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Yang–Mills Fields, Volume II

Anastasios Mallios

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Modern Differential Geometry in Gauge Theories

Yang–Mills Fields, Volume II

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Anastasios Mallios
Department of Mathematics
University of Athens Panepistimioupolis
GR-157 84, Athens, Greece
amallios@math.uoa.gr

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Part III General Relativity

General Preface

"What can be said at all can be said clearly."

L. Wittgenstein in *Tractatus Logico-Philosophicus* (Routledge, 1997), p. 3.

"Σοφόν τοί τό σαφές οὐ τό μή σαφές."

Εὐριπ., Ὀρέστης

It is nowadays generally accepted that the theory of *principal fiber bundles* is the appropriate mathematical framework for describing one of the most beautiful, as well as important, physical theories, viz. the so-called *gauge field theory*, or gauge theories, being in effect to quote M. F. Atiyah, "*physical theories of a geometrical character*."

Now, in this context, a *principal fibration* is defined by the (local) *gauge group* (or *internal symmetry group*) of the physical system (*particle field*) under consideration. Yet, the particular physical system at issue is carried by, or lives on, a "*space*" (vacuum) that in the classical case is usually a *smooth* (viz. $\mathcal{C}^{\infty}$-) *manifold*. Within our abstract framework, instead, this is in general *an arbitrary topological space*, being also the *base space* of all the *fiber spaces* involved.

Accordingly, *we do not use any notion of calculus* (smoothness) in the classical sense, though we can apply, most of the powerful machinery of the standard *differential geometry*, in particular, the *theory of connections, characteristic classes*, and the like. However, all this is done *abstractly*, which constitutes an *axiomatic treatment of differential geometry* in terms of *sheaf theory* and *sheaf cohomology* (see A. Mallios [VS: Vols I, II]), while, as already noted, *no calculus is used at all*! So the present study can be construed as a further application of that abstract (i.e., axiomatic) point of view in the realm of gauge theories, given, as mentioned before, the intimate connection of the latter theories with (differential) geometry.

Thus, working within the aforementioned abstract set-up, we essentially replace all the previous fiber spaces (viz. principal and/or vector bundles) by the corresponding *sheaves of sections*, the former being, of course, just our model (motivation), while our study is otherwise, as has already explained above, *quite abstract*(!), that is, axiomatic. Of course, *in the classical case the two perspectives are* certainly mathematically speaking (categorically!) *equivalent;* however, the *sheaf-theoretic language*, to which we also stick throughout the present treatment, is even in the standard case, in common usage in the recent physics literature (cf., for instance, Yu. I. Manin [2] or even S. A. Selesnick [1]). Thus, it proves that the same language is at least *physically more transparent*, while finally being *more practical*. In addition, *wave functions* are considered as *sections* (i.e., functions whose *domain is varied* as well as their *range*, along *with the point of application*) of appropriate bundles (this volume, Chapt. IV; Section 10). Furthermore, it is still very likely that the kind of common base space of the sheaves involved herewith can also be thought of as corresponding to recent aspects of the *"vacuum,"* for instance, *"... the structure of such spaces is governed by topology, rather than geometry"* (cf. P. J. Braam [1: p. 279]).

On the other hand, a significant advantage of the present abstract formulation of the classical gauge field theory (i.e., the smooth case) lies in the possibility of employing the standard conceptual machinery of the usual (smooth) differential geometry, even for base *spaces* (of the fiber spaces, as above) *that* (i) *are not smooth enough,* (ii) *include a large amount of singularities* in the classical sense, and (iii) *are not smooth at all* (!), but provide the appropriate framework for the exploitation of the axiomatic theory [VS], as this happens in certain important cases (see concrete examples throughout the sequel). Of course, this potential generalization of the classical theory might very likely be of a particular significance to (mathematical) physicists, who long ago were already aware of, as well as tantalized by, the aforesaid type of spaces. Furthermore, the same abstract approach, has certainly theoretical/pedagogical advantages, being namely, of greater perspective, clarity and unification. It is thus more akin to the nowadays generally accepted aspect that *"the basic ideas of modern physics are quite simple"* (see, for instance, H. Fritzcsh [1: p. 211]), or even that *"... the problems of quantum gravity are much more than purely technical ones; they touch upon very essential philosophical issues"* (cf. G. 't Hooft [1: p. 2]). So it is quite natural to try to manufacture a similar situation pertaining to the mathematics involved; thus, something like this would also be in concord with the apostrophes, as stated in the epigraph of this preface.

Further details about each of the two individual volumes are given by separate prefaces.

Preface to Volume II

This second volume of the present treatise continues our study of *gauge theories*, in the framework of *abstract differential geometry*, by referring now to *Yang–Mills fields* in general, the particular case of Maxwell fields being covered already by the relevant exposition in the first volume.

We start with Chapter I, pertaining to the *general theory of Yang–Mills fields*, in what in particular concerns the corresponding (always, abstract) *Yang–Mills equations*, along with the material connected with the relevant *Yang–Mills functional* and its *variation*. To this end, we develop systematically all the necessary *abstract differential-geometric machinery*, as defined by a given $\mathcal{A}$-metric, and the associated abstract *Laplace–Beltrami operators* and their consequences, such as the *Dirac–Kähler operator* and *Green's formula*. We use the latter in the sequel, when studying the geometry of the *moduli space* of $\mathcal{A}$-connections (cf. Chapt. III).

One of our main conclusions here, and in complete analogy with the classical (smooth) case, is that (cf. Section 8 of the present chapter)

(∗) the solution of the Yang–Mills equations, which correspond to a given vector sheaf $\mathcal{E}$, are exactly the critical points of the Yang–Mills functional that can be associated with $\mathcal{E}$.

Yet, by analogy with our study in Chapter IV of Volume I, we also give a corresponding *cohomological classification of Yang–Mills fields*.

In Chapter II we deal with the space of Yang–Mills $\mathcal{A}$-connections (*viz.*, those $\mathcal{A}$-connections (in point of fact, their curvatures) that satisfy the Yang–Mills equations corresponding to our abstract setting). Furthermore, due to the physical importance of considering *gauge invariant $\mathcal{A}$-connections*, one finally considers the quotient of the previous space, through the *gauge group* (*viz.*, in our case the group of ($\mathcal{A}$-)automorphisms of the particular vector sheaf (locally free $\mathcal{A}$-module of finite rank) $\mathcal{E}$ under consideration)—in other words, the so-called moduli space (or *orbit space*) of $\mathcal{E}$. We thus begin in the first section the rudiments of the "geometry of Yang–Mills $\mathcal{A}$-connections," which is further specialized in the subsequent Chapter. The corresponding moduli space of *self dual $\mathcal{A}$-connections* is also considered in

Section 4. The chapter ends with two more sections that are mainly connected with "*second quantization*" and its *intrinsic commutative character*; see e.g. "*Morita equivalence*", or even Finkelstein's aphorism, pertaining to "non-commutativity", in connection with the quantum deep (see Section 5, (5.32)). Indeed, the former situation affects negatively, in effect, still the "*relativistic aspect of quantization*" (ibid.). In this context, one can further remark that,

(∗∗) the innate character of Nature is really "*bosonic*" (: *symmetric*): everything is light; see also, for instance, Chapter IV, Section 10, (10.21).

Within the same context, more recent developments, started from a *topos-theoretic* perspective of ADG, can be found in E. Zafiris [1], A. Mallios [14], [16], [17], A. Mallios – E. Zafiris [1], and A. Mallios – P.P. Ntumba [1], [2], [3], P.P. Ntumba [1], the latter items referring, in effect, to a *symplectic aspect* of ADG (yet, to what we may call, *Abstract Symplectic Geometry*). Yet, a topos-theoretic aspect of ADG can also be found in I. Raptis [3], [5].

As already mentioned, we continue in Chapter III our study of the geometry of the moduli space of a given vector sheaf, following, always within the advocated *abstract setup*, the corresponding classical pattern, mainly as indicated by I.M. Singer [1]. See, for instance, Section 7 of the chapter, referring to (the abstract form of) *Gribov's ambiguity* à la Singer. In particular, we consider the notion of the *tangent space* at a point ($\mathcal{A}$-connection) of the space of $\mathcal{A}$-connections, appropriately expressed according to Singer as those *tangent vectors*, at the point at issue, of suitably defined curves through the same point in the space of $\mathcal{A}$-connections; we have thus here a classical analogue of a *Newtonian* description of the notion of a tangent vector. The same aspect, suitably localized, is finally transferred to the orbit of the point concerned (*viz.*, to the corresponding moduli space). A brief account of this has been presented in A. Mallios [6].

Finally, Chapter IV of the present volume is concerned with *general relativity*, when this is considered as a gauge theory, thus, to recall A. Einstein himself (see, for instance, the relevant apophthegm at the beginning of the chapter), as a *field theory*—that is, according to the terminology of the present treatise, a theory, pertaining to a Yang–Mills field; in particular, for the case at issue, to a *Maxwell field* (*viz.*, to the *massless* 2-*spin graviton* (boson), which might be called an *Einstein field* to distinguish it from other bosons). One of our main results is the *Einstein equations* (*in vacuo*), which can be obtained within our abstract framework by following the classical pattern of the "variation of the Lagrangian density," alias that of the so-called *Einstein–Hilbert action* (functional); all this, suitably formulated in terms of the *abstract differential-geometric setup*, has been applied throughout this study.

What is of particular interest here, and is also very likely to have potential applications in problems related, for instance, with *quantum gravity*, is the possibility of using as *sheaf of coefficients* (our generalized arithmetic) the sheaf of (differential) algebras of generalized functions, à la E.E. Rosinger, functions that contain by definition a large number of singularities, in effect the largest one dealt with so far. So it is a natural hunch that one can apply

(∗ ∗ ∗) a gauge theory (field theory/graviton) to understand the "atomistic and quantum structure" of reality

(cf. A. Einstein [1: p. 165]) as a result of the abstract differential-geometric machinery developed so far, this being independent, as already pointed out, of any set of singularities (cf., for instance, Rosinger's algebras), in the classical sense. A more detailed discussion is given in the Section 9 of this chapter. Moreover, for the sake of completeness, a brief account concerning *Rosinger's algebra* (as well as that of *multifoam algebra*) sheaf is provided in Section 5 of the same chapter.

Finally, the fact that general relativity, although referred to a Maxwell field (graviton/boson), is treated in this second volume is due just to technical reasons, having to do with the way of describing it by analogy with the classical case. So this description is still afforded by means of a *Lorentzian $\mathcal{A}$-metric* (cf. Section 2 of Chapter IV, along with Section 2 of Chapter I); again, no "manifold" concept is needed, this sort of $\mathcal{A}$-metrics, together with necessary relevant notions, being studied in the same volume of our treatise.

Acknowledgments

The following lines represent only a small part of my indebtedness to all those people who in several ways contributed, by their contact or personal communication, the present material as well as all of the present consideration of ADG [*abstract* (≡ modern) *differential geometry*], together with its potential physical applications thus providing an indispensable and corroborative factor of the whole project at issue: Thus it was Elemér Rosinger who some years ago, during one of my visits to the University of Pretoria in South Africa, heard about my intention to present general relativity, the mathematical part of course [e.g., Einstein's equation (*in vacuo*) in terms of ADG, and in particular using his (sheaf of) algebras of generalized functions]; the reaction then was more than enthusiastic, so that project was finally realized in A. Mallios [8]. Somewhat earlier, I had already started to think of the possibility of presenting Yang–Mills theory in terms of ADG, motivated here by the relevant remark (M.F. Atiyah) that the same, being a gauge theory, is, in effect, of a geometrical character (hence, ADG), yet supported by the common aspect that "basic ideas of modern physics are quite simple" (H. Fritzsch) [ADG is, in principle, a *naive* theory; *viz.*, *axiomatic* (S. Mac Lane)]. So the first relevant ideas were already presented in A. Mallios [6], in full details in the same Volume II of this work, Chapters I–II, thanks, concerning the latter reference, to kind and lively interest in my whole work of K. Iséki and the late T. Ishihara. So Elemér Rosinger, in that context, post-anticipated me, in point of fact, while supporting me too, at the same time, concerning the idea of Yang–Mills theory, when he asked for an analogous abstract formulation of the Yang–Mills equations, yet this in his characteristic, for the whole enterprise enthusiastic, stimulating, and always lively manner.

On the other hand, the continued moral and quite definitive support of Steve Selesnick was certainly alive always and perceptible. What I call in this exposition *Selesnick's correspondence* (Vol. I; Chapter II) was the guiding principle, throughout the text, pertaining to its connection with physics, in spite of his usual reservations, referring to the usefulness of that otherwise extremely nice, very convenient, and workable (!) idea; later I met an analogous point of view, related with the electromagnetic field, in Yu. Manin's Springer book on *Gauge Field Theory* while quite recently, by that same author, concerning now any other field, in his article in [3]

(I owe this last quotation to Yannis Raptis). It was actually also Steve Selesnick who was responsible for a delightful collaboration in the last few years with Raptis, something that has led to an especially fruitful and substantial result, referring in particular to potential physical consequences of ADG for *quantum relativity* and the problem of the so-called *singularities* in general.

The beautiful and very informative recent work of Stathis Vassiliou on the *Geometry of Principal Sheaves*, to appear in the MIA series of Kluwer, came at the right time to vindicate and further extend the scope as well as the applicability of ADG. The ongoing work of Maria Papatriantafillou comes to cover the quite natural *formally categorical* treatise of ADG, both of the aforesaid recent two aspects of ADG being altogether definitive and necessary complements of the whole, thus far, enterprise on the matter. Within an analogous vein of ideas the recent work of Elias Zafiris comes already to test the ADG point of view in a *topos-theoretic* environment for the subject, yet with possible applications to *quantum gravity* as well.

During the time of several visits in the last few years to Rabat (Fès, included), Morocco, I had the opportunity to talk about ADG and its potential physical consequences mainly with Mohamed Oudadess and, in effect, with the whole "équipe d' analyse fonctionnelle" that thrives there, in particular, as it concerns topological algebras theory, thus having always an eager and also critical audience, being a test, of my own perceptions on the subject. Indeed, a very pleasant atmosphere, still inspiring too, Mohamed Oudadess, at least, being steadily a prompt and critical listener (!) providing me thus with a precious experience of having first reactions of a thoughtful "amateur" (the last denomination is, of course, his own) to the matter that often led me to greater elaborations of the ideas discussed and to increase understanding.

I have had in similar supporting and inspiring reactions in the past from contact with Nelu Colojoăra, the late Gerd Lassner, Konrad Schmüdgen, Susanne Dierolf, the late Klaus Floret, Franek Szafraniec, Jan de Graaf, Fredy van Oystaeyen, Roman Zapatrin, and last, but not least, with Chris Isham for his incisive corroborative critique, especially concerning our relevant joint work on the subject with Yannis Raptis. The reaction of my Russian editors Vassia Lyubetsky and Sasha Zarelua was supportive, vindicative, and much enlightening, as well.

My special thanks here are due too, for partial financial support during the recent few years, to the office of the Special Research Program conducted by the University of Athens and, in particular, to the Vice-Rector, at that time, Prof. Michael Dermitzakis for his lively and very kind support to my own research work.

The realization and appearance of the material contained in the present two volumes would have not been accomplished without the skilful and, really wonderful, typing (LaTeX) talent of our secretary in the Section of Algebra and Geometry of our Department, the late Popi Bolioti. It is to her fond memory to record here too the excellent job she has done.

The present two-volume work owes its appearance to the enthusiasm, eager interest, and prompt reaction of Prof. George A. Anastassiou (University of Memphis, USA), *as well as* to the editorial help and extremely kind attention of the Executive Editor of Birkhäuser, Boston, Ms. Ann Kostant, and her so efficient and competent editorial staff at Birkhäuser production. It is a particular pleasure to express at this place my heartfelt thanks and deep appreciation as well to all of the above people for their kindness and the warm attitude they showed toward my work.

[It is really amazing that the whole story began simply from one source: the *Math. Z.* (146 (1976)) article of Stephen Allan Selesnick (!); see also the Acknowledgments of the first two volumes on ADG. Then the enterprise has been continued by pointing out the quite instrumental role the notion of *connection* has had in the whole development of CDG, along with its physical applications.]

Contents of Volume I

Part II

Yang–Mills Theory: General Theory

1

Abstract Yang–Mills Theory

> "Today there is an amazing confluence of the gauge theories in physics (for the Yang–Mills equations) and the geometrical theory of connections on fiber bundles."
>
> S. Mac Lane in *Mathematics: Form and Function* (Springer-Verlag, New York, 1986). p. 259.

> "Gauge theories [have] a direct differential-geometric interpretation in terms of fiber bundles with connection."
>
> M. F. Atiyah in *Geometry of Yang–Mills Fields* (Accademia Nazionale dei Lincei, SNS, Pisa, 1979). p. 42.

The objects of our study in this chapter belong to what we may call the *Yang–Mills category* (see Section 4.2 for concrete definitions), while the corresponding morphisms are suitable connection-preserving sheaf morphisms (ibid. (4.16)). Now, since the necessary background material for the subject matter at issue has not been systematically developed so far, within the framework of abstract differential geometry (see A. Mallios [VS: Vols. I and II]), which is employed by the present treatise, we give below a detailed exposition of all the relevant issues that will be needed in the sequel. In this context, see also, however, A. Mallios [6: p. 164, Appendix II] for a brief account on the same material. Among the various standard presentations of this subject, see, for instance, T. Petrie–J. Randall [1]. So we start with the ensuing fundamental notions for all the subsequent discussion.

1 The Differential Setting

As was the case in Volume I of the present treatise, here too, the adjective "differential" has, of course, only a formal meaning, referring to a particular type of ("differential") operator connected with the subject matter under consideration, given that no "smooth structure" at all (!) is assumed on our base space X. Thus, the ensuing discussion of this section aims, in effect, to collect together all the "differential operators" that have been employed in the preceding (see [VS: Vols. I and II], yet, Volume I of this treatise), to the extent that they provide the corresponding to our case (generalized, alias abstract) *de Rham complex*, along with the respective "connection operators" (see below). All of this will be necessary for our subsequent considerations in Section 2, where we shall define the corresponding "*dual* (alias, *adjoint*) *differential setup*."

A. Mallios, *Modern Differential Geometry in Gauge Theories: Yang–Mills Fields, Volume II*, DOI: 10.1007/978-0-8176-4634-9_1,

So, as usual, we start with a $\mathbb{C}$-*algebraized space*

$$(X, \mathcal{A}), \tag{1.1}$$

which is further assumed to be endowed with a given differential triad (see also Chapt. I, Section 1),

$$(\mathcal{A}, \partial, \Omega^1). \tag{1.2}$$

Indeed, one can assume, depending on the particular case under consideration, the existence of a sequence of exterior differentials (or also differential operators, or else differentials of the first kind)

$$(d^n)_{n\in\mathbb{Z}_+}, \tag{1.3}$$

where we still set

$$d^0 \equiv \partial, \tag{1.3'}$$

as in (1.2) (see also (1.4) and (1.7) below). In this context, see, for instance, A. Mallios [5: pp. 17ff], or even [VS: Chapt. VIII, pp. 226ff] as it concerns the above sequence of differentials. So one defines

$$d^n : \Omega^n \longrightarrow \Omega^{n+1}, \;\; n \in \mathbb{Z}_+, \tag{1.4}$$

where we set

$$\Omega^n := \bigwedge_{i=1}^{n} (\Omega^1)^i \equiv \underbrace{\Omega^1 \wedge \cdots \wedge \Omega^1}_{n \text{ times}}, \;\; n \in \mathbb{N}, \tag{1.5}$$

and we also set

$$\Omega^0 := \mathcal{A}, \tag{1.5'}$$

as in (1.2). Here, each one of the d^n's, as above, is, by definition, a $\mathbb{C}$-linear morphism of the $\mathcal{A}$-modules concerned, such that the following (defining) relation is further assumed to be valid; namely one has

$$d^{p+q}(s \wedge t) = d^p(s) \wedge t + (-1)^p s \wedge d^q(t), \tag{1.6}$$

for any $s \in \Omega^p(U)$ and $t \in \Omega^q(U)$, and any open $U \subseteq X$, with p and q in $\mathbb{Z}_+$. On the other hand, supposing that the relations

$$d^1 \circ d^0 \equiv d^1 \circ \partial = 0 \tag{1.7}$$

as well as

$$d^2 \circ d^1 = 0 \tag{1.7'}$$

are valid, one further proves that

$$d^{p+1} \circ d^p = 0, \quad \text{for any} \quad p \geqq 2. \tag{1.8}$$

Yet, concerning the equations (1.7), (1.7′), and (1.8), we also set, for convenience, as already done in [VS] and Volume I too, simply

$$d \circ d \equiv dd \equiv d^2 = 0. \tag{1.9}$$

Thus, one arrives at the following generalized (alias, *abstract*) de Rham complex—in point of fact, a *cochain complex of* $\mathbb{C}$*-vector space sheaves* on the space X (see (1.1)):

$$\begin{aligned} 0 \longrightarrow \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \equiv \Omega^0 \xrightarrow{d^0 \equiv \partial} \Omega^1 \xrightarrow{d^1} \Omega^2 \\ \longrightarrow \cdots \Omega^n \xrightarrow{d^n} \Omega^{n+1} \longrightarrow \cdots . \end{aligned} \tag{1.10}$$

Of course, in the previous sequence the complexes $\mathbb{C}$ stand, in effect, for the constant ($\mathbb{C}$-vector space) sheaf of the complexes, while one still obtains the relation

$$\partial \circ \varepsilon = 0, \tag{1.11}$$

according to our hypothesis for $\mathcal{A}$ and (1.6), along with (1.3′) (see also Chapt. I, (1.16)). However, the above complex (1.10) is not, in general, exact! (In other words, the *"abstract Poincaré lemma"* is lacking, in general, but on the other hand, see also, for instance, Chapter IV, Section 5 in the sequel.)

So, to summarize, we have so far considered a $\mathbb{C}_X$-complex on X associated with the given $\mathbb{C}$-*algebraized space*, as in (1.1)—that is, the cochain complex of $\mathbb{C}$-vector space sheaves on X, of positive degree,

$$(\Omega^*, d) \equiv \left\{(\Omega^n, d^n)\right\}_{n \in \mathbb{Z}_+} \tag{1.12}$$

(see (1.5) and (1.5′), along with (1.3′) and (1.4), as well as (1.6)). Furthermore, we also call (1.12) the abstract de Rham complex of X, yet, occasionally, apart from (1.10), as the case may be. In this connection, we still note that in the particular case that (1.10) is exact, while further suitable assumptions are imposed on the pair $(X, \mathcal{A})$, as in (1.1), it is, in point of fact, through (1.12) that one gets at the (abstract de Rham) cohomology of X, with complex coefficients that can still be expressed by means of our "structure sheaf" $\mathcal{A}$, a fact that might be of fundamental importance in the applications (loc. cit.). In this regard, see also [VS: Chapt. III] for a detailed account of the relevant terminology employed herewith.

1.1 Vectorization of the Abstract de Rham Complex (Prolongations)

We come now to what one might call a "vectorization" of the preceding, with respect to a given $\mathcal{A}$-module $\mathcal{E}$ on X, by considering the following sequence of "differentials of the second kind":

$$(D^n)_{n \in \mathbb{Z}_+}, \tag{1.13}$$

such that one has

$$D^n : \Omega^n(\mathcal{E}) \longrightarrow \Omega^{n+1}(\mathcal{E}),\ n \in \mathbb{Z}_+, \tag{1.14}$$

where we set

$$\Omega^n(\mathcal{E}) := \Omega^n \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \Omega^n \tag{1.15}$$

for any $n \in \mathbb{Z}_+$ (see also Vol. I: Chapt. I, (2.1), concerning the last $\mathcal{A}$-*isomorphism* in (1.15) of the $\mathcal{A}$-module involved). On the other hand, the same "differential operators" as in (1.14) are actually given, through the following relation (definition):

$$D^n := 1_{\mathcal{E}} \otimes d^n + (-1)^n \Omega^n \wedge D,\ n \in \mathbb{Z}_+, \tag{1.16}$$

where we have still set

$$D^0 \equiv D \tag{1.17}$$

(however, see also (1.22) in the sequel).

Thus, by looking at (1.16), in terms of (local) sections of the $\mathcal{A}$-modules involved therein, one obtains (see also (1.14) and (1.15))

$$D^n(s \otimes t) = s \otimes d^n(t) + (-1)^n t \wedge D(s),\ n \in \mathbb{Z}_+, \tag{1.18}$$

for any $s \in \mathcal{E}(U)$ and $t \in \Omega^n(U)$, with U open in X. Yet, we also take into account here that (*viz.*, (1.15), for $n = 0$)

$$\Omega^0(\mathcal{E}) \equiv \Omega^0 \otimes_{\mathcal{A}} \mathcal{E} \equiv \mathcal{A} \otimes_{\mathcal{A}} \mathcal{E} = \mathcal{E} \tag{1.19}$$

(see (1.5′), along with [VS: Chapt. II, (5.15)]). In particular, by further applying (1.18), for $n = 0$, one obtains, in view also of (1.3′), (1.17), (1.19), and (1.14), the following $\mathbb{C}$-*linear morphism*:

$$D : \mathcal{E} \longrightarrow \Omega^1(\mathcal{E}), \tag{1.20}$$

such that one has (a result, in effect, of (1.18), in view of (1.19))

$$D(\alpha \cdot s) = \alpha \cdot D(s) + s \otimes \partial(\alpha) \tag{1.21}$$

for any $\alpha \in \mathcal{A}(U)$, $s \in \mathcal{E}(U)$, and open $U \subseteq X$. Accordingly, one thus concludes that

(1.22) the above map, as in (1.20) (see also (1.17)),

$$D \equiv D^0, \tag{1.22.1}$$

is, in effect, an $\mathcal{A}$-connection of $\mathcal{E}$. Thus, by still extending the classical terminology, we can consider the rest of the "differential operators", as in (1.13) (*viz.*, for $n \geqq 1$), as the *prolongations* of $D(\equiv D^0)$. (See also (1.25) in the sequel.)

Consequently, based on the preceding, we can say that the vectorization of (1.12), with respect to a given $\mathcal{A}$-module $\mathcal{E}$ on X, as above, is given now by the relation (see (1.14) and (1.15))

$$(\Omega^*(\mathcal{E}), D) \equiv \{(\Omega^n(\mathcal{E}), D^n)\}_{n\in\mathbb{Z}_+}. \tag{1.23}$$

Yet, the relation of (1.12) with (1.23) is further explained by the subsequent discussion; indeed, one can say that

(1.24) (1.12) is obtained from (1.23), by simply putting $\mathcal{E} = \mathcal{A}$ in the latter;

namely, based on (1.18), in conjunction also with (1.15) and (1.19), one first obtains that

$$D^n\big|_{\Omega^n(\mathcal{A})\equiv\mathcal{A}\otimes_{\mathcal{A}}\Omega^n=\Omega^n} = d^n, \ n \in \mathbb{N}. \tag{1.25}$$

On the other hand, by looking at (1.6), for $p = 0$ and $q = 1$, one has

$$d^1(\alpha \cdot s) = \alpha \cdot d^1(s) - s \wedge \partial(\alpha) \tag{1.26}$$

for any $\alpha \in \mathcal{A}(U)$, $s \in \Omega^1(U)$, and open $U \subseteq X$, which thus, in view of (1.18), when the latter is applied for $n = 1$, can be construed as yielding that

$$d^1 \text{ is the first prolongation of } d^0 \equiv \partial. \tag{1.27}$$

(In this concern, see also, for instance, [VS: Chapt. VIII, p. 192, Remark 2.1]). So the above relation (1.25), along with (1.27), justifies our assertion in (1.24). ■ However, the same relations, as before, prove also that

(1.28) the *exterior differentials* d^n*, with* $n \in \mathbb{Z}_+$, as in (1.3) (see (1.3′)), or else "differentials of the first kind" can be viewed simply as the successive prolongations of the given (see (1.2)) standard (flat) $\mathcal{A}$-connection $\partial \equiv d^0$ on $\mathcal{A}$.

In this regard, see also A. Mallios [5: p. 19, (1.19)].

So the preceding, as given by (1.3) and (1.13), constitute our abstract differential setup, in terms of which one can recast, while at the same time extend, within the present axiomatic framework, several fundamental notions and results of the classical theory (e.g., differential geometry of smooth manifolds). We proceed in Section 2 to the *dual* framework of the above under the only proviso that one is supplied with an appropriate $\mathcal{A}$-*metric* on a given $\mathcal{A}$-module $\mathcal{E}$ on X, as before. Indeed, the problem is actually reduced, for suitable $(X, \mathcal{A})$, to a similar one for $\mathcal{A}$! See, thus, for example, (2.19) below.

2 The Dual Differential Setting

Our purpose in the present section is to define, within the abstract framework of this treatise, the so-called *dual differential operators* of those already considered in Section 1, something that one can really achieve, provided we put up the necessary setting there. So, what one actually needs here is the notion of an $\mathcal{A}$-metric on a given $\mathcal{A}$-module $\mathcal{E}$ on X; we continue to assume herewith that we are given the context, as in (1.1) of Section 1. Thus, motivated by the standard situation, the latter notion concerns, in effect, a *sheaf morphism*, say,

$$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}, \tag{2.1}$$

such that the following conditions are fulfilled:

(i) ρ is an $\mathcal{A}$-*bilinear morphism* between the $\mathcal{A}$-modules concerned, as in (2.1).

(ii) ρ is symmetric; that is, one has

$$\rho(s, t) = \rho(t, s) \tag{2.2}$$

for any s and t in $\mathcal{E}(U)$, with U open in X.

There is another condition that we impose on ρ (*viz.*, its *positive definiteness*). However, to formulate the latter notion, we need to have on our *structure sheaf* $\mathcal{A}$, as in (1.1), a richer structure; so we further suppose that

(2.3) the underlying $\mathbb{R}$-*algebra sheaf* $\mathcal{A}$, as in (1.1), is a (partially) ordered ($\mathbb{R}$-)algebra sheaf on X; that is, we assume the existence of a subsheaf

$$\mathcal{P} \subseteq \mathcal{A}, \tag{2.3.1}$$

defining (sectionwise) the preorder in $\mathcal{A}$.

For details on the terminology employed in (2.3), see [VS: Chapt. IV, pp. 316ff].

In this connection, whenever we have the situation described by (2.3), as above, we then also speak of our previous pair, as in (1.1),

$$(X, \mathcal{A}), \tag{2.4}$$

as a (partially) ordered algebraized space. Thus, under the hypothesis that (2.3) holds true, we further assume, concerning the conditions we impose on ρ, that

(iii) ρ is *positive definite*, in the sense that one has

$$\rho(s, s) \in \mathcal{P}(U) \quad \text{(see (2.3.1)), such that} \tag{2.5}$$

$$\rho(s, s) = 0 \quad \text{(if, and) only if} \quad s = 0, \tag{2.6}$$

where $s \in \mathcal{E}(U)$ (see (2.1)) and U is open in X.

Note 2.1 The relation $\rho(s, s) \in \mathcal{P}(U)$, as applied in (2.5), will also be denoted in the sequel by

$$\rho(s, s) \geqq 0 \tag{2.5$'$}$$

for any s and U, as in (2.5).

Now, before we come to the next property of the map ρ that we are going to employ, we first consider another map, say $\tilde{\rho}$, deduced from ρ by virtue of property (i); indeed, one gets the existence of a map

$$\tilde{\rho} : \mathcal{E} \longrightarrow \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \tag{2.7}$$

in fact, an $\mathcal{A}$-morphism of the $\mathcal{A}$-modules in (2.7), such that one actually defines

$$\tilde{\rho}(s) \equiv \rho_s : \mathcal{E} \longrightarrow \mathcal{A} : t \longmapsto \tilde{\rho}(s)(t) \equiv \rho_s(t) := \rho(s,t) \tag{2.8}$$

for any s and t in $\mathcal{E}(U)$ and any open $U \subseteq X$.

Thus, assuming now that we are given the framework of (2.3), as above, we first remark that

(2.8) property (iii) of ρ (*viz.*, the positive definiteness of ρ, see (2.5)) entails that $\tilde{\rho}$, as defined by (2.7), becomes an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules involved in (2.7); namely, at the following (canonical) imbedding ($\mathcal{A}$-isomorphism into) one gets

$$\mathcal{E} \underset{\overrightarrow{\tilde{\rho}}}{\subseteq} \mathcal{E}^*. \tag{2.8.1}$$

Now, the previous imbedding is not, in general, onto, as is virtually the case in the classical theory (i.e., when considering finite-dimensional $\mathbb{C}$ (or even $\mathbb{R}$)-vector spaces, hence, corresponding vector bundles, or their associated sheaves of sections).

Thus, whenever we have the relation

$$\mathcal{E} \underset{\tilde{\rho}}{\cong} \mathcal{E}^* \tag{2.9}$$

within an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned, we say that ρ (see (2.1)) is *strongly nondegenerate*.

Note 2.2 (Terminological) As already remarked (see (2.8)), the positive definiteness of ρ (see (2.5) or else (2.5$'$)) entails (2.8.1); yet, as explained above (see the comments following (2.8.1)), when referring to the classical theory, (2.8.1) is equivalent to (2.9). Thus, classically speaking, in the *semi-Riemannian* (or else *pseudo-Riemannian*) case (see B. O'Neill [1: pp. 54f], or even A.L. Besse [1: p. 29, Definition 1.33]), one generalizes, as it happens (e.g., in the *General Theory of Relativity*), by considering the nondegeneracy of ρ [*viz.*, (2.9), or, equivalently (for the finite-dimensional case), (2.8.1), in place of (2.5) ("positive definiteness" of ρ)].

Now, in this treatise, we first consider the Riemannian case (see Definition 2.1), while, later (see Chapt. IV, Section 2), we also employ the semi-Riemannian (in particular, Lorentz) case, where ρ is not necessarily positive definite, as in (2.5) above (see Chapt. IV, Definition 2.1, in particular, (2.7): *Lorentz condition*), but is always (loc. cit. (2.6)) strongly nondegenerate. Thus, we define the notion of an $\mathcal{A}$-valued inner product ρ on a given $\mathcal{A}$-module $\mathcal{E}$ by extending the classical situation to our abstract framework, according to the following.

Definition 2.1 Let $(X, \mathcal{A})$ be a (partially) ordered algebraized space (see (2.3)) and let $\mathcal{E}$ be an $\mathcal{A}$-module on X. Now, an $\mathcal{A}$-metric, or even a *Riemannian $\mathcal{A}$-metric*, on $\mathcal{E}$ is an *$\mathcal{A}$-bilinear symmetric positive definite* and strongly nondegenerate map ρ, as in (2.1). We also then speak of the pair

$$(\mathcal{E}, \rho) \tag{2.10}$$

as a metrized $\mathcal{A}$-module, or even as a Riemannian $\mathcal{A}$-module $\mathcal{E}$ on X.

Thus, given a (partially) ordered algebraized space $(X, \mathcal{A})$, as in (2.3), and an $\mathcal{A}$-module $\mathcal{E}$ on X, an $\mathcal{A}$-valued inner product on $\mathcal{E}$ is, by definition, an $\mathcal{A}$-bilinear symmetric and positive definite map (sheaf morphism), as in (2.1). Consequently, in that sense, an $\mathcal{A}$-metric ρ on $\mathcal{E}$ (see Definition 2.1) is a strongly nondegenerate $\mathcal{A}$-valued inner product on $\mathcal{E}$.

