Catastrophe Theory

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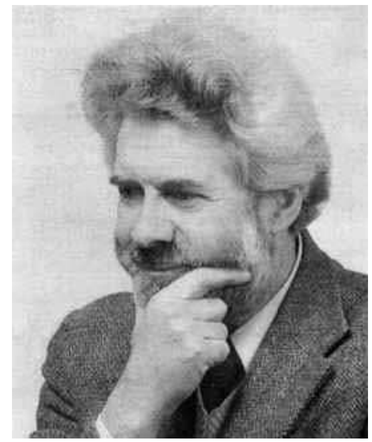
Vladimir I. ARNOLD

Rene THOM



Christopher ZEEMANN

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Catastrophe Theory studies and classifies phenomena characterized by sudden shifts in behavior arising from small changes in circumstances.

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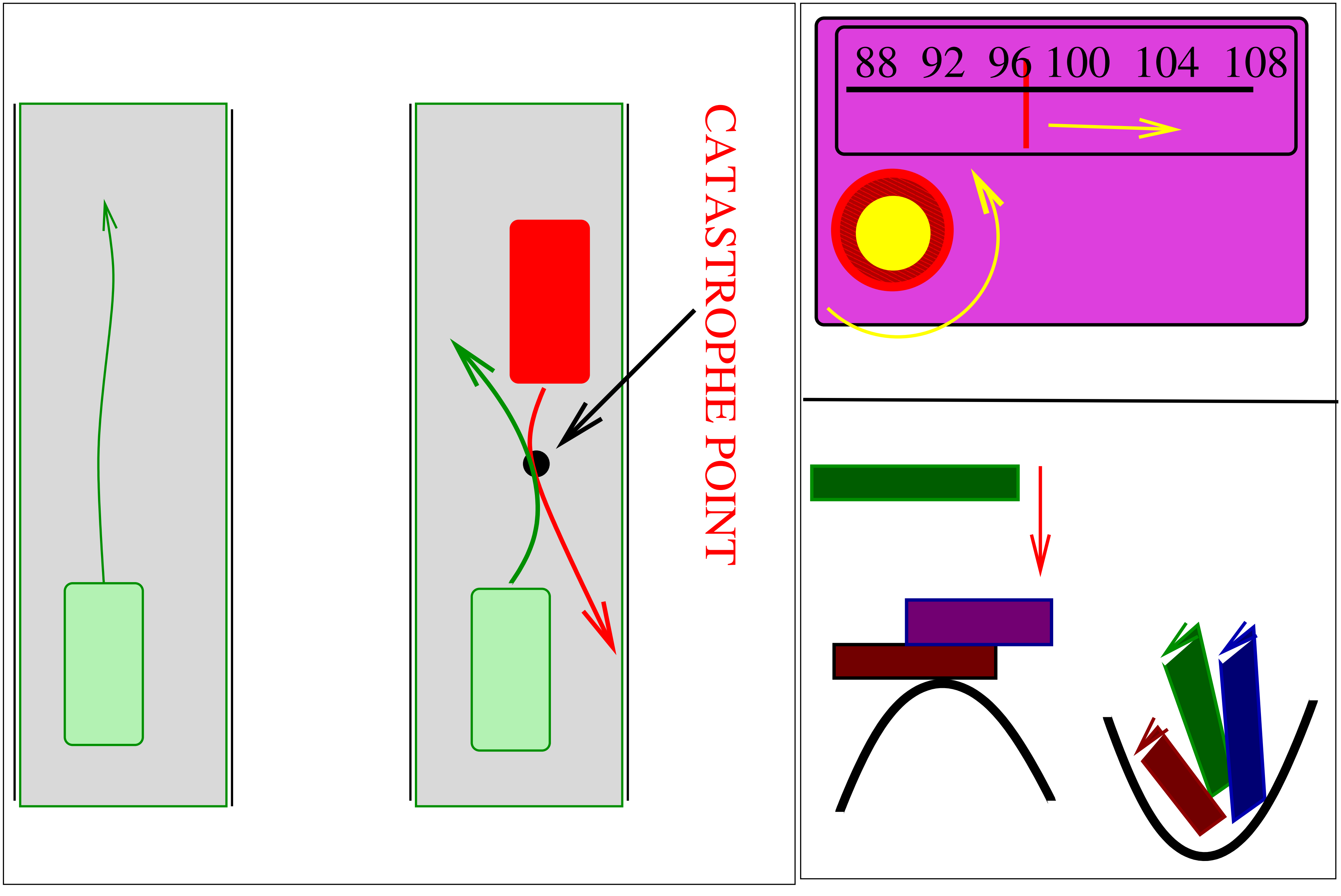
Catastrophe Theory studies and classifies phenomena characterized by sudden shifts in behavior arising from small changes in circumstances.

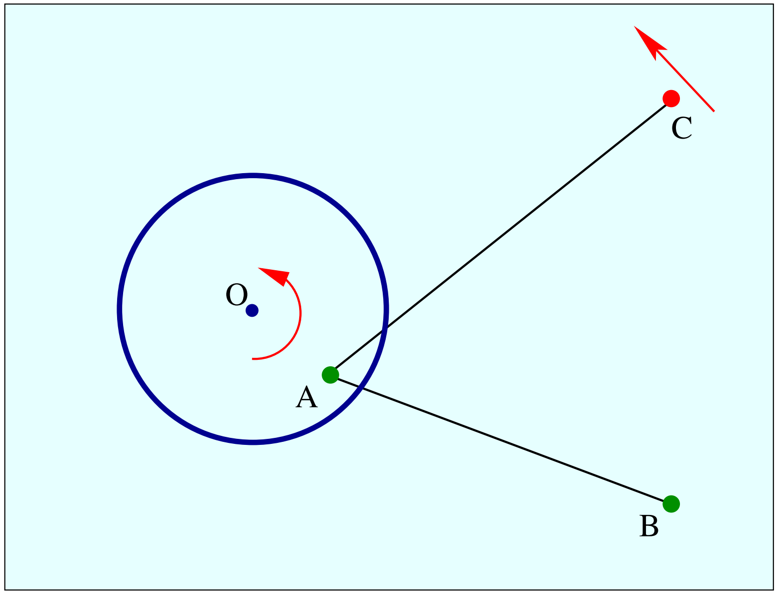
Small changes in certain parameters of a nonlinear system can cause equilibria to appear or disappear, or to change from attracting to repelling and vice versa, leading to large and sudden changes of the behaviour of the system. Catastrophe theory has been applied to a number of different physical, chemical, biological, and even social phenomena.

First Examples



Zeemann machine

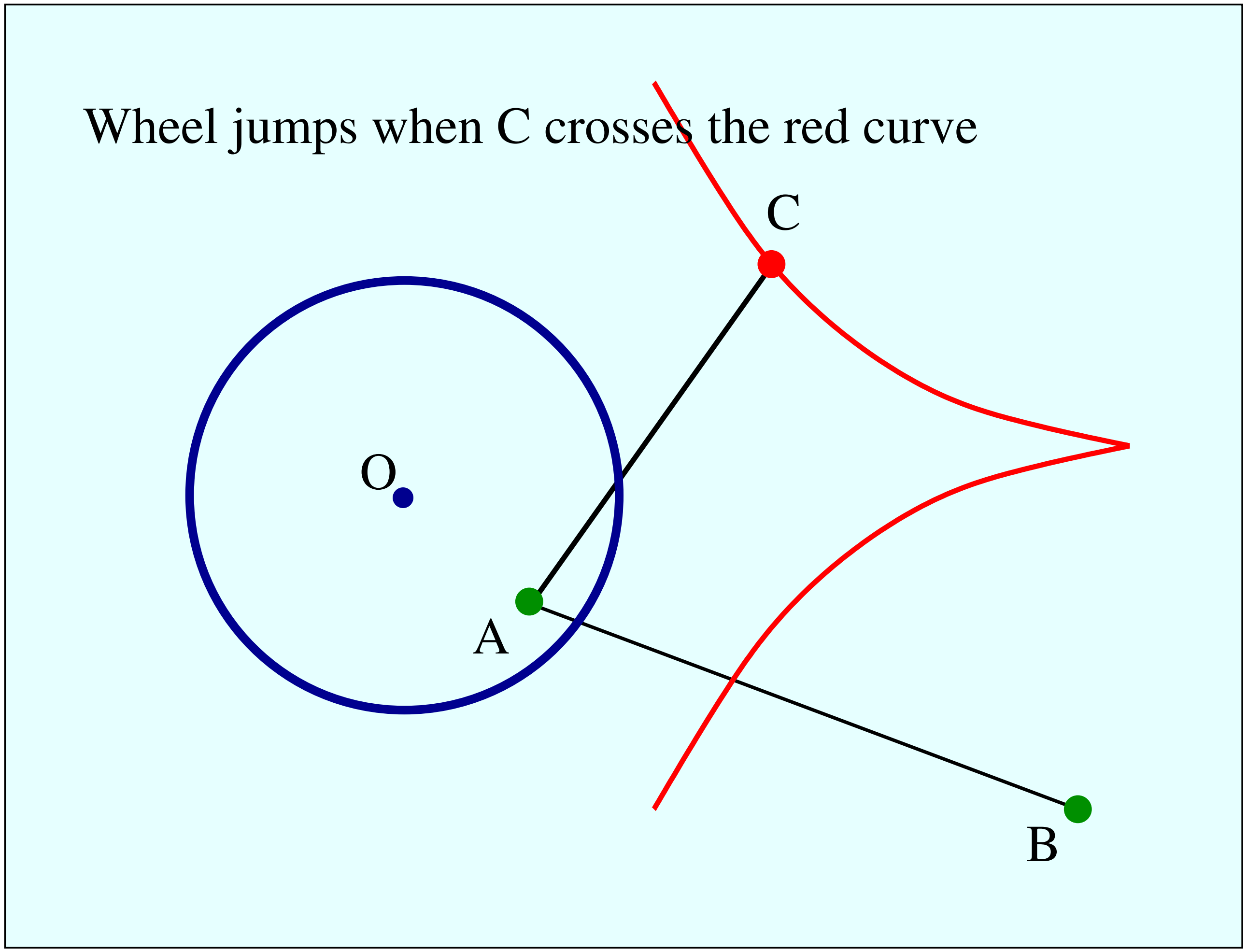
This device, invented by the mathematician Christopher Zeeman consists of a wheel which is tethered by an elastic to a fixed point B in its plane. The control input to the system is another elastic attached to the same point A as the first and roughly of the same length. The other end C of the elastic can be moved about an area diametrically opposite to the fixed point.



