for $t < \frac{1}{r} \log \frac{C}{C-1}$, we have

$$R \leq \frac{Cr}{C - (C-1)e^{rt}}.$$

Proof: . Since $R(0) > 0$, by the maximum principle, we have $R > 0$ along the flow.

In the case (1), we have at the minimum point of R, denote by $f := R_{min}$ the minimum of R, we have

$$1 \geq R_{min}/r \geq c.$$

Since at time $t = 0$, R is not a constant, we have $R_{min} < r$, and we get $c < 1$. Note that for $t > 0$, at the minimum point of R, we have the differential inequality

$$\frac{d}{dt} f \geq f(f - r).$$

Hence we have

$$-(f^{-1})_t \geq 1 - rf^{-1}.$$

Integrating this we find

$$f = R_{min} \geq \frac{rc}{c + (1-c)e^{rt}}.$$

By the same method we can prove that $C > 1$ in the case (2) and the estimate

$$R \leq \frac{Cr}{C - (C-1)e^{rt}}.$$

Then we are done. □

1.4.3 Scalar curvature positive somewhere

In this part, we assume that the scalar curvature R at $t = 0$ is positive somewhere on M. In this case, we introduce a potential function f.

Definition 1.4.4 *Let (M, g) be a compact surface with a riemannian metric g. The potential function f is the solution of the Poisson equation*

$$\Delta f = R - r$$

with mean value zero, that is, $\int_M f d\mu = 0$.

We have to consider the existence of such a potential. In fact, we can introduce the variational functional

$$J(u) = \int_M (|Du|^2/2 - (R - r)u)$$

on the space

$$\mathcal{A} := \{H^1(M); \int_M f d\mu = 0\},$$

where D is the covariant derivative of g. Using Poincare's inequality we can easily see that

$$\inf_{\mathcal{A}} J(u) > -\infty.$$

Then one can minimize $J(\cdot)$ over the Hilbert space $\mathcal{A}$ and shows that the minimum is attained at some point in $\mathcal{A}$. Obviously, the minimum point of $J(\cdot)$ is a solution of the Poisson equation

$$\Delta f = R - r.$$

By the maximum principle, we know that the solution with zero mean value property is unique. For more detail, see Th.Aubin's book [3].

We now compute the evolution equation for the potential function f.

Proposition 1.4.5 *Let $g = g(t)$ where $g(t)$ is a solution of the Ricci-Hamilton flow. Then the potential function satisfies the following evolution equation:*

$$f_t = \Delta f + rf - b$$

on M, where

$$b = \frac{1}{A} \int_M |Df|^2$$

is only a function of the time variable t.

Proof: We do computation.

Note that

$$\begin{aligned}\Delta(\Delta f + rf) &= \Delta(R - r + rf) \\ &= \Delta R + r\Delta f \\ &= \Delta R + r(R-r) \\ &= R_t + rR - R^2 + rR - r^2 \\ &= R_t - (R-r)^2\end{aligned}$$

We also have

$$R_t = \partial_t \Delta f$$

Since in local coordinate, we have

$$\Delta f = g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k.$$

We will write

$$f_{ij} := \frac{\partial^2 f}{\partial x^i \partial x^j}$$

for short notation in this part. Notice that $g^{ij}\partial_t \Gamma_{ij}^k = 0$. Then we have

$$\partial_t \Delta f = g_t^{ij} f_{ij} + \Delta f_t = \Delta f + (R-r)^2.$$

By this we obtain that

$$\Delta(\Delta f + rf) = \Delta f_t.$$

This implies that

$$\Delta f + rf = \Delta f_t + b$$

for some constant function b only in space variables. In other word, we have

$$\Delta f_t = \Delta f + rf - b.$$

Differentiating the relation

$$\int_M f d\mu = 0,$$

we get

$$\int_M (f_t + f(r-R))d\mu = 0.$$

By this we conclude that

$$\int_M (rf - b - f\Delta)d\mu = 0,$$

which clearly implies that

$$b = \frac{1}{A}\int_M |Df|^2$$

□

Definition 1.4.6 *We define a P-function*

$$h = \Delta f + |Df|^2$$

and a trace-free part of the second order derivative of the potential f:

$$M_{ij} = D_i D_j f - \frac{1}{2}\Delta f \cdot g_{ij}.$$

We remark that by definition, a P-function is a smooth function which can be applied by the maximum principle. As pointed out by R.Hamilton [17], once we have a nice bound for h, we can bound R because of the formula

$$R = h - |DF|^2 + r.$$

Note also that

$$h = e^{-f} div(e^f Df).$$

Proposition 1.4.7 *Along the Ricci-Hamilton flow, we have*

(1).

$$h_t = \Delta h - 2|M|^2 + rh$$

where at the normal coordinate at a given point, $|M|^2 = \sum_{ij} M_{ij}^2$.

(2).

$$(\partial_t - \Delta)|M|^2 = -2|DM|^2 - 2R|M|^2.$$

Here we just compute the equation for h and we postpone the proof for (2) in the later section (see Lemma 1.9.2).

Proof: We do computation. First, we have

$$|M|^2 = |M_{ij}|^2 = |D_i D_j f|^2 - \frac{1}{2}(R - r)^2.$$

Secondly we have

$$\begin{aligned} \partial_t h &= \partial_t \Delta f + 2Df \cdot Df_t + (R - r)|Df|2 \\ &= \Delta f_t + (R - r)2 + 2Df \cdot Df_t + (R - r)|Df|2. \end{aligned}$$

Thirdly we have

$$\begin{aligned} \Delta h &= \Delta(\Delta f + |Df|^2) \\ &= \Delta(\Delta f) + \Delta|Df|^2 \\ &= \Delta(\Delta f) + 2|D^2 f|^2 + 2\Delta Df \cdot Df \\ &= \Delta(\Delta f) + 2|D^2 f|^2 + 2D\Delta f \cdot Df + R|Df|^2. \end{aligned}$$

Finally we have

$$\begin{aligned}
(\partial_t - \Delta)h &= \Delta(f_t - \Delta f) + (R-r)^2 - 2|D^2 f|^2 \\
&+ 2Df \cdot D(f_t - \Delta f) - r|Df|^2 \\
&= \Delta(rf - b) + (R-r)^2 - 2|D^2 f|^2 + 2Df \cdot D(rf - b)) - r|Df|^2 \\
&= r\Delta f + (R-r)^2 - 2|D^2 f|^2 + r|Df|^2 \\
&= r(R-r)(R-r)^2 - 2|D^2 f|^2 + r|Df|^2 \\
&= (R-r)^2 - 2|D^2 f|^2 + rh \\
&= -2|M|^2 + rh.
\end{aligned}$$

□

Applying the maximum principle to the equation

$$h_t = \Delta h - 2|M|^2 + rh$$

we immediately find that

$$\partial_t h_{max} \leq r h_{max},$$

where h_{max} is the maximum value at time t. And this implies that there is a constant $C > 0$ such that

$$h \leq Ce^{rt}.$$

Importantly we have

$$R = h - |Df|^2 + r \leq h + r \leq Ce^{rt} + r.$$

This gives us a upper bound on R for all t. Combining this with Theorem 1.4.3 we have the following estimate

Theorem 1.4.8 *For any initial riemannian metric on the compact surface M, there is a uniform constant C such that along the flow, we have*

$$-C \leq R \leq Ce^{rt} + r.$$

Furthermore, if $r \leq 0$, then the scalar curvature remains bounded both above and below.

Using this Theorem and the short-time existence of the flow, we can obtain a global solution of the Ricci-Hamilton flow. This follows from the standard continuity method for heat flow. So we conclude

Theorem 1.4.9 *For any initial riemannian metric on the compact surface M, the Ricci-Hamilton flow has a solution for all time. Furthermore, if $r < 0$, the global solution converges at infinity to a metric with constant negative curvature.*

1.4.4 average scalar curvature flat case

Basically we just give some remarks in this part.

Assume that $r = 0$ at $t = 0$. We solve the Poisson equation

$$\Delta_0 \bar{u} = R_0, \quad \text{on } M.$$

Here Δ_0 and R_0 are the Laplacian operator and scalar curvature of the initial metric g_0. It is always solvable since $r(0) = 0$, where

$$r(0) = \int_M R_0 d\mu / A.$$

Set

$$\bar{g} = e^{\bar{u}} g_0.$$

Then we have

$$\bar{R} = e^{-u}(-\Delta_{g_0} + R_0)) = 0.$$

In this case, $\bar{g}$ is a stable point of the Ricci-Hamilton flow. So we may expect that the Ricci-Hamilton flow converges to a flat metric at infinity. Using this $\bar{g}$ to replace the initial metric g_0, we restrict the metric family $(g(t)$ in the conformal class of the initial metric $\bar{g}$ to solve the Ricci-Hamilton flow. Write

$$g(t) = e^u \bar{g}.$$

Then the scalar curvature of $g(t)$ is

$$R = e^{-u}(-\Delta_{\bar{g}} u)$$

and the Ricci-Hamilton flow is reduced into

$$u_t = e^{-u}(\Delta_{\bar{g}} u) \tag{1.4.2}$$

In the computation of this section, we will always write

$$\bar{\Delta} = \Delta_{\bar{g}} \quad \bar{D} = D_{\bar{g}}.$$

and $d\bar{\mu} = d\mu_{\bar{g}}$.

Using the maximum principle we get that there is a uniform constant $C > 0$ such that $|u| \leq C$. Therefore, the metrics $g(t)$ are uniformly equivalent for all time t. This gives us the uniform control of diameter, injectivity radius, and the best constant in the Sobolev inequality.

Theorem 1.4.10 *The Ricci-Hamilton flow exists for all time, and it converges at infinity to a flat metric.*

Proof: Using the Ricci flow we have

$$\frac{d}{dt}\int |\bar{D}u|^2 d\bar{\mu} + 2\int e^{-u}(\bar{\Delta}u)^2 d\bar{\mu} = 0, \qquad (*)$$

and using the bounded-ness of u and the inequality

$$\int (\bar{\Delta}u)^2 d\bar{\mu} \geq c\int |\bar{D}u|^2 d\bar{\mu}$$

(a proof of this inequality can be obtained by arguing by contradiction), we have

$$\frac{d}{dt}\int |\bar{D}u|^2 d\bar{\mu} + c\int |\bar{D}u|^2 d\bar{\mu} \leq 0$$

By the Gronwall inequality we have

$$\int |\bar{D}u|^2 d\bar{\mu} \leq Ce^{-ct}.$$

Going back to $(*)$ again and integrating it over time we find

$$2\int_T^{+\infty} dt \int e^{-u}(\bar{\Delta}u)^2 d\bar{\mu} \leq \int |\bar{D}u|^2 d\bar{\mu}(T)$$

which implies that

$$2\int_T^{+\infty} dt \int R^2 d\mu \leq Ce^{-cT}.$$

By this we can have at least one point $\bar{t} \in [T, T+1]$ such that

$$\int R^2 d\mu \leq Ce^{-cT}.$$

On the other hand, using the evolution of R we have

$$\frac{d}{dt}\int R^2 d\mu + 2\int |DR|^2 d\mu = \int R^3 d\mu. \qquad (**)$$

Since R is uniformly bounded, we have

$$\frac{d}{dt}\int R^2 d\mu \leq C\int R^2 d\mu.$$

Using the Gronwall inequality again we have

$$\int R^2 d\mu(t) \leq C\int R^2 d\mu(\bar{t}),$$

for any $t \in [T, T+1]$, which clearly implies that

$$\int R^2 d\mu \le Ce^{-ct}.$$

all t. That is to say that $R \to 0$ at infinity in $L^p(M, g_0)$ for any $p > 1$.

Integrating (**) we have

$$\int_T^\infty dt \int |DR|^2 d\mu \le e^{-cT},$$

and arguing as before we find some t in each $[T, T+1]$ such that

$$\int |DR|^2 d\mu \le Ce^{-ct}.$$

Note that

$$\frac{d}{dt}\int |DR|^2 d\mu + 2\int |\Delta R|^2 d\mu \le -2\int R^2 \Delta R d\mu.$$

Using the Holder inequality we have

$$2|\int R^2 \Delta R d\mu| \le \int (\Delta R)^2 d\mu + \int R^4 d\mu.$$

By this we have

$$\frac{d}{dt}\int |DR|^2 d\mu + \int |\Delta R|^2 d\mu \le \int R^4 d\mu.$$

Using the inequality

$$\int |\Delta R|^2 d\mu \ge c\int |DR|^2 d\mu$$

we have

$$\frac{d}{dt}\int |DR|^2 d\mu + c\int |DR|^2 d\mu \le \int R^4 d\mu.$$

Using this we can get

$$\int |DR|^2 d\mu \le Ce^{-ct}.$$

for some uniform constant c and for all t. Again integrating over time we have

$$\int_T^\infty dt \int (\Delta R)^2 d\mu \le Ce^{-cT}$$

so we have

$$\int (\Delta R)^2 d\mu \le Ce^{-ct}$$

at least in some point in each $[T, T+1]$. Compute the evolution of ΔR we can get the bound

$$\int (\Delta R)^2 d\mu \le Ce^{-ct}$$

for all t. Using this bound, we can bound the L^2 norm of D^2R. In fact, we have

$$\int |D^2R|^2 d\mu = \int (\Delta R)^2 - \frac{1}{2}\int R|DR|^2 d\mu.$$

Using the L^∞ bound of u, we have

$$\int |D^2R|^2 d\mu \leq Ce^{-ct}$$

We now can bound the maximum of R by the L^2 norm of R, DR, and D^2R. Thus we get that R converges to zero exponentially.

Using the formula $u_t = -R$ and the L^∞ bound of u again, we know that u converges to a positive constant at infinity. □

To conclude this section we make a remark. To understand the equation (4.1) better, we need to study the corresponding ODE:

$$s_t = s(s-r).$$

It is easy to see that the unique solution is of the form:

$$s(t) = \frac{r}{1-ce^{rt}},$$

where $c > 1$. Note that, for $r > 0$, $s(t) < 0$.

From the equations for R and s we conclude that

$$\partial_t(R-s) = \Delta(R-s) + (R-s)(R-r+s). \tag{1.4.3}$$

By the maximum principle, it follows that if $s(0) < min_M R(x,0)$, then $R - s > 0$ for all time.

1.5 Li-Yau-Hamilton's Harnack estimates

1.5.1 case one:$R_{min} > 0$

Set $L = \log R$ and define

$$Q = \Delta L + R - r$$

We will call Q the Harnack quantity for our Ricci-Hamilton flow. Then we compute

$$R_t = RL_t,$$

and

$$\begin{aligned} L_t &= \frac{R_t}{R} \\ &= \frac{\Delta R + R(R-r)}{R} \\ &= \Delta L + |\nabla L|^2 + R - r \end{aligned}$$

Hence, we have

$$Q = L_t - |\nabla L|^2$$

Using

$$Q = \Delta L + R - r$$

we compute

$$\begin{aligned} Q_t &= (R-r)\Delta L + \Delta L_t + RL_t \\ &= \Delta Q + \Delta|\nabla L|^2 \\ &+ (R-r)\Delta L + R(\Delta L + |\nabla L|^2 + R - r) \\ &= \Delta Q + 2(\langle \nabla \Delta L, \nabla L\rangle + D^2L|^2) \\ &+ 2R|\nabla L|^2 + (2R-r)\Delta L + R(R-r) \\ &= \Delta Q + 2\langle \nabla Q, \nabla L\rangle + 2|D^2L|^2 \\ &+ 2(R-r)\Delta L + (R-r)^2 + rQ \\ &= \Delta Q + 2\langle \nabla Q, \nabla L\rangle + 2|D^2L + \frac{1}{2}(R-r)g|^2 + rQ. \end{aligned}$$

Therefore, we obtain the following

Lemma 1.5.1 *Under the Ricci-Hamilton flow with $R > 0$, we have*

$$Q_t \geq \Delta Q + 2\langle \nabla Q, \nabla L\rangle + Q^2 + rQ.$$

Proof: Using the Hadamard inequality we have

$$2|T|^2 \geq (tr_g T)^2$$

for any symmetric 2-tensor on the surface M. Then we get the inequality from the equation above. □

Consider the ODE

$$s_t = s^2 + rs$$

with initial data $s = s_0 < -r < 0$ at $t = 0$. Solving this ODE we get

$$s(t) = -\frac{Cr}{C - e^{-rt}}$$

where $C = s_0/(s_0 + r) > 1$. Applying the maximum principle we get

$$Q(t) \geq s(t)$$

for all $t \geq 0$. Hence, we obtain the following Li-Yau-Hamilton Harnack estimate:

$$\partial_t \log R - |\nabla \log R|^2 \geq s(t).$$

This estimate is also called the differential Harnack inequality.

Take two points (x_1, t_1) and (x_2, t_2) in the space-time, where $t_1 < t_2$, $x_1, x_2 \in M$. Let

$$\gamma : [t_1, t_2] \to M^2$$

be a smooth curve joining point x_1 and x_2. Define

$$A(x_1 t_1, x_2 t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\gamma'|^2 dt$$

Then we have

$$\begin{aligned}
\log \frac{R(x_1, t_1)}{R(x_2, t_2)} &= \int_{t_1}^{t_2} \frac{d}{dt} \log R(\gamma(t), t) dt \\
= &= \int_{t_1}^{t_2} (\frac{\partial}{\partial t} \log R(\gamma(t), t) + \langle \nabla \log R, \gamma' \rangle) dt \\
&\geq \int_{t_1}^{t_2} (|\nabla \log R|^2 - s(t0 - |\nabla \log R| \cdot |\gamma'|) dt \\
&\geq \int_{t_1}^{t_2} (s(t) - \frac{1}{4}(|\gamma'|^2) dt \\
&= -\frac{1}{4} \int_{t_1}^{t_2} |\gamma'|^2 dt - \log \frac{Ce^{rt_2} - 1}{Ce^{rt_1} - 1}.
\end{aligned}$$

Therefore, we get

$$\log \frac{R(x_1, t_1)}{R(x_2, t_2)} \geq -\frac{1}{4} A(x_1 t_1, x_2 t_2) - \log \frac{Ce^{rt_2} - 1}{Ce^{rt_1} - 1}.$$

It is easy to see that there is a uniform constant $B > 0$ such that

$$\log \frac{Ce^{rt_2} - 1}{Ce^{rt_1} - 1} \leq B(t_2 - t_1).$$

So we conclude

Theorem 1.5.2 *Along the Ricci-Hamilton flow with $R > 0$, there is a uniform constant $C > 0$ such that for all $x_1, x_2 \in M$ and $t_1 < t_2$ we have*

$$\frac{R(x_1, t_1)}{R(x_2, t_2)} \geq e^{-\frac{1}{4}A(x_1 t_1, x_2 t_2) - C(t_2 - t_1)}$$

1.5.2 case two:$R_{min} \leq 0$

Following the method of B.Chow [9], we only assume that $R > 0$ only somewhere. We make the choice $s = 0$ when $R \geq 0$ on all M.

Set $L = \log(R - s)$. We have the following important formula:

$$L_t = \Delta L + |\nabla L|^2 + R - r + s. \tag{5.1}$$

We now define the Harnack quantity

$$Q = L_t - |\nabla L|^2 - s = \Delta L + R - r.$$

Compute

$$\begin{aligned}
\partial_t Q &= \Delta L_t + (\Delta)_t L + R_t \\
&= \Delta(\Delta L + |DL|^2 + R) + (R - r)\Delta L + \Delta R + R(R - r) \\
&+ \Delta Q + 2|D^2 L|^2 + 2\langle DL, D\Delta L\rangle \\
&+ R|DL|^2 + (R - r)\Delta L \\
&+ \Delta L + |DL|^2 + R(R - r) \\
&= \Delta Q + 2\langle DL, DQ\rangle + 2|D^2 L|^2 \\
&+ 2(R - r)\Delta L + (R - r)^2 + (R - r)^2 \\
&+ s|DL|^2 + (r - s)\Delta L + r(R - r).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
Q_t &= \Delta Q + 2\langle DL, DQ\rangle + 2|D^2 L + \frac{1}{2}(R - r)g|^2 + (r - s)Q \\
&+ s|DL|^2 + s(R - r).
\end{aligned}$$

Using the upper bound $R \leq e^{rt}$, we have $sR \geq -C$. Thus using the inequality

$$|D^2 L + \frac{1}{2}(R - r)g|^2 \geq \frac{1}{2}(\Delta L + R - r)^2 = \frac{1}{2}Q^2,$$

we have the following inequality

$$Q_t \geq \Delta Q + 2\langle DL, DQ\rangle + 2|D^2 L + \frac{1}{2}(R - r)g|^2 + (r - s)Q + s|DL|^2 - C$$

Note that

$$(sL)_t = \Delta(sL) + s|DL|^2 + s(R - r + s) + s(s - r)L.$$

Since we have

$$L \geq -C - Ct,$$

we obtain that

$$(sL)_t \geq \Delta(sL) + 2\langle DL, D(sL)\rangle - s|DL|^2 - C.$$

Define

$$P = Q + sL$$

Then we have

$$P_t \geq \Delta P + 2\langle DL, DP\rangle + Q^2 + (r - s)Q - C.$$

Since $sL = P - Q$ is bounded, we have foe some constant $C > 0$ such that for t large enough, we have

$$P_t \geq \Delta P + 2\langle DL, DP\rangle + \frac{1}{2}(P^2 - C^2).$$

Applying the maximum principle we have

$$P \geq C\frac{1 + ce^{Ct}}{1 - ce^{Ct}}$$

for some $c > 1$. Therefore, for t large enough, we have

$$P \geq -2C$$

and

$$Q \geq -3C.$$

This is our Li-Yau-Hamilton Harnack estimate.

We now give an application of this Harnack estimate. Define, for two point (x_1t_1) and (x_2t_2) in our space-time,

$$\mathbb{L}(x_1t_1, x_2t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\gamma'(t)|^2 dt$$

where γ is any path in M connecting x_1 and x_2. Note that, for $t_2 > t_1$,

$$L(x_1t_1) - L(x_2t_2) = \int_{t_1}^{t_2} \frac{d}{dt} L(\gamma(t), t)dt = \int_{t_1}^{t_2} (L_t + \langle DL, \gamma'\rangle)dt$$

Since

$$L_t - |DL|2 \geq -C(t_2 - t_1) - \mathbb{L}/4,$$

Integrating we get

$$R(x_2t_2) - s(t_2) \geq e^{\mathbb{L}/4 - C(t_2 - t_1)}(R(x_1t_1) - s(t_1)).$$

1.6 Hamilton's Entropy estimate

In this section, we assume $R > 0$ somewhere and we study the following important quantity

$$\int_M R \log R d\mu$$

which is called the entropy for the Ricci-Hamilton flow, and we will show that this function is decreasing in t.