Warning! In this connection, and, in conjunction with our previous scholium in Note 2.2, we still remark here that, in our case, an $\mathcal{A}$-valued inner product on $\mathcal{E}$, is not necessarily an $\mathcal{A}$-metric in the sense of Definition 2.1; only conversely, given that (2.8.1) does not always imply (2.9), *viz.*, the "strong nondegeneracy" of ρ, as in (2.1) (see the comments following (2.8.1)), by contrast with what happens in the classical theory (finite-dimensional case).

Now, further details pertaining to a local treatment in terms of local frames of Riemannian vector sheaves, which will also be considered in the sequel, are given in [VS: Chapt. IV, Section 8] (see, for instance, ibid., p. 322, Theorem 8.1).

Thus, after the preceding preliminary material, we come now to our main object.

2.1 Dual Differential Operators

We start by assuming that we have a differential triad

$$(\mathcal{A}, \partial, \Omega^1), \tag{2.11}$$

which, as usual, is associated with a given $\mathbb{C}$-*algebraized space*

$$(X, \mathcal{A}) \tag{2.12}$$

(see [VS: Vol. II], or Volume I, Chapter I, Section 1 of this treatise). Furthermore, assume that we are given the sequence of differentials of the first kind (see Section 1),

$$(d^n)_{n\in\mathbb{Z}_+} \tag{2.13}$$

(to the extent, of course, that (1.7) and (1.7′) are valid, the rest of d^i's being entailed, according to (1.6), while (1.8) holds always true). Hence, we can further consider the concomitant abstract de Rham complex (see (1.10) and (1.12))

$$\{(\Omega^n, d^n)\}_{n\in\mathbb{Z}_+}. \tag{2.14}$$

On the other hand, we still consider a pair

$$(\mathcal{E}, D) \tag{2.15}$$

consisting of an $\mathcal{A}$-module $\mathcal{E}$ on X and an $\mathcal{A}$-*connection* D of $\mathcal{E}$. (We may call (2.15) a generalized Yang–Mills field by extending the respective terminology for a Yang–Mills field, referring actually to a similar pair, as in (2.15), with $\mathcal{E}$ being, in particular, a vector sheaf on X; see Section 4, (4.13)). Accordingly, we can finally consider the corresponding "vectorization" of (2.14), with respect to $\mathcal{E}$, that is, one gets at the sequence (1.23), yet one obtains the following $\mathbb{C}$-*sequence*, alias a sequence of $\mathbb{C}$-vector space sheaves on X,

(2.16)
$$\begin{aligned} &\mathcal{E} \xrightarrow{D\equiv D^0} \Omega^1(\mathcal{E}) \xrightarrow{D^1} \Omega^2(\mathcal{E}) \longrightarrow \cdots \\ &\longrightarrow \Omega^n(\mathcal{E}) \xrightarrow{D^n} \Omega^{n+1}(\mathcal{E}) \longrightarrow \cdots , \end{aligned}$$

such that one still has (see also (1.19) above), concerning the first term of the previous sequence,

(2.17)
$$\Omega^0(\mathcal{E}) := \Omega^0 \otimes_{\mathcal{A}} \mathcal{E} \equiv \mathcal{A} \otimes_{\mathcal{A}} \mathcal{E} \equiv \mathcal{A}(\mathcal{E}) = \mathcal{E}$$

(modulo an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned, regarding the last equality above). Moreover, one also defines

(2.18)
$$D^n \in \mathcal{H}om_{\mathbb{C}}(\Omega^n(\mathcal{E}), \Omega^{n+1}(\mathcal{E})), \; n \in \mathbb{Z}_+,$$

as the differentials of the second kind, or yet prolongations of those in (2.13), being still, by definition, $\mathbb{C}$-morphisms of the $\mathbb{C}$-vector space sheaves involved (see also our discussion in Section 1).

Now, to proceed further, we make use of an $\mathcal{A}$-metric on $\mathcal{E}$, in the sense of Definition 2.1, so that one can then define what we call the "dual $\mathbb{C}$-sequence of (2.16) (see (2.31)). So our aim is to give, first, appropriate conditions on a pair $(X, \mathcal{A})$, as in (2.12), so that a generalized Yang–Mills field (see (2.15)) has $\mathcal{E}$ as a Riemannian $\mathcal{A}$-module on X (see Definition 2.1). In fact, we consider vector sheaves on X for technical reasons. Thus, first we assume that

(2.19) we are given a (partially) ordered algebraized space

(2.19.1)
$$(X, \mathcal{A})$$

(see (2.3)), for which the underlying space X is *paracompact* (*Hausdorff*), while we also suppose that $\mathcal{A}$ is a fine sheaf on X, endowed with an $\mathcal{A}$-metric ρ; namely,

(2.19.2)
$$(\mathcal{A}, \rho)$$

is a Riemannian $\mathcal{A}$-module on X (see Definition 2.1).

Now, as an immediate consequence of our hypothesis in (2.19), one gets the conclusion that for any given vector sheaf $\mathcal{E}$ on X, one defines "on it" (precisely speaking, on $\mathcal{E} \times \mathcal{E} = \mathcal{E} \oplus \mathcal{E}$) a ($\mathcal{A}$-valued) symmetric and $\mathcal{A}$-*bilinear morphism*

(2.20) (2.20.1)
$$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}.$$

For convenience, we still retain in (2.20.1), by an obvious abuse of notation, the same symbol ρ as in (2.19.2).

The assertion follows straightforwardly by applying a standard argument, in view of our hypothesis in (2.19) that $\mathcal{A}$ is fine and X paracompact; hence, one is supplied with an appropriate partition of unity of $\mathcal{A}$, while $\mathcal{E}$ is still supposed to be (loc. cit.) locally free (of finite rank). ■

For further details, see [VS: Chapt. IV, p. 324, (8.36), and p. 325, Theorem 8.2].

However, to proceed further and prove that the $\mathcal{A}$-morphism ρ, as given by (2.20.1), is an $\mathcal{A}$-metric of $\mathcal{E}$, we have to reinforce our assumption for $\mathcal{A}$, in relation to (2.19). Thus, we further assume that

(2.21) in addition to (2.19), our structure sheaf $\mathcal{A}$ is strictly positive: By definition, this means that

(2.21.1) for every locally finite open covering of X, there exists a strictly positive partition of unity of $\mathcal{A}$, subordinate to the given covering.

In this context, we also note that to formulate (2.21.1), we do not actually need X to be a paracompact (Hausdorff) space.

Yet, for convenience, we recall that (see [VS: Chapt. IV, p. 326, Definition 8.4])

(2.22) by a strictly positive partition of unity of $\mathcal{A}$ subordinate to a locally finite open covering of X, say,

(2.22.1)
$$\mathcal{U} = (U_\alpha)_{\alpha \in I},$$

with X being a (partially) ordered algebraized space (see (2.5)), one means a family of $\mathcal{A}$-*endomorphisms* of $\mathcal{A}$,

(2.22.2)
$$(\phi_\alpha) \subseteq End\mathcal{A} = (\mathcal{E}nd\mathcal{A})(X) = \mathcal{A}(X),$$

such that the following conditions are satisfied:

(2.22.3)
(i) $\mathrm{Supp}(\phi_\alpha) \subseteq U_\alpha, \ \alpha \in I.$
(ii) $\sum_\alpha \phi_\alpha = 1.$
(iii) $\phi_\alpha\big|_{U_\alpha} \in \mathcal{P}(U_\alpha) \cap \mathcal{A}^{\bullet}(U_\alpha) = \mathcal{P}(U_\alpha) \cap \mathcal{A}(U_\alpha)^{\bullet}$, for any $\alpha \in I$,

(see also (2.5)).

In other words, one considers a partition of unity of $\mathcal{A}$ consisting, according to the definitions (see (2.22.2)), of global (continuous) sections of $\mathcal{A}$, which are also subordinate to the open covering $\mathcal{U}$ of X of the particular type considered (conditions (i) and (ii)), while the same family provides a family of "strictly positive" (local continuous) sections of $\mathcal{A}$, in the sense that condition (iii), as above, holds true.

Now, we still express the above property of $\mathcal{A}$, as in (2.21.1), by simply saying that $\mathcal{A}$ is a strictly positive fine sheaf on X; indeed, due to conditions (i) and (ii) in (2.22.3), it is clear that such a sheaf on X is *a fortiori* fine (see also [VS: Chapt. III, p. 238, Definition 8.1]).

We come now to supplement our hypothesis for $(X, \mathcal{A})$, as in (2.19), by just assuming henceforth that

(2.23) we are given a (partially) ordered algebraized space

(2.23.1) $$(X, \mathcal{A}),$$

with X a paracompact (Hausdorff) space and $\mathcal{A}$ a strictly positive fine sheaf on X. Moreover, we assume that $\mathcal{A}$ is endowed with an $\mathcal{A}$-metric ρ, so that

(2.23.2) $$(\mathcal{A}, \rho)$$

is a Riemannian $\mathcal{A}$-module on X.

The significance of the previous setup, as described by (2.23), lies exactly in the following conclusion; that is, one obtains that

(2.24) every vector sheaf $\mathcal{E}$ on X, the latter space being associated with data, as in (2.23), admits an $\mathcal{A}$-metric ρ (see also the comments following (2.21.1). Therefore,

(2.24.1) $$(\mathcal{E}, \rho)$$

is a Riemannian vector sheaf on X.

See also [VS: Chapt. IV, p. 328, Theorem 8.3]. ■

This fundamental result will be standard throughout our discussion.

Thus, suppose now that we are given a differential triad

(2.25) $$(\mathcal{A}, \partial, \Omega^1)$$

on X, with $(X, \mathcal{A})$ satisfying (2.23). Yet, we further suppose that Ω^1 is a vector sheaf on X. Accordingly,

(2.26) the sequence

(2.26.1) $$(\Omega^n)_{n \in \mathbb{Z}_+}$$

(see also (1.5) and (1.5′)) consists of vector sheaves on X.

See [VS: Chapt. IV, p. 311, Lemma 7.1]. ■

Now, assume finally that we are given a pair

(2.27) $$(\mathcal{E}, D),$$

consisting of a vector sheaf $\mathcal{E}$ on X and an $\mathcal{A}$-connection D on $\mathcal{E}$, thus, a Yang–Mills field on X (see (4.13)), within the context of (2.23) and (2.25), as above. Consequently, by virtue of (2.24) and (2.26), with $\mathcal{E}$ as in (2.27), one concludes that

(2.28) all the $\mathcal{A}$-modules appearing in the $\mathbb{C}$-sequence (2.16) become, in effect, Riemannian vector sheaves on X.

Thus, we are now in the position to write the dual $\mathbb{C}$-sequence of (2.16), along with the corresponding dual differential operators of (2.18); that is, one gets the following sequence of (Riemannian) vector sheaves on X (see (1.15)):

(2.29) $$\cdots \longrightarrow \Omega^{n+1}(\mathcal{E}) \xrightarrow{\delta^{n+1}} \Omega^{n}(\mathcal{E}) \xrightarrow{\delta^{n}} \cdots,$$

(2.30) $$\cdots \longrightarrow \Omega^{2}(\mathcal{E}) \xrightarrow{\delta^{2}} \Omega^{1}(\mathcal{E}) \xrightarrow{\delta^{1}} \Omega^{0}(\mathcal{E}) \cong \mathcal{E}$$

(see also (2.17)), where we still have, by definition,

(2.31) $$\rho(D^n(s), t) = \rho(s, \delta^{n+1}(t)), \; n \in \mathbb{Z}_+,$$

for any $s \in \Omega^n(\mathcal{E})(U)$ and $t \in \Omega^{n+1}(\mathcal{E})(U)$ and any given open $U \subseteq X$. Before we explain the idea inherent in the previous defining relation (2.30), we briefly recall certain technical issues implicit in the notation employed in the aforesaid relation. So we illuminate them through the following.

Note 2.3 The open set $U \subseteq X$ is a common local gauge of the vector sheaves $\mathcal{E}$ and Ω^1; hence, analogously, for $\mathcal{E}$ and Ω^n as well, for all $n \in \mathbb{Z}_+$ (see also (2.26)), one actually obtains

(2.31) $$\begin{aligned} \Omega^n(\mathcal{E})(U) &\equiv (\mathcal{E} \otimes_{\mathcal{A}} \Omega^n)(U) = ((\mathcal{E} \otimes_{\mathcal{A}} \Omega^n)|_U)(U) \\ &= (\mathcal{E}|_U \otimes_{\mathcal{A}|_U} \Omega^n|_U)(U) = \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \Omega^n(U). \end{aligned}$$

In this connection, see also [VS: Chapt. I, p. 55, (11.40), and Chapt. II, p. 132, Lemma 5.1] as well as [VS: Chapt. VII, p. 100, (1.9) and (1.10)]. ■

Thus, we have employed in (2.31) the following two basic relations (see the last of the previous quotations), valid for any vector sheaf $\mathcal{E}$, with $rk\mathcal{E} = m$, and any $\mathcal{A}$-module Ω on X; that is, one has

(2.32) $$\Omega(\mathcal{E})|_U \equiv (\Omega \otimes_{\mathcal{A}} \mathcal{E})|_U = \Omega^m|_U = (\Omega|_U)^m$$

(modulo $\mathcal{A}|_U$-isomorphisms), where U is any local gauge of $\mathcal{E}$, so that one then obtains

(2.33) $$\begin{aligned} \Omega(\mathcal{E})(U) &\equiv (\Omega \otimes_{\mathcal{A}} \mathcal{E})(U) = \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{E}(U) \\ &= \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \Omega(U), \end{aligned}$$

within $\mathcal{A}(U)$-isomorphisms of the $\mathcal{A}(U)$-modules concerned for the last relations.

Consequently, as a result of (2.33), one concludes that

(2.34) (2.31) is valid, simply, with respect to any local gauge $U \subseteq X$ of either one of the two given vector sheaves Ω^n and $\mathcal{E}$ on X (not necessarily a common one!).

The above also clarifies the type of local sections that appeared in (2.30). Now we come to the justification of (2.30). Indeed, we prove right away that

(2.35) (2.30) provides, in effect, the definition of the "differential operators"

(2.35.1) $$\delta^n, \ n \in \mathbb{N},$$

"duals" of the D^n's, $n \in \mathbb{Z}_+$ (see (2.18), or even (1.12) in the preceding section). In fact,

(2.35.2) the operators δ^n are actually defined as the "transpose" of the given operators D^n's,

when based on the canonical identifications (2.9) associated with the corresponding $\mathcal{A}$-metrics of the (Riemannian; see (2.28)) vector sheaves involved in (2.29).

Thus, by virtue of (2.30), one obtains (making use of the symmetry of ρ)

(2.36) $$\rho(\delta^n(t), s) \equiv \rho_{\delta^n(t)}(s) \equiv \tilde{\rho}(\delta^n(t))(s) := \rho(D^{n-1}(s), t), \ n \in \mathbb{N},$$

for any $s \in \Omega^{n-1}(\mathcal{E})(U)$ and $t \in \Omega^n(\mathcal{E})(U)$, with U open in X, in such a manner that one finally gets (see also (2.9)) at the map (i.e., $\mathcal{A}(U)$-*linear morphism*)

(2.37) $$\tilde{\rho}(\delta^n(t)) \equiv \rho_{\delta^n(t)} \in (\Omega^{n-1}(\mathcal{E})(U))^* \underset{\tilde{\rho}}{\cong} \Omega^{n-1}(\mathcal{E})(U)$$

for any $t \in \Omega^n(\mathcal{E})(U)$ and $U \subseteq X$, with $n \in \mathbb{N}$. In other words, as already stated in (2.35),

(2.38) one may look at the operator δ^n, $n \in \mathbb{N}$, as the dual (alias, transpose via ρ) of D^n, as the latter is given by (2.18).

Thus, to repeat it again, one obtains (locally, for any open $U \subseteq X$) the map, in point of fact, $\mathbb{C}$-linear morphism of the $\mathcal{A}(U)$-modules (hence, of the $\mathbb{C}$-vector spaces, too) involved:

(2.39) $$\delta^n : \Omega^n(\mathcal{E})(U) \longrightarrow \Omega^{n-1}(\mathcal{E})(U) : t \longmapsto \delta^n(t) \underset{\tilde{\rho},1-1}{\overset{\subseteq}{\rightarrow}} \rho_{\delta^n(t)} \in (\Omega^{n-1}(\mathcal{E})(U))^* \underset{\tilde{\rho},\,\text{onto}}{\cong} \Omega^{n-1}(\mathcal{E})(U).$$

(So we do make use of (2.9) in the above definition of δ^n. Yet, we employed earlier an obvious abuse of notation concerning the $\mathcal{A}$-morphism $\tilde{\rho}$ [see (2.6) and (2.7), along with [VS: Chapt. I, p. 63 and p. 75, (13.18)]]). Thus, the preceding justifies now our claim in (2.35), hence, in turn, definition (2.30), as well. ■

The previous "duality," as expressed by (2.35.2), or even by (2.38), as above is given by the following "correspondence":

(2.40) $$D^n \longrightarrow \delta^{n+1}, \quad n \in \mathbb{Z}_+,$$

which is further depicted by the following diagram:

(2.41) $$\begin{aligned} &\Omega^0(\mathcal{E}) \equiv \mathcal{A}(\mathcal{E}) \\ &\cong \mathcal{E} \underset{\delta^1}{\overset{D^0 \equiv D}{\rightleftarrows}} \Omega^1(\mathcal{E}) \underset{\delta^2}{\overset{D^1}{\rightleftarrows}} \Omega^2(\mathcal{E}) \cdots \underset{\delta^n}{\overset{D^{n-1}}{\rightleftarrows}} \Omega^n(\mathcal{E}) \underset{\delta^{n+1}}{\overset{D^n}{\rightleftarrows}} \Omega^{n+1}(\mathcal{E}) \underset{\delta^{n+2}}{\overset{D^{n+1}}{\rightleftarrows}} \cdots, \end{aligned}$$

a (double) $\mathbb{C}$-sequence of $\mathbb{C}$-linear morphisms (see also (2.17)), which is a combination of the previous two $\mathbb{C}$-sequences (2.16) and (2.29).

Thus, one gets the sequence

(2.42) $$(\delta^n)_{n \in \mathbb{N}}$$

as the dual of (2.18), consisting of $\mathbb{C}$-linear morphisms between the $\mathcal{A}$-modules (in effect, vector sheaves see (2.26) and (2.27)) concerned. Its terms—as given by (2.35.1); see also (2.39)—are still called *adjoint exterior derivative operators*, or even *contraction operators* (see (2.39)), thus extending the classical terminology.

We come now to consider the necessary ingredients, in order to define, within our abstract setting, the *"Laplacian" of an $\mathcal{A}$-connection*. As we shall presently see, we have actually elaborated all the relevant necessary material.

3 The Abstract Laplace–Beltrami Operators

Suppose we have the framework of (2.25), while for the $\mathcal{A}$-metrics involved we assume the validity of (2.29) (see also Definition 2.1). Moreover, the $\mathcal{A}$-modules considered are, by assumption (see (2.26) and (2.27)), vector sheaves on X, so that all the vector sheaves now, appearing in (2.41), are thus, according to our hypothesis (see also (2.28)), Riemannian vector sheaves on X, in the sense of Definition 2.1.

Thus, based now on the $\mathcal{A}$-sequence (2.41), one can further define a corresponding *sequence of Laplace–Beltrami operators*,

(3.1) $$(\Delta^n)_{n \in \mathbb{N}},$$

according to the following relation;

(3.2) $$\begin{aligned} \Delta^n \equiv \Delta &:= \delta^{n+1} \circ D^n + D^{n-1} \circ \delta^n \equiv \delta \circ D + D \circ \delta \\ &\equiv \delta D + D\delta : \Omega^n(\mathcal{E}) \longrightarrow \Omega^n(\mathcal{E}), \; n \in \mathbb{N}, \end{aligned}$$

each one of the preceding operators thus being a $\mathbb{C}$-linear morphism of the $\mathcal{A}$-modules (actually, vector sheaves) concerned, as in (3.2). Yet, by extending to our case the corresponding classical situation, we shall also refer to any one of the above "differential" operators simply as the Laplacian (operator), its "order" (*viz.*, the index $n \in \mathbb{N}$, as in (3.1)), being determined from the context.

Precisely speaking, by the latter term, one actually refers to the Laplacian (or even, Laplace–Beltrami operator) Δ, corresponding to a given $\mathcal{A}$-connection D on a vector sheaf $\mathcal{E}$ on X, the latter space being the carrier of the appropriate framework, as, for example, in (2.25), within which the preceding have a meaning. Thus, according to (3.2), one has, in particular for $n = 1$,

$$\begin{aligned} \Delta \equiv \Delta^1 &:= \delta^2 \circ D^1 + D^0 \circ \delta^1 \equiv \delta^2 \circ D^1 + D \circ \delta^1 \\ &\equiv \delta D + D\delta : \Omega^1(\mathcal{E}) \longrightarrow \Omega^1(\mathcal{E}), \end{aligned} \tag{3.3}$$

so that, by the very definitions, one thus obtains that

$$\Delta^1 \equiv \Delta \in End_{\mathbb{C}}(\Omega^1(\mathcal{E})), \tag{3.4}$$

while, quite generally, as already mentioned in the preceding (see (3.2)), one still has

$$\Delta^n \equiv \Delta \in End_{\mathbb{C}}(\Omega^n(\mathcal{E})), \; n \in \mathbb{N}. \tag{3.5}$$

In the last relation, hence in (3.4), one considers the $\mathcal{A}$-modules (in effect, vector sheaves) as being, due to our hypothesis for $\mathcal{A}$ (see Volume I: Chapt. I, (1.3) and (1.5)), $\mathbb{C}$-vector space sheaves on X, the last member of (3.5) or of (3.4) being a $\mathbb{C}$-algebra sheaf on X. (In this regard, see also [VS: Chapt. II, pp. 138f].)

The same Laplacian operators can still be viewed by definition (see (3.5)) as global sections of the corresponding $\mathbb{C}$-algebra sheaves. We shall also do this occasionally throughout the ensuing discussion. Therefore, one still obtains that (by an obvious abuse of notation, we retain the same symbol for Δ, as in (3.5))

$$\Delta^n \equiv \Delta \in (\mathcal{E}nd_{\mathbb{C}}(\Omega^n(\mathcal{E})))(X) \equiv End_{\mathbb{C}}(\Omega^n(\mathcal{E})), \; n \in \mathbb{N}, \tag{3.6}$$

the last member of (3.6) being thus a $\mathbb{C}$-algebra. See also loc. cit., p. 139, (6.30) and (6.31). Yet, ibid. Chapt. I, p. 73, (13.7), and p. 27, (6.2) and (6.3).

For simplicity, we shall also employ concerning the $\mathcal{A}$-metric ρ considered, the following notation:

$$\rho(s, t) \equiv (s, t), \tag{3.7}$$

for any s and t in $\mathcal{E}(U)$, and any open $U \subseteq X$, as above. By employing the last simplified notation, as in (3.7), it is still very convenient to write the previous defining relation (2.30) in the form

$$(Ds, t) = (s, \delta t), \tag{3.7'}$$

with s and t in $\Omega^*(\mathcal{E})(U)$ (see (3.15.2)) and U open in X.

Now, concerning the $\mathbb{C}$-sequence (2.16) (indeed, for the case at hand, this is, in particular, an $\mathcal{A}$-sequence of vector sheaves on X), one obtains the relation (see also (2.33))

$$(D^n \circ D^{n-1})(s \otimes t) = t \wedge R(s), \ n \in \mathbb{N}, \tag{3.8}$$

for any s and t in $\mathcal{E}(U)$ and U open in X, with

$$R \equiv R(D) \tag{3.9}$$

denoting the curvature of the given $\mathcal{A}$-connection $D(\equiv D^0)$ of the vector sheaf $\mathcal{E}$ on X: See [VS: Chapt. VIII, p. 229, (8.22)]. Therefore, based on (3.8), one concludes that

(3.10) the $\mathbb{C}$-sequence (2.16) is a complex (of $\mathbb{C}$-vector space sheaves and $\mathbb{C}$-linear morphisms); that is, one has

$$D^n \circ D^{n-1} = 0, \ n \in \mathbb{N} \tag{3.10.1}$$

(with $D^0 \equiv D$, see (1.15)), if, and only if, one has

$$R(D) \equiv R = 0 \tag{3.10.2}$$

(*viz.*, whenever the given $\mathcal{A}$-connection D on $\mathcal{E}$ is flat).

On the other hand, based on (2.36), one obtains (we apply the simplified notation (3.7) concerning the $\mathcal{A}$-metric ρ by also taking into account the symmetry of the same)

$$(\delta^n \delta^{n+1} t, s) = (\delta^{n+1} t, D^{n-1} s) = (D^n D^{n-1} s, t) \tag{3.11}$$

for any s and t, as in (2.36) in the preceding section. Therefore, one thus concludes that

(3.12) (3.10.1) is equivalent to the relation

$$\delta^n \circ \delta^{n+1} = 0, \ n \in \mathbb{N}; \tag{3.12.1}$$

hence, in turn (see (3.10)), the latter relation is still equivalent with the flatness of the $\mathcal{A}$-connection D as well.

In this context, by extending the classical situation, we can look further at the so-called *Dirac–Kähler operator*

$$D + \delta, \tag{3.13}$$

with D and δ being any one of the items (terms) of the two sequences (2.18) and (2.42), respectively. Thus, for any given flat $\mathcal{A}$-connection D on $\mathcal{E}$, one obtains for the corresponding Laplacian (see (3.2)) the relation

(3.14) (3.14.1)
$$\Delta = (D + \delta)^2,$$

while we still set

(3.14.2)
$$\sqrt{\Delta} \equiv D + \delta.$$

The assertion follows from (3.10.1) and (3.2). ■

On the other hand,

the Dirak–Kähler operator is self-adjoint, with respect to the given $\mathcal{A}$-metric ρ (see (3.7), along with (2.39)); that is, one has:

(3.15) (3.15.1)
$$((D + \delta)s, t) = (s, (D + \delta)t),$$

for any $s, t \in \Omega^*(\mathcal{E})(U)$, with U open in X, where we have set;

(3.15.2)
$$\Omega^*(\mathcal{E}) := \bigoplus_{n \in \mathbb{Z}_+} \Omega^n(\mathcal{E}).$$

Indeed, our assertion in (3.15.1) is an immediate consequence of (2.36), or even of (3.7′), as above, along with the bi-additivity and symmetry of ρ (see (2.1) conditions (i) and (ii)). ■

An analogous conclusion, as above, is also still valid for the Laplacian; see (3.16) below.

3.1 Positivity of the Laplacian and the Green's Formula

For the terminology in the heading of this subsection, we still refer to (2.7 ′). Thus, our first task is to prove that

the Laplacian is a self-adjoint operator, with respect to the $\mathcal{A}$-metric ρ; that is, one obtains

(3.16) (3.16.1)
$$(\Delta s, t) \equiv (\Delta^n s, t) = (s, \Delta^n t) \equiv (s, \Delta t)$$

for any $n \in \mathbb{N}$ and s and t in $\Omega^n(\mathcal{E})(U)$, with U open in X.

Indeed, by virtue of (3.2) and (2.36), along with the bi-additivity and symmetry of ρ, one has

(3.17) (3.17.1)
$$\begin{aligned}(\Delta s, t) \equiv (\Delta^n s, t) &= ((\delta^{n+1} D^n + D^{n-1} \delta^n)s, t) \\ &= (\delta^{n+1} D^n s, t) + (D^{n-1} \delta^n s, t) \\ &= (D^n s, D^n t) + (\delta^n s, \delta^n t) \\ &\equiv (Ds, Dt) + (\delta s, \delta t) \equiv (D + \delta)(s, t) \\ &= (s, \delta^{n+1} D^n t) + (s, D^{n-1} \delta^n t) \\ &= (s, (\delta^{n+1} D^n + D^{n-1} \delta^n)t) = (s, \Delta^n t) \equiv (s, \Delta t),\end{aligned}$$

which thus proves (3.16). ■

On the other hand, by omitting the precise indication of the order of the Laplacian involved, along with operators D and δ, and by further applying the simplified formula for Δ (see (3.2) as well as (3.7′)), one can carry out the previous calculations as follows:

$$(3.17') \qquad \begin{aligned}(\Delta s, t) &= ((\delta D + D\delta)s, t) = (\delta D s, t) + (D\delta s, t)\\ &= (Ds, Dt) + (\delta s, \delta t) = (s, \delta D t) + (s, D\delta t)\\ &= (s, (\delta D + D\delta)t) = (s, \Delta t),\end{aligned}$$

that is, (3.17), as before, while (3.18) is also implicit, within the previous relations. ■

Now, as a byproduct of the previous proof (see (3.17.1)), one obtains

$$(3.18) \qquad (\Delta s, t) = (Ds, Dt) + (\delta s, \delta t),$$

or, more precisely, one has

$$(3.19) \qquad (\Delta^n s, t) = (s, \Delta^n t) = (D^n s, D^n t) + (\delta^n s, \delta^n t)$$

for any s and t in $\Omega^n(\mathcal{E})(U)$, as in (3.16.1). The last two equations constitute, by definition, the *Green's formula*. (Concerning the classical case, see, for instance, W. Greub et al. [1: Vol. I, p. 172].)

Thus, by virtue of (3.18), or (3.19) and for $s = t \in \Omega^n(\mathcal{E})(U)$ (see (3.16.1)), one obtains

$$(3.20) \qquad (\Delta s, s) = (s, \Delta s) = (Ds, Ds) + (\delta s, \delta s) \geqq 0$$

(see (2.7), along with (2.7′)); that is, equivalently,

$$(3.21) \qquad (\Delta^n s, s) = (s, \Delta^n s) = (D^n s, D^n s) + (\delta^n s, \delta^n s) \geqq 0.$$

In other words,

(3.21′) the Laplacian (operator) is also non-negative ("positive").

Consequently, as a byproduct of (3.20) or of (3.21), one concludes that

$$(3.22) \qquad \Delta s = 0 \text{ if, and only if, } Ds = 0 \text{ and } \delta s = 0;$$

that is, equivalently, by definition,

$$(3.22') \qquad \Delta^n s = 0 \text{ if, and only if, } D^n s = 0 \text{ and } \delta^n s = 0,$$

such that $s \in \Omega^n(\mathcal{E})(U)$, U open in X, and $n \in \mathbb{N}$.

A particular application of our previous conclusion in (3.22) is given below by considering the Yang–Mills equations. Yet, in this regard and by still extending the classical terminology, (3.23) a (local) section

(3.23.1) $$s \in \Omega^n(\mathcal{E})(U),$$

with $n \in \mathbb{N}$ and U open in X, such that

(3.23.2) $$\Delta^n s = 0 \text{ (or even } \Delta s = 0)$$

is called an ($\mathcal{E}$-valued) harmonic (n-)form on U.

Thus, roughly speaking, elements of

(3.24) $$\ker \Delta \text{ or } \ker \Delta^n \subseteq \Omega^n(\mathcal{E}), \qquad n \in \mathbb{N}$$

(see also (3.2)), characterize, by definition, the harmonic forms on X. As we shall see (see Section 4), "instantons" may be viewed as such forms on X. On the other hand, by still employing the above terminology, we can express our conclusion in (3.22′) by just saying, in complete analogy with the classical case, that

(3.24′) a given $\mathcal{E}$-valued n-form, say s, on U [*viz.*, $s \in \Omega^n(\mathcal{E})(U)$], is harmonic ($\Delta s = 0$) if and only if it is simultaneously closed ($Ds = 0$) and co-closed ($\delta s = 0$).

Now, in anticipation of Section 4, consider the case of the 2-form on X, which is defined by the curvature $R(D) \equiv R$ of a given $\mathcal{A}$-connection D on a vector sheaf $\mathcal{E}$ on X. Thus, one has

(3.25) $$R \in \Omega^2(\mathcal{E}nd\mathcal{E})(X),$$

where $\mathcal{E}nd\mathcal{E}$ is still a vector sheaf on X (see [VS: Chapt. II, p. 137, Lemma 6.1]). Hence, in place of $\mathcal{E}$, as above, one can consider the vector sheaf $\mathcal{E}nd\mathcal{E}$ on X. On the other hand, according to *Bianchi's identity* (see [VS: Chapt. VII, p. 224, Theorem 7.1]), one has

(3.26) $$D^2_{\mathcal{E}nd\mathcal{E}}(R) = 0.$$

Thus, by combining Green's formula, for $n = 2$ (see (3.19)), and Bianchi's identity, as in (3.26), one obtains

(3.27) $$(\Delta R, R) = (R, \Delta R) = (\delta R, \delta R)$$

(we have set above, for convenience, $\Delta^2_{\mathcal{E}nd\mathcal{E}} \equiv \Delta$ and $\delta^2_{\mathcal{E}nd\mathcal{E}} \equiv \delta$; see also Section 4). Therefore, one concludes that

(3.28) $$\Delta R = 0 \text{ if and only if } \delta R = 0,$$

a fact that will be used below. See Section 4 for applications of this material.

In other words,

R, as in (3.25), is an $\mathcal{E}nd\mathcal{E}$-valued harmonic 2-form on X (see (3.23)), one has

(3.29) (3.29.1) $$\Delta^2_{\mathcal{E}nd\mathcal{E}}(R) = 0 \text{ (or simply, } \Delta(R) = 0)$$

if and only if one has

(3.29.2) $$\delta^2_{\mathcal{E}nd\mathcal{E}}(R) = 0 \text{ (or } \delta(R) = 0).$$

Yet, by applying our previous terminology in (3.24′) on the vector sheaf $\mathcal{E}nd\mathcal{E}$ on X, as in (3.29), we can still express our latter conclusion by saying that

(3.30) the curvature $R \equiv R(D)$ of a given $\mathcal{A}$-connection D on $\mathcal{E}$ is a harmonic ($\mathcal{E}nd\mathcal{E}$-valued 2-form on X) if and only if it is co-closed [being, according to Bianchi's identity (see (3.26)), already closed].

Now, within this same vein of ideas, one can further conclude, quite generally, that

(3.31) there are no nontrivial harmonic forms on X that are also exact.

Indeed (arguing locally), suppose that $\omega \in \Omega^p(\mathcal{E})(U)$, with $\Delta\omega = 0$, while we also assume that $\omega = D\alpha$, with $\alpha \in \Omega^{p-1}(\mathcal{E})(U)$. Now, since, by hypothesis (see (3.24′)), one still obtains that $\delta\omega = 0$, one has $\delta D\alpha = 0$, as well, so that one obtains

(3.32) $$(\delta D\alpha, \alpha) = (D\alpha, D\alpha) = 0;$$

that is (see (2.5)), $\omega = 0$. ■

An equivalent formulation of (3.31) is the relation

(3.33) $$\ker \Delta \cap \Omega^p(\mathcal{E})_{ex} = \{0\}$$

for any $p \in \mathbb{N}$. Therefore,

(3.34) two harmonic (p-)forms that differ by an exact form are actually equal.

As already stated, it is now the above (harmonic) 2-form, as in (3.29.1), that will be important to us in the subsequent discussion.