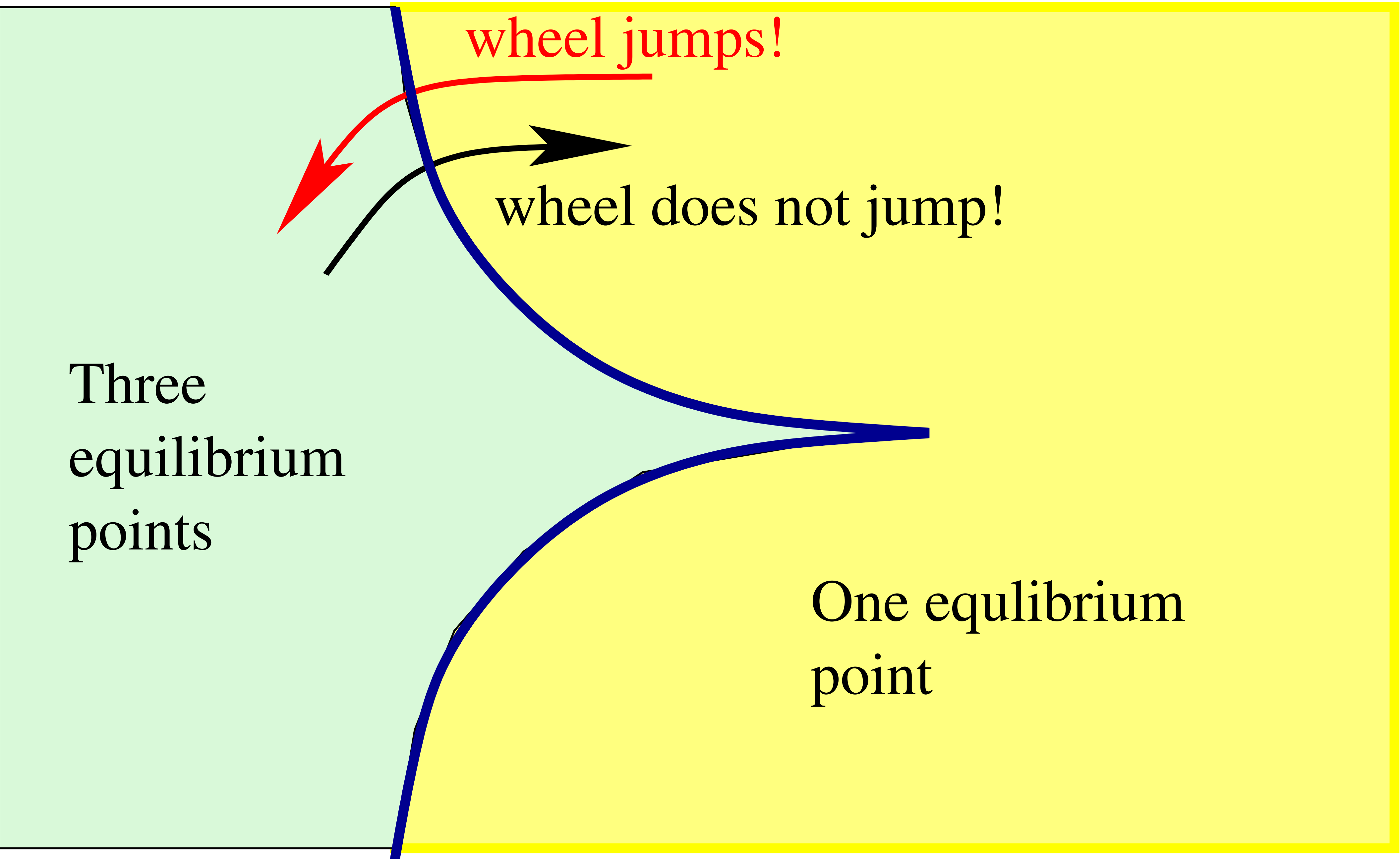
The experiment shows that:

- To each position of C there correspond 1 or 3 equilibrium points of the system.
- If we slowly move C then usually the wheel rotates slowly (this means that the equilibrium point slowly changes), however sometimes the wheel (the equilibrium point) suddenly jumps!

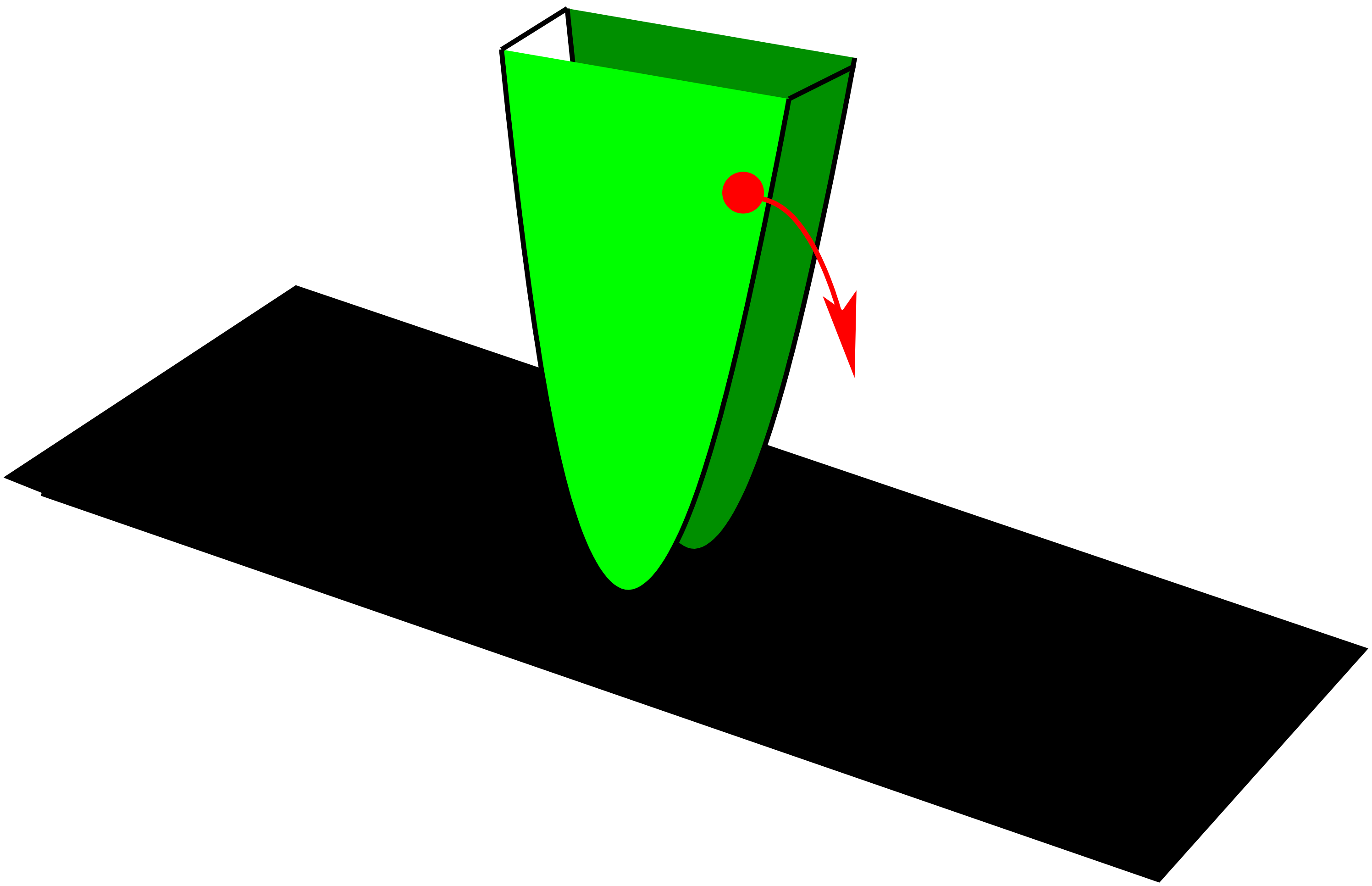
- The positions of C where the equilibrium point jumps lie on a cusp.



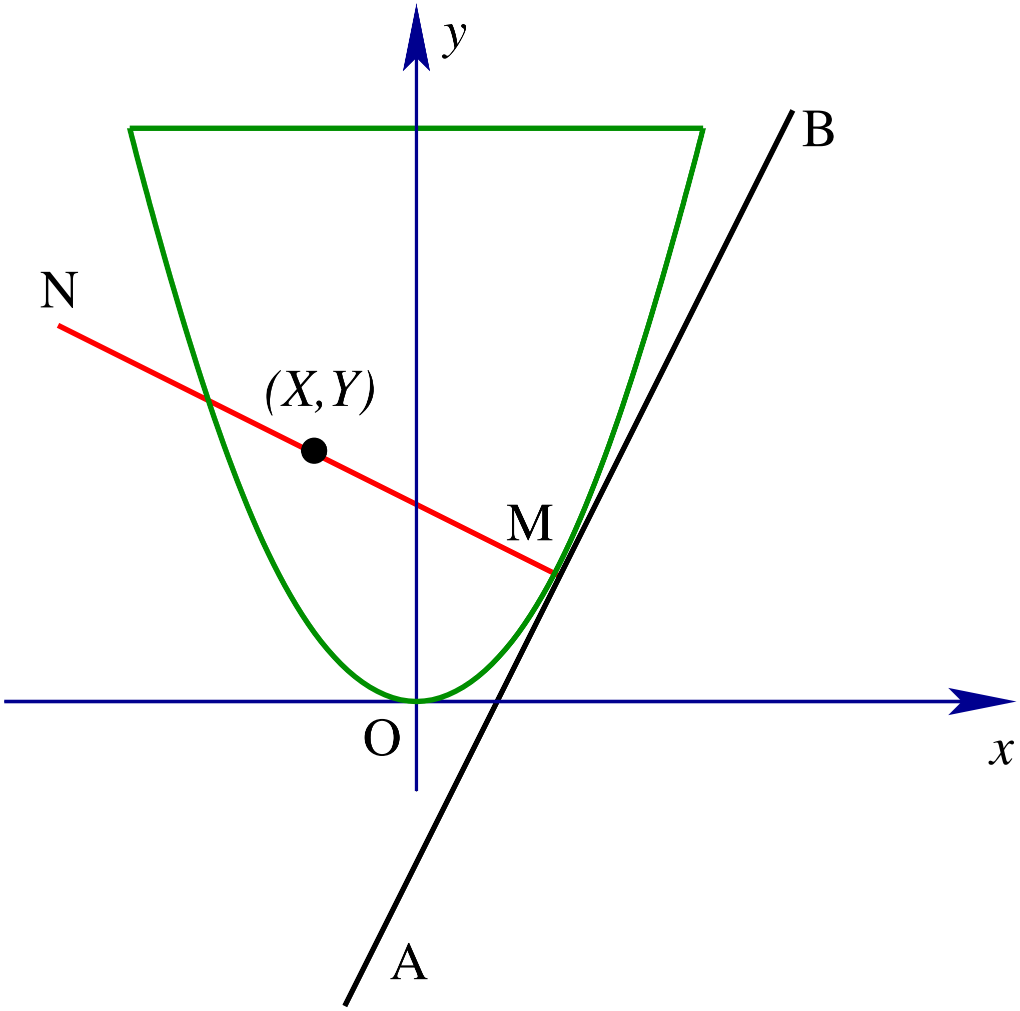
- The cusp is the common boundary of two regions: I) the set of C such that the system has one equilibrium point; II) the set of C such that the system has three equilibrium points;
- **Hysteresis behavior.** If the equilibrium point jumps while C crosses the boundary, then while C is moving in the opposite direction the equilibrium point does not jump!


wheel jumps!
wheel does not jump!
Three
equilibrium
points
One equlibrium
point

Let us consider another machine: “parabolic rock”.



