Linear Connections on Fuzzy Manifolds

J. Madore

*Laboratoire de Physique Théorique et Hautes Energies**
Université de Paris-Sud, Bât. 211, F-91405 ORSAY

Abstract: Linear connections are introduced on a series of noncommutative geometries which have commutative limits. Quasicommutative corrections are calculated.

* *Laboratoire associé au CNRS* URA D0063

1 Introduction and Motivation

It is possible that the representation of space-time by a differential manifold is only valid at length scales larger than some fundamental length and that on smaller scales the manifold must be replaced by something more fundamental. One alternative is a noncommutative geometry. If a coherent description could be found for the structure of space-time which was pointless on small length scales, then the ultraviolet divergences of quantum field theory could be eliminated. In fact the elimination of these divergencies is equivalent to course-graining the structure of space-time over small length scales; if an ultraviolet cut-off Λ is used then the theory does not see length scales smaller than Λ^{-1}. It is also believed that the gravitational field could serve as a universal regulator, a point of view which can be made compatible with noncommutative geometry by supposing that there is an intimate connection between (classical and/or quantum) gravity and the noncommutative structure of space-time. To compare the two it is necessary to have a valid definition of a linear connection in noncommutative geometry. There have been several examples given of differential calculi on noncommutative geometries (Connes 1986, Dubois-Violette 1988, Wess & Zumino 1990). Recently a general definition of the noncommutative equivalent of a linear connection has been proposed in noncommutative geometry which makes full use of the bimodule structure of the space of 1-forms (Dubois-Violette & Michor 1995, Mourad 1995). It has been applied to the quantum plane (Dubois-Violette *et al.* 1995) and to matrix geometries (Madore *et al.* 1995).

A differential manifold can always be imbedded in a flat euclidean space of sufficiently high dimension and a linear (metric) connection on the manifold can be considered as defined by the imbedding in terms of the standard flat connection in the enveloping space. We shall show that noncommutative approximations to a large class of differential manifolds can be obtained by a similar procedure and corresponding linear connections can be constructed as a restriction of the unique metric connection on the enveloping matrix geometry. In the limit, when the length parameter which determines the noncommutativity tends to zero, first-order corrections to the commutative linear connection can be calculated. It is these terms which must eventually be compared with the quasiclassical corrections to the connection in quantum gravity.

Some basic formulae from previous articles are given in this Section and in Section 2 a basic universal linear connection is introduced from which linear connections can be constructed in a way similar to that in which connections can be induced on an ordinary manifold when it is imbedded in a flat space of higher dimension. The quasicommutative limit is considered in Section 3.

Let V be a differential manifold and $\mathcal{C}(V)$ the algebra of smooth functions on V. For simplicity we suppose V to be parallelizable and we choose θ^α to be a globally defined moving frame on V. Let $(\Omega^*(V), d)$ be the ordinary differential calculus on V. A linear connection on V can be defined as a connection on the cotangent bundle to V. It can be characterized as a linear map

$$\Omega^1(V) \xrightarrow{D} \Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V) \tag{1.1}$$

which satisfies the condition

$$D(f\xi) = df \otimes \xi + fD\xi \tag{1.2}$$

for arbitrary $f \in \mathcal{C}(V)$ and $\xi \in \Omega^1(V)$.

The connection form $\omega^\alpha{}_\beta$ is defined in terms of the covariant derivative of the moving frame:

$$D\theta^\alpha = -\omega^\alpha{}_\beta \otimes \theta^\beta. \tag{1.3}$$

Let π be the projection of $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ onto $\Omega^2(V)$. The torsion form Θ^α can be defined as

$$\Theta^\alpha = (d - \pi \circ D)\theta^\alpha. \tag{1.4}$$

The module $\Omega^1(V)$ has a natural structure as a right $\mathcal{C}(V)$-module and the corresponding condition equivalent to (1.2) is determined using the fact that $\mathcal{C}(V)$ is a commutative algebra:

$$D(\xi f) = D(f\xi). \tag{1.5}$$

By extension, a linear connection over a general noncommutative algebra $\mathcal{A}$ with a differential calculus $(\Omega^*(\mathcal{A}), d)$ can be defined as a linear map

$$\Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{1.6}$$

which satisfies the condition (1.2) for arbitrary $f \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$. The module $\Omega^1(\mathcal{A})$ has again a natural structure as a right $\mathcal{A}$-module but in the noncommutative case it is impossible in general to consistently impose the condition (1.5) and a substitute must be found. We must decide how it is appropriate to define $D(\xi f)$ in terms of $D(\xi)$ and df. It has been proposed (Mourad 1995, Dubois-Violette & Michor 1995) to introduce as part of the definition of a linear connection a map σ of $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ into itself and to define $D(\xi f)$ by the equation

