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Symmetry group classification of ordinary differential equations: Survey of some results

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SUMMARY

After the initial seminal works of Sophus Lie on ordinary differential equations, several important results on point symmetry group analysis of ordinary differential equations have been obtained. In this review, we present the salient features of point symmetry group classification of scalar ordinary differential equations: linear nth-order, second-order equations as well as related results. The main focus here is the contributions of Peter Leach, in this area, in whose honour this paper is written on the occasion of his 65th birthday celebrations. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: ordinary differential equations; symmetry; transformations

1. INTRODUCTION

Ordinary differential equations are a fertile area of study. In particular, the Lie algebraic properties of these equations have attracted considerable attention since the initial seminal works of Lie [1–4]. One of Lie's profound results was on the complete complex classification of all possible continuous groups acting in the plane.

Lie [2] presented, among other things, a list of all continuous groups of transformations in the complex plane. He further stressed that this be made the basis of a classification and integration of scalar ordinary differential equations which he implicitly carried out (see also [5]).

The main focus of this review, apart from being devoted to point symmetry group classification of scalar linear nth-order and second-order ordinary differential equations, revolves around Peter Leach's contributions in this fascinating and well-researched area. So my bias, besides this being a personal account, is made obvious up front.

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We first state the main results on the point symmetry group properties of linear nth-order ($n\geqslant 1$) equations in Section 2 and then in Section 3 we review the point symmetry group classification of scalar second-order ordinary differential equations both in the complex and real domains. In the concluding remarks, we make further references to the works of Leach on well-researched equations and related works on classification and integrability as well as mention some open problems in this vast area.

2. SCALAR LINEAR nth-ORDER DIFFERENTIAL EQUATIONS

The general, homogeneous, form is

$$y^{(n)}+\sum_{i=0}^{n-1} a_i(x)y^{(i)}=0, \quad n\geqslant 1 \tag{1}$$

Earlier studies of the transformation properties of (1) were in the works of Kummer [6], Laguerre [7], Brioschi [8], Halphen [9], Lie [3] and Forsyth [10]. All transformations utilized in the present paper are local. In particular, here we deal with local equivalence of Equations (1). Global transformation properties of the linear equations (1) are important as well (see, e.g. [11]).

2.1. *Equivalent differential equations*

Two differential equations are (locally) equivalent *via* an invertible point transformation if one can be transformed into the other by the transformation.

2.1.1. First-order differential equations. First-order equations, $y'=f(x,y)$, are equivalent to each other. An equation of order 1 can be transformed into the simplest one $y'=0$ *via* a point transformation.

Example
The Ricatti equation $y'+y^2=0$ is transformable to $Y'=0$ by means of $X=x$, $Y=1/y-x$.

2.1.2. Linear second-order differential equations [*3, 12–18*]. Linear second-order equations are equivalent to each other. Specifically, they can be transformed to the simplest equation $y''=0$ by invertible maps.

Example
The simple harmonic oscillator equation $y''+y=0$ can be reduced to $Y''=0$ *via* the invertible transformation $X=\tan x$, $Y=y\sec x$.

Remark
The simple, time-dependent, repulsive and forced oscillators were investigated for $SL(3,R)$ symmetry group in various papers (see, e.g. [19–24]).

A linear equation of order $n\geqslant 3$ need not be transformable into its simplest form. Therefore, one needs to have knowledge of the transformation properties of (1) for $n\geqslant 3$.

2.2. Laguerre canonical forms [7]

The equivalence group of Equation (1) comprised the transformations

$$x \mapsto f(x), \quad f'(x) \neq 0 \tag{2}$$

and

$$y \mapsto yg(x), \quad g(x) \neq 0 \tag{3}$$

One can invoke (3) to deduce that the transformation

$$y \mapsto y \exp\left(\frac{1}{n}\int_{x_0}^{x} a_{n-1}(s)\,\mathrm{d}s\right)$$

reduces Equation (1) for $n \geqslant 2$ to

$$y^{(n)} + \sum_{i=0}^{n-2} a_i(x) y^{(i)} = 0 \tag{4}$$

One can further annull the a_{n-2} coefficient in Equation (1) for $n \geqslant 2$ *via* the mappings (2) and (3) with $f(x)$ and $g(x)$ defined in terms of h as $f' = h^{-2}$, $g = h^{1-n}$, where $h(x)$ satisfies the second-order equation

$$\frac{(n+1)!}{(n-2)!3!} h'' + a_{n-2} h = 0$$

The corresponding differential equations $y'' = 0$ for $n = 2$ and

$$y^{(n)} + \sum_{i=0}^{n-3} a_i(x) y^{(i)} = 0, \quad n \geqslant 3 \tag{5}$$

are the *Laguerre canonical forms* and are characterized by the vanishing of the coefficients a_{n-1} and a_{n-2} in Equation (1). These are sometimes referred to as the Laguerre–Forsyth [10] canonical forms. The transformations that reduce Equation (1) to Equation (5) are the *Laguerre transformations*. We initially use Equation (4) as it is no more difficult for group classification purposes than Equation (5).