1.6.1 case one:$R_{min} > 0$

Following R.Hamilton [17], we define

$$Z = r^{-1} \int_M QR d\mu,$$

which is a global function in time variable t. Then by the definition of Q, it is easy to see that

$$Z = r^{-1} \int (-\frac{|\nabla R|^2}{R} + R(R-r)).$$

Compute

$$\begin{aligned} Z_t &= r^{-1} \int \left(-\frac{(r-R)|\nabla R|^2 + 2\langle \nabla R, \nabla R_t \rangle}{R} + R_t(R-r) + RR_t \right) d\mu + \\ & \int (-\frac{|\nabla R|^2}{R} + R(R-r))(r-R) d\mu \end{aligned}$$

Using the equation

$$R_t = \Delta R + R^2 - rR$$

we find that

$$Z_t \geq Z^2 + rZ.$$

If Z is positive at some time t_0, then Z is blow up after a finite time. This is impossible by our definition of Z, which always exists along the flow. Then we have that $Z \leq 0$. Using the expression of Z we find that

$$\int (-\frac{|\nabla R|^2}{R} + R(R-r)) \leq 0.$$

This means that

$$\int \frac{|\nabla R|^2}{R} \geq \int (R-r)^2.$$

Note that

$$\frac{d}{dt} \int R \log R d\mu = \int (R-r)^2 - \int \frac{|\nabla R|^2}{R}.$$

Then we have

Theorem 1.6.1 *Assume $R > 0$ at initial time $t = 0$. Then along the global Ricci-Hamilton flow, we have that*

$$\int R \log R d\mu$$

is decreasing. In particular, we have a uniform constant $C > 0$ such that

$$\int R \log R d\mu \leq C.$$

1.6.2 case two: $R_{min} \leq 0$

Note that
$$s_t = s(s-r).$$
So, for $v = 1/(r-s)$, we have
$$v_t = -vs$$
We compute and get
$$(Rd\mu)_t = \Delta R d\mu$$
and
$$(sd\mu)_t = s(s-r)d\mu.$$
Hence, we have
$$[v(R-s)d\mu]_t = v(\Delta R + s(r-R))d\mu.$$
This implies that
$$\begin{aligned} & \partial_t \left(v \int_M (R-s)\log(R-s)d\mu \right) \\ = & -v \int_M \frac{|DR|^2}{R-s} d\mu + v \int_M (R-s)(R-r+s)d\mu \\ \leq & \ v \int_M (R-r)^2 d\mu \end{aligned}$$
Integrating this inequality in time we find
$$\int_M (R-s)\log(R-s) \leq C \int_0^\infty dt \int_M (R-r)^2 + C.$$
Hence, using the estimate in section 2, we get

Lemma 1.6.2 *Along the flow we have a uniform constant $C > 0$ such that*
$$\int_M (R-s)\log(R-s) \leq C.$$

1.7 Boundedness of R

Following R.Hamilton [17], we first study the case when $R > 0$ in all M

1.7.1 case one:$R > 0$ in all M

Take a point (x, τ), where $\tau \geq 1$, in the space-time such that the curvature R at (x, τ) is the maximum of R at time τ. Let

$$T = \tau + R_{max}(\tau)^{-1}/2$$

Then at time T, using the evolution equation of R we have

$$\partial_t R_{max} \leq R_{max}^2$$

Then we have

$$-(1/R_{max})_t \leq 1.$$

This implies that for $t \in [\tau, T]$,

$$R_{max}^{-1}(\tau) - R_{max}^{-1}(t) \leq T - \tau = R_{max}(\tau)^{-1}/2.$$

This gives us that

$$R_{max}(t) \leq 2R_{max}(\tau).$$

Note, on $[\tau, T]$, since

$$\partial_t g_{ij} = (r - R)g_{ij} \leq r g_{ij}$$

and

$$\partial_t g_{ij}(R - r)g_{ij} \geq (r - 2R_{max}(\tau)g_{ij}$$

we know that for any tangential vector X, $g(X, X)$ will grow at most by a constant factor.

Let $d_T(x, y)$ be the geodesic distance between the points x and y at time T. Then we have

$$\mathbb{L}(x\tau, yT) \leq C d_T(x, y)^2/(T - \tau).$$

Then by the Li-Yau-Hamilton's Harnack inequality we have

$$R(x, \tau) \leq R(y, T)$$

for all $y \in B_\rho(x)$ where $\rho = \pi\sqrt{R_{max(T)}/2}$.

By theorem 5.9 in [7] we know that the injectivity radius of M is at least ρ at time T. In the ball B of radius ρ at x we have that the scalar curvature R is always comparable to $R_{max}(\tau) \geq \frac{1}{2}R_{max}(T)$. Hence we have the estimate

$$\int_B R \log R d\mu \geq c\rho^2 R_{max}(T) \log R_{max}(T) = c \log R_{max}(T)$$

at time T for some uniform constant $c > 0$. Since

$$z \log z + e^{-1} \geq 0$$

for all $z \in \mathbb{R}$, we have

$$\int_B R \log R d\mu \leq \int_B (R \log R + e^{-1}) d\mu \int_M (R \log R + e^{-1}) d\mu \leq C.$$

Then the entropy estimate show that $R_{max}(T)$ is bounded, and then $R_{max}(\tau)$ is bounded. This implies that for all $\tau \geq 1$, R is bounded.

Using this bound on R, we can bound from below the injectivity radius. Since the volume is bounded, this also gives us an upper bound on diameter. Using the diameter bound and the fact that the growth of distances is bounded, we can conclude that for $T - \tau \leq 1$ the estimate

$$\mathbb{L}(x\tau, yT) \leq C/(T - \tau).$$

By the Harnack inequality again we get for all $t \geq 1$, any two point x and y,

$$R(x, t) \leq CR(y, t + 1).$$

Therefore we get a lower uniform bound of the scalar function R.

1.8 derivative estimates of R

Once we have a uniform C^0 bound of R, we can get the uniform bounds for all derivatives of R for all time. In fact, since we have a uniform bound on volume, diameter and the injectivity radius, we have the uniform control of the Sobolev constant. So one may use the energy bound method to proceed as follows.

By induction, we have the following evolution equation for the derivative $|D^nR|^2$ of R:

$$(\partial_t - \Delta)|D^nR|^2 = -2|D^{n+1}R|^2 + \Sigma_{i+j=n} D^iR * D^jR * D^nR.$$

Integrating over M, we can find some uniform constant $_n$ such that

$$\frac{d}{dt}\int |D^nR|^2 + \int 2|D^{n+1}R|^2 \leq C_n \int |D^nR|^2.$$

Using the well-known interpolation inequality we have that

$$\int |D^nR|^2 \leq C_n{}^{-1} \int |D^{n+1}R|^2 + C\int R^2.$$

Then we have

$$\frac{d}{dt}\int |D^nR|^2 + \int |D^nR|^2 \leq C.$$

By the Gronwall inequality we have that

$$\int |D^nR|^2 \leq C(n),$$

for all n and all time. By Sobolev imbedding theorem, we then have the uniform bound of all derivatives of R.

In the interesting paper, W.X.Shi [32] find the following Bernstein type estimate

Theorem 1.8.1 *Along the Ricci-Hamilton flow, if R is uniformly bound, so are the derivatives of R. In fact, If $|R| \leq C$, we have, on $0 < t \leq C^{-1}$, for each ≥ 1, there exists a uniform constant $C_n > 0$ such that*

$$|D^nR|^2 \leq C_nC/t^{n/2}.$$

Proof: Recall the following equations for the scalar curvature and its derivatives:

$$(\partial_t - \Delta)R = R(R - r).$$

and

$$(\partial_t - \Delta)|D^nR|^2 = -2|D^{n+1}R|^2 + \Sigma_{i+j=n} D^iR * D^jR * D^nR.$$

Assume that $|R| \leq C$ and consider $tC \leq 1$. Define

$$F = t|DR|^2 + AR^2.$$

Then we have

$$(\partial_t - \Delta)F \le (Ct|R| - 2A)|DR|^2 + C^2 A$$

Take $A \ge C$. We have

$$(\partial_t - \Delta)F \le C^2 A$$

Since $F \le AC^2$ at the initial time $t = 0$. By the maximum principle, we have

$$F \le (1 + Ct)AC^2 \le 2AC^2$$

This gives us

$$t|DR|^2 \le 2AC^2.$$

By induction we can get that there exists a constant $C_n > 0$ such that

$$|D^n R|^2 \le C_n C/t^{n/2}.$$

□

1.8.1 case two: $R > 0$ only in some where

In this case we first try to get a uniform upper bound of the scalar curvature function R. According to the lesson learned from the case $R > 0$, we need only to need to bound the injectivity radius of $g(t)$. In fact, once we have injectivity radius bound, we can combine the Harnack inequality with entropy estimate to get a uniform supremum bound of R. The injectivity radius bound also implies a uniform bound for the diameter of $g(t)$. From the diameter bound we can get the uniform positive lower bound of $R - s$. Then we can see that $R > 0$ at large time. Then we obtain the following result of B.Chow [9]

Theorem 1.8.2 *Assume that $R(0)$ positive somewhere. Along the Ricci-Hamilton flow, we have for large $t > 0$, $R(t) > 0$.*

So we need only to prove the following

Theorem 1.8.3 *Along the Ricci-Hamilton flow, we have for any $t > 0$, we have the following injectivity radius bound:*

$$i(g(t)) \geq \min\{i(g(0)), \min_{\tau \in [0,t]} \frac{\pi}{\sqrt{K_{max}(\tau)}}\},$$

where $i(g(t))$ is the injectivity radius of $g(t)$.

To prove this result, we need the following three lemmas.

Lemma 1.8.4 *Let γ be a shortest closed geodesic on (M, g). If the length of γ is less than $\frac{2\pi}{\sqrt{K_{max}}}$, then γ is stable. That is to say, the second variation of the length functional at γ is non-negative.*

Proof: We argue by contradiction. Assume γ is unstable. Choose a closed curve $\hat{\gamma}$ near to γ such that

$$l(\hat{\gamma}) < l(\gamma).$$

By choosing two points p and q in γ, we break γ into two piece γ_1 and γ_2 of equal length. We can also choose two points $\hat{p}$ and $\hat{q}$ in $\hat{\gamma}$, and break $\hat{\gamma}$ into two piece $\hat{\gamma}_1$ and $\hat{\gamma}_2$ of with their lengths $< \frac{\pi}{\sqrt{K_{max}}}$. We can find unique geodesics β_i (which is near by $\hat{\gamma}_i$) joining $\hat{p}$ and $\hat{q}$. Then we have

$$l(\beta_i) < l(\hat{\gamma}_i).$$

Then we should have

Assertion: There exists a smooth closed geodesic β with

$$l(\beta) \leq l(\beta_1 U \beta_2).$$

In fact, let $\mathbb{A}$ be the space of all non-degenerate pairs $(\alpha_1\alpha_2)$ joining two points $x \in M$ and $y \in M$ with

$$l(\alpha_i) < \frac{\pi}{\sqrt{K_{max}}}$$

for $i = 1, 2$. Then $\mathbb{A}$ is an open 4-manifold locally parametrized by points in $M \times M$. Define

$$m = \inf_{\mathbb{A}}(l(\alpha_1) + l(\alpha_2)).$$

Clearly, by using the fact that the exponential map of M being a diffeomorphism, we have

$$m \geq i(g).$$

As in [28], we can find that the infimum m is achieved by some $\beta \in \mathbb{A}$. If β is not smooth, we would shorten β inside $\mathbb{A}$. Hence, we have that β is a smooth closed geodesic curve. This proves the Assertion. Using this Assertion, we get a contraction with our assumption at the beginning of the proof.

□

Lemma 1.8.5 *Let γ be a geodesic loop in $g(t)$ as in the above Lemma. Then along the Ricci-Hamilton flow, we have, at time t,*

$$\frac{d}{dt}l(\gamma) \geq rl(\gamma)$$

Proof: . Let T be the unit tangent vector fiend on γ and let N be its unit normal. Note

$$\nabla_T N = 0.$$

Then we have the second variational for the length functional at γ:

$$\int_\gamma \langle R(N,T)N,T\rangle = -\frac{1}{2}\int R \geq 0.$$

Then we have

$$\int R \leq 0.$$

Hence

$$\frac{d}{dt}l(\gamma) = \frac{1}{2}\int(r - R) \geq \frac{r}{2}l(\gamma).$$

□

Lemma 1.8.6 *Suppose γ_t is a shortest closed geodesic in the metric $g(t)$. If $l(\gamma) < \frac{2\pi}{\sqrt{K_{max}(\tau)}}$, then for $\epsilon > 0$ small enough, there exists a geodesic $\gamma_{t-\epsilon}$ in $(g(t-\epsilon))$ with*

$$l(\gamma_{t-\epsilon}) < l(\gamma),$$

where $l(\gamma_{t-\epsilon})$ is the length of $\gamma_{t-\epsilon}$ in the metric $(g(t-\epsilon))$.

Proof: For notation more clear, we write $\gamma = \gamma_t$. Using the lemma above we have some $\epsilon > 0$ small such that

$$l_{t-\epsilon}(\gamma_t) < l_t(\gamma_t),$$

where $l_{t-\epsilon}$ is the length functional in the metric $(g(t-\epsilon))$. By choosing two points p and q, we break γ_t into two piece γ_1 and γ_2 of equal length in the metric $(g(t-\epsilon))$. Since

$$l_{t-\epsilon}(\gamma_i) < \frac{\pi}{\sqrt{K_{max}(t-\epsilon)}}$$

for sufficient small ϵ, we can find unique geodesics β_i (which is near by γ_i) joining p and q in the metric $(g(t-\epsilon))$. Then we have

$$l_{t-\epsilon}(\beta_i) < l_{t-\epsilon}(\gamma_i)$$

Using the Assertion above we have a smooth closed geodesic β such that

$$l_{t-\epsilon}(\beta) \leq l_{t-\epsilon}(\beta_1 U \beta_2) < l_t(\gamma_t),$$

Then we are done. □

Proof: According to the lemma above, along the Ricci-Hamilton flow, if the length functional of shortest closed geodesic is less than $\frac{2\pi}{\sqrt{K_{max}(\tau)}}$, it is increasing. Then by using the Klingerberg lemma [7], we have

$$i(g(t)) \geq min\{i(g(0)), min_{\tau\in[0,t]}\frac{\pi}{\sqrt{K_{max}(\tau)}}\},$$

□

1.9 solitons are limit at infinity

Definition 1.9.1 *A solution $g(t)$ to the normalized Ricci-Hamilton flow is called a Ricci- soliton if there exists a one-parameter family of diffeomorphism $\phi(t) : M \to M$ such that*

$$g(t) = \phi(t)^* g(0).$$

Let $X = \frac{d}{dt}\phi(t)$. If $X = -\nabla f$ for some smooth function, then we call $g(t)$ the gradient soliton.

Clearly we have

$$\partial_t g = L_X g.$$

Then we have we have

$$L_X g = (r - R)g.$$

This means that X is a conformal vector field. Hence $\phi(t)$ are conformal diffeomorphisms. Note that

$$(L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

we get

$$(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i.$$

If $g(t)$ is a gradient soliton, then $X = -\nabla f$, and we have

$$(r - R)g_{ij} = -2D_{ij} f.$$

Taking the trace of both sides we get

$$\Delta f = R - r.$$

Clearly in the case when $M = S^2$, the conformal group of S^2 gives the gradient solitons on S^2.

Recall

$$M_{ij} = D_i D_j f - \frac{1}{2}(R - r)g_{ij}$$

Then we have the following evolution for M_{ij}

Lemma 1.9.2 *Along the normalized Ricci-Hamilton flow, we have*

$$(\partial_t - \Delta)M_{ij} = -2RM_{ij} + rM_{ij}.$$

Proof: We do computation in normal coordinates. Recall that

$$\Delta f = R - r$$

and

$$f_t - \Delta f = Rf + b$$

where b is a constant. Then we compute

$$\partial_t \Gamma_{ij}^k = -\frac{1}{2} g^{kl}(\nabla_i R g_{lj} + \nabla_j R g_{il} - \nabla_l R g_{ij})$$

and

$$\partial_t f_{ij} = D_{ij}^2 f_t - (\Gamma_{ij}^k)_t f_k.$$

Using the relation that

$$R_{ijkl} = \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl})$$

we obtain

$$\begin{aligned}
D_{ij}\Delta f &= f_{kkji} = f_{kjki} - \nabla_i(R_{jl}f_l) \\
&= f_{kjik} - R_{ikj}^l f_{kl} + R_{il}f_{lj} - R_{jl}f_{il} - \nabla_i R_{jl} f_l \\
&= f_{jkik} + R_{ikj}^l f_{kl} + R_{il}f_{lj} - R_{jl}f_{il} - \nabla_i R_{jl} f_l \\
&= f_{jikk} - \nabla_k(R_{jki}^l f_l) - R_{ikj}^l fkl \\
&\quad +R_{il}f_{lj} - R_{jl}f_{il} - \nabla_i R_{jl} f_l \\
&= \Delta f_{ji} - \frac{1}{2}(R_i f_j + f_i R_j - R_k f_k g_{ij}) \\
&- \quad 2R(f_{ij} - \frac{1}{2}\Delta f g_{ij})
\end{aligned}$$

By the definition of M_{ij}, we have

$$M_{ij} = f_{ij} - \frac{1}{2}(R - r)g_{ij}.$$

Compute,

$$\begin{aligned}
\partial_t M_{ij} &= \partial_t(D_{ij}^2 f - \frac{1}{2}(R-r)g_{ij}) \\
&= D_{ij}^2 f_t - (\Gamma_{ij}^k)_t f_k - \frac{1}{2}R_t g_{ij} + \frac{1}{2}(r-R)\partial_t g_{ij} \\
&= D_{ij}(\Delta f + rf) + \frac{1}{2}(\nabla_i R\delta_j^k + \nabla_j R\delta_i^k - \nabla^k R g_{ij}) f_k \\
&\quad -\frac{1}{2}(\Delta R + R(R-r))g_{ij} + \frac{1}{2}(R-r)^2 g_{ij} \\
&= D_{ij}\Delta f + \frac{1}{2}(\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \nabla_k R \nabla_k f g_{ij}) \\
&\quad -\frac{1}{2}\Delta R g_{ij} + rM_{ij}.
\end{aligned}$$

Combining these relations together we get

$$\partial_t M_{ij} = \Delta f_{ij} - \frac{1}{2}\Delta R g_{ij} + (r - 2R)M_{ij}$$

This implies that

$$\partial_t M_{ij} = \Delta\left(f_{ij} - \frac{1}{2}(R-r)g_{ij}\right) + (r-2R)M_{ij}.$$

□

From this lemma we can easily get

Proposition 1.9.3 *Along the normalized Ricci-Hamilton flow, we have*

$$(\partial_t - \Delta)|M_{ij}|^2 = -2R|M_{ij}|^2 - 2|D_k M_{ij}|^2.$$

Proof: It is easy to see that

$$\begin{aligned}\partial_t|M_{ij}|^2 &= 2M_{ij}\partial M_{ij}\\ &= 2M_{ij}(\Delta M_{ij} + (r-2R)M_{ij})\\ &= \Delta|M_{ij}|^2 - 2R|M_{ij}|^2 - 2|D_k M_{ij}|^2\end{aligned}$$

□

This is our result mentioned in section 1.4.

By the maximum principle we get the important estimate

Proposition 1.9.4 *Along the normalized Ricci-Hamilton flow, there exist constants* $C_1, C_2 > 0$ *such that*

$$|M_{ij}|^2 \le C_1 e^{-C_2 t}.$$

Recall that our Ricci-Hamilton flow is a family of the metric $g(t)$ satisfying

$$\partial_t g_{ij} = (r-R)g_{ij}.$$

Set

$$\overline{g} = \phi_t^* g$$

where ϕ_t is the diffeomorphism generated by the gradient vector field ∇f.

Then we have

$$\partial_t \overline{g}_{ij} = 2M_{ij}(\overline{g}) := \overline{M}_{ij}$$

From the estimate above we immediately know that

$$|\overline{M}_{ij}|^2 \le C_1 e^{-C_2 t}.$$

Then we have the limit at infinity for $\overline{M}_{ij}$:

$$(\overline{M}_\infty)_{ij} = 0.$$

So we have

$$\overline{R}_\infty = r.$$

This means that $\overline{R}$ converges exponentially to the constant r. From this we have that the scalar curvature R converges exponentially to the constant r. This eventually implies that the metric $g(t)$ converges exponentially to the constant curvature metric g_∞.

Chapter 2

Bartz-Struwe-Ye estimate

We first introduce the Moving plane method. Then we discuss the beautiful argument of the new Harnack inequality found in [4] for the Ricci-Hamilton flow on the sphere S^2. In this kind of argument, they have to use the uniformazation theorem on the sphere S^2. Note that in the first Chapter, the uniformazation theorem is a by-product of the convergence theorem proved in the last section.

2.1 Moving plane method

In this section, we introduce an application of the maximum principle. By now it is called the moving plane method. This tool was first introduced by A.D.Alexanderov, was used by J.Serrin to study a free boundary problem. More than 20 years ago, it has been used by Gidas, Ni, and Nirenberg [16] to prove the following result:

Theorem 2.1.1 *Given a locally Lipschitz function $f : R \to R$. Assume that*

$$u \in C^2(B_1) \cap C^1(\overline{B_1})$$

satisfies the following partial differential equation:

$$-\Delta u = f(u), \quad u > 0 \ \ in \ \ B_1$$

and

$$u = 0, \quad on \ \ \partial B_1.$$

Then $u = u(r)$ for $r = |x|$, i.e., u is a radially symmetric, so that $u'(r) < 0$ in $(0, 1)$.

If $n = 1$, the theorem above implies that u is a even function in $[-1, 1]$.

Proof: We will only prove the difficulty case when $n \geq 3$.
Choose a constant $K > 0$ such that

$$|f(y) - f(z)| \leq K|y - z|$$

for all $y, z \in [-M, M]$ where $M = \max_{B_1} |u|$. Since our equation is rotationally invariant, we need only to prove that u is reflection symmetric. We will prove that u is symmetric with respect to the plane $x_1 = 0$.

We now consider the reflection point x^λ of the point $x \in B_1$ with respect to the plane $x_1 = \lambda$. Then for $x = (x_1, x_2, ..., x_n)$, we have

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n).$$

Let

$$\Sigma_\lambda = \{x \in B_1; x_1 < \lambda\}.$$

and define

$$u_\lambda(x) = u(x^\lambda).$$

Then we have the equation

$$-\Delta u_\lambda = f(u_\lambda), \ \text{ in } \ \Sigma_\lambda.$$

Hence we have

$$-\Delta(u_\lambda - u) = f(u_\lambda) - f(u) := C_\lambda(x)(u_\lambda - u)$$

on Σ_λ, where $|C_\lambda(x)| \leq K$. Note that

$$u_\lambda - u = 0, \ \text{ on } \partial\Sigma_\lambda$$

Let $v = sup\{u - u_\lambda, 0\}$. Then we have

$$\begin{aligned}
\int_{\Sigma_\lambda} |\nabla v|^2 &= -\int_{\Sigma_\lambda} \Delta v \cdot v \\
&= -\int_{\Sigma_\lambda} C_\lambda(x) v^2 \\
&= K \int_{\Sigma_\lambda} v^2
\end{aligned}$$

The left side is bounded by

$$K|\Sigma_\lambda|^{\frac{2}{n}} (\int_{\Sigma_\lambda} v^{2n/(n-2)})^{(n-2)/n}$$

By the Sobolev inequality we can bound it from above by

$$\leq K|\Sigma_\lambda|^{2/n} \int_{\Sigma_\lambda} |\nabla v|^2$$

So for $|\Sigma_\lambda|$ sufficient small, we have the upper bound

$$\frac{1}{2} \int_{\Sigma_\lambda} |\nabla v|^2.$$

Hence we have $v = 0$ on such a Σ_λ. So we have

$$u_\lambda \geq u, \text{ on } \Sigma_\lambda.$$

Thus we have proved the following

Assertion 2.1.2 *There exists a $\epsilon_1 > 0$ such that for all $\lambda \in (-1, -1 + \epsilon_1)$, we have*

$$u_\lambda \geq u, \textit{ on } \Sigma_\lambda.$$

Define

$$\overline{\lambda} = sup\{\mu \in (-1, 0); u_\lambda \geq u; \text{ in } \Sigma_\lambda \text{ for all } \lambda \leq \mu\}$$

from the assertion above we have

$$-1 < \overline{\lambda} \leq 0.$$

Assertion 2.1.3

$$\overline{\lambda} = 0.$$

In fact, if $\overline{\lambda} < 0$, then we will show that there is a $\epsilon > 0$ such that for all $\lambda \in [\overline{\lambda}, \overline{\lambda} + \epsilon)$ we have

$$u_\lambda \geq u, \text{ on } \Sigma_\lambda.$$

This implies that $\overline{\lambda}$ can not be the supremum. Thus we get a contradiction.

We now choose a compact set $D_\delta \subset \Sigma_{\overline{\lambda}}$ for some $\delta > 0$ such that

$$|\Sigma_{\overline{\lambda}} - D_\delta| < \delta.$$

Then we have

$$u_{\overline{\lambda}} - u \geq a(\delta) > 0$$

on D_δ. So by continuity, we can find a $\epsilon_1 = \epsilon_1(\delta) > 0$ such that for all

$$\lambda \in [\overline{\lambda}, \overline{\lambda} + \epsilon_1)$$

it holds

$$u_\lambda - u \geq a(\delta)/2$$

on D_δ and

$$|\Sigma_\lambda - D_\delta| < 2\delta.$$

Since

$$-\Delta(u_\lambda - u) = C_\lambda(x)(u_\lambda - u)$$

on $\Sigma_\lambda - D_\delta$ and

$$u_\lambda - u \geq 0$$

on $\partial(\Sigma_\lambda - D_\delta)$, we can use the argument for Assertion 4.7.2 to prove that

$$u_\lambda - u \geq 0$$

on $\Sigma_\lambda - D_\delta$ for $\delta > 0$ small enough. Thus we have $\overline{\lambda} = 0$. $\square$

We close this part by recalling the Hopf boundary point lemma.