Consider the mathematical model:



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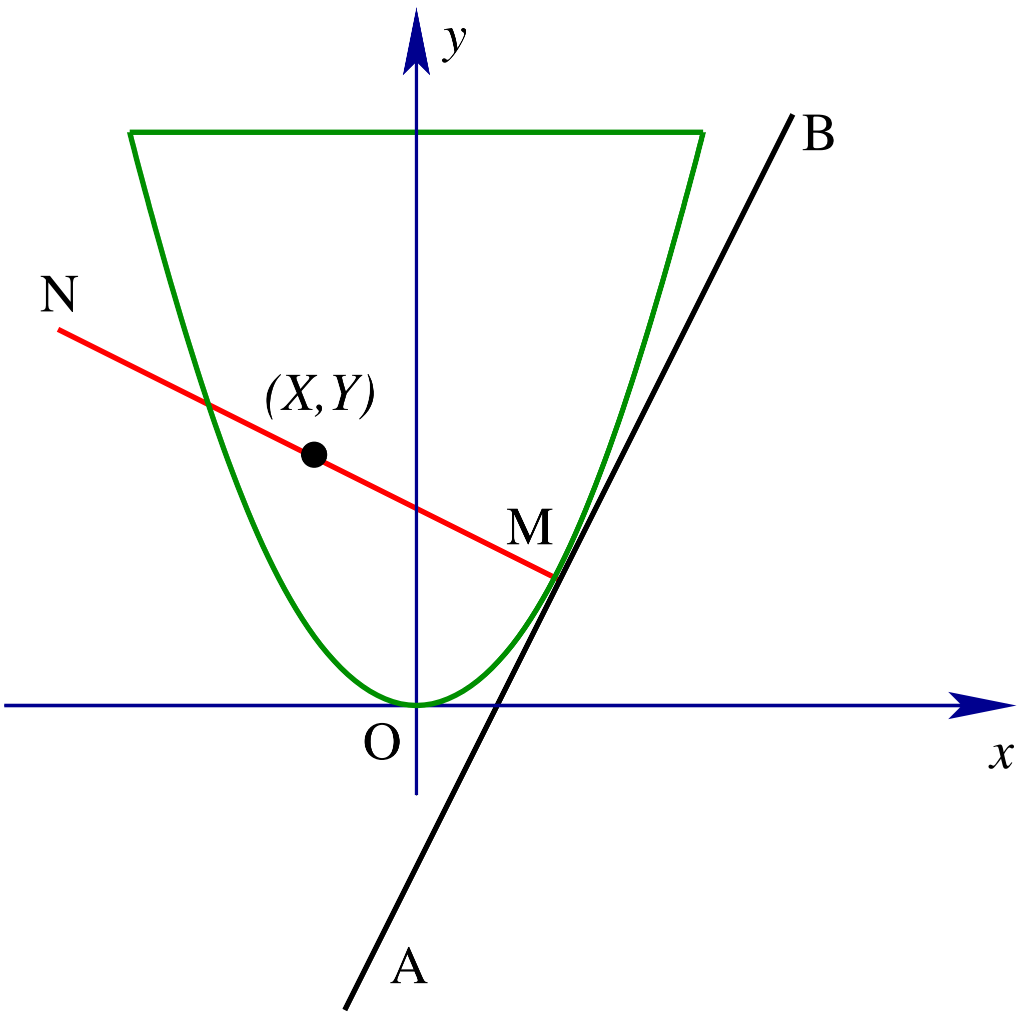

$$y = x^2 \Rightarrow$$

$$x = t,\ y = t^2 \Rightarrow$$

$$MN : (x - t) + 2t(y - t^2) = 0$$

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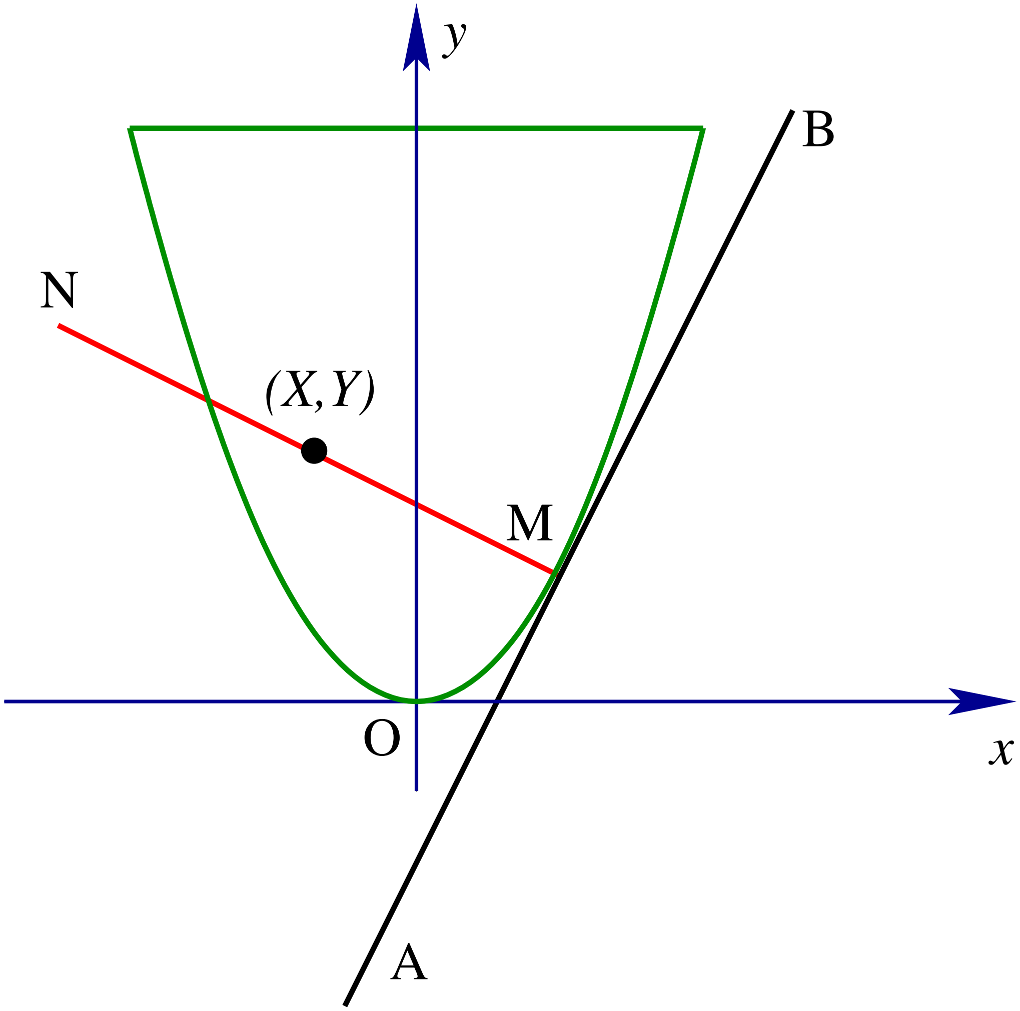


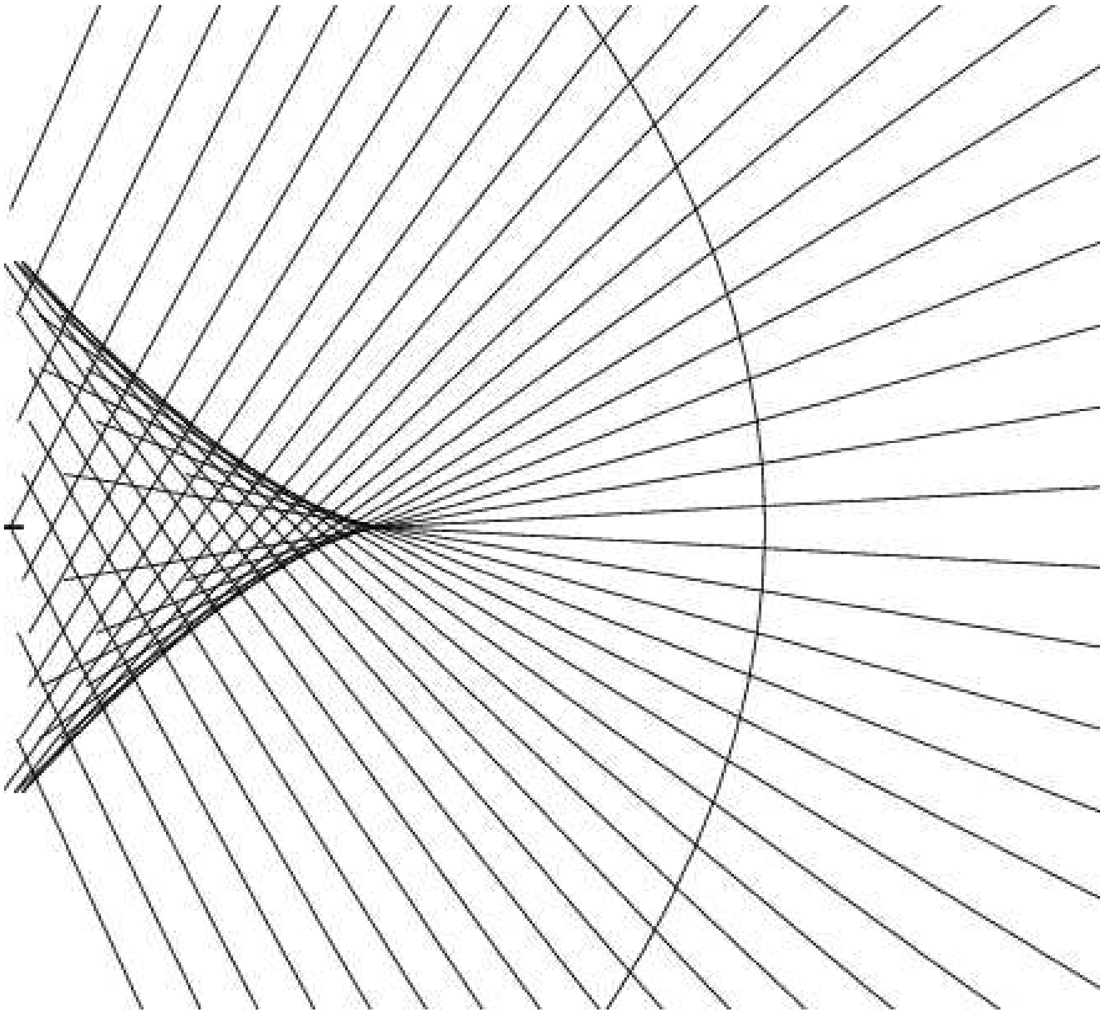
$$y = x^2 \Rightarrow$$

$$x = t,\ y = t^2 \Rightarrow$$

$$MN : (x - t) + 2t(y - t^2) = 0$$

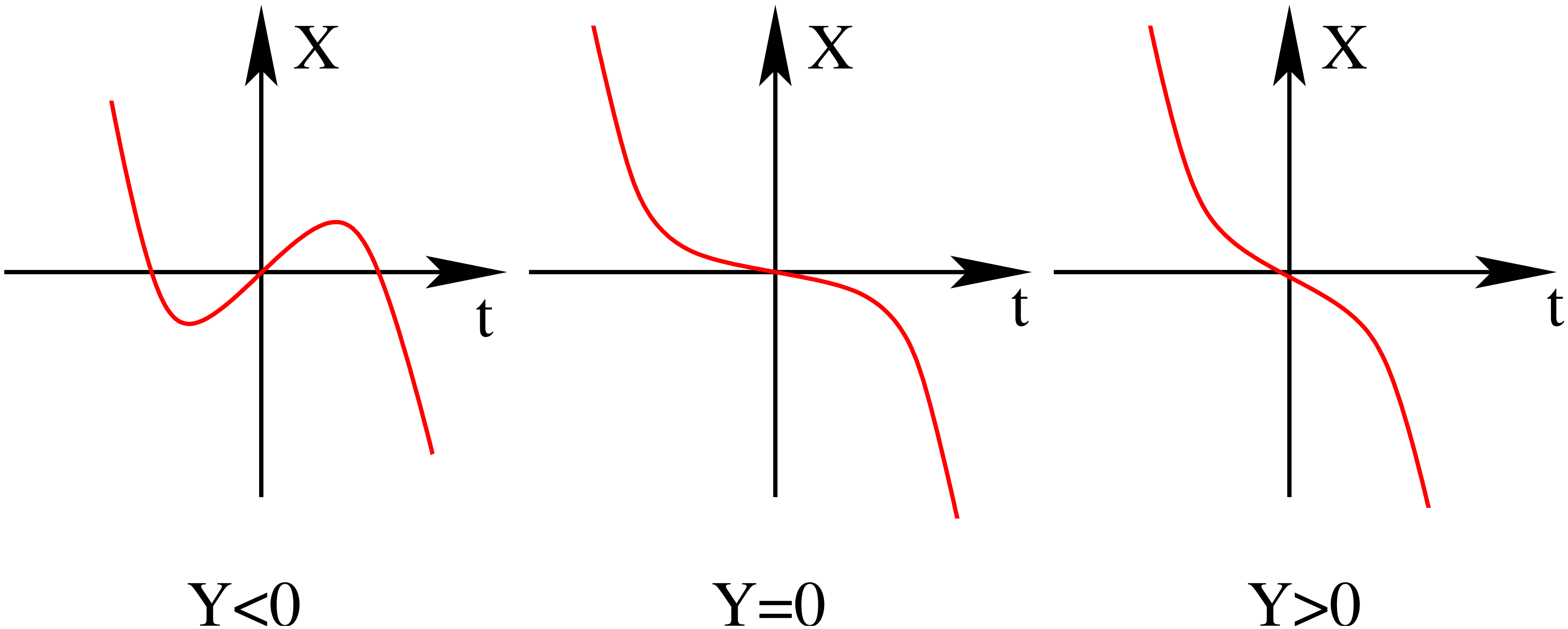
Then, setting $X = -\frac{1}{2}x$, $Y = \frac{1}{2}(1 - 2y)$, we get the equation $t^3 + Yt + X = 0$

The set of normals to parabola.