2.3. Determining equation for $n \geqslant 3$ [25]

Theorem 1

The Lie point symmetry generator

$$X = \xi(x)\frac{\partial}{\partial x} + \left[\left(\frac{n-1}{2}\xi' + c_0\right) y + \eta(x)\right]\frac{\partial}{\partial y} \tag{6}$$

is admitted by Equation (4) for $n \geqslant 3$ and is the most general, where c_0 is a constant, $\eta(x)$ satisfies Equation (4) and $\xi(x)$ is determined by the relations

$$\frac{(n+1)!(i-1)}{(n-i)!(i+1)!}\xi^{(i+1)} + 2i\xi' a_{n-i} + 2\xi a'_{n-i} + \sum_{j=2}^{i-1} a_{n-j} \frac{(n-j)![n(i-j-1)+i+j-1]}{(n-i)!(i-j+1)!}\xi^{(i-j+1)} = 0, \quad i = 1, \ldots, n \tag{7}$$

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It is known from the works of Lie [3] that the maximal point symmetry Lie algebra of an nth-order ($n\geqslant 3$) ordinary differential equation has a dimension $r\leqslant n+4$. Hence, Equations (7) can have at most three independent solutions.

2.4. *Principal Lie algebra* [25, 26]

For arbitrary coefficients $a_i(x)$, Equation (4) admits the Lie algebra spanned by the $n+1$ homogeneous and superposition operators

$$X_1 = y\frac{\partial}{\partial y} \tag{8}$$

$$X_{i+1} = \eta_i(x)\frac{\partial}{\partial y}, \quad i=1,\ldots,n \tag{9}$$

where $\eta_i(x)$ are n linearly independent solutions of Equation (4). This algebra is referred to as the principal Lie algebra of Equation (4).

2.5. *Example of extension of the principal algebra* [2, 25–27]

Consider the simplest nth-order equation

$$y^{(n)}=0, \quad n\geqslant 3 \tag{10}$$

We have $(n+1)$ symmetry generators given by (8) and (9) with

$$\eta_i(x)=C_i x^{i-1}, \quad i=1,\ldots,n$$

where the C_i's are n arbitrary constants. Moreover, the use of Equation (7), since $a_i=0$, gives

$$\xi=A_0+A_1x+A_2x^2, \quad A_i \text{ are constants}$$

The extension is maximum, i.e. three dimensional. Therefore, the maximum symmetry algebra of Equation (10) is spanned by (8), (9) with η_i given above and

$$X_{n+2}=\frac{\partial}{\partial x}, \quad X_{n+3}=x\frac{\partial}{\partial x}, \quad X_{n+4}=x^2\frac{\partial}{\partial x}+(n-1)xy\frac{\partial}{\partial y} \tag{11}$$

2.6. *Iterative linear equations* [11, 25, 26, 28]

Suppose that y_1 and y_2 are two linearly independent solutions of a linear homogeneous second-order equation and define the n functions

$$u_k=y_1^{n-k}y_2^{k-1}, \quad k=1,\ldots,n \tag{12}$$

Then $u_1,\ldots,u_n$ are n linearly independent solutions of a linear nth-order equation of form (4). This linear nth-order equation is referred to as the *iterative linear equation* and its solution set is spanned by two solutions y_1 and y_2 of the linear homogeneous second-order equation.

Example
Suppose that

$$y''+a_0(x)y=0$$

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has two independent solutions y_1 and y_2. Then, the iterative third-order linear equation is

$$u''' + 4a_0 u' + 2a_0' u = 0 \tag{13}$$

This is obtained by the use of Equation (12) as

$$u_1 = y_1^2, \quad u_2 = y_1 y_2, \quad u_3 = y_2^2$$

The next theorem gives Lie algebraic criteria for a linear nth-order equation to be iterative.

Theorem 2
A necessary and sufficient condition that a linear nth-order Equation (4), $n \geqslant 3$, be iterative is that

(a) it has the maximum, $n+4$, dimension Lie symmetry algebra or
(b) it is reducible by the Laguerre transformations to the simplest equation $y^{(n)} = 0$.

Example
Equation (13) has $n+4=7$ point symmetries and is iterative.