4 The Abstract Yang–Mills Equations

The purpose of the ensuing discussion, as the title of this section indicates, is to formulate the standard Yang–Mills equations within the framework of the "abstract differential geometry," as the latter is described by the language of vector sheaves. Thus, to facilitate the reading and to fix the notation employed in the sequel, we first give, according to the preceding, the precise setup within which we are going to formulate the equations under consideration. So we start with the assumption that we are given the following data:

(4.1) a partially ordered algebraized space (see (2.4)),

(4.1.1) $$(X, \mathcal{A}),$$

such that X is a paracompact (Hausdorff) space and $\mathcal{A}$ is a strictly positive fine sheaf on X (see (2.21) and (2.22)). Moreover, we suppose that

(4.1.2) $$(\mathcal{A}, \rho)$$

is a Riemannian $\mathcal{A}$-module on X (Definition 2.1), so that, by assumption,

(4.1.3) $$\rho : \mathcal{A} \oplus \mathcal{A} \longrightarrow \mathcal{A}$$

is an $\mathcal{A}$-*bilinear symmetric positive-definite* and strongly nondegenerate (sheaf) morphism of the $\mathcal{A}$-modules concerned; that is, in other words, ρ is a Riemannian $\mathcal{A}$-metric on $\mathcal{A}$ (ibid.).

Yet, we assume that

(4.1.4) $$(\mathcal{A}, \partial, \Omega^1)$$

is a given differential triad on X, with Ω^1 being a vector sheaf on X, while we still accept that we are given the whole sequence of "differentials of the first kind" (exterior derivative operators in other words),

(4.1.5) $$(d^n)_{n \in \mathbb{N}_+}$$

(see (1.3)), such that (1.7) and (1.7′) are valid.

Now, for convenience, we shall also refer to the previous data as an abstract Riemannian (differential) space, or, for short, simply as a Riemannian space X.

Thus, as a consequence of our previous assumptions for X, or by just considering a Riemannian space X, according to the latter terminology, as in (4.1), one first concludes that

(4.2) if $\mathcal{E}$ is a given vector sheaf on a Riemannian space X, as above, then $\mathcal{E}$ is still endowed with an $\mathcal{A}$-metric

(4.2.1) $$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A},$$

having analogous properties to those of (4.1.3), so that

(4.2.2) $$(\mathcal{E}, \rho)$$

becomes a Riemannian vector sheaf on X (see Definition 2.1).

See also (2.24) as well as (2.28). Furthermore, based on our hypothesis for X, as in (4.1), one still obtains that

(4.3) every vector sheaf $\mathcal{E}$ on a Riemannian space X (see (4.1)) admits an $\mathcal{A}$-connection D.

In this regard, see also [VS: Chapt. VI, p. 85, Theorem 16.1], as well as [VS: Chapt. III, p. 247, (8.56)]. ■

On the other hand, based on the preceding, we also understand that, within the context of a Riemannian space X, as above, one can still define, via a given vector sheaf $\mathcal{E}$ on X, the "vectorization" of the sequence

$$(\Omega^n)_{n\in\mathbb{Z}_+} \tag{4.4}$$

(see (1.5) and (1.6)), that is, the sequence

$$\Omega^n(\mathcal{E}) \equiv \Omega^n \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \Omega^n, \qquad n \in \mathbb{Z}_+ \tag{4.5}$$

(see (1.15) and (1.19)), in such a manner that all the terms in (4.4) and (4.5) become vector sheaves on X (see also (4.1.4)). Yet, one further defines the sequences of the "differentials of the second kind"

$$(D^n)_{n\in\mathbb{Z}_+} \tag{4.6}$$

(see (1.16) and (1.17)). Finally, one gets at the sequence of the "adjoint differentials" (see (2.35) and (2.39))

$$(\delta^n)_{n\in\mathbb{N}}, \tag{4.7}$$

hence, in conjunction with (4.6) (see (3.1) and (3.2)), at that one of the Laplacians,

$$(\Delta^n)_{n\in\mathbb{N}}. \tag{4.8}$$

One thus obtains the following framework, which will be of constant use throughout the subsequent discussion: One concludes that

(4.9) given a Riemannian space X, as above, every vector sheaf $\mathcal{E}$ on X admits an $\mathcal{A}$-connection, while one can still define the concomitant Laplacians on the "vectorized," via $\mathcal{E}$, $\mathcal{A}$-modules (in fact, vector sheaves) of "differential forms" on X.

Indeed, as we shall see, the previous setup, as it is recapitulated, for instance, by (4.9), is an appropriate one for ADG (abstract differential geometry), within which we can deal with Yang–Mills fields in such a manner that one can still formulate therein the corresponding to the latter Yang–Mills equations.

4.1 Yang–Mills Fields

Suppose we are given a differential triad

$$(\mathcal{A}, \partial, \Omega^1) \tag{4.10}$$

on a $\mathbb{C}$-algebraized space

$$(4.11) \qquad (X, \mathcal{A})$$

(see [VS: Vol. II] or Volume I, Chapt. I, (1.4) and (1.13)). Then, a pair

$$(4.12) \qquad (\mathcal{E}, D)$$

consisting of a vector sheaf $\mathcal{E}$ on X, with

$$(4.13) \qquad rk_{\mathcal{A}}\mathcal{E} \equiv rk\mathcal{E} \geqq 2$$

(see [VS: Vol. I] or Volume I, Chapt. I, (2.41.1)), and an $\mathcal{A}$-connection D of $\mathcal{E}$, is, in general, called a Yang–Mills field on X.

Now, concerning the previous terminology, we first remark that

(4.14) to formulate the above general notion of a Yang–Mills field, as in (4.12), what we really need is the general framework of a "differential triad," as in (4.10) and (4.11); see also Chapter I, (2.1) and (2.3).

Indeed, the preceding terminology is still in agreement with a common usage of the term in other more classical contexts; see, for instance, Yu.I. Manin [2: p. 72, §2.17]. Moreover, we insisted above on the rank of $\mathcal{E}$, as in (4.13), in that the latter should be, by definition, in the case of a Yang–Mills field $(\mathcal{E}, d)$, as before, greater than 1; on the other hand, the particular case of $rk\mathcal{E} = 1$ has been already considered in the preceding (see Volume I, Chapt. III) for Maxwell fields (ibid. Definition 1.1). This last term, as it has been already mentioned, still constitutes the extension to the present abstract setting of analogous terminology, which has also been applied to the present treatment of more classical frameworks; thus, see again, for instance, the above quoted work of Manin [2: p. 71, §1.16].

On the other hand, the same term "Yang–Mills field" as above, concerning a pair $(\mathcal{E}, D)$ like in (4.12), will be applied for the curvature $R(D) \equiv R$ of an $\mathcal{A}$-connection D (ibid.), where the latter satisfies, through its curvature (always!), the so-called Yang–Mills equations. Of course, this situation is the outcome of the corresponding one in physics, where, by employing an obvious abuse of terminology, we often identify the cause, or even causality (*viz.*, in other words, the $\mathcal{A}$-connection/potential), with the result (i.e., curvature/field strength), whereas all these are still with the carrier (vector sheaf $\mathcal{E}$) that is always accompanied by the corresponding ($\mathcal{A}$-)connection D, as in (4.12): See, for instance, the relevant situation in the case of the electromagnetic field, as this has been annotated in Chapter III, (3.55) and (3.56) of Volume I; see also (3.57.1) therein, as well as in Chapter IV, (6.8) of Volume I. Anyhow, the aforesaid distinction of the relevant terminology will always be made clear, throughout the subsequent discussion, from the context.

Now, before we proceed to the formulation of the fundamental equations, which are, for that matter, our main objective in this section, we discuss the particular objects, introduced above by (4.12) and (4.13), still within the quite general context of (4.10) and (4.11), that will be also of use presently.

4.2 The Yang–Mills Category

As already mentioned, the framework within which we continue to work in this and also the following subsection is still the general one, as described by (4.10) and (4.11). So the objects of the category, as in the heading of this subsection, are the Yang–Mills fields, as they were defined above by (4.12) and (4.13).

On the other hand, a morphism of the category at issue is a map

$$\phi : (\mathcal{E}, D) \longrightarrow (\mathcal{E}', D'), \tag{4.15}$$

which is (i) a sheaf morphism of the vector sheaves involved in (4.15)—that is, one has

$$\phi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}') \equiv \mathcal{H}om_{A}(\mathcal{E}, \mathcal{E}')(X). \tag{4.16}$$

(by an obvious abuse of notation, we retain here the same symbol for the above two maps)—while (ii) we still accept that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad D \quad} & \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{E}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi\otimes 1_{\Omega^1}} \\ \mathcal{E}' & \xrightarrow{\quad D' \quad} & \mathcal{E}' \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{E}') \end{array} \tag{4.17}$$

that is, the two given $\mathcal{A}$-connections D and D', as in (4.15), are ϕ-related; equivalently (commutativity of the diagram (4.17)), one has

$$D' \circ \phi = (\phi \otimes 1_{\Omega^1}) \circ D \equiv (\phi \otimes 1) \circ D \tag{4.18}$$

(see the relevant situation in the case of Maxwell fields in Volume I, Chapter III, (1.8) and (1.9)).

We denote the above category by

$$\mathcal{YM}_X \tag{4.19}$$

and call it the Yang–Mills category of the topological space X considered, as in (4.11), the latter space being the carrier of the given differential triad (see (4.10)).

Now, the same category, as above, has tensor products; that is, for any two objects of the category at issue, say $(\mathcal{E}, D)$ and $(\mathcal{E}', D')$, one defines their tensor product by the relation

$$(\mathcal{E}, D) \otimes (\mathcal{E}', D') := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}', D \otimes D'), \tag{4.20}$$

where the second member of (4.20) is, of course, still a Yang–Mills field, as in (4.12) and (4.13), according to the very definitions (see [VS: Chapt. II, p. 132, (5.27), along with Chapt. VI, p. 18, Section 5.2]). In this context, one further sets, by definition (loc. cit., p. 18, (5.10)),

(4.21) $$D_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{F}} \equiv D \otimes D' := D \otimes 1_{\mathcal{F}} + 1_{\mathcal{E}} \otimes D' \equiv D \otimes 1 + 1 \otimes D'.$$

On the other hand, the category, under consideration, has also an *internal $\mathcal{H}om$ functor* in the sense that for any two objects of $\mathcal{YM}_X$, as above, one still defines a new object of the same category according to the relation

(4.22) $$\mathcal{H}om((\mathcal{E}, D), (\mathcal{E}', D')) := (\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}'), D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}')}),$$

where one further sets, by definition,

(4.23) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}')}(\phi) := D' \circ \phi - (\phi \otimes 1_{\Omega^1}) \circ D \equiv D' \circ \phi - (\phi \otimes 1) \circ D$$

for any (local) section

(4.24) $$\phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')(U) = Hom_{A|_U}(\mathcal{E}|_U, \mathcal{E}'|_U)$$

and any open $U \subseteq X$. In this regard, see also [VS: Chapt. VI, p. 19, (5.12) and (5.13), along with Chapt. II, p. 135, (6.8)].

Now, for the particular case that one has just one Yang–Mills field $(\mathcal{E}, D)$ on X, as above, then one has

(4.25) $$\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \equiv \mathcal{E}nd\mathcal{E}$$

(loc. cit., Chapt. II, p. 138, (6.28)), so that, in view of (4.23), one further obtains

(4.26) $$D_{\mathcal{E}nd\mathcal{E}}(\phi) = D \circ \phi - (\phi \otimes 1) \circ D$$

for any

(4.27) $$\phi \in (\mathcal{E}nd\mathcal{E})(U) = Hom_{A|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \equiv End(\mathcal{E}|_U)$$

and any open $U \subseteq X$. Yet, for convenience, we still write, in place of (4.26), the formal relation

(4.28) $$D_{\mathcal{E}nd\mathcal{E}}(\phi) = D \circ \phi - \phi \circ D \equiv D\phi - \phi D \equiv [D, \phi],$$

with ϕ, as in (4.27), or even the formula

(4.29) $$D_{\mathcal{E}nd\mathcal{E}} = \mathcal{L}_D,$$

with an obvious meaning of the second member of (4.29) (*Lie operator/derivation*), that will also be of use in the sequel.

Furthermore, concerning the morphisms, in general, of the previous category, as these are defined by (4.16) and (4.18), it is still useful to remark, based on (4.23) and in view of later applications, that the morphisms of the Yang–Mills category $\mathcal{YM}_X$, as above, when referring to two given objects of it, say $(\mathcal{E}, D_{\mathcal{E}})$ and $(\mathcal{F}, D_{\mathcal{F}})$, are those (global) sections (hence, sheaf morphisms; see (4.16))

$$\phi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(X), \tag{4.30.1}$$

which commute with the individual $\mathcal{A}$-connections $D_{\mathcal{E}}$ and $D_{\mathcal{F}}$ or, equivalently, are horizontal with respect to the $\mathcal{A}$-connection $D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}$ (they annihilate it); that is, one has

(4.30)

$$\begin{aligned} D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}(\phi) &= D_{\mathcal{F}} \circ \phi - (\phi \otimes 1_{\Omega^1}) \circ D_{\mathcal{E}} \\ &\equiv D_{\mathcal{F}} \circ \phi - \phi \circ D_{\mathcal{E}} \equiv D_{\mathcal{F}}\phi - \phi D_{\mathcal{E}} = 0. \end{aligned} \tag{4.30.2}$$

Therefore, one has (see also (4.18)) the relation

$$D_{\mathcal{F}} \circ \phi = \phi \circ D_{\mathcal{E}} \tag{4.30.3}$$

or just, for simplicity,

$$D_{\mathcal{F}}\phi = \phi D_{\mathcal{E}} \tag{4.30.4}$$

(commutativity of ϕ, with respect to $D_{\mathcal{E}}$ and $D_{\mathcal{F}}$), being thus the analogon, herewith, of the required commutativity of the diagram (4.17).

Thus, by specializing now, as in the preceding (see (4.25)), to the case

$$\mathcal{E} = \mathcal{F}, \tag{4.31}$$

one has, in view of (4.30.2), the relation

$$D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0 \tag{4.32}$$

or, equivalently (see (4.30.4)),

$$D\phi = \phi D, \tag{4.33}$$

yet, alias, the relation (see (4.28) and (4.29))

$$[D, \phi] \equiv \mathcal{L}_D(\phi) = 0, \tag{4.34}$$

such that

$$\phi \in End\mathcal{E} = (\mathcal{E}nd\mathcal{E})(X) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(X), \tag{4.35}$$

characterizing thus, by anyone of them, along with (4.35), the endomorphisms of a given Yang–Mills field $(\mathcal{E}, D)$, within the category $\mathcal{YM}_X$.

We recapitulate the preceding, through the following two relations, whose meaning is thus clear from the context, according to the previous discussion. So we have

(4.36)
$$\begin{aligned} &Mor((\mathcal{E}, D_{\mathcal{E}}), (\mathcal{F}, D_{\mathcal{F}})) \\ &\quad := \{\phi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(X) : \\ &\qquad\quad D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}(\phi) = 0 \Longleftrightarrow D_{\mathcal{F}}\phi = \phi D_{\mathcal{E}}\} \\ &\quad = \ker D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}. \end{aligned}$$

In particular, for $\mathcal{E} = \mathcal{F}$, one has

(4.37)
$$\begin{aligned} &End((\mathcal{E}, D)) \\ &\quad := \{\phi \in End\mathcal{E} = (\mathcal{E}nd\mathcal{E})(X) : \\ &\qquad\quad D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0 \Longleftrightarrow D\phi = \phi D \Longleftrightarrow [D, \phi] = 0\} \\ &\quad = \ker D_{\mathcal{E}nd\mathcal{E}}. \end{aligned}$$

Indeed, to be more precise (!), a prefix "$\mathcal{YM}_X$-" should actually be put before the first members of the above two formulas; however, it has been omitted for simplicity's sake.

Note 4.1 The preceding relation

(4.38)
$$[D, \phi] \equiv \mathcal{L}_D(\phi) = 0$$

(see (4.34)) enables one to look at ϕ, as the latter map is given by (4.35), acting, as (or at least participating in) a "flow," in the sense that ϕ thus appears to be as "causally stationary" (where "causality" is meant with respect to the $\mathcal{A}$-connection D of $\mathcal{E}$). In this connection, see also Scholium 4.1.

4.3 Gauge Equivalent Yang–Mills Fields

We further specialize in this subsection in the case that the morphisms of the category $\mathcal{YM}_X$, as considered in the preceding, are, in particular, isomorphisms of the same category. Thus, by definition,

(4.39) gauge equivalences between Yang–Mills fields are isomorphisms of the corresponding Yang–Mills category $\mathcal{YM}_X$.

Therefore, given two Yang–Mills fields $(\mathcal{E}, D_{\mathcal{E}})$ and $(\mathcal{F}, D_{\mathcal{F}})$ (see (4.12) and (4.13)), a gauge equivalence between them is, by definition (see (4.39), as above), an isomorphism of them, when they are considered objects of the Yang–Mills category to which they belong. In other words, according to (4.36), this refers to a map (in point of fact, to a global section of the pertinent sheaf, herewith)—namely

(4.40)
$$\phi \in \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(X) = Isom_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$$

such that (cf. (4.30.4))

(4.41)
$$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}(\phi) = 0 \text{ or, equivalently, } D_{\mathcal{F}}\phi = \phi D_{\mathcal{E}}.$$

Now, in view of (4.40), the last relation in (4.41) can still be written in the form (see also (4.30.2))

$$D_{\mathcal{F}} = (\phi \otimes 1_{\Omega^1}) \circ D_{\mathcal{E}} \circ \phi^{-1} \tag{4.42}$$

or simply, by a convenient abuse of notation,

$$D_{\mathcal{F}} = \phi D_{\mathcal{E}} \phi^{-1} \equiv Ad(\phi) D_{\mathcal{E}} \equiv \phi_*(D_{\mathcal{E}}). \tag{4.43}$$

Therefore, one concludes that

(4.44) a gauge equivalence, as above (see (4.39)), between two given Yang–Mills fields $(\mathcal{E}, D_{\mathcal{E}})$ and $(\mathcal{F}, D_{\mathcal{F}})$ is, according to (4.41) and (4.43), (i) an isomorphism between $\mathcal{E}$ and $\mathcal{F}$ (namely of their respective carriers) that (ii) preserves (see (4.43)) the corresponding $\mathcal{A}$-connections $D_{\mathcal{E}}$ and $D_{\mathcal{F}}$ of the Yang–Mills fields at issue.

We then speak also of gauge equivalent Yang–Mills fields $(\mathcal{E}, D_{\mathcal{E}})$ and $(\mathcal{F}, D_{\mathcal{F}})$, through the map ϕ, as above, and write

$$(\mathcal{E}, D_{\mathcal{E}}) \underset{\phi}{\sim} (\mathcal{F}, D_{\mathcal{F}}). \tag{4.45}$$

Indeed, the previous relation is an equivalence between the objects of the category $\mathcal{YM}_X$ so that one can further consider, by analogy with the Maxwell group in the preceding (see Volume I, Chapt. III, Theorem 2.1), the set

$$\Phi^n_{\mathcal{A}}(X)^{\nabla} \tag{4.46}$$

—namely, the set of equivalence classes of Yang–Mills fields on X, of rank $n \in \mathbb{N}$ (with $n \geqq 2$, see (4.13), that is, precisely speaking, all those, modulo (4.45), whose carriers are vector sheaves of the said rank). Yet, an appropriate cohomological classification of the above set will be also given in the sequel (see Section 9 of the present chapter) by extending the corresponding situation that we have already considered in the particular case of Maxwell fields ($n = 1$, as above; see Volume I of this treatise, Chapt. IV).

On the other hand, one can still consider the set

$$\mathcal{YM}(X) \equiv \sum_{n \geqq 2} \Phi^n_{\mathcal{A}}(X)^{\nabla}, \tag{4.47}$$

called the Yang–Mills set of X. In point of fact, it is an (associative) semigroup, with respect to the tensor product operation, as defined by (4.20) and (4.21). (Evidence about our previous claim for the set (4.47) will be supplied by our considerations in Section 9 in the sequel; see also [VS: Chapt. V, p. 353, Lemma 2.1], referring to a local aspect of (4.36), as well as [Volume I, Chapt. III, (2.33), (2.34)).

In particular, by taking just one Yang–Mills field

$$(\mathcal{E}, D) \tag{4.48}$$

on X (see (4.12) and (4.13)), and according to the very definitions (see (4.37) and (4.42), or even (4.43), for $\mathcal{F} = \mathcal{E}$) one concludes that

(4.49) the gauge equivalences of $(\mathcal{E}, D)$ (see (4.45)) are the connection-preserving automorphisms of $\mathcal{E}$ (see also (4.44), for $\mathcal{F} = \mathcal{E}$); that is, those maps

$$\phi \in \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(X) \equiv (\mathcal{A}ut_{\mathcal{A}}\mathcal{E})(X) = Aut_{\mathcal{A}}\mathcal{E} \equiv Aut\mathcal{E} \equiv Isom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \tag{4.49.1}$$

(*viz.*, the $\mathcal{A}$-automorphisms of $\mathcal{E}$) for which one still obtains

$$\phi_*(D) \equiv Ad(\phi)D = D \tag{4.49.2}$$

("$\mathcal{A}$-connection preserving"; see also (4.43), for the particular case at issue).

Yet, by still employing the previous notation, as in (4.37) (see also the comments following it), the above maps, considered by (4.49), constitute the subgroup of $Aut\mathcal{E}$ (see also (4.49.2)) consisting of the "transfigurations" of $(\mathcal{E}, D)$ (automorphisms of $(\mathcal{E}, D)$; see (4.39)) in the Yang–Mills category $\mathcal{YM}_X$ (see (4.19)); that is, one sets

$$\mathcal{YM}_X - Aut_{\mathcal{A}}((\mathcal{E}, D)) \equiv Aut(\mathcal{E}, D) < Aut\mathcal{E}. \tag{4.50}$$

In this connection, see also Scholium 4.1 pertaining to a potential "physical transcription" of the above.

Applications of (4.49.2), as well as further restrictions apart from the noted condition on the maps involved in (4.49), that is, equivalently, in (4.50) will be also considered throughout the subsequent discussion when taking, for example, Riemannian vector sheaves on X (see Definition 2.1 or even (4.2)), always within, of course, the appropriate abstract setting of the present treatise.

Scholium 4.1 By commenting a bit more on the preceding relations (4.36), or equivalently (4.37) for the case at issue, on the relation (4.38), or even on (4.49.2), one could say here that the same relations might be construed as another formulation of the principle that

(4.51) *flow is causally stationary.*

Now, by a "flow" one means, as already hinted at in Note 4.1, an $\mathcal{A}$-isomorphism between the fields, or, in particular, an $\mathcal{A}$-automorphism of a given field, that remains "stationary" (horizontal) with respect to the "causality" (*viz.*, the $\mathcal{A}$-connection (gauge potential)), supported by the physical system, under consideration (i.e., by a pair of fields, or a given field with itself)—in other words, ("stationary") with respect to the $\mathcal{A}$-connections,

(4.52) $D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}$ or, in particular (for $\mathcal{F} = \mathcal{E}$), $D_{\mathcal{E}nd\mathcal{E}}$, as before (see (4.23) and (4.29)), this being expressed by the relations (equations)

(4.53) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})}(\phi) = 0 \text{ and } D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0,$$

respectively (see (4.36) and (4.37)).

Yet, within the same setup as above, one can still understand as

"flow, the various gauge equivalences" given by the sets

(4.54) (4.54.1)
$$\mathcal{YM}_X - \mathcal{I}som_{\mathcal{A}}((\mathcal{E}, D_{\mathcal{E}}); (\mathcal{F}, D_{\mathcal{F}})) \equiv \mathcal{I}som_{\mathcal{A}}((\mathcal{E}, D_{\mathcal{E}}), (\mathcal{F}, D_{\mathcal{F}}))$$
or, in particular,
$$\mathcal{YM}_X - Aut_{\mathcal{A}}(\mathcal{E}, D) \equiv Aut(\mathcal{E}, D)$$

(however, see also (4.50) concerning the last set—in effect, group—as above).

On the other hand, this same "flow" as before (*viz.*, the previous sets (4.54.1)) might also be conceived further as the carrier itself of the field(s) at issue.

Of course, one can still remark here that the previous "flow/carrier" may also be construed as being generated by the $\mathcal{A}$-connection (gauge potential) that defines the field(s) according to a standard argument (thus, differential equations/solutions), that is, again, the same sets (4.54.1) as above.

In this connection, we can still make the following remarks, according to the following.

Scholium 4.2 ("Gel'fand duality") By looking, just, for convenience, at the second relation (equation) in (4.53)—namely at the relation,

(4.55) $$D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0 \quad \text{or simply} \quad D(\phi) = 0$$

—we can still write it, as we did in the preceding, in the equivalent form

(4.56) $$Ad(\phi)D = D \quad \text{or} \quad \phi_*(D) = D$$

(see (4.37) and (4.49.2), as above); that is, the $\mathcal{A}$-connection D (in point of fact, $D_{\mathcal{E}nd\mathcal{E}}$) now appears, according to (4.56), as a fixed point of ϕ_*. Thus, one concludes that

(4.57)

$\phi \in Aut\mathcal{E}$ is a solution of D; that is, one has

(4.57.1) $$D(\phi) = 0 \quad \text{or, equivalently,} \quad \phi \in \mathcal{YM}_X - Aut_{\mathcal{A}}(\mathcal{E}, D)$$

(see (4.54.1), along with (4.37)), if and only if, after the interchange herewith of

(4.57.2) $$\text{functions} \rightleftarrows \text{variable}$$

(what actually one might consider here, as a type of *Gel'fand duality,* according to a standard argument), D is a fixed point of $\phi_* \equiv Ad(\phi)$; namely, one has

(4.57.3) $$\phi_*(D) = D.$$

Yet, as an outcome of the preceding, one can still write the following relations (see also (4.50)) pertaining to a given Yang–Mills field $(\mathcal{E}, D)$ on X, as before; that is, one gets, the "flow" of $(\mathcal{E}, D)$ within $\mathcal{YM}_X$,

(4.58) $$\{\phi \in Aut\mathcal{E} : D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0\} = \ker(D_{\mathcal{E}nd\mathcal{E}}|_{Aut\mathcal{E}}) = \mathcal{YM}_X - Aut(\mathcal{E}, D) \equiv Aut(\mathcal{E}, D) < Aut\mathcal{E}.$$

Accordingly, to say it another way, one thus concludes that

(4.59) given a Yang–Mills field $(\mathcal{E}, D)$ on X, by compelling an automorphism, say ϕ, of $\mathcal{E}$ in the category (of vector sheaves on X) $\mathcal{V}ect\mathcal{S}h_X$ to become an automorphism of $\mathcal{E}$ in the (Yang–Mills) subcategory $\mathcal{YM}_X$ is equivalent with ϕ being a solution (zero-place) of $D_{\mathcal{E}nd\mathcal{E}}$—in other words (see (4.51)), "causally stationary."

On the other hand, within the same vein of ideas and still based on the foregoing (see, e.g. (4.57.2)), one can equivalently express (4.51) by saying that

(4.60) a flow is "self-dual," hence "symmetric"(!) (*viz.*, "commutative") with respect to the corresponding "Gel'fand duality."

In that context, one can thus virtually construed the

(4.61) "Gel'fand duality," as another way of identifying a "generalized symmetry" (cf. also (4.57.2)). So the "symmetries" (see (4.57.3)) of an $\mathcal{A}$-connection specify its "flow," an equivalent, in effect, version of (4.51).

Yet, in anticipation of our considerations in Chapter IV, Section 5, which are actually found in a similar context to the above, by referring, in particular, to Riemannian vector sheaves (see Definition 2.1), we also give the relations (see (2.9))

(4.62) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0,$$

such that

(4.63) $$\tilde{\rho} \in Isom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*).$$

Accordingly, based on (4.59), along with (4.50) and (4.36), for the case at issue, one concludes that

(4.64) the above two relations are (together) equivalent to the condition that

(4.64.1) $$\tilde{\rho} \in \mathcal{YM}_X - Isom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*).$$

We are now ready to come properly to our main objective herewith—that is, to the equations referred to in the heading of the present subsection.

4.4 Yang–Mills Equations

The equations at issue refer to a Yang–Mills field

$$(\mathcal{E}, D), \tag{4.65}$$

as this was defined in the foregoing (see (4.12) and (4.13)), on a given (arbitrary, in principle) topological space X, being, by definition, the carrier of the vector sheaf $\mathcal{E}$, as above. However, the same space X is now supposed to be properly structured so that when looking at the particular properties of X—in point of fact, strictly speaking, at the particular framework carried by X—the equations under discussion acquire a meaning. Thus, we assume henceforward that

(4.66) we are given that framework on X, as defined by (4.1), at the beginning of this section, so that the space X itself is, in particular, paracompact (Hausdorff).

Now, our next step is to consider the sheaf

$$\mathcal{E}nd\mathcal{E} \equiv \mathcal{E}nd_{\mathcal{A}}\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \tag{4.67}$$

—in effect, another vector sheaf on X, as well, in view of our hypothesis for $\mathcal{E}$, as above (see also [VS: Chapt. II, p. 137, Lemma 6.1]); that is, by definition (loc. cit., p. 134, Definition 6.1), the sheaf of germs of $\mathcal{A}$-endomorphisms of $\mathcal{E}$. (Concerning the vector sheaf (4.67), as above, we also remark that this is, in effect, an $\mathcal{A}$-algebra sheaf on X; hence, in particular, a $\mathbb{C}$-algebra sheaf on X, as well. We still recall herewith that, in view of the hypothesis for $\mathcal{A}$ (see Volume I, Chapt. I, (1.3)), one has (ibid. (1.5) and (1.6))

$$\mathbb{C} \underset{\rightarrow \varepsilon}{\subset} \mathcal{A}. \tag{4.68}$$

In this regard, see also [VS: Chapt. II, p. 138, (6.29)].

Note 4.2 (Physical significance of (4.67)) By looking at the vector sheaf $\mathcal{E}nd\mathcal{E}$ on X, as before, we further remark, in anticipation of what we shall see by the ensuing discussion, that although we start with a given Yang–Mills field $(\mathcal{E}, D)$, as in (4.65), below we are virtually dealing with the previous vector sheaf $\mathcal{E}nd\mathcal{E}$, which, by virtue of our hypothesis for X (see (4.6), along with (4.9)), is finally providing the concomitant pair

$$(\mathcal{E}nd\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}}), \tag{4.69}$$

therefore (see also (4.71)) another Yang–Mills field on X. Now, this might certainly be related to the *matrix-theoretic framework*, dominated the early stages of quantum mechanics (Heisenberg). Indeed, by looking at the things locally (thus, in terms of a local gauge, say U of $\mathcal{E}$), one has

$$(\mathcal{E}nd\mathcal{E})|_U = M_n(\mathcal{A})|_U = M_n(\mathcal{A}|_U), \tag{4.70}$$

where $n = rk\mathcal{E}$ ($\geqq 2$; see (4.13)); see also [VS: Chapt. II, p. 137, (6.23), and Chapt. IV, p. 294, (3.23)]. Accordingly,

(4.71) by studying $(\mathcal{E}, D)$, one usually replaces the latter by its "transfigurations" sheaf, still, a Yang–Mills field, (4.69), as above.

So it seems that the above is, at least, more convenient, if not more pragmatic! Yet, in this connection, let us also recall that

$$\mathcal{A}ut\mathcal{E} < \mathcal{E}nd\mathcal{E}, \tag{4.72}$$

so that one can restrict (pull back) on $\mathcal{A}ut\mathcal{E}$ the corresponding $\mathcal{A}$-connection $D_{\mathcal{E}nd\mathcal{E}}$ of $\mathcal{E}nd\mathcal{E}$, in effect getting an *$\mathcal{A}ut\mathcal{E}$-principal sheaf* on X, in the sense, mainly, that is recently advocated by the respective work of E. Vassiliou [3]. In this regard, see also Volume I, Chapt. II, Section 9 of this treatise as well as [VS; Chapt. VI, p. 28, Section 6.1]. Thus, $\mathcal{E}nd\mathcal{E}$ is actually the natural (canonical) "representation (vector) sheaf" of $\mathcal{A}ut\mathcal{E}$ (identity endomorphisms; hence, isomorphisms of $\mathcal{E}$).

On the other hand, by further looking at our previous comments in (4.71), we shall still realize, presently below, that

(4.73) it is the (vector) sheaf $\mathcal{E}nd\mathcal{E}$ (see (4.67)) that is virtually involved in our calculations below pertaining to the Yang–Mills equations of $\mathcal{E}$, as $\mathcal{E}$ is given by (4.65).

Yet, the above should also be related with our preceding remarks in Scholium 4.1 (see, in particular, (4.54) therein).

Thus, based now on our hypothesis, as in (4.66), on the above comments in (4.73), and on (4.5) in the foregoing, as a matter of fact, we actually consider throughout the subsequent discussion

(4.74) the Yang–Mills field

$$(\mathcal{E}nd\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}}) \tag{4.74.1}$$

on X, that is naturally associated with $(\mathcal{E}, D)$, the one initially given on X, as in (4.65).

Now, as already remarked in the foregoing (see (4.26) and (4.29), in conjunction with (4.3)),

$$D_{\mathcal{E}nd\mathcal{E}} \tag{4.75}$$

is the $\mathcal{A}$-connection of the vector sheaf $\mathcal{E}nd\mathcal{E}$ on X, that is (canonically) entailed on the latter (ibid.), by the initially given one $D_{\mathcal{E}} \equiv D$ of $\mathcal{E}$, as in (4.65).

Consequently (see also (4.9)), all the machinery of the (abstract) "Laplacians" developed thus far, referring to a given Yang–Mills field $(\mathcal{E}, D)$ on X, as above, is still valid for the particular Yang–Mills field on X, as in (4.74.1). So we are now in a position to look at the following fundamental relations, which is our main objective in this section. However, for technical reasons connected with the same formulation of the said relations, (see, e.g., (4.76)), we first give, for convenience, the following.

Definition 4.1 By a Yang–Mills space X one means an *enriched ordered algebraized space* (X, A) (see (5.1)), which is also Riemannian (see (4.1) and (5.2)) as well as a curvature space (see Volume I, Chapt. I, (7.19) and (7.20)).

So, practically speaking, as we shall also realize by Definition 4.2, a Yang–Mills space is a topological space X with respect to which Yang–Mills equations can be defined; thus, the nomenclature employed through Definition 4.1. We come next to define the fundamental equations under discussion according to the subsequent definition.

Definition 4.2 Suppose we are given a Yang–Mills space X (see Definition 4.1) and let $(\mathcal{E}, D)$ be a Yang–Mills field on X. Then, we call the Yang–Mills equation(s) of $(\mathcal{E}, D)$ (or even, for brevity's sake, simply of $\mathcal{E}$), any one of the following two equivalent relations (see below for the proof of the claimed herewith equivalence):

$$\delta_{\mathcal{E}nd\mathcal{E}}(R) = 0 \tag{4.76}$$

or

$$\Delta_{\mathcal{E}nd\mathcal{E}}(R) = 0, \tag{4.77}$$

where, as usual, we set

$$R \equiv R(D_{\mathcal{E}} \equiv D) \in \Omega^2(\mathcal{E}nd\mathcal{E})(X) \tag{4.78}$$

(see also Volume I, Chapt. I, (7.22)).