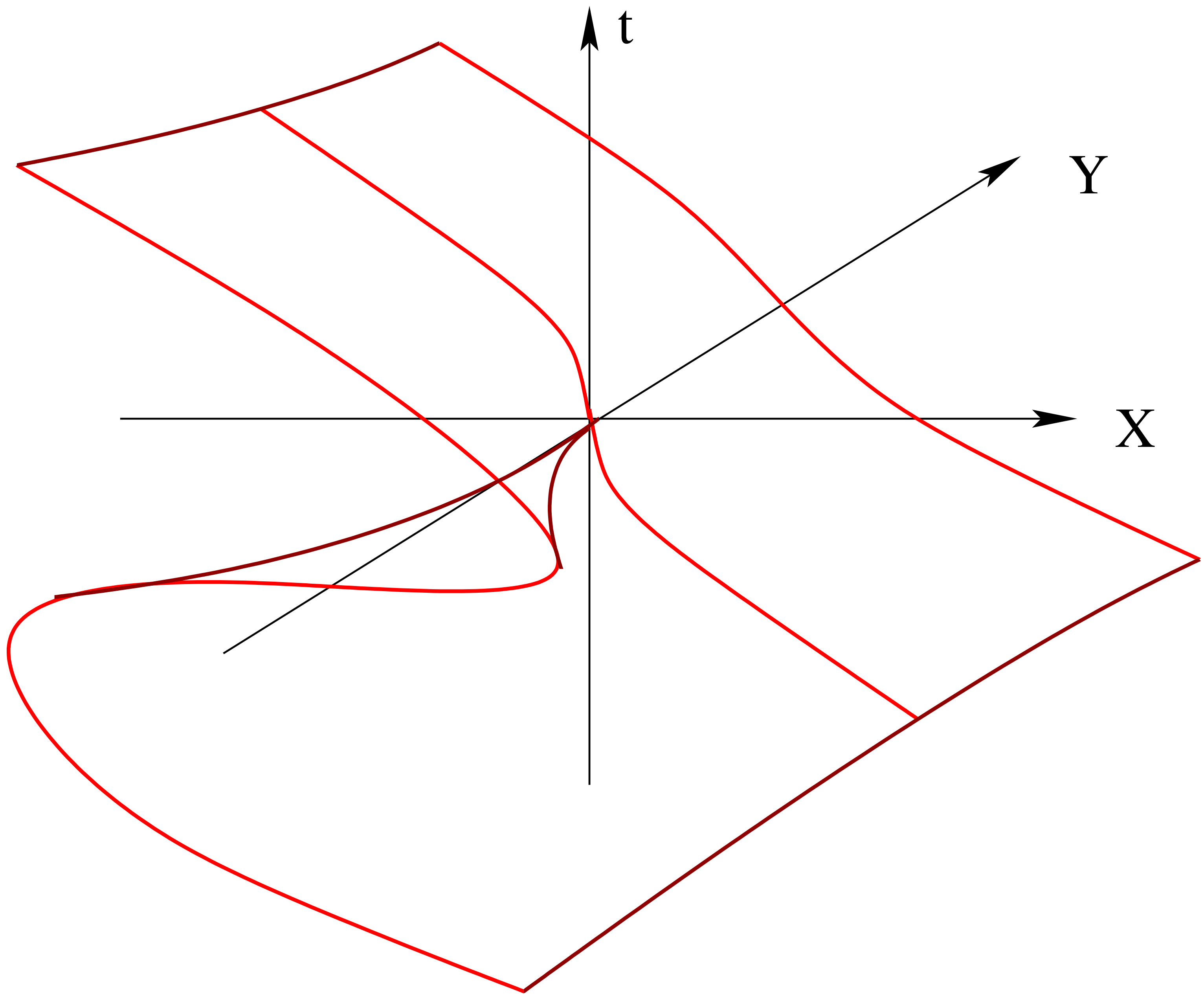


Let us construct the graph of the surface $\Sigma : t^3 + Yt + X = 0$ in the three-dimensional space with coordinates (X, Y, t).

Consider the graph of Σ as the family of graphs of the function $X = -t^3 - Yt$ while Y varies.

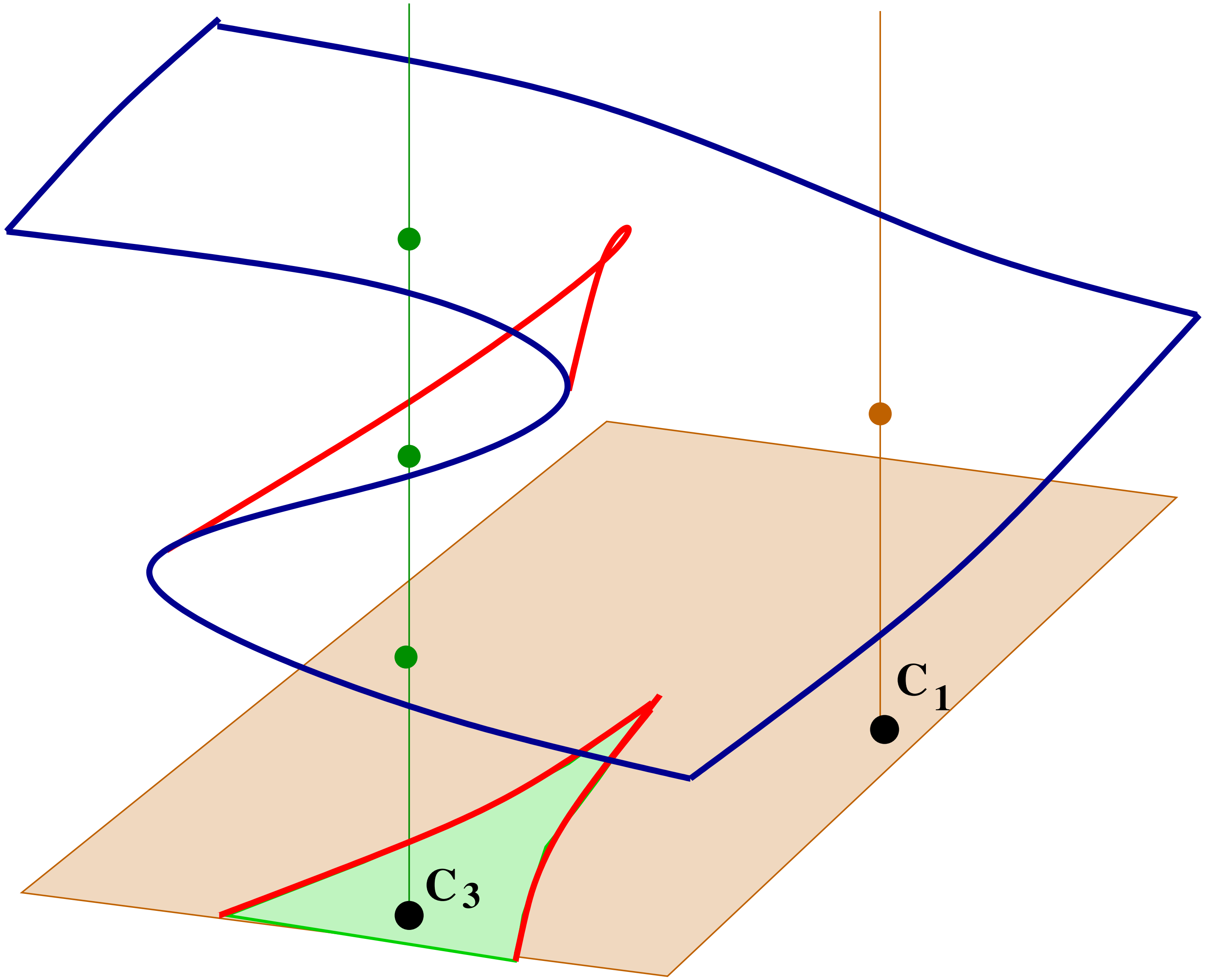


The graph of $\Sigma : t^3 + Yt + X = 0$.

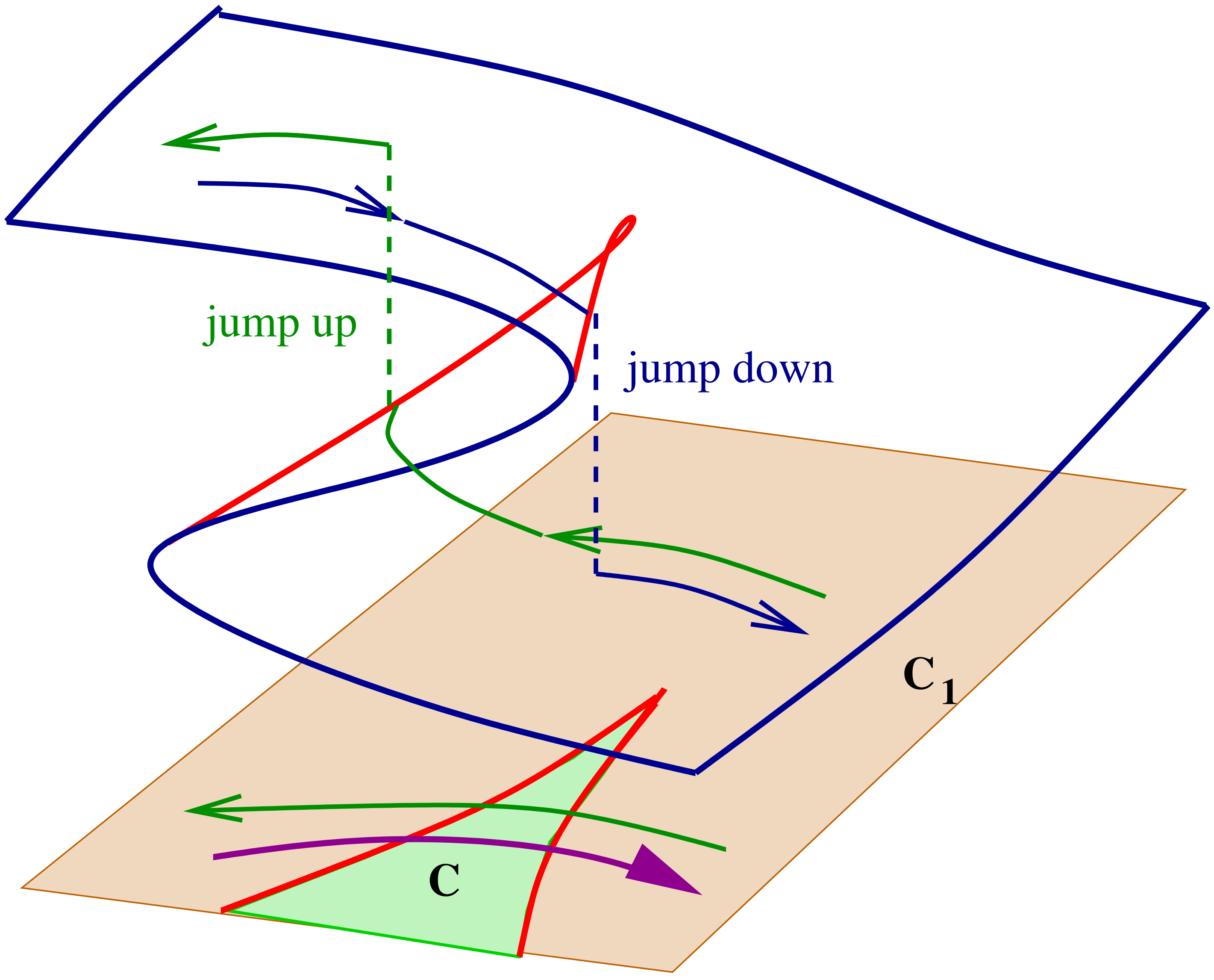


Explanation of experimental phenomena.