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \tag{1.7}$$

If the algebra is commutative this is equivalent to (1.5). The curvature R can be defined as the map

$$\Omega^1(\mathcal{A}) \stackrel{R}{\rightarrow} \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{1.8}$$

given, in the case that the torsion vanishes, by $R = (\pi \otimes 1) \circ D^2$.

A metric g on V can be defined as a $\mathcal{C}(V)$-linear, symmetric map of $\Omega^1(V) \otimes_{\mathcal{C}} \Omega^1(V)$ into $\mathcal{C}(V)$. This definition makes sense if one replaces $\mathcal{C}(V)$ by an algebra $\mathcal{A}$ and $\Omega^1(V)$ by any differential calculus $\Omega^1(\mathcal{A})$ over $\mathcal{A}$. By analogy with the commutative case we shall say that the covariant derivative (1.6) is metric if $(1 \otimes g) \circ D = d \circ g$.

We shall use the conventions that lower-case Greek indices take the values from 1 to d, lower-case Latin indices at the beginning of the alphabet take the values from 1 to $m^2 - 1$ and the lower-case Latin indices from p to the end of the alphabet take the values from 1 to $n^2 - 1$. The integers d, m, n satisfy the inequalities

$$d < m^2 - 1, \qquad m < n.$$

3 Induced Linear Connections

Noncommutative geometry is based on the fact that one can formulate (Koszul 1960) much of the ordinary differential geometry of a manifold in terms of the algebra of smooth functions defined on it. It is possible to define a finite noncommutative geometry based on derivations by replacing this algebra by the algebra M_n of $n \times n$ complex matrices (Dubois-Violette *et al.* 1989, 1990). Since M_n is of finite dimension as a vector space, all calculations reduce to pure algebra. Matrix geometry is interesting in being similar is certain aspects to the ordinary geometry of compact Lie groups; it constitutes a transition to the more abstract formalism of general noncommutative geometry (Connes 1986, 1994). Our notation is that of Dubois-Violette *et al.* (1989). We first recall some important formulae.

Let λ_r, for $1 \leq r \leq n^2 - 1$, be an anti-hermitian basis of the Lie algebra of the special unitary group SU_n in n dimensions. The λ_r generate M_n as an algebra and the derivations $e_r = \mathrm{ad}\,\lambda_r$ form a basis for the Lie algebra of derivations $\mathrm{Der}(M_n)$ of M_n. In order for the derivations to have the correct dimensions we must introduce a mass parameter μ and replace λ_r by $\mu\lambda_r$. We shall set $\mu = 1$. We define df for $f \in M_n$ by

$$df(e_r) = e_r(f). \tag{2.1}$$

In particular

$$d\lambda^r(e_s) = -C^r{}_{st}\lambda^t. \tag{2.2}$$

We raise and lower indices with the Killing metric g_{rs} of SU_n and we use the Einstein summation convention.

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form fdg with f and g in M_n. The set of all differential forms is a differential algebra $\Omega^*(M_n)$. The couple $(\Omega^*(M_n), d)$ is a differential calculus over M_n. There is a convenient system of generators of $\Omega^1(M_n)$ as a left- or right-module completely characterized by the equations

$$\theta^r(e_s) = \delta^r_s. \tag{2.3}$$

The θ^r are related to the $d\lambda^r$ by the equations

$$d\lambda^r = C^r{}_{st}\,\lambda^s\theta^t, \qquad \theta^r = \lambda_s\lambda^r d\lambda^s. \tag{2.4}$$