2.7. Symmetry algebra of maximum order $n+4$ [25, 27]

We provide a list of three linear equations of orders $n=3$, 4 and 5, that admit $n+4$ point symmetries:

$$y^{(3)} + a_1 y^{(1)} + \tfrac{1}{2} a_1^{(1)} y = 0$$

$$y^{(4)} + a_2 y^{(2)} + a_2^{(1)} y^{(1)} + \left(\tfrac{3}{10} a_2^{(2)} + \tfrac{9}{100} a_2^2\right) y = 0$$

$$y^{(5)} + a_3 y^{(3)} + \tfrac{3}{2} a_3^{(1)} y^{(2)} + \left(\tfrac{9}{10} a_3^{(2)} + \tfrac{16}{100} a_3^2\right) y^{(1)} + \left(\tfrac{1}{5} a_3^{(3)} + \tfrac{16}{100} a_3 a_3^{(1)}\right) y = 0$$

If a_i are constants ($a_i^{(j)} = 0$, for $j = 1, 2, \ldots$), then the equations of odd order contain odd derivatives of y and equations of even order even derivatives of y.

For n odd, $n+4$ point symmetries occur if and only if

$$\frac{\mathrm{d}}{\mathrm{d}x} \left\{ \prod_{i=1}^{(n-1)/2} \left(\frac{(n+1)!}{(n-2)!3!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2i)^2 a_{n-2} \right) \right\} y = 0 \tag{14}$$

and for n even one has

$$\left\{ \prod_{i=1}^{n/2} \left(\frac{(n+1)!}{(n-2)!3!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2i-1)^2 a_{n-2} \right) \right\} y = 0 \tag{15}$$

which has $n+4$ point symmetries.

Both are derivable from a second-order linear equation and are thus iterative.

2.8. Nonexistence of $(n+3)$-dimensional symmetry Lie algebra [25]

Theorem 3
A linear nth-order ($n \geqslant 3$) Equation (4) does not possess a submaximal symmetry Lie algebra of dimension $n+3$.

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Remark
A nonlinear nth-order ($n\geqslant 3$) equation, however, can admit a submaximal algebra of dimension $n+3$.

2.9. The $(n+2)$*-dimensional symmetry Lie algebra* [*25, 26*]

Example
The linear equation

$$y^{(n)}+\sum_{i=0}^{n-2} b_i y^{(i)}=0, \quad n\geqslant 3 \tag{16}$$

where the b_i's are n independent constants, has $n+2$ Lie point symmetries. The principal Lie algebra of this equation extends by one and its symmetry algebra is spanned by operators (8) and (9), with $\eta_i(x)$ being n independent solutions of Equation (16), and by the translation generator

$$X_{n+2}=\frac{\partial}{\partial x}$$

Theorem 4
A necessary and sufficient condition that linear nth-order ($n\geqslant 3$) Equation (4) admits a submaximal symmetry algebra of dimension $n+2$ is that it is point transformable into a constant coefficient linear equation which is neither of the form (14) nor (15).

2.10. Invariants of equivalence groups ([*7, 9, 10, 29, 30*]*, see also* [*31*])

Differential invariants of equivalence groups or subgroups of linear nth-order equations were first considered in the late 1870s (see Berkovich [32] for another approach on invariants of linear equations).

It is well known that the linear second-order equation

$$L_2(y)\equiv y''+2a_1(x)y'+a_0(x)y=0$$

has one seminvariant

$$I(x)=a_1'+a_1^2-a_0$$

which remains invariant under the subgroup of the equivalence group which consists of linear changes in the dependent variable.

Theorem 5
The equation $\bar{L}_2(\bar{y})\equiv\bar{y}''+2\bar{a}_1(x)\bar{y}'+\bar{a}_0(x)\bar{y}=0$ is equivalent to $L_2(y)=0$ by means of the transformations (3) if and only if $\bar{I}(x)=I(x)$, where $\bar{I}(x)=\bar{a}_1'+\bar{a}_1^2-\bar{a}_0$ is the seminvariant of the equation $\bar{L}_2(\bar{y})=0$.

Example
The variable coefficient equation

$$y''+2xy'+(1+x^2)y=0$$

has $I=0$ and can be reduced to $\bar{y}''=0$ (which has $\bar{I}=0$) *via* $\bar{y}=y\exp(x^2/2)$.

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2.11. *Symmetries associated with first integrals* [*28, 33–39*]

Suppose that J is a first integral of a scalar nth-order ordinary differential equation $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$. Then an operator $X = \xi\partial/\partial x + \eta\partial/\partial y + \sum_{s\geqslant 1}\zeta_s\partial/\partial y^{(s)}$, where $\zeta_s = D_x(W) + \xi y^{(s+1)}$ with $W = \eta - \xi y'$, is said to be a symmetry generator associated with the first integral J if $X(J) = 0$.

It is known that a symmetry algebra of a first integral is a subalgebra of the underlying equation that gave rise to the first integral [38].

Symmetry Lie algebras of first integrals of second- and higher-order linear equations have been investigated with a number of interesting properties. We look at one such property taken from Moyo and Leach [39].

Theorem 6

If y is an integrating factor or characteristic of $y^{(n)} = 0$, $n\geqslant 3$, then the first integral obtained *via* this characteristic has $sl(2, R)$ symmetry algebra.