Theorem 2.1.4 *Let Ω be a bounded domain in R^n. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the following partial differential inequality:*

$$(\partial - \Delta)u + c(x,t)u \leq 0$$

where $c(x,t) \geq 0$ is a bounded function on $\overline{\Omega} \times [0,T)$. Assume that there is a point $x^0 \in \partial\Omega$ and a time $t_0 \in (0,T)$ such that

$$u(x^0, t_0) > u(x,t)$$

for all $(x,t) \in \Omega \times (0,T)$ and $u(x^0, t_0) \geq 0$. Suppose that Ω satisfies the interior sphere condition, that is, there is an open ball $B \subset \Omega$ such that $x^0 \in \partial B$. Then we have

$$\frac{\partial u}{\partial \nu}(x^0, t_0) > 0$$

where ν is the exterior unit normal of $\partial\Omega$ at x^0.

2.2 A new Harnack estimate

Recall that we study the Ricci-Hamilton flow on the sphere S^2,

$$g_t = (r - R)g, \tag{1}$$

with g denotes the evolving metric, R the scalar curvature and r the average of R. Let $c := g_{S^2}$ be the standard metric on S^2 of volume 4π. Let

$$g = e^u c.$$

Then the flow (1) becomes

$$\frac{\partial e^u}{\partial t} = \Delta_c u - 2 + re^u, \tag{2}$$

on S^2 where Δ_c is the Laplacian on the standard metric c.

We consider S^2 as the standard unit sphere in R^3 and let

$$F : S^2 \to R^2$$

be the stereographic projection , whose inverse is given by

$$F^{-1}(x) = (2x, |x|^2 - 1)/(1 + |x|^2), \quad x \in R^2$$

We introduce the following coordinates

$$x = (x_1, x_2)$$

around the north pole $p_0 = (0, 0, 1)$:

$$G : R^2 \to S^2 - \{p_0\}$$

$$G(x) = (2x, 1 - |x|^2)/(1 + |x|^2), \quad |x| \leq 1$$

For a given smooth function f on S^2, we set

$$(F^{-1})^*(fg_{S^2}) = \overline{f} g_{R^2},$$

where g_{R^2} denotes the Euclidean metric. Then $\overline{f}$ is given by

$$\overline{f}(x) = f(F^{-1}(x))4/(1 + |x|^2)^2 \tag{3}$$

A simple computation leads to

Lemma 2.2.1 *Define*

$$\begin{aligned} a_0 &= f(p_0) = (f \cdot G)(0, \\ a_i &= \partial(f \cdot G)/\partial x_i(0) \end{aligned}$$

and

$$a_{ij} = \partial^2(f \cdot G)/\partial x^i \partial x^j(0).$$

Then we have the following expansions near ∞*:*

$$\begin{aligned} \overline{f}(x) &= 4/|x|^4 \left(a_0 + a_i x_i/|x|^2 + (\frac{1}{2}a_{ij} - 2a_0\delta_{ij})x_i x_i/|x|^4 + O(1/|x|^3) \right) \\ \partial\overline{f}/\partial x_i &= -\frac{16x_i}{|x|^6}\left(a_0 + \frac{a_j x_j}{|x|^2} \right) \\ &+ \frac{4a_i}{|x|^6} - \frac{8x_i}{|x|^8} a_j x_j + O(1/|x|^7) \end{aligned}$$

We will call these expansion as (4).

Let u_0 be a smooth function on S^2, and let$u$ be the unique smooth solution of (2) with initial value u_0 on a maximal time interval $[0, T^*)$. We let $f = e^u$, define w, $\overline{f}$ in terms of

$$(F^{-1})^*(f g_{S^2}) = \overline{f} g_{R^2}, \quad \overline{f} = e^w$$

and let

$$\begin{aligned} a_0(t) &= f(p_0, t), \\ a_i(t) &= \frac{\partial(f(,t) \cdot G)}{\partial(x_i)}(0), \\ a_{ij}(t) &= \frac{\partial^2(f(.,t) \cdot G)}{\partial x^i \partial x^j}(0) \end{aligned}$$

Note that

$$w(x,t) = u(F^{-1}(x,t)) + log\frac{4}{(1+|x|^2)^2} \tag{5}$$

and that w satisfies the following flow equation

$$e^w \frac{\partial w}{\partial t} = \Delta w + r e^w \tag{6}$$

where r is the average scalar curvature of the metric

$$e^u g_{S^2}.$$

By Lemma 2.2.1, the expansion above holds for $\overline{f}(.,t)$ with $a_0 = a_0(t), a_i = a_i(t)$ and $a_{ij}(t)$. we notice that the expansion is uniform for all $x \in [0, T]$, where T is any given number in $[0, T^*)$. Our purpose is to estimate Du. We introduce the *mass center* $y(t)$ of $w(., t)$ as $y(t) = (y_1(t), y_2(t))$, where

$$y_i(t) = \frac{a_i(t)}{4a_0(t)}.$$

Then we have the following

Proposition 2.2.2 *The is a constant $C > 0$ depending only on $||u_0||_{C^4}$ such that*

$$||y(t)|| \leq C$$

for all $t \in [0, T^*)$.

Proof: Given $T \in (0, T^*)$. Doing a rotation of coordinates and the transformation $x_2 \to -x_2$ if necessary , we may assume $y_2(T) = max_i|y_i(T)|$. By the expansion (4) and the arguments for Lemma 4.2 in [16] we derive that for some $\lambda_0 \geq 1$ depending only on $||u_0||_{C^4}$ the following holds:

For each $\lambda \geq \lambda_0$,

$$\overline{f}(x, 0) > \overline{f}(x^\lambda, 0)$$

whenever $x_2 < \lambda$, where $x^\lambda = (x_1, 2\lambda - x_2)$ for $x = (x_1, x_2)$. (Note that x^λ is the reflection of x about the plane $x_2 = \lambda$.) Hence,

$$w(x, 0) > w(x^\lambda, 0) \tag{7}$$

whenever $x_2 < \lambda$, $\lambda \geq \lambda_0$.

Using the same argument and the fact that the expansion (4) for $\overline{f}(., t)$ is uniform for all $t \in [0, T]$, we can find some $\lambda_1 \geq \lambda_0$ such that for each $\lambda \geq \lambda_1$

$$w(x, t) > w(x^\lambda, t) \quad \text{whenever} t \in [0, T] \quad \text{and} \quad x_2 < \lambda. \qquad 8)$$

We are going to show that $y_2(T) \leq \lambda_0$. To prove this, we consider the function $w^\lambda(x, t) := w(x^\lambda, t)$ on the region $x_2 \leq \lambda$, $0 \leq t \leq T$ and define

$$I = \{\lambda : \lambda > \lambda_0, \lambda > max_{0 \leq t \leq T} y_2(t), w^\lambda \leq \lambda\}$$

Note that w^λ solves the equation (6) and coincides with w along the plane $x_2 = \lambda$. By (8), I is nonempty.

I is also open. Indeed, $w^\lambda := w$ can never happen for $\lambda \geq \lambda_0$ because of (5). Hence for a given $\lambda \in I$, the maximum principle implies

$$w^\lambda \leq \lambda \quad \text{for} \quad x_2 < \lambda, \tag{9}$$

and the proof of the parabolic version of the Hopf boundary point lemma implies that

$$\frac{\partial w}{\partial x_2} < 0 \tag{10}$$

along the plane $x_2 = \lambda$. Consequently,

$$\overline{f}^{\lambda} < \overline{f} \quad for \quad x_2 < \lambda \tag{11}$$

and

$$\frac{\partial \overline{f}}{\partial x_2} < 0 \tag{12}$$

along the plane $x_2 = \lambda$, where

$$\bar{f}^{\lambda}(x,t) := \bar{f}(x^{\lambda}, t)$$

is defined on $x_2 \leq \lambda$. For each fixed $t \in [0,T]$, we can move the origin to $y(t)$ and find the new expansion for $\bar{f}$ at time t in the following form:

$$\bar{f}(x,t) = \frac{4}{|x|^4}(a_0 + \frac{a_{ij}x_i x_j}{|x|^4} + 0(|x|^{-3})), \tag{13}$$

and

$$\partial_{x^i}\bar{f} = -\frac{16}{|x|^6}a_0 x_i + 0(|x|^{-7}), \tag{14}$$

with different coefficients a_{ij}. The plane $x_2 = \lambda$ becomes the plane $x_2 = \lambda - y_2(t)$ in the new gauge. Since $\lambda \in I$, we have $\lambda - y_2(t) > 0$. Hence we can argue as in [16] to show that there is an $\epsilon(t) > 0$ with the property that:

if $s \in (\lambda - \epsilon, \lambda + \epsilon)$, then $\bar{f}^s(.,t) \leq \bar{f}(.,t)$. Using the facts that $\lambda > max_{0\leq t\leq T} y_2(t)$ and the expansion above is uniform for all $t \in [0,T]$, we can choose $\epsilon(t)$ for all $t \in [0,T]$. Clearly

$$\bar{f}^{\lambda} \leq \bar{f}$$

is equivalent to $w^{\lambda} \leq w$. By this we obtain that

$$(\lambda - \epsilon, \lambda + \epsilon) \subset I$$

for some ϵ. Thus we have proved the openness of I.

W now try to prove that I is closed in $[\lambda_0, \infty)$. Let $\lambda > \lambda_0$ be in the closure of I. By continuity of w, we have $w^{\lambda} \leq w$ and $\lambda \geq max_{0\leq t\leq T} y_2(t)$. If $\lambda = max_{0\leq t\leq T} y_2(t)$, then we have $\lambda = y_2(t_2)$ for some $t_0 \in [0,T]$. Consider now the point $y_0(t_0)$ as the origin and the corresponding stereographic projection $F : S^2 \to R^2$. Define z and z^{λ} by

$$F^*(e^w g_{R^2}) = e^z g_{S^2}, \quad F^*(e^{w^{\lambda}} g_{R^2}) = e^{z^{\lambda}} g_{S^2}.$$

Then z, z^{λ} are defined on $S^2_+ \times [0,T]$ for a semi-sphere S^2_+. The functions z and z^{λ} satisfy the equation (00) and we also know that $z^{\lambda} \leq z$ and z^{λ} and z are the same on ∂S^2_+. Using the Lemma above and the expansion (13-14) above at t_0, we have at the north pole N that

$$\frac{\partial e^z}{\partial \nu} = \frac{\partial e^{z^{\lambda}}}{\partial \nu} = 0$$

where ν is the inward unit normal of ∂S^2_+. this implies that

$$\frac{\partial z}{\partial \nu} = \frac{\partial z^\lambda}{\partial \nu} = 0$$

at the point (N, t_0). By the Hopf boundary point lemma we have that $z^\lambda = z$ on ∂S^2_+. This gives us that $w^\lambda = w$, which is a contradiction with the property that $w(x, 0) > w(x^\lambda, 0)$. From this we conclude that whenever $\lambda \in I$, it holds that

$$\lambda > max_{0 \le t \le T} y_2(t).$$

This shows that I is closed. Hence we have that $I = (\lambda_0, \infty)$. This means that

$$y_2(T) \le \lambda_0.$$

Then $|y(T)| \le C$. Since T is arbitrary, we proved what we wanted. $\square$

It is also clear that the proposition above implies that $|\nabla u(N, t)| \le C$ for all $t \in [0, T^*)$. By using a rotation, we can transform any point in the sphere into the north pole. Thus the proposition gives us the following uniform gradient estimate

Theorem 2.2.3 *Along the Ricci-Hamilton flow (2), we have a uniform constant* $C > 0$ *such that*

$$|\nabla_{S^2} u(x, t)| \le C$$

for all $x \in S^2$.

Integrating along a great circle connecting the maximum point and minimum point of $e^{u(t)}$, we get that

$$\inf_{M_t} e^u \ge C \sup_{M_t} e^u.$$

where $M_t = M \times \{t\}$. This is a new Harnack inequality for the Ricci-Hamilton flow (2).

With this Harnack inequality we can prove the global existence and convergence at infinity as before.

Chapter 3

Hamilton's another proof on S^2

In this part, we describe the new proof of R.Hamilton [19] for the convergence of the Ricci-Hamilton flow. Usually this is called a proof via an isoperimetric estimate and a singularity analysis.

We study a metric $g = \{g_{ij}\}$ on the two-sphere S^2 evolving under the Ricci-Hamilton flow, which is not normalized and on a surface takes the simple form

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij}$$

where R is the scalar curvature of g. This flow is different from the normalized Ricc-Hamilton flow just by a rescaling.The reason for this change is that we are considering geometric quantities which are invariant under the delation. So these changes does not affect any any results claimed true for the normalized Ricc-Hamilton flow. We point out that in this section, we call this un-normalized flow the Ricci-Hamilton flow, which is the only place that is different from the previous sections.

We now consider any curve Λ of length L on S^2 dividing the total area A into two connected parts $A_1 + A_2 = A$, and take the smooth curve $\overline{\Lambda}$ of least length $\overline{L}$ on the round sphere of the same total area $\overline{A} = A$ dividing it into two connected parts $\overline{A}_1 + \overline{A}_2 = \overline{A}$ with $\overline{A}_1 = A_1$ and $\overline{A}_2 = A_2$ the same as before. We form the isoperimetric ratio

$$C_S(\Lambda) = L/\overline{L},$$

and we let

$$\overline{C}_S = \inf_{\Lambda} C_S(\Lambda)$$

taking the infimum over all smooth curves Λ on S^2. It is well-know that the shortest curve $\overline{\Lambda}$ cutting off areas $\overline{A}_1$ and $\overline{A}_2$ on the round sphere is a circle of latitude, and its length $\overline{L}$ is given by

$$\frac{4\pi}{\overline{L}^2} = \frac{1}{\overline{A}_1} + \frac{1}{\overline{A}_2}.$$

In fact, in the stereographic coordinates, we can write the metric on the standard sphere as

$$g = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\theta^2).$$

Let $\gamma(\theta) = (r_0, \theta)$ be a circle of latitude. Then its length is

$$L(\gamma) = \int_0^{2\pi} \frac{2r_0}{1+r_0^2} d\theta = \frac{4\pi r_0}{1+r_0^2}.$$

We have

$$\overline{A}_1 = \int_0^{r_0} dr \int_0^{2\pi} \frac{4r}{(1+r^2)^2} r d\theta = \frac{4\pi r_0^2}{1+r_0^2}$$

and

$$\overline{A}_2 = 4\pi - \overline{A}_1 = \frac{4\pi}{1+r_0^2}.$$

Combining these together, we find that

$$\frac{1}{\overline{L}^2} = \frac{1}{4\pi}(\frac{1}{\overline{A}_1} + \frac{1}{\overline{A}_2}).$$

Using this formula, we find that the isoperimetric ratio of Λ can be expressed as

$$C_S^2(\Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right) / 4\pi$$

By definition, on the round sphere we have $\overline{C}_S = 1$, and in any other metric we can come as close to this as we wish by taking a very short curve like a small circle. Hence $\overline{C}_S \leq 1$ in any metric. When $\overline{C}_S < 1$ we will show that the value of $\overline{C}_S$ is attained by $C_S(\overline{\Lambda})$ for some $\overline{\Lambda}$ which is a single simple closed loop of constant curvature. The constant $\overline{C}_S$ is also the Sobolev constant, defined as the best constant in the following inequality

$$\left\{ \inf_c \int (f-c)^2 da \right\}^{1/2} \leq \overline{C}_S \int |\nabla f| da$$

where the infimum is taken over all constants c. In fact equality is attained precisely when f has a jump discontinuity along the optimal curve $\overline{\Lambda}$ and is constant on either side. For a outline of a proof, see the nice survey article by Yau [37].

Theorem 3.0.4 (Main Theorem) *The isoperimetric ratio $\overline{C}_S$ increases under the Ricci flow on the two-sphere.*

This estimate gives us a new proof of a theorem by Ben Chow [9].

Corollary 3.0.5 *([9]). Under the Ricci flow on the two-sphere, any metric approaches positive constant curvature.*

Proof: As we already met in previous sections, by the Klingerberg lemma [7], the injectivity radius can be controlled by the maximum of the curvature and the length of the shortest closed geodesic circle. The isoperimetric ratio controls the length of a short geodesic circle in terms of the maximum curvature (see [8]). Therefore, if we take a sequence of points in the space-time where the curvature is as large as it has ever been and dilate so the curvature there is one, we can extract a convergent subsequence.

If the curvature times the area is unbounded, the limit will be complete but not compact, and we can arrange that it is an eternal solution to the Ricci flow, that is a solution defined for all time $t \in (-\infty, +\infty)$. Since the scalar curvature is bounded below, after dilating the limit will have nonnegative curvature. Since it is not flat, the strong maximum principle shows it is strictly positive. By the Gromoll-Meyer theorem which says that any n-dimensional complete non-compact Riemannian manifold with positive sectional curvature must be diffeomorphic to R^n, we know that the limit is (R^2, ds^2). It then follows that the limit is the cigar solution

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

In fact, we can write the limit metric as

$$ds^2 = g(x, y)(dx^2 + dy^2).$$

Let

$$\begin{aligned} x &= e^u \cos u \\ y &= e^u \sin v \end{aligned}$$

for $u \in (-\infty, +\infty)$ and $v \in (0, 2\pi]$. Then we have the metric on the cylinder,

$$ds^2 = g(u, v)(du^2 + dv^2)$$

where

$$g(x, y) = g(u, v)e^{-2u}.$$

Since the limit metric is a gradient Ricci soliton (see also the Main theorem in [25]), the soliton is moved down by translating in the coordinate u. Then we have a smooth function f defined on the cylinder such that the gradient of f is just $a\frac{\partial f}{\partial u}$ for some constant $a \in R$, which generates the soliton metric. Hence we have

$$f_u = ag, \quad f_v = 0.$$

This implies that $g = g(u)$ is a function of u and $a \neq 0$. Thus $g(u)e^{-2u}$ is a smooth function of $x^2 + y^2 = e^{2u}$ and $g(u + at)$ defines the soliton. Note that the scalar curvature of ds^2 is

$$R = \frac{1}{g}\left(\frac{g_u}{g}\right)_u.$$

Using the equation of the Ricci-Hamilton flow we get

$$ag_u = \left(\frac{g_u}{g}\right)_u$$

Solve this and we obtain that

$$g = \frac{Ce^{Cu}}{C_1 \pm ae^{Cu}}$$

Since $g > 0$, we must have

$$g = \frac{Ce^{Cu}}{C_1 + ae^{Cu}}.$$

After rescaling of the constants, we have

$$g = \frac{e^{Cu}}{1 + e^{Cu}}.$$

Since $g(u)e^{-2u}$ is a function of e^{2u} we can arrange the constant C such that

$$g = \frac{e^{2u}}{1 + e^{2u}},$$

and this gives us that

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

But the cigar has isoperimetric ratio zero; hence it cannot occur as a limit of surfaces whose isoperimetric ratio is bounded away from zero.

The other possibility is that the curvature times the area is bounded. In this case the limit of the dilations will be compact. Again it will have strictly positive curvature. Since the normalized entropy

$$E = \int R \log(RA) da$$

is monotone decreasing for the Ricci flow, it must be constant on the limit flow. But then Ben Chow's proof (see section 1.6.2) of the entropy estimate shows that a certain integral of a positive expression is zero, which implies that the limit is a compact homothetically shrinking solution. But we know this can only be the sphere. □

We now discuss the existence and smoothness of the optimal curve $\overline{\Lambda}$.

Theorem 3.0.6 *On any surface with Sobolev constant $\overline{C}_S^2 < 1$ (the isoperimetric ratio on the sphere) the minimum is attained on a smooth curve $\overline{\Lambda}$.*

Proof: If the length L is sufficiently short, the curve lies in a single coordinate patch where the metric g_{ij} is bounded above and below as closely as we like by the Euclidean metric δ_{ij}, say for any $\epsilon > 0$

$$(1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij}.$$

Let L_0 be the length of the curve measured in Euclidean metric and let A_0 is the enclosed area in the coordinate patch measured too in Euclidean metric. Then we have

$$\sqrt{1-\epsilon}L_0 \leq L \leq \sqrt{1+\epsilon}L_0.$$

If A_1 is the enclosed area in the coordinate patch, then

$$(1-\epsilon)A_0 \leq A_1 \leq (1+\epsilon)A_0.$$

Hence, by the isoperimetric inequality in the plane

$$L_0^2 \geq 4\pi A_0,$$

we have

$$L^2 \geq \frac{1-\epsilon}{1+\epsilon} 4\pi A_1$$

and hence

$$C_S^2(\Lambda) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right) / 4\pi \geq \frac{1-\epsilon}{1+\epsilon}.$$

Assertion 3.0.7 *For every $\eta > 0$ there is a $\lambda > 0$ and $\alpha > 0$ such that if $C_S^2(\Lambda) \leq 1-\eta$ then $L \geq \lambda$ and $A_1 \geq \alpha$ and $A_2 \geq \alpha$.*

In fact, if $\overline{C}_S < 1$, we can approximate it as closely as we wish by $C_S(\Lambda) \leq 1-\eta$ for some Λ. Then this Λ has length $L \geq \lambda$. Since L is not too small, neither area can be too small either, so we can find an $\alpha > 0$ such that $A_1 \geq \alpha$ and $A_2 \geq \alpha$.

Both A_1 and A_2 are no more than the total area A, so they are also bounded above. Then L is also bounded above; in fact

$$L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right) / 4\pi = C_S(\Lambda) < 1$$

and

$$\frac{1}{A_1} + \frac{1}{A_2} \geq \frac{2}{A}$$

so

$$L < \sqrt{2\pi A}.$$

This restricts the geometry of any curve Λ with $C_S(\Lambda) < 1$. □

The fact that the lengths are bounded does not tell us a lot. If we parametrize the curve by arc length s then

$$L = \int \sqrt{g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}} ds$$

so in a coordinate patch where g_{ij} is comparable to the Euclidean metric, the coordinate functions x^1 and x^2 have one derivative with respect to s in L^1, which makes them continuous, but this does not help much. Since the dimension is low, standard techniques will apply. But here we follow a new approach of R.Hamilton.

The idea is to take our approximating curve and improve it by running the curve shrinking flow on the surface for a short time. The curve shrinking/shortening flow is

$$\frac{\partial \Gamma}{\partial t} = kN$$

where N is the unit normal defined by normalizing dT/ds and k is the curvature of the curve Γ. The existence of the flow can be proved by expressing the nearby curve as a graph of the initial curve (see appendix C). Here we move each point on the curve in the normal direction with velocity equal to the geodesic curvature k at that point. The enclosed areas A_1 and A_2 evolves at a rate

$$\frac{dA_1}{dt} = -\int kds, \frac{dA_2}{dt} = +\int kds$$

while the length evolves at a rate

$$\frac{dL}{dt} = -\int k^2 ds.$$

By the Gauss-Bonnet theorem

$$\int kds + \int Kda = 2\pi$$

on A_1 (or A_2) since a circle S^1 on S^2 encloses a disk D^2. Now K and A_1 (or A_2) are bounded, so $\int kds$ is bounded. The Sobolev constant evolves at a rate

$$\frac{d}{dt} \log C_S(\Lambda) = \frac{2}{L}\frac{dL}{dt} - \frac{1}{A_1}\frac{dA_1}{dt} - \frac{1}{A_2}\frac{dA_2}{dt} + \frac{1}{A}\frac{dA}{dt}$$

and since L is bounded above and A_1 and A_2 are bounded below, we have

$$\frac{d}{dt} \log C_S(\Lambda) < 0$$

provided

$$\frac{dL}{dt} << -1.$$

Hence, we can find a constant E such that $C_S(\Lambda)$ decreases when Λ moves by the curve shrinking flow unless

$$\int k^2 ds \leq E.$$

We now recall a basic property of curve shortening flow.