The θ^r satisfy the same structure equations as the components of the Maurer-Cartan form on the special unitary group SU_n:

$$d\theta^r = -\frac{1}{2}C^r{}_{st}\,\theta^s\theta^t. \tag{2.5}$$

The product on the right-hand side of this formula is the product in $\Omega^*(M_n)$. We shall refer to the θ^r as a frame or Stehbein. If we define $\theta = -\lambda_r\theta^r$ we can write the differential df of an element $f \in \Omega^0(M_n)$ as a commutator:

$$df = -[\theta, f]. \tag{2.6}$$

¿From (2.5) we see that the linear connection defined by

$$D\theta^r = -\omega^r{}_s \otimes \theta^s, \qquad \omega^r{}_s = -\frac{1}{2}C^r{}_{st}\,\theta^t \tag{2.7}$$

has vanishing torsion. With this connection the geometry of M_n looks like the invariant geometry of the group SU_n. Since the elements of the algebra commute with the frame θ^r, we can define D on all of $\Omega^*(M_n)$ using (1.2) or (1.7). The map σ is given by

$$\sigma(\theta^r \otimes \theta^s) = \theta^s \otimes \theta^r. \tag{2.8}$$

¿From the formula (1.8) we see that $R(\theta^r) = -\Omega^r{}_s \otimes \theta^s$ where the curvature 2-form $\Omega^r{}_s$ is given by

$$\Omega^r{}_s = \frac{1}{8}C^r{}_{st}C^t{}_{pq}\theta^p\theta^q. \tag{2.9}$$

¿From Equation (2.4) we find that $D(d\lambda^r)$ is given by

$$D(d\lambda^r) = C^r{}_{st}\big(d\lambda^s \otimes \theta^t - \frac{1}{2}\lambda^s C^t{}_{pq}\theta^p \otimes \theta^q\big).$$

A short calculation yields

$$D(d\lambda^r) = -\frac{1}{2}C^r{}_{s(p}C^s{}_{q)t}\lambda^t\theta^p \otimes \theta^q. \tag{2.10}$$

¿From this formula it is obvious also that the torsion vanishes.

The connection (2.7) is the unique torsion-free metric connection on $\Omega^1(M_n)$ (Madore *et al.* 1995). It has been used (Dubois-Violette *et al.* 1989, Madore 1990, Madore & Mourad 1993, 1994, Madore 1995) in the construction of noncommutative generalizations of Kaluza-Klein theories.

Let $\{\lambda^\alpha\}$ be a set of d matrices which generate M_n as an algebra and which are algebraically independent. By this we mean that the λ^α do not satisfy any polynomial relation of order p with $p << n$. Since each λ^r can be written as a polynomial $\lambda^r = \lambda^r(\lambda^\alpha)$ in the λ^α we have

$$d\lambda^r = A^r_\alpha(d\lambda^\alpha), \tag{2.11}$$

where $A^r_\alpha(d\lambda^\alpha)$ is a polynomial in λ^α and $d\lambda^\alpha$ which is linear in the latter. Since the λ^α generate M_n it follows that the equations $e_\alpha f = 0$ can have only $f \propto 1$ as solutions. The algebraic independence implies that there is no relation of the form

$$A^\alpha_\beta(d\lambda^\beta) = 0, \tag{2.12}$$

with $A^\alpha_\beta(d\lambda^\beta)$ a polynomial of order $p-1$ in the λ^β.

Each choice of $\{\lambda^\alpha\}$ defines M_n as a n^2-dimensional approximation to the algebra of functions on a d-dimensional submanifold V of $\mathbb{R}^{n^2-1}$. Let $\Omega^*_{\mathcal{C}}$ be the associated differential calculus. We shall argue in the next section that a differential subalgebra of $\Omega^*_{\mathcal{C}}$ has a limit as $n \to \infty$ which can be considered as the de Rham differential calculus over V.