Concerning the notation employed in the above definition, we actually use the following simplifications, as we also do later, by setting

$$\delta_{\mathcal{E}nd\mathcal{E}} \equiv \delta^2_{\mathcal{E}nd\mathcal{E}} : \Omega^2(\mathcal{E}nd\mathcal{E}) \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E}) \tag{4.79}$$

as well as

$$\Delta_{\mathcal{E}nd\mathcal{E}} \equiv \Delta^2_{\mathcal{E}nd\mathcal{E}} : \Omega^2(\mathcal{E}nd\mathcal{E}) \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E}) \tag{4.80}$$

(see also (2.39), along with (3.2) in the foregoing). Therefore, by analogy, one obtains (loc. cit.);

$$\begin{aligned} \Delta_{\mathcal{E}nd\mathcal{E}} \equiv \Delta^2_{\mathcal{E}nd\mathcal{E}} &:= \delta^3_{\mathcal{E}nd\mathcal{E}} \circ \Delta^2_{\mathcal{E}nd\mathcal{E}} + \Delta^1_{\mathcal{E}nd\mathcal{E}} \circ \delta^2_{\mathcal{E}nd\mathcal{E}} \\ &\equiv \delta_{\mathcal{E}nd\mathcal{E}} \circ D_{\mathcal{E}nd\mathcal{E}} + D_{\mathcal{E}nd\mathcal{E}} \circ \delta_{\mathcal{E}nd\mathcal{E}} \equiv \delta D + D\delta. \end{aligned} \tag{4.81}$$

So we come now to the above-promised proof.

Equivalence of (4.76) with (4.77): The asserted equivalence in Definition 4.2 of the two equations at issue has been, in effect, already established by our previous argument in (3.28) (or even (3.29)); thus, as it was explained therein, the noted equivalence is a straightforward combination of Green's formula with Bianchi's identity (see (3.19) and (3.26)). ■

We continue by extending to the present abstract setting the classical terminology connected with the previous equations. Thus, in this connection we set the following.

Definition 4.3 Suppose that an $\mathcal{A}$-connection D (on a given vector sheaf $\mathcal{E}$ on X, as above) has curvature $R \equiv R(D)$ (see (4.78)) which satisfies any one of the (two equivalent) relations (4.76) or (4.77). Then, we call D a Yang–Mills $\mathcal{A}$-connection, or even a *Yang–Mills potential*.

Yet, it is also common, classically, to call the curvature (field strength) of a given Yang–Mills potential, as above, a Yang–Mills field; so this, by contrast with the terminology, which we have applied in the foregoing for a pair $(\mathcal{E}, D)$, as in (4.65), is something that we shall continue to employ in the sequel as well.

Within the same setup as above, another issue of particular significance is the application of the so-called *Hodge operator*, alias, $*$-*operator* (see Volume I, Chapt. I, Section 10), when looking for solutions of the Yang–Mills equations by employing the aforementioned classical terminology in the present framework when studying Yang–Mills fields. Indeed, we are dealing with the corresponding situation, that appears in the next Section 4.5, where, as we shall see, one still makes full use of our hypothesis in (2.9) (however, see also (4.14)).

4.5 Self-Dual Gauge Fields

The term in the heading of this subsection is an abbreviation of the more complete term a *self-dual Yang–Mills field*—that is, a pair

(4.82) $$(\mathcal{E}, D)$$

as in (4.12) and (4.13) whose curvature $R(D) \equiv R$ satisfies the relation

(4.83) $$*R = R.$$

Note 4.3 (Terminological) We employed above the term *gauge field* for a Yang–Mills field (see (4.12) and (4.13)), a practice that will also be applied, occasionally, in the sequel as well. On the other hand, this same practice will still be employed, as the case may be, for a pair $(\mathcal{L}, D)$, with $\mathcal{L}$ a line sheaf on X, along with an $\mathcal{A}$-connection D on it (*viz.*, for a Maxwell field), on the particular space X considered.

Now, concerning the relation (4.83), as above, we recall that the $*$-operator, or Hodge operator, is, according to the general abstract theory (see [VS: Vols. I and II; in particular, Chapt. IV, Section 12] and Volume I, Chapter I, Section 10), an

$\mathcal{A}$-automorphism of the $\mathcal{A}$-module defined by the exterior algebra of $\mathcal{E}^*$ (see [VS: Chapt. IV, p. 308, (7.9)])—namely, one has

(4.84) $$* \in \mathcal{A}ut_{\mathcal{A}}(\wedge\mathcal{E}^*) \equiv \mathcal{A}ut(\wedge\mathcal{E}^*)$$

(see also loc. cit., p. 344, (12.10)), where $\mathcal{E}$ stands for a Riemannian (free) $\mathcal{A}$-module (of finite rank) on X (see Definition 2.1). Thus, if

(4.85) $$rk\mathcal{E} = n \in \mathbb{N},$$

then, in particular, one has the following $\mathcal{A}$-isomorphisms of the $\mathcal{A}$-modules involved:

(4.86) $$*_p \equiv * : \wedge^p \mathcal{E}^* \longrightarrow \wedge^{n-p} \mathcal{E}^*$$

for any $0 \leqq p \leqq n$ (see also [VS: Chapt. IV, p. 311, Lemma 7.1, and p. 344, (12.3)]). Furthermore, we still assume, concerning (4.84), that, along with the $\mathcal{A}$-metric ρ on $\mathcal{E}$, one has

(4.87) $$\mathcal{E} \underset{\tilde{\rho}}{\cong} \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$$

within an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned (see also (4.1) at the beginning of this section); that is,

(4.88) the $\mathcal{A}$-metric ρ on the $\mathcal{A}$-module $\mathcal{E}$ on X, as before, entails (see (4.2.1)) a

(4.88.1) strongly nondegenerate $\mathcal{A}$-valued inner product on $\mathcal{E}$.

See also Definition 2.1 together with the ensuing comments therein.

On the other hand, we also recall here that the $\mathcal{A}$-automorphism $*$, as in (4.84), is actually referred to as a (Riemannian) free $\mathcal{A}$-module $\mathcal{E}$, of finite rank, on X. Therefore,

(4.89) by considering now, more generally, a vector sheaf $\mathcal{E}$ on X, then, in view of our hypothesis for $\mathcal{E}$ (see, e.g., Volume I, Chapt. II, (6.5); *viz.*, $\mathcal{E}$ is, by definition, "locally free"), one can look, yet, locally(!), at the corresponding situation, as was described by (4.84) and (4.87).

Thus, as a consequence of (4.89), in conjunction with [VS: Chapt. I, p. 68, Theorem 12.1, along with Chapt. II, p. 125, (4.6)], one finally concludes that

(4.90) by assuming, for simplicity, (4.1) (however, see also (4.88.1), as well as [VS: Chapt. IV, p. 328, Theorem 8.3]), one gets (4.87) for any vector sheaf $\mathcal{E}$ on X. Therefore,

(4.90.1) (4.84) is in force, for every vector sheaf $\mathcal{E}$ on X.

On the other hand, as a byproduct of the preceding argument, one also obtains that

(4.91) an $\mathcal{A}$-valued inner product on a given vector sheaf $\mathcal{E}$ that is locally, strongly nondegenerate, becomes globally such as well, hence, finally, an $\mathcal{A}$-metric on $\mathcal{E}$.

Of course, we assume above that (the base space) X provides the *appropriate setup*, the previous assertion, as in (4.91) to acquire a meaning. (See (4.1), or even (2.3), (2.5), and (2.9), along with Definition 2.1 and subsequent comments therein.)

Indeed, our last conclusion in (4.90.1), as well as the corresponding general setup in (4.91), supplies an extension to the case of vector sheaves on X of our previous relevant considerations in [VS: Chapt. IV, Section 12]; thus, in point of fact, a generalization to the present abstract setting of the corresponding classical framework of (Hodge) $*$-operator theory.

Thus, as a first application of the preceding, within the abstract setup of the present treatise, we obtain, by analogy with the standard case, the ensuing relations, connecting the above $*$-operator (see (4.84) and (4.86)) with the so-called "dual differential operators," as the latter were defined in Section 2—that is, given the aforesaid operators

$$(\delta^n)_{n\in\mathbb{N}} \tag{4.92}$$

(see (2.35.1)), one obtains (see also (1.16) and (1.17), as well as (2.18))

$$\delta^p = (-1)^{n(p+1)+1} * D^{n-p} * \tag{4.93}$$

for any $1 \leqq p \leqq n = rk_{\mathcal{A}}\mathcal{E}$, with $\mathcal{E}$ a given vector sheaf on X, of rank $n \in \mathbb{N}$, as indicated. Yet, in abbreviated form, one has

$$\delta = (-1)^{n(p+1)+1} * D* \equiv \pm * D * . \tag{4.94}$$

Concerning the classical counterpart of the preceding, see, for example, F. W. Warner [1: p. 220] or even C. von Westenholz [1: p. 337, (7.4)] or M. Nakahara [1: p. 253, (7.184a)]. Yet, one can depict (4.93), or its abbreviated form, as in (4.94), through the following commutative diagram (see also (2.18), as well as (2.39)):

$$\begin{array}{ccc} \Omega^{p}(\mathcal{E}) & \xrightarrow{\delta^p \equiv \delta} & \Omega^{p-1}(\mathcal{E}) \\ {\scriptstyle *}\big\downarrow & & \nwarrow{\scriptstyle *} \\ \Omega^{n-p}(\mathcal{E}) & \xrightarrow{(-1)^{n(p+1)+1} D^{n-p} \equiv \pm D^{n-p}} & \Omega^{n-p+1}(\mathcal{E}) \end{array} \tag{4.95}$$

such that $1 \leqq p \leqq n$, as before, with $\mathcal{E}$ any vector sheaf on X having $rk_{\mathcal{A}}\mathcal{E} \equiv rk\mathcal{E} = n \in \mathbb{N}$.

So, based on the preceding, we are now in a position to state the following.

Proposition 4.1 Suppose we have a Yang–Mills space X. Then, any self-dual Yang–Mills field $(\mathcal{E}, D)$ on X (see (4.82) and (4.83)—in point of fact, strictly speaking, its curvature $R(D) \equiv R$) is a solution of the corresponding Yang–Mills equation (see (4.76) and (4.79)).

Before we come to the proof of the previous assertion, we have first to comment a bit more on the notation connected, for the case at hand, with (4.94); in this context, see also our previous remarks in (4.73): Thus, in view of (4.78), one has first to think of an appropriate "vectorization of the $*$-operator," as the latter is defined by (4.86) (see also (4.84)), when referring, in particular, to vector sheaves on X of the form

$$\Omega^1(\mathcal{F}) = \Omega^1 \otimes_{\mathcal{A}} \mathcal{F} = \mathcal{F} \otimes_{\mathcal{A}} \Omega^1, \tag{4.96}$$

where $\mathcal{F}$ is a given vector sheaf on X, while we still assume herewith that Ω^1 is a vector sheaf on X too (see, e.g., (4.1.4)). So one defines the $\mathcal{A}$-isomorphism

$$*_p \otimes 1_{\mathcal{F}} : \wedge^p \Omega^1(\mathcal{F}) \equiv \Omega^p(\mathcal{F}) \longrightarrow \Omega^{m-p}(\mathcal{F}) \equiv \wedge^{m-p} \Omega^1(\mathcal{F}) \tag{4.97}$$

for any $1 \leqq p \leqq m = rk\Omega^1$ (see also (4.98) below) by a straightforward analogy to (4.86), while it can be still based on (4.2), for the case at hand, as well as on the corresponding relation to (4.87). Now, for technical reasons that will become clear below, from the subsequent proof of Proposition 4.1, as above (see (4.99)), we further assume that

$$rk_{\mathcal{A}}\Omega^1 \equiv rk\Omega^1 \equiv m = 4. \tag{4.98}$$

We note here that the previous condition on Ω^1 is satisfied, of course, in the classical case when considering a 4-dimensional (Riemannian or pseudo-Riemannian (e.g., Lorentzian)) manifold X (see, e.g., Volume I, Chapt. I, Section 2.1, in particular, (2.22) of this treatise). So we come now to the following:

Proof of Proposition 4.1 Setting in (4.93) $p = 2$ and by also taking (4.97) for $\mathcal{F} \equiv \mathcal{E}nd\mathcal{E}$ as well as (4.98) into account, one obtains

$$\begin{aligned} \delta^2_{\mathcal{E}nd\mathcal{E}}(R) &= ((-1)^{3m+1} * D^{m-2}_{\mathcal{E}nd\mathcal{E}} *)(R) \\ &= ((-1)^{3m+1} * D^{m-2}_{\mathcal{E}nd\mathcal{E}})(*R) \\ &= -(*)D^2_{\mathcal{E}nd\mathcal{E}}(R) = 0, \end{aligned} \tag{4.99}$$

the last two equalities, as above, being the result of our hypothesis for R (see (4.83); *viz.*, of the self-duality of $(\mathcal{E}, D)$) of the Bianchi's identity (see Volume I, of the present treatise, Chapt. I, (8.15.1) and (8.16), or even [VS: Chapt. VIII, p. 225, (7.34)]) and of the $\mathcal{A}$-linearity of the (Hodge) $*$-operator, as in (4.86), that, of course, finishes the proof. ■

On the other hand, it is now quite clear, on the basis of (4.99), that

(4.100) Proposition 4.1 is still valid, for any *anti-self-dual Yang–Mills field* $(\mathcal{E}, D)$ on X; namely, for such a one, for which one has

(4.100.1) $$*R = -R,$$

where, as usual, we put $R \equiv R(D)$.

Remark 4.1 Referring to Proposition 4.1, and hence, in turn, to our previous conclusion in (4.100) as well, we should still notice that the aforesaid propositions are valid under the assumption (4.98); in this context, we further note that the said condition refers, in principle, to the differential triad $(\mathcal{A}, \partial, \Omega^1)$ concerned (see (4.1.4), independently of the Yang–Mills field $(\mathcal{E}, D)$ involved.

Note 4.4 (Terminological, continued) By further commenting on the relevant terminology of the standard theory (see Note 4.3), we also remark that the terms

(4.101) self-dual connections (respectively, anti-self-dual connections) or even instantons (respectively, anti-instantons) are still in use classically, pertaining to the corresponding situations described by Proposition 4.1 and (4.100).

So, by extension to the present case, one can still employ the previous terminology when referring to Yang–Mills fields satisfying (4.83) and (4.100.1), respectively, being thus, according to the preceding (Proposition 4.1, resp. (4.100); see also (4.98)), solutions of the corresponding Yang–Mills equations. The same equations are still called in the classical theory the *Euler–Lagrange equations*.

Now, our next objective is to prove that solutions of the Yang–Mills equations can be construed, as we say, as "critical points of the Yang–Mills functional." Thus, we start by first explaining straightforwardly the relevant terminology, within the present abstract setting, in Section 5.

5 Yang–Mills Functional

Before we come to the formal definition, within the present abstract setting, of the classical notion in the title of this section, we have, as usual, to first clear up the corresponding framework that is appropriate to that aim. Thus, by supplementing now our previous hypothesis in (4.1), as it concerns the notions related to the "partial order" in $\mathcal{A}$, we further assume herewith that

(5.1) $(X, \mathcal{A})$ is an *enriched ordered algebraized space*, or else an *ordered algebraized space with square root*.

The above terminology refers, in effect, to our requirement for the structure sheaf $\mathcal{A}$ involved to supply the appropriate mechanism, so that one can formulate all the necessary notions of "positivity" that one usually applies in the standard case. So concepts like "positive square roots" or "absolute value" and the like are formulated here in terms of sections of the (vector) sheaves involved, related, in particular, according to their correspondence to the previous classical concepts sheaf morphisms, while the latter are also $\mathcal{A}$-valued (see, e.g., (2.1) or even (2.7)). For technical details, including the precise definition of the terms appearing in (5.1), we refer to [VS: Chapt. IV, p. 336, Definition 10.1, see also p. 327, Examples 8.1]. Thus,

(5.2) we assume henceforth that (4.1) holds true in the restricted sense of (5.1), as above, concerning the "partial order-structure" of $\mathcal{A}$.

So, in particular, we consider in the sequel a $\mathbb{C}$-algebraized space

$$(X, \mathcal{A}), \tag{5.3}$$

for which the underlying $\mathbb{R}$-algebraized space satisfies (5.2) (loc. cit.). Thus, according to our hypothesis for X, as in (5.2) (see also (4.1)), one concludes that

(5.4) any given vector sheaf $\mathcal{E}$ on X admits a (Riemannian) $\mathcal{A}$-metric ρ (see Definition 2.1) as well as an $\mathcal{A}$-connection D (see (4.12)), in effect, compatible with ρ (see (4.61)).

Indeed, our extra hypothesis in (5.2), referring to the (partial) order structure of $\mathcal{A}$ in comparison to our previous conditions in (4.1), has actually been made in favor of the following result in (5.5): As a consequence of our hypothesis in (5.2) and of the general abstract theory (loc. cit.), one now concludes that

(5.5) for any vector sheaf $\mathcal{E}$ on X, one gets an $\mathcal{A}$-*valued norm*, or simply an $\mathcal{A}$-*norm* on $\mathcal{E}$, according to the map

$$\|\cdot\| : \mathcal{E} \longrightarrow \mathcal{A} \tag{5.5.1}$$

—in effect, a sheaf morphism of the sheaves concerned, having the analogous properties of a usual ($\mathbb{C}$-vector space) norm, derived from an inner product. Thus, based on (5.2) and the definition of the "square root" (see [VS: Chapt. IV, p. 335, (10.5)]), one defines

$$\|\cdot\|^2 := \rho|_\Delta, \tag{5.5.2}$$

where Δ stands for the "diagonal" of $\mathcal{E} \oplus \mathcal{E}$ (see (4.2.1)); hence, equivalently (loc. cit.), one sets

$$\|\cdot\| := \sqrt{\rho|_\Delta} \tag{5.5.3}$$

(see also (5.11) in the sequel).

See, for instance, L. Smith [1: pp. 249ff], along with (4.2.1) in the preceding subsection. Of course, (5.5.1) being, as it is, a sheaf morphism, it is actually expressed through the corresponding sections of the sheaves involved; see, for instance, (5.11), along with [VS: Chapt. I, p. 75, (13.19), and p. 27, (6.3)].

In particular, based on our hypothesis for the $\mathcal{A}$-module Ω^1 (see (4.1)), we conclude that the preceding is valid for the vector sheaf

$$\Omega^2(\mathcal{E}nd\mathcal{E}) \tag{5.6}$$

on X; the latter sheaf is of a special interest to us, since whenever we have a Yang–Mills field on X (see 4.12)),

$$(\mathcal{E}, D), \tag{5.7}$$

such that X is also a curvature space (see Chapt. I, (7.19)), one gets

$$R(D) \equiv R \in \Omega^2(\mathcal{E}nd\mathcal{E})(X). \tag{5.8}$$

Thus, we are now in the position to set the following.

Definition 5.1 Suppose we are given a Yang–Mills space X (see Definition 4.1) and let

$$(\mathcal{E}, D) \tag{5.9}$$

be a given Yang–Mills field on X. Moreover, let

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{5.10}$$

be the (affine) space of $\mathcal{A}$-connections of $\mathcal{E}$ (see Volume I: Chapt. I, (5.4), in conjunction with (4.14) in the preceding subsection). Then, one defines the Yang–Mills functional of $\mathcal{E}$, denoted by $\mathcal{YM}_{\mathcal{E}}$, or simply by $\mathcal{YM}$, according to the following map (see also (5.5)):

$$\begin{aligned} \mathcal{YM}_{\mathcal{E}} : \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \longrightarrow \mathcal{A}(X) : D \longmapsto \mathcal{YM}_{\mathcal{E}}(D) \\ \equiv \mathcal{YM}(D) := \frac{1}{2}\|R\|^2 := \frac{1}{2}\rho(R, R). \end{aligned} \tag{5.11}$$

Concerning the terminology employed in the above definition, the terms Yang–Mills Lagrangian and even *Yang–Mills action* are also in use in the classical case for the map (5.11); we shall also apply it, occasionally, in the sequel, within the present abstract setting, by thus extending the standard terminology.

On the other hand, based on the fact that $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ is an affine space, modeled on the $\mathcal{A}(X)$-module

$$\Omega^1(\mathcal{E}nd\mathcal{E})(X) \tag{5.12}$$

(see Volume I, Chapt. I, (5.7) or even [VS: Chapt. VI, p. 32, Theorem 7.1]), one has

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_0 = \Omega^1(\mathcal{E}nd\mathcal{E})(X), \tag{5.13}$$

within a bijection, established by the map

$$D \longmapsto D_0 + u, \tag{5.14}$$

with D and D_0 in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ and $u \equiv u(D) \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$. Thus, in (5.13) we have set (see also Chapt. I, (5.8))

$$\begin{aligned} \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_0 = \{D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) : D = D_0 + u, \\ \text{with } u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)\} \cong \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \end{aligned} \tag{5.15}$$

for some fixed element

$$D_0 \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}). \tag{5.16}$$

Now, by virtue of (5.14), one may also refer to another form of (5.11) (*viz.*, to the following correspondence), applying, for convenience, an obvious abuse of notation,

$$\begin{aligned} \mathcal{YM} : \Omega^1(\mathcal{E}nd\mathcal{E})(X) \longrightarrow \mathcal{A}(X) : u \longmapsto \mathcal{YM}(u) \\ := \frac{1}{2}\|u\|^2 \equiv \frac{1}{2}\rho(u, u) \end{aligned} \tag{5.17}$$

by taking into account (5.4) along with the ensuing comments therein. For a classical account see also, for instance, J. Baez–J.P. Muniain [1: pp. 274ff]. Use of the above correspondence, as in (5.17), will also be made in the sequel.

Now, another aspect of the same map, as in (5.11) or even in (5.17), that will be considered in our subsequent discussion as well is its invariance with respect to a gauge transformation, which we deal with presently below.

5.1 Group of Gauge Transformations

Suppose we are given a $\mathbb{C}$-algebraized space (see Chapt. I, (1.4))

$$(X, \mathcal{A}) \tag{5.18}$$

and let $\mathcal{E}$ be a vector sheaf on X. Then, by definition, the group of gauge transformations of $\mathcal{E}$, denoted by

$$Aut_{\mathcal{A}}\mathcal{E} \equiv Aut\mathcal{E}, \tag{5.19}$$

is given, according to the relations, by

$$Aut\mathcal{E} := (\mathcal{A}ut\mathcal{E})(X) = (\mathcal{E}nd\mathcal{E})^{\bullet}(X) = (\mathcal{E}nd\mathcal{E})(X)^{\bullet} \equiv (End\mathcal{E})^{\bullet}, \tag{5.20}$$

the last group in (5.20) being the group of units (invertible elements) of the $\mathcal{A}(X)$-algebra of endomorphisms (in effect, $\mathcal{A}$-endomorphisms) of $\mathcal{E}$. In this connection, see also Volume I of this treatise, Chapt. II, Section 9, as well as [VS: Chapt. II, pp. 138 and 139, (6.28) and (6.31), along with Chapt. V, p. 391, (8.44)]. Yet, according to the preceding, we still speak of the same group (5.19) as the group of $\mathcal{A}$-automorphisms of $\mathcal{E}$, alias, of $\mathcal{A}$-isomorphisms of $\mathcal{E}$ onto itself, being thus, as already said, the group of units of the $\mathcal{A}(X)$-algebra of $\mathcal{A}$-endomorphisms of $\mathcal{E}$,

$$End\mathcal{E} \equiv (\mathcal{E}nd\mathcal{E})(X) \tag{5.21}$$

(loc. cit.). The same group as above has also been employed in the preceding, as already mentioned (see Volume I, Chapt. I, Section 9), in Section 4 (4.49.1) and (4.72), and it will still be of use several times in the sequel as well (see Chapt. 2, Section 1). On the other hand, it is also useful to look at the above situation, as described by (5.20) locally. (In point of fact, we have already considered this earlier.) Thus,

(5.22) by taking a vector sheaf $\mathcal{E}$ on X, as above, and looking further at a local frame of $\mathcal{E}$, say,

$$\mathcal{U} = (U) \tag{5.22.1}$$

(see Vol. I: Chapt. I, (2.29) and (2.53)), the previous group, as in (5.19), is actually reduced locally to the analogous group of (the free $\mathcal{A}$-module) $\mathcal{A}^n$ (*viz.*, of the "local model" of $\mathcal{E}$), where we posit

$$n = rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} \in \mathbb{N} \tag{5.22.2}$$

(by definition, the finite rank of $\mathcal{E}$); namely, one thus gets the group

$$GL(n, \mathcal{A}(U)) = \mathcal{GL}(n, \mathcal{A})(U) \tag{5.22.3}$$

for any open $U \in \mathcal{U}$, as in (5.22.1).

Indeed, one obtains (see also Volume I: Chapt. II, Section 9),

$$\begin{aligned}(\mathcal{E}nd\mathcal{E})(U) &\equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E})(U) = Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \\ &= Hom_{\mathcal{A}|_U}(\mathcal{A}^n|_U, \mathcal{A}^n|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A}^n)(U) = (\mathcal{E}nd\mathcal{E})(U) \\ &= M_n(\mathcal{A})(U) = M_n(\mathcal{A}(U)) = End(\mathcal{A}(U)^n) = End(\mathcal{A}^n(U))\end{aligned} \tag{5.23}$$

for any (open) $U \in \mathcal{U}$ (*viz.*, local gauge of $\mathcal{E}$). Therefore, in particular, one has (see also (5.20))

$$\begin{aligned}(\mathcal{A}ut\mathcal{E})(U) &= (\mathcal{E}nd\mathcal{E})^{\bullet}(U) = (\mathcal{E}nd\mathcal{E})(U)^{\bullet} = M_n(\mathcal{A}(U))^{\bullet} \\ &= GL(n, \mathcal{A}(U)) = \mathcal{GL}(n, \mathcal{A})(U) = M_n(\mathcal{A})(U)^{\bullet} \\ &= M_n(\mathcal{A})^{\bullet}(U) = (\mathcal{E}nd\mathcal{A}^n)^{\bullet}(U) = (\mathcal{A}ut\mathcal{A}^n)(U)\end{aligned} \tag{5.24}$$

for any U, as before, which thus proves our assertion in (5.22). ■

Yet, as a byproduct of (5.23) and (5.24), one still obtains the following useful relations, for any local gauge U of $\mathcal{E}$, as above. Thus, one has

$$(\mathcal{A}ut\mathcal{E})(U) = Aut(\mathcal{E}|_U) = Aut(\mathcal{A}^n|_U) = (\mathcal{A}ut\mathcal{A}^n)(U). \tag{5.25}$$

Indeed (loc. cit.), one obtains

$$\begin{aligned}(\mathcal{A}ut\mathcal{E})(U) &= (\mathcal{E}nd\mathcal{E})^{\bullet}(U) = (\mathcal{E}nd\mathcal{E})(U)^{\bullet} = \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E})(U)^{\bullet}\\ &= Hom_{\mathcal{A}|_U}(\mathcal{E}|_U,\mathcal{E}|_U)^{\bullet} = (End(\mathcal{E}|_U))^{\bullet} = Aut(\mathcal{E}|_U)\\ &= Aut(\mathcal{A}^n|_U) = (\mathcal{E}nd\mathcal{A}^n)^{\bullet}(U) = (\mathcal{A}ut\mathcal{A}^n)(U),\end{aligned} \tag{5.26}$$

that, of course, proves (5.25). ■

As a result of the preceding calculations, one thus concludes that

(5.27) any *local automorphism* of a given vector sheaf $\mathcal{E}$ over a local gauge U of $\mathcal{E}$ is virtually given by a similar one of $\mathcal{A}^n$, where $n = rk\mathcal{E}$ (*viz.*, by a local automorphism of $\mathcal{A}^n$ over U).

Indeed, a very convenient result (!), as we shall have the opportunity to realize several times, in the sequel, being, in effect, a straightforward consequence of the same definition of a vector sheaf, as a locally free $\mathcal{A}$-module (of finite rank) over X, and related outcomes thereof, as exhibited in the preceding discussion.

On the other hand, within the same vein of ideas, one still obtains the following useful relations, vindicating further (5.27); that is, one has

$$(\mathcal{A}ut\mathcal{E})|_U = (\mathcal{A}ut\mathcal{A}^n)|_U = \mathcal{GL}(n,\mathcal{A})|_U \tag{5.28}$$

within isomorphisms of the group sheaves involved, for any local gauge U of $\mathcal{E}$, as in (5.22.1): Thus, one actually gets (see also Volume I, Chapt. II, Lemma 9.2, as well as [VS: Chapt. II, p. 137, (6.23)])

$$\begin{aligned}(\mathcal{A}ut\mathcal{E})|_U &= (\mathcal{E}nd\mathcal{E})^{\bullet}|_U = ((\mathcal{E}nd\mathcal{E})|_U)^{\bullet}\\ &= (\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E})|_U)^{\bullet} = Hom_{\mathcal{A}|_U}(\mathcal{E}|_U,\mathcal{E}|_U)^{\bullet} = (\mathcal{E}nd(\mathcal{E}|_U))^{\bullet}\\ &= \mathcal{A}ut(\mathcal{E}|_U) = \mathcal{A}ut(\mathcal{A}^n|_U) = (\mathcal{A}ut\mathcal{A}^n)|_U,\end{aligned} \tag{5.29}$$

which, of course, proves (5.28). ■

Now, the preceding discussion justifies also the abuse of terminology, that, in point of fact, is often employed when

(5.30) we usually speak of the group sheaf

$$\mathcal{GL}(n,\mathcal{A}) \tag{5.30.1}$$

as the gauge group of a given vector sheaf $\mathcal{E}$ on X, with $rk\mathcal{E} = n \in \mathbb{N}$, in place virtually of

$$\mathcal{A}ut\mathcal{E}. \tag{5.30.2}$$

Finally, as a byproduct of the preceding proof of (5.28), as given by (5.29), and also in conjunction with (5.23), one further gets the following relations:

$$(\mathcal{E}nd\mathcal{E})|_U = \mathcal{E}nd(\mathcal{E}|_U) = \mathcal{E}nd(\mathcal{A}^n|_U) = (\mathcal{E}nd\mathcal{A}^n)|_U, \tag{5.31}$$

within $\mathcal{A}|_U$-isomorphisms of the $\mathcal{A}|_U$-algebra sheaves concerned (see [VS: Volume I, Chapt. II, p. 138, Definition 6.2]), for any (open) U, as above. (In this regard, see also Chapter II of Volume I of the present treatise, proof of Lemma 9.2 therein).

We come now to our main objective—to the establishment of the *gauge invariance of the Yang–Mills functional* that is naturally associated with any given Yang–Mills field $(\mathcal{E}, D)$ on X, with respect to any one of the above local gauge transformations of $\mathcal{E}$, that are, however, ($\mathcal{A}$-)*metric preserving,* as we are going to explain below. Yet, in this connection, we also remark, for the sake of generality, that we did not make use of any $\mathcal{A}$-metric (see (5.2)) throughout our discussion in this section, something that we shall do straightforwardly in the following subsection.

5.2 Gauge Invariance of the Yang–Mills Functional

Suppose we are given a Yang–Mills space X (see Definition 4.1) and let

$$(\mathcal{E}, D) \tag{5.32}$$

be a given Yang–Mills field on X, while we further assume that the family of the open subsets of X, as in (5.22.1), stands for a local frame of $\mathcal{E}$. Thus, by looking at the field strength of $\mathcal{E}$ (in effect, of $(\mathcal{E}, D)$; *viz.*, of the "field" at issue), that is, by definition, at the curvature of the given $\mathcal{A}$-connection D of $\mathcal{E}$, one has

$$\begin{aligned} R(D) \equiv R \in \Omega^2(\mathcal{E}nd\mathcal{E})(X) &\equiv (\Omega^2 \otimes_{\mathcal{A}} \mathcal{E}nd\mathcal{E})(X) \\ &= (\mathcal{E}nd\mathcal{E} \otimes_{\mathcal{A}} \Omega^2)(X) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(X) \\ &= Z^0(\mathcal{U}, \Omega^2(\mathcal{E}nd\mathcal{E})) \end{aligned} \tag{5.33}$$

with $\mathcal{U}$ a local frame of $\mathcal{E}$ (see, e.g., (5.22.1)). Therefore, locally, one obtains, for any local gauge $U \in \mathcal{U}$ of $\mathcal{E}$,

$$\begin{aligned} R|_U \in \Omega^2(\mathcal{E}nd\mathcal{E})(U) &= \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(U) \\ &= Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \Omega^2(\mathcal{E})|_U) \end{aligned} \tag{5.34}$$

in such a manner that, for any open $V \subseteq U$ (hence, in effect, still a local gauge of $\mathcal{E}$ (see Volume I, Chapt. I, (2.29)) and for every

$$s \in (\mathcal{E}|_U)(V) = \mathcal{E}(V) \tag{5.35}$$

(see also ibid., (2.31)), one gets, by virtue of (5.34),

$$(R|_V)(s) \equiv R(s) \in \Omega^2(\mathcal{E})(V) = \Omega^2(V) \otimes_{\mathcal{A}(V)} \mathcal{E}(V) \tag{5.36}$$

(see also [VS: Chapt. I, p. 27, (6.3), and p. 28, Definition 6.1, along with Volume II, Chapt. VII, p. 100, (1.8) and (1.10)]).

Thus, we come now to look at the interference of a given $\mathcal{A}$-metric ρ, in principle, on $\mathcal{A}$ (see (5.2), along with (4.1)), hence (see (4.2)) on every vector sheaf $\mathcal{E}$ on X, as well, in the preceding considerations. Therefore, taking a Yang–Mills field $(\mathcal{E}, D)$ on X, as above, we can still look at it (see (4.2.2)) as a Riemannian vector sheaf on X:

(5.37) $$(\mathcal{E}, \rho) \equiv \{(\mathcal{E}, D); \rho\}$$

(in this regard, see also (4.61)). So we further set the following basic definition, throughout the subsequent discussion:

A given element

(5.38.1) $$\phi \in \mathcal{A}ut_{\mathcal{A}}(\mathcal{E}) \equiv \mathcal{A}ut\mathcal{E}$$

is said to be a metric-preserving gauge transformation of $\mathcal{E}$ if the following relation holds true:

(5.38) (5.38.2) $$\rho \circ (\phi, \phi) \equiv \phi^*(\rho) = \rho,$$

that is, whenever ρ is a "fixed point" of ϕ^*. Yet, equivalently, one has (5.38.2), in terms of (local) sections of $\mathcal{E}$, in the form

(5.38.3) $$(\rho \circ (\phi, \phi))(s, t) = \rho(\phi(s), \phi(t)) = \rho(s, t)$$

for any (continuous local) sections s and t in $\mathcal{E}(U)$, with U open in X.

Thus, we have explained so far the necessary background terminology to state the following basic result.

Lemma 5.1 Suppose we have a Yang–Mills space X and let

(5.39) $$(\mathcal{E}, D)$$

be a given Yang–Mills field on X. Then, the Yang–Mills functional associated with $\mathcal{E}$ (see (5.11)) is invariant under the action of any metric-preserving gauge transformation of $\mathcal{E}$ (see (5.38.2)); that is (by employing a convenient abuse of notation that will be justified in the course of the ensuing proof), one has (see also (5.11))

(5.40) $$\phi \circ \mathcal{YM}_{\mathcal{E}} \circ \phi^{-1} \equiv Ad(\phi)\mathcal{YM}_{\mathcal{E}} \equiv \phi^*(\mathcal{YM}_{\mathcal{E}}) = \mathcal{YM}_{\mathcal{E}}$$

for any

(5.41) $$\phi \in \mathcal{A}ut\mathcal{E}, \text{ with } \phi^*(\rho) = \rho$$

(see (5.38.2)).

