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THE LIE DERIVATIVE AND COHOMOLOGY OF G-STRUCTURES

(submitted by B.N.Shapukov)

Abstract. In [1], J.F. Pommaret constructed the so-called Spencer P-complex for a differential operator. Applying this construction to the Lie derivative associated with a general pseudogroup structure on a smooth manifold, he defined the deformation cohomology of a pseudogroup structure. The aim of this paper is to specify this complex for a particular case of pseudogroup structure, namely, for a first-order G-structure, and to express this complex in differential geometric form, i. e., in terms of tensor fields and the covariant derivative. We show that the Pommaret construction provides a powerful tool for associating a differential complex to a G-structure. In a unified way one can obtain the Dolbeaut complex for the complex structure, the Vaisman complex for the foliation structure [2], and the Vaisman–Molino cohomology for the structure of manifold over an algebra [3].

1. Preliminaries

Let M be a smooth n-dimensional manifold, $L(M) \to M$ the frame bundle of M, and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M.

For a Lie subgroup $G \subset GL(n)$, consider the bundle $E_G(M) = L(M)/G \to M$. Let $s\colon M \to E_G(M)$ be a section. The Lie derivative of s with respect to a vector field $X \in \mathfrak{X}(M)$ is defined in the following way. For each $X \in \mathfrak{X}(M)$, the flow ϕ_t of X induces the flow $d\phi_t$ on $L(M)$ whose projection onto $E_G(M)$ gives a flow $\overline{\phi}_t$ on $E_G(M)$. The tangent vector field $d\overline{\phi}_t/dt$ is the complete lift $\overline{X} \in \mathfrak{X}(E_G(M))$ of X. We denote by VE_G the vertical subbundle of $TE_G(M)$; then the pullback bundle $s^*(VE_G)$ is a vector bundle over M, and for each $p \in M$ there is defined an isomorphism $\Pi_p\colon (VE_G)_{s(p)} \to (s^*(VE_G))_p$. Then

$$(\mathcal{L}_X s)_p = \Pi(\overline{X}(s(p)) - ds_p(X(p))),$$

is the Lie derivative of s with respect to X at a point $p \in M$.

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From the definition it immediately follows that $\mathcal{L}_X s = 0$ if and only if X is an infinitesimal automorphism of s, i.e., if the flow $\overline{\phi}_t$ preserves s.

Now let $Q \subset L(M)$ be an integrable G-structure on M. Then Q determines a section $s\colon M \to E_G(M)$, and $\mathcal{L}_X s = 0$ if and only if $X \in \mathfrak{X}_Q(M)$, where $\mathfrak{X}_Q(M)$ is the Lie algebra of infinitesimal automorphisms of Q. We consider the Lie derivative $\mathcal{L}_X s$ as a first-order differential operator $\mathcal{D}_{\mathcal{L}}\colon TM \to s^*(VE)$.

In [1] for a differential operator $\mathcal{D}\colon \Gamma(\xi) \to \xi'$, where ξ, ξ' are vector bundles and $\Gamma(\xi)$ is the sheaf of sections of ξ, a differential complex was constructed (the Spencer P-complex)

$$0 \to \Theta \to \Gamma(\xi) \xrightarrow{\mathcal{D}} F^0 \to F^1 \to \dots,$$

where Θ is the kernel of $\mathcal{D}$. This construction, applied to a Lie derivative associated with a general pseudogroup structure, gives the deformation complex of this structure. We shall consider a particular case of pseudogroup structure, a first-order G-structure, and express this complex in terms of tensor fields and covariant derivative.

2. P-complex for the Lie derivative

Let us recall the definition of the P-complex [1] associated to an involutive linear differential operator $\mathcal{D} = \Phi \circ j^q\colon \xi \to \xi'$, where ξ, ξ' are vector bundles over M, and $\Phi\colon J^q\xi \to \xi'$ is a morphism of vector bundles.

Let $\rho^1(\Phi)$ be the first prolongation of Φ, $R^1 = \operatorname{im} \rho^1(\Phi)$, $\sigma^1(\Phi)$ the first prolongation of the symbol of Φ, $G^1 = \operatorname{im} \sigma^1(\Phi)$, and $\pi^1\colon R^1 \to \operatorname{im}(\Phi)$ the canonical projection. Let us denote by $\Omega^*(M)$ the algebra of differential forms on M. Then the groups of the P-complex are

$$F^0 = \operatorname{im} \Phi, \quad F^q = \frac{\Omega^q(M) \otimes F^0}{\delta(\Omega^{q-1}(M) \otimes G^1)},$$

where δ is induced by the Spencer algebraic operator. The differential $D\colon F^q \to F^{q+1}$ is defined as $D([\omega]) = [D_S\omega^1]$, where $\omega \in \Omega^q \otimes F_0$, $\omega^1 \in \Omega^q \otimes R^1$ are such that $\pi^1(\omega^1) = \omega$, D_S is the Spencer differential operator (for details we refer the reader to [1]).

Now let us find the P-complex for the operator $\mathcal{D}_{\mathcal{L}}$.

Let us denote by $E_{\mathfrak{g}}$ the subbundle of the affinor bundle $T^1_1(M)$ consisting of affinors whose matrices with respect to the frames from Q lie in the Lie algebra $\mathfrak{g}$ of G, and let $F_{\mathfrak{g}} = T^1_1(M)/E_{\mathfrak{g}}$.