Also, maximal symmetries associated with first integrals of maximally symmetric equations have been explored.

Example

For $y''' = 0$, if y is the integrating factor, then $yy''' = 0$. Integration of this last equation results in the first integral

$$J = yy'' - \tfrac{1}{2}y'^2$$

which has $sl(3, R)$ symmetry algebra. As an equation this appears as an equivalence class for second-order equations (see the next section).

2.12. *Complete symmetry group* [*40–42*]

Krause [40] introduced the notion of complete symmetry group of an equation. In simple words, it is the group required to specify completely the equation up to inessential constants.

The economy of complete symmetry groups has been demonstrated in Andriopoulus and Leach [42] amongst other results. We observe the following:

The Lie algebra spanned by the three operators

$$X_1 = x\frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

generates a complete symmetry group of the simplest second-order equation $y'' = 0$.

The representation of the complete symmetry group for an equation need not be unique, e.g. $y'' = 0$ has other representations. Algebras of different dimensions can generate complete symmetry groups of an equation. Hence, a minimum dimension criterion was introduced. Another result is contained in the following theorem.

Theorem 7

The dimension of the complete symmetry group of an nth-order, $n\geqslant 2$, linear ordinary differential equation (4) is $n+1$.

We next survey group classification of scalar second-order equations.

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3. SCALAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

The normal form is

$$y'' = f(x, y, y') \tag{17}$$

3.1. Lie's point symmetry group classification in the complex plane [*2, 3*]

Lie presented a complete (explicit) classification of scalar second-order equations which admit non-similar complex Lie algebras L_r of dimension r, where $r = 0, \dots, 8$.

Lie showed that the complex Lie algebra of vector fields acting in the plane admitted by a given second-order equation are of dimension $0, 1, 2, 3$ or 8. Lie proved that a second-order equation cannot admit a maximal four-, five-, six- or seven-dimensional symmetry Lie algebra. He also showed that if a second-order equation admits an eight-dimensional algebra, it is linearizable by a point transformation (see also Theorem 8) and equivalent to the simplest equation $y'' = 0$.

3.2. Point symmetry group classification in the real domain [*16, 43, 44*]

One- and two-dimensional algebras have identical structures over the reals as well as over the complex numbers. Consequently, symmetry algebra classification of second-order equations over the reals is the same as that over the complex numbers for one- and two-dimensional Lie algebras. However, the situation is not the same for three- or higher-dimensional Lie algebras as there are fewer complex than real algebras of dimension 3 or more.

In the real plane, two of the complex Lie algebras, of dimension 3, each split up into two real non-isomorphic Lie algebras (see Table I). Therefore, there are two more non-isomorphic real three-dimensional Lie algebras, viz. $L_{3;7}$ and $L_{3;9}$, than complex algebras. As a result, there are more three-dimensional algebra of vector fields acting on the real plane than in the complex plane. These yield additional non-similar scalar second-order equations that admit real Lie algebras as one can observe from Table IV.

Algebra labels: When one deals with more than one Lie algebra of the same dimension, one usually distinguishes one from the other by means of two indices. The first here refers to the dimension and the second to the number of the algebra. Hence, here $L_{3;4}$ denotes the fourth Lie algebra of dimension 3 and so on for the other algebras. In contemporary works, the notation $A^a_{r;j}$ is also used (see [46–48]).

Table I. Non-isomorphic real three-dimensional Lie Algebras [2, 45].

Algebra	Non-zero commutation relations
$L_{3;1}$	
$L_{3;2}$	$[X_2, X_3] = X_1$
$L_{3;3}$	$[X_1, X_3] = X_1$, $[X_2, X_3] = X_1 + X_2$
$L_{3;4}$	$[X_1, X_3] = X_1$
$L_{3;5}$	$[X_1, X_3] = X_1$, $[X_2, X_3] = X_2$
$L_{3;6}$	$[X_1, X_3] = X_1$, $[X_2, X_3] = aX_2$, $a \neq 0, 1$
$L_{3;7}$	$[X_1, X_3] = bX_1 - X_2$, $[X_2, X_3] = X_1 + bX_2$
$L_{3;8}$	$[X_1, X_2] = X_1$, $[X_2, X_3] = X_3$, $[X_3, X_1] = -2X_2$
$L_{3;9}$	$[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$

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Remark

If we change to the complex basis with $\bar{X}_1 = X_2 + \mathrm{i}X_1$, $\bar{X}_2 = X_1 + \mathrm{i}X_2$, $\bar{X}_3 = (\mathrm{i}+b)X_3$, where $a = (b-\mathrm{i})/(b+\mathrm{i})$, the Lie algebra $L_{3;6}$ takes the form $L_{3;7}$. In the complex basis with $\bar{X}_1 = \frac{1}{2}X_1 + \frac{1}{2}X_3$, $\bar{X}_2 = \mathrm{i}X_2$, $\bar{X}_3 = (\mathrm{i}/2)X_1 - (\mathrm{i}/2)X_3$, $L_{3;8}$ becomes $L_{3;9}$. Lie's complex classification of non-isomorphic Lie algebras is determined from Table I by excluding $L_{3;7}$ and $L_{3;9}$.