Lemma 3.0.8 *For the curve shrinking flow on a closed surface, there exists a constant $C < \infty$ such that if we have a solution for $0 \leq t \leq 1/M$ with $k^2 \leq M$ with any $M \geq 1$ then*

$$\left(\frac{\partial k}{\partial s}\right)^2 \leq CM/t.$$

Proof: Apply the maximum principle to

$$t\left(\frac{\partial k}{\partial s}\right)^2 + k^2$$

and the result follows. One can find more detail in Appendix C. □

We can flow the solution until k becomes unbounded, and hence as long as

$$\sup_p tk^2(p,t) \leq 1$$

because all the derivatives also stay bounded by the same sort of argument. Suppose the equality above is reached first at some time $\tau \leq 1$ at a point ξ where

$$\tau k^2(\xi, \tau) = 1.$$

[If not, L is bounded and k^2 is bounded and we get $\int k^2 ds \leq E$ as before.] Then by the first derivative estimate we get

$$\tau k^2(p, \tau) \geq 1/2$$

for all p whose distance to ξ is less than $c\sqrt{\tau}$ for some $c > 0$. Then

$$\int k^2 ds \geq c/\sqrt{\tau}$$

for some other $c > 0$. This shows $C_S(\Lambda)$ decreases unless τ is not too small, and then $\int k^2 ds \leq E$ for some E anyway.

Lemma 3.0.9 *For every closed surface and every $\eta > 0$ there exists a constant E such that if $C_S(\Lambda^0) \leq 1 - \eta$ then under the curve shrinking flow Λ_0 evolves into a curve Λ_t with*

$$C_S(\Lambda^t) \leq C_S(\Lambda^0)$$

and Λ^t has

$$\int k^2 ds \leq E.$$

Proof: By the previous argument, as long as $\int k^2 ds \geq E$ the Sobolev constant $C_S(\Lambda^t)$ continues to decrease. By the previous lemma, we can decrease the isoperimetric constant and bound the L^2 norm of the curvature of the curve along the curve shortening flow. This proves the result. □

For use of proving our theorem, we also need an algebra lemma below:

Lemma 3.0.10 *For any positive numbers L_1, L_2, A_1, A_2, A_3 we have*

$$(L_1 + L_2)^2 (\frac{1}{A_2} + \frac{1}{A_1 + A_3}) \geq$$
$$\min\{{L_1}^2 (\frac{1}{A_1} + \frac{1}{A_2 + A_3}), {L_2}^2 (\frac{1}{A_3} + \frac{1}{A_2 + A_1})\}$$

Proof: If not, we have

$$(L_1 + L_2)^2 (\frac{1}{A_2} + \frac{1}{A_1 + A_3}) \leq {L_1}^2 (\frac{1}{A_1} + \frac{1}{A_2 + A_3})$$

and

$$(L_1 + L_2)^2 (\frac{1}{A_2} + \frac{1}{A_1 + A_3}) \leq {L_2}^2 (\frac{1}{A_3} + \frac{1}{A_2 + A_1}).$$

Rearranging we get

$$\frac{A_1(A_2 + A_3)}{A_2(A_1 + A_3)} \leq \frac{L_1^2}{(L_1 + L_2)^2}$$

and

$$\frac{A_3(A_2 + A_1)}{A_2(A_1 + A_3)} \leq \frac{L_2^2}{(L_1 + L_2)^2}.$$

Adding these two inequality we get

$$1 + \frac{2A_1A_3}{A_2(A_1 + A_3)} \leq 1 - \frac{2L_1L_2}{(L_1 + L_2)^2}.$$

Then we have

$$\frac{2L_1L_2}{(L_1 + L_2)^2} + \frac{2A_1A_3}{A_2(A_1 + A_3)} \leq 0$$

So we have to have that $L_1 = 0$ or $L_2 = 0$,and $A_1 = 0$ or $A_3 = 0$. A contradiction with L_1, L_2, A_1, A_2, A_3 being positive.

□

Now to finish the proof of the theorem, take a sequence of curves Λ_j^0) with

$$C_S(\Lambda_j^0) \to \overline{C}_S$$

as $j \to \infty$. Deform each by the curve shrinking flow to a curve Λ_j with

$$C_S(\Lambda_j) \leq C_S(\Lambda_j^0)$$

and

$$\int k_j^2 ds \leq E$$

where k_j is the curvature of Λ_j. Then we still have

$$C_S(\Lambda_j) \to \overline{C}_S$$

so the Λ_j are a much better approximating sequence.

Since each Λ_j has

$$\int k^2 ds \leq E$$

we see that the curves are reasonably well behaved in local coordinate charts. Take a chart where the metric g_{ij} is comparable to the Euclidean metrics δ_{ij}, say $\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij}$, and where the connection forms are bounded, say $|\Gamma_{ij}^k| \leq 2$. This happens for example in geodesic coordinates or in harmonic coordinates. If s is the arc length parameter in the g_{ij} metric and T^i and N^i the unit tangent and normal vectors then

$$\frac{dx^i}{ds} = T^i$$

and

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds}\frac{dx^k}{ds} = kN^i.$$

Both T and N have bounded Euclidean length, so in the Euclidean metric

$$\left|\frac{d^2x^i}{ds^2}\right| \leq C(|k|+1).$$

Since the length L is bounded, we get a bound

$$\int \left|\frac{d^2x^i}{ds^2}\right|^2 ds \leq C$$

from the bound on k.

We try to bound the curvature k_c of the curve in the Euclidean metric. Let l be the Euclidean length parameter. Then

$$\left(\frac{dl}{ds}\right)^2 = \delta_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

and dl/ds is bounded above and below since the metrics are comparable. Differentiating again

$$2\frac{dl}{ds}\frac{d^2l}{ds^2} = 2\delta_{ij}\frac{d^2x^i}{ds^2}\frac{dx^j}{ds}$$

and it follows that d^2l/ds^2 is bounded. Then by the rule for inverse functions, we have

$$\frac{d^2l}{ds^2}\frac{ds}{dl} = -(\frac{dl}{ds})^2\frac{d^2s}{dl^2}.$$

Hence, d^2s/dl^2 is bounded also, and we get

$$\int \left|\frac{d^2x^i}{dl^2}\right|^2 dl \leq C$$

in our coordinate patch. This is just a bound

$$\int k_c^2 ds \leq C$$

for the curvature and arc length of the Euclidean metric. Since in the place

$$k_c = \frac{d\theta}{ds}$$

where θ is the angle of the tangent line, we see that

$$\begin{aligned} \theta_2 - \theta_1 &= \int_1^2 k_c ds \\ &\leq \left(\int_1^2 k_c^2 ds\right)^{1/2} \left(\int_1^2 1 ds\right)^{1/2} \\ &\leq C\sqrt{s_2 - s_1} \end{aligned}$$

so the angle θ is of class $C^{1/2}$ with respect to arc lengths. Then for a short enough segment we can write the curve as the graph of a function $y = f(x)$ with $|dy/dx| \leq 1$. Since

$$k_c = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

we now have

$$\int \left(\frac{d^2y}{dx^2}\right)^2 dx \leq C$$

and we see y is of class H^2, and hence $C^{1+\frac{1}{2}}$ with respect to x.

It is now easy to see that since the Λ_j are locally uniformly bounded in L_2^1 and $C^{1+\frac{1}{2}}$ we can extract a subsequence which converges in L_p^1 for $p < 2$ or $C^{1+\alpha}$ for $\alpha < 1/2$. The limit $\overline{\Lambda}$ will be a genuine immersed curve. A limit of embedded curves may not be embedded, but at least it cannot cross itself; at worst it will be self-tangent. The limit will still have bounded norm in L_2^1 or $C^{1+\frac{1}{2}}$.

Moreover the limit $\overline{\Lambda}$ attains the minimal ratio $C_S(\overline{\Lambda}) = \overline{C}_S$. From this and the last lemma above it easily follows that it cannot be tangent to itself, for it would be more efficient to take just one part of the curve or the other. Therefore $\overline{\Lambda}$ is embedded. Now the usual first variation argument shows $\overline{\Lambda}$ has constant curvature

$$k = \frac{L}{2}\left(\frac{1}{A_1} - \frac{1}{A_2}\right)$$

and hence $\overline{\Lambda}$ is smooth.

We are now in the right position to give the proof of the Main theorem, which is a straightforward calculation. Start with the optimal curve $\overline{\Lambda}$ at time $\bar{t}$ where $\overline{C}_S = C_S(\overline{\Lambda})$, and construct the one-parameter family of parallel curves Λ_r at distance r from $\overline{\Lambda} = \Lambda_0$ on either side. We take $r > 0$ when the curve moves from A_1 into A_2, and $r < 0$ when it moves the other way. We then regard L, A_1, A_2 and $C_S = C_S(\Lambda_r)$ as functions of r, and t also by considering the same curves Λ_r at times t near $\bar{t}$.

First we clearly have

$$\frac{dA_1}{dr} = L \text{ and } \frac{dA_2}{dr} = -L$$

and by a standard formula

$$\frac{dL}{dr} = \int kds$$

where k is the geodesic curvature of the curve Λ_r. (Of course by a standard variational argument k is constant on $\overline{\Lambda}$, but we do not seem to use this, except to check $\overline{\Lambda}$ is smooth.) Thus we also get

$$\frac{d^2}{dr^2}A_1 = \int kds \text{ and } \frac{d^2}{dr^2}A_2 = -\int kds.$$

Now we use the Gauss-Bonnet formula

$$\int_1 Kda + \int kds = 2\pi$$

where the first integral is over the part with area A_1. Differentiating the Gauss-Bonnet formula with respect to r gives

$$\frac{d^2L}{dr^2} = \frac{d}{dr}\int kds = -\int Kds.$$

So much for the space derivatives.

Now we compute the time derivatives under the Ricci flow. This gives

$$\frac{dL}{dt} = -\int Kds$$

and

$$\begin{aligned}\frac{dA_1}{dt} &= -2\textstyle\int_1 Kda = -4\pi + 2\int kds\\ \frac{dA_2}{dt} &= -2\textstyle\int_2 Kda = -4\pi - 2\int kds\end{aligned}$$

and of course $dA/dt = -8\pi$. So much for the time derivatives.

Recall that the Sobolev constant of Λ_r is

$$C_S^2 = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right) / 4\pi$$

and taking logarithms for simplicity, we obtain

$$\log C_S^2 = 2 \log L - \log A_1 - \log A_2 + \log A - \log 4\pi.$$

First we compute

$$\frac{d}{dr} \log C_S^2 = \frac{2}{L} \frac{dL}{dr} - \frac{1}{A_1} \frac{dA_1}{dr} - \frac{1}{A_2} \frac{dA_2}{dr}$$

which gives

$$\frac{d}{dr} \log C_S^2 = \frac{2}{L} \int kds - L \left(\frac{1}{A_1} - \frac{1}{A_2} \right).$$

At the infimum $\overline{\Lambda}$ when $t = \bar{t}$ and $r = 0$ we must have the first variation equal to zero. This gives the useful relation

$$\int kds = \frac{L^2}{2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right)$$

which we can substitute in future calculations. The second space derivative is

$$\frac{d^2}{dr^2} \log C_S^2 = \frac{2}{L} \frac{d^2 L}{dr^2} - \frac{2}{L^2} \left(\frac{dL}{dr} \right)^2 - \frac{1}{A_1} \frac{d^2 A_1}{dr^2} + \frac{1}{A_1^2} \left(\frac{dA_1}{dr} \right)^2 - \frac{1}{A_2} \frac{d^2 A_2}{dr^2} + \frac{1}{A_2^2} \left(\frac{dA_2}{dr} \right)^2$$

which can be worked out to

$$\frac{d^2}{dr^2} \log C_S^2 = \frac{2}{L} \frac{d^2 L}{dr^2} - L^2 \left(\frac{1}{A_1} - \frac{1}{A_2} \right)^2 + L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right).$$

The time derivative is

$$\frac{d}{dt} \log C_S^2 = \frac{2}{L} \frac{dL}{dt} - \frac{1}{A_1} \frac{dA_1}{dt} - \frac{1}{A_2} \frac{dA_2}{dt} + \frac{1}{A} \frac{dA}{dt}$$

which can be worked out to

$$\frac{d}{dt} \log C_S^2 = \frac{2}{L} \frac{dL}{dt} - L^2 \left(\frac{1}{A_1} - \frac{1}{A_2} \right)^2 + 4\pi \frac{A_1^2 + A_2^2}{A_1 A_2 (A_1 + A_2)}.$$

Since we have a "heat" equation

$$\frac{dL}{dt} = - \int Kds = \frac{d^2 L}{dr^2},$$

so we can get the formula below.

Formula 3.0.11 *The Sobolev constant C_S satisfies*

$$\frac{d}{dt}\log C_S^2 = \frac{d^2}{dr^2}\log C_S^2 + \frac{4\pi(A_1^2+A_2^2)}{A_1A_2(A_1+A_2)}[1-C_S^2].$$

Corollary 3.0.12 *If $\overline{C}_S^2 < 1$ it will increase.*

Proof: It suffices to show that $\overline{C}_S^2 + \epsilon t$ increases for all $\epsilon > 0$ no matter how small. If not, we can find a time $\bar{t} > 0$ when it is no larger than at previous times, hence at previous times $\overline{C}_S^2 + \epsilon t$ was no smaller. Hence at previous times $\overline{C}_S^2$ was larger by $\epsilon(\bar{t} - t)$.

Now pick the optimal $\overline{\Lambda}$ at time $\bar{t}$, and construct the family Λ_r as before. Since $\overline{\Lambda}$ is a minimum over all r at $t = \bar{t}$, we get

$$\frac{d^2}{dr^2}\log C_S^2 \geq 0$$

and hence for $\overline{C}_S^2 < 1$

$$\frac{d}{dt}\log C_S^2 \geq 0$$

which means at times $t < \bar{t}$, $\overline{C}_S^2$ was not larger than its value at $\bar{t}$ by more than $\delta(\bar{t}-t)$ where δ is as small as we like for t near $\bar{t}$. When $\delta < \epsilon$ we get a contradiction, proving the Main Theorem. □

Chapter 4

Perelman's W-functional and its applications

Using the trick used in [30], we can give a third proof of convergence of the Ricci-Hamilton flow on surfaces. However, we will not do that here. We prefer to give some easy but important introduction to Perelman's work [30]. This work is really a ground breaking! Before reading this chapter, we suggest the readers look at our introduction of works of R.Hamilton in our appendix A of this book.

4.1 Is the Ricci-Hamilton flow a gradient flow?

Some years ago, many people believed that the Ricci-Hamilton flow is not a gradient flow. In [30], G.Perelman proved that it is a gradient flow. Actually he introduced two functionals. One is called $\mathcal{F}$ functional. The other is the W- functional. Both depend on the existence of the Ricci-Hamilton flow. This means that, we first prove the existence of the Ricci-Hamilton flow. Then we solve a non-linear back-ward scalar heat equation. The functionals depend on the solutions of the non-linear back-ward scalar heat equations. Hence, once we know that it is a gradient flow, we can expect that there is any period orbit except the fixed points in the space of riemannian metrics mod diffeomorphisms. Here the fixed points are Ricci solitons for the Ricci-Hamilton flow. In general, classifying Ricci solitons is a difficulty question.

Let M^n be a closed manifold.

Let g be a Riemannian metric on M and let dV be the volume form of g. Let R be the scalar curvature of g. We follow the idea of G.Perelman [30] introducing a new functional $\mathcal{F}$ as follows

$$\mathcal{F} = \int (R + |Df|^2)e^{-f}dV \tag{4.1.1}$$

where f is a smooth function on the manifold M. Set

$$dm = e^{-f}d\mu.$$

We consider the variation for the metric g and for the function f. Write by $\delta g_{ij} = v_{ij}$, $\delta f = h$ the variations respectively. Let $v = g^{ij}v_{ij}$, where (g^{ij}) be the inverse matrix of (g_{ij}). Let R_{ij} be the Ricci tensor of g.

Recall that

$$\begin{aligned} \delta R &= -\Delta v + D_iD_jv_{ij} - R_{ij}v_{ij}, \\ \delta|Df|^2 &= v^{ij}D_ifD_jf + 2\langle Df, Dh\rangle, \\ \delta(e^{-f}dV) &= (\frac{v}{2} - h)e^{-f}dV. \end{aligned}$$

Let $dm = e^{-f}dV$. We make dm be fixed. Then we have $\frac{v}{2} - h = 0$ on M.

Then we get

$$\begin{aligned} \delta\mathcal{F} = & \int_M [-\Delta v + D_iD_jv_{ij} - R_{ij}v_{ij} + \\ & 2\langle Df, Dh\rangle + (R + |Df|^2)(\frac{v}{2} - h)]dm \end{aligned}$$

Using the facts that

$$\int_M e^{-f}D_iD_jv_{ij}dV = \int_M D_jD_ie^{-f}v_{ij}dV = \int_M (f_if_j - f_{ij})v_{ij}dV$$

$$\int_M e^{-f}\Delta v dV = \int_M \Delta e^{-f} v dV = \int_M e^{-f}[|Df|^2 - \Delta f] v dV$$

and

$$\int_M e^{-f}\langle Df, Dh\rangle dV = \int_M e^{-f}[|Df|^2 - \Delta f] h dV$$

we get

$$\delta \mathcal{F} = -\int_M v_{ij}(R_{ij} + D_i D_j f) dm.$$

Recall that

$$\int_M dm = 1.$$

then the evolution equations for the gradient flow of $\mathcal{F}$ is

$$\partial_t g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$$

and

$$(\partial_t + \Delta) f = -R.$$

Let

$$X = \nabla f$$

be the gradient vector field on M. Let $\phi(t)$ be the one-parameter family of diffeomorphisms generated by X. Set

$$\hat{g}_{ij} = (\phi^* g)_{ij}, \quad \hat{f} = f \circ \phi.$$

Then, one can easily obtain that the metric $\hat{g}$ and $\hat{f}$ satisfy

$$\partial_t g_{ij} = -2R_{ij}, \quad (\partial_t + \Delta) f = |\nabla f|^2 - R.$$

where all geometric quantities are from the metric $\hat{g}$ In this way we have

$$\mathcal{F}_t = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 dm \geq 0.$$

This means that along the gradient flow $\mathcal{F}$ is non-decreasing in time t.

For fixed g, we define

$$\lambda(g) = \inf \mathcal{F}(g, f).$$

where the infimum is taken over all smooth function with the property

$$\int_M e^{-f} dV = 1.$$

Using the monotonicity of $\mathcal{F}$, we know that $\lambda(g)$ is non-decreasing in time t. This property can be used to show that there is no breathers in some special case. For detail we ask the reader look at the remarkable paper of G.Perelman [30].

4.2 W-functional and its property

In this section we introduce the W-functional of G.Perelman. Then we use it to get a lower bound for the injectivity radius of the flow. This conclusion is the little loop lemma of R.Hamilton, which is guessed to be by him. This is also one of the most difficulty obstruction to Hamilton's program on Ricci-Hamilton flow. With this result, one can see that in a compact manifold of dimension three, we can not have blow up narrow necks like the shape $S^1 \times B^2$.

Introduce the scale parameter $\tau > 0$. We define

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(|Df|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f} dV$$

where f is restricted to satisfy

$$\int_M (4\pi\tau)^{-n/2} e^{-f} dV = 1.$$

Note that $\mathcal{W}$ is invariant under the simultaneous scaling of τ and g_{ij}.

Let $\sigma = \delta\tau$. Using the results from the previous section, we get

$$\delta((4\pi\tau)^{-n/2} e^{-f} dV) = \frac{1}{2}(v - 2h - \frac{n\sigma}{\tau})(4\pi\tau)^{-n/2} e^{-f} dV$$

Then we obtain that

$$\begin{aligned} \delta\mathcal{W} \quad &= \int_M [\sigma(R + |\nabla f|^2) + \tau(v - 2h)(\Delta f - |\nabla f|^2) \\ &-\tau v_{ij}(R_{ij} + D_{ij} f) + h \\ &\frac{1}{2}(\tau(R + |\nabla f|^2) + f - n)(v - 2h - \frac{n\sigma}{\tau}](4\pi\tau)^{-n/2} e^{-f} dV \end{aligned}$$

Take

$$\begin{aligned} v_{ij} &= -2(R_{ij} + D_{ij} f,) \\ h &= -\Delta f - R + \frac{n}{2\tau}, \\ \sigma &= -1. \end{aligned}$$

Taking the trace of the first equation, and then using the second equation, we have

$$v = -2(R + \Delta f) = 2(h - \frac{n}{2\tau})$$

Plugging this into the first variation formula for $\mathcal{W}$, we obtain that

$$\begin{aligned} \mathcal{W}_t = \int_M [-\sigma(R + |\nabla f|^2 - n(\Delta f + |\nabla f|^2) + \\ 2\tau |R_{ij} + D_{ij} f|^2 - \Delta f - R + \frac{n}{2\tau}](4\pi\tau)^{-n/2} e^{-f} dV \end{aligned}$$

Since

$$\int_M \Delta e^{-f} dV = \int_M (|\nabla f|^2 - \Delta f) e^{-f} dV,$$

we find that

$$\mathcal{W}_t = \int_M 2\tau |R_{ij} + D_{ij} f - \frac{1}{2\tau} g_{ij}|^2 (4\pi\tau)^{-n/2} e^{-f} dV$$

Define

$$\mu(g_{ij}, \tau) = \inf \mathcal{W}(g_{ij}, f, \tau)$$

where the infimum is taken over all smooth functions satisfying

$$\int_M (4\pi\tau)^{-n/2} e^{-f} dV = 1.$$

We remark that for fixed g and $\tau > 0$, we can use the Sobolev compactness imbedding theorem, we can find

Assertion 4.2.1 *The minimization problem* $\inf \mathcal{W}$ *is solvable.*

Proof: In fact, let

$$\phi = e^{-f}.$$

Fix the metric g. Then we have

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M [4\tau |\nabla \phi|^2 - \phi^2 \log \phi^2 + (\tau R - n)\phi^2] dV.$$

with *phi* satisfying

$$(4\pi\tau)^{-n/2} \int_M \phi^2 dV = 1.$$

Without loss of generality we may assume that

$$4\pi\tau = 1.$$

So we need only to prove that the following functional

$$J(\phi) = \int_M [4\tau |\nabla \phi|^2 - \phi^2 \log \phi^2 + (\tau R - n)\phi^2] dV$$

has a minimizer in the subset

$$\mathbb{A} = \{\phi \in H^1(M); \int_M \phi^2 dV = 1\}$$

Note that by Sobolev compactness imbedding, $\mathbb{A}$ is weakly closed in H^1 and J is weakly lower semi-continuous. So to get a minimizer of HJ over $\mathbb{A}$, we only to to prove that J is coercive on $\mathbb{A}$. Let $u \in \mathbb{A}$. Then using the simple inequality

$$\log x \leq n x^{1/n}, \quad \forall \ x > 1,$$

the interpolation inequality, and Holder inequality, we have

$$\begin{aligned}
\int_M u^2 \log u^2 dV \quad & \leq \int_{\{u>1\}} u^2 \log u^2 dV \\
& \leq \int_{\{u>1\}} u^{2+2/n} dV \\
& \leq \int_M u^{2+2/n} dV \\
& \leq \epsilon \int_M u^{2+4/n} dV + C(\epsilon) \int_M u^2 dV \\
& \leq \epsilon (\int_M u^{\frac{2n}{n-2}} dV)^{\frac{n-2}{n}} + C(\epsilon) \\
& \leq \epsilon (\int_M |\nabla u|^2 dV + C(\epsilon)
\end{aligned}$$

Choosing $\epsilon = \tau$ we obtain that

$$\begin{aligned}
J(\phi) \quad = \quad & \int_M [4\tau|\nabla\phi|^2 - \phi^2 \log \phi^2 + (\tau R - n)\phi^2] dV \\
\geq & \int_M [3\tau|\nabla\phi|^2 + (\tau R - n)\phi^2] dV - C \\
\geq & \int_M 3\tau|\nabla\phi|^2 dV + \min_M (\tau R - n) - C
\end{aligned}$$

This proves the coercivity of $J(\cdot)$. By the direct method in calculus of variation and evenness of $J(\cdot)$, we can find positive minimizer of

$$J(\phi) \to \min$$

in the subset

$$\mathbb{A} = \{\phi \in H^1(M); \int_M \phi^2 dV = 1\}$$

One can also minimize $J(\cdot)$ over the subset

$$\mathbb{B} = \{\phi \in H^1(M); \phi > 0, \int_M \phi^2 dV = 1\}$$

Here $\phi > 0$ means that $\phi(x) \geq 0$ on M but ϕ is non-trivial. As in our earlier work [27], we can prove that the minimizer is in the interior of $\mathbb{B}$. □

The regularity of the minimizer can be obtained by standard elliptic theory [15] One may see [27] and [34] for more about the variation theory on convex subset in uniform Banach space.