¿From (2.10) we have

$$D(d\lambda^\alpha) = -\frac{1}{2}C^\alpha{}_{s(p}C^s{}_{q)t}\lambda^t\theta^p \otimes \theta^q. \tag{2.13}$$

¿From (2.4) each θ^r on the right-hand side of this equation can in turn be expressed in terms of the $d\lambda^\alpha$:

$$\theta^r = \lambda_s(\lambda^\alpha)\lambda^r(\lambda^\alpha)A^s_\beta(d\lambda^\beta). \tag{2.14}$$

Equations (2.13) and (2.14) define a covariant derivative on the differential calculus $\Omega^*_{\mathcal{C}}$. For finite n it is a restriction of (2.7). By construction it satisfies the Leibniz rules (1.2) and (1.7). The right-hand side however cannot be written in the form (1.3); there is no corresponding connection form in general. The map σ, which is given on $\theta^r \otimes \theta^s$ by the simple expression (2.8), becomes very complicated when defined on $d\lambda^\alpha \otimes d\lambda^\beta$.

3 Fuzzy manifolds

To discuss the commutative limit it is convenient to change the normalization of the generators λ^α. Recall that the λ^α have the dimensions of mass. We introduce the parameter $\hbar$ with the dimensions of (length)^2 and define 'coordinates' x^α by

$$x^\alpha = i\hbar\lambda^\alpha. \tag{3.1}$$

We define matrices $L^{\alpha\beta}$ by the equations

$$[x^\alpha, x^\beta] = i\hbar L^{\alpha\beta}. \tag{3.2}$$

By our assumption the $L^{\alpha\beta}$ can be expressed as polynomials in the x^α, normally of order n. By taking higher-order commutators of the x^α the algebra will eventually close as a Lie algebra to form an irreducible n-dimensional representation of the Lie algebra of SU_m for some $m \leq n$. By assumption $m^2 - 1 - d$ must be at least as large as the number of Casimir relations of SU_m. We shall assume that $m << n$. Let x^a be the extended set of matrices:

$$\{x^a\} = \{x^\alpha, L^{\alpha\beta}, [x^\alpha, L^{\beta\gamma}], \dots\}$$

Globally the limit manifold V will be then a submanifold of the sphere of some radius r in $\mathbb{R}^{m^2-1}$. A metric on it would necessarily have euclidean signature. We shall have the relation

$$\hbar \sim \frac{r^2}{n}, \tag{3.3}$$

and so $\hbar \to 0$ as $n \to \infty$. This is the commutative limit.

For each $l \geq 1$ let $\mathcal{C}_l$ be the vector space of l^{th}-order symmetric polynomials in the x^α and $\mathcal{L}_l$ the vector space of l^{th}-order symmetric polynomials in the x^a. Then we have

$$\mathcal{C}_l \subset \mathcal{C}_{l+1}, \qquad \mathcal{L}_l \subset \mathcal{L}_{l+1},$$

and the set $\{\mathcal{L}_l\}$ is a filtration of M_n:

$$\bigcup_l \mathcal{C}_l \subseteq M_n, \qquad \bigcup_l \mathcal{L}_l = M_n. \tag{3.4}$$

For fixed l the set $\mathcal{C}_l$ tends to the set of l^{th}-order polynomials in the x^α in the limit $n \to \infty$. We shall refer to the algebra M_n with the set of $\{\mathcal{C}_l\}$ as a fuzzy manifold. The $\{\mathcal{C}_l\}$ do not form a graded algebra but from the definition of the $\{\mathcal{L}_l\}$ we have

$$\mathcal{C}_k\mathcal{C}_l \subset \mathcal{C}_{k+l} + \hbar\mathcal{L}_{k+l-1}.$$

A specific example is the fuzzy 2-sphere (Madore 1992). Consider $\mathbb{R}^3$ with coordinates x^a, $1 \leq a \leq 3$, and euclidean metric $g_{ab} = \delta_{ab}$. Let V be the sphere S^2 defined by

$$g_{ab}x^a x^b = r^2. \tag{3.5}$$

Consider the algebra $\mathcal{P}$ of polynomials in the x^a and let $\mathcal{I}$ be the ideal generated by the relation (3.5). That is, $\mathcal{I}$ consists of elements of $\mathcal{P}$ with $g_{ab}x^a x^b - r^2$ as factor. Then the quotient algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is dense in the algebra $\mathcal{C}(S^2)$. Any element of $\mathcal{A}$ can be represented as a finite multipole expansion of the form

$$f(x^a) = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \cdots, \tag{3.6}$$

where the $f_{a_1 \ldots a_i}$ are completely symmetric and trace-free. We obtain a vector space of dimension n^2 if we consider only polynomials of order $n-1$. We can redefine the product of the x^a to make this vector space into the algebra of $n \times n$ matrices.