Proof Applying the defining relation of $\mathcal{YM}_{\mathcal{E}}$, the Yang–Mills functional of $\mathcal{E}$, as in (5.11), and taking into account also the form of the curvature of D, $R(D) \equiv R$, after (the action on it of) a gauge transformation $\phi \in \mathcal{A}ut\mathcal{E}$ (*viz.*, the "2-form" on X),

(5.42) $$\phi R \phi^{-1} \equiv Ad(\phi)(R) \equiv \phi^*(R)$$

(see Chapt. I, (7.37): "transformation law of curvature"), one obtains (see also (5.38.3))

(5.43) $$\|\phi R \phi^{-1}\|^2 = \rho(\phi R \phi^{-1}, \phi R \phi^{-1}) = \rho(R\phi^{-1}, R\phi^{-1}) = \rho(R, R) = \|R\|^2$$

(modulo a multiple of 2), where at the end we also applied the fact that, in view of $\phi \in \mathcal{A}ut$, one gets

(5.44) $$\mathcal{E} = \phi^{-1}(\mathcal{E}),$$

within an $\mathcal{A}$-isomorphism, given, just by ϕ—namely, equivalently, one has

(5.45) $$s = \phi^{-1}(\phi(s))$$

for any $s \in \mathcal{E}(U)$. The preceding also explain the sort of abusing notation in (5.40), while the same argument as in (5.43) terminates the proof of the lemma. ■

Now, before we close the present section, we still comment, a bit more, on the previous sort of gauge transformations, as in (5.41), in the form of the following remark, which will be of use in the following discussion as well. So it is easy to see, according to (5.38.2), that

(5.46) given a vector sheaf $\mathcal{E}$ on X, with the space X satisfying (5.2), so that (see (4.2.2)) $\mathcal{E}$ can be viewed as a Riemannian vector sheaf on X,

(5.46.1) $$(\mathcal{E}, \rho),$$

the set of metric-preserving gauge transformations of $\mathcal{E}$ (see (5.38)) provides a subgroup of $\mathcal{A}ut\mathcal{E}$, denoted in the sequel by

(5.46.2) $$(\mathcal{A}ut\mathcal{E})_\rho < \mathcal{A}ut\mathcal{E}.$$

6 First Variational Formula

Our aim in this and Section 7 is actually to pave the way to the final statement of the formula in the title of this section, which will be treated in Section 8 (see (8.8)). So we start here with evaluating the corresponding variation of the field strength. Now, as we shall presently see, the relevant framework that one would need can be quite general. So we just assume that

(6.1) we are given a curvature space X—namely, the following triad of objects:

(6.1.1) $$(\mathcal{A}, \partial, d^1)$$

(see Vol. I, Chapt. I, (7.19) and (7.20))—while, for simplicity's sake, we also assume that all the $\mathcal{A}$-modules involved herewith are, in particular, vector sheaves on X. Furthermore, let

(6.1.2) $$(\mathcal{E}, D)$$

be a given Yang–Mills field on X, with

(6.1.3) $$rk\mathcal{E} = n \in \mathbb{N} \text{ (at least 2)}$$

(see (4.12) and (4.13)), while we still consider a local frame of $\mathcal{E}$, say,

(6.1.4) $$\mathcal{U} = (U_\alpha)_{\alpha \in I}.$$

Thus, according to the general theory, the corresponding field strength (= curvature) of the given $\mathcal{A}$-connection (gauge potential) D of $\mathcal{E}$ (see (6.1.2)) is given by the relations

(6.2) $$\begin{gathered} R(D) \equiv R = (R_\alpha) \in \Omega^2(\mathcal{E}nd\mathcal{E})(X), \\ Z^0(\mathcal{U}, \Omega^2(\mathcal{E}nd\mathcal{E})) = \prod_{\alpha \in I} \Omega^2(\mathcal{E}nd\mathcal{E})(U_\alpha), \end{gathered}$$

so that, in particular, one has

(6.3) $$\begin{aligned} R_\alpha \in \Omega^2(\mathcal{E}nd\mathcal{E})(U_\alpha) &\equiv (\Omega^2 \otimes_{\mathcal{A}} \mathcal{E}nd\mathcal{E})(U_\alpha) \\ &= \Omega^2(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} (\mathcal{E}nd\mathcal{E})(U_\alpha) = \Omega^2(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} M_n(\mathcal{A})(U_\alpha) \\ &= (\Omega^2 \otimes_{\mathcal{A}} M_n(\mathcal{A}))(U_\alpha) \equiv M_n(\Omega^2)(U_\alpha) = M_n(\Omega^2(U_\alpha)) \end{aligned}$$

for any $\alpha \in I$, as in (6.1.4). See also, concerning the above calculations in (6.3), Chapter I, (2.42) of Volume I of this treatise or even [VS: Chapt. VII; p. 100, (1.9) and (1.10)]. Yet, for convenience, we further evaluate R_α, as above, when appropriately considered, as a (sheaf) morphism (see (6.5) below) at a particular (local) section,

(6.4) $s \in \mathcal{E}(V)$, for a given open $V \subseteq U_\alpha$, with U_α, a local gauge of $\mathcal{E}$, as in (6.1.4).

Now, in this connection, we also remark that

(6.5) an open $V \subseteq X$, with $V \subseteq U_\alpha$, as above, is still a local gauge of $\mathcal{E}$; therefore, by virtue of our hypothesis in (6.1), concerning the $\mathcal{A}$-modules on X, considered herewith, V may also be viewed as a common local gauge of (the vector sheaves) Ω^1 and $\mathcal{E}nd\mathcal{E} \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ as well.

See also [VS: Chapt. II, p. 137, Lemma 6.1]. So, by further looking at (6.3), one first obtains the following relations:

(6.6)
$$\begin{aligned} R_\alpha \in \Omega^2(\mathcal{E}nd\mathcal{E})(U_\alpha) &= (\mathcal{E}nd\mathcal{E} \otimes_{\mathcal{A}} \Omega^2)(U_\alpha) \\ &= (\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \otimes_{\mathcal{A}} \Omega^2)(U_\alpha) \\ &= (\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(U_\alpha) \\ &= Hom_{\mathcal{A}|_{U_\alpha l}}(\mathcal{E}|_{U_\alpha}, \Omega^2(\mathcal{E})|_{U_\alpha}) \end{aligned}$$

(see also [VS: Chapt. IV, p. 304, Corollary 6.1, along with Chapt. II, p. 134, (6.4.1)]. Accordingly, we may look at

(6.7) R_α, $\alpha \in I$ (see (6.2)), as a sheaf morphism, [precisely speaking, as an $\mathcal{A}|_{U_\alpha}$-morphism of the $\mathcal{A}|_{U_\alpha}$-modules (in effect, vector sheaves over U_α)] involved, as in the last term of (6.6).

Thus, by considering now any (local) section (see (6.4))

(6.8)
$$s \in \mathcal{E}(V) = (\mathcal{E}|_{U_\alpha})(V),$$

one gets, in view of (6.6) and of (6.5) as well as of our calculations in (6.3), the relations

(6.9)
$$\begin{aligned} R_\alpha(s) \in (\Omega^2(\mathcal{E})|_{U_\alpha})(V) &= \Omega^2(\mathcal{E})(V) \\ &\equiv (\Omega^2 \otimes_{\mathcal{A}} \mathcal{E})(V) \\ &= \Omega^2(V) \otimes_{\mathcal{A}(V)} \mathcal{E}(V) = \Omega^2(V) \otimes_{\mathcal{A}(V)} \mathcal{A}^n(V) \\ &= (\Omega^2 \otimes_{\mathcal{A}} \mathcal{A}^n)(V) = (\Omega^2)^n(V) = (\Omega^2(V))^n; \end{aligned}$$

that is, one has

(6.10)
$$R_\alpha(s) = (\omega_1, \ldots, \omega_n),$$

such that $\omega_i \in \Omega^2(V)$, $1 \leqq i \leqq n$, with $s \in \mathcal{E}(V)$. So, we come to our main aim—namely, to identify the variation of $R \equiv R(D)$ in $\Omega^2(\mathcal{E}nd\mathcal{E})(X)$ (see (6.2)) that is entailed by a corresponding variation of D in the set of $\mathcal{A}$-connections of $\mathcal{E}$.

6.1 Variation of the Field Strength, Caused by a Variation of the Gauge Potential

What we are going to do is to

(6.11) find the field strength curve (*viz*., the curvature curve) that corresponds to a given gauge potential curve—in other words to an ($\mathcal{A}$-)connection curve in the (affine) space of $\mathcal{A}$-connections on $\mathcal{E}$,

(6.11.1)
$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E});$$

see also (6.1.2), along with Volume I, Chapt. I, Section 5, concerning the latter space, as in (6.11.1).

Thus, by considering the given $\mathcal{A}$-connection D of the Yang–Mills field $(\mathcal{E}, D)$, under discussion (see (6.1.2)) as fixed, one gets for the space (6.11.1) the relation

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) = D + \Omega^1(\mathcal{E}nd\mathcal{E})(X) = D + Hom_{\mathcal{A}}(\mathcal{E}, \Omega^1(\mathcal{E})), \tag{6.12}$$

within a bijection of the sets appeared in the first relation of (6.12); see also Volume I, Chapt. I, (5.4) and (5.7) or even [VS: Chapt. VI, p. 32, Theorem 7.1].

Now, a curve in the space $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, starting (*viz.*, for $t = 0$) at $D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ (see (6.1.2)), can be certainly supplied, by virtue of (6.12), by the relation

$$\alpha_D(t) \equiv D_t := D + t \cdot \tilde{\omega} \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \qquad t \in \mathbb{R}, \tag{6.13}$$

such that (Volume I, Chapt. I, (5.8.2))

$$\tilde{\omega} \in \Omega^1(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) \tag{6.14}$$

(see also (6.1.4)), or, in common parlance, one has an *"$\mathcal{E}nd\mathcal{E}$-valued 1-form"* on X, in the usual extended terminology of the classical theory, still applied for illustrative reasons in the present abstract setting as well. (Let us recall here that our supporting space X does not carry, up front, any "smooth structure" at all!) Thus, by definition, the curve (6.13) in the affine space of $\mathcal{A}$-connections of $\mathcal{E}$ is, in particular, just a (real) 1-dimensional "linear variety" (subspace) of the affine space $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, as above.

Accordingly, our relevant problem in (6.11) is now reduced to identifying the corresponding curvature curve in the $\mathcal{A}(X)$-module $\Omega^2(\mathcal{E}nd\mathcal{E})(X)$ (hence, in particular, a $\mathbb{C}$-vector space):

$$R(D_t) \equiv R(D + t \cdot \tilde{\omega}) \equiv R_t \in \Omega^2(\mathcal{E}nd\mathcal{E})(X), \qquad t \in \mathbb{R}. \tag{6.15}$$

We depict the above through the following diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\alpha_D(t) \equiv D_t} & \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) = D + \Omega^1(\mathcal{E}nd\mathcal{E})(X) \\ & \underbrace{R(D_t) \equiv R_t}_{\substack{||| \\ \alpha_R(t)}} \searrow & \Big\downarrow R(D) \equiv R \\ & & \Omega^2(\mathcal{E}nd\mathcal{E})(X) \end{array} \tag{6.16}$$

Now, to calculate (6.11), we simply look at

(6.17) the effect on the curvature R (see (6.2)) of a given translation in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ of the $\mathcal{A}$-connection D of $\mathcal{E}$ (see (6.1.2)) by an element

$$\omega' \in \Omega^1(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})). \tag{6.17.1}$$

In this connection, taking into account (6.17.1), we also recall, for convenience of the subsequent discussion, the following relations from the general theory (see [VS:

Chapt. II]); that is, concerning an $\mathcal{E}nd\mathcal{E}$-valued 1-form ω on X, one obtains (see also (6.1.4), as well as, e.g., (6.3))

$$\begin{aligned}\omega \in \Omega^1(\mathcal{E}nd\mathcal{E})(X) &= Z^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) \\ &\subseteq C^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) = \prod_\alpha \Omega^1(\mathcal{E}nd\mathcal{E})(U_\alpha) \\ &\cong \prod_\alpha M_n(\Omega^1)(U_\alpha) = C^0(\mathcal{U}, M_n(\Omega^1)) \\ &= \prod_\alpha M_n(\Omega^1(U_\alpha)),\end{aligned} \tag{6.18}$$

so that, in other words, one gets the one-to-one correspondence

$$\omega \longleftrightarrow (\omega^{(\alpha)}) \in C^0(\mathcal{U}, M_n(\Omega^1)) = \prod_\alpha M_n(\Omega^1(U_\alpha)) \tag{6.19}$$

—namely, an identification of an $\mathcal{E}nd\mathcal{E}$-valued 1-form on X with a 0-cochain of $n \times n$ matrices of 1-forms, "locally defined on X" ($n \times n$ matrices of local sections of Ω^1, the latter being, by definition, the $\mathcal{A}$-module, in point of fact (see (6.1)), vector sheaf, of "1-forms" on X); in this concern, see also [VS: Chapt. VII, p. 119, Theorem 3.2]. Therefore, in the case of an $\mathcal{A}$-connection D of a vector sheaf $\mathcal{E}$ on X, as, for example, in (6.1.2), as well as of a local frame $\mathcal{U}$ of $\mathcal{E}$ (see (6.1.4)),

(6.20) the given $\mathcal{A}$-connection D of $\mathcal{E}$ can be identified with the so-called $\mathcal{A}$-connection 0-cochain matrix form of D, as in (6.19), associated with a given local frame $\mathcal{U}$ of $\mathcal{E}$ (see also Volume I, Chapt. I, (2.50), along with (2.56) therein). Of course, the latter case is characterized by the corresponding *"transformation law of potentials."* See [VS]: Chapt. VII, p. 119, Theorem 3.2 along with p. 112, (2.39) and p. 103, (1.24) and (1.32), or even p. 104, (1.35).

On the other hand, the previous argument in (6.20) is actually connected with formula (6.12), perse, defining an $\mathcal{A}$-connection D' (uniquely), through an initially given one D and an element ω', as in (6.19), something that will be still of use presently below. However, first, we deviate a bit from the main flux of this discussion, by the subsequent comments, indeed, a straightforward (notwithstanding, useful) by-product of the type of argument, employed in (6.18), as above:

Note 6.1 In connection with our previous argument in (6.18), we remark that, more generally,

(6.21) given a vector sheaf $\mathcal{E}$ on (a topological space) X and a local frame

$$\mathcal{U} = (U_\alpha)_{\alpha \in I} \tag{6.21.1}$$

of $\mathcal{E}$, one gets

$$C^0(\mathcal{U}, \mathcal{E}) = C^0(\mathcal{U}, \mathcal{A}^n), \tag{6.21.2}$$

where $n = rk\mathcal{E}$.

See also [VS: Chapt. I, p. 55, (11.40)]. In fact, the same argument as before holds true:

(6.22) for any two sheaves $\mathcal{E}$ and $\mathcal{F}$ on X, for which one has

(6.22.1) $$\mathcal{E} = \mathcal{F}|_U,$$

within an isomorphism of the sheaves concerned, for any element U of a given open covering, say $\mathcal{U}$, of X (common "local frame" of $\mathcal{E}$ and $\mathcal{F}$). One then obtains

(6.22.2) $$C^0(\mathcal{U}, \mathcal{E}) = C^0(\mathcal{U}, \mathcal{F}).$$

Indeed, under the above circumstances, one actually gets

(6.22.3) $$C^p(\mathcal{U}, \mathcal{E}) = C^p(\mathcal{U}, \mathcal{F})$$

for any $p \in \mathbb{Z}_+$. (See also [VS: Chapt. III, p. 175, (4.11)].)

We can proceed now, concerning the response to our problem in (6.11). Thus, let us further consider another $\mathcal{A}$-connection of $\mathcal{E}$, say (see (6.12)),

(6.23) $$\tilde{D} := D + \omega' \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

which locally, in terms of the corresponding $\mathcal{A}$-connection 0-cochain matrix forms (see (6.19) and (6.20)), takes the form

(6.24) $$\begin{aligned} \tilde{D} \longleftrightarrow \tilde{\omega} &\equiv (\tilde{\omega}^{(\alpha)}) = \omega + \omega' \equiv (\omega^{(\alpha)}) + (\omega'^{(\alpha)}) \\ &= (\omega^{(\alpha)} + \omega'^{(\alpha)}) \in C^0(\mathcal{U}, M_n(\Omega^1)) = \prod_{\alpha} M_n(\Omega^1(U_\alpha)). \end{aligned}$$

Accordingly, by looking now at the corresponding local expression of the curvature of the $\mathcal{A}$-connection $\tilde{D}$,

(6.25) $$\tilde{R} \equiv R(\tilde{D}) = (R(\tilde{D})_\alpha) \equiv (\tilde{R}_\alpha)$$

(see also (6.2) and (6.3)), one obtains, in view of (6.24) and *Cartan's structural equation* (see Volume I, Chapt. I, (7.27) and (7.32)),

(6.26) $$\begin{aligned} \tilde{R}_\alpha \equiv R(\tilde{D})_\alpha := R(\tilde{D})|_{U_\alpha} &\equiv R(\tilde{D}_\alpha) \\ &= d(\omega^{(\alpha)} + \omega'^{(\alpha)}) + (\omega^{(\alpha)} + \omega'^{(\alpha)}) \wedge (\omega^{(\alpha)} + \omega'^{(\alpha)}) \\ &= R_\alpha + d\omega'^{(\alpha)} + [\omega^{(\alpha)}, \omega'^{(\alpha)}] + \omega'^{(\alpha)} \wedge \omega'^{(\alpha)}, \end{aligned}$$

where, of course, we set (see (6.2) and (6.3))

(6.27) $$R_\alpha \equiv R(D)_\alpha = d\omega^{(\alpha)} + \omega^{(\alpha)} \wedge \omega^{(\alpha)} \in M_n(\Omega^2(U_\alpha))$$

(Cartan's structural equations, loc. cit.), for any $\alpha \in I$, as in (6.1.4), while we still employ in (6.26) the familiar notation (*Lie bracket*)

$$[\omega^{(\alpha)}, \omega'^{(\alpha)}] \equiv \omega^{(\alpha)} \wedge \omega'^{(\alpha)} - \omega'^{(\alpha)} \wedge \omega^{(\alpha)} \tag{6.28}$$

for any $\alpha \in I$, as before.

On the other hand, by further referring to (6.24), we also write, in view of (6.27), the curvature of the $\mathcal{A}$-connection (see (6.23))

$$\tilde{D} := D + \omega' \equiv \omega + \omega' \tag{6.29}$$

by still employing, for convenience, an obvious abuse of notation, concerning the particular local gauge of $\mathcal{E}$, say U_α, $\alpha \in I$, as above, in the form

$$R(\omega + \omega') = R(\omega) + d\omega' + [\omega, \omega'] + \omega' \wedge \omega' \tag{6.30}$$

(see also (6.26)), which will be of use below.

Furthermore, the same relation (6.30) can still be distilled by using the $\mathcal{A}$-connection of the vector sheaf $\mathcal{E}nd\mathcal{E}$, in particular, in view of (6.18), its "first prolongation" (*viz.*, the $\mathbb{C}$-linear morphism) (see also, e.g., Volume I, Chapt. I, (7.13)),

$$D^1_{\mathcal{E}nd\mathcal{E}} : \Omega^1(\mathcal{E}nd\mathcal{E}) \longrightarrow \Omega^2(\mathcal{E}nd\mathcal{E}). \tag{6.31}$$

So we look further at the relevant situation, even within a more general setting (see (6.35.1) below).

6.2 Covariant Differential Operators (Prolongations) for $\mathcal{E}nd\,\mathcal{E}$

We assume, as in (6.1), that we are given a curvature space X (see (6.1.1)), along with a Yang–Mills field on X,

$$(\mathcal{E}, D), \tag{6.32}$$

such that $rk\mathcal{E} = n \in \mathbb{N}$.

Based now on the proof of the ("differential") Bianchi's identity, see [VS: Chapt. VIII, p. 225], one concludes, concerning the "differential operator" (6.31), the relation

$$D^1_{\mathcal{E}nd\mathcal{E}}(\omega') = d^1(\omega') + [\omega, \omega'] \tag{6.33}$$

for any

$$\omega \in C^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) \quad \text{and} \quad \omega' \in C^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})), \tag{6.34}$$

with $\mathcal{U}$ a local frame of $\mathcal{E}$ (see (6.1.4)). In point of fact, one gets the following situation:

(6.35) more generally, one has the relation

(6.35.1) $$D^{p}_{\mathcal{E}nd\mathcal{E}}(\omega') = d^{p}(\omega') + [\omega, \omega']$$

for any $p \in \mathbb{N}$, such that

(6.35.2) $$\omega \in C^{0}(\mathcal{U}, \Omega^{1}(\mathcal{E}nd\mathcal{E})) \quad \text{and} \quad \omega' \in C^{0}(\mathcal{U}, \Omega^{p}(\mathcal{E}nd\mathcal{E})),$$

whenever, of course, the "differential operators" in (6.35.1) have a meaning; namely, any time one has the corresponding to the aforesaid formula "differential setup."

Accordingly, combining now (6.33) with (6.30), one obtains

(6.36) $$R(D + \omega') \equiv R(\omega + \omega') = R(\omega) + D^{1}_{\mathcal{E}nd\mathcal{E}}(\omega') + \omega' \wedge \omega'$$

with ω and ω' as in (6.33). Thus, by further considering the (curvature) curve (6.15), one gets

(6.37) $$\begin{aligned} R(D_t) &\equiv R(D + t \cdot \tilde{\omega}) \equiv R(\omega + t \cdot \tilde{\omega}) \equiv R_t \\ &= R(\omega) + D^{1}_{\mathcal{E}nd\mathcal{E}}(t \cdot \tilde{\omega}) + (t \cdot \tilde{\omega}) \wedge (t \cdot \tilde{\omega}); \end{aligned}$$

that is, one finally obtains

(6.38) $$R_t = R + D^{1}_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega}) + t^{2} \cdot (\tilde{\omega} \wedge \tilde{\omega})$$

for any $t \in \mathbb{R}$, which is thus, in view of (6.15), the curve in $\Omega^{2}(\mathcal{E}nd\mathcal{E})(X)$ that we were looking for, according to (6.11), corresponding to the given $\mathcal{A}$-connection-curve (6.13) in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$.

Now, for convenience, referring to the discussion in Section 8, we summarize the preceding into the form of the following.

Lemma 6.1 Suppose we are given a curvature space X (see (6.1.1)) and let

(6.39) $$(\mathcal{E}, D)$$

be a Yang–Mills field on X. Moreover, consider an $\mathcal{A}$-connection-curve in the (affine) space of $\mathcal{A}$-connections of $\mathcal{E}$,

(6.40) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

of the form ("variation of D")

(6.41) $$D_t := D + t \cdot \tilde{\omega}, \; t \in \mathbb{R}$$

such that

(6.42) $$\tilde{\omega} \in \Omega^{1}(\mathcal{E}nd\mathcal{E})(X).$$

Then, the corresponding (curvature) curve in

(6.43) $$\Omega^2(\mathcal{E}nd\mathcal{E})(X)$$

is given (in matrix form) by the relation ("first variation formula")

(6.44) $$R_t \equiv R(D_t) = R + t \cdot D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega}) + t^2 \cdot (\tilde{\omega} \wedge \tilde{\omega})$$

with $t \in \mathbb{R}$ and $R \equiv R(D)$. ■

We close the present section with the subsequent remarks in Scholium 6.1 pertaining to the nature of the preceding results, as summarized by Lemma 6.1 that will still be of use in the sequel.

Scholium 6.1 The information one gets, through Lemma 6.1, concerning the variation (curve) of the field strength of a given Yang–Mills field $(\mathcal{E}, D)$ on a curvature space X (see (6.39) and (6.1.1)), corresponding to a variation (curve) of the respective gauge potential ($\mathcal{A}$-connection D, as above, is actually only local (!). In this regard, we can still remark, however, along with R. Haag [1: p. 326], for instance, that

(6.45) "...the central message of *Quantum Field Theory* [is] that all information characterizing the theory is strictly local ..."

(the emphasis is ours). Yet, on the other hand, we also note, herewith, that

(6.45′) "In a field theory it is much simpler to use strictly local functions."

(Emphasis ours). See J.M. Ziman [1: p. 18, footnote]. Thus, (6.44) is the outcome of an exploitation of Cartan's structural equation, the latter being given, in effect, in terms of local data (see (6.27)), namely, by restricting ourselves to a local gauge, say U_α, of $\mathcal{E}$ (see (6.1.4)). That is, equivalently,

(6.46) the relation (6.44) ("first variational formula") is a result that is valid on a local gauge, say U_α, of $\mathcal{E}$, hence, of a point-character too, yielding information at each particular instant of time $t \in \mathbb{R}$, along the corresponding curve at issue, determined through the $\mathcal{A}$-connection-curve $\alpha_D(t)$, $t \in \mathbb{R}$ (see (6.13)) and the local gauge U_α of $\mathcal{E}$, as above.

The same result (local information) as in (6.44) is virtually needed (see Section 8), in order to look at the "principle of least action" within the present abstract setting, thus to evaluate

(6.47) $$\delta\mathcal{YM}_{\mathcal{E}}(R) \equiv \widehat{\mathcal{YM}_{\mathcal{E}}(R_t)}^{\cdot}(0) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}_{\mathcal{E}}(R_t)$$

(see also (5.11), as well as (8.24) below), hence, a matter *eo ipso* local! Therefore, our information, already supplied via (6.44), is actually quite enough, to the extent, of course, that one wants to know the relevant situation, for $t \to 0$, in $\mathbb{R}$ (*viz.*, in turn, in a neighborhood of

$$R(D + t\tilde{\omega})_{t=0} = R(D) \equiv R, \tag{6.48}$$

where now the latter quantity can be viewed, of course, within the context of (6.2) and (6.3), or even of (6.10)).

Now, classically speaking, all the above local information can usually be put together through an integration process (in terms of functions, thus, in point of fact, within the present abstract (sheaf-theoretic) setup, via sections, with "compact support"). Yet, this actually amounts to the same thing, by also reducing the problem to a finite number of local data (consider, e.g., the case of a compact (Hausdorff) space X). We will address this matter again in Sections 7 and 8.

7 Volume Element

Our aim in the following discussion is to supply, within the present abstract setting, the appropriate background material so that one can formulate the corresponding classical notion, as in the title of this section. As an outcome, one can further obtain, here too, an $\mathcal{A}$-metric, in point of fact, an $\mathcal{A}$-valued inner product, starting from any given vector sheaf $\mathcal{E}$ on X, precisely speaking, from a suitable "structure sheaf" $\mathcal{A}$ on (the pertinent topological space X, this latter point actually being a preponderant issue, permeating the whole of the present treatise. Now, the aforesaid notions are determined in effect via a suitably defined $\mathcal{A}(X)$-valued "integral" of compactly supported sections of $\mathcal{E}$, as above, by extending the analogous situation of the standard theory. Yet, by analogy to the classical pattern, one employs here the "Hodge $*$-operator" that we already considered in Section 4.5; see also Volume I, Chapter I, Section 10 of this treatise.

Now, for simplicity, we thus assume that we are given

(7.1)

an enriched ordered algebraized space

$$(X, \mathcal{A}) \tag{7.1.1}$$

(see (5.1)) in such a manner that

$$(\mathcal{E}, \rho) \tag{7.1.2}$$

is a Riemannian $\mathcal{A}$-module on X (see, e.g., (4.1.2)). In this connection, X is, thus far, simply an arbitrary topological space.

So, based on our hypothesis in (7.1.2), consider now the pair

$$(\mathcal{A}, \rho) \tag{7.2}$$

as a Riemanian $\mathcal{A}$-module on X, the latter providing (see (7.1.1)) an ordered algebraized space

$$(X, \mathcal{A}), \tag{7.3}$$

"with square root" (see [VS: Chapt. IV, p. 336, Definition 10.1]: "enriched ordered algebraized space"). Thus, for convenience of reference concerning the subsequent discussion, we recall that one can further define on X a

$$\text{volume element, say, } \omega, \tag{7.4}$$

given by the following global (continuous) section of our "structure sheaf" $\mathcal{A}$; that is, one sets

$$\omega := \sqrt{|\rho|} \cdot \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \in (\wedge^n \mathcal{A}^n)(X) \equiv (\det \mathcal{A}^n)(X) = \mathcal{A}(X). \tag{7.5}$$

See Volume I, Chapter I, Section 10 of the present treatise, as well as [VS: Chapt. IV, Section 11]. Here one still sets

$$\varepsilon \equiv (\varepsilon_i)_{1 \leqq i \leqq n} \subseteq \mathcal{A}^n(X) = \mathcal{A}(X)^n \tag{7.6}$$

for the canonical coordinate (global) sections of $\mathcal{A}^n$, or even the *Kronecker gauge of* $\mathcal{A}^n$; that is, one has

$$\varepsilon_i := (\delta_{ij})_{1 \leqq j \leqq n} \in \mathcal{A}^n(X) = \mathcal{A}(X)^n \tag{7.7}$$

for any $1 \leqq i \leqq n$, while we also define

$$\delta_{ij} \in \mathcal{A}(X), \quad 1 \leqq i, j \leqq n, \tag{7.8}$$

as the (canonical) *Kronecker sections* of $\mathcal{A}$ over X, the whole picture being actually depicted, in "matrix form," by the relation

$$\varepsilon \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_n(\mathcal{A}(X)) = M_n(\mathcal{A})(X), \tag{7.9}$$

which can be considered as the Kronecker (global section) matrix of $\mathcal{A}$ over X, of order $n \in \mathbb{N}$, or, for short, the $n \times n$ Kronecker (section) matrix of $\mathcal{A}$ over X. See also [VS: Chapt. IV, p. 290, along with Chapt. II, p. 123, (3.22)]. Yet, by still referring to (7.5), we also set

$$\sqrt{|\rho|} := \sqrt{|\det(\rho(\varepsilon_i, \varepsilon_j))|}, \tag{7.10}$$

where ρ denotes the (canonical) extension of the given $\mathcal{A}$-metric ρ, as in (7.2), to (the free $\mathcal{A}$-module) $\mathcal{A}^n$ (see also loc. cit., Chapt. IV, p. 341, (11.4), as well as p. 324, (8.37) and (8.37′), therein). Now, on the other hand, one further concludes that

(7.11) the volume element ω on X, as defined by (7.5), can still be given by the relation

$$\omega = s_1 \wedge \cdots \wedge s_n, \tag{7.11.1}$$

such that

(7.11.2) $$s \equiv (s_i)_{1 \leqq i \leqq n} \subseteq \mathcal{A}^n(X) = \mathcal{A}(X)^n$$

is an orthonormal gauge of $\mathcal{A}^n$. In this connection, we also remark that such a gauge of $\mathcal{A}^n$ is always available as a simple application of the *Gram–Schmidt orthonormalization process*; that is, here too, also in force, by virtue of our hypothesis, as in (7.1); see also [VS: Chapt. IV, p. 340, Theorem 10.1] as well as Volume I, Chapter I, Section 10, (10.7) of the present treatise.

Note 7.1 The preceding argument is still valid for any free $\mathcal{A}$-module $\mathcal{E}$ over X, of rank $n \in \mathbb{N}$; that is, in the case one has

(7.12) $$\mathcal{E} = \mathcal{A}^n,$$

modulo an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules involved. Indeed, more generally,

(7.13) the previous argument is also in force for any vector sheaf $\mathcal{E}$ on X of rank $n \in \mathbb{N}$.

In this regard, see also Volume I, Chapter I of the present study, Scholium 10.1, as well as the relevant comments therein pertaining to the $*$-operator; (loc. cit., Section 10; (10.23)). However, according to that same discussion, one is then led to consider a

(7.14) paracompact (Hausdorff) space X, while $\mathcal{A}$ is still a strictly positive fine sheaf on X.

See also (4.1) in Section 4. Therefore, in other words, we actually

(7.15) assume, henceforward, that we have the situation described by (5.2) in the foregoing.

Thus, for convenience of reference, we assume henceforth that

(7.16) we are given an enriched ordered algebraized space

(7.16.1) $$(X, \mathcal{A})$$

(see (5.1)) in such a manner that X is a paracompact (Hausdorff) space and $\mathcal{A}$ is a *strictly positive fine sheaf* on X. Yet, we suppose that

(7.16.2) $$(\mathcal{A}, \rho)$$

is a Riemannian $\mathcal{A}$-module on X.

Now, as a result of our previous hypothesis in (7.16), concerning the space X considered, we can still apply the preceding argument in (4.84) as well as in (4.87), in principle, locally (!), while, at the very end, one gets, by virtue of the same hypothesis for X as before, that

(7.17) the relations (4.84) and (4.87) are still in force for every vector sheaf $\mathcal{E}$ on X (see also (4.90.1)).

Furthermore, in view of (7.16), hence of (4.91) as well, one also obtains that

(7.18) every vector sheaf $\mathcal{E}$ on X becomes a Riemannian vector sheaf; namely, one gets a pair

(7.18.1) $$(\mathcal{E}, \rho),$$

where ρ stands for a (Riemannian) $\mathcal{A}$-metric on $\mathcal{E}$. (Of course, we generically employed here the same symbol ρ by abusing notation in connection with (7.16.2)).

Accordingly, one concludes, here too, the relation

(7.19) $$\mathcal{E} \underset{\tilde{\rho}}{\cong} \mathcal{E}^*,$$

within an isomorphism of the $\mathcal{A}$-modules (in fact, of the vector sheaves) involved. Thus, taking into account (7.18), one can consider

(7.20) $$\wedge^p \mathcal{E}^*, \ 1 \leqq p \leqq n = rk\mathcal{E},$$

an $\mathcal{A}$-valued inner product, determined through the corresponding $\mathcal{A}$-metric ρ on it, so that, if we further set, for simplicity's sake,

(7.21) $$(\alpha, \beta) \equiv \rho(\alpha, \beta)$$

with α, β in $(\wedge^p \mathcal{E}^*)(U)$ and for any open $U \subseteq X$, one gets

(7.22) $$\alpha \wedge *\beta := \rho(\alpha, \beta) \cdot \omega \equiv (\alpha, \beta) \cdot \omega = (\beta, \alpha) \cdot \omega = \beta \wedge *\alpha.$$

Therefore, one has

(7.23) $$(*\alpha, \beta) = \alpha \wedge \beta$$

for any α, β in $(\wedge^p \mathcal{E}^*)(U) = (\wedge^p \mathcal{E})(U)$, as above.

Now, in order to extend the previously defined inner product, as in (7.21) and (7.22), to the so-called *integration inner product* in the classical case (see Y. Choquet-Bruhat et al. [1: p. 296] or even W.A. Poor [1: p. 153]), it would be more appropriate to assume, concerning our structure $\mathbb{C}$-algebra sheaf $\mathcal{A}$, certain complementary topological-algebraic conditions that we explain below. This type of integration will be considered in Scholium 7.1 below.

