Lie symmetry groups of high dimensional non-integral nonlinear systems

Hongcai Ma[1*], Senyue Lou[2,3], and Aiping Deng[1]

1.Department of Mathematics, Donghua University, Shanghai, 200051, P. R. China

2. Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, P. R. China

3. Center of Nonlinear Science, Ningbo University, Ningbo, 315211, P. R. China

E-mail: hongcaima@hotmail.com

Abstract. A modified direct method is developed to find finite symmetry groups of nonlinear mathematical physics systems. Applying the modified direct method to the well known (3+1)-dimensional KP equation and the (2+1)-dimensional KdV equation. The Lie symmetry groups obtained via traditional Lie approaches are only special cases. Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches.

1. Introduction

The symmetry study plays a very important role in almost every branch of natural science especially in integrable systems for the existence of infinitely many symmetries. Lie's theory [1] gives us a standard method to find the Lie point symmetry group of a nonlinear system. From then on, the standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known integrable systems. However, it is very difficult to find non-Lie point symmetry groups and there are few literatures for the topic. And the final known expressions of the Lie point symmetry groups obtained by using the standard method may be quite complicated and difficult to real applications especially for physicists and other non-mathematical scientists.

Fortunately, the so called CK direct method which was first introduced by Clarkson and Kruskal (CK) [2] can be used to derive symmetry reductions of a nonlinear system without using any group theory. The CK's method can be used for most types of nonlinear systems to find ALL the possible similarity reductions. The fact means that there is a simple method to find generalized symmetry groups for many types of nonlinear systems.

In the reference [3], we have modified the CK's direct method to find the generalized Lie and non-Lie symmetry groups of the (2+1)-dimensional Kadomtsev-Petviashvili equation and the Davey-Stewartson system. We believe that the method will be valid for other nonlinear systems.

In the section 2 of this paper is an simple outline of the modified method. The section 3 is used to find generalized Lie symmetry groups of the known (3+1)-dimensional KP equation. The general

[*] To whom any correspondence should be addressed.

symmetry groups of the (2+1)-dimensional KdV equation are given in section 4. The last section is a short summary and discussion.

2. A direct method to derive symmetries of nonlinear system

The main idea of the CK's direct method is to seek a reduction of a given partial differential equation (PDE) in the form

$$u(x_1,\ x_2,\ ...,\ x_n) = W(x_1,\ x_2,\ ...,\ x_n,\ U(z_1,\ z_2,\ ...,\ z_m)), \tag{1}$$

$$(z_i = z_i(x_1,\ x_2,\ ...,\ x_n),\ i = 1,\ 2,\ ...,\ m < n)$$

which is the most general form for a similarity reduction [4]. For a given PDE

$$F(x_i,\ u,\ u_{x_i},\ u_{x_i x_j},\ ...,\ i,\ j = 1,\ 2,\ ...,\ n) = 0, \tag{2}$$

substituting (1) into (2) and demanding that the result be a lower dimensional partial (or ordinary) differential equation for $U(z_1,\ z_2,\ ...,\ z_m)$ imposes conditions upon $W,\ z_i$ and their derivatives that enable one to solve for W and z_i.

In fact, for many real physical systems, it is enough to seek the symmetry reductions in a simple form

$$u(x_1,\ x_2,\ ...,\ x_n) = \alpha(x_1,\ x_2,\ ...,\ x_n) + \beta(x_1,\ x_2,\ ...,\ x_n)U(z_1,\ z_2,\ ...,\ z_m) \tag{3}$$

instead of the general form (1).

The CK's direct symmetry reduction method implies that it is possible to find full symmetry groups of (3+1)-dimensional equation and (2+1)-dimensional KdV equation by a simple direct method without using any group theory. It is quite easy to realize this idea. The only thing one has to do is to substitute

$$u(x_1,\ x_2,\ ...,\ x_n) = W(x_1,\ x_2,\ ...,\ x_n,\ U(X_1,\ X_2,\ ...,\ X_n)), \tag{4}$$

$$(\ X_i = X_i(x_1,\ x_2,\ ...,\ x_n),\ i = 1,\ 2,\ ...,\ n)$$

into a given PDE (2) and require U satisfies the same PDE under the transformation

$$\{u,\ x_1,\ ...,\ x_n\} \to \{U,\ X_1,\ ...,\ X_n\}$$

Generally, it is enough to suppose the group transformation has some simple forms, say,

$$u(x_1,\ x_2,\ ...,\ x_n) = \alpha(x_1,\ x_2,\ ...,\ x_n) + \beta(x_1,\ x_2,\ ...,\ x_n)U(X_1,\ X_2,\ ...,\ X_n) \tag{5}$$

instead of (4) for various nonlinear systems.