Remark

The Lie algebras $L^{\mathrm{I}}_{3;7}$, $L^{\mathrm{II}}_{3;7}$, $L^{\mathrm{II}}_{3;8}$ and $L_{3;9}$ (see Table II) do not occur in Lie's realizations (up to similarity) in the complex plane. Lie obtained his three-dimensional algebra realizations according to the dimensions of the derived algebra which can be 0, 1, 2 or 3. The realizations in the real plane presented in Table II are from Mahomed and Leach [43]. These concur with the results of González-Lopéz *et al.* [49] (see also [50]).

Table II. Realizations of one-, two- and three-dimensional algebras in the real plane ($p = \partial/\partial x$ and $q = \partial/\partial y$).

Algebra	Realizations in the (x, y) plane
L_1	$X_1 = p$
$L^{\mathrm{I}}_{2;1}$	$X_1 = p, X_2 = q$
$L^{\mathrm{II}}_{2;1}$	$X_1 = q, X_2 = xq$
$L^{\mathrm{I}}_{2;2}$	$X_1 = q, X_2 = xp + yq$
$L^{\mathrm{II}}_{2;2}$	$X_1 = q, X_2 = yq$
$L_{3;1}$	$X_1 = q, X_2 = xq, X_3 = h(x)q$
$L_{3;2}$	$X_1 = q, X_2 = p, X_3 = xq$
$L^{\mathrm{I}}_{3;3}$	$X_1 = q, X_2 = p, X_3 = xp + (x+y)q$
$L^{\mathrm{II}}_{3;3}$	$X_1 = q, X_2 = xq, X_3 = p + yq$
$L^{\mathrm{I}}_{3;4}$	$X_1 = p, X_2 = q, X_3 = xp$
$L^{\mathrm{II}}_{3;4}$	$X_1 = q, X_2 = xq, X_3 = xp + yq$
$L^{\mathrm{I}}_{3;5}$	$X_1 = p, X_2 = q, X_3 = xp + yq$
$L^{\mathrm{II}}_{3;5}$	$X_1 = q, X_2 = xq, X_3 = yq$
$L^{\mathrm{I}}_{3;6}$	$X_1 = p, X_2 = q, X_3 = xp + ayq,\ a \neq 0, 1$
$L^{\mathrm{II}}_{3;6}$	$X_1 = q, X_2 = xq, X_3 = (1-a)xp + yq,\ a \neq 0, 1$
$L^{\mathrm{I}}_{3;7}$	$X_1 = p, X_2 = q, X_3 = (bx+y)p + (by-x)q$
$L^{\mathrm{II}}_{3;7}$	$X_1 = xq, X_2 = q, X_3 = (1+x^2)p + (x+b)yq$
$L^{\mathrm{I}}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + y^2q$
$L^{\mathrm{II}}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 - x^2)q$
$L^{\mathrm{III}}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 + x^2)q$
$L^{\mathrm{IV}}_{3;8}$	$X_1 = q, X_2 = yq, X_3 = y^2q$
$L_{3;9}$	$X_1 = (1+x^2)p + xyq, X_2 = xyp + (1+y^2)q$ $X_3 = yp - xq$

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Table III. Lie canonical forms for scalar second-order equations ($p=\partial/\partial x$ and $q=\partial/\partial y$).

Algebra	Canonical forms of generators	Representative equations
$L^{\mathrm{I}}_{2;1}$	$X_1=p,\ X_2=q$	$y''=f(y')$
$L^{\mathrm{II}}_{2;1}$	$X_1=q,\ X_2=xq$	$y''=f(x)$
$L^{\mathrm{I}}_{2;2}$	$X_1=q,\ X_2=xp+yq$	$xy''=f(y')$
$L^{\mathrm{II}}_{2;2}$	$X_1=q,\ X_2=yq$	$y''=y'f(x)$

If a second-order equation has a single generator of symmetry, then in general its order can be reduced by 1 [2]. In addition, Lie [2] showed that the second-order equations possessing two generators of symmetry have four canonical forms. These are presented in Table III together with their representative equations.

Remark
Lie showed that the algebras $L^{\mathrm{II}}_{2;1}$ and $L^{\mathrm{II}}_{2;2}$ (of Table III) result in linearization (see Theorem 8) of the underlying second-order equation.

A number of studies of second-order equations invariant under two symmetries have been undertaken. For example, in Bouquet *et al.* [51], the invariance under time and self-similar transformations are considered and solutions are obtained in parametric form. The Emden–Fowler type equations have also been of special interest (see, e.g. [52, 53]).