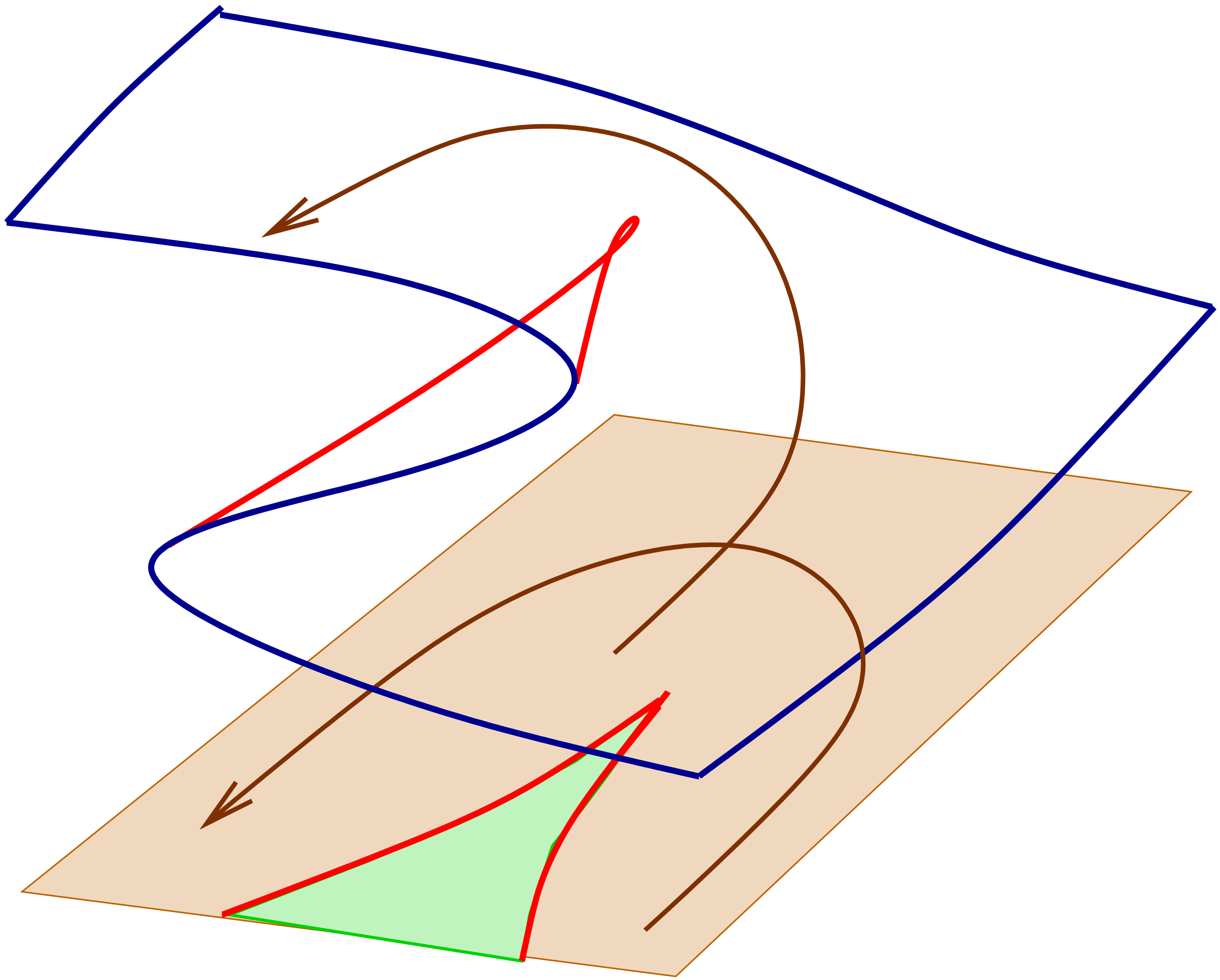
1. The number of equilibrium points depends of the position of C.



2. The equilibrium point jumps when C crosses the cusp.



3. One can avoid the jump (the catastrophe)



The mathematical formulation.

We are given a system with one degree of freedom (one-dimensional configuration space) and m-dimensional control space.

t is a coordinate in the configuration space (a parameter of the system);

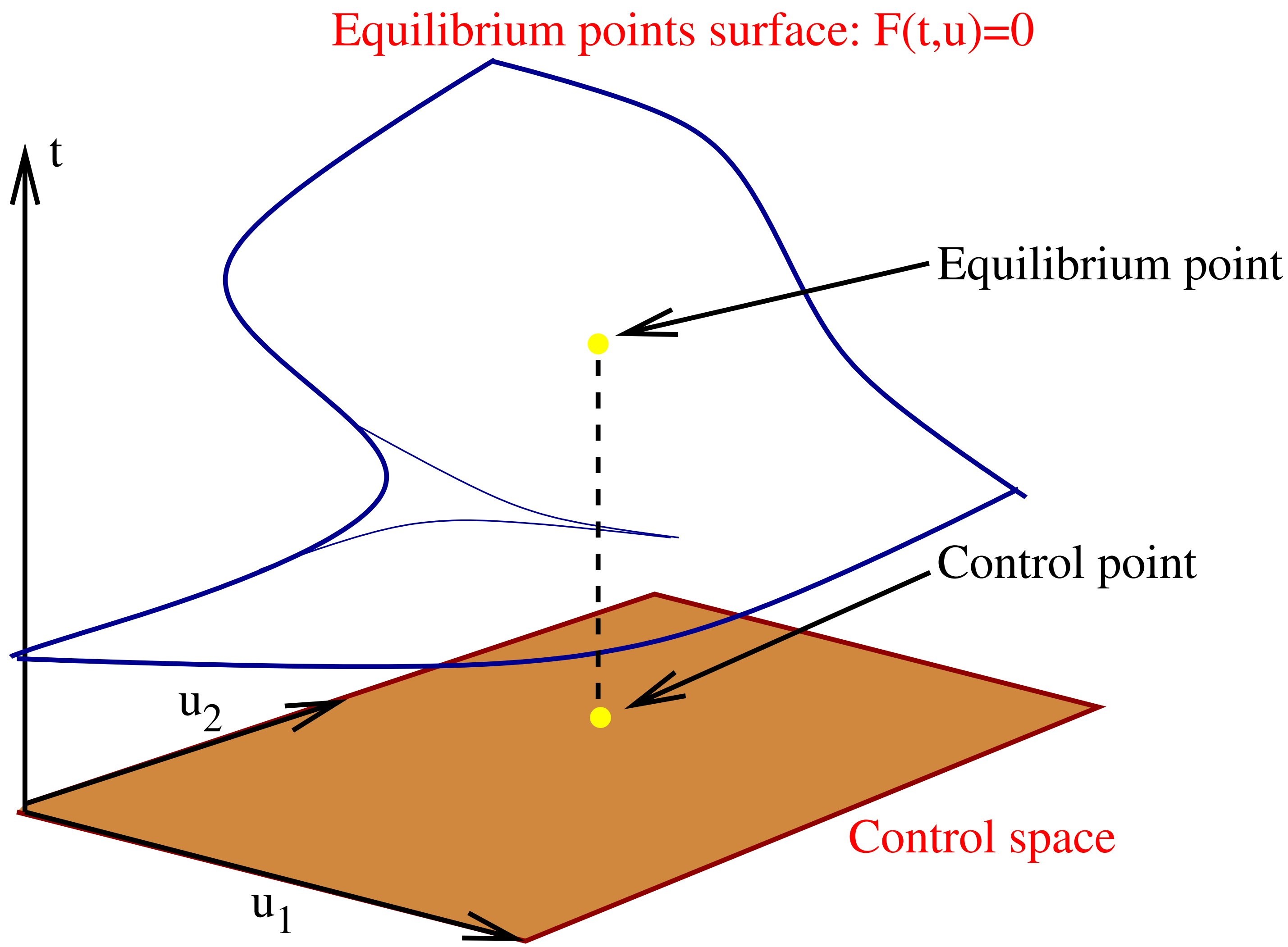
$u_1, u_2, \ldots, u_m$ are coordinates in the control space (control parameters);

$U(t, u_1, \ldots, u_m)$ is the "potential energy" of the system;

The main problem: How does an equilibrium point of the system behave when we vary control parameters?

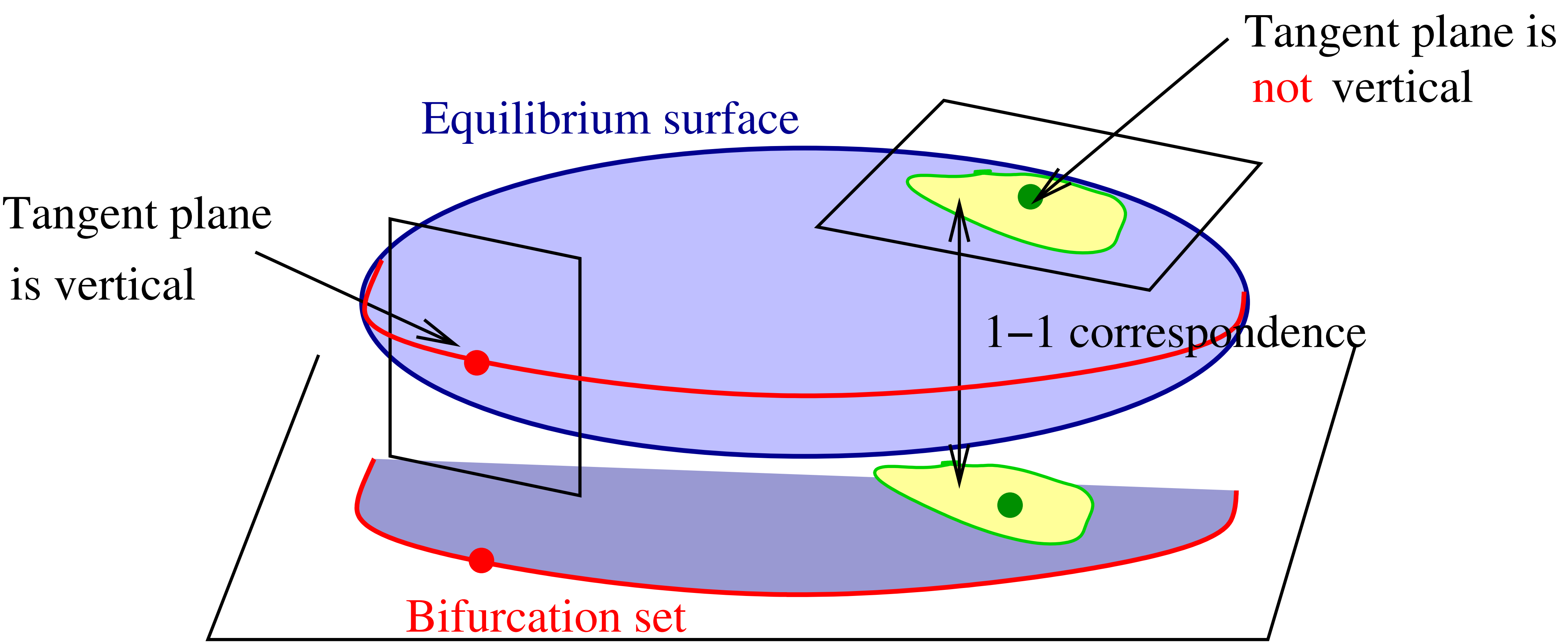
The surface of "equilibrium points":

$$F(t, u_1, \ldots, u_m) = \frac{\partial}{\partial t} U(t, u_1, \ldots, u_m) = 0$$



Given $F(t, u_1, \ldots, u_n)$ we define **the bifurcation set**:

$$B_F = \{u \ \ | \ \ \exists t_0 : \frac{\partial}{\partial t} F(t_0, u) = 0\}$$



Theorem. Assume that $\frac{\partial^j}{\partial t^j}U(t, u_1, \ldots, u_m) = 0$, for all $j = \overline{1, k}$, and $\frac{\partial^{k+1}}{\partial t^{k+1}}U(t, u_1, \ldots, u_m) \neq 0$.