Suppose that we suppress the terms $n^{\rm th}$ order in the expansion (3.6) of every function f. The resulting set is a vector space $\mathcal{A}_n$ of dimension n^2. We can introduce a new product in the x^a which will make it into the algebra M_n. We make the identification

$$x^a = \kappa J^a \tag{3.7}$$

where the J^a generate the n-dimensional irreducible representation of the Lie algebra of SU_2 with $[J_a, J_b] = i\epsilon_{abc}J^c$. Since the J^a satisfy the quadratic Casimir relation $J_a J^a = (n^2-1)/4$ the parameter κ must be related to r by the equation $4r^2 = (n^2-1)\kappa^2$. Introduce the constant

$$\bar{k} = \kappa r. \tag{3.8}$$

The x^a satisfy the commutation relations

$$[x_a, x_b] = i\bar{k}C^c{}_{ab}x_c, \qquad C_{abc} = r^{-1}\epsilon_{abc}. \tag{3.9}$$

The two length scales r and $\bar{k}$ are related through the integer n:

$$4r^4 = (n^2-1)\bar{k}^2. \tag{3.10}$$

In particular (3.3) is satisfied. The space $\mathcal{L}_l$ is the space of symmetric polynomials of order l in the x^a. Define x^α as the first two of the x^a. Then $L^{12} = r^{-1}x^3$. Because of the Casimir relation we have

$$\bigcup_l \mathcal{C}_l = \bigcup_l \mathcal{L}_l = M_n.$$

For $n >> l$ $\mathcal{L}_l$ can be identified as the space of polynomials of order l on S^2 and $\mathcal{C}_l$ as the space of polynomials of order l on the coordinate patch.

The fuzzy sphere with three generators is not a good example for the construction of linear connections since the limit manifold is not parallelizable. Global frames must be constructed on the U_1 bundle S^3 over S^2. From them connections can be constructed on S^2 using a Kaluza-Klein-type decomposition. (Grosse & Madore 1991). A more convenient example is obtained by taking only two generators. It is known (Weyl 1931) that the algebra M_n can be generated by two matrices u and v which satisfy the relations

$$u^n = 1, \qquad v^n = 1, \qquad uv = qvu, \qquad q = e^{2\pi i/n}.$$

The space $\mathcal{C}_l$ becomes then the space of symmetric polynomials of order l in u and v. For $n >> l$ it can be identified as the space of polynomials of order l on the torus.

One sees from these two examples that the structure of the limit manifold is determined by the filtration. The dimension of the manifold is encoded in the dimension of $\mathcal{C}_1$. The manifolds differ in global topology because the vector spaces $\mathcal{C}_l$ differ. A polynomial in the x^α of order l, with $n >> l$, can of course be always written as a polynomial in u and v but will then in general be of order n. The transformation in no way respects the filtration. This corresponds to the fact that a map from the torus onto the sphere is necessarily singular. A physical theory expressed in terms of the matrix approximation would detect the difference between the topologies through the dependence of the action on the derivations $e^\alpha = \mathrm{ad}\, x^\alpha$.

Let $\{x^\alpha\}$ be an arbitrary subset of generators of M_n. If we rewrite (2.11) in terms of x^α we see that in the commutative limit

$$A^r_\alpha(dx^\alpha) = \frac{\partial x^r}{\partial x^\alpha}dx^\alpha + o(\bar{k}).$$

This gives the differential of an arbitrary function in terms of the differential of the coordinates. The forms θ^r are singular in the limit $\bar{k} \to 0$ (Madore 1992). No conclusions can be drawn directly from Equation (2.13) concerning this limit unless (2.14) is used first to eliminate the θ^r.