Assertion 4.2.2 *For an arbitrary metric g on a closed manifold M. the function $\mu(g_{ij}, \tau)$ is negative for small $\tau > 0$.*

Proof: We follow the argument of G.Perelman [30]. Let $\bar{\tau} > 0$ be a small number such that the Ricci-Hamilton flow exists on the time interval $[0, \bar{\tau}]$ for the initial metric g_{ij}. Let $u = (4\pi\tau)^{-n/2}e^{-f}$ be the solution of the conjugate heat equation, starting from a δ-function at $t = \bar{\tau}$, $\tau(t) = \bar{\tau} - t$. Then we have

$$\mathcal{W}_\tau = -\mathcal{W}_t \leq 0$$

and

$$\mathcal{W}(g(t), f(t), \tau(t)) \to 0$$

as $t \to \bar{\tau}$. Therefore, by the monotonicity of $\mathcal{W}$, we get

$$\mu(g, \bar{\tau}) \leq \mathcal{W}(g(0), f(0), \tau(0)) = \mathcal{W}(g(0), f(0), \bar{\tau}) < 0.$$

□

4.3 Uniform injectivity radius bound

Given a solution $g_{ij}(t)$ of the Ricci-hamilton flow. We begin with

Definition 4.3.1 *We say that $g_{ij}(t)$ is locally collapsing at T if there is a sequence of times $t_k \to T$ and a sequence metric balls $B_k = B(p_k, r_k)$ at time t_k such that r_k^2/t_k is bounded,*

$$|Rm|(g_{ij}(t_k)) \leq r_k^2, \ \ in \ B_k \ \ r_k^{-n} Vol(B_k) \to 0.$$

G.Perelman [30] proved the following

Theorem 4.3.2 *If M is closed and $T < \infty$, then the solution $g_{ij}(t)$ of the Ricci-Hamilton flow is not collapsing at T.*

Proof: Assume that there is a sequence of collapsing balls $B_k = B(p_k, t_k)$ at times $t_k \to T$. Then we claim that

$$\mu(g(t_k), r_k^2) \to -\infty.$$

Indeed, we can take

$$f_k(x) = -\log \phi(r_k^{-1} dist_{t_k}(x, p_k)) + c_k,$$

where $\phi = \phi(\rho)$ equals to one on $[0, 1/2]$, decreasing on $[1/2, 1]$, and very close to zero on $[1, \infty)$, and c_k is a constant. Note that using

$$(4\pi r_k^2)^{-n/2} \int_M f_k^2 dV = 1.$$

we get

$$e^{c_k} = \int_M (4\pi r_k^2)^{-n/2} \phi^2(r_k^{-1} dist_{t_k}(x, p_k)) dV \leq (4\pi r_k^2)^{-n/2} Vol(B_k).$$

Clearly, since $r_k^{-n} Vol(B_k) \to 0$, we have

$$c_k \to -\infty.$$

Let $A_k(s)$ be the area of the sphere $S_k := \partial B(p_k, r_k s)$. Let

$$\bar{R}_k(s) = r_k^2 A_k(s)^{-1} \int_{S_k} R d\sigma$$

be the average of the scalar curvature over S_k. Then we have

$$\mathcal{W}(g(t_k), f_k, r_k^2) = \frac{\int_0^1 [4|f'(s)|^2 + (\bar{R}_k(s) + c_k - \log f^2 - n) f^2(s)] A_k(s)}{\int_0^1 f(s)^2 A_k(s) ds}$$

As in the last section, we can find a minimizer f of the functional

$$\frac{\int_0^1 [4|f'(s)|^2 - (\log f^2 + n) f^2(s)] A(s)}{\int_0^1 f(s) A(s) ds}$$

with boundary condition $f(0) = 1$ and $f(1) = 0$. Here $A(s)$ is area of the unit sphere in the Poincare hyperbolic space $H^n(-1)$. Let m be the corresponding minimum value. We now fix this minimizer function f in the all computation in this argument. Just as in proving S.Y.Cheng's eigenvalue comparison theorem (the philosophy is that the large the domain (or measure), the smaller the first eigenvalue), we can use the curvature bound $|Rm|(g_{ij}(t_k)) \leq r_k^2$ in B_k to show that

$$\frac{\int_0^1 [4|f'(s)|^2 - (\log f^2 + n) f^2(s)] A_k(s)}{\int_0^1 f(s) A_k(s) ds} \leq m.$$

Then we have

$$\mathcal{W}(g(t_k), f_k, r_k^2) \leq m + n(n-1) + c_k.$$

Hence, applying the monotonicity of $\mathcal{W}$, we get

$$\mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2) \to -\infty.$$

Since $t_k + r_k^2$ is bounded, this is impossible. □

Definition 4.3.3 *We say that a metric g_{ij} is κ-non-collapsed on the scale ρ, if every metric ball B of radius $r < \rho$, which satisfies $|Rm|(x)$ for every $x \in B$, has volume at least κr^n.*

Clearly this concept is scaling invariant up to a scale. This means that if a metric g_{ij} is κ-non-collapsed on the scale ρ, then $\alpha^2 g_{ij}$ is κ-non-collapsed on the scale $\alpha\rho$.

With this concept and the theorem above, we can conclude that

Theorem 4.3.4 *Given a metric g_{ij} on a closed manifold M and $T < \infty$, one can find a $\kappa = \kappa(g_{ij}, T)$, such that if we have the solution $g_{ij}(t)$ of the Ricci-Hamilton flow on $[0, T]$ starting at g_{ij} , then $g_{ij}(t)$ is κ-non-collapsed on the scale $T^{1/2}$ for all $t \in [0, T]$.*

Using the convergence theorem of R.Hamilton (see appendix A below) we can conclude that

Corollary 4.3.5 *Fix $T < +\infty$. Let g_{ij}, $t \in [0, T)$ be a solution of the Ricci-Hamilton flow on a closed manifold M. Assume that for some sequence $t_k \to T$, $p_k \in M$ and some constant C we have*

$$Q_k := |Rm|(g_{ij}(p_k, t_k)) \to \infty,$$

and

$$|Rm|(g_{ij}(x,t)) \leq CQ_k,$$

whenever $t < t_k$. *Then up to a scaling factor* Q_k *of metric* $g_{ij}(t_k)$ *at* p_k *there is a subsequence of the metrics converges to a complete ancient solution to the Ricci-Hamilton flow, which is* κ*-non-collapsed on all scales for some* $\kappa > 0$.

This result immediately implies that we have a uniform injectivity radius bound for the metric sequence involved in the convergent (see also Theorem 4.7in [8]). This implies that the little loop lemma (see [21]) of R.Hamilton is true. According to R.Hamilton [21], one can accomplish Hamilton program if one can improve the Harnack differential inequality.

Before closing this section, we give some comments on remaining sections of [30]: From the section 7 to section 10, G.Perelman generalized the Li-Yau-Hamilton Harnack inequality and the Bishop-Gromov volume comparison theorem for Ricci-Hamilton flow; from section 11 to section 12, he concentrates to dimension three and tries to classify the ancient solution based on Hamilton's pinching estimate. These two sections are really hard and really good. His section 13 is about the geometrization conjecture of W.Thurston.

Chapter 5

Appendix A: Ricci-Hamilton flow on Riemannian manifolds

5.1 preliminary material

Let (M^n, g) be a Riemniannian manifold of dimension n. In the local coordinates (x^i), the metric can be written as

$$g = g_{ij}dx^i dx^j$$

In short, we just write $g = (g_{ij})$ or $g = g_{ij}$. Let

$$(g^{ij}) = (g_{ij})^{-1}$$

be the inverse of the matrix (g_{ij}). define

$$d\mu = \sqrt{det(g_{ij})}dx$$

be the induced measure (which is the volume form on oriented manifold) on M.

Let

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})$$

be the Christoffel symbols which give us the Levi-Civita connection and covariant derivatives of tensor on M.

The Riemannian curvature tensor is

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p.$$

Let

$$R_{ijkl} = g_{hk} R_{ijl}^h,$$

We also call this tensor the Riemannian tensor for an obvious reason.

We now define the Ricci tensor:

$$R_{ik} = g^{jl} R_{ijkl}$$

and the scalar curvature function of M:

$$R = g^{ik} R_{ik}.$$

The Riemannian tensor R_{ijkl} is anti-symmetric in pairs i, j and k, l, and symmetric in their interchange. So we can define the curvature operator on 2-forms:

$$Rm : \wedge^2 M \to \wedge^2 M$$

and

$$Rm(U)_{ij} = g^{pr} g^{qs} R_{ijpq} U_{rs} = R_{ijpq} U_{pq}$$

for the 2-form $U = U_{pq} dx^p \wedge dx^q$. We have from the symmetry of R_{ijkl} that Rm is self-adjoint, that is,

$$\langle Rm(U), V \rangle = \langle U, Rm(V) \rangle.$$

for the 2-forms U and V.

Definition 5.1.1 *We say that (M, g) has positive curvature operator if Rm is positive definite in $\wedge^2 M$.*

Note that positive curvature operator is stronger than positive sectional curvature.

The Riemannian tensor R_{ijkl} satisfies the first Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

and the second Bianchi identity:

$$\nabla_i R_{jklh} + \nabla_j R_{kilh} + \nabla_k R_{ijlh} = 0$$

Taking contraction on the second Bianchi identity, we get the following useful formula:

$$g^{ij} \nabla_i R_{jk} = \frac{1}{2} \nabla_k R,$$

which is called the second contracted Bianchi identity.

In dimension three, we have the following important property that

$$R_{ijkl} = g_{ik} R_{jl} - g_{il} R_{jk} - g_{jk} R_{il} + g_{jl} R_{ik} - \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}).$$

This result implies that the full Riemannian curvature tensor is determined by the Ricci tensor. Assume that we diagonalize R_{ij} at a point so that $\lambda \le \mu \le \nu$ are the eigenvalues. Then the only non-zero components of R_{ijkl} are those of the form

$$R_{1212} = \frac{1}{2} (\lambda + \mu - \nu),$$

and those derived from it by permutation. Hence, the positive sectional curvature condition is equivalent to the one that each eigenvalue of the Ricci tensor is smaller than the sum of the other two, which is in turn equivalent to the following condition:

$$R_{ij} < \frac{1}{2} R g_{ij}.$$

5.2 Hamilton's program

In 1982, R.Hamilton defined the following flow for metrics:

$$\partial_t g_{ij} = -2R_{ij}.$$

It is by now called Ricci-Hamilton flow for the metric family $(g(t))$. It is clear that this flow is invariant under the diffeomorphism group of M. Therefore, the symmetry of the initial metric is preserved along the flow. The difficulty one has to face is how one can prove its short time existence of the flow. However, R.Hamilton [18] proved the local existence of this flow on any compact manifold. The proof is hard and more involved, and was simplified later by De Turck [13]

Example: Let $M = S^n$ be the standard sphere in R^{n+1}. Let c_{ij} be its metric. Recall that the Ricci tensor for the metric c_{ij} is $(n-1)c_{ij}$. Let

$$g_{ij} = \rho^2 c_{ij}.$$

Since the Ricci tensor is scale invariant, we have $R_{ij}(g) = (n-1)c_{ij}$. Then we have

$$\frac{d\rho^2}{dt} = -2(n-1).$$

Hence we have

$$\rho^2(t) = 1 - 2(n-1)t.$$

note the spheres shrink to a point at time $T = \frac{1}{2(n-1)}$.

Definition 5.2.1 *A solution $(g(t))$ of the Ricci-Hamilton flow is called a Ricci soliton if there exists a one-parameter family of diffeomorphisms $(\Phi(t))$ such that*

$$g(t) = \Phi^* g(0)$$

Clearly the example above is a Ricci soliton. Another famous example of Ricci soliton is called *cigar soliton* on R^2:

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

The generating vector field is

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Using the Ricci-Hamilton flow, R.Hamilton proved in [18] the following remarkable theorem:

Theorem 5.2.2 (Hamilton's Theorem) *Let M be a compact 3-manifold which admit a riemannian metric with strictly positive Ricci- curvature. Then M also admit a metric of constant positive curvature.*

We make a remark here about the metric of positive scalar curvature. It was asked by S.T.Yau [37] to find a criterion for existence of a Riemannian metric of positive scalar curvature on a compact manifold. It is well-know that there is some topological obstruction for such a metric. However, the progress is not large if we forget the resolution of the Yamabe problem. With this understanding, people know that finding a nice Riemannian metric of positive Ricci curvature on a compact manifold is a very difficult task.

R.Hamilton [18] observed that

Lemma 5.2.3 *Along the Ricci-Hamilton flow, we have*

$$\partial_t \Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}).$$

and

$$\partial_t \log \mu = -R.$$

where $\mu = \sqrt{det(g_{ij})}$.

We define $B_{ijkl} = R_{ipjq}R_{kplq}$. Using the Bianchi identities we can compute that

Lemma 5.2.4 *Along the Ricci-Hamilton flow we have*

$$(\partial_t - \Delta)R_{ijkl} = 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}).$$

We can define the square of Rm as an operator

$$Rm^2 : \wedge^2 M \to \wedge^2 M$$

such that

$$Rm^2(U)_{ij} = R_{ijpq}R_{pqrs}U_{rs}.$$

Then we can write

$$(Rm^2)_{ijkl} = R_{ijpq}R_{pqkl}.$$

In the space of 2-forms on M, we have the Lie bracket defined by

$$[U, V]_{ij} = U_{ip}g^{pq}V_{qj} - V_{ip}g^{pq}U_{qj}$$

Fix a point $x \in M$. This bracket gives us an isomorphism between $\wedge_x^2 M$ and the Lie algebra $so(n)$. So $so(n)$ has an induced metric which comes from $\wedge^2 M$. Let $\phi^a = \{\phi_{pq}^a\}$ be any orthonormal basis in $so(n)$ and let c_c^{ab} denote the structure constants:

$$[\phi^a, \phi^b] = c^{abc}\phi^c.$$

We also write $c_{abc} = c^{abc}$. For any linear operator

$$L : so(n) \to so(n)$$

we define its *Lie square* as

$$L^\sharp = c_{ace}c_{bdf}L_{cd}L_{ef}.$$

Computations give that

$$(Rm^2)_{ijkl} = 2(B_{ijkl} - B_{ijlk})$$

and

$$(Rm^\sharp)_{ijkl} = 2(B_{ikjl} - B_{iljk}).$$

From these, we obtain that

$$(\partial_t - \Delta)R_{ijkl} = (Rm^2)_{ijkl} + (Rm^\sharp)_{ijkl}$$

This formula is conceptual more clear than the one above. Using this formula and the maximum principle we can conclude that

Proposition 5.2.5 *On the compact manifold M, the positivity of the curvature operator is preserved along the Ricci-Hamilton flow.*

Corollary 5.2.6 *Under the Ricci-Hamilton flow, we have*

$$(\partial_t - \Delta)R_{jk} = 2g^{pr}g^{qs}R_{pjqk}R_{rs} - 2g^{pq}R_{pj}R_{qk}.$$

We also have

Corollary 5.2.7 *Along the Ricci-Hamilton flow, we have*

$$(\partial_t - \Delta)R = 2|Rc|^2.$$

In dimension three, we have

Lemma 5.2.8 *Along the Ricci-Hamilton flow, it holds*

$$(\partial_t - \Delta)R_{ij} = -Q_{ij}$$

where

$$Q_{ij} := 6S_{ij} - 3RR_{ij} - (2S - R^2)g_{ij}$$

with

$$S_{ij} = R^2_{ij} = R_{ik}g^{kl}R_{lj}$$

and

$$S = g^{ij}S_{ij}.$$

We remark that the tensor $Q_{ij} = 0$ on M^3 if and only if (M^3, g) is a three dimensional symmetric Riemannian manifold.

In the rest of this section we assume that M is a compact manifold. Let

$$r = \int_M R d\mu / \int_M d\mu.$$

be the average of the scalar curvature.

We now choose a time normalization factor $\varphi = \varphi(t)$ such that the metric $\bar{g} = \varphi g$ has unit volume, i.e.,

$$\int d\bar{\mu} = 1.$$

Then we choose the new time scale $\bar{t} = \int \varphi(t) dt$. It is easy to know that

$$\bar{R}_{ij} = R_{ij}, \quad \bar{R} = \varphi^{-1} R, \quad \bar{r} = \varphi^{-1} r$$

Since

$$\int d\bar{\mu} = 1,$$

we have

$$\int_M d\mu = \varphi^{-n/2}$$

so

$$\frac{d}{dt} \log \int_M d\mu = -r$$

and

$$\frac{d}{dt} \log \varphi = \frac{2}{n} r.$$

Then we can compute that

$$\frac{\partial}{\partial \bar{t}} \bar{g}_{ij} = \frac{\partial}{\partial t} g_{ij} + \frac{d}{dt} \log \varphi = \frac{2}{n} \bar{r} \bar{g}_{ij} - 2\bar{R}_{ij}.$$

We will call this flow as normalized Ricci-Hamilton flow. It is easy to see that these two flows are the same except that a change of scale in space and a change of coordinate in time. Along this normalized flow, the volume is preserved. This flow is very useful in the blow up analysis for the Ricci-Hamilton flow. Let Rc^o be the trace free part of the Ricci tensor. that is

$$Rc^o_{ij} = R_{ij} - \frac{1}{n} R g_{ij}$$

Diagonalize the matrix R_{ij} at a point, and we see that

$$|Rc|^2 = |Rc^o|^2 + \frac{1}{n} R^2.$$

R.Hamilton [18] proved that

Theorem 5.2.9 *In dimension three, the positivity of the Ricci curvature is preserved along Ricci-Hamilton flow, and there exists positive constants* δ *and* C *such that the following pinching estimate is true:*

$$\frac{|Rc^o|^2}{R^2} \leq CR^{-\delta}$$

Using this, he proved the famous result mentioned at the beginning of this section. In dimension four, he refined the maximum principle, and then proved a similar pinching estimate under the assumption that the initial metric has positive curvature operator. In general dimensions, he proved that the positivity of the curvature operator is preserved along the Ricci-Hamilton flow. R.Hamilton conjectured that the global existence and convergence of the Ricci-Hamilton flow holds for positive curvature operator. Interestingly H.Chen [11] improved Hamilton's result in dimension four where only 2 positive curvature operator is assumed. In dimension large than four, the conjecture is open.

In the study of long time existence of the Ricci-Hamilton flow in any dimension, we need to do the singularity analysis by using the blow up technique. Clearly the Ricci soliton will play a important role in the global problem about convergent of the solution metric. It is generally believed that if we have a global solution of the Ricci-Hamilton flow, then it converges at infinity to either a Ricci-soliton or a metric with constant curvature.

In analysis of non-linear partial differential equations, blow-up technique is that we choose a point and rescale the measure in space-time. Considering it as the origin, we then make the following variable change:

$$(x, t) \to (\lambda x, \lambda^2 t)$$

such that the origin is not blow up point anymore.

In geometry, for the Ricci-Hamilton flow defined on the time interval $A \leq t \leq T_2$ where $A < 0$ and $T_2 > 0$, we choose a marking on M. By definition, a marking is a choice of a point $Q \in M$ which we call the origin, and an orthonormal frame q at Q at at time $t = 0$ with respect to the metric $g(0)$. We regard the collection (M, g, Q, q) as an evolving complete marked Riemannian manifold. The blow up technique can now be defined.

Definition 5.2.10 *We say that a sequence* (M_k, g_k, Q_k, q_k) *of evolving complete marked Riemannian manifolds converges to the evolving complete marked Riemannian manifold* $M = (M, g, Q, q)$ *if there exists a sequence of open sets* U_k *in* M *containing* Q *and a sequence of diffeomorphisms* F_k *of the sets* U_k *in* M *to open sets* V_k *in* M_k *mapping* Q *to* Q_k *and* q *to* q_k*, such that any compact subset in* M *eventually lies in all* U_k *and the pull-backs* $\hat{g}_k$ *of the metrics* g_k *by the mappings* F_k *converges to* g *on every compact subset of* $M \times (A, T_2)$ *uniformly with all their derivatives.*

Remark: We remark that using cut-off of vector fields, we can complete each F_k into a global diffeomorphism.

Example: If $F_k(x,t) = F_\lambda(x,t) = (\lambda x, \lambda^2 t)$, then

$$(F_\lambda^* g)_{ij}(x,t) = \lambda^2 g_{ij}(\lambda x, \lambda^2 t).$$

Using the delation invariant of Ricci tensor, it is clear that if g is a solution of Ricci-Hamilton flow, so is $F_\lambda^* g$. Since at the blow up point, we just require that the quantity $|Rm|$ goes to infinity, we have to develop a comparison tool for the behavior of the solution at nearby points. R.Hamilton [23] find such a tool for solutions with non-negative curvature operator. This tool is called the Li-Yau-Hamilton Harnack inequality.

R.Hamilton ([22]) proved the following result

Theorem 5.2.11 . *Let (M_k, g_k, Q_k, q_k) be a sequence of evolving complete marked Riemannian manifolds which are solutions to the Ricci flow. Suppose that (1) the absolute value of the Riemannian sectional curvature of the M_k at time interval $A < t < T_2$ are uniformly bound above by a uniform constant $B < +\infty$, and (2) the injectivity radii of the M_k at the origin Q_k at $t = 0$ are uniformly bounded below by a constant $\delta > 0$. Then there is a subsequence which converges to an evolving complete marked Riemannian manifold $M = (M, g, Q, q)$ which is also a solution of the Ricci-Hamilton flow, with the sectional curvature and injectivity bound as M_k.*

In section 4 in [30], G.Perelman used the W-functional get the injectivity radius bound required in the Theorem above.

The limiting metric is an ancient solution in the sense that the flow is defined in the time interval $(-\infty, T)$ for some finite T. So the classification of these ancient solutions is a big deal.

In [24], R.Hamilton studied the non-singular solution of the Ricci-Hamilton flow on a compact manifold. By definition, a *non-singular solution* of the Ricci-Hamilton flow is the one which is the solution of the normalised flow existing for all time $0 \le t < \infty$, and the curvature remains bounded

$$|Rm| \le C < \infty$$

for all time with some constant C independent of t. R.Hamilton showed that

Theorem 5.2.12 *Any non-singular solution to the normalised Ricci-Hamilton flow on a compact three manifold M satisfies one and only one of the following properties:*

C). the solution collapses (that is, the maximum injectivity radius on M at time t goes to zero as $t \to \infty$ (we remark that this case is excluded by G.Perelman [30]); or

P). the solution converges to the metric of constant positive sectional curvature; or

Z). the solution converges to a metric of zero sectional sectional curvature; or

H). the solution converges to a metric of constant negative sectional sectional curvature; or

H'). we can find a finite collection of complete non-compact hyperbolic three manifolds with finite volume $\mathcal{V}_1, ..., \mathcal{V}_n$, and for all t beyond some time $T < \infty$ we can find diffeomorphisms

$$\phi_i(t) : \mathcal{H}_i \to M$$

of these manifolds into the manifold M with the solution so that the pull-back of the solution metric $g(t)$ by $\phi_i(t)$ converges to the hyperbolic metric as $t \to \infty$; and moreover if we call the exceptional part of M those points where either the point is not in the image of any ϕ_i, or where it is but the pull-back metric is not as close to the hyperbolic metric as we like, we can make the volume of the exceptional part as small as we like by taking t large enough; and each $\mathcal{H}_i$ is topological essential in the sense that each ϕ_i injects $\pi_1(\mathcal{H}_i)$ into $\pi_1(M)$.

R.Hamilton's argument is clever. R.Hamilton obtained a better scalar curvature pinching estimate and played the volume comparison of solutions between normalised and un-normalised Ricci-Hamilton flow in case C),P),Z),and H). In the case H'), He used a special parametrization given by harmonic maps.

In dimension three, G.Perelman [30] (see sections 11 and 12 in [30]) tried to improved R.Hamilton's result above. He can give more delicate analysis including the volume comparison and other beautiful local and global monotonicity formulae of the Ricci-Hamilton flow. It is clear that G.perelman made a great progress in the study of the Rici-Hamilton flow. For the developments of Ricci- Hamilton flow before G.Perelman's work, one may see the nice survey of H.D.Cao and B.Chow [6].

5.3 The proof of Hamilton's Theorem

In this section we give a brief proof of Theorem 5. 2.2.

Let's recall three famous theorems in Riemannian geometry.