In Table IV, g is an arbitrary function of its argument(s) and A an arbitrary constant.

Remark
There are algebra realizations that do not appear in Table IV. We comment on these. Neither the Abelian Lie algebra $L_{3;1}$ nor the algebra $L^{\mathrm{IV}}_{3;8}$ are admitted by a second-order equation. The algebras $L_{3;2}$, $L_{3;4}$ or $L_{3;5}$ are not maximal algebras of a second-order equation meaning that if any one of these is admitted by such an equation then it has the eight-dimensional algebra L_8. In the real classification, the Lie algebra $L_{3;8}$ has three non-similar realizations, viz. $L^{\mathrm{I}}_{3;8}$, $L^{\mathrm{II}}_{3;8}$ and $L^{\mathrm{III}}_{3;8}$ that result in three non-equivalent real equations. In the complex classification, $L^{\mathrm{I}}_{3;7}$, $L^{\mathrm{II}}_{3;8}$ and $L_{3;9}$, and their corresponding equations do not occur.

Scalar second-order equations, under general point transformations, have been investigated by Tressé [54] according to the differential invariants of the equivalence group. He used the Lie infinitesimal approach. Invoking Tressé's invariants, a method to solve the equivalence problem for the cases of zero and one symmetry was presented in Berth and Czichowski [55]. Classes of second-order equations (in the complex plane) equivalent to second-order equations possessing three-dimensional algebra of point symmetries are described in Ibragimov and Meleshko [56].

Second-order equations were also studied *via* fibre preserving point transformations [57] invoking the Cartan equivalence method. This classification gives restricted classes of equations due to the nature of the transformations. This approach was also employed for obtaining descriptions of the equivalence classes to the first and second Painlevé transcendents [58]. In this context, it is worth mentioning the works [59, 60] in which equivalence to Painlev'e equations are also observed. Moreover, Leach and co-workers have also employed the Painlevé analysis (see, e.g. [61]). Notwithstanding, the Janet bases have been used in the representation of the determining equations of the symmetry (Schwarz [62]) for second-order equations as well.

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Table IV. Lie group classification of scalar second-order equations in the real plane ($p=\partial/\partial x$ and $q=\partial/\partial y$).

Algebra	Canonical forms of generators	Representative equations
L_1	$X_1=p$	$y''=g(y,y')$
$L^{\mathrm{I}}_{2;1}$	$X_1=p,\ X_2=q$	$y''=g(y')$
$L^{\mathrm{I}}_{2;2}$	$X_1=q,\ X_2=xp+yq$	$xy''=g(y')$
$L^{\mathrm{I}}_{3;3}$	$X_1=p,\ X_2=q$ $X_3=xp+(x+y)q$	$y''=A\mathrm{e}^{-y'}$
$L^{\mathrm{I}}_{3;6}$	$X_1=p,\ X_2=q$ $X_3=xp+ayq$	$y''=Ay'^{(a-2)/(a-1)}, a\neq 0, \frac{1}{2}, 2$
$L^{\mathrm{I}}_{3;7}$	$X_1=p,\ X_2=q$ $X_3=(bx+y)p+(by-x)q$	$y''=A(1+y'^2)^{3/2}\mathrm{e}^{b\arctan y'}$
$L^{\mathrm{I}}_{3;8}$	$X_1=q,\ X_2=xp+yq$ $X_3=2xyp+y^2q$	$xy''=Ay'^3-\frac{1}{2}y'$
$L^{\mathrm{II}}_{3;8}$	$X_1=q,\ X_2=xp+yq$ $X_3=2xyp+(y^2-x^2)q$	$xy''=y'+y'^3+A(1+y'^2)^{3/2}$
$L^{\mathrm{III}}_{3;8}$	$X_1=q,\ X_2=xp+yq$ $X_3=2xyp+(y^2+x^2)q$	$xy''=y'-y'^3+A(1-y'^2)^{3/2}$
$L_{3;9}$	$X_1=(1+x^2)p+xyq$ $X_2=xyp+(1+y^2)q$ $X_3=yp-xq$	$y''=A\left[\frac{1+y'^2+(y-xy')^2}{1+x^2+y^2}\right]^{3/2}$
L_8	$X_1=p,\ X_2=q,\ X_3=xq$ $X_4=xp,\ X_8=yp,\ X_6=yq$ $X_7=x^2p+xyq$ $X_8=xyp+y^2q$	$y''=0$

3.3. Linearization [*2, 3, 16, 30, 44, 54, 63–66*]

Theorem 8

The following are equivalent statements:

1. A scalar second-order equation (17) is linearizable *via* a point transformation.
2. Equation (17) has the maximum eight-dimensional Lie algebra.
3. The Tressé relative invariants

$$I_1 = f_{y'y'y'y'}$$
$$I_2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f_{y'y'} - 4\frac{\mathrm{d}}{\mathrm{d}x} f_{y'y} - 3f_y f_{y'y'} + 6f_{yy} + f_{y'}\left(4f_{y'y} - \frac{\mathrm{d}}{\mathrm{d}x} f_{y'y'}\right) \tag{18}$$

   both vanish identically for Equation (17).
4. Equation (17) has the cubic in derivative form

$$y''=A(x,y)y'^3+B(x,y)y'^2+C(x,y)y'+D(x,y) \tag{19}$$

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with the coefficients A to D satisfying the Lie conditions

$$
\begin{aligned}
w_x &= wW - AD + \tfrac{1}{3}C_y - \tfrac{2}{3}B_x \\
w_y &= -w^2 - Bw - AW - A_x - AC \\
W_x &= W^2 + Dw + CW - D_y + BD \\
W_y &= -Ww + AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y
\end{aligned} \tag{20}
$$

where w and W are auxiliary functions.

5. Equation (17) has the cubic in derivative form (19) with the coefficients A to D satisfying the two invariant conditions

$$
\begin{aligned}
3A_{xx} + 3A_xC - 3A_yD + 3AC_x + C_{yy} - 6AD_y + BC_y - 2BB_x - 2B_{xy} = 0 \\
6A_xD - 3B_yD + 3AD_x + B_{xx} - 2C_{xy} - 3BD_y + 3D_{yy} + 2CC_y - CB_x = 0
\end{aligned} \tag{21}
$$

6. Equation (17) has two commuting symmetries X_1, X_2, with $X_1 = \rho(x, y)X_2$, for a non-constant function ρ, such that a point transformation $X = X(x, y)$, $Y = Y(x, y)$ which brings X_1 and X_2 to their canonical form

$$X_1 = \frac{\partial}{\partial Y}, \quad X_2 = X\frac{\partial}{\partial Y}$$

reduces the equation to the linear form $Y'' = F(X)$.

7. Equation (17) has two non-commuting symmetries X_1, X_2, in a suitable basis with $[X_1, X_2] = X_1$ and $X_1 = \rho(x, y)X_2$ for a non-constant function ρ, such that a point change of variables $X = X(x, y)$, $Y = Y(x, y)$ which brings X_1 and X_2 to their canonical form

$$X_1 = \frac{\partial}{\partial Y}, \quad X_2 = Y\frac{\partial}{\partial Y}$$

reduces the equation to the linear form $Y'' = Y'F(X)$.

8. Equation (17) has two commuting symmetries X_1, X_2, with $X_1 \neq \rho(x, y)X_2$, for any non-constant function ρ, such that a transformation $X = X(x, y)$, $Y = Y(x, y)$ which brings X_1 and X_2 to their canonical form

$$X_1 = \frac{\partial}{\partial X}, \quad X_2 = \frac{\partial}{\partial Y}$$

reduces the equation to one which is at most cubic in the first derivative.

9. Equation (17) has two non-commuting symmetries X_1, X_2, in a suitable basis with $[X_1, X_2] = X_1$ and $X_1 \neq \rho(x, y)X_2$ for any non-constant function ρ, such that a point change of variables $X = X(x, y)$, $Y = Y(x, y)$ which brings X_1 and X_2 to their canonical form

$$X_1 = \frac{\partial}{\partial Y}, \quad X_2 = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}$$

reduces the equation to

$$XY'' = aY'^3 + bY'^2 + \left(1 + \frac{b^2}{3a}\right)Y' + \frac{b}{3a} + \frac{b^3}{27a^2} \tag{22}$$

where $a(\neq 0)$ and b are constants.

Remark 1
The Tressé invariants (18) were derived by Tressé [54] using the Lie infinitesimal approach. He also obtained the invariant conditions (21) for Equation (17) [30, Chapter III]. Conditions (21) were re-derived in Mahomed and Leach [65] wherein it is shown that conditions (21) are equivalent to the vanishing of the Tressé invariants (18). Notwithstanding, Grissom *et al.* [66] invoked the Cartan equivalence method to deduce conditions (21). In a recent paper, Ibragimov and Magri [67] gave a geometric proof to determine the Lie conditions (20). Conditions (21) are also the compatibility conditions of the Lie conditions (20), i.e. $w_{xy}=w_{yx}$ and $W_{xy}=W_{yx}$.

Remark 2
The invertible transformation

$$\bar{X}=Y+\frac{b}{3a}X, \quad \bar{Y}=\frac{1}{2}Y^2+\frac{b}{3a}XY+\frac{b^2}{18a^2}X^2+\frac{1}{2a}X^2 \tag{23}$$

transforms Equation (22) of condition 9 to $\bar{Y}''=0$. One can also obtain changes of variables (see [64]) that reduce the cubic in first derivative equations of condition 8 to linear equations.