7.1 A Topological ($\mathbb{C}$-)Algebra (Structure) *sheaf* $\mathcal{A}$

Henceforth, we posit that we are given

> a topological space X, satisfying (7.16), while we still assume that our structure ($\mathbb{C}$-algebra) sheaf $\mathcal{A}$ is a topological algebra sheaf on X, in the sense that
>
> (7.24) for every open $U \subseteq X$, the respective (local) section algebra of $\mathcal{A}$,
>
> (7.24.1) $$\Gamma(U, \mathcal{A}) \equiv \mathcal{A}(U),$$
>
> is a unital commutative complete topological ($\mathbb{C}$-)algebra, having continuous multiplication.
>
> Furthermore, we also suppose that
>
> the "spectrum" (see Note 7.2 below) of each one of the topological algebras $\mathcal{A}(U)$, as above, satisfies the relation
>
> (7.24.2) $$\mathfrak{M}(\mathcal{A}(U)) \underset{\rightarrow}{\subset} U$$
>
> as a topological subspace, while we still assume that there is defined on U a *Radon-like measure* (see Scholium 7.1).

Now, concerning the relevant terminology on topological algebra theory, we refer to A. Mallios [TA]; yet, see also [VS: Chapt. XI, pp. 300ff] for details on topological algebra (pre)sheaves.

Note 7.2 Given a topological algebra E (loc. cit.), its spectrum (i.e., Gel'fand space of E), denoted by

(7.25) $$\mathfrak{M}(E),$$

is, by definition, the set of nonzero continuous algebra morphisms of E into $\mathbb{C}$ (in effect, onto)—in short, continuous characters of E, topologized as a subspace of E'_s (*viz.*, of the weak topological dual of E).

An important issue is the following homeomorphism:

(7.26) $$\mathfrak{M}(\mathcal{C}_c(X)) = X.$$

Here, $\mathcal{C}_c(X)$ stands for the algebra of $\mathbb{C}$-valued continuous functions on a completely regular (Hausdorff) topological space X equipped with the compact-open topology c (see [TA], p. 223, Theorem 1.2). A similar result also holds true; that is, one has

(7.27) $$\mathfrak{M}(\mathcal{C}^{\infty}(X)) = X$$

within a homeomorphism of the topological spaces concerned, referring to the ($\mathbb{C}$-) algebra

$$\mathcal{C}^{\infty}(X) \tag{7.28}$$

of $\mathbb{C}$-valued $\mathcal{C}^{\infty}$-functions on a "smooth" (*viz.*, $\mathcal{C}^{\infty}$-) manifold X, endowed with the so-called (canonical) *Schwartz topology*; see [TA: p. 227, Theorem 2.1, along with Scholium 2.1 therein].

Now, assuming in the relation (7.22) that either one of the "p-forms" α or β, appearing therein, has compact support (*viz.*, it vanishes in the complement of a compact subset of its domain of definition), one obtains the relation

$$(\alpha, \beta) := \int_X \alpha \wedge *\beta = \int_X (\alpha, \beta) \cdot \omega, \tag{7.29}$$

thus getting an $\mathcal{A}$-valued inner product on the $\mathcal{A}$-algebra sheaf (see also (4.87)),

$$\wedge\mathcal{E}^* := \bigoplus_{p=0}^{\infty} \wedge^p \mathcal{E}^* \cong \wedge\mathcal{E}, \tag{7.30}$$

for any given vector sheaf $\mathcal{E}$ on X. (In this connection, see also [VS: Chapt. IV, p. 308, (7.9) and (7.10)]; yet, notice that since $\mathcal{E}$ is, by hypothesis, a vector sheaf on X, the (direct) sum in (7.30) is actually finite, terminated at $n = rk\mathcal{E} \in \mathbb{N}$ see also [loc. cit., p. 312, (7.23)]. On the other hand, the inner product in (7.30) is taken, of course, as in (7.22), among pairs in the same sum, as before).

Scholium 7.1 (Radon-like measures) The "measures," in point of fact, "integrals," of the type alluded to in the title of the present scholium, were already mentioned in connection with (7.24), while they were also considered in (7.29), in the sense that we explain in subsequent discussion: Thus, based on our hypothesis in (7.24.2), one concludes, in particular, the relation

$$\mathfrak{M}(\mathcal{A}(X)) \underset{\rightarrow}{\subset} X \tag{7.31}$$

(*viz.*, the spectrum of (the topological algebra) $\mathcal{A}(X)$ carries the relative topology from X (more generally, a continuous injection in (7.31), along with the analogous condition for (7.24.2), would suffice, concerning the ensuing discussion)). Now, we call

(7.32)

a "Radon-like measure on X" any element of

$$\mathcal{C}_c(X)', \tag{7.32.1}$$

—namely, any element of the topological dual of (the topological vector space)

$$\mathcal{C}_c(X) \tag{7.32.1}$$

((7.26) for the notation applied) or any continuous ($\mathbb{C}$-) linear form on the latter space.

Now, classically speaking, elements of (7.32.1) can be viewed as “integrals” on $\mathcal{C}(X)$ (i.e., “Radon measures on X”), à la Riesz (or even, Markov–Riesz); see N. Bourbaki [1: Chapters. 1–4], A. Mallios [TA: p. 474, Lemma 2.1], or even R. I. Hadjigeorgiou [1: Chapt. II, Theorem 1.1]; here one has a really elaborated and detailed proof of that classical result, as above, the so-called *Riesz representation theorem.* Yet, M. Fragoulopoulou [1: p. 74]. Hence, the terminology applied.

On the other hand, by further suitably specializing on the type of the topological algebras $\mathcal{A}(U)$ considered, as in (7.24.1), hence, according to the very definition, on that one of the topological ($\mathbb{C}$-)algebra sheaf $\mathcal{A}$ (see (7.24)) (e.g., continuity of the relevant “Gel’fand maps,” or even, occasionally, as is usually the case in the applications, (functional) “semi-simplicity” of the algebras involved, see A. Mallios [TA] concerning the terminology applied herewith), then, a “Radon-like measure on X” gives rise, à la Hahn–Banach, for instance (when suitable complementary conditions are put on (7.31)), to a similar one on

$$\mathcal{C}_c(\mathfrak{M}(\mathcal{A}(X))), \tag{7.33}$$

so that, finally, even to a “state” of (*viz.*, a continuous linear form on) the (topological) algebra $\mathcal{A}(X)$ itself. Analogous considerations to the above will be employed in Section 8. The situation described by the previous argument can, of course, be specified to the important particular case that one has

$$\mathcal{A}(X) = \mathbb{A}, \tag{7.34}$$

within an isomorphism of topological algebras, such that our structure sheaf $\mathcal{A}$ is, in effect, the Gel’fand sheaf of a given “geometric topological algebra” $\mathbb{A}$, whose spectrum $\mathfrak{M}(\mathbb{A})$ is thus, in particular, (homotopic to) X. See, for instance, A. Mallios [10]. The preceding have, of course, a special bearing on the classical case of differential geometry on smooth (*viz.*, $\mathcal{C}^\infty$-) manifolds, concerning the corresponding in that case structure sheaf $\mathcal{C}^\infty_X$ (see Volume I, Chapt. I, (1.15) of this treatise) and the topological algebra $\mathcal{C}^\infty(X)$ (see (7.28) in the foregoing, along with A. Mallios [10]). Yet, applications of the above considerations are also found in connection with the recently employed algebra sheaf of Rosinger (“generalized functions”), within the abstract framework of the present study, as well the one of I. Raptis (*finitary algebra sheaves*). Both of the latter types of algebra sheaves very likely have potential applications in problems connected with quantum gravity; see A. Mallios–E. E. Rosinger [1, 2], along with A. Mallios–I. Raptis [1–4].

Now, as another potential application of the preceding, we further remark that, according to the above general setup, a “Radon-like measure” on the “moduli space”

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/(\mathcal{A}ut\mathcal{E})_\rho \tag{7.35}$$

(see also (5.46.2), along with Chapt. II, (2.9) and (3.4)), of a given vector sheaf $\mathcal{E}$ on X, the latter space being endowed with the pertinent structure, so that (7.35) get a meaning (loc. cit.), while the same set (7.35) is considered, if appropriately topologized (see Note 7.3 below), as the spectrum

(7.36) $$\mathfrak{M}(\mathbb{A})$$

of a suitable topological algebra $\mathbb{A}$, might be, in view of the preceding discussion, a continuous linear form ("state") on

(7.37) $$\mathcal{C}_c(\mathfrak{M}(\mathbb{A})) \text{ or even on } \mathbb{A};$$

that is, an element of the topological dual space of the latter spaces, namely, of

(7.38) $$\mathcal{C}_c(\mathfrak{M}(\mathbb{A}))' \text{ or of } \mathbb{A}', \text{ respectively.}$$

Note 7.3 By further commenting on our previous remarks concerning (7.35), we first note that the set

(7.39) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

being an affine space, in view of Chapter I, (5.4) and (5.7) ($\mathcal{E}$ is, by hypothesis, a vector sheaf on X, as also Ω^1) can be naturally topologized via its model $\Omega(\mathcal{E}nd\mathcal{E})(X)$, when the latter is appropriately construed as a topological vector space (see, e.g., Chapt. III, Section 4), hence, a completely regular (topological) space and thus the quotient space (7.35) as well; see also W. Roelcke–S. Dierolf [1] concerning "quotient uniform structures" in general); see also E. Papatriantafillou [1–3].

The rest of our argument, pertaining to (7.36), is now standard, connected with topological algebra theory; see, for example, A. Mallios [TA: p. 223, Theorem 1.2].

The above are, indeed, akin also to recent considerations in the "theory of moduli spaces," within the framework of the classical differential geometry on smooth manifolds. See, for instance, J.C. Baez [1], A. Ashtekar–C.J. Isham [1], and A. Ashtekar–J. Lewandowski [1]; yet, see also J.N. Tavares [1] for another connection of the above with the theory of the so-called *loop representations* within the context of gauge field theory.

8 Yang–Mills Functional (continued): The Variation Formula

Our aim here is to formulate, within the abstract setting that is advocated by the present study, the formula which connects the Yang–Mills functional, as the latter was defined by (5.11), with its variation (status) "at the limit" $t \to 0$ (i.e., with the "derivative at $t = 0$" of the respective *"Yang–Mills curve"*; (*viz.*, of what we called the "first variational formula"), as given by (6.44), that corresponds to an analogous curve (variation in time) of the "dynamics" [*viz.*, of the given $\mathcal{A}$-connection D of the Yang–Mills field

(8.1) $$(\mathcal{E}, D)$$

under consideration (see (6.41), along with (6.46))].

Of course, the above Yang–Mills field is taken over a suitable (topological) space X, satisfying the pertinent conditions so that the terminology applied herewith has a meaning; thus, take, for instance,

(8.2) a curvature space X (see (6.1)) such that (6.44) is valid, while we still assume that the same space X satisfies (5.2); hence, in effect, we consider what we called in the preceding a Yang–Mills space (see Definition 4.1). Therefore, (5.11) and (5.17) acquire a meaning; that is, in other words, the corresponding Yang–Mills functional (or Yang–Mills Lagrangian, yet Yang–Mills action) can be construed, along with the relevant Yang–Mills equations (see Definition 4.2).

Thus, by assuming the above framework of (8.2), one concludes that the pair $(\mathcal{E}, D)$, as in (8.1), can be viewed as a *Riemannian Yang–Mills field*

$$(\mathcal{E}, \rho). \tag{8.3}$$

Consequently, looking now at the Yang–Mills functional, as in (5.11) one gets the respective Yang–Mills curve, strictly speaking (!) according to the relation

$$\gamma(t) := \mathcal{YM}_{\mathcal{E}}(D_t) \equiv \mathcal{YM}_{\mathcal{E}}(R(D_t)) \equiv \mathcal{YM}_{\mathcal{E}}(R_t) := \frac{1}{2}\rho(R_t, R_t), \ t \in R, \tag{8.4}$$

so that, based further on (6.44) and the $\mathcal{A}$-bilinearity of ρ in (8.3) (see (4.1.3) and (4.2.1)) (see also (7.21) for the notation applied below), one obtains

$$\begin{aligned} \mathcal{YM}_{\mathcal{E}}(R_t) = {} & t \cdot (R, D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega})) + t^2[(R, \tilde{\omega} \wedge \tilde{\omega}) \\ & + (D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega}), \tilde{\omega} \wedge \tilde{\omega}) + (\tilde{\omega} \wedge \tilde{\omega}, \tilde{\omega} \wedge \tilde{\omega}) \\ & + \frac{1}{2}(D^1, \ldots, D^1, \ldots)] + t^3(D^1, \ldots, \tilde{\omega} \wedge \tilde{\omega}) \\ & + \frac{1}{2}t^4(\tilde{\omega} \wedge \tilde{\omega}, \tilde{\omega} \wedge \tilde{\omega}) + \frac{1}{2}(R, R). \end{aligned} \tag{8.5}$$

Therefore, one finally gets the relation

$$\begin{aligned} \dot{\widehat{\mathcal{YM}_{\mathcal{E}}(R_t)}}(0) \equiv \frac{d}{dt}\Big|_{t=0} \mathcal{YM}_{\mathcal{E}}(R_t) := \lim_{t \to 0} \frac{1}{t}(\mathcal{YM}_{\mathcal{E}}(R_t) - \mathcal{YM}_{\mathcal{E}}(R_0)) \\ = \rho(R, D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega})) \equiv (R, D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega})). \end{aligned} \tag{8.6}$$

Note 8.1 We comment below on the meaning of the term “lim” as it appeared in relation (8.6). Thus, by looking at the curve

$$\gamma(t) := \frac{1}{2}\rho(R_t, R_t) \equiv \frac{1}{2}(R_t, R_t), \tag{8.7}$$

$t \in \mathbb{R}$, as in (8.4) above (see also (7.21) for the simplification of the notation applied), such that (see (6.15))

(8.8) $$R_t \equiv R(D_t) \equiv R(D + t \cdot \tilde{\omega}) \in \Omega^2(\mathcal{E}nd\mathcal{E})(X)$$

for any $t \in \mathbb{R}$, we conclude that

(8.9) $$\gamma(t) \in \mathcal{A}(X), \ t \in \mathbb{R}$$

(see also (4.2.1)); that is,

(8.10) the Yang–Mills curve, as this is given by (8.4) and (8.7), is actually an

(8.10.1) $\mathcal{A}(X)$*-valued curve.*

So, by assuming now that

(8.11) our structure sheaf $\mathcal{A}$ is, in particular, a topological $\mathbb{C}$-vector space sheaf on X, in the sense that

(8.11.1) $$\Gamma(U, \mathcal{A}) \equiv \mathcal{A}(U)$$

is a topological $\mathbb{C}$-vector space for any open $U \subseteq X$ with the respective "connecting (restriction) maps" (see [VS: Chapt. I, p. 38, (9.4) and (9.7)]) being continuous $\mathbb{C}$-linear maps,

one can understand completely the "limit" in (8.6)—namely, the derivative at $t = 0$ of the (Yang–Mills) curve (8.4). Yet, we also note, in this case, that

(8.12) γ is a continuous $\mathcal{A}(X)$*-valued curve.*

Thus, let us now look at the point

(8.13) $$\gamma(0) = \frac{1}{2}(R, R) \equiv z \in \mathcal{A}_x$$

of the curve at issue (see also (6.38) for $t = 0$), such that

(8.14) $$x \equiv \pi(z) \equiv \pi\left(\frac{1}{2}(R, R)\right) \in X,$$

where, as usual, so far

(8.15) $$(\mathcal{A}, \pi, X)$$

stands for the given structure sheaf $\mathcal{A}$ on X. Therefore, for any open neighborhood U of $x \in X$, one can further consider an appropriate (open) neighborhood, say $I \subseteq \mathbb{R}$, of $0 \in \mathbb{R}$ such that

(8.16) $$\gamma(t) \in \pi^{-1}(U), \ t \in I.$$

Of course, we have employed herewith the continuity of the map

$$\pi \circ \gamma : \mathbb{R} \longrightarrow X \tag{8.17}$$

(see also (8.12), (8.15)). Consequently, one can still consider a corresponding restriction of $\mathcal{A}$ on U, that is,

$$\mathcal{A}|_U = \pi^{-1}(U) \tag{8.18}$$

(see, e.g., [VS: Chapt. I, p. 5, (1.10)]), along with a relevant restriction of the curve γ, as in (8.7), according to (8.9) and (8.16).

Therefore, one can further look at whatever restriction of our previous argument, concerning (8.10) and (8.11), by referring to any open $V \subseteq U \subseteq X$, as well as to the respective section (topological $\mathbb{C}$-vector) space

$$\mathcal{A}(V) = (\mathcal{A}|_U)(V). \tag{8.19}$$

Yet, in this connection, see also, for instance, [VS: Chapt. I, p. 10, (2.10) and (2.12), along with p. 55, (11.40)].

8.1 Lagrangian Density and Its Variation

For convenience, we still mention throughout the subsequent discussion that we actually assume that

(8.20) we are given a Yang–Mills space X (see Definition 4.1) that satisfies (8.11).

Now, by extending within the present abstract setting the classical terminology, we further define as the *Yang–Mills action* the following "integral form" of the Yang–Mills functional, as the latter was defined by (5.11) (we retain the relevant notation here, for simplicity); thus, we set

$$\mathcal{YM}_{\mathcal{E}}(D) := \frac{1}{2}\int_X (R(D), R(D))vol = \frac{1}{2}\int_X tr(R \wedge *R) \tag{8.21}$$

(see also (7.22) in the foregoing). Yet, we still refer to (8.21), as the Lagrangian density of $\mathcal{E}$, relative to the given $\mathcal{A}$-connection D of $\mathcal{E}$.

Note 8.2 Concerning the notation employed in (8.21), the term

$$vol \tag{8.22}$$

therein stands for the "volume element" ω on X, as defined by (7.5)—a global (continuous) section of $\mathcal{A}$. On the other hand, the integral in (8.21) may be viewed in the sense of our previous comments in Scholium 7.1. Thus, we assume that

(8.23) the integral in (7.21) is understood as a continuous linear form on the topological vector space (algebra) $\mathcal{A}(X)$, while we also suppose that the integrand therein, being actually an element of $\mathcal{A}(X)$, has compact support.

Now, by looking again at the *principle of least action* (see (6.47)) and based on the continuity of the linear form (integral), as in (8.21), as well as on (8.6) and (2.36) for $n = 2$, one obtains (*variation of the Lagrangian density*)

$$\begin{aligned} \delta\mathcal{YM}_{\mathcal{E}}(D) &= \frac{1}{2}\delta\int_X (R, R)vol = \int_X \delta\frac{1}{2}(R, R)vol \\ &= \int_X (R, D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega}))vol = \int_X (\delta^2_{\mathcal{E}nd\mathcal{E}}(R), \tilde{\omega})vol. \end{aligned} \tag{8.24}$$

Accordingly, we finally conclude that

$$\delta\mathcal{YM}_{\mathcal{E}}(D) = 0 \quad \text{if and only if} \quad \delta^2_{\mathcal{E}nd\mathcal{E}}(R) = 0. \tag{8.25}$$

Therefore, in other words, one obtains that

(8.26) the critical points of the Yang–Mills functional—namely, those

(8.26.1) $$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

for which one has

(8.26.2) $$\delta\mathcal{YM}_{\mathcal{E}}(D) = 0$$

—are exactly the solutions of the Yang–Mills equations. Yet, this actually amounts to the same thing:

(8.26.3) the zeros of the variation of the Lagrangian density (see (8.24) and (8.26.2)) are exactly the solutions of the Yang–Mills equations.

We comment below, for clarity's sake, concerning the notation we applied in the preceding—in particular, in what refers to (8.21) and (8.24). So we now terminate the present section.

Scholium 8.1 By looking at the "variation of the Lagrangian density" we further remark that, by definition,

(8.27) the relation (8.24) is valid for any "integral"

(8.27.1) $$\int_X$$

(*viz.*, a continuous linear form on the topological vector space $\mathcal{A}(X)$), assuming that the latter space has a "separating topological dual," $\mathcal{A}(X)'$.

As a consequence, one thus infers that the first relation in (8.25), true for any element in $\mathcal{A}(X)'$, entails that the integrand in (8.24) is zero. Therefore, based further on the nondegeneracy of (the $\mathcal{A}$-metric) ρ on $\mathcal{A}$, one finally concludes the second relation in (8.25), while the converse is, of course, true. So this proves (8.25) and equivalently, (8.26) as well. ■

On the other hand, the preceding can still be related, of course, with our previous comments in Scholium 7.1, pertaining to a potential meaning of the "integral" as in (8.27.1).

9 Cohomological Classification of Yang–Mills Fields

Our purpose in this final section of the present chapter is to obtain, as the title indicates, an analogous cohomological classification for Yang–Mills fields to that already attained for Maxwell fields (see Volume I, Chapter IV of this study). Thus, more specifically, we intend to prove, by analogy with Chapter IV, (5.51), the following (set-theoretic) bijection:

$$\Phi^n_{\mathcal{A}}(X)^{\nabla} = \check{\mathbb{H}}^1(X, \mathcal{GL}(n, \mathcal{A}) \xrightarrow{\tilde{\partial}} M_n(\Omega^1)), \tag{9.1}$$

the particular items of which, along with the corresponding notation involved, we explain here.

Thus, the first members of (9.1) stands for the set of equivalence classes of Yang–Mills fields on X of rank $n \in \mathbb{N}$ (see (4.46), along with (4.14) therein), X being, as usual, an arbitrary, in principle, topological space, a common carrier space of the vector sheaves concerned (ibid.). Now, the second member of (9.1) denotes the first Čech hypercohomology set of X with respect to the 2-term $\mathbb{Z}$-complex on X:

$$0 \longrightarrow \mathcal{GL}(n, \mathcal{A}) \xrightarrow{\tilde{\partial}} M_n(\Omega^1) \longrightarrow 0 \longrightarrow \cdots, \tag{9.2}$$

the particular meaning of which we discuss below.

Thus, first we recall, concerning the *logarithmic derivation* $\tilde{\partial}$ as in (9.2) (see, e.g., Volume I, Chapt. I), that one sets

$$\tilde{\partial} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow M_n(\Omega^1), \tag{9.3}$$

such that

$$\tilde{\partial}(\alpha) := \alpha^{-1} \cdot \partial(\alpha) \tag{9.4}$$

for any $\alpha \in \mathcal{GL}(n, \mathcal{A})(U) = GL(n, \mathcal{A}(U))$, with U an open set in X. On the other hand, one still obtains (see [VS: Chapt. VI, p. 7, (1.33))

$$\tilde{\partial}(st) = Ad(t^{-1}) \cdot \tilde{\partial}(s) + \tilde{\partial}(t) \tag{9.5}$$

for any $s \equiv (s_{ij})$ and $t \equiv (t_{ij})$ in $GL(n, \mathcal{A}(U))$, as above, where we also set

$$Ad(s)(\tilde{\partial}(t)) \equiv Ad(s) \cdot \tilde{\partial}(t) := s \cdot \tilde{\partial} \cdot s^{-1}, \tag{9.6}$$

with s and t, as before. In particular, based on (9.5), one has

$$\tilde{\partial}(\alpha^{-1}) = -Ad(\alpha) \cdot \tilde{\partial}(\alpha) \tag{9.7}$$

for any $\alpha \equiv (\alpha_{ij}) \in \mathcal{GL}(v, \mathcal{A})(U)$, given that

$$\tilde{\partial}|_{\mathcal{GL}(n,\mathbb{C})=GL(n,\mathbb{C})} = 0 \tag{9.8}$$

(see also Chapt. I, (1.33), along with [VS: Chapt. VI, p. 7, (1.30)]). [Of course, the same relation (9.7) can still be obtained as a consequence of (9.4), in conjunction with the "quotient rule"; [loc. cit. p. 3, Lemma 1.2, or even p. 4, (1.12)]].

Thus, based further on our previous terminology in Volume I of this treatise (Chapt. IV, Section 1.2, as well as, on Sections 2–4 therein) pertaining to Čech hypercohomology, we are going to establish our present notation concerning the "Čech hypercohomology of the $\mathbb{Z}$-complex" (9.2), the latter being still non-abelian (!) due to the presence of the non-abelian group sheaf on X,

$$\mathcal{GL}(n, \mathcal{A}), \ n \geqq 2. \tag{9.9}$$

Now, taking into account Volume I: Chapter IV (3.11), and in view of (9.2) of this section, one further defines, relative to a given open covering of X, or even with respect to a local frame of any finite number of $\mathcal{A}$-modules involved,

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}, \tag{9.10}$$

the following $\mathbb{Z}$-modules, here non-abelian, in general (see (9.9), as above), groups:

$$\begin{aligned} \mathcal{F}^0 &= \check{C}^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})), \\ \mathcal{F}^1 &= \check{C}^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \oplus \check{C}^0(\mathcal{U}, M_n(\Omega^1)), \\ \mathcal{F}^2 &= \check{C}^2(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \oplus \check{C}^1(\mathcal{U}, M_n(\Omega^1)); \end{aligned} \tag{9.11}$$

the corresponding "differentials" between the $\mathbb{Z}$-modules (groups) concerned, as above, are given by the following relations (see Volume I, Chapt. IV, (4.17)–(4.19)):

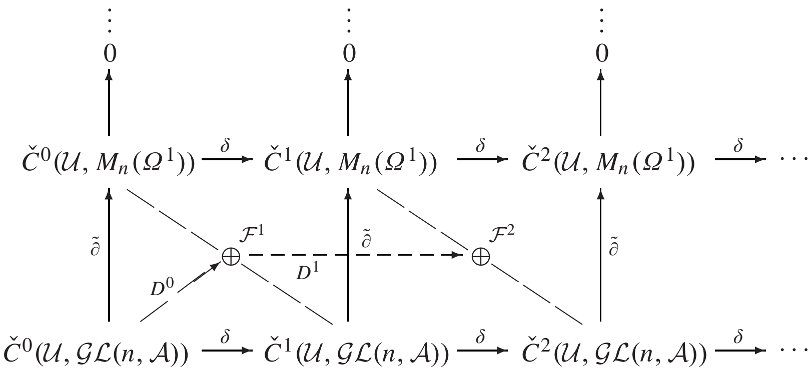
$$D^0 = \delta \oplus \tilde{\partial} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \tag{9.12}$$

as well as

$$D^1 = \delta \oplus (\delta - \tilde{\partial}) : \mathcal{F}^1 \longrightarrow \mathcal{F}^2. \tag{9.13}$$

Thus, one can depict the above in the following diagram (see also Volume I, Chapt. IV, (4.15))

(9.14)



Now, our next goal is to prove that

(9.15)
> the "solution space" of the operator D^1 (see (9.13) and (9.14)—namely, the space (subgroup)
>
> (9.15.1) $\ker D^1$
>
> —characterizes the set
>
> (9.15.2) $\Phi^n_{\mathcal{A}}(X)^{\nabla}$
>
> (i.e., the (set of equivalence classes of) Yang–Mills fields on X).

Accordingly—what amounts to the same thing—a given pair

(9.16) $$(g, \omega) \in \mathcal{F}^1 \equiv Dom D^1 := \check{C}^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \times \check{C}^0(\mathcal{U}, M_n(\Omega^1))$$

defines a Yang–Mills field on X if and only if one has

(9.17) $$(g, \omega) \in \ker D^1,$$

and, equivalently, if and only if

(9.18) $$D^1(g, \omega) = 0.$$

In point of fact, as we are going to see,

(9.19) relation (9.18) is "equivalence preserving" with respect to equivalent Yang–Mills fields on X.

Yet, the above explains the true meaning of our initial assertion pertaining to the bijection (9.1), the latter being the formal analogue of our previous conclusion for Maxwell fields in Volume I of this treatise (Chapt. IV, Lemma 4.1).

9.1 Local Characterization of Yang–Mills Fields

For terminological questions, related to the ensuing discussion, we shall occasionally refer to Section 4. Thus, given a Yang–Mills field

$$(\mathcal{E}, D) \tag{9.20}$$

on X, or rank $n \geqq 2$ ((4.12) and (4.13)), let

$$[(\mathcal{E}, D)] \in \Phi^n_{\mathcal{A}}(X)^{\nabla} \tag{9.21}$$

be its corresponding equivalence class, an element of the set (9.15.2), as indicated (see also (4.45) and (4.46)). On the other hand, by looking at things locally (i.e., in terms of a given local frame of $\mathcal{E}$, as, e.g., in (9.10)), one gets the following:

(9.22) local identification (characterization) of the pair $(\mathcal{E}, D)$, as above, according to a fundamental principle (*viz.*, the so-called *transformation law of potentials*), so that one obtains the one-to-one correspondence, locally (!),

$$\begin{aligned}(\mathcal{E}, D) \longleftrightarrow (g, \omega) &\equiv ((g_{\alpha\beta}), (\omega_\alpha)) \\ &\in \check{Z}^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \times \check{C}^0(\mathcal{U}, M_n(\Omega^1)) \\ &\underset{\rightarrow}{\subseteq} \check{C}^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \times \check{C}^0(\mathcal{U}, M_n(\Omega^1))\end{aligned} \tag{9.22.1}$$

if and only if the following holds true:

$$\omega^{(\beta)} = Ad(g^{-1}_{\alpha\beta}) \cdot \omega^{(\alpha)} + \tilde{\partial}(g_{\alpha\beta}). \tag{9.22.2}$$

Equivalently, we further set, for the last relation,

$$\delta(\omega^{(\alpha)}) \equiv \omega^{(\beta)} - Ad(g^{-1}_{\alpha\beta}) \cdot \omega^{(\alpha)} = \tilde{\partial}(g_{\alpha\beta}), \tag{9.23}$$

which we still write in the succinct form

$$\delta(\omega) = \tilde{\partial}(g). \tag{9.24}$$

See also Volume I, Chapter I, (2.56) and (2.71) as well as [VS: Chapt. VI, p. 113, Theorem 2.1, and p. 116, Theorem 3.1, or even p. 119, Theorem 3.2] for further details.

Note 9.1 By referring to the preceding discussion, we further notice here the formal resemblance of (9.24) to the analogous local characterization of Maxwell fields; see Volume I, Chapter III, Lemma 2.1, along with the corresponding scholium therein, as in (2.33), pertaining to a physical interpretation of the same relation, which has, of course, an obvious analogous substance for the case considered herewith. Yet, see the relevant comments therein, following (2.36). *In toto*, one thus concludes that, in general, (9.25) for any given Yang–Mills field

(9.25.1) $$(\mathcal{E}, D)$$

of rank $n \in \mathbb{N}$ (Maxwell fields correspond here to $n = 1$),

(9.25.2) the variance of the carrier agrees always with the one of the field itself (*viz.*, of the $\mathcal{A}$-connection concerned).

Thus, we arrive herewith, through (9.25.2), at what may still be construed as a physical interpretation of (9.24) referring to a Yang–Mills field, in general (*viz.*, including Maxwell fields), as in (9.25.1).

Therefore, to recapitulate our conclusions in terms of (9.24), we can say, once more, that

(9.26) one gets a one-to-one correspondence,

(9.26.1) $$(\mathcal{E}, D) \longleftrightarrow (g, \omega),$$

in the sense of (9.22), hence when $(\mathcal{E}, D)$ is locally determined (!) (this, of course, still reminds us of the "drastically local" character of quantum field theory, see, e.g., R. Haag [1: p. 326]), if and only if one has

(9.26.2) $$\delta(\omega) = \tilde{\partial}(g)$$

(however, see also (9.23) concerning the notation employed herewith).

Now, as already said in the preceding, we turn next to verify that

(9.27) equivalence of Yang–Mills fields amounts to an equivalence with respect to (the group) $\mathrm{im} D^0$; that is, one gets

(9.27.1) $$(\mathcal{E}, D) \sim (\mathcal{E}', D')$$

if and only if one has (locally)

(9.27.2) $$(g', \omega') - (g, \omega) = D^0(\eta),$$

where $\eta \in \check{C}^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$; see also (9.11) as well as (9.36.5) and (9.45).

In this context, we first recall that

(9.28) $$(\mathcal{E}, D) \sim (\mathcal{F}, D')$$

(*viz.*, one has an equivalence of Yang–Mills fields (or even, "gauge equivalent" Yang–Mills fields)), if and only if there exists (by definition) an $\mathcal{A}$-isomorphism

(9.29) $$\varphi \in \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{F}),$$

such that

$$D' = (\varphi \otimes 1) \circ D \circ \varphi^{-1} \equiv \varphi D \varphi^{-1} \equiv Ad(\varphi) \cdot D \equiv \varphi_*(D) \tag{9.30}$$

(*viz.*, D and D' are φ-related $\mathcal{A}$-connections of $\mathcal{E}$ and $\mathcal{F}$, respectively); see Section 4.3. Yet, we still have (see [VS: Chapt. V, p. 353, Lemma 2.1])

$$g' g^{-1} = \delta(\varphi), \tag{9.31}$$

where we set (see also (9.22.1)

$$\mathcal{E} \longleftrightarrow g \equiv (g_{\alpha\beta}) \quad \text{and} \quad \mathcal{E}' \longleftrightarrow g' \equiv (g'_{\alpha\beta}), \tag{9.32}$$

such that (9.31) is actually of the form (ibid. p. 354, (2.11))

$$g'_{\alpha\beta} = \varphi_\alpha g_{\alpha\beta} \varphi_\beta^{-1}, \tag{9.33}$$

which we may write, for convenience, in the succinct form

$$g' = Ad(\varphi) \cdot g \equiv \varphi^*(g), \tag{9.34}$$

such that

$$\varphi \equiv (\varphi_\alpha) \in \check{C}^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})). \tag{9.35}$$

Accordingly, to sum up the preceding,

(9.36)

gauge equivalence of two given Yang–Mills fields,

$$(\mathcal{E}, D) \sim (\mathcal{E}', D'), \tag{9.36.1}$$

when locally considered—namely, when one has

$$(g, \omega) \sim (g', \omega') \tag{9.36.2}$$

—is virtually referred to an $\mathcal{A}$-isomorphism

$$\varphi \in \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{E}'), \tag{9.36.3}$$

which is also $\mathcal{A}$-connection preserving—that is, in the sense that one has

$$D' = \varphi_*(D) \equiv Ad(\varphi) D. \tag{9.36.4}$$

Thus, locally and taking $\mathcal{E} = \mathcal{E}'$, one concludes that there exists a 0-cochain

$$\eta \equiv (\eta_\alpha) \in \check{Z}^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \underset{\rightarrow}{\subseteq} \check{C}^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})), \tag{9.36.5}$$

such that one gets

$$g' = Ad(\eta) \cdot g \equiv \delta(g) \tag{9.36.6}$$

as well as

$$\omega' = Ad(\eta)\omega + \tilde{\partial}(g). \tag{9.36.7}$$

Yet, by setting

$$\delta(\omega) \equiv \omega' - Ad(\eta)\omega, \tag{9.36.8}$$

one gets, equivalently, concerning (9.36.6), the relation

$$\delta(\omega) = \tilde{\partial}(g). \tag{9.36.9}$$

9.2 The Map (9.1)

Now, suppose we have a locally determined Yang–Mills field [*viz.*, the (one-to-one) correspondence (see also (9.22) and (9.24))]

$$(\mathcal{E}, D) \longleftrightarrow (g, \omega) \in \check{C}^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}) \times \check{C}^0(\mathcal{U}, M_n(\Omega^1)), \tag{9.37}$$

such that

$$(g, \omega) \in \ker D^1 \ [\textit{viz.},\ D^1(g, \omega) = 0]. \tag{9.38}$$

Hence, equivalently (see (9.13)), one has

$$D^1(g, \omega) \equiv (\delta \oplus (\delta - \tilde{\partial}))(g, \omega) = (\delta g, \delta\omega - \tilde{\partial}(g)) = 0, \tag{9.39}$$

so that one finally obtains, equivalently, for (9.38),

$$\delta(g) = 0 \quad \text{and} \quad \delta(\omega) = \tilde{\partial}(g), \tag{9.40}$$

where we recall that (see (9.36.6) and (9.36.8))

$$\delta(g) \equiv Ad(\eta) \cdot g \text{ as well as } \delta(\omega) \equiv \omega' - Ad(\eta) \cdot \omega. \tag{9.41}$$

On the other hand, one obtains (see also (9.36.2), (9.36.6), and (9.41))

$$(g', \omega') - (g, \omega) = (g'g^{-1}, \omega' - \omega) = (Ad(\eta), \tilde{\partial}(\eta)) \equiv (\delta\eta, \tilde{\partial}\eta); \tag{9.42}$$

hence,

$$g'g^{-1} = \delta(\eta),\ \text{thus } g' = \delta(\eta) \cdot g, \tag{9.43}$$

and

$$\omega' - \omega \equiv \omega' - Ad(\eta)\omega = \tilde{\partial}(g). \tag{9.44}$$

Therefore, in view of (9.12), one gets

$$(g', \omega') - (g, \omega) = D^0(\eta) = (\delta \oplus \tilde{\partial})(\eta) = (\delta(\eta), \tilde{\partial}(\eta)). \tag{9.45}$$

Thus, the last relation, in conjunction with (9.38) and (9.40), still entails that

$$D^1 D^0 = 0, \tag{9.46}$$

hence, equivalently,

$$\mathrm{im} D^0 \subseteq \ker D^1. \tag{9.47}$$

Now, we further set

$$[(g, \omega)] := (g, \omega) + \mathrm{im} D^0, \tag{9.48}$$

the first member of (9.48) standing for an equivalence class of Yang–Mills fields, of order, say, $n \in \mathbb{N}$—namely, for an element of the set

$$\Phi^n_{\mathcal{A}}(X)^{\nabla}. \tag{9.49}$$

The above also explains our notation in (9.1), the second member of the same relation denoting, according to (9.48), the quotient set

$$\ker D^1 / \mathrm{im}\, D^0; \tag{9.50}$$

precisely speaking, the corresponding inductive limit set, relative to the (upward directed) set of (proper) local frames of a given Yang–Mills field $(\mathcal{E}, D)$ on X. ■

This also terminates our discussion on the asserted (set-theoretic) bijection (9.1), along with the notation that was actually employed therein, by extending the analogous simpler case for Maxwell fields, due to the abelianess of the corresponding groups involved there, as exposed in Volume I, Chapter IV of this treatise.