Theorem 5.3.1 (Klingerberg's Theorem) *Let M be an even-dimensional, compact, simply connected Riemannian manifold with strictly positive sectional curvature K_M. Then the injectivity radius inj on M is bounded from below by $\pi/\sqrt{\max_M K_M}$.*

Theorem 5.3.2 (Meyers' Theorem) *Let (M^n, g) be a complete Riemannian manifold with its Ricci curvature bounded below, i.e.,*

$$Ric(g) \geq (n-1)c^2 g > 0$$

Then we have the following diameter bound:

$$diam(g) \leq diam(S^n(c^{-1})).$$

Theorem 5.3.3 (Bishop-Gunther-Gromov Comparison Theorem) *Let (M^n, g) be a complete Riemannian manifold with its Ricci curvature bounded below, i.e.,*

$$Ric(g) \geq (n-1)c^2 g > 0.$$

Then

$$V(B_x(r))/V_c(r)$$

is decreasing in r, where $V_c(r)$ is the volume function in the space form of constant sectional curvature c.

We now assume that $n = 3$. Let (e_i) be a local moving frame on M^3. Using the volume element form (or Hodge 8 operator) we can define an isomorphism between $\wedge^1 M$ and $\wedge^2 M$ such that

$$\begin{aligned} \theta^1 = \frac{1}{\sqrt{2}} * e_1 &= \frac{1}{\sqrt{2}} e_2 \wedge e_3 \\ \theta^2 = \frac{1}{\sqrt{2}} * e_2 &= \frac{1}{\sqrt{2}} e_3 \wedge e_1 \\ \theta^3 = \frac{1}{\sqrt{2}} * e_3 &= \frac{1}{\sqrt{2}} e_1 \wedge e_2 \end{aligned}$$

form an orthonormal basis in $\wedge^2 M$ locally.

Write

$$Rm = M_{pq}\theta^p \wedge \theta^q.$$

Using the isomorphism above can choose (e_i) such that

$$Rm = (M_{pq}) = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

Then

$$R_{1212} = \langle Rm(e_1, e_2)e_1, e_2 \rangle = \frac{1}{2}M_{33} = \frac{1}{2}m_3,$$

etc. By this we have the following expression of the Ricci tensor

$$cm = \frac{1}{2}\begin{pmatrix} m_2+m_3 & 0 & 0 \\ 0 & m_1+m_3 & 0 \\ 0 & 0 & m_1+m_2 \end{pmatrix}$$

The scalar curvature R can be expressed by

$$R = m_1 + m_2 + m_3.$$

By direct computation we find

$$Rm^\sharp = \begin{pmatrix} m_2m_3 & 0 & 0 \\ 0 & m_1m_3 & 0 \\ 0 & 0 & m_1m_2 \end{pmatrix}$$

Hence the evolution equation for curvature operator c

$$(\partial_t - \Delta)Rm = Rm^2 + Rm^\sharp$$

an be written as

$$(\partial_t - \Delta)\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + \begin{pmatrix} m_2m_3 & 0 & 0 \\ 0 & m_1m_3 & 0 \\ 0 & 0 & m_1m_2 \end{pmatrix}$$

According Hamilton's maximum principle we need only to analyze the following ODE:

$$\frac{d}{dt}\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + \begin{pmatrix} m_2m_3 & 0 & 0 \\ 0 & m_1m_3 & 0 \\ 0 & 0 & m_1m_2 \end{pmatrix}$$

We can assume that $m_1 \geq m_2 \geq m_3$ such that

$$m_1 = \sup_{|X|=1} < MX, X >$$

and

$$m_3 = \inf_{|X|=1} < MX, X >$$

where $M = Rm$. Note that m_i are globally defined functions. We will consider m_i as functions of the matrix M. From these definitions we know that m_1 is a convex function of the matrix M and m_3 is a concave function of M. Since

$$R = m_1 + m_2 + m_3 = tr_g M$$

is a linear function of M.(Here we assume that the metric g is fixed). Hence $m_2 + m_3 = R - m_1$ is a concave function of M and $m_1 + m_2 = R - m_3$ is a convex function of M. We have the following results.

Theorem 5.3.4 *For $\epsilon \in [0, \frac{1}{3}]$, the pinching condition $R \geq 0$ and $Rc \geq \epsilon R g$ is preserved along the Ricci-Hamilton flow in dimension three.*

The proof of this theorem is a easy application of Hamilton's maximum principle, so we omit it.

Theorem 5.3.5 *For any $\beta > 0$ and $B < +\infty$, we can find uniform positive constants $A < +\infty$ and $\delta \in (0,1)$ such that pinching condition $\beta g \leq Rc \leq Bg$ and*

$$m_1 - m_3 \leq A(m_2 + m_3)^{1-\delta}$$

is preserved along the Ricci-Hamilton flow in dimension three.

Proof: By Hamilton's maximum principle, the condition $\beta g \leq Rc \leq Bg$ is preserved along the Ricci-Hamilton flow in dimension three.

We now choose $\delta \in (0,1)$ such that

$$\delta m_1 \leq m_2 + m_3$$

at $t = 0$. note that this condition defines a convex, closed subset, so we can use Hamilton's maximum principle to conclude that

$$\delta m_1 \leq m_2 + m_3$$

is true for all $t > 0$. Choose $A > 0$ such that:

$$m_1 - m_3 \leq A(m_2 + m_3)^{1-\delta}, \qquad (*)$$

at $t = 0$. This is possible since

$$m_1 - m_3 \leq m_1 + m_2 \leq B$$

and

$$m_2 + m_3 \geq \beta > 0$$

at $t = 0$.

We claim that $(*)$ is preserved along the Ricci-Hamilton flow. In fact, we can see that $(*)$ defines a closed convex subset of matrices with

$$m_1 \geq m_2 \geq m_3$$

and

$$m_2 + m_3 \geq 0.$$

So we only need to check that the inequality is preserved by the ODE.

Compute,

$$\begin{aligned} \frac{d}{dt} \log(m_1 - m_3) &= \frac{m_1^2 - m_3^2 + m_2 m_3 - m_1 m_2}{m_1 - m_3} \\ &= m_1 - m_2 + m_3 \end{aligned}$$

while,

$$\begin{aligned}\frac{d}{dt}\log(m_2+m_3) &= \frac{m_2^2-m_3^2+m_1m_3+m_1m_2}{m_2+m_3}\\ &\geq \frac{m_1(m_2+m_3)}{m_2+m_3}=m_1\end{aligned}$$

Hence

$$\frac{d}{dt}\log\frac{m_1-m_3}{(m_2+m_3)^{1-\delta}}\leq \delta m_1-(m_2+m_3)\leq 0.$$

Therefor $(*)$ is preserved by the ODE. □

Obviously we can deduce the Theorem 5.2.9 from the relation

$$Rc^o=\frac{1}{6}\begin{pmatrix} m_2+m_3-m_1 & 0 & 0\\ 0 & m_1+m_3-m_2 & 0\\ 0 & 0 & m_1+m_2-m_3\end{pmatrix}$$

Using the pinching estimate, we know that the blow time of $|Rm|$, $|Rc|$, and R are the same. We write the blow up time as $T>0$.

In the following we consider the normalized Ricci-Hamilton flow and we show that the normalized metric $\hat{g}(t)$ converges to a metric of constant sectional curvature. Clearly, we need only to show that the normalized metric $\hat{g}(t)$ converges to a metric $\bar{g}$ with zero trace free Ricci tensor:

$$Rc^o(\bar{g})=0.$$

Lemma 5.3.6 *Let $\hat{g}=\phi g$ with $\phi>0$. Then we have*
1). $\hat{\Gamma}_{ij}^k=\Gamma_{ij}^k$.
2). $\hat{R}_{ijk}^l=R_{ijk}^l$.
3). $\hat{R}_{ijkl}=R_{ijkl}$.
4). $\hat{R}_{ij}=R_{ij}$.
5). $\hat{R}=\phi^{-1}R$.
6). $d\hat{\mu}=\phi^{n/2}d\mu$.

Choose $\phi=\phi(t)$ such that

$$\int_M d\hat{\mu}=1.$$

Then we have

$$0=\int_M(-\phi\hat{R}+\frac{n}{2\phi}\frac{d\phi}{dt})d\mu$$

and

$$\frac{n}{2\phi}\frac{d\phi}{dt}=\phi\hat{\rho}.$$

where

$$\hat{\rho} = \int_M \hat{R} d\hat{\mu}.$$

Put

$$\hat{t} = \int_0^t \phi(\tau) d\tau.$$

Then we have the normalized Ricci-Hamilton flow:

$$\partial_{\hat{t}} \hat{g} = -2\hat{Rc} + \frac{2}{n}\hat{\rho}\hat{g}.$$

Assertion 5.3.7

$$\int_0^T R_{max} dt = +\infty.$$

Proof: Define ρ by the flow

$$\frac{d\rho}{dt} = 2R_{max}\rho$$

with initial data

$$\rho(0) = R_{max}(0).$$

Then we have

$$\begin{aligned}
(\partial_t - \Delta)(R - \rho) &= 2(|Rc|^2 - R_{max}\rho) \\
&\leq 2(R^2 - R_{max}\rho) \\
&\leq 2R_{max}(R - \rho)
\end{aligned}$$

By the maximum principle we get

$$R \leq \rho$$

on M. Hence we have

$$\rho \geq R_{max} \to +\infty$$

as $t \to T$. Since for any $\tau < T$, we have

$$\log \frac{\rho(\tau)}{\rho(0)} = \int_0^\tau \frac{d}{dt} \log \rho = 2\int_0^T R_{max} dt$$

Hence, sending $\tau \to T$, we have

$$\int_0^T R_{max} dt = +\infty.$$

□

Assertion 5.3.8 *There exists a uniform constant $C > 0$ such that*

$$\hat{R}_{max} \leq C.$$

Proof: Let

$$\hat{L}(t) = diam(\hat{g}(t)).$$

Note that

$$\hat{V}(t) := vol(\hat{g}(t)) = 1.$$

Since $\hat{Rc} > 0$, by Bishop-Gunther-Gromov comparison theorem we have

$$1 \leq \frac{4}{3}\pi \hat{L}^3.$$

On the other hand we have

$$\hat{Rc} = Rc \geq 2\beta^2 R_{min} g = 2\beta^2 \hat{R}_{min} \hat{g}.$$

So by Meyers' Theorem we have

$$\hat{L} \leq \frac{\pi}{\beta\sqrt{\hat{R}_{min}}}.$$

By the pinching estimate we have

$$\frac{\hat{R}_{max}}{\hat{R}_{min}} = \frac{R_{max}}{R_{min}} \to 1,$$

and hence, there is a uniform constant $C > 1$ such that

$$\frac{\hat{R}_{max}}{\hat{R}_{min}} \geq C^{-1}.$$

Therefore, we have

$$\hat{R}_{max} \leq C\hat{R}_{min} \leq C(\frac{\pi}{\beta\hat{L}})^2 \leq \bar{C}.$$

□

Using this estimate, we can bound $\nabla^\alpha \hat{Rm}$ for any α. Then we can can bound the local derivatives of the metric $\hat{g}$. Hence we can assume that

$$\hat{g} \to \bar{g},$$

and

$$\hat{R} \to c > 0$$

a s $\hat{t} \to \hat{T}$.

Note that
$$\int_0^\tau r(t)dt = \int_0^{\hat{\tau}} \hat{r}(\hat{t})\hat{t}$$
where
$$\hat{\tau} = \int_0^\tau \phi(t)dt.$$
Since
$$R_{min} \le r(t) \le R_{max}$$
and by pinching estimate again, we have
$$\int_0^{\hat{\tau}} \hat{r}(\hat{t})\hat{t} \to +\infty.$$
as $\tau \to T$. Hence
$$\hat{T} = +\infty,$$
and by pinching estimate we have
$$Rc(\bar{g}) = 0.$$
Actually we can show the exponential convergence. We refer the reader to [17] for this.

Chapter 6

Appendix B: the maximum principles

The classical maximum principle is for the scalar heat equation. It says that for any smooth solution of the heat flow, point-wise control for at $t = 0$ is preserved along the flow. Assume that M is a compact manifold. A well-known result is

Theorem 6.0.9 The maximum principle 1. *Assume that u is a smooth (C^2) solution of the semi-linear heat equation*

$$(\partial_t - \Delta)u = \langle V, Du \rangle$$

where Δ is the Laplacian with respect to a time-dependent Riemannian metric $G(t)$, and V is a smooth vector field on $M \times [0, T)$. If we assume $|u| \leq C$ at time $t = 0$, then we have

$$|u| \leq C, \quad \forall t > 0$$

A little more advanced maximum principle is the following

Theorem 6.0.10 The maximum principle 2. *Assume that*

$$u : M \times [0, T) \to \mathbb{R}$$

is a smooth (C^2) solution of the semi-linear heat equation

$$(\partial_t - \Delta)u = \langle V, Du \rangle + f(u)$$

where D and V are as before, and f is a smooth function on $\mathbb{R}$. If we assume $C_1 u \leq C_2$ at time $t = 0$, then we have

$$\phi_1 \leq u \leq \phi_2, \quad \forall t > 0$$

where ϕ_i for $i = 1, 2$ are smooth solutions of the ode

$$\phi_{i_t} = f(\phi_i), \quad \phi_i(0) = C_i.$$

The maximum principles can be generalized to complete non-compact Riemannian manifolds with some restriction (see [1].

6.1 Hamilton's maximum principle

If $f(t)$ is a Lipschitz function of time variable t. We define

$$\frac{df}{dt} \leq c$$

if and only if

$$\limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h} \leq c.$$

Let $F = F(t, y)$ is a smooth function of the time variable and space variable $y \in R^k$. Let Y be a compact subset of R^k. Let

$$f(t) = \sup\{F(t, y); y \in Y\}.$$

Then $f)t)$ is a Lipschitz function. Then we have the following

Lemma 6.1.1 (Basic Lemma)

$$\frac{df}{dt} \sup\{\frac{\partial}{\partial t} F(t, y); y \in Y(t)\}.$$

where $Y(t) = \{y; F(t, y) = f(t)\}$.

Proof: Choose a sequence of $t_j \to t^+$ so that

$$\lim_{t_j \to t^+} \frac{f(t_j) - f(t)}{t_j - t}$$

equals the lim sup. Since Y is compact, we can choose $y_j \in Y$ with

$$f(t_j) = F(t_j, y_j).$$

By passing to a subsequence y_j we can assume that

$$y_j \to y.$$

By continuity we have

$$f(t) = F(t, y),$$

this implies that $y \in Y(t)$. Hence

$$F(t, y_j) \leq F(t, y).$$

By this we have

$$f(t_j) - f(t) \leq F(t_j, y_j) - F(t, y_j).$$

By the mean value theorem we can find r_j between t_j and t such that

$$F(t_j, y_j) - F(t, y_j) = \frac{\partial}{\partial t} g(r_j, y_j)(t_j - t)$$

Sending $t_j \to t$ we have $r_j \to t$ and then

$$\frac{f(t_j) - f(t)}{t_j - t} \leq \frac{\partial}{\partial t} g(t, y).$$

This gives us the conclusion. □

Let M be a compact manifold with a Riemannian metric g , and let

$$f = \{f^a\} : M \to R^k$$

be a vector-valued function on M . Let U be an open subset of R^k and let

$$\phi : U \subset R^k \to R^k$$

be a smooth vector field on U. We let g and ϕ depend on time also. Then we consider the nonlinear heat equation (PDE):

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f)$$

with initial data $f = f_0$ at t=0, and we suppose it has a solution for some time interval $0 \leq t \leq T$. We let $\mathcal{X}$ be a closed convex subset of mappings $f : U \to R^k$ containing the initial data f_0, and we ask when the solution remains in $\mathcal{X}$. To answer this we study the ordinary differential equation (ODE):

$$\frac{df}{dt} = \phi(f)$$

on U, and ask when its solutions remains in $\mathcal{X}$. We define the tangent cone $T_f\mathcal{X}$ to closed convex set $\mathcal{X}$ at a point f as the smallest closed convex cone with vertex at f which contains $\mathcal{X}$. Then it is the intersection of all the closed half-spaces containing $\mathcal{X}$ with f on the boundary of the half-space.

We say that a linear function l on R^k is a *support function* for $\mathcal{X}$ at $f \in \partial\mathcal{X}$ and write $l \in S_f\mathcal{X}$ if $l(f) \geq l(k)$ for all other $k \in \mathcal{X}$. Then $\phi(f) \in T_f\mathcal{X}$ if and only if $l(\phi(f)) \leq 0$ for all $l \in S_f\mathcal{X}$.

Lemma 6.1.2 . *The solutions of the ODE*

$$\frac{df}{dt} = \phi(f)$$

which start in the closed convex set $\mathcal{X}$ *will remain in* $\mathcal{X}$ *if and only if* $\phi(f) \in T_f\mathcal{X}$ *for all* $f \in \mathcal{X}$.

Proof: Suppose $l(\phi(f)) > 0$ for some $l \in S_f\mathcal{X}$. Then

$$\frac{dl(f)}{dt} = l(\frac{df}{dt}) = l(\phi(f)) > 0$$

So $l(f)$ is increasing and f cannot remain in $\mathcal{X}$. To see the converse, first note that we may assume $\mathcal{X}$ is compact. This is because we can modify the vector field $\phi(f)$ by multiplying by a cutoff function which is everywhere nonnegative, equals one on a large ball , and equals zero on a larger ball. The paths of solutions are unchanged inside the first ball.Then we can intersect X with the second ball to make $\mathcal{X}$ convex and compact . If there were a counterexample before the modification there would still be one afterward. Let $s(f)$ be the distance from f to $\mathcal{X}$, with $s(f) = 0$ if $f \in \mathcal{X}$. Then

$$s(f) = supl(f - k),$$

where the sup is over all $k \in \partial\mathcal{X}$ and all $l \in S_k\mathcal{X}$. This defines a compact subset Y of $R^k \times R^k$. Hence by the basic lemma

$$\frac{ds(f)}{dt} \leq sup\{l(\phi(f))$$

where the sup is over all pairs (k, l) with $k \in \partial\mathcal{X}$, $l \in S_k\mathcal{X}$, and

$$s(f) = l(f - k).$$

This can happen only when k is the unique closest point in $\mathcal{X}$ to f and l is the linear function of length 1 with gradient in the direction $f - k$.

We now use the fact that ϕ is smooth on the compact subset $\mathcal{X}$ to get the bound

$$|\phi(f) - \phi(k)| \leq C|f - k|$$

for some constant and for all f and k in $\mathcal{X}$. Then since we have

$$l(\phi(k)) \leq 0$$

by the assumption and

$$|f - k| = s(f)$$

where

$$\frac{ds(f)}{dt} \leq s(f).$$

Since $s(f) = 0$ at $t = 0$, it must remain 0 for all time. □

The key observation of R.Hamilton [17] is that the ODE controls the parabolic partial differential equation (PDE).

Theorem 6.1.3 *If the solution of the ODE stays in $\mathcal{X}$, then so does the solution of the PDE.*

Proof: As before, we can take $\mathcal{X}$ as a compact sunset. Let $s(f)$ be the distance of $f \in R^k$ from $\mathcal{X}$ and let

$$s(t) = \sup_{x \in M} s(f(x,t)) = \sup l(f(x,t) - k)$$

where the later sup is taken over all $x \in M$, all $k \in \partial\mathcal{X}$ and all $l \in S_k(\mathcal{X})$. Since the sup is taken over a compact set,we can use the basic lemma to get

$$\frac{d}{dt}s(t) \leq sup\frac{d}{dt}l(f(x,t) - k),$$

where the sup is taken over all x, k, l as above with

$$l(f(x,t) - k) = s(t).$$

Since x is the maximum point of $l(f)$, we have

$$l(\Delta f) = \Delta l(f) \leq 0.$$

Now we have

$$\frac{d}{dt} l(f(x,t) - k) = l(\Delta f) + l(\phi(f)) \leq l(\phi(f)).$$

Since $l(\phi(k)) \leq 0$ by assumption, we have some constant $C > 0$ so that

$$l(\phi(f)) \leq l(\phi(f)) - l(\phi(k)) \leq |\phi(f) - \phi(k)| \leq C|f - k| = Cs(t).$$

Thus we have

$$\frac{d}{dt} s(t) \leq Cs(t).$$

Since $s(t) = 0$ at $t = 0$, we have $s(t)$ for all $t > 0$. This shows that $f(x,t)$ remains in $\mathcal{X}$. $\square$

We can generalize this result to vector bundles. Let V be a vector bundle over the compact manifold M, and suppose V has a fixed Riemannian metric h. Let g be the Riemannian metric on M, and let A be a connection on V compatible with h. Both g and A may depend on time variable t. We can form the Laplacian of a section f of V as the trace of the second covariant derivative with respect to g , using the connection A on V and the Levi-Civita connection Γ on TM for g. Let W be an open subset of V and let ϕ be a vector field on W tangent to the fibers . Then we consider the following nonlinear heat equation (PDE):

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f)$$

Let X be a closed subset of $U \subset W$. We ask when solutions of the PDE which start in X will stay in X. We need to impose the conditions that X is invariant under parallel translation by the connection A at each time , and that each fiber of X is convex. Then we can judge the behavior of the PDE by comparing to the following ordinary differential equation (ODE):

$$\frac{df}{dt} = \phi(f)$$

in each fiber.

Theorem 6.1.4 *If the solutions of the ODE's in each fiber remain in X, then the solutions of the PDE remain in X too.*

Proof: Again modifying the equation we can assume that X is compact . Using the metric h in the fiber and writing $|f - k|$ for the Euclidean distance from f to k in the metric h , we let $s(t)$ be the maximum distance of any $f(x, t)$ from the set X. Then

$$s(t) = supl(f(x, t) - k),$$

where the sup is taken over all $x \in M$, all $k \in \partial X$ in the fiber over x , and all support functions $l \in S_k X$ at k in the fiber at x , The set of all such pairs (k, l) is a compact subset of $V \oplus V^*$. Then as before, we have

$$\frac{ds(t)}{dt} \leq sup \frac{d}{dt} l(f(x, t) - k)$$

where the sup is taken over all x where the distance in the fiber from $f(x, t)$ to x is maximal, k is the unique closest point in X to $f(x, t)$, and l is the linear function of length 1 on the fiber of V at x with gradient in the direction from k to $f(x, t)$. Again

$$\frac{d}{dt} l(f(x, t) - k) = l(\Delta f) + l(\phi(f))$$

and since $l(\phi(k)) \leq 0$ by assumption

$$l(\phi(f)) \leq |\phi(f) - \phi(k)| \leq C|f - k| = Cs(t),$$

where C is some constant bounding the first derivative of ϕ on a neighborhood of X. If we extend a vector in a bundle from a point x by parallel translation along geodesics emanating radially out of x, we get a smooth section of the bundle such that all the symmetrized covariant derivatives at x are zero .We extend $k \in V$ and $l \in V^*$ in this manner . Since the metric in V is invariant we continue to have $|l| = 1$, and since X is invariant under parallel translation we continue to have $k \in \partial X$ and l a support function for X at k . Therefore

$$l(f(x, t) - k) \leq s(t)$$

in the neighborhood . It follows that $l(f(x, t) - k)$ has its maximum at x ,so

$$\Delta l(f(x, t) - k) \leq 0$$

at x . But k and l have all their symmetrized covariant derivatives equal to zero at x , so $l(\Delta f) \leq 0$ at x . From this at x we get that

$$\frac{ds(f)}{dt} \leq s(f).$$

Since $s(f) = 0$ at $t = 0$, it must remain 0 for all time. This completes the proof . □

In our applications to the Ricci-Hamilton flow, we have a principal G-bundle P over M where G is a compact Lie group , E is a vector space with a metric and G acts on E preserving the metric , ϕ is a G-invariant vector field on E, and Z is a closed convex subset of E invariant under ϕ . Then solutions of the equation

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f)$$

for sections f in $P \times_G E$ remain in the set $X = P \times_G Z$.

Another important observation is that R.Hamilton [20] generalized them to parabolic systems for tensors. As in the matrix case, we introduce the following concept.

Definition 6.1.5 *We say that a symmetric tensor $M_{ij} \geq 0$ if $M_{ij}v^iv^j \geq 0$ for all vectors v^j.*

As usual, we let

$$\Delta M_{ij} = g^{pq}\nabla_p\nabla_q M_{ij}$$

be the Laplacian of the tensor M_{ij} on the Riemannian manifold (M, g).