3. Transformation group of the (3+1)-dimensional KP equation

Firstly, we take the (3+1)-dimensional KP equation as a simple illustration model to show the validity of the method. The (3+1)-dimensional KP equation

$$u_{xt} + 6u_{xx}u + 6u_x^2 + u_{xxxx} + 3u_{yy} + 3u_{zz} = 0, \tag{6}$$

which describes the dynamics of solitons and nonlinear waves in plasmas and superfluids is the generalization of (2+1)-dimensional KP equation. If u is z-independent, equation (6) is completely integrable and then many kinds of solutions can be obtained by different kinds of methods.

At first, let

$$u = \alpha + \beta U(\xi,\ \eta,\ \tau), \tag{7}$$

where $\alpha,\ \beta,\ \xi,\ \eta,\ \tau$ are the functions of $x,\ y,\ t$.

Substituting (7) into (6) and requiring $U(\xi,\ \eta,\ \tau)$ being also a solution of the (3+1)- dimensional KP equation (6) but with the independent variables (eliminating U_{ξ^4} by means of the (3+1)-dimensional KP equation), we have

$$4\beta\eta_x^3\tau_x U_{\tau\eta^3}+4\beta\eta_x^3\zeta_x U_{\zeta\eta^3}+12\beta\xi_x^2\eta_x\zeta_x U_{\zeta\xi^2\eta}+4\beta\eta_x\zeta_x^3U_{\zeta^3\eta}+4\beta\xi_x^3\eta_x U_{\xi^3\eta}$$

$$+4\beta\xi_x^3\zeta_x U_{\zeta\xi^3}+6\beta\xi_x^2\zeta_x^2U_{\zeta^2\xi^2}+F(x,\ y,\ t,\ U,\ U_\xi,\dots)=0, \tag{8}$$

where $U_{\xi^n}=\partial_\xi^n U,\ U_{\eta^n}=\partial_\eta^n U,\ U_{\zeta^n}=\partial_\zeta^n U$ and F is the complexity function of U and independent of $U_{\tau\eta^3},\ U_{\zeta\eta^3},\ U_{\zeta\xi^2\eta},\ U_{\zeta^3\eta},\ U_{\xi^3\eta},\ U_{\zeta\xi^3},\ U_{\zeta^2\xi^2}$. (8) means

$$\tau_x=0,\ \eta_x=0,\ \zeta_x=0.$$

after tedious calculation, we have

$$\tau=c_1^3t+C_2,\ \beta=c_1^2, \tag{10}$$

$$\xi=c_1x-\tfrac{1}{6c_1}\left[\left(C\eta_{0t}-\sqrt{1-C^2}\zeta_{0t}\right)y+\left(C\zeta_{0t}+\sqrt{1-C^2}\eta_{0t}\right)z\right]+\xi_0(t), \tag{11}$$

$$\eta=Cc_1^2y+c_1^2\sqrt{1-C^2}z+\eta_0, \tag{12}$$

$$\zeta=Cc_1^2z-c_1^2\sqrt{1-C^2}y+\zeta_0, \tag{13}$$

$$\alpha=\tfrac{1}{36c_1^2}\left[\sqrt{1-C^2}(z\eta_0-y\zeta_0)+C(y\eta_0+z\zeta_0)\right]_{tt}-\tfrac{1}{72c_1^4}\left[\eta_{0t}^2+\zeta_{0t}^2+c_1^3\xi_{0t}\right], \tag{14}$$

where $c_1,\ c_2$ and C are constant $\xi_0\equiv\xi_0(t),\ \eta_0\equiv\eta_0(t)$ and $\zeta_0\equiv\zeta_0(t)$ are all arbitrary functions of t. Obviously we have the following theorem:

Theorem 1. If $u=u(x,\ y,\ z,\ t)$ is a solution of (3+1)-dimensional KP equation (6), then

$$u_1=\alpha+\beta u(\xi,\eta,\zeta,\tau) \tag{15}$$

Where $\tau,\xi,\eta,\zeta,\alpha,\beta$ are decided by (10)–(14) is also the solution of equation (6). If the arbitrary functions and constants in (10)–(14) are selected as

$$C=1-\tfrac{1}{2}\epsilon^2r_1^2,\ c_1=1+3\epsilon t_1,\ c_2=\epsilon t_0, \tag{16}$$

$$\eta_0(t)=\epsilon y_0(t),\ \zeta_0(t)=\epsilon z_0(t),\ \xi_0(t)=\epsilon x_0(t) \tag{17}$$

where $\{r_1,\ s_1,\ t_0\}$ are arbitrary constant and $\{x_0(t),\ y_0(t),\ z_0(t)\}$ are arbitrary functions of t, the general Lie point symmetry of (3+1)-dimensional KP equation is obtained:

$$\sigma\equiv C_1\{zu_y-yu_z\}+t_0u_t+t_1\{xu_x+2yu_y+2zu_z+3tu_t+2u\}$$

$$+\left\{x_0(t)u_x-\tfrac{1}{6}x_{0t}(t)\right\}+\left\{y_0(t)u_y-\tfrac{1}{6}yy_{0t}(t)u_x+\tfrac{1}{36}yy_{0tt}(t)\right\}$$

$$+\left\{z_0(t)u_z-\tfrac{1}{6}zz_{0t}(t)u_x+\tfrac{1}{36}zz_{0tt}(t)\right\}.$$

4. Transformation group of the (2+1)-dimensional potential KdV equation

In this section, we will discuss (2+1)-dimensional potential KdV equation

$$v_t - v_{xxy} - 4vv_y - 4v_x\partial_x^{-1}v_y = 0, \tag{18}$$

where ∂_x^{-1} is a operator of indefinite integrate dx, it has a single dromion solution driven by not only two perpendicular line ghost solitons [5], but also one ghost straight line soliton and one ghost curved line soliton[6]. If we choose $v = u_x$, then (18) have the following potential form,

$$u_{xt} - u_{xxxy} - 4u_x u_{xy} - 4u_{xx}u_y = 0$$

The general form

$$u_{xt} - u_{xxxy} + au_x u_{xy} + bu_{xx}u_y = 0, \tag{19}$$

has been studied by many authors [7]. Equation (19) is a generalization of the shallow water wave (SWW) equation.

$$u_{xxxt} + au_x u_{xt} + bu_{xx}u_t - u_{xt} - u_{xx} = 0$$

It was obtained in the classical shallow water theory by using Boussinesq approximation method [8]. There exits two special integrable cases $a = 2b$ and $a = b$. However, in [7], Clarkson and Mansfield found though SWW is integrable both for $a = 2b$ and $a = b$, yet (2+1)-dimensional equation (19) has the Painlev´e property only for $a = 2b$. We just discuss the non-integral one, say, we choose $a = 2b$ in equation (19).

Assume

$$u = \alpha + \beta U(\xi,\ \eta,\ \tau) \tag{20}$$

where $\alpha,\ \beta,\ \xi,\ \eta,\ \tau$ are all function of $x,\ y,\ t$, and substitute (20) into (19)with restricting U satisfy the same equation with (19) :

$$U_{\xi\tau} - U_{\xi\xi\xi\eta} + aU_\xi U_{\xi\eta} + bU_{\xi\xi}U_\eta = 0 \tag{21}$$

Eliminating $U_{\xi\tau}$ by means of (21), we have,

$$-\beta\tau_x^2(\xi_y\tau_x + 3\xi_x\tau_y)U_{\xi^7\eta^3} - 3\beta\xi_x\eta_x^2\tau_x U_{\xi^3\eta^3} - 3\beta\xi_x^2\eta_x\tau_y U_{\xi^4\eta_2}$$

$$-\beta\xi_x^3\xi_y U_{\xi^4} - \beta\eta_x^3\eta_y U_{\eta^4} - \beta_y\xi_x^3 U_{\xi^3} + F(x,\ y,\ t,\ U_\xi, \ldots) = 0. \tag{22}$$

where $U_{\xi^n} = \partial_\xi^n U,\ U_{\eta^n} = \partial_\eta^n U$ and $F(x,\ y,\ t,\ U_\xi, \ldots)$ is the complex function of $x,\ y,\ t,\ U$ and derivative of U, and independent of $U_{\xi^7\eta^3},\ U_{\xi^3\eta^3},\ U_{\xi^4\eta_2},\ U_{\xi^4},\ U_{\eta^4}$ and U_{ξ^3}. Equation (22) means