Then, "in the general situation", one can change coordinates:

$$t = t(s, v) \quad u = u(v)$$

in such a way that

$$U(t(s, v), u(v)) = s^{k+1} + v_{k-1}s^{k-1} + \cdots + v_1 s$$

Note that $k < m$!

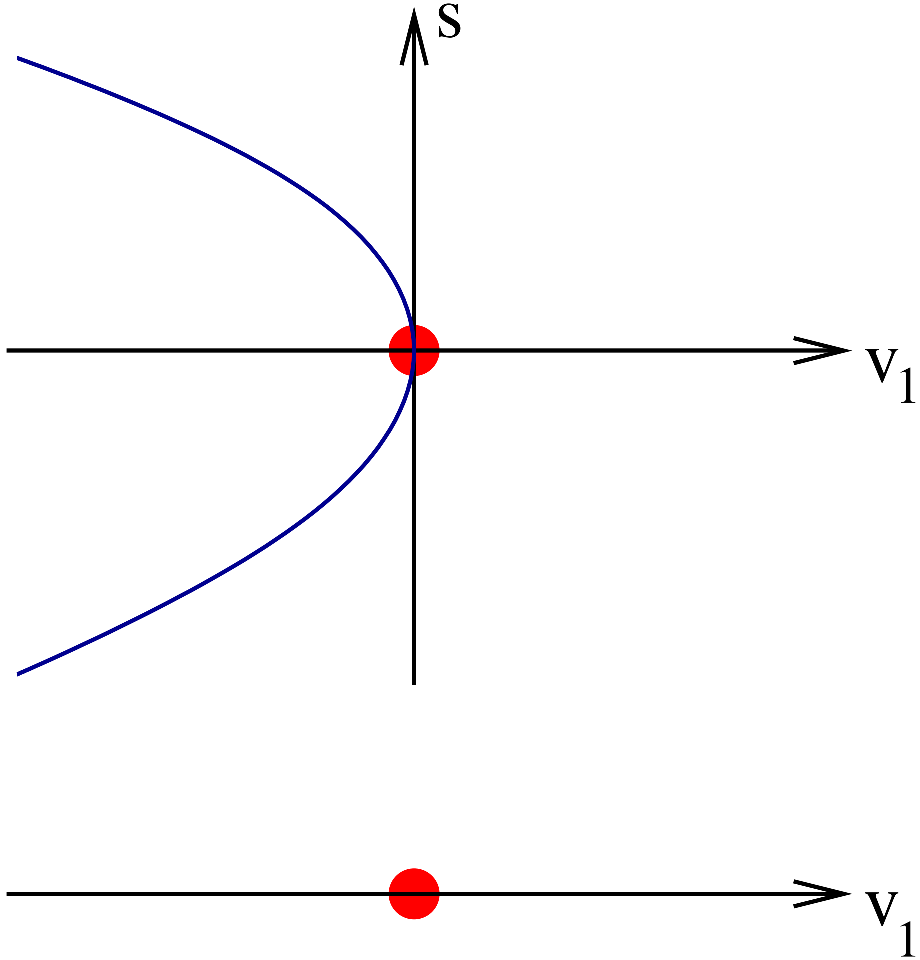
This means that, in order to solve our problem, we can study concrete surfaces!

Case $k = 2$;

Potential energy: $U(s, v_1) = s^3 + v_1 s$

Equilibrium surface: $U(s, v_1) = 3s^2 + v_1$

Bifurcation set: $v^1 = 0$

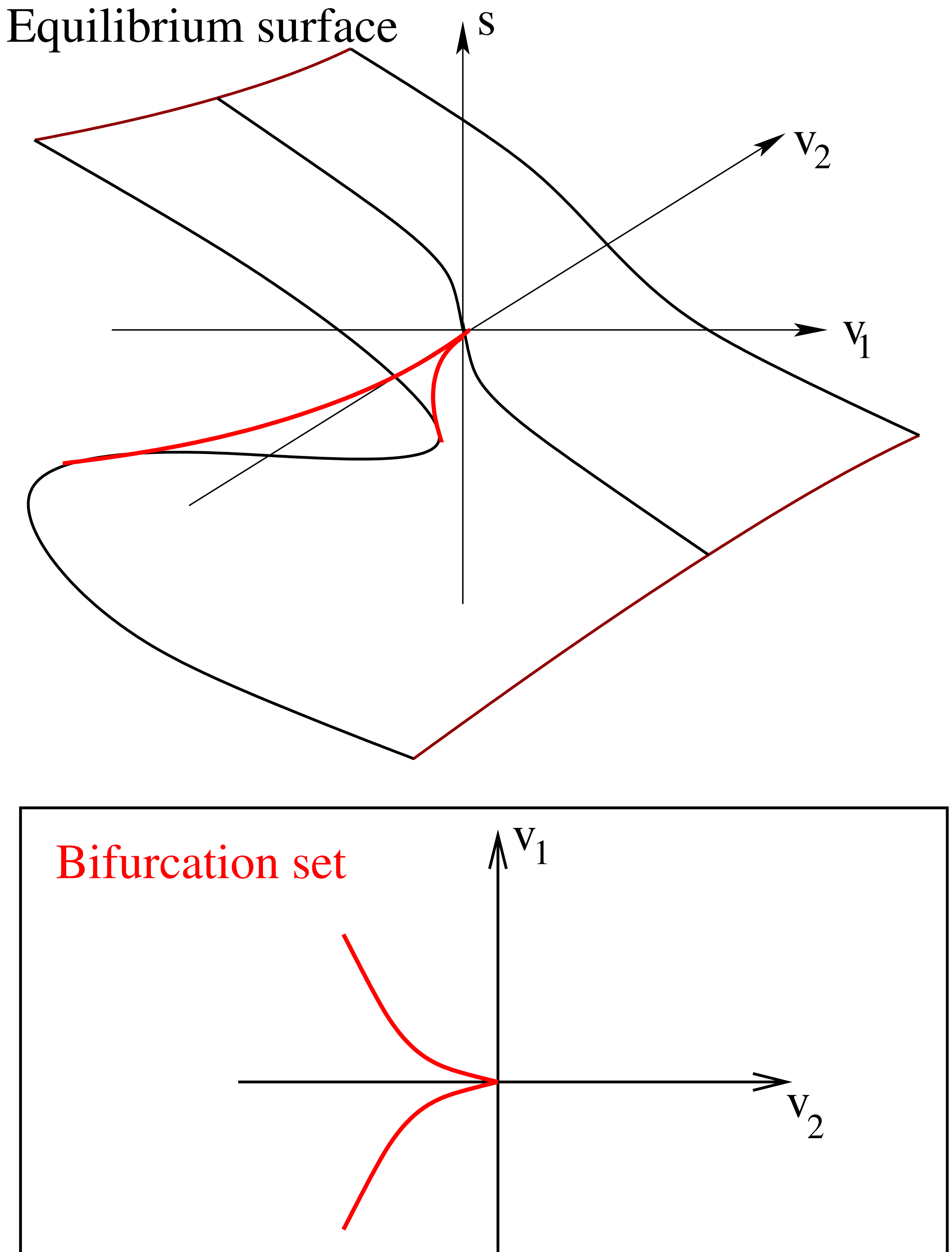


Case $k = 3$;

Potential energy: $U(s, v_1) = s^4 + v_2 s^2 + v_1 s$

Equilibrium surface: $F(s, v_1) = 4s^3 + 2v_2 s + v_1$

Bifurcation set: $\frac{v_1^2}{8} + \frac{v_2^3}{27} = 0$



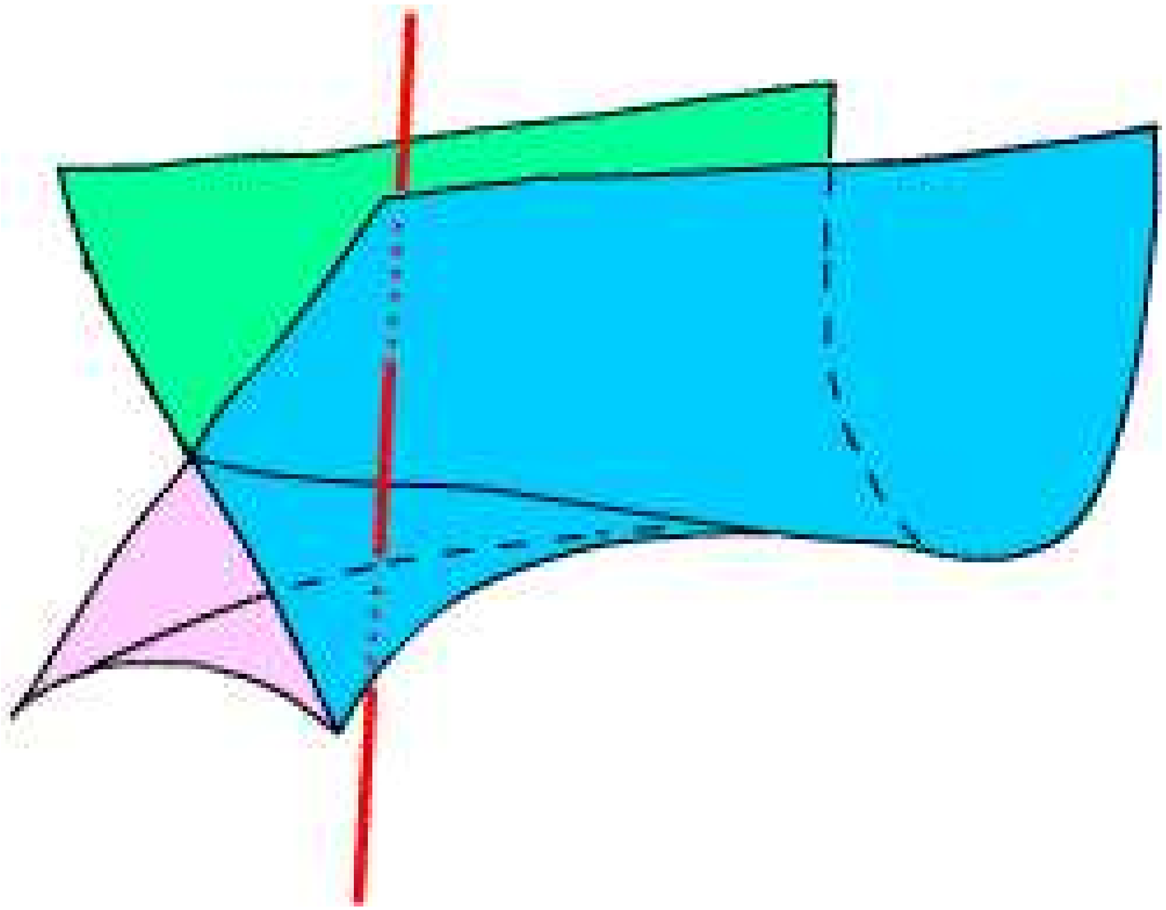
Equilibrium surface
s
v_2
v_1
Bifurcation set
v_1
v_2