Consider the 1-form $[f, dg]$. It satisfies

$$[f, dg](X) = [f, Xg]. \tag{3.11}$$

In the limit $\hbar \to 0$ define a Poisson bracket $\{f, g\}$ on V by

$$i\hbar\{f, g\} = [f, g]. \tag{3.12}$$

By taking the limit of (3.11) we can define the extension $\{f, dg\}$ by

$$\{f, dg\}(X) = \{f, Xg\}. \tag{3.13}$$

It is obvious that $\{f, dg\}$ is not an element of $\Omega^1(V)$. It is a $\mathbb{C}$-linear map of the derivations into the functions but it cannot be $\mathcal{C}(V)$-linear, because Poisson vector fields do not form a $\mathcal{C}$-module. The 1-form defined by (3.11) contains a term of order $\hbar$ which cannot be approximated by an element of $\Omega^1(V)$. Define $\Omega^1_{\mathcal{C}}(V)$ to be the 1-forms of a new differential calculus on V defined by (3.13). We have seen then that

$$\Omega^1_{\mathcal{C}}(V) \neq \Omega^1(V). \tag{3.14}$$

In a sense the left-hand side is smaller since it is only defined on Poisson vector fields. However since

$$d\{f, g\} = \{df, g\} + \{f, dg\} \tag{3.15}$$

every element of $\Omega^1(V)$ defines by restriction an element of $\Omega^1_{\mathcal{C}}(V)$. So in a sense the left-hand side is larger. The map d of $\Omega^1_{\mathcal{C}}(V)$ into $\Omega^2_{\mathcal{C}}(V)$ is defined by $d\{f, dg\} = \{df, dg\}$ with

$$\{df, dg\}(X, Y) = \{Xf, Yg\} - \{Yf, Xg\}.$$

The image is also not $\mathcal{C}(V)$-linear and would not coincide with the bracket of 1-forms defined, for example, by Koszul (1985).

We define the element dx^{ab} of $\Omega^1_{\mathcal{C}}(V)$ as

$$dx^{ab} = \{x^\alpha, dx^\beta\}. \tag{3.16}$$

We can write the induced connection in the quasicommutative limit in the form

$$\begin{aligned} D(dx^\alpha) &= -\Gamma^\alpha{}_{\beta\gamma} dx^\beta \otimes dx^\gamma - \hbar\Gamma^\alpha_{(1)} + o(\hbar^2), \\ D(dx^{\alpha\beta}) &= -\Gamma^{\alpha\beta}_{(1)} + o(\hbar), \end{aligned} \tag{3.17}$$

where

$$\Gamma^\alpha_{(1)} = \Gamma_L{}^\alpha{}_{\beta\gamma\delta} dx^{\beta\gamma} \otimes dx^\delta + \Gamma_R{}^\alpha{}_{\beta\gamma\delta} dx^d \otimes dx^{\beta\gamma}. \tag{3.18}$$

The $\Gamma_L{}^\alpha{}_{\beta\gamma\delta}$ and $\Gamma_R{}^\alpha{}_{\beta\gamma\delta}$ can be considered as functions on the limit manifold V. Although the right-hand side of (2.13) is symmetric in p and q, in general because of our convention of placing all coefficients of forms to the left of the differential,

$$\Gamma_L{}^\alpha{}_{\beta\gamma\delta} \neq \Gamma_R{}^\alpha{}_{\beta\gamma\delta}.$$

The right-hand side of the second equation (3.17) is an element of $\Omega^1_{\mathcal{C}}(V) \otimes \Omega^1_{\mathcal{C}}(V)$.

We have deduced the form of the Equations (3.17) from (2.13) and (2.14). They depend however only on the Poisson structure, through the differential calculus $\Omega^*_{\mathcal{C}}(V)$. The Poisson structure is the unique 'shadow' of the original noncommutative algebra and the extra terms on the right-hand side of (3.17) the unique 'shadow' of the noncommutative linear connection. As we have mentioned the manifolds we can approximate in this way are compact with metrics necessarily of euclidean signature. They are of interest in that their algebra of functions can be approximated by algebras of finite dimension. Of more physical relevance for relativistic physics are noncompact manifolds which can support metrics of Minkowski signature. The first example along the lines indicated by the relation (3.2) was given by Snyder (1947). See also Madore (1988, 1995). Doplicher *et al.* (1995) have given an analysis of several possible noncommutative extensions of Minkowski space within the context of relativistic quantum field theory.

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