Example
The second-order equation

$$y''+3yy'+y^3=0 \tag{24}$$

is linearizable *via* a point transformation since invariant conditions (21) hold. One can easily obtain the two non-commuting symmetry generators

$$X_1=\frac{\partial}{\partial x}, \quad X_2=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y} \tag{25}$$

of Equation (24). One can note here that $X_1\neq\rho(x,y)X_2$. So we invoke condition 9 to find a linearizing transformation. The point transformation that reduces the symmetries (25) to their canonical form is

$$X=\frac{1}{y}, \quad Y=x+\frac{1}{y} \tag{26}$$

Equation (24) reduces to

$$XY''=-Y'^3+6Y'^2-11Y'+6$$

by means of transformation (26). Equation (24) linearizes to $\bar{Y}''=0$ *via* transformations (23) with $a=-1$, $b=6$ and by using (26). That is

$$\bar{X}=x-\frac{1}{y}, \quad \bar{Y}=\frac{x^2}{2}-\frac{x}{y}$$

Remark
Equation (24) was also looked at from a dynamical point of view in Leach *et al.* [68]. The fact that this equation is linearizable does not play a critical role from a physical point of view. Furthermore, Equation (24) arises in the study of first integrals for the modified Emden equation (see [69]).

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3.4. Symmetry breaking

An explanation of symmetry breaking for second-order equations is provided in the following result (see [17]).

Theorem 9
A submaximal dimensional point symmetry algebra of a second-order equation is a subalgebra of the eight-dimensional point symmetry algebra $sl(3, R)$ of the free particle equation $y''=0$.

This result does not apply to third-order equations [70].

3.5. Equations with three symmetries as first integrals of third-order equations [71]

Equivalence classes of second-order equations possessing maximally three Lie point symmetries were interpreted as first integrals of third-order equations. Two of the equivalence classes have the same parent equation. This is given as follows.

The representative equation corresponding to $L^{\mathrm{II}}_{3;8}$ of Table IV after eliminating the A by differentiation gives

$$y'''(1+y'^2)-3y'y''^2=0$$

which has six point symmetries. This is also shown to be the parent equation of the representative equation of $L^{\mathrm{I}}_{3;7}$ with $b=0$.

3.6. Complete symmetry groups of equations with three symmetries [41]

The complete symmetry group is realized in the standard three-dimensional algebra for each of the equivalence classes of second-order equations.

4. CONCLUDING REMARKS

There have been several contributions on group analysis to differential equations such as the Hénon–Heiles problem, Kepler problem, Ermakov systems and their variants as well as several others by a number of researchers, including Leach and co-workers. A sample of results can be found by referring at Leach [72], Leach and Gorringe [73], Govinder and Leach [74], Maharaj *et al.* [75], and Moyo and Leach [76].

The (explicit) point symmetry group classification problem for third-order equations, including linearization, has also been studied (see [25, 44, 77–81]. Linearization for such equations *via* other than point transformations has been investigated by many authors (see, e.g. [59, 82–84]). Algebra bounds for systems were considered by González-Gascón and González-Lopéz [14]. The point symmetry group classification for linear systems of two second-order equations was first considered by Gorringe and Leach [85] and later by Wafo and Mahomed [86]. The Abel–Forsyth formulas for scalar equations have been extended to systems of linear equations using symmetry arguments [87]. There have also been works on the algebraic criteria for linearization *via* point transformations for a system of second-order equations (see [88]). Invariant criteria for linearization *via* invertible maps in the case of quadratic and cubic semi-linear systems of second-order equations have been studied recently as well (see [89, 90]). Notwithstanding, the canonical forms for systems of two second-order equations according to their symmetry properties were also examined [91].

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The equivalence problem for systems of second-order equations has been investigated by Fels [92]. The complete symmetry group properties for systems have been explored in Andriopoulos and Leach [42].

The investigation of symmetry breaking for systems of second-order linear equations in general remains an open problem. The complete explicit group classification for arbitrary second-order systems is also an open problem, even for the case of a system of two second-order equations. The complete explicit group classification for scalar nth-order ($n>3$) equations too is an open problem. Some inroads have been made for $n=4$ by Cerquetelli *et al.* [93]. Also of great interest is the study of the classification problem of scalar third-order equations *via* differential invariants of equivalence groups.

There are essentially two main approaches to integrability of systems of ordinary differential equations invoking Lie point symmetries. One is reduction *via* differential invariants or integrals and the other is the approach of integrating canonical forms. The interested reader is referred to Lie [3], Bianchi [45], Eisenhart [94], Ovsiannikov [95], Stephani [96], Bluman and Anco [97], Leach and Mahomed [98], Ibragimov and Nucci [99], Olver [5, 100], Ibragimov [31], Hydon [101], and Wafo and Mahomed [91, 102], for details.

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