2

Moduli Spaces of $\mathcal{A}$-Connections of Yang–Mills Fields

> "Gauge theories possess an infinite-dimensional symmetry group . . . and all physical, or geometric properties are gauge invariant"
>
> M. Atiyah in *The Geometry and Physics of Knots* (Cambridge University Press, Cambridge, 1990). p. 3.

Our purpose in this chapter is to expose, in the abstract language that we employ throughout this treatise, the fundamentals of the classical theory indicated by the subject in the title. Roughly speaking, we want to put into perspective the classical and physically, yet mathematically, important(!) theme of the so-called *geometry of Yang–Mills equations*. This was first advocated by I.M. Singer [1] (see also, for instance, M.F. Atiyah [1: p. 2]). Equivalently, one considers the corresponding space of solutions of the said equations, thus, by definition (see Chapt. I, Definitions 4.1 and 4.2), the *space of the Yang–Mills $\mathcal{A}$-connections*. However, in view of the physical significance of the "gauge invariant ($\mathcal{A}$-)connections" (see Atiyah's phrasing in the epigraph above), the same space is finally divided out by the corresponding "gauge group," so that it is, in effect, the resulting quotient space ("moduli space," or even "orbit space") that is under consideration.

Our treatment of this material in this chapter is in accordance with our general goal—to do physics in terms of the standard differential geometry of smooth manifolds by means of the "abstract ("modern") differential geometry," since the latter aspect has been started and further exposed in A. Mallios [VS: Vols. I and II]. In this connection, the standard work of Singer is still made in a differential geometric manner, within the context of (smooth) fiber bundle theory.

On the other hand, in Section 7 of Chapter 3 we further consider, according to the standard patterns, (global) *gauge fixing* (Gribov's ambiguity) by still following the classical account thereof of I.M. Singer (loc. cit.): Indeed, Singer's pioneering work was also the main motivation for all the subsequent discussion. Our primary objective here is to put into an abstract perspective the relevant classical material.

1 Preliminaries: The Group of Gauge Transformations or Group of Internal Symmetries

As the title of this section indicates, we consider here the group of transformations. The same group has been discussed in Chapter I, Section 5.1, to consider (Chapt. I, Section 5.2) the "gauge invariance" of the Yang–Mills functional, as well as in

A. Mallios, *Modern Differential Geometry in Gauge Theories: Yang–Mills Fields, Volume II*, DOI: 10.1007/978-0-8176-4634-9_2,

Chapter II, Section 9 of Volume I in connection with the concept of a "local gauge" of a given vector sheaf (loc. cit. (9.8), (9.9) and (9.11)). In this context, see also [VS: Chapt. VI, Section 17], where the same theme is examined.

So, to start with, and in accordance with our general point of view—that is, impose the least possible preassumptions on the particular framework employed—suppose that

(1.1) we are given a $\mathbb{C}$-algebraized space,

(1.1.1) $$(X, \mathcal{A}),$$

while we also let $\mathcal{E}$ be a vector sheaf on X, such that

(1.1.2) $$rk_{\mathcal{A}}\mathcal{E} \equiv rk\mathcal{E} = n \in \mathbb{N}.$$

Now, in this context, we first recall that, by definition (see Chapt. I, (5.19)),

(1.2) the group of gauge transformations of $\mathcal{E}$ is given as follows:

(1.2.1) $$\begin{aligned} Aut_{\mathcal{A}}(\mathcal{E}) \equiv Aut\mathcal{E} &:= (\mathcal{A}ut\mathcal{E})(X) := (\mathcal{E}nd\mathcal{E})^{\bullet}(X) \\ &= (\mathcal{E}nd\mathcal{E})(X)^{\bullet} \equiv (End\mathcal{E})^{\bullet}. \end{aligned}$$

In other words, and according to our previous convention, as in (1.2.1), the group under discussion,

(1.3) $$Aut_{\mathcal{A}}(\mathcal{E}) \equiv Aut\mathcal{E},$$

is that one of the $\mathcal{A}$-automorphisms of $\mathcal{E}$ ($\mathcal{A}$-isomorphisms of the given vector sheaf $\mathcal{E}$ onto itself). So, by taking the very definition of a sheaf morphism into account (see [VS: Chapt. I, p. 11, Proposition 2.1]), one gets an equivalent and, occasionally, more convenient expression of the latter notion, through a (uniquely defined) morphism of the corresponding (complete) presheaves of (continuous local) sections of the sheaves concerned (loc. cit., Chapt. I, p. 75, (13.19)). Thus, by looking at a particular element (e.g., "gauge transformation" of $\mathcal{E}$), say,

(1.4) $$\phi \in Aut\mathcal{E} = (\mathcal{A}ut\mathcal{E})(X),$$

that is, by definition (see (1.3)), an $\mathcal{A}$-automorphism of the given vector sheaf $\mathcal{E}$, one obtains, equivalently, a map (i.e., a morphism of (complete) presheaves, as explained above),

(1.5) $$\phi = (\phi_U),$$

in such a manner that one defines

(1.6) $$\begin{aligned} \phi_U \equiv \phi|_U \in \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(U) &= Isom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \\ \equiv Aut_{\mathcal{A}|_U}(\mathcal{E}|_U) &\equiv Aut(\mathcal{E}|_U) = (\mathcal{A}ut\mathcal{E})(U) \end{aligned}$$

for any open set $U \subseteq X$. Therefore, in other words, one concludes that

(1.7) a local $\mathcal{A}$-automorphism of $\mathcal{E}$, that is, what we also call a "local gauge" of $\mathcal{E}$, is technically speaking just a

(1.7.1) local (continuous) section (over an open $U \subseteq X$) of $\mathcal{A}ut\mathcal{E}$, the latter group being still called, besides the one in (1.3), the "group of gauge transformations of $\mathcal{E}$." (See also Volume I, Chapter I, (6.12).)

The above rather strange terminology, when referring to the word gauge (the latter being usually associated with "coordinates"), is, indeed, fully explained by the very definition of $\mathcal{E}$ as a vector sheaf on X, of finite rank $n \in \mathbb{N}$ (see (1.10.1) below, along with (1.14.2)). Yet, concerning the notation applied in (1.4), or even in (1.2.1), see also Volume I, Chapter I of the present treatise—in particular, (6.10) and (6.12). In this context, see also (6.38).

Note 1.1 (Terminological) The term

(1.8) $$\textit{local gauge} \text{ of } \mathcal{E},$$

with $\mathcal{E}$ a given vector sheaf on X (see (1.1.1)) will also be used below, and this was similar to the case in the preceding, for an

(1.8′) open $U \subseteq X$, for which (1.10.1) in the sequel holds.

The distinction between these two notions (*viz.*, domain of definition of a function and the function itself) becomes certainly clear from the context (!). See also [VS: Chapt. II, p. 126, Definition 4.2, along with Chapt. V, p. 351, (1.15), or even Chapt. VII, p. 99, (1.5)]). Yet, see also Volume I, Chapter II—in particular (9.19)–(9.23).

Now, without loss of generality, and more importantly (yet, naturally), based on our hypothesis for $\mathcal{E}$ (see for instance, Volume I, Chapt. I of this treatise, (6.24)), one can instead, when looking at (1.5), a local frame of $\mathcal{E}$ (in place of the whole topology of X), say,

(1.9) $$\mathcal{U} = (U)$$

in the sense that

(1.10) for any local gauge (see (1.8) and (1.8′)), $U \in \mathcal{U}$, one obtains

(1.10.1) $$\mathcal{E}|_U = \mathcal{A}^n|_U$$

within an $\mathcal{A}|_U$-isomorphism of the $\mathcal{A}|_U$-modules concerned (see also (1.1.2)).

Of course, (1.9) still yields, by assumption, an open covering of X, for which (1.10.1) holds true; thus, one has, by definition, what we call a "local frame" of $\mathcal{E}$ (see [VS: Chapt. II, p. 126, Definition 4.2]). Accordingly, one thus concludes that

(1.11) by considering an $\mathcal{A}$-automorphism of $\mathcal{E}$, that is, an element

(1.11.1) $$\phi \in Aut\mathcal{E} \equiv (\mathcal{A}ut\mathcal{E})(X),$$

one can look at it locally, as in (1.5) and (1.6), by means of a local frame of $\mathcal{E}$, as in (1.9), such that (1.10.1) is valid.

Our conclusion in (1.11) exemplifies the special role that the local frames of a given vector sheaf $\mathcal{E}$, as above, can play in any argument pertaining to calculations in terms of $\mathcal{E}$. Yet, in this context, we have already noted in the preceding, based on our remark in Volume I, Chapter I, (6.24), the basic fact that

(1.12) any given local frame of a vector sheaf $\mathcal{E}$ on an (arbitrary) topological space X may be converted into a basis of the topology of X.

On the other hand, in the particular case where one has a paracompact (Hausdorff) space X, one further concludes that

(1.13) the local frames of a given vector sheaf $\mathcal{E}$ on X provide a cofinal subset of the set of locally finite open coverings of X.

The above is, indeed, a straightforward consequence of the very definitions (see also [VS: Chapt. II, p. 127, (4.9)]). ■

On the other hand, it might also be useful in some other context, pertaining, for example, to topologies defined through the open covering of the type considered in (1.13) (*Sorkin's topology*; see R.D. Sorkin [1]), in conjunction with applications of ADG (abstract differential geometry, as it is advocated by the present study) in questions, related even with problems of quantum relativity ("finitary algebra sheaves" and the like); see, for instance, I. Raptis [1, 2] as well as H.F. de Groote [1].

Consequently, when looking at

(1.14) a local gauge of $\mathcal{E}$, in the sense of (1.7)—namely, at an element,

(1.14.1) $$\phi|_U \equiv \phi_U \in (\mathcal{A}ut\mathcal{E})(U) = Aut(\mathcal{E}|_U)$$

(see (1.6))—one finally concludes by virtue of (1.10.1) and (1.6) that

(1.14.2) $$\begin{aligned} \phi|_U \in Aut(\mathcal{E}|_U) &= Aut(\mathcal{A}^n|_U) = (\mathcal{A}ut\mathcal{A}^n)(U) \\ &= \mathcal{GL}(n, \mathcal{A})(U) = GL(n, \mathcal{A}(U)), \end{aligned}$$

for any open $U \subseteq X$ for which (1.10.1) is valid—in other words, a "local gauge" of $\mathcal{E}$—in the sense of (1.8), while we still take into account (1.12).

Therefore, one comes to the conclusion that

(1.15) when looking at a "local gauge" of a given vector sheaf $\mathcal{E}$ on X, in either the sense of (1.10.1) or even of (1.6), one is led, in effect, to deal with (see also (1.1.2))

(1.15.1) an $n \times n$ matrix of "generalized coordinates"—namely, local (continuous) sections of the "structure sheaf" alias of our (generalized) "arithmetics" $\mathcal{A}$.

That is, one has, in view of (1.14.2),

(1.16)
$$\phi|_U \equiv \phi_U \equiv (\alpha_{ij}^{(U)}) \in GL(n, \mathcal{A}(U)),$$
such that
$$\alpha_{ij}^{(U)} \in \mathcal{A}(U), \quad 1 \leq i, \quad j \leq n = rk\mathcal{E},$$

for any open $U \in \mathcal{U}$, as in (1.9), while (1.12) is taken into account as well. Yet, in this context, see also Volume I, Chapter II, Section 9, where we considered a local gauge of $\mathcal{E}$ in terms of a map, in point of fact, an $\mathcal{A}|_U$-isomorphism of $\mathcal{A}|_U$-modules,

(1.17)
$$\phi|_U \equiv \phi_U : \mathcal{E}|_U \longrightarrow \mathcal{A}^n|_U \cong (\mathcal{A}|_U)^n,$$

that is an analogous situation to (1.10.1). However, (1.17) is actually referred now to a "local instance" (section) of a corresponding local isomorphism of the group sheaves of units $\mathcal{A}ut\mathcal{E}$ and $\mathcal{GL}(n, \mathcal{A})$ of (the $\mathcal{A}$-algebra sheaves) $\mathcal{E}nd\mathcal{E}$ and $\mathcal{E}nd\mathcal{A}^n = M_n(\mathcal{A})$, respectively. On the other hand, one is still led, locally, to a similar conclusion, as in (1.16); see Volume I Sect. 9, (9.19)–(9.22), along with (9.33), therein; see also [VS: Chapt. V, (1.13) and (1.16)]. The preceding is thus an outcome of the basic relations,

(1.18)
$$(\mathcal{A}ut\mathcal{E})(U) = Aut(\mathcal{E}|_U) = Aut(\mathcal{A}^n|_U) = (\mathcal{A}ut\mathcal{A}^n)(U),$$

which is valid once (1.17) is in force ! (See also Volume I, Chapt. II, (9.33).)

Now, the above, in conjunction with our previous comments following (1.13), points out still further the significance that finally has our structure sheaf $\mathcal{A}$, concerning our calculations (see Chapt. IV, Sections 6 and 8), a ($\mathbb{C}$-algebra) sheaf that we adopt each time! Therefore, to state it once again (!), one concludes that

(1.19) locally, all our calculations are actually reduced—namely, they are made, in effect, through (continuous) local sections of

(1.19.1)
$$\mathcal{A} \underset{\varepsilon \leftarrow}{\supset} \mathbb{C}$$

(see Volume I, Chapt. II, (9.40))—that is, via local sections of our "generalized arithmetics," which, at the very end, we have chosen (it is important now, by employing ADG, that the choice is actually ours (!)) each time.

Equation (1.19.1) certainly reminds us of the general principle that

(1.20) "*...classical observables ...and quantum observables ...are* [always] *on the same structural footing*." (!)

See for instance, H.F. de Groote [1]. Yet, in this context, we can still refer to our previous relevant remarks in Volume I, Chapter III, (1.22) and (1.23), in conjunction with the corresponding description of photons, within the present abstract setting (ibid. (1.19)).

Now, concerning the relevance of the foregoing to the classical theory (of differential geometry), referred to as smooth (*viz.*, $\mathcal{C}^\infty$-) manifolds (CDG), the previous setup is expressed in terms of differentiable ($\mathcal{C}^\infty$-, yet "smooth") functions—in point of fact, sections of the respective (classical) "structure sheaf" of the theory:

(1.21)
$$\mathcal{A} = \mathcal{C}^\infty_X.$$

See Volume I, Chapter II, (6.2′) and/or (6.25), along with (3.4) and (3.4′), therein.

Consequently, as we have already remarked, one usually employs—and this still happens in the classical theory rather exclusively (!)—an abuse of terminology by referring to the (non-abelian, in general, unless $n = 1$) group sheaf on X,

(1.22)

(1.22.1)
$$\mathcal{GL}(n, \mathcal{A})$$

(see Volume I, Chapt. I, (1.31), (1.35)), as the "gauge group" of a given vector sheaf $\mathcal{E}$ on X, of rank $n \in \mathbb{C}$, in place, of course, of the group sheaf on X,

(1.22.2)
$$\mathcal{A}ut\mathcal{E},$$

as the latter group was given by (1.3) or (1.2) in the foregoing. (In point of fact, we considered therein the respective group of global sections of the above; see also [VS: Chapt. I, p. 75, (13.19)]). However, the abuse of language is certainly clear and occasionally, in practice, acceptable, in the sense always of (1.14) and in also conjunction with (1.11) and (1.12).

Thus, on the basis of (1.22), we shall also employ the practice of abusing—namely, the relevant terminology applied.

1.1 The Internal Symmetry Group, as the Group of Gauge Transformations

The group at issue in this section is actually connected with the "interpretation" (description!) of elementary physical systems (particles) in terms of "principal sheaves," which was already considered in Volume I, Chapter II, Section 8. Indeed, as we have mentioned, the (internal) symmetry group, or the group of (internal) symmetries of the physical system under consideration, refers to the "structure group (sheaf)"

(1.23.1) $$\mathcal{G}$$

(1.23) of the principal ($\mathcal{G}$-)sheaf, say $\mathcal{P}$, on X (the latter space being our standard arbitrary, in general, topological space, carrier of all the sheaves involved, herewith), describing the physical system at issue. So, by definition (Chapt. II, Definition 8.1),

(1.23.2) $\mathcal{P}$, as above, is a principal homogeneous $\mathcal{G}$-space.

Now, the same group (sheaf) $\mathcal{G}$ (*viz.*, sheaf of groups on X; see [VS: Chapt. II, p. 86, Definition 1.1]) as in (1.23.1) above is still called the gauge group (sheaf) of $\mathcal{P}$, or of the physical system concerned. The latter terminology is still supported in view of the natural—according to the very definition of $\mathcal{P}$—action of $\mathcal{G}$ on $\mathcal{P}$ (see also Volume I, Chapter II of the present study, Section 8, in particular (8.9)).

However, we actually consider, and this is also the case in the classical theory, a certain representation of $\mathcal{G}$—that is, a group sheaf morphism,

$$\rho : \mathcal{G} \longrightarrow \mathcal{A}ut\mathcal{E} = (\mathcal{E}nd\mathcal{E})^{\bullet} \subseteq \mathcal{E}nd\mathcal{E} \tag{1.24}$$

(Chapt. II, (9.2)), where $\mathcal{E}$ is a vector sheaf on X "associated with $\mathcal{P}$" via the given representation ρ, as in (1.24) above (see also Chapt. II, (9.5)). Thus, henceforth, by referring to the

(1.25) group (in point of fact, sheaf of groups, or even group sheaf) of gauge transformations of a physical system, we virtually mean, unless otherwise stated, the one defined by (1.2.1), thus corresponding to a given representation of the symmetry group of the system at issue, as in (1.24).

Yet, in this connection, we also recall the so-called "symmetry axiom" according to which

(1.26) the symmetry group (sheaf) is the same at the classical and the quantum régime alike.

See also Volume I, Chapter II, the remarks following (3.13). The same might also be related with our previous comments in (1.20).

Now, as already said in the introduction to this chapter, our main objective throughout the present discussion is to put into perspective, within our abstract setup, the classical material pertaining to the so-called "geometry of Yang–Mills fields"—that is, to examine, in terms mainly of differential-geometric notions (classical or not), the space of $\mathcal{A}$-connections of a given Yang–Mills field,

$$(\mathcal{E}, D) \tag{1.27}$$

(see, for instance, Chapt. I, (2.15), or (4.12)–(4.13), therein) on a topological space X—in other words, the "geometry" of what we have called in the preceding, space of $\mathcal{A}$-connections of $\mathcal{E}$,

(1.28) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$$

(see, e.g., Chapt. I, (5.10)). However, in view of the paramount importance of the "gauge invariance" of the means (notions), through which one looks at a particular system, we actually consider the quotient space

(1.29) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E},$$

which will be, by definition (see also (2.9) and (2.14)), the "moduli space" of $\mathcal{E}$. We deal with this matter in the next section, while we continue this study in Chapter III, as it particularly concerns "structural properties" (in effect, differential-geometric ones) of the same space.

On the other hand, within the same vein of ideas, we can further remark, in accordance with *Klein's point of view*, that

(1.30) the "geometry" of any substance (object, alias structure) is actually characterized by an (abstract) group

such that for the case under consideration (*viz.*, the "geometry of Yang–Mills fields"), the respective situation is reduced to that of the moduli space of $\mathcal{E}$, as given by (1.29).

Accordingly, by looking at the given vector sheaf $\mathcal{E}$ (substance/"space") as before, this is (can be) "identified," in effect, through its group of automorphisms,

(1.31) $$\mathcal{A}ut_{\mathcal{A}}(\mathcal{E}) \equiv \mathcal{A}ut\mathcal{E}$$

(see also, e.g., (1.24)), to which the "type" of any particular (structural) quantity—for instance, an $\mathcal{A}$-connection D of $\mathcal{E}$—should obey, being thus characterized, in the case at issue, by means of the corresponding "transformation law" (of potentials), when looking at D, equivalently, via a local frame of $\mathcal{E}$ (see, for instance, Volume I, Chapt. I of this study, (2.54), (2.56.1), or even (2.71) therein). In this context, see also A. Mallios [9: (3.26)] as well as, H. Weyl [1: p. 16].

On the other hand, concerning the "group action" of (1.31) on the set (1.28) (see Section 2), being already indicated by (1.29), and in conjunction with our previous comments in (1.23), we really have, in complete analogy with what actually happens in the classical case, the next basic result, being also in accord with our assumption in (1.25) (yet, see (2.5)). Thus, one concludes that

(1.32) there does exist an "identification" (bijection) between the set of $\mathcal{A}$-connections of a given vector sheaf $\mathcal{E}$ on X and those of the principal ($\mathcal{GL}(n, \mathcal{A})$-)sheaf of the "local frames" (local gauges) of $\mathcal{E}$, where

(1.32.1) $$n = rk\mathcal{E} \in \mathbb{N}.$$

For details of the proof of (1.32) and the relevant terminology applied above, see E. Vassiliou [1: p. 246, Theorem 5.5]. ■

Yet, in this context, see also our previous discussion in Volume I, Chapter II, Sections 8 and 9 of this treatise concerning "principal $\mathcal{G}$-sheaves" and corresponding physical interpretations.

2 Moduli Space of $\mathcal{A}$-Connections

To start with, consider a vector sheaf $\mathcal{E}$ on X, for which one has

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \neq (0); \tag{2.1}$$

that is, we assume that $\mathcal{E}$ admits nontrivial $\mathcal{A}$-connections, the first member of (2.1) denoting, as usual, the affine space of $\mathcal{A}$-connections of $\mathcal{E}$ modeled on the vector sheaf (we still assume here that Ω^1 is a vector sheaf on X, too)

$$(\mathcal{E}nd\mathcal{E}) \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{E}nd\mathcal{E}) \tag{2.2}$$

(see Chapt. I, (5.6)). Of course, we already assumed that we were also given, as always, a differential triad

$$(\mathcal{A}, \partial, \Omega^1) \tag{2.3}$$

on a topological space X—the base space of a given $\mathbb{C}$-algebraized space

$$(X, \mathcal{A}) \tag{2.4}$$

—this same X being still the base space of any $\mathcal{A}$-module (in particular, vector sheaf) involved in our discussion. In this connection, see also Volume I, Chapter I for the terminology. Yet, concerning our previous assertion in (2.1), see Volume I, Chapter I, Sections 5 and 6.

On the other hand, "gauge theories" are not, by their very nature, sensible against objects that differ by a gauge transformation (see (1.2.1), (1.4), (1.5), or even (1.11)), or to put it, differently, they just seem to recognize objects that transform exactly, according to an appropriate changing of local gauges. Yet, in other words,

(2.5) we shall have isolated a (physical) conservation law, hence, the eventual substance of a physical object, whenever we have recognized a gauge invariant physical process. (See, for instance, the standard *isotopic spin conservation* (Yang–Mills).)

In this connection, see also S.A. Selesnick [2: p. 225] as well as the epigraph of M.F. Atiyah at the start of the chapter. Yet, what seems probably to be of a particular significance is the fact that such conservation laws appear to be inherent to the physical object itself; namely, one also might say, they are dependent on the particular symmetry (gauge) group of the object under consideration, as, for instance, on

(2.6)

(2.6.1) $$\mathcal{GL}(n, \mathcal{A})|_U = \mathcal{A}ut\mathcal{E}|_U$$

(see Volume I, Chapt. I, (6.26)), and not on the "surrounding space" (as, e.g., "space–time manifold").

Accordingly, what we call

(2.7) "gauge invariance" refers to something that remains invariant with respect to the group of automorphisms (see (2.6.1)) of the object (field) itself at issue, and not relative to anything else outside of it (!).

The phenomenal entanglement of the "structure sheaf" $\mathcal{A}$ in (2.6.1) does not essentially refer to the space X, but to our "arithmetics"—that is, to our own choice of a mechanism to describe matters—while this (see also [VS: Vols. I and II]) does not, have much to do (if anything at all (!), see, for instance, cohomology, loc. cit.) with the space X itself! Instead, the same mechanism (as it concerns differential-geometric concepts, at least) seems to depend just on $\mathcal{A}$ (!), along, of course, with the "differential" paraphernalia that accompany it, as the case each time might demand.

Thus, as a first consequence of the preceding, we identify

(2.8) $\mathcal{A}$-connections that are gauge equivalent;

that is, for any given vector sheaf $\mathcal{E}$ on X (see (2.3) and (2.4)), for which (2.1) holds true, we consider the space—namely, quotient set (see also (2.10) below)

(2.9) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E} = \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/(\mathcal{E}nd\mathcal{E})^{\bullet}.$$

Of course, we have set above, according to the very definitions,

(2.10) $$\mathcal{A}ut\mathcal{E} = (\mathcal{E}nd\mathcal{E})^{\bullet};$$

that is, the group sheaf (of germs) of $\mathcal{A}$-automorphisms of $\mathcal{E}$ is exactly the group sheaf of units of the $\mathcal{A}$-algebra $\mathcal{E}nd\mathcal{E}$ (see [VS: Chapt. II, p. 138, (6.29), along with Chapt. I, p. 87, (17.3) and Chapt. V, p. 390, Scholium 8.2. Yet, see Volume I, Chapter II, p. 138, Definition 6.2 of this treatise]. In particular, for $\mathcal{E} = \mathcal{A}^n$, one obtains (see also ibid., Chapt. I, (1.24) and (1.35)),

(2.11) $$(\mathcal{E}nd\mathcal{A}^n)^{\bullet} \equiv M_n(\mathcal{A})^{\bullet} = \mathcal{GL}(n, \mathcal{A}),$$

a fact that will also be of use below, as one still has, by applying in (2.11) the global section functor,

(2.12) $$M_n(\mathcal{A})^{\bullet}(X) = M_n(\mathcal{A}(X))^{\bullet} = GL(n, \mathcal{A}(X)) = \mathcal{GL}(n, \mathcal{A})(X).$$

On the other hand, as already explained in the foregoing (see our remarks in (1.11) and (1.12)),

(2.13) one still employs, in practice, the group sheaf

(2.13.1) $\mathcal{GL}(n, \mathcal{A})$, in place of $\mathcal{A}ut\mathcal{E}$, with $n = rk\mathcal{E}$.

See also (1.22), as above. Accordingly, one further applies a (conscious!) abuse of notation (loc. cit.) by assuming, in place of (2.9), the relation (see also (2.11))

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{GL}(n, \mathcal{A}) = \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/M_n(\mathcal{A})^{\bullet} \tag{2.14}$$

as a definition for the moduli space of $\mathcal{E}$.

Of course, to say it, once more (!), by virtue of (2.6.1),

(2.15) the above two definitions (2.9) and (2.14), referring to the moduli space of $\mathcal{E}$, are locally equivalent;

that is, for any local gauge U of $\mathcal{E}$ (see (1.10.1)), one has

$$(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E})|_U = (\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{GL}(n, \mathcal{A}))|_U, \tag{2.16}$$

within a bijection of the sets concerned. Indeed, our assertion in (2.16) is, in effect, an immediate consequence of our previous considerations, as in (1.5), (1.6), and (1.14); see also (2.20). ■

Thus, the above, in conjunction with our preceding remark in (1.13), justifies, in practice, the

(2.17) use of either one of the two definitions (2.9) or (2.14) as the moduli space of $\mathcal{E}$.

Now, we still recall the rudiments of the relevant notation in (2.9) by remarking that, as already said in the preceding (see also Volume I, Chapt. I, Section 6),

(2.18) the group sheaf $\mathcal{A}ut\mathcal{E}$ acts on the set (affine space) of $\mathcal{A}$-connections of $\mathcal{E}$, $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$.

—that is, one has the map

$$\mathcal{A}ut\mathcal{E} \times \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \longrightarrow \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{2.19}$$

given by the relation (see also loc. cit. (6.45))

$$\begin{aligned}(\phi, D) \mapsto \phi \cdot D &:= (\phi \otimes 1) \circ D \circ \phi^{-1} \\ &\equiv \phi D \phi^{-1} \equiv Ad(\phi) \cdot D \equiv \phi_*(D) \qquad (1 \equiv 1_{\Omega^1})\end{aligned} \tag{2.20}$$

for any pair (ϕ, D) in the source of the map (2.19). Indeed, the map (2.19) yields a group action, given that, by virtue of (2.20), it also satisfies the two following relations: (i)

$$1_{\mathcal{E}} \cdot D = D, \text{ with } D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{2.21}$$

and (ii)

$$(\phi \circ \psi) \cdot D = \phi \cdot (\psi D) \tag{2.22}$$

for any

$$\phi \text{ and } \psi \text{ in } \mathcal{A}ut\mathcal{E} := \mathcal{I}som_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \tag{2.23}$$

and $D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, where $1_{\mathcal{E}}$ in (2.21) stands, of course, for the "identity $\mathcal{A}$-automorphism" of $\mathcal{E}$. (Equations (2.21) and (2.22) are indeed immediate consequences of the very definitions and (2.20).) Yet, we prove that (2.20) is well defined in the sense that one really has

$$\phi \cdot D := (\phi \otimes 1) \circ D \circ \phi^{-1} \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{2.24}$$

for any (ϕ, D), as in (2.20). Thus, by the hypothesis for D and the very definition of (2.24), one obtains

$$\phi \cdot D \in \mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \Omega^1(\mathcal{E})). \tag{2.25}$$

On the other hand, we further verify that

(2.26) the $\mathbb{C}$-linear morphism $\phi \cdot D$, as in (2.25), fulfills also the *Leibniz condition*—namely, one has the relation (see Volume I: Chapt. I, (2.3))

$$(\phi \cdot D)(\alpha \cdot s) = \alpha \cdot (\phi \cdot D)(s) + s \otimes \partial(\alpha) \tag{2.26.1}$$

for any $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$, with U open in X.

Indeed, one has, in view of (2.24),

$$\begin{aligned}(\phi \cdot D)(\alpha \cdot s) &\equiv ((\phi \otimes 1)D\phi^{-1})(\alpha \cdot s) = ((\phi \otimes 1)D)(\phi^{-1}(\alpha \cdot s))\\ &= ((\phi \otimes 1)D)(\alpha \cdot \phi^{-1}(s)) = (\phi \otimes 1)(D(\alpha \cdot \phi^{-1}(s)))\\ &= (\phi \otimes 1)(\alpha \cdot D(\phi^{-1}(s) + \phi^{-1}(s) \otimes \partial(\alpha))\\ &= \alpha \cdot (\phi \otimes 1)(D(\phi^{-1}(s))) + (\phi \otimes 1)(\phi^{-1}(s) \otimes \partial(\alpha))\\ &= \alpha \cdot ((\phi \otimes 1)D\phi^{-1})(s) + s \otimes \partial(\alpha),\end{aligned} \tag{2.27}$$

which, of course, proves (2.26.1), and therefore, along with (2.25), the desired equation (2.24) as well. In our previous calculations in (2.27) we have employed the obvious relation

$$\phi \otimes 1 \in \mathcal{E}nd(\Omega^1(\mathcal{E})) \equiv \mathcal{H}om_{\mathcal{A}}(\Omega^1(\mathcal{E}), \Omega^1(\mathcal{E})), \tag{2.28}$$

according to the very definitions. ■

In point of fact, by further looking at the preceding relation (2.28), given that ϕ and $1 \equiv 1_{\Omega^1}$ are, in effect, $\mathcal{A}$-automorphisms of the vector sheaves concerned (see (2.23) and (2.20) as well as (2.2)), one actually obtains, more precisely, concerning (2.28), that

$$\phi \otimes 1 \in \mathcal{A}ut(\Omega^1(\mathcal{E})) = (\mathcal{E}nd(\Omega^1(\mathcal{E})))^{\bullet}. \tag{2.29}$$

In particular, $\phi \otimes 1$ is thus an $\mathcal{A}$-endomorphism of the $\mathcal{A}$-module (in fact, vector sheaf)

(2.30) $$\Omega^1(\mathcal{E}) \equiv \Omega^1 \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \Omega^1,$$

what we just used, for that matter, in (2.27), as it was, of course, already hinted at, by (2.28).

Now, as an outcome of the above discussion (see (2.18)), we conclude that

(2.31) given a vector sheaf $\mathcal{E}$ on X, the group (sheaf) of $\mathcal{A}$-automorphisms of $\mathcal{E}$,

(2.31.1) $$\mathcal{A}ut\mathcal{E},$$

acts on the set of $\mathcal{A}$-connections of $\mathcal{E}$,

(2.31.2) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}).$$

Therefore, one can further consider, according to the general theory of "transformation group sheaves," the corresponding orbit space of the action at issue, denoted by

(2.31.3) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E}.$$

Thus, we also say that

(2.32) the set $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ becomes an $\mathcal{A}ut\mathcal{E}$-set, or even an $\mathcal{A}ut\mathcal{E}$-space.