Definition 6.1.6 *Assume that $N_{ij} = p(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using the metric g_{ij}. We say that N_{ij} satisfies* the null-eigenvector condition *if whenever v^j is a null-vector of M_{ij}, i.e.,*

$$M_{ij}v^i = 0,$$

for all j, then we have

$$N_{ij}v^iv^j \geq 0.$$

R.Hamilton [17] established the following

Theorem 6.1.7 *Let (M, g) be a compact Riemannian manifold. Let u^i be a vector field on M. Suppose that on $[0, T]$,*

$$(\partial_t - \Delta)M_{ij} = u^k\nabla_k M_{ij} + N_{ij},$$

where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvalue condition above. If $M_{ij} \geq 0$ at $t = 0$, then it remains so on $[0, T]$.

For a proof of this result, one may consult [17]. Using this maximum principle, R.Hamilton [17] proved that in dimension three, non-negativity of the Ricci tensor (and positivity of sectional curvature) is preserved along the Ricci-Hamilton flow.

Chapter 7

Appendix C: Curve shortening flow on manifolds

We first derive the standard first and second variations of arc-length functional. Then we study the curve shrinking/shortening problem in a Riemannian manifold.

7.1 first and second variations of arc-length

Let (M^n, g) be a Riemannnian manifold. Let $\gamma : [a, b] \to M$ be a smooth curve parametrized proportional to arc-length, i.e., $|\gamma'| = l$.

A *variation* of the smooth curve $\gamma : [a, b] \to M$ is a smooth mapping:

$$F : [a, b] \times (-\epsilon, \epsilon) \to M$$

such that $F(s, 0) = \gamma(s)$.

Let

$$X = F_* D_1, \quad Y = F_* D_2,$$

where $D_1 = \frac{\partial}{\partial s}$ and $D_2 = \frac{\partial}{\partial \mu}$. Let

$$\gamma_\mu(\cdot) = F(\cdot, \mu).$$

We call Y *the variation field* of γ.

Then we have

$$\begin{aligned} \frac{d}{d\mu} L(\gamma_\mu) &= \int_a^b \frac{\partial}{\partial \mu} |\frac{\partial}{\partial s} \gamma_\mu(s)| ds \\ &= \int_a^b \frac{\partial}{\partial \mu} \langle X, X \rangle^{1/2} ds \\ &= \int_a^b \frac{1}{|X|} \langle \nabla_Y X, X \rangle ds \\ &= \int_a^b \frac{1}{|X|} \langle D_1 Y, X \rangle ds \end{aligned}$$

Hence, at $\mu = 0$, we have

$$\begin{aligned}
\frac{d}{d\mu}L(\gamma_\mu) &= \frac{1}{l}\int_a^b \langle D_1 Y, X\rangle ds \\
&= \frac{1}{l}\int_a^b \langle D_1 Y, \gamma'\rangle ds \\
&= \frac{1}{l}(\langle Y, \gamma'\rangle|_a^b - \int_a^b \langle Y, \nabla_{\gamma'}\gamma'\rangle ds)
\end{aligned}$$

This formula is called *the first variation of arc-length.* The critical points of arc-length is geodesics in M.

Assume that γ is a geodesic. Consider a two-parameter variation of γ:

$$F : [a, b] \times (-\epsilon, \epsilon)^2 \to M$$

such that $F(s, 0, 0) = \gamma(s)$. Let

$$X = F_* D_1, \quad Y_1 = F_* D_2, \quad Y_2 = F_* D_3.$$

and let

$$\gamma_{\mu_1\mu_2}(\cdot) = F(\cdot, \mu_1, \mu_2).$$

Then we have

$$\frac{\partial}{\partial\mu_2}L(\gamma_{\mu_1\mu_2}) = \int_a^b \frac{1}{|X|}\langle \nabla_X Y_2, X\rangle ds$$

and

$$\begin{aligned}
\frac{\partial}{\partial\mu_1}\frac{\partial}{\partial\mu_2}L(\gamma_{\mu_1\mu_2}) &= \int_a^b \frac{1}{|X|}(\langle \nabla_{Y_1}\nabla_X Y_2, X\rangle + \langle \nabla_X Y_2, \nabla_{Y_1} X\rangle) ds \\
&- \int_a^b \frac{1}{|X|^3}\langle \nabla_X Y_2, X\rangle\langle \nabla_{Y_1} X, X\rangle ds \\
&= \int_a^b \frac{1}{|X|}(\langle R(Y_1, X)Y_2, X\rangle + \langle \nabla_X \nabla_{Y_1} Y_2, X\rangle + \langle \nabla_X Y_2, \nabla_X Y_1\rangle) ds \\
&- \int_a^b \frac{1}{|X|^3}\langle \nabla_X Y_2, X\rangle\langle \nabla_X Y_1, X\rangle ds
\end{aligned}$$

Thus at $(\mu_1, \mu_2) = (0, 0)$, we have

$$\begin{aligned}
\frac{\partial}{\partial\mu_1}\frac{\partial}{\partial\mu_2}L(\gamma_{\mu_1\mu_2}) &= \frac{1}{l}((\langle \nabla_{Y_1} Y_2, X\rangle|_a^b \\
&+ \int_a^b \langle \nabla_X Y_2, \nabla_X Y_1\rangle - \langle R(Y_1, X)X, Y_2\rangle \\
&- \langle \nabla_X Y_2, \frac{X}{|X|}\rangle\langle \nabla_X Y_1, \frac{X}{|X|}\rangle ds
\end{aligned}$$

This is the second variation formula for the arc-length.

7.2 Curve shortening flow in Riemannian manifolds

We collect some basic facts about curve shortening flow in Riemannian manifolds . We believe a good understanding for this flow is also helpful in studying Ricci-Hamilton flow on manifolds. One may consult [14] and [12] for more material.

On an n-dimensional *Riemannian* manifold (M, g), let

$$\gamma : S^1 \times (a, b) \to M$$

be an evolving immersed curve. Denote by γ_t the associated trajectory,i.e.

$$\gamma_t(\cdot) =: \gamma(\cdot, t).$$

Then the length of γ_t is

$$L(\gamma_t) = \int_{S^1} |\frac{d}{du}\gamma_t| du = \int_{S^1} |\frac{\partial \gamma}{\partial u}| du = \int_{S^1} v du,$$

where

$$v =: |\frac{\partial \gamma}{\partial u}|$$

is the speed of the curve γ_t. We can define the arc length parameter s by

$$\frac{\partial}{\partial s} =: \frac{1}{v}\frac{\partial}{\partial u},$$

which implies

$$ds = v du.$$

As usual,we denote by T the unit tangent vector field of γ_t,i.e.

$$T =: \frac{\partial \gamma}{\partial s} = \frac{1}{v}\frac{\partial \gamma}{\partial u}.$$

Then the time derivative of length is

$$\begin{aligned}
\frac{d}{dt}L(\gamma_t) &= \int_{S^1} \frac{\partial v}{\partial t} du \\
&= \int_{S^1} < \nabla_t \frac{\partial \gamma}{\partial u}, T > du \\
&= \int_{S^1} < \nabla_u \frac{\partial \gamma}{\partial t}, T > du \\
&= \int_{S^1} \{\frac{\partial}{\partial u} < \frac{\partial \gamma}{\partial t}, T > - < \frac{\partial \gamma}{\partial t}, \nabla_u T >\} du \\
&= -\int_{S^1} < \frac{\partial \gamma}{\partial t}, \nabla_u T > du \\
&= -\int_{S^1} < \frac{\partial \gamma}{\partial t}, \frac{DT}{\partial s} > ds.
\end{aligned}$$

If we require γ to evolve according to the equation

$$\frac{\partial \gamma}{\partial t} = \frac{DT}{\partial s},$$

then we find that

$$\frac{d}{dt}L(\gamma_t) = -\int_{\mathbf{S}^1} k^2 ds \leq 0,$$

where

$$k^2 =: |\frac{DT}{\partial s}|^2$$

is the curvature squared. This leads us to give the following

Definition 7.2.1 *For a curve shortening flow, we mean an evolving immersed curve $\gamma(\cdot, t)$ satisfying the evolution equation*

$$\frac{\partial \gamma}{\partial t} = \frac{DT}{\partial s}.$$

Obviously, we can regard $\gamma_t(S^1)$ as a 1-dimensional sub-manifold of M. With the induced metric from M, its mean curvature vector field is

$$H = (\nabla_T T)^{\perp}.$$

Note that $< T, T > \equiv 1$. So we have $< \nabla_T T, T > \equiv 0$,which implies $\nabla_T T \perp T\gamma_t$. Therefore $H = \nabla_T T$. This shows that a curve shortening flow is a mean curvature flow of a one dimensional sub-manifold in M.

Next we will give some fundamental formulae for the curve shortening flow.

Lemma 7.2.2 *The evolution of v is*

$$\frac{\partial v}{\partial t} = -k^2 v.$$

Proof: By definition,

$$v^2 =< \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} > .$$

Differentiating it with respect to t,we get

$$\begin{aligned}
\frac{\partial v}{\partial t} &= 2 < \nabla_t \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} > \\
&= 2 < \nabla_u \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial u} > \\
&= 2v^2 < \nabla_T \frac{DT}{\partial s}, T > \\
&= -2v^2 < \frac{DT}{\partial s}, \frac{DT}{\partial s} > \\
&= -2k^2 v^2
\end{aligned}$$

□

We often need to exchange the order of derivatives. We have

Lemma 7.2.3 *Covariant differentiation with respect to s and t are related by the equation*

$$\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s + R(T, \frac{DT}{\partial s}),$$

where R is the curvature operator on M.

Proof: We compute

$$\nabla_t \nabla_u = \nabla_u \nabla_t + R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t})$$

and

$$\nabla_s = \frac{1}{v} \nabla_u,$$

so

$$\begin{aligned}
\nabla_t \nabla_s &= \frac{\partial}{\partial t}(\frac{1}{v}) \nabla_u + \frac{1}{v} \nabla_t \nabla_u \\
&= k^2 \frac{1}{v} \nabla_u + \frac{1}{v} \nabla_u \nabla_t + \frac{1}{v} R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}) \\
&= \nabla_s \nabla_t + k^2 \nabla_s + R(T, \frac{DT}{\partial s}).
\end{aligned}$$

Here we have used Lemma 7.2.2. □

Lemma 7.2.4 *The covariant differentiation of the unit tangent vector to the curve with respect to time t is*

$$\nabla_t T = k^2 T + \frac{D^2 T}{\partial s^2}.$$

Proof: The proof is a straightforward calculation.

$$\begin{aligned}
\nabla_t T &= \nabla_t (\frac{1}{v} \frac{\partial \gamma}{\partial u}) \\
&= k^2 \frac{1}{v} \frac{\partial \gamma}{\partial u} + \frac{1}{v} \nabla_u \frac{\partial \gamma}{\partial t} \\
&= k^2 T + \frac{D^2 T}{\partial s^2}.
\end{aligned}$$

□

7.3 Bernstein type estimates

In this section, we will assume M is a locally symmetric space,i.e. $\nabla R = 0$. For a locally symmetric space,we have

$$\begin{aligned}
\nabla R(X,Y,Z,W) &= (\nabla_X R)(Y,Z,W) \\
&= \nabla_X(R(Y,Z,W)) - R(\nabla_X Y, Z, W) - R(Y, \nabla_X Z, W) \\
&\quad -R(Y,Z,\nabla_X W) = 0
\end{aligned}$$

$\Rightarrow$

$$\nabla_X(R(Y,Z,W)) = R(\nabla_X Y, Z, W) + R(Y, \nabla_X Z, W) + R(Y, Z, \nabla_X W), \qquad (1)$$

for all $X, Y, Z, W \in TM$. We will also assume that M satisfies $Condition(\Lambda)$, i.e. there exists a positive constant Λ,s.t.

$$R(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) \leq \Lambda,$$

for all unit vectors $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}$.

With these assumptions,we can give some precise estimates which will bound the evolution of

$$|\frac{D^n T}{\partial s^n}|^2.$$

This type of estimate also appears in [32] for the Ricci-Hamilton flow.

First, let us compute the time derivative of

$$|\frac{D^n T}{\partial s^n}|^2$$

as follows:

$$\begin{aligned}
& \frac{\partial}{\partial t}(|\frac{D^n T}{\partial s^n}|^2) \\
= \;& 2 < \frac{D}{\partial t}\frac{D^n T}{\partial s^n}, \frac{D^n T}{\partial s^n} > \\
= \;& 2 < \frac{D}{\partial s}\frac{D}{\partial t}\frac{D^{n-1} T}{\partial s^{n-1}} + k^2 \frac{D^n T}{\partial s^n} + R(T, \frac{DT}{\partial s})\frac{D^{n-1} T}{\partial s^{n-1}}, \frac{D^n T}{\partial s^n} > \\
= \;& 2 < \frac{D}{\partial s}\frac{D}{\partial t}\frac{D^{n-1} T}{\partial s^{n-1}}, \frac{D^n T}{\partial s^n} > +2k^2 |\frac{D^n T}{\partial s^n}|^2 + 2R(T, \frac{DT}{\partial s}, \frac{D^{n-1} T}{\partial s^{n-1}}, \frac{D^n T}{\partial s^n}) \\
= \;& 2 < \frac{D}{\partial s}(\frac{D}{\partial s}\frac{D}{\partial t}\frac{D^{n-2} T}{\partial s^{n-2}} + k^2 \frac{D^{n-1} T}{\partial s^{n-1}} + R(T, \frac{DT}{\partial s})\frac{D^{n-2} T}{\partial s^{n-2}}), \frac{D^n T}{\partial s^n} > \\
+ \;& 2k^2 |\frac{D^n T}{\partial s^n}|^2 + 2R(T, \frac{DT}{\partial s}, \frac{D^{n-1} T}{\partial s^{n-1}}, \frac{D^n T}{\partial s^n}) \\
= \;& 2 < \frac{D^2}{\partial s^2}\frac{D}{\partial t}\frac{D^{n-2} T}{\partial s^{n-2}}, \frac{D^n T}{\partial s^n} > +2 < \frac{D}{\partial s}(k^2 \frac{D^{n-1} T}{\partial s^{n-1}}), \frac{D^n T}{\partial s^n} > +2k^2 |\frac{D^n T}{\partial s^n}|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2 < \frac{D}{\partial s}(R(T, \frac{DT}{\partial s})\frac{D^{n-2}T}{\partial s^{n-2}}), \frac{D^nT}{\partial s^n} > +2R(T, \frac{DT}{\partial s}, \frac{D^{n-1}T}{\partial s^{n-1}}, \frac{D^nT}{\partial s^n}) \\
&= \quad \cdots = 2 < \frac{D^n}{\partial s^n}\frac{D}{\partial t}T, \frac{D^nT}{\partial s^n} > +2\sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i}(k^2\frac{D^{n-i}T}{\partial s^{n-i}}), \frac{D^nT}{\partial s^n} > \\
&+ \quad 2\sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i}(R(T, \frac{DT}{\partial s})\frac{D^{n-1-i}T}{\partial s^{n-1-i}}), \frac{D^nT}{\partial s^n} > .
\end{aligned}
$$

Using Lemma 7.2.3, we get

$$
\begin{aligned}
& \quad 2 < \frac{D^n}{\partial s^n}\frac{D}{\partial t}T, \frac{D^nT}{\partial s^n} > \\
&= \quad 2 < \frac{D^{n+2}T}{\partial s^{n+2}}, \frac{D^nT}{\partial s^n} > +2 < \frac{D^nT}{\partial s^n}(k^2T), \frac{D^nT}{\partial s^n} > \\
&= \quad \frac{\partial^2}{\partial s^2}(|\frac{D^nT}{\partial s^n}|^2) - 2|\frac{D^{n+1}T}{\partial s^{n+1}}|^2 + 2 < \frac{D^nT}{\partial s^n}(k^2T), \frac{D^nT}{\partial s^n} > .
\end{aligned}
$$

So

$$
\begin{aligned}
& \quad \frac{\partial}{\partial t}(|\frac{D^nT}{\partial s^n}|^2) \\
&= \quad \frac{\partial^2}{\partial s^2}(|\frac{D^nT}{\partial s^n}|^2) - 2|\frac{D^{n+1}T}{\partial s^{n+1}}|^2 + 2\sum_{i=0}^{n} < \frac{D^i}{\partial s^i}(k^2\frac{D^{n-i}T}{\partial s^{n-i}}), \frac{D^nT}{\partial s^n} > \\
&+ \quad 2\sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i}(R(T, \frac{DT}{\partial s})\frac{D^{n-1-i}T}{\partial s^{n-1-i}}), \frac{D^nT}{\partial s^n} > .
\end{aligned}
$$

It is easy to see that

$$
\frac{D^i}{\partial s^i}(k^2\frac{D^{n-i}T}{\partial s^{n-i}}) = \sum_{j+k\leq i} \mathcal{C}_{ijk} < \frac{D^{j+1}T}{\partial s^{j+1}}, \frac{D^{k+1}T}{\partial s^{k+1}} > \frac{D^{n-j-k}T}{\partial s^{n-j-k}}, \tag{2}
$$

and

$$
\frac{D^i}{\partial s^i}(R(T, \frac{DT}{\partial s})\frac{D^{n-1-i}T}{\partial s^{n-1-i}}) = \sum_{j+k\leq i} \mathcal{C}_{ijk}R(\frac{D^jT}{\partial s^j}, \frac{D^{k+1}T}{\partial s^{k+1}})\frac{D^{n-1-j-k}T}{\partial s^{n-1-j-k}}, \tag{3}
$$

where the coefficients C_{ijk} are constants. To obtain (3), we have repeatedly used (1).

Noting that M satisfies Condition (Λ), and then putting above equations together, we obtain

$$
\begin{aligned}
& \frac{\partial}{\partial t}(|\frac{D^nT}{\partial s^n}|^2) \leq \frac{\partial^2}{\partial s^2}(|\frac{D^nT}{\partial s^n}|^2) - |\frac{D^{n+1}T}{\partial s^{n+1}}|^2 + C_1|\frac{D^nT}{\partial s^n}|^2 + C_2|\frac{DT}{\partial s}|^2|\frac{D^nT}{\partial s^n}|^2 \\
&+ \quad C_3|\frac{D^2T}{\partial s^2}||\frac{D^nT}{\partial s^n}|^2 + C_4\sum_{0\leq i,j,k<n} |\frac{D^iT}{\partial s^i}||\frac{D^jT}{\partial s^j}||\frac{D^kT}{\partial s^k}||\frac{D^nT}{\partial s^n}|,
\end{aligned}
$$

where C_i are positive constants depending on n and Λ. In the last term, the range of indices satisfies in addition either

$$i + j + k = n + 2$$

or

$$i + j + k = n.$$

Theorem 7.3.1 *Fix $t_0 \in [0, +\infty)$. Let*

$$M_{t_0} =: \max k^2(\cdot, t_0).$$

Assume

$$M_{t_0} < +\infty.$$

Then there exist constants $\tilde{c}_l < +\infty$ independent of t_0 such that for

$$t \in (t_0, t_0 + \frac{1}{2\Lambda} \log(1 + \frac{\Lambda}{4M_{t_0} + \Lambda + 1})],$$

we have

$$|\frac{D^l T}{\partial s^l}|^2 \leq \frac{\tilde{c}_l M_{t_0}}{(t - t_0)^{l-1}}.$$

Proof: Without loss of generality, we may assume that $t_0 = 0$, and then translate the estimates.

(1) For $l = 1$, we have

$$\begin{aligned} \frac{\partial}{\partial t}(|\frac{DT}{\partial s}|^2) &= 2 < \frac{D}{\partial s}\frac{D}{\partial t}T + k^2 \frac{DT}{\partial s} + R(T, \frac{DT}{\partial s})T, \frac{DT}{\partial s} > \\ &= 2 < \frac{D^3 T}{\partial s^3}, \frac{DT}{\partial s} > +4k^4 + 2k^2 R(T, N, T, N) \\ &\leq \frac{\partial^2}{\partial s}(|\frac{DT}{\partial s}|^2) - 2|\frac{D^2 T}{\partial s^2}|^2 + 4k^4 + 2\Lambda k^2 \end{aligned}$$

It follows from the maximum principle that M_t satisfies

$$\log \frac{M_t}{\frac{2}{\Lambda}M_t + 1} - \log \frac{M_0}{\frac{2}{\Lambda}M_0 + 1} \leq 2\Lambda t.$$

If

$$t \leq \frac{1}{2\Lambda} \log(1 + \frac{\Lambda}{4M_{t_0} + \Lambda + 1}),$$

then

$$M_t \leq 2M_0.$$

So we may choose $\tilde{c}_1 = 2$.

(2) For $l=2$, we have

$$\begin{aligned}
\frac{\partial}{\partial t}(|\frac{D^2T}{\partial s^2}|^2) &= 2 < \frac{D}{\partial s}\frac{D}{\partial t}\frac{DT}{\partial s} + k^2\frac{D^2T}{\partial s^2} + R(T,\frac{DT}{\partial s})\frac{DT}{\partial s}, \frac{D^2T}{\partial s^2} > \\
&= 2 < \frac{D^2}{\partial s^2}(\frac{DT}{\partial t}), \frac{D^2T}{\partial s^2} > +2 < \frac{D}{\partial s}(k^2\frac{DT}{\partial s}), \frac{D^2}{\partial s^2} > \\
&+ 2 < \frac{D}{\partial s}(R(T,\frac{DT}{\partial s})T), \frac{D^2}{\partial s^2} > +2k^2|\frac{D^2T}{\partial s^2}|^2 \\
&+ 2R(T,\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^2T}{\partial s^2}) \\
&\le \frac{\partial^2}{\partial s^2}(|\frac{D^2T}{\partial s^2}|^2) - |\frac{D^3T}{\partial s^3}|^2 + 18|\frac{DT}{\partial s}|^2|\frac{D^2T}{\partial s^2}|^2 \\
&+ 2\Lambda|\frac{D^2T}{\partial s^2}|^2 + 2\Lambda|\frac{DT}{\partial s}|^4.
\end{aligned}$$

So

$$\begin{aligned}
& \frac{\partial}{\partial t}(t|\frac{D^2T}{\partial s^2}|^2 + 3|\frac{DT}{\partial s}|^2) \\
\le & \frac{\partial^2}{\partial s^2}(t|\frac{D^2T}{\partial s^2}|^2 + 3|\frac{DT}{\partial s}|^2) - t|\frac{D^3T}{\partial s^3}|^2 + [t(18|\frac{DT}{\partial s}|^2 + 2\Lambda) - 5]|\frac{D^2T}{\partial s^2}|^2| \\
+ & (2\Lambda t + 12)|\frac{DT}{\partial s}|^4 + 6\Lambda|\frac{DT}{\partial s}|^2.
\end{aligned}$$

Since

$$t \le \frac{1}{2\Lambda}\log(1+\frac{\Lambda}{4M_0+\Lambda+1}) \le \frac{1}{2(4M_0+\Lambda+1)},$$

we have

$$\frac{\partial}{\partial t}(t|\frac{D^2T}{\partial s^2}|^2 + 3|\frac{DT}{\partial s}|^2) \le \frac{\partial^2}{\partial s^2}(t|\frac{D^2T}{\partial s^2}|^2 + 3|\frac{DT}{\partial s}|^2) + 52M_0^2 + 12\Lambda M_0.$$

Thus it follows that

$$t|\frac{D^2T}{\partial s^2}|^2 + 3|\frac{DT}{\partial s}|^2 \le 16M_0,$$

and we may conclude on this time interval that

$$|\frac{D^2T}{\partial s^2}|^2 \le \frac{16M_0}{t}.$$

So we may choose $\tilde{c}_2 = 16$.

The induction hypothesis and repeated usage of the Peter-Paul inequality, i.e.,

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2,$$

allow us to find constants a_i and A, B on our time interval such that

$$\frac{\partial}{\partial t}(\sum_{i=1}^{m} a_i t^{i-1}|\frac{D^iT}{\partial s^i}|^2) \le AM_0^2 + BM_0.$$

Thus we obtain $\tilde{c}_m$ as before. □

Note that these estimates prove the long time existence result. That is, as long as the curvature remains bounded on time interval $[0, \alpha)$,one can define a smooth limit for the tangent vector T at time α. Thus,by integrating the tangent vector T, one can obtain a smooth limit curve.

Chapter 8

Appendix D: Selected topics in Nirenberg's problem

We have two aims in this appendix. One is a density result of the prescribed gaussian curvature functions for Nirenberg's problem. This result says that any smooth function, which is positive somewhere, on S^2 can be approximated in $C^{1,\sigma}$ (for any $\sigma < 1$) norm by a sequence of smooth positive somewhere Morse functions, which are the prescribed gaussian curvature functions in Nirenberg's problem. A similar result in the scalar curvature problem on the sphere of higher dimension was already obtained by Yanyan Li [46], and R.Schoen and D.Zhang [48]. The L^p dense result in Nirenberg's problem was obtained by J.P.Bourguignon and J.P.Ezin [42].