Case $k = 4$;

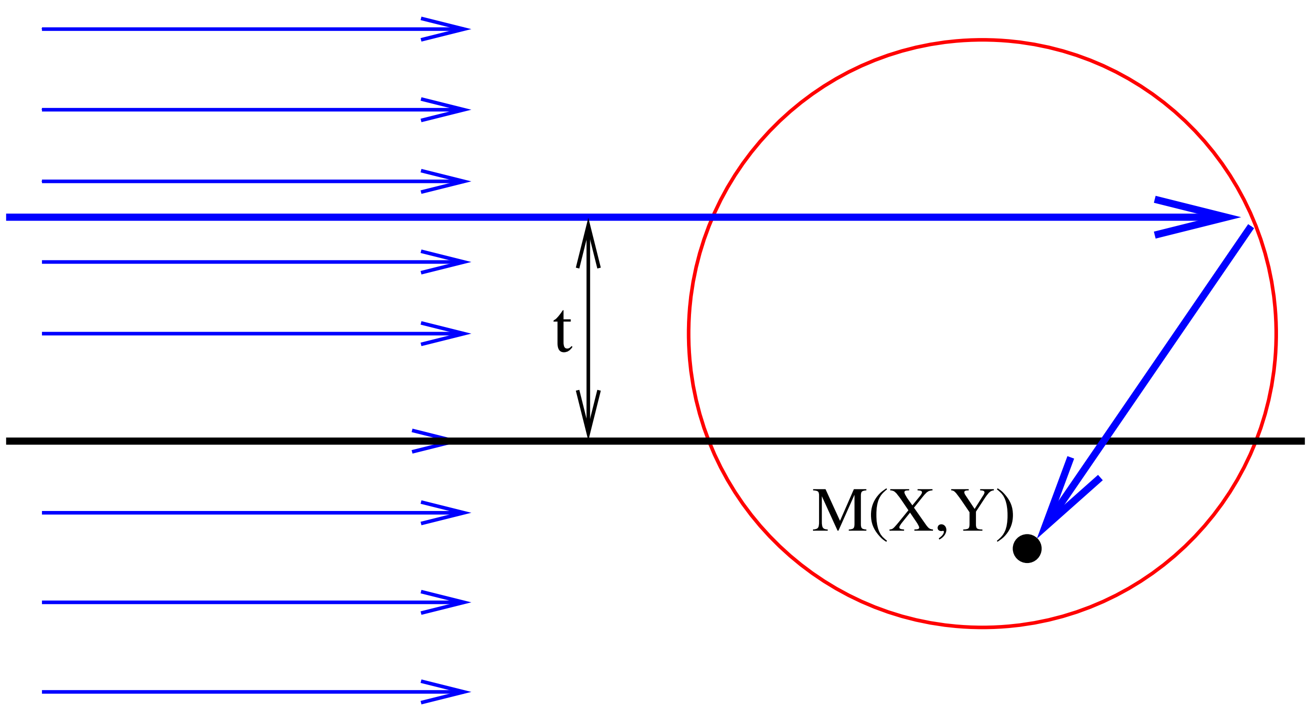
Potential energy: $U(s, v_1) = s^5 + v_3 s^3 + v_2 s^2 + v_1 s$

Equilibrium surface: $F(s, v_1) = 5s^4 + 3v_3 s^2 + 2v_2 s + v_1$

Bifurcation set:



Optics

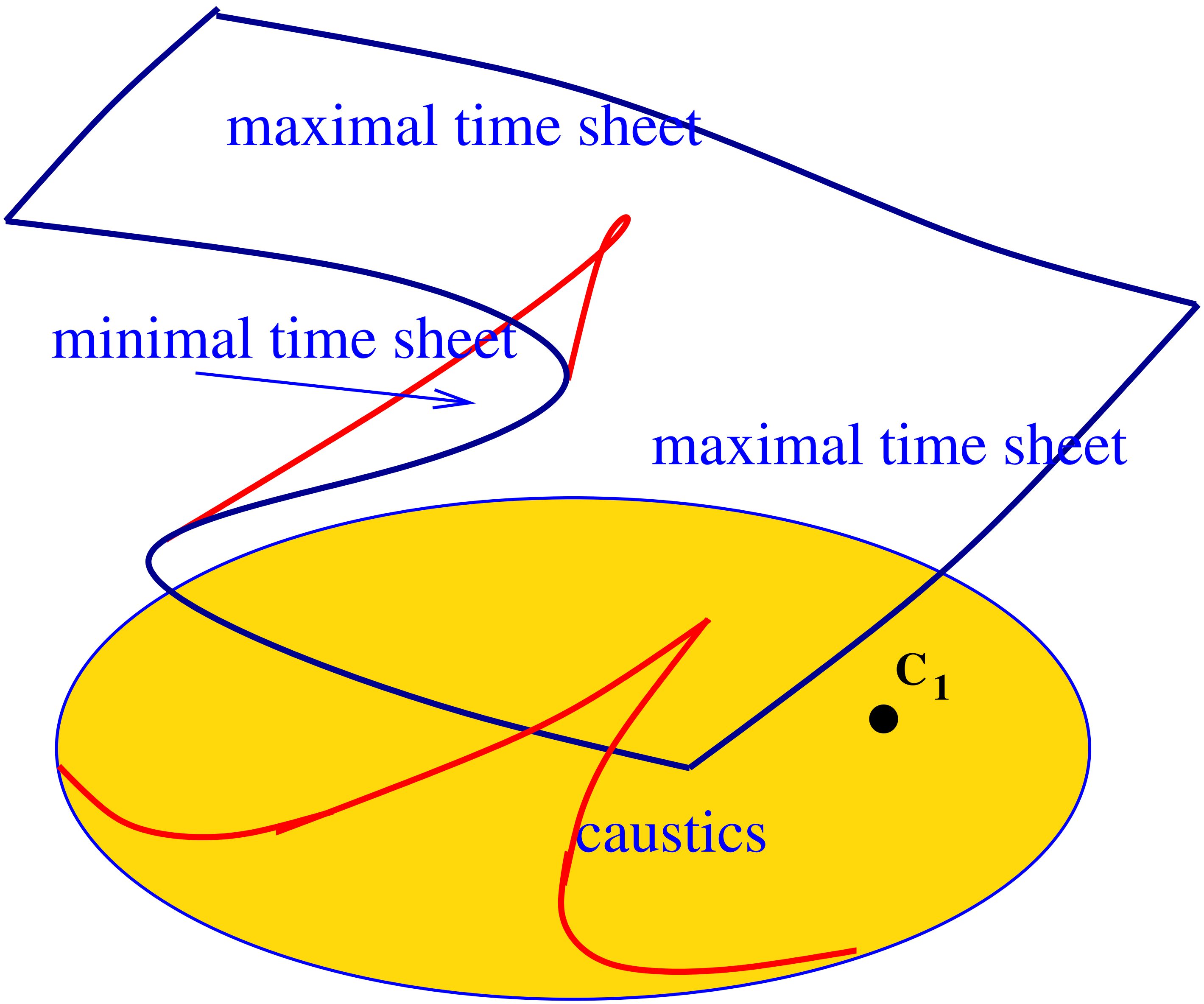


Let $L(X, Y, t)$ be the length of the path joining ∞ and $M(X; Y)$.

$$L(X, Y, t) \approx -\frac{1}{4}(1 + 5X)t^4 + \frac{1}{2}Yt^3 + Xt^2 - 2Yt$$

Fermat's principle $\Rightarrow \frac{\partial L}{\partial t}(X, Y, t) = 0$.

Then $L(X, Y, t)$ is equivalent to $t^4 + v_2 t^2 + v_1 t$.



Chemistry

Van der Waals Equation of State

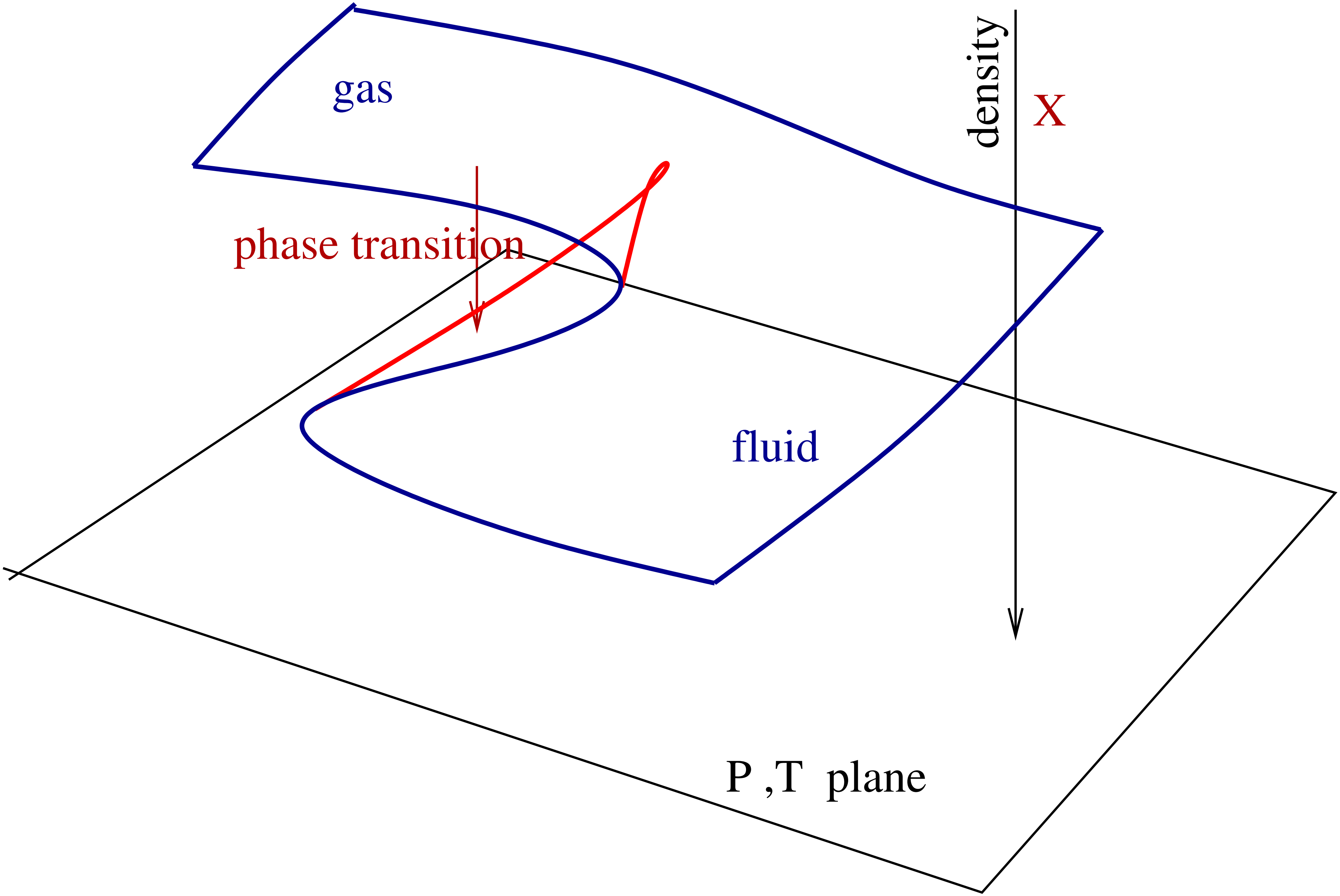
$$(P + \frac{\alpha}{V^2})(V - \beta) = RT$$

Here P is the gas pressure, V is the gas volume, and T is the gas temperature, and α, β are parameters depending on the gas molecule properties.