$$\tau_x = 0,\ \tau_y = 0,\ \xi_y = 0,\ \eta_x = 0,\ \beta_y = 0 \tag{23}$$

Substituting (23) into (19) and calculating step by step, we arrive,

$$\tau = C_2 t + C_5,\ \xi = C_1 x + \xi_0(t),\ \eta = \frac{C_2 y}{C_1^2} + C_3 t + C_4, \tag{24}$$

$$\alpha = \frac{C_1^2 x C_3}{bC_2} + \frac{y\xi_{0t}(t)}{aC_1} + F_2(t),\ \beta = C_1.$$

and the following finite transport theorem:

Theorem 2. If $u = u(x,\ y,\ t)$ is a solution of the (2+1)-dimensional potential KdV equation (19), then

$$u_1 = \frac{C_1^2 x C_3}{bC_2} + \frac{y\xi_{0t}(t)}{aC_1} + F_2(t) + C_1 u(\xi,\ \eta,\ \tau)$$

With (24) is also a solution. If the arbitrary function and constant in theorem 2 have the following form:

$$\xi_0(t)=\epsilon f(t),\ F_2(t)=\epsilon h(t),\ C_1=1+\epsilon c_1,\ C_2=1+\epsilon c_2,\ C_3=\epsilon c_3,\ C_4=\epsilon y_0,\ C_5=\epsilon t_0$$

Where ϵ is the infinitesimal parameter, we obtain the general Lie point symmetry:

$$\sigma\equiv h(t)+[f(t)u_x+\tfrac{y}{a}f_t]+c_1[xu_x-2yu_y+u]$$

$$+c_2(yu_y+tu_t)+c_3\left(tu_y+\tfrac{x}{b}\right)+t_0u_t+y_0u_y.$$

5. Summary and discussions

In summary, the Lie point symmetry group of many nonlinear systems can be generated by some types of simple direct ansatz. If all the symmetry reductions of a nonlinear system can be obtained by the CK's direct method, then at least the full Lie point symmetry group of the model can be obtained by the simple ansatz (5). Furthermore, after some concrete analysis we find that for the single component models, such as the KP equation, the KdV equation, the Boussinesq equation etc., the ansatz (5) is also enough to find the general non-Lie symmetry groups. While for the multi-component nonlinear systems, some minor modifications are needed. For instance, for the (2+1)-dimensional NNV system, the basic ansatz (5) has to be modified in order to find its generalized non-Lie symmetry group. The similar property is valid for other types of known (2+1)-dimensional integrable models such as the Kadomtsev-Petviashvili equa-tion and the Davey-Stewartson system [3].

Though the generalized Lie point symmetry group of the nonlinear systems can be obtained from the standard Lie algebra theory, the final expressions obtained by means of the simple direct method proposed in this letter are much simpler than those obtained from the traditional approaches.

The author is in debt to thank the helpful discussions with the Drs. X. Y. Tang, X.Q. Wu. The work was supported by the National Outstanding Youth Foundation of China (No 19925522) , the National Natural Science Foundation of China under Grant (No 90203001, No 10475055, No 10647112) and the Foundation of Donghua University.

References

[1] P. J. Olver 1986 Application of Lie Group to Differential Equation(New York: Springer). G. W. Bluman and S. C. Anco 2002 Symmetry and Integration Methods for Differential Equations, Appl. Math. Sci. Vol. 154, (New York: Springer).
[2] P. A. Clarkson and M. Kruskal 1989 J. Math. Phys. **30** 2201
[3] S. Y. Lou and H. C. Ma 2005 J. Phys. A: Math. Gen. **38**, L129
[4] G. W. Bluman and J. d. Cole 1974 Similarity Methods for Differential Equations, Applied Mathematical Sciences, ol. 13(Berlin: Springer)
[5] R.Radha, and Lakshmanan 1995 Phys. Lett. A **197**, 7
[6] S.Y.Lou 1996 Commun.Theor.Phys. **26**, 487
[7] P.A.Clarkson and E.L. Mansfield 1995 Acta Appl. Math.39, 245 ;E. L. Mansfield and P.A. Clarkson 1997 Math. Comput. Simul. **43,** 39
[8] Espinosa and J. Fujioka 1994 J. Phys. Soc. Jpn. **63**, 1289