The second result concerns the structure of the solution set of the scalar curvature problems (and also Nirenberg's problem) as the prescribed scalar curvature varied in a suitable way. The result shows that there are many more free directions at a scalar curvature function so that we can find more solutions for the near-by prescribed scalar curvature functions. We introduce two methods in studying this problem separately. Although these methods are known to many people, it is not apparent that one can apply them to our problem here. In fact, to formulate a good condition is a difficult and important point. We only formulated a very simple condition to scalar curvature problems and to Nirenberg's problem. See our Main Theorem and Theorem 3 for the statements. We think that we are the pioneers in such a study from the view-point of Catastrophe theory or Bifurcation Theory. Our results in this section are new, but we have obtained them many years ago.

8.1 A $C^{1,\sigma}$ dense result in Nirenberg's problem

We begin with some basic material. Let (D, ds^2)be a 2-dimensional manifold. In local coordinates (u^i), write

$$ds^2 = g = g_{ij}du^i du^j.$$

Let

$$|g| = det(g_{ij}).$$

For the vector field $X = X^j\partial_j$, we define the length of it as

$$|X| = \sqrt{g_{ij}X^iX^j}.$$

We define the gradient operator $\bigtriangledown$ of the metric g as follows. For $f \in C^2(D)$, let

$$\bigtriangledown f = (g^{ij}\frac{\partial f}{\partial u^j}).$$

Hence,

$$|\bigtriangledown f|^2 = g^{ij}\frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}.$$

Define

$$\partial_j = \frac{\partial}{\partial u^j}$$

We define the divergence operator div of g as follows. For the vector field $X = X^j\partial_j$, we let

$$divX = \frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}X^j).$$

For $f \in C^2(D)$, define

$$\triangle f = div(\bigtriangledown f).$$

Hence,

$$\triangle_g f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial u^j})$$

We call $\triangle_g$ the Laplacian operator of (D, g).

In the local moving frame we define the Laplacian operator of (D, g) in the following way:

$$g = (\theta^1)^2 + (\theta^2)^2.$$

Recall the structure equations:

$$d\theta^i = \theta^i_j \wedge \theta^j, d\theta^1_2 = -K\theta^1 \wedge \theta^2.$$

Define

$$du = u_i\theta^i.$$

Taking the exterior differentiation we have

$$0 = du_i \wedge \theta^i + u_i\theta^j \wedge \theta^i_j.$$

Define u_{ij} by

$$u_{ji}\theta^i = du_j - u_i\theta^i_j.$$

By this we have;

$$u_{ji}\theta^i \wedge \theta^j = 0.$$

Hence $u_{ij} = u_{ji}$.
Define

$$\triangle_g u = u_{ii}.$$

Obviously one can define $\triangle_g$ in higher dimensions.
In the isothemal corrdinates

$$g = exp(2f)(dx^2 + dy^2),$$

Let

$$\theta^1 = e^f dx, \theta^2 = e^f dy.$$

It is easy to verify that these two definitions are the same.
In higher dimensions, we can also verify the equivalent of these two definitions.
For $u \in C^2(D)$, let $\hat{g} = exp(2u)g$. Let$k$be the Gauss curvature of the metric g, Let K be the Gauss curvature of the metric $\hat{g}$. Then, we have

Assertion 8.1.1

$$K = exp(-2u)(-\triangle_g u + k).$$

Proof: We use the local moving frame to prove this result. Let

$$g = (\theta^1)^2 + (\theta^2)^2.$$

Recall the structure equation:

$$d\theta^i = \theta^i_j \wedge \theta^j, d\theta^1_2 = -K\theta^1 \wedge \theta^2.$$

Let

$$\hat{\theta^i} = e^u \theta^i.$$

By

$$d\hat{\theta^i} = \hat{\theta^i_j} \wedge \hat{\theta^j}$$

we obtain that

$$\hat{\theta^i_j} = \theta^i_j + u_i\theta^j - u_j\theta^i.$$

So

$$d(\hat{\theta^1_2}) = d(\theta^1_2) + d(u_1\theta^2 - u_2\theta^1).$$

By the equation

$$d\hat{\theta^i_2} = -K\hat{\theta^1} \wedge \hat{\theta^2}$$

we know that

$$-Kexp(2u)\theta^1 \wedge \theta^2 = -k\theta^1 \wedge \theta^2 + du_1 \wedge \theta^2 - du_2 \wedge \theta^1 + u_1 d\theta^2 - u_2 d\theta^1.$$

Note that

$$du_i = u_{ij}\theta^j + u_j\theta_i^j,$$

Hence we have

$$-Kexp(2u)\theta^1 \wedge \theta^2 = -k\theta^1 \wedge \theta^2 + \triangle_g u\theta^1 \wedge \theta^2.$$

□

We now recall the Nirenberg problem in S^2. Let (S^2, c) be the standard unit 2-sphere in Euclidean space R^3. Then Nirenberg's problem is to find functions $K(x)$ on S^2 such that they are the Gaussian curvature functions of metrics g, which are point-wise conformally equivalent to c on S^2. This amounts to finding a smooth function $u : S^2 \to R$ satisfying the following equation on S^2:

$$\Delta u = 1 - Ke^{2u}, \qquad (N).$$

Here Δ is the Laplacian with respect to c. Assume that u is a solution of (N). Then the metric $g = e^{2u}c$ is conformal to c and has K as its Gaussian curvature. Integrating both sides of (N) on S^2, we find

$$\int_{S^2} Ke^{2u} d\sigma = 4\pi.$$

So a necessary condition for (N) to be solvable is that K be positive somewhere and we write such function class by $C^2_+(S^2)$.

We have the following density result.

Theorem 8.1.2 *On S^2 any smooth positive somewhere function can be approximated in $C^{1,\sigma}$-norm by a sequence of Gaussian curvature functions.*

To begin a proof, let's assume the well-known existence result of A.Chang and P.Yang [39], Han [51], and Chang and Liu [52]. The result says that

Theorem 8.1.3 *Assume K is a Morse function on S^2 satisfying $\Delta K(x) \neq 0$ whenever $\nabla K(x) = 0$ and $K(x) > 0$. Let p =number of local maxima and q =number of saddle points with $\Delta K < 0$. If $p \neq q + 1$, then (N) admits at least one solution.*

So we need only to prove that any smooth positive somewhere Morse function can be arbitrarily approximated in $C^{1,\sigma}$-norm by a smooth positive somewhere Morse function K on S^2 with nonzero Laplacian at each critical points, and with $|1+p-q|$ nonzero. Here p is the number of the critical points of K at which $\triangle K < 0$ and at which the Morse index of K is one, and q is the number of local maximum points of K.

The construction of such a function can be carried out as follows. Choose any minimum point x_0 of a given Morse function H. Then by the standard Morse lemma, we know that in some neighborhood of x_0 there is a local coordinate (x^1, x^2) such that our function H can be written as $H(x) = H(0)+(x^1)^2+(x^2)^2$. So we can create a near-by Morse function H_t with only two critical points, one is a local minimum, and the other is a local maximum, in this neighborhood, and H_t is H outside. This H_t has one more maximum point than H. Such a construction was also used in [54] in the scalar curvature problem in the sphere of higher dimensions (so one may see [54] for more detail), and it can be made again at any minimum point of H_t. By repairing this process we may find an approximated Morse function H_t with q as large as we want, but with fixed p. Here we measure the approximation in $C^{1,\sigma}$ norm on S^2. We remark that the Laplacian of H_t at any critical point can be slightly modified to be nonzero by enlarging the scale in just one direction, for example, the x^1-direction. Now Chang-Yang-Han-Chang-Liu's existence result above (see also [39][51][52]) tells us that H_t is the prescribed gaussian curvature in Nirenberg's problem. Hence we get the $C^{1,\sigma}$ density result for Gaussian curvature functions on S^2. Such a result could not be expected to improve to C^2 norm because Morse functions are stable in C^2 function space.

We point out that our argument for the density result depends on the existence result of this geometrical problem. Once we have an existence result for some kind of Morse functions, we can use the above construction to get the density result; for example, we could prove the density result for the scalar curvature problem in dimensions three and four by using the above construction.

8.2 Bifurcations in the scalar curvature problem

Suppose (M^n, g_0) is a compact Riemannian manifold with positive scalar curvature function and $n \geq 3$. Given a smooth function K on it. The *usual scalar curvature problem* on M^n is to find a conformal metric g to g_0 such that g has the scalar curvature function K. We will let $N+1 = (n+2)/(n-2)+1$ be the Sobolev exponent. Let $L = -c_n \triangle +R_g$ be the conformal Laplacian so that the first eigenvalue of it is positive, where $c_n = 4(n-1)/(n-2)$. Suppose K_0 is the scalar curvature function of the metric $u_0^{4/n-2} g_0$, where u_0 is some positive smooth function on M.

When studying the scalar curvature problem on S^n, $n \geq 3$, we want to understand the structure of the solution set. Recall that the scalar curvature problem on S^n is to find a pair $(u, K) \in C^2_+(S^n) \times C^2(S^n)$ such that the function K is the scalar curvature function of the metric $u^{\frac{4}{n-2}} c$, where c is the standard metric on S^n. If the problem is non-degenerate, then we can try to do the Morse theory (see the famous work of A.Bahri and J.M.Coron [40]) or use the standard implicit function theorem. We will do some work in this part if it is not in the above case. So we are led to the situation where there occurs a “bifurcation”. Following the general direction of the lecture note of L.Nirenberg [38], that is, using the Lyapunov-Schmidt process to reduce the problem to a finite dimensional one, we try to use the “standard bifurcation theory” and “catastrophe theory” of Thom-Mather for our problem. The difficulty lies in finding good conditions on the prescribed function such that we can use the singularity theory. Our study is motivated by our reading of the works of A.Bahri [54], A.Bahri and J.M.Coron [40], A.S.Y.Chang and P.Yang [39] and trying to get some solution of small Morse index.

Suppose (M^n, g_0) is a compact Riemannian manifold with positive scalar curvature function and $n \geq 3$. We study the scalar curvature problem on M. Our main concern in this part is to find some condition on K_0 such that there is a class of functions near K_0 being scalar curvature functions of conformal metrics of g_0. In short, we call such a function a scalar curvature function. It is clear, if K is a composition of a conformal diffeomorphism with K_0, then it is a scalar curvature function. So, in the sense of trying to use the "catastrophe theory", our problem may be considered as finding a suitable condition on K such that K is a composition of a conformal diffeomorphism with our K_0. Note in catastrophe/singularity theory, they use coordinates substitution to get a normal form. Hence we can not apply the catastrophe/singularity theory directly. However, we may first consider this problem from the nonlinear functional analysis point of view. Then we will reduce the problem to a finite dimension one.

For simplicity, using a change of conformal metric, we may assume at this moment that $u_0 = 1$.

Let $N(u, K)$ be the operator $-Lu + K(p)u^N$ from a second order Holder space $(C^{2,\alpha}(M))$ times another Holder space $(C^{0,\alpha}(M))$ to a Holder space $(C^{0,\alpha}(M))$. If the linearized operator $T = N'_u(u_0, K_0)$ is a surjective between the tangent spaces (so it is an isomorphism by Fredholm alternativity), then by the implicit function theorem, we know every function near K_0 is a scalar curvature function. So we assume the kernel of T is a finite dimensional vector space $E = sp(v_1, ..., v_d)$, where each v_i has unit L^2-norm. In fact, this is the only case which is of geometrical interest. If we introduce the operator $F(u, K) = N(u, K) + P(u - u_0)$, where $P(u-u_0)$ is the L^2 projection of $u-u_0$ to E, then $F_u(u_0, K_0)$ is an isomorphism and we can safely use the implicit function theorem. So we find neighborhoods G_0 and G_2 of 0, and a neighborhood G_1 of K_0 in the Holder spaces, and a smooth function $u = u(f, K)$ from $G_0 \times G_1$ to G_2 such that $u(f, K)$ satisfies:

$$F(u(f, K), K) = f. \qquad (D1)$$

Furthermore, our map $u(., K)$ is a diffeomorphism from G_0 to G_2 for every $K \in G_1$. Hence, obtaining a solution for the scalar curvature equation is equivalent to obtaining a solution of the equation $P(u(f, K) - u_0) = f$ on the intersection of E with G_0 if K is in G_1. Now, it seems that for every $K \in G_1$, we have a $d-$family of solutions $u(f, K)$, where f is the parameter with range as the intersection of G_2 with E, of our scalar curvature problem. So this is the local structure of the solution set of the scalar curvature problem if the prescribed scalar curvature function is near K_0 in the above sense. In fact, this is not true because of the Kazdan-Warner condition. For example, if we take

$$M = S^n = \{x = (x_1, ..., x_{n+1} \in R^{n+1}; |x| = 1\}$$

be the standard sphere and take $K = 1 + \epsilon x_1$, then the scalar curvature problem has no solution.

In the following paragraph, we want to learn more about solving our system (D1) from the view-point of singularity theory. For this purpose, we write

$$f = \sum x_i v_i, \quad x = (x_1, ..., x_d),$$

and $u(x, K) = u(\sum x_i v_i, K)$. Hence we need only to solve the nonlinear system

$$< u(x, K) - u_0, v_i >= x_i.$$

for every $i = 1, ..., d$. This kind of reduction is well-known, see section two of L.Simon [33], for example. We remark that the key to solving this problem lies in understanding the map $u(x, K)$. But it is very difficult to achieve such a understanding. Let's go back to solving the nonlinear system.

So to understand this system, we consider the Taylor expansion of the function $u(f, K)$ near $(0, K_0)$. First note that (u_0, K_0) is in the zero set of $F(u, K)$. By the implicit function theorem we have $u(0, K_0) = u_0$ and $u_{x_i}(0, K_0) = 1$ for every i.

Now, we take the derivatives in the equation (1) and find

$$-Lu_x + NK(p)u^{N-1}u_x + P(u_x) = v,$$

and

$$-Lu_K + \delta K(p)u^N + NK(p)u^{N-1}u_K + P(u_K) = 0.$$

From this we find that at (u_0, K_0),

$$< \delta K(p), v_i > +N < Ku_K, v_i > + < u_K, v_i >= 0.$$

Let's compute the second derivatives and we find that

$$-Lu_{xx} + NK(p)u^{N-1}u_{xx} + N(N-1)K(p)u^{N-2}u_x^2 + P(u_{xx}) = 0,$$

$$-Lu_{xK} + NK(p)u^{N-1}u_{xK} + N\delta K u^{N-1}u_x$$

$$+N(N-1)K(p)u^{N-2}u_x u_K + P(u_{xK}) = 0,$$

and

$$-Lu_{KK} + NK(p)u^{N-1}u_{KK} + \delta^2 K(p)u^N$$

$$+N\delta K(p)u^{N-1}u_K +$$

$$N(N-1)K(p)u^{N-2}u_K^2 + P(u_{KK}) = 0.$$

So at (u_0, K_0), we have:

$$N(N-1) < Ku_x^2, v > + < u_{xx}, v >= 0, \qquad (D2)$$

$$N < \delta K u_x, v > + N(N-1) < K u_x u_K, v > + < u_{xK}, v >= 0,$$

and

$$\begin{aligned} & < \delta^2 K, v > + N < \delta K u_K, v > \\ + \quad & N(N-1) < K u_K^2, v > + < u_{KK}, v >= 0. \end{aligned}$$

Again taking the third derivatives, we find more relations at (u_0, K_0). But these relations are not useful to us.

Now going back to our finite dimensional problem and developing the Taylor expansion at (u_0, K_0) we find that

$$\begin{aligned} 0 \quad = \quad & 2 < u_K, v_i > + < u_{xx}, v_i > x^2 + < u_{xK}, v_i > x \\ + \quad & < u_{KK}, v_i > + h.o.t.. \end{aligned}$$

We can consider the right hand as a perturbational system of the first two terms because we can let $\delta K = t^2 h$ and $x = ty$ with $|t|$ suitable small. Then we get

$$0 = 2 < u_K h, v_i > + < u_{xx}, v_i > y^2 + t < u_{xK} h, v_i > y$$

$$+ t^2 < u_{KK} h.h, v_i > + h.o.t.. \qquad (D3)$$

So it is a small perturbation of a "quadratic" equation, and by using (D2) we can conclude:

Theorem 8.2.1 *Assume $d = 1$.*

(i) Suppose there is a smooth function h such that $A := < K u_x^2, v >< u_K.h, v >$ is nonzero. Then by (D2) and (D3), either $A < 0$, we have two solutions of the scalar curvature problem for such K, or $A < 0$, we have none.

(ii) Suppose there is a smooth function h such that $< u_K.h, v >= 0$ but $B := < K u_x^2, v >< u_{xK} h, v >$ is nonzero, Then by (D2) and (D3) we are in the standard Whitney fold case. This means, either $B > 0$, and we have two solutions of the scalar curvature problem for such K, or $B < 0$, we have none.

If $d = 1$ and $B = 0$, we need to consider the third derivatives. See M.S.Berger [54] or Martin Golubitsky and David G.Schaeffer [47] for discussion. We further theorize that Theorem 2 (above) is only useful in compact Riemannian manifolds which are not conformally diffeomorphic to the standard sphere.

In the remaining paragraph, we discuss the case when $M = S^n$ equipped with the standard metric and $K = n(n-1) + H$ with H is small in the Holder space. Here if we take $u_0 = 1$ again, then we have our $d = (n+2)(n+1)/2$ because of the conformal group on S^n. So to obtain some interesting result we need some facts from the conformal geometry on S^n. Now we discuss the conformal Killing vector fields on S^n first (see [42]. It is well known that any vector $w \in R^{n+1}$ gives a conformal Killing field W on S^n by

$$W(v) = w - (w.v)v$$

and this W is the projection of w onto the tangent space of S^n at v. Conformal Killing vector fields on S^n form a Lie group, which is isomorphic to the Lie algebra $o(n+1,1)$ of the group $0(n+1,1)$ of all linear isometries of Minkowski $(n+2)$-space. By definition $\{v_1, ..., v_d\}$ is the basis of $o(n+1,1)$. Given a conformal diffeomorphism $F : S^n \to S^n$. Then we have $F^*(u^{N-1}c) = T_F(u)^{N-1}c$ with $T_F(u) = |F'|^{\frac{n-2}{2}} u \circ F$. Note that $F^*(c) = |F'|^2 c$. This T_F is an isometry of the Hilbert space $H^1(S^n)$. Let Σ be the positive part of the sphere of $H^1(S^n)$ containing $u_0 = 1$. Note the orbit of $u_0 = 1$ in Σ under the conformal group action is $\{u_F := |F'|^{\frac{n-2}{2}}\}$. More explicitly, u_F are of the forms

$$u_F(w) = d_n\left(\frac{\lambda}{\lambda^2 + 1 + (\lambda^2 - 1) cosd(a, w)}\right)^{(n-2)/2}$$

where d_n is a uniform constant, d is the geodesic distance on (S^n, c), $a \in S^n$, and $\lambda > 0$. For such a function, we will write $u_{a,\lambda} = u_F$.

Let

$$J_K(u) = \frac{1}{N+1} \int_{S^n} K u^{N+1} d\mu.$$

Then it is well-known that the critical points of J_K on Σ uniquely correspond with the solutions of the problem of prescribing scalar curvature function K on S^n. As shown by A.Bahri (see [15]) and O.Rey (see the appendix D of [17]) that every nontrivial u_F is non-degenerate on the orthogonal part $E^\perp$ of the subspace E of $H^1(S^n)$ with

$$E = span\{D_1, ..., D_n, D_\lambda\}.$$

Here D_i is the Lie derivative of u_F along the conformal vector field generated by the orthonormal complement e_i of $a \in S^n$ (so $\{a, e_1, ..., e_n\}$ forms a basis of R^{n+1}) and D_λ is the Lie derivative of u_F along the scale λ.

Now, we discuss the bifurcation at the point u_F if $H = t.h$ with t being a small scalar and h being a suitable function. We assume without loss of generality that the C^2 norm of h is not bigger than one. Because, for $t = 0$, u_F is a non-degenerate critical point of J_K on $E^\perp$, we can use the implicit function theorem to find a positive constant ϵ such that for $|t| < \epsilon$ there is a solution curve $u(t) = u_F(t)$ of $dJ_K|_{exp(E^\perp)}(u) = 0$ for any smooth function h. Here exp is the exponential map of $H^1(S^n)$. Now we will find the conditions on h such that

$$dJ_K|_{exp(E)}(u(t)) = 0. \qquad (D*)$$

This means that $u(t)$ is the solution of the problem of prescribing scalar curvature function $K = n(n-1) + t.h$. The easy condition for solving $(D*)$ is another (higher order) non-degenerate condition for using the implicit function theorem again. Let

$$j(t,y) = J_{K(t)}(u(t,y)).$$

Here $y = (a,\lambda) \in S^n \times R_+$ are the parameters for u_F and we write $u(t,y) = u_F(t)$. Note $u(0,y) = u_F(0) = u_F$. Then we are looking for the conditions for h such that $u(t,.)$ is a critical point of $j(t,.)$ for each small t. Note, by the Lagrange multiplier method we find that there are some constants $l_1,...,l_{n+1}$ such that

$$dJ_K(u(t,y)) = l_1 D_1 + ... + l_n D_n + l_{n+1} D_\lambda.$$

In this form we are in the early consideration before Theorem 2. Therefore $j_y(t,y) := d_y j(t,y) = 0$ is equivalent to $l_i = 0$ for all $i = 1,...,n+1$. By construction, $\int_{S^n} u_F^{N+1} d\mu$ and we have $j_y(0,y) = 0$. Hence we obtain by using Taylor's expansion of j at $t = 0$ that

$$j_y(t,y) = t d_t j_y(0,y) + 0(t^2).$$

So, if our functions $d_t j_y(0,y)$ has simple zero at $y = y_0$, then we get a solution $u(t,y)$ for the problem of prescribed scalar curvature $K = n(n-1) + t.h$ by the standard bifurcation method. In general, one should use the Thom-Mather theory to study the zero set of $j_y(t,y)$. We leave this open at this moment and perhaps we will return to it later. Because $u_F(0,y)$ is a critical point of J_K at $t = 0$ and $u_F(t,y)$ is a curve in Σ we obtain that

$$\int_{S^n} u_F^N (u_F)_t(0,y) d\mu = 0.$$

Now, by a direct computation, we find that

$$\begin{aligned} d_t j_y(0,y) &= d_y d_t j(0,y) \\ &= \int_{S^n} h u_F^N (u_F)_y d\mu + d_y \int_{S^n} u_F^N (u_F)_t(0,y) d\mu, \\ &= \int_{S^n} h u_F^N (u_F)_y d\mu. \end{aligned}$$

Hence, we get the following result.

Theorem 8.2.2 *If $\{u = u_F = u_{a,\lambda}\}$ is a family of non-constant Yamabe solutions and h is a smooth function on S^n such that for the orthonormal basis $\{a, e_1, ..., e_n\}$, the vector-valued function*

$$\{\int_{S^n} h u^p u_w d\mu; w = \lambda, e_1, ..., e_n\}$$

has a simple zero or other standard zero point (a_0, λ_0) in the bifurcation theory or Catastrophe theory, then there is a positive constant ϵ such that the problem of prescribed scalar curvature function $n(n-1) + t.h$ has a solution, which is near to u_{a_0,λ_0} for each small $t : |t| < \epsilon$.

We remark here that one can also introduce higher order non-degeneracy if $\int_{S^n} hu^N u_y d\mu$ is trivial. And the other remark is that this kind of argument can also be used to study bifurcation in Nirenberg's problem. The result is

Theorem 8.2.3 *Suppose* $u_F = \frac{1}{2} log|J_F|$ *is a nontrivial, where* F *is a conformal transformation* S^2*. If* h *is a smooth function such that* $\int_{S^2} he^{2u_F}(u_F)_y d\mu$ *has a simple zero, where* y *is the parameter for the conformal group on* S^2*, then there is a small positive constant* ϵ *such that Nirenberg's problem of prescribed gaussian curvature* $1 + t.h$ *has a solution for each small* $t : |t| < \epsilon$*.*

For the detail of our proof of this Theorem, one may see [53]. For more about the Yamabe problem, one may see Th.Aubin's book [3].

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