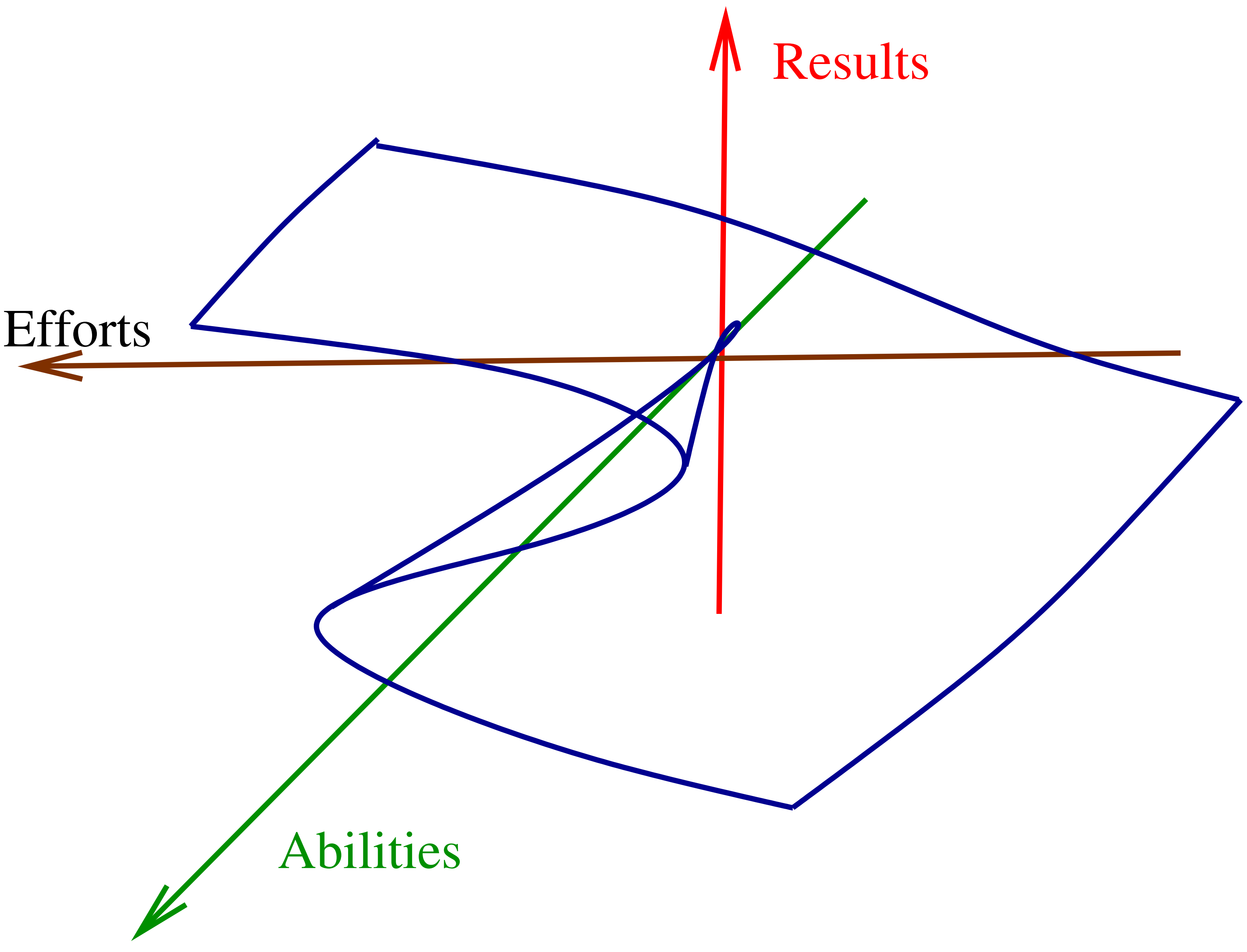
Set $X = \frac{1}{V}$, then after a linear change of coordinates we appear at the equation

$$X^3 + aX + b = 0,$$

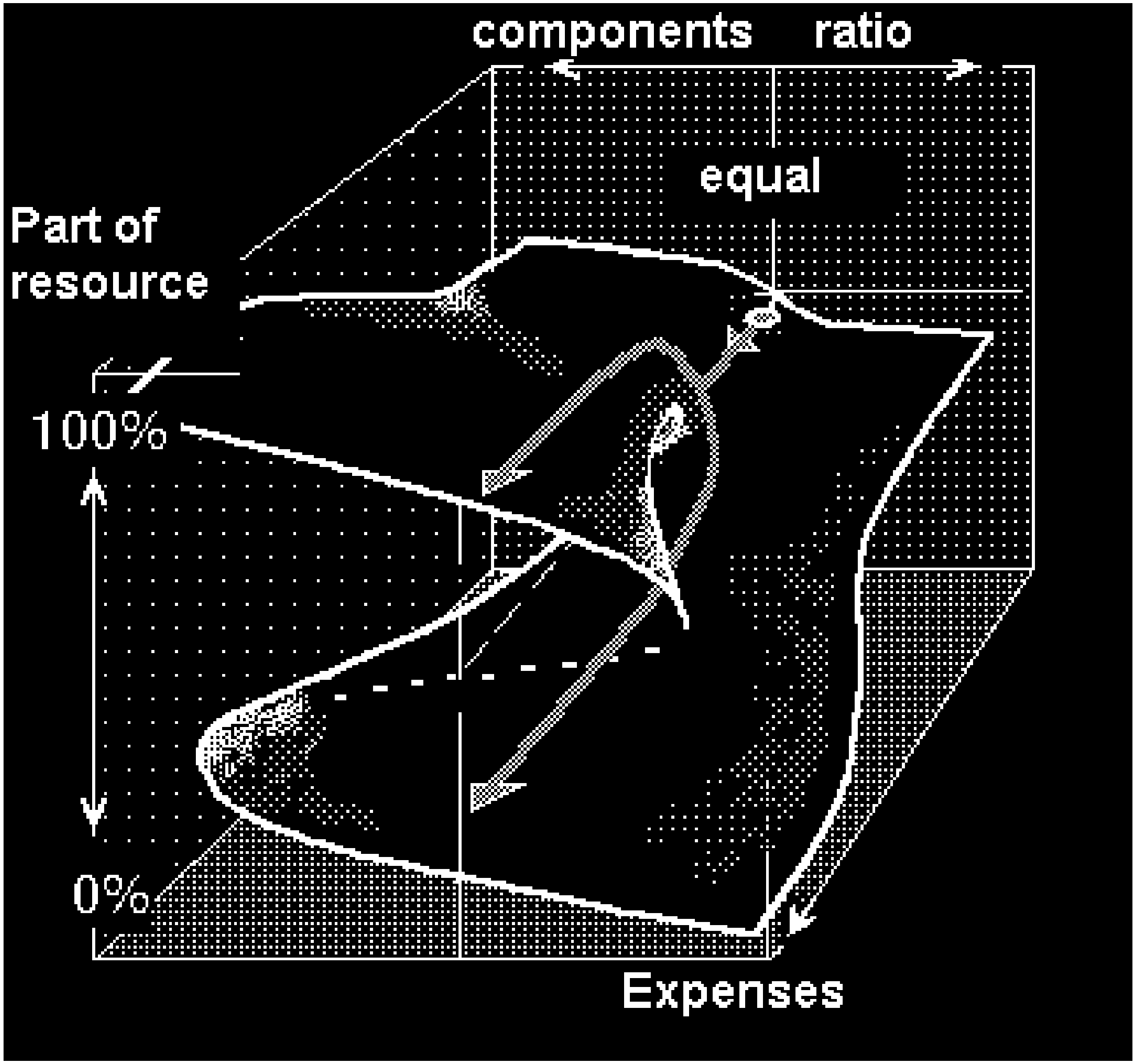
where a, b are linearly expressed in terms of P, T.


gas
density
X
phase transition
fluid
P ,T plane

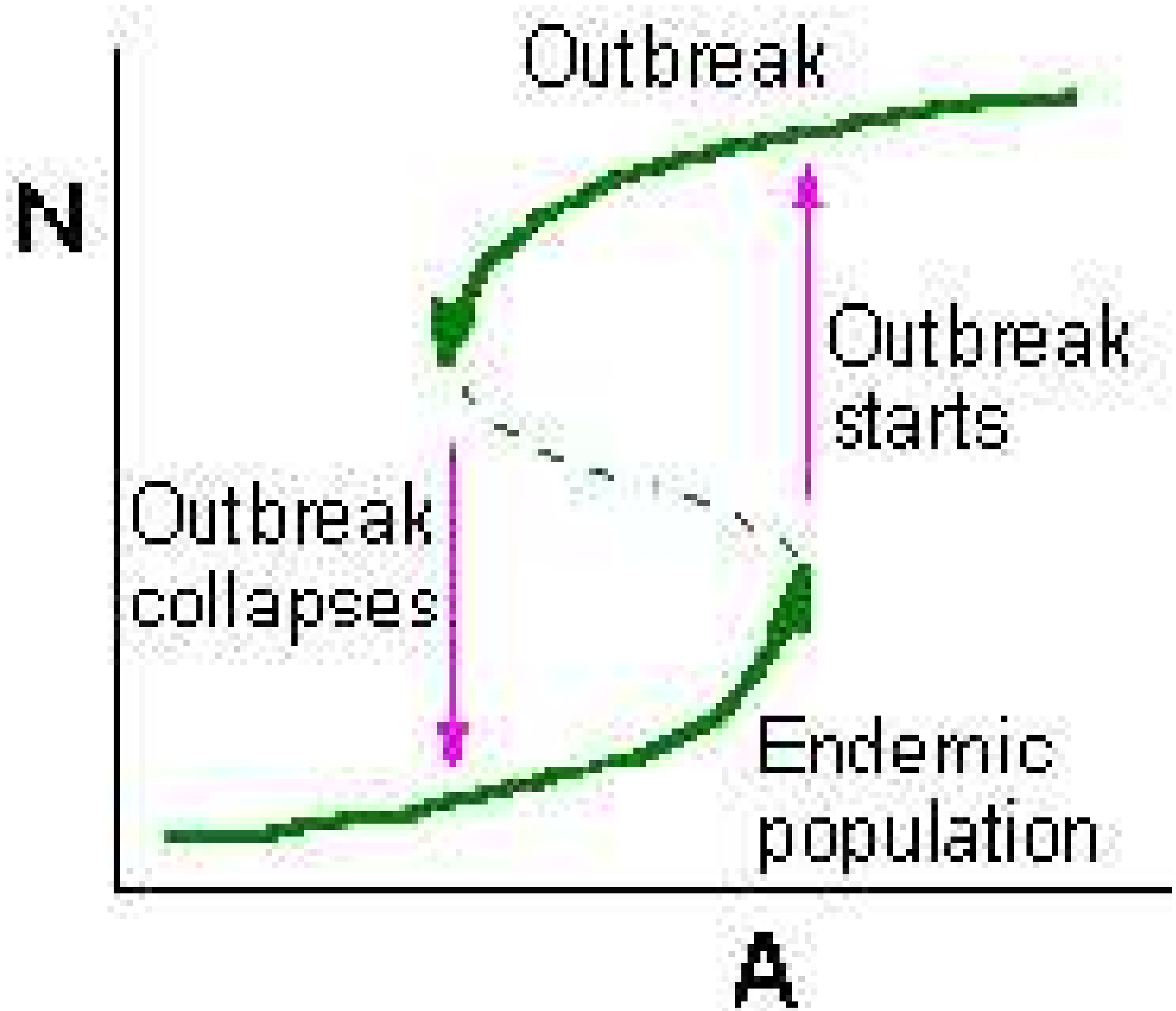
Psychology



Finances



Ecology



N
Outbreak
Outbreak
starts
Outbreak
collapses
Endemic
population
A

Real surface

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