Theorem 1. *The P-complex of the Lie derivative is isomorphic to the complex $(C^q(P), d)$, where*

$$C^q(P) = \frac{\Omega^k(M) \otimes TM}{\operatorname{Alt}(\Omega^{k-1}(M) \otimes E_{\mathfrak{g}})},$$

and the differential $d\colon C^q(P) \to C^{q+1}(P)$ is induced by the differential operator $D = \operatorname{Alt} \circ \nabla$, where Alt *is the alternation and ∇ is the covariant derivative of a torsion-free connection adapted to Q (i. e. $(D\omega)_{i_1 \dots i_{q+1}}^{\;j} = \nabla_{[i_1}\omega_{i_2 \dots i_{q+1}]}^{\;j}$ with respect to local coordinates adapted to Q).*

Proof. First we note that $s^*(VE) \cong F_{\mathfrak{g}}$. Since Q is integrable, there exists an atlas $\mathcal{A} = \{(U_\alpha, x^i_\alpha)\}$ adapted to Q. Then the section $s\colon M \to E(M)$ has the form $s|_{U_\alpha} = [\{\partial_i\}]$, where $\{\partial_i\}$ is the natural frame field of (U_α, x^i_α) and brackets denote the frame equivalence class. For the Lie derivative, we have $L_X s|_{U_\alpha} = [\partial_i X^j dx^i \otimes \partial_j]$. Therefore $\mathcal{D}_{\mathcal{L}} = \Phi \circ j^1$, where $\Phi\colon J^1(TM) \to F_{\mathfrak{g}}$, $\Phi(X^j, X^j_i) = [X^j_i]$.

Let $\pi\colon T^1_1(M) \to F_{\mathfrak{g}}$ be the projection, and we denote by the same letter π all the induced projections. From our calculation it follows that $G^1 = \pi(S^2(M) \otimes TM) \subset T^*(M) \otimes F_{\mathfrak{g}}$. Therefore one can easily see that the projection π induces an isomorphism between the group $C^q = \Omega^k(M) \otimes TM/\mathrm{Alt}(\Omega^{k-1}(M) \otimes E_{\mathfrak{g}})$ and the group $\Omega^{k-1}(M) \otimes F_{\mathfrak{g}}/\delta(\Omega^{k-2}(M) \otimes G^1)$ of the P-complex.

Since Q is integrable, there exists a torsion-free linear connection ∇ adapted to Q. By simple calculation, one can verify that the differential of the P-complex can be written in terms of ∇ in the following way: $d[\omega] = [D\omega]$, where $(D\omega)_{i_1 \dots i_{q+1}}^{\;j} = \nabla_{[i_1} \omega_{i_2 \dots i_{q+1}]}^{\;j}$. Note that this definition does not depend on the choice of ∇, because for two adapted connections ∇, ∇' the deformation tensor $T = \nabla' - \nabla$ is a section of the bundle $S^2(M) \otimes T(M) \cap T^*(M) \otimes E_{\mathfrak{g}}$. Also using properties of the curvature tensor R and the fact that R is a section of the bundle $\Omega^2(M) \otimes E_{\mathfrak{g}}$, one can directly verify that $D^2 = 0$.

3. Examples

1. Let Q be a foliation structure on a smooth manifold M, and let Δ be the corresponding integrable distribution. Then $E_{\mathfrak{g}} = \{A \in T^1_1(M) \mid A(\Delta) \subset \Delta\}$. Therefore $C^p = \Omega^p(\Delta) \otimes (TM/\Delta)$. If (x^i, x^α) are adapted local coordinates, i.e., if Δ is given by the equations $dx^i = 0$, then d can be written locally as $(d\omega)_{\alpha_1 \dots \alpha_{q+1}}^{\;i} = \partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{q+1}]}^{\;j}$. Thus we arrive at Vaisman's foliated cohomology [2].

2. Let Q be a symplectic structure given by a symplectic form θ. Then the subbundle $E_{\mathfrak{g}}$ consists of those affinors that are skew-symmetric with respect to θ, and using the isomorphism $T^1_1(M) \to T^2(M)$ determined by θ, we obtain

$$C^q = \frac{\Omega^q(M) \otimes T^*(M)}{\delta(\Omega^{q-1} \otimes S^2(M))},$$

where $S^2(M)$ is the bundle of symmetric tensors of type $(2,0)$ and

$$(\delta\omega)_{i_1 \dots i_{q-1} i_q i_{q+1}} = \omega_{[i_1 \dots i_{q-1} i_q] i_{q+1}}.$$

The adapted connection ∇ satisfies $\nabla\omega = 0$ (a symplectic connection), and the differential $d\colon C^q \to C^{q+1}$ is induced by the operator $(\widetilde{D}\omega)_{i_1 \dots i_q i_{q+1} i_{q+2}} = \nabla_{[i_1} \omega_{i_2 \dots i_q i_{q+1}] i_{q+2}}$. In particular, $C^0 \cong \Omega^1(M)$, $C^1(M) \cong \Omega^2(M)$, and $d\colon C^0 \to C^1$ is the exterior differential. Thus we find that the kernel of d is the Lie algebra of (locally) Hamiltonian vector fields.

3. For a complex structure, the complex (C^q, d) is the Dolbeaut complex [1]. And it was shown in [4] that for the structure of manifold over algebra, (C^q, d) is the Vaisman–Molino complex constructed in [3].

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