Now, the notation in (2.31.3) is justified by the fact that

(2.33) the above established action (2.19) yields, in point of fact, an equivalence relation on $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ (*viz.*, on the corresponding *"action space"*); yet, the same equivalence relation coincides, in effect, with the one defined, by means of the "gauge equivalence" of $\mathcal{A}$-connections of $\mathcal{E}$ (see Volume I, Chapt. I, (6.2)).

The assertion is certainly an immediate consequence of the same definitions; see (2.24), along with Volume I, Chapter I, (6.5), (6.7), or even (6.43) therein. ■ Hence, to repeat it again (loc. cit. (6.48)), one concludes that

(2.34) "gauge equivalent" $\mathcal{A}$-connections of a given vector sheaf $\mathcal{E}$ on X, alias "ϕ-related" $\mathcal{A}$-connections of $\mathcal{E}$,

(2.34.1) $$\text{with } \phi \in \mathcal{A}ut\mathcal{E}$$

(ibid. (6.2) and (6.4)), and, on the other hand, "equivalent" $\mathcal{A}$-connections of $\mathcal{E}$, through the action (2.19), are, in point of fact, synonymous terms!

Accordingly, the justification of the respective common notation, thus far, concerning what we have called above the moduli space of $\mathcal{E}$, as in (2.9) and (2.14), or even (2.31.3). It is actually the latter aspect, as in (2.31.3), that we are going to consider straightaway below, with an eye also to applications in Chapter 3.

2.1 The Orbit Space of $\mathcal{A}$-Connections

Our purpose in the following discussion is to consider, as already mentioned above, the moduli space of a given vector sheaf $\mathcal{E}$ on X (see (2.9)) as the "orbit space" of the set of $\mathcal{A}$-connections of $\mathcal{E}$ under the action on the latter set of the group (sheaf)

$$\mathcal{A}ut\mathcal{E}, \tag{2.35}$$

according to (2.19). Thus, given an $\mathcal{A}$-connection of $\mathcal{E}$,

$$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{2.36}$$

we denote by $\mathcal{O}_D$ the so-called orbit of D in the (quotient) set (2.31.3) (alias, the equivalence class of D, an element of the latter set) under the equivalence relation on $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, defined on it by (2.24). Hence, one has, by definition,

$$\mathcal{O}_D := \{\phi \cdot D : \phi \in \mathcal{A}ut\mathcal{E}\} \equiv [D] \tag{2.37}$$

such that (see (2.20)), one sets

$$\phi \cdot D \equiv \phi D \phi^{-1} := (\phi \otimes 1) \circ D \circ \phi^{-1} \equiv Ad(\phi) \cdot D \tag{2.38}$$

for any $\phi \in \mathcal{A}ut\mathcal{E}$, while the last term in (2.37) stands for the equivalence class of D, as explained above. Furthermore, in view of (2.34.1), one still obtains

$$\mathcal{O}_D = \{D' \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) : D' \underset{\phi}{\sim} D, \text{ for some } \phi \in \mathcal{A}ut\mathcal{E}\}, \tag{2.39}$$

where we set (see Volume I, Chapt. I, (6.46) and (6.47))

$$D' \underset{\phi}{\sim} D :\underset{\text{def}}{\Longleftrightarrow} D' = \phi D \phi^{-1} \equiv \phi \cdot D, \tag{2.40}$$

which also explains (2.34.1) as well. In other words,

(2.41) the orbit of an $\mathcal{A}$-connection D in (2.31.3) is the set of those $\mathcal{A}$-connections of $\mathcal{E}$ which are "gauge equivalent" (see (2.40)) to D, modulo $\mathcal{A}ut\mathcal{E}$ *viz.*, for ϕ varying in $\mathcal{A}ut\mathcal{E}$).

In the subsequent discussion we shall use either one of the two (equivalent) defining relations $\mathcal{O}_D$ as the particular case in hand may demand. Yet, there is still another equivalent expression for $\mathcal{O}_D$, which will be also of use below (see, e.g., Chapter III). Thus, one obtains

$$\mathcal{O}_D = \{D - D_{\mathcal{E}nd\mathcal{E}}(\phi)\phi^{-1} : \phi \in \mathcal{A}ut\mathcal{E}\}. \tag{2.42}$$

Indeed, applying (4.26) of Chapter 1, one has

$$D_{\mathcal{E}nd\mathcal{E}}(\phi) = D \circ \phi - (\phi \otimes 1) \circ D \equiv D\phi - \phi D \equiv [D, \phi] \tag{2.43}$$

for any $\phi \in (\mathcal{E}nd\mathcal{E})(U) = End(\mathcal{E}|_U)$ (see, e.g., (1.18) or even (1.6)) and open $U \subseteq X$. Therefore, one further gets at the relation

$$(D_{\mathcal{E}nd\mathcal{E}}(\phi))\phi^{-1} = D - \phi D \phi^{-1} \equiv D - \phi \cdot D \tag{2.44}$$

for any

$$\phi \in Aut\mathcal{E} \equiv (\mathcal{A}ut\mathcal{E})(X) \subseteq End\mathcal{E}, \tag{2.45}$$

so that one finally obtains

$$\phi \cdot D = D - D_{\mathcal{E}nd\mathcal{E}}(\phi) \cdot \phi^{-1} \tag{2.46}$$

for any ϕ, as in (2.45), which thus explains (2.42). ■

Note 2.1 Based on our discussion in Section 1 (see for instance, (1.2.1) therein), we consider in (2.42), as a group of gauge transformations of $\mathcal{E}$, the group (see (1.4))

$$Aut\mathcal{E} := (\mathcal{A}ut\mathcal{E})(X). \tag{2.47}$$

Of course, in a manner similar to (2.18), one proves that

$$Aut\mathcal{E} \text{ acts on } \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{2.48}$$

as well, so that

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \text{ can be viewed as an } Aut\mathcal{E}\text{-set too.} \tag{2.49}$$

Consequently, one can further consider

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/Aut\mathcal{E} \tag{2.50}$$

as the orbit space of the previous action. In point of fact, it is still the latter set that we usually consider in calculations, and even more its further restriction, occasionally, to any local gauge of $\mathcal{E}$ (see also, e.g., (2.16))—that is, to think, by analogy with (2.47), in terms of the group (see also (1.18))

$$(\mathcal{A}ut\mathcal{E})(U) = Aut(\mathcal{E}|_U) \tag{2.51}$$

—therefore, the entanglement finally of the group (sheaf),

$$\mathcal{GL}(n, \mathcal{A}), \qquad \text{with } n = rk\mathcal{E}, \tag{2.52}$$

as already explained in the foregoing (see also Volume I, Chapt. I, (6.33) therein; yet, see (1.14) in Section 1).

On the other hand, as an application of the "geometric" point of view that is provided by the notion of the orbit space, one has the relation

(2.53) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/Aut\mathcal{E} = \sum_{D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})} \mathcal{O}_D,$$

the second member of (2.53) denoting the partition of the orbit space in terms of the individual orbits (equivalence classes) of the $\mathcal{A}$-connections of $\mathcal{E}$. In this context, see also Volume I, Chapter I, Section 6. Finally, based on our previous remarks in (2.34), we may further consider

(2.54) the orbit space

(2.54.1) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/Aut\mathcal{E} = \sum_{D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})} \mathcal{O}_D$$

as the moduli space of $\mathcal{E}$.

Yet, the appearance of the group

(2.55) $$Aut\mathcal{E} := (\mathcal{A}ut\mathcal{E})(X)$$

in (2.50), hence, in (2.53) too, in place of the group (sheaf) $\mathcal{A}ut\mathcal{E}$ is in accord, as already hinted at several times in the preceding, with the

(2.56) more convenient point of view especially in calculations, namely, of looking at a given sheaf (along with J. Leray) in terms of its complete presheaf of sections.

However, by employing (2.55), as above, the result is always the same; see [VS: Chapt. I; p. 73, Theorem 13.1 as well as p. 75, (13.19)]. ■

Yet, concerning the previous formulas for $\mathcal{O}_D$, as in (2.37) or (2.39), we shall also employ the same practice as in (2.42).

As already said in the preceding, applications of the above—in particular, of the notion of the orbit space (see (2.53)) or even of the individual orbit $\mathcal{O}_D$ of a given $\mathcal{A}$-connection D of $\mathcal{E}$ in any one of the equivalent forms appearing in the foregoing—will be made in Chapter III, pertaining to the "geometry of the Yang–Mills fields."

As a preamble to that discussion, we further comment on the previous defining relation $\mathcal{O}_D$, as in (2.42): Thus, as an immediate consequence of the very definitions (see (2.43)), one concludes that

(2.57) given an $\mathcal{A}$-connection D of a vector sheaf $\mathcal{E}$, there always exists the $\mathcal{A}$-connection

(2.57.1) $$D_{\mathcal{E}nd\mathcal{E}}$$

of the vector sheaf

(2.57.2) $$\mathcal{E}nd\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}).$$

In this context, see also Chapter I, Section 4, concerning the meaning of the same definition of $D_{\mathcal{E}nd\mathcal{E}}$, as well as Section 5 of the present chapter, pertaining to a potential physical significance of the $\mathcal{A}$-algebra sheaf $\mathcal{E}nd\mathcal{E}$, as above.

On the other hand, we still look at a number of certain basic properties of the same $\mathcal{A}$-connection (2.57.1) that will also be of use in the ensuing discussion: Thus, based on Chapter I, (4.23), in the preceding, one gets the relation

(2.58) $$D_{\mathcal{E}nd\mathcal{E}}(1) = D' - D \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$$

for any two $\mathcal{A}$-connections D and D' of the vector sheaf $\mathcal{E}nd\mathcal{E}$ on X, where

(2.59) $$\mathbf{1} \equiv \mathbf{1}_{\mathcal{E}} \in Aut\mathcal{E} \equiv (\mathcal{E}nd\mathcal{E})^{\bullet}(X) \subseteq (\mathcal{E}nd\mathcal{E})(X) \equiv End\mathcal{E}$$

stands for the identity $\mathcal{A}$-automorphism of $\mathcal{E}$.

Now, since $\mathcal{E}nd\mathcal{E}$ is an $\mathcal{A}$-algebra sheaf (see [VS: Chapt. II, p. 138, Definition 6.2]), $\mathcal{A}$ being, by assumption, a unital $\mathbb{C}$-algebra sheaf on X, one obtains

(2.60) $$\mathbb{C} \underset{\to_{\varepsilon}}{\subset} \mathcal{A} \underset{\to_{j}}{\subset} \mathcal{E}nd\mathcal{E},$$

the previous isomorphisms being given by the obvious exploitation of the identity $\mathcal{A}$-automorphism $\mathbf{1}_{\mathcal{E}}$, as above; namely, we have

(2.61) $$\varepsilon(\lambda) := \lambda \cdot \mathbf{1}_{\mathcal{A}} \quad \text{and} \quad j(\alpha) := \alpha \cdot \mathbf{1}_{\mathcal{E}}$$

for any $\lambda \in \mathbb{C}$ and $\alpha \in \mathcal{A}(U)$, U open in X. Thus, in particular, one gets

(2.62) $$\mathbb{C}^{\bullet} \underset{\to}{\subset} \mathcal{A}^{\bullet} \underset{\to}{\subset} (\mathcal{E}nd\mathcal{E})^{\bullet} = \mathcal{A}ut\mathcal{E}$$

concerning the respective group (sheaf) of units of the $\mathbb{C}$-algebra sheaves involved, as in (2.60).

On the other hand, based on relation (2.60), one can look at the "induced $\mathcal{A}$-connections" of (2.57.1) on $\mathbb{C}$ and $\mathcal{A}$, respectively (i.e., precisely speaking, the pull-backs of the latter, via the maps ε and j, respectively, as in (2.60), on the sheaves concerned (see also Volume I, Chapt. I, (3.48) and (8.32) in Chapter I of this treatise). Thus, one first obtains

(2.63) $$D_{\mathcal{E}nd\mathcal{E}}|_{\mathbb{C}} = 0$$

(in fact, an immediate consequence of (2.58)), for $D' = D$; as a matter of fact, one concludes that

(2.64)

(2.64.1) $$D_{\mathcal{E}nd\mathcal{E}}(\mathbf{1}) = 0$$

if, and only if, $D' = D$ in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, where, of course, one has, concerning (2.58), the relation

2.64.2) $$D^{(D',D)}_{\mathcal{E}nd\mathcal{E}}(\mathbf{1}) = D' - D,$$

the previous notation referring just to the dependence of the respective formula on the given $\mathcal{A}$-connections D and D' of $\mathcal{E}$.

In point of fact, our previous assertion in (2.63) is a particular case of the more general fact that

$$D_{\mathcal{E}nd\mathcal{E}}|_{\mathcal{A}} = \partial. \tag{2.65}$$

Indeed, the preceding relation follows from the very definitions and (2.61), as above. ■

Thus, one further obtains the following relation, which also explains our last assertion above, in view of the respective behavior of ∂ on $\mathbb{C}$; that is, one has

$$\mathbb{C} \underset{\rightarrow}{\subset} \ker\partial = \ker(D_{\mathcal{E}nd\mathcal{E}}|_{\mathcal{A}}) \underset{\rightarrow}{\subset} \ker(D_{\mathcal{E}nd\mathcal{E}}) \subseteq \mathcal{E}nd\mathcal{E} \tag{2.66}$$

(see also Chapt. I, (1.16)).

As an application of the previous discussion, we further consider the important case of Riemannian vector sheaves on X, along with their corresponding moduli spaces, the latter being now defined by taking into account the respective $\mathcal{A}$-metrics ρ of the (vector) sheaves concerned.

2.2 The Orbit Space of a Maxwell Field

Suppose we have a Maxwell field

$$(\mathcal{L}, D) \tag{2.67}$$

on X, a given topological space, as above. Therefore, by considering the given $\mathcal{A}$-connection D of $\mathcal{L}$, as in (2.67), we can further look at the corresponding orbit of D, in accordance with (2.37); namely, we have

$$\mathcal{O}_D := \{\phi \cdot D : \phi \in \mathcal{A}ut\mathcal{L}\} \equiv [D]. \tag{2.68}$$

On the other hand, since, by hypothesis, $rk\mathcal{L} = \mathbf{1}$, one gets (see also Volume I, Chapt. I, (5.10))

$$\mathcal{A}ut\mathcal{L} := (\mathcal{E}nd\mathcal{L})^{\bullet} = \mathcal{A}^{\bullet}, \tag{2.69}$$

within an isomorphism of the group sheaves concerned. Thus, an element of the first member of (2.68) may be identified through a 0-cochain of $\mathcal{A}^{\bullet}$; namely, one has

$$\mathcal{A}ut\mathcal{L} \ni \phi \longleftrightarrow (s_\alpha) \in C^0(\mathcal{U}, \mathcal{A}^{\bullet}), \tag{2.70}$$

with $\mathcal{U}$ a local frame of $\mathcal{L}$. In this connection, see also Volume I, Chapter III, Lemma 2.2, in particular (2.44), (2.45), and (2.48) therein.

Therefore, by employing herewith our previous terminology (loc. cit. (3.11)), see also (2.46) therein), one concludes that

(2.71) the orbit of D, as above, may be construed as a *light ray,* alias a *beam of photons* (see also ibid. (2.45)), so that one may still consider the following identification:

$$\mathcal{O}_D \equiv [D] \longleftrightarrow [(\mathcal{L}, D)] \in \Phi^1_{\mathcal{A}}(X)^{\nabla}_R, \text{ with } R \equiv R(D). \tag{2.71.1}$$

In this connection, see also loc. cit. (3.16), (3.71), (3.109), as well as Chapter IV, Theorem 5.1.

3 Moduli Space of $\mathcal{A}$-Connections of a Yang–Mills Field

We come to examine our main theme in this chapter—namely, the moduli space of a given Yang–Mills field

(3.1) $$(\mathcal{E}, D),$$

as this has been already defined in Section 2 (see (2.9) or even (2.50)) for any vector sheaf $\mathcal{E}$ in general, admitting $\mathcal{A}$-connections; however, now, in the particular case that such an $\mathcal{A}$-module $\mathcal{E}$, as before, is actually given within a setup, which can be described, by virtue of our discussion in Chapter 1, as an abstract Yang–Mills framework. Thus, in particular,

(3.2) we assume that the setup, as described in Chapter I, (4.1) is in force.

The above framework—by referring to the topological space X, in particular the common base space of all the sheaves involved—has been called an abstract Riemannian (differential) space, or, for convenience, just a "Riemannian space." However, we still recall that by complete contrast with what actually happens in the classical theory, the previous attribute to the space X involved does not refer at all (!) to the space X itself but simply to the type of sheaves that are considered on it.

Therefore, following the same vein of ideas as those that were stated, for instance, in Section 2 (see, e.g., (2.7)) and since now, according to our hypothesis in (3.2), we are given an $\mathcal{A}$-metric ρ on $\mathcal{A}$ (see Chapt. I, (4.1.2)), which further, by virtue of the same hypothesis, is inherited on every vector sheaf $\mathcal{E}$ on X (ibid. (4.2), for convenience, we retain here the same symbol ρ for the $\mathcal{A}$-metric, thus defined on $\mathcal{E}$), the important $\mathcal{A}$-automorphisms of $\mathcal{E}$ are now, of course,

(3.3) those $\phi \in Aut\mathcal{E}$, which are "metric preserving"—that is, the ones for which one has

(3.3.1) $$\phi^*(\rho) \equiv \rho \circ (\phi, \phi) = \rho$$

(see also Chapt. I, (5.38)). Accordingly, one now considers as an appropriate group of gauge transformations the subgroup of $Aut\mathcal{E}$ defined by relation (3.3.1). We denote it by

(3.3.2) $$(Aut\mathcal{E})_\rho = (\mathcal{A}ut\mathcal{E})_\rho(X),$$

within an obvious meaning of the second member of the last relation, in view of (3.3.1).

Of course, it is easy to see, based on the very definitions and (3.3.1), that

(3.4) $$(Aut\mathcal{E})_\rho,$$

as is given by the same relation (3.3.1), is really a group, indeed, a subgroup of $Aut\mathcal{E} = (\mathcal{A}ut\mathcal{E})(X)$, hence, the relevant notation, as applied in (3.3.2). Therefore, by considering a given Yang–Mills field $(\mathcal{E}, D)$, in general, as in (3.1), within the framework of (3.2), that is, as a triplet,

$$\{(\mathcal{E}, D); \rho\}, \tag{3.5}$$

(alias, as a Riemannian vector sheaf on X, and as a Riemannian Yang–Mills field on X) then, based on our previous remarks in (3.3), one thus defines, as a moduli space of (3.5), the quotient set

$$\mathcal{M}(\mathcal{E})_\rho \equiv \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/(Aut\mathcal{E})_\rho. \tag{3.6}$$

To distinguish from previously employed relevant terminology, we may call the latter set the moduli space of the Riemannian Yang–Mills field (3.5).

On the other hand, we are going to consider a further specialized instance of (3.6), pertaining, in particular, to that associated with Yang–Mills equations (3.5), as the latter have been considered in Chapter I, Section 4. In this context, we further note, in anticipation (see (3.10) below), that the aforementioned specialization of (3.6) now concerns the more particular sort of $\mathcal{A}$-connections involved, and not the gauge group $Aut\mathcal{E}$ or even $(Aut\mathcal{E})_\rho$, as in (3.3.2), the latter remaining the same as in (3.6). So, precisely speaking, the aforesaid restriction is referred, in effect, to the domain of the definition of the Yang–Mills functional which can be associated with (3.5) (see Chapt. I, (5.11)); yet, as is already known (loc. cit. Lemma 5.1), the latter functional is invariant, relative to the gauge group $(Aut\mathcal{E})_\rho$, as above, and hence is the final definition of the same functional on the respective quotient space, as in (3.6). Accordingly, the specification of (3.6) is virtually referred to as the nominator of the fraction, appearing in (3.6) (see (3.8.2), along with (3.10)), as we explain straightforwardly in the following discussion.

3.1 Moduli Space of Yang–Mills $\mathcal{A}$-Connections

First, to cope with the framework implicit in the various notions involved, we adopt the following more restricted setup than that in (3.2). Thus,

(3.7) we accept that we have the framework of Chapter I, (5.1).

Now, suppose that

(3.8) we are given a Yang–Mills field on X,

$$(\mathcal{E}, D), \tag{3.8.1}$$

which, in effect, is a solution of the pertinent Yang–Mills equations (see Chapt. I, Definition 4.2); that is, we assume that the corresponding field strength (curvature) of $(\mathcal{E}, D)$ (thus, in point of fact, of D), $R \equiv R(D)$, satisfies the said equations (ibid. (4.76) and (4.77)). We denote the set of $\mathcal{A}$-connections of $\mathcal{E}$ as above by

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\mathcal{YM}}, \tag{3.8.2}$$

the elements of which are still called, to distinguish them from other cases, the Yang–Mills $\mathcal{A}$-connections of $\mathcal{E}$.

Now, in this connection and being also in accord with our previous relevant comments in Section 2 (see, e.g., (2.7)), our main conclusion herewith is that

(3.9) a Yang–Mills field

(3.9.1) $$(\mathcal{E}, D),$$

in the sense of (3.8) (*viz.*, in effect, as an element of the set (3.8.2), in other words, as a solution of the Yang–Mills equations (Chapt. I, Definition 4.2) is such when one moves across the orbit $\mathcal{O}_D$ of the given $\mathcal{A}$-connection (solution) D of $\mathcal{E}$ (see also, e.g., (2.37) or even (2.42) in the preceding). In this context, we also recall that $\mathcal{O}_D$ is understood here, with respect to the action of $(Aut\mathcal{E})_\rho$ on $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, in point of fact, on its subset $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\mathcal{YM}}$.

Accordingly, one can further consider the naturally entailed, in view of the preceding argument, quotient space

(3.10) $$\mathcal{M}(\mathcal{E})_{\mathcal{YM}} := \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\mathcal{YM}}/(Aut\mathcal{E})_\rho,$$

which we call the moduli space of the Yang–Mills $\mathcal{A}$-connections of $\mathcal{E}$, in agreement with the corresponding nomenclature as it applied for the set (3.8.2) or even, occasionally, for convenience and when no confusion is risked, simply the moduli space of $\mathcal{E}$.

Yet, the same space (quotient set) is also named the "space of solutions" or "solution-space" of the Yang–Mills equations that are associated with a given Yang–Mills field $(\mathcal{E}, D)$, as in (3.9.1).

On the other hand, one can still refer to the preceding by also saying, equivalently, that

(3.11) the space of solutions of the Yang–Mills equations (*viz.*, the set (3.8.2)), as above, is $(Aut\mathcal{E})_\rho$-invariant.

Thus, as already noted, this is another equivalent expression of the same relation (3.10). Yet, what amounts to the same thing, one thus infers the

invariance of the solutions of the Yang–Mills equations under the action of $(Aut\mathcal{E})_\rho$. In other words,

(3.12) if $R(D) \equiv R$ is a solution of the Yang–Mills equations of $\{(\mathcal{E}, D); \rho\} \equiv \mathcal{E}$ (see (3.5)), then the same holds true for

(3.12.1) $$\phi^*(R) \equiv Ad(\phi)R \equiv \phi R \phi^{-1}$$

for any $\phi \in (Aut\mathcal{E})_\rho$.

Indeed, all the above equivalent assertions are, in effect, an immediate consequence of the characterization of the aforesaid solutions, as "critical points" of the Yang–Mills functional (see Chapt. I, (8.26)), along with the "gauge invariance" of the latter; namely, of the fact that

(3.13) the Yang–Mills functional (ibid. (5.11)) is still $(Aut\mathcal{E})_\rho$-invariant

(see also loc. cit. Lemma 5.1). ■

Scholium 3.1 Looking at our main conclusions in the present section as well as in Chapter I, Section 5, one infers the gauge invariance (i.e., equivalently, the $(Aut\mathcal{E})_\rho$-invariance of two fundamental issues of our relevant argument, as they certainly are (i) the Yang–Mills functional (Chapt. I, (5.11)) and (ii) the (space of) solutions of the Yang–Mills equations for a given Yang–Mills field (see (3.8.2) as well as (3.12)). Now, this is, as already noted, in accord with our previous comments in (2.7); yet, one can look at these conclusions as a farther indication/corroboration of what may construed as the

(3.14) *"functoriality" of Nature's function* (unfolding), to the extent, of course, we can perceive it (i.e., relative to our own measurements, through which we detect it); therefore, one comes here to another aspect of the classical *principle of general relativity.*

On the other hand, strictly speaking, the above "functoriality of Nature" is more general/intimate than the aforementioned classical principle, due to the full absence from our relevant argument of any notion of "space" (manifold), in the classical perspective of the term, according to the very nature of the abstract setting employed, throughout the present treatise.

4 Moduli Space of Self-Dual $\mathcal{A}$-Connections

We continue our reference to the "moduli space of $\mathcal{A}$-connections," the subject matter of this chapter, in the particular case of what we may call, within the present abstract context, *self-dual $\mathcal{A}$-connections*. We are actually concerned with the "moduli space of Yang–Mills fields" for the case of the self-dual Yang–Mills fields. Thus, given a Yang–Mills field

(4.1) $$(\mathcal{E}, D)$$

on X, of order $n \geq 2$, we further consider, according to what we have said in the preceding (see (1.29)), the quotient space

(4.2) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E}$$

(we still refer to Sections 1 and 2 of this chapter concerning the relevant notation, as well as the corresponding framework employed herewith).

Now, to proceed further, we need to supplement our framework, thus far, with that of Volume I, Chapter I, Section 10 of this treatise; namely, with the corresponding context, pertaining to the relevant discussion in our case on the classical issue, concerning the so-called Hodge $*$-operator (see also [VS: Vol. I]). So we set the following.

Definition 4.1 Given a Yang–Mills field $(\mathcal{E}, D)$, as above, we say that we have a *self-dual Yang–Mills field* or even (and this is also usually the practice, classically) a self-dual $\mathcal{A}$-connection D whenever the curvature of D, as in (4.1) above, satisfies the relation

(4.3) $$*R = R,$$

where, as usual, we set $R \equiv R(D)$ for the curvature of the given $\mathcal{A}$-connection D.

In relation (4.3), "$*$" denotes the Hodge $*$-operator, as has already been defined in Volume I: Chapter I, (10.13) of the present treatise, where we refer the reader for further details.

Now, in this context, we also note that this concept of a Hodge $*$-operator is, by definition, connected with the one of an $\mathcal{A}$-metric, say ρ, on the Yang–Mills field $(\mathcal{E}, D)$ concerned, precisely on the corresponding "carrier" (vector sheaf) $\mathcal{E}$ of it, loc. cit. Accordingly, the appropriate context for our abstract setting is one of a

(4.4) "Riemannian space" X, in the sense of Chapter I, (4.1), which we thus assume in the sequel, pertaining to the common base space of the vector sheaves involved.

Thus, looking at the quotient space (4.2), as above, and, in particular, at the denominator (action group) $\mathcal{A}ut\mathcal{E}$, referring to a given vector sheaf $\mathcal{E}$ on X, as in (4.4), we actually consider in the sequel its subgroups, consisting of the $\mathcal{A}$-metric-preserving $\mathcal{A}$-automorphisms of $\mathcal{E}$—namely, the group

(4.5) $$(\mathcal{A}ut\mathcal{E})_\rho < \mathcal{A}ut\mathcal{E},$$

(see (3.3) in Section 3, as well as in Chapter I, (5.38) and (5.46)).

On the other hand, based on the very definitions concerning the Hodge $*$-operator (see Volume I, Chapt. I, Section 10), one concludes that

(4.6) $$\phi \circ * = * \circ \phi$$

for any $\phi \in (\mathcal{A}ut\mathcal{E})_\rho$. In other words,

(4.7) the Hodge $*$-operator commutes with every element $\phi \in (\mathcal{A}ut\mathcal{E})_\rho$ (*viz.*, with any $\mathcal{A}$-metric preserving $\mathcal{A}$-automorphism of $\mathcal{E}$).

As a consequence of the above, and by still referring to a self-dual $\mathcal{A}$-connection D, in the sense of Definition 4.1, as above, one obtains, by virtue of (4.3) and (4.6), the relations

(4.8) $$*(\phi(R)) = (* \circ \phi)(R) = (\phi \circ *)(R) = \phi(*R) = \phi(R)$$

for any $\phi \in (\mathcal{A}ut\mathcal{E})_\rho$, where we set

$$\phi(R) := \phi R \phi^{-1} \equiv Ad(\phi) \cdot R \tag{4.9}$$

with $R \equiv R(D)$, the curvature of the given sulf-dual $\mathcal{A}$-connection D. Therefore, in other words, in view of (4.8), one thus concludes that

(4.10) self-duality is retained by $\mathcal{A}$-metric-preserving gauge-transformations ($\mathcal{A}$-automorphisms) of $\mathcal{E}$.

Accordingly, one can further consider the moduli space of self-dual $\mathcal{A}$-connections of a given Yang–Mills field $(\mathcal{E}, D)$ on X—namely, the quotient set

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\text{self-dual}}/(\mathcal{A}ut\mathcal{E})_\rho, \tag{4.11}$$

a generic element of which is of the form

$$[(\mathcal{E}, D)] := \{\phi_*(D) \equiv \phi D \phi^{-1} : \phi \in (\mathcal{A}ut\mathcal{E})_\rho\} \tag{4.12}$$

(see also, for instance, (2.20)). Yet, self-dual $\mathcal{A}$-connections are also known in the classical literature as instantons, so that we may still write (4.11) in a self-explanatory form as the quotient set

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\text{inst}}/(\mathcal{A}ut\mathcal{E})_\rho \tag{4.13}$$

(*viz.*, the moduli space of instantons associated with a given Yang–Mills field $(\mathcal{E}, D)$ on X, as above (see also Chapt. I, Note 4.4)). The same (quotient) set (4.13) can be expressed by just saying that

(4.14) instantons are preserved, by any $\mathcal{A}$-metric-respecting gauge transformation ($\mathcal{A}$-automorphism) of $\mathcal{E}$.

Of course, the previous quotient set (4.11) (or, equivalently, (4.13)) may still be construed, according to the very definitions and our results in the preceding, as a subspace of the quotient set

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})_{\mathcal{YM}}/(\mathcal{A}ut\mathcal{E})_\rho; \tag{4.15}$$

namely, of the moduli space of the Yang–Mills $\mathcal{A}$-connections of $\mathcal{E}$, or just the moduli space of $\mathcal{E}$, or even "solution space" of the Yang–Mills equations, associated with a given Yang–Mills field $(\mathcal{E}, D)$, as above (see (3.10) in the preceding, along with the relevant comments after it). In this context, see also Chapter I, Proposition 4.1, along with (4.100) and Note 4.4 therein.

5 Quantized Moduli Spaces

"Quantization is provided by the physical law itself."

C. von Westenholz in *Differential Forms in Mathematical Physics* (North-Holland, Amsterdam, 1981) p. 323.

We can refer to the same author, as above, in that (again, the emphasis below is ours),

(5.1) *"... the structure underlying an intrinsic approach to physics is 'essentially' de Rham-coholomogy ... "*

(loc. cit. p. 321), or even

(5.2) *"The mathematical structure underlying field quantities ... is essentially de Rham cohomology"*

(ibid. p. 323). Now, the "field quantities" à la von Westenholz, as above, are actually, in our case, what we have called in the preceding Yang–Mills fields,

(5.3) $$(\mathcal{E}, D),$$

so that the previous two phrases of the same author, as in (5.1) and (5.2), might still relate to our "cohomological classification of Maxwell and/or Yang–Mills fields," which we have already considered in the foregoing (see Volume I, Chapt. IV, Section 5, and Chapt. I, Section 9 of this volume).

On the other hand, within the same vein of ideas advocated by von Westenholz, one can still mention D. Finkelsten's dictum that

(5.4) *"All is quantum"*

(our emphasis; see D.R. Finkelstein [2: p. 477]). Yet, within the present abstract framework, we may also refer to our conclusion in the preceding work (see Volume I, Chapt. V, (5.128)):

(5.5) Every (free) elementary particle is (pre)quantizable.

In this context, we further note that a "(free) elementary particle," according to the terminology applied by the present treatise, is, in general, a Yang–Mills field $(\mathcal{E}, D)$, as in (5.3), where

(5.6) $$rk\mathcal{E} = n \in \mathbb{N},$$

which may also be particularized to a Maxwell field

(5.7) $$(\mathcal{L}, D)$$

in the case that $rk\mathcal{L} = 1$. See the corresponding classification in Volume I, Chapter II.

Now, a moral of the relevant considerations is that one is actually dealing

(5.8) not much with $\mathcal{E}$, or even $(\mathcal{E}, D)$, that is, with the (free) "particle," itself, as, in effect, with its endomorphism $\mathcal{A}$-algebra $\mathcal{E}nd\mathcal{E}$, or, even more precisely, with the "Yang–Mills field"

(5.8.1) $$(\mathcal{E}nd\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}});$$

that is, with what we may define as the

(5.8.2) “matrix” representation or even “matrix” disguise of $\mathcal{E}$.

In this context, concerning the terminology used in (5.8.2), we further remark that, “locally”—that is, with respect, say, to a local gauge U of $\mathcal{E}$—when (5.6) is in force one obtains the relations

$$(\mathcal{E}nd\mathcal{E})|_U = M_n(\mathcal{A})|_U = M_n(\mathcal{A}|_U) \tag{5.9}$$

so that, for any open $V \subseteq U$, one also gets at the relations

$$((\mathcal{E}nd\mathcal{E})|_U)(V) = (\mathcal{E}nd\mathcal{E})(V) = M_n(\mathcal{A})(V) = M_n(\mathcal{A}(V)), \tag{5.10}$$

which finally remind us of *Heisenberg’s point of view* (*matrix-mechanics,* following the classical nomenclature). According to the foregoing, we may call the Yang–Mills field (5.8.1) the “Heisenberg form” of the initially given Yang–Mills field $(\mathcal{E}, D)$.

Consequently, we are thus temped to say that

(5.11)

by considering the transition

$$(\mathcal{E}, D) \rightsquigarrow (\mathcal{E}nd\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}}), \tag{5.11.1}$$

one virtually gets, so to speak, at a

(5.11.2) *“second-quantization functor,”*

which thus can further cope, directly, with the *second-quantized objects,* as they actually are already physical objects of the form (5.3), or even (5.7), as appeared in the disguised form (5.8.1); namely, a “canonical,” so to say, “matrix representation” of the initial object ((free) elementary particle)

$$(\mathcal{E}, D). \tag{5.11.3}$$

In this regard, we still notice that in the case of a Maxwell field $(\mathcal{L}, D)$, one gets

$$(\mathcal{E}nd\mathcal{L}, D_{\mathcal{E}nd\mathcal{L}}) = (\mathcal{A}, D_{\mathcal{A}}) = (\mathcal{A}, \partial). \tag{5.11.4}$$

Furthermore, the same transition (5.11.1) is still already encoded in the moduli space of $\mathcal{E}$, as well, that is given by

$$\mathcal{M}(\mathcal{E}) := \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E} = \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/(\mathcal{E}nd\mathcal{E})^{\bullet}. \tag{5.12}$$

Yet, the inner structure of the latter space is examined again in terms of the aforesaid matrix representation of $\mathcal{E}$ (*viz.*, by means, in effect, of the pair (5.8.1), as above); see Chapter III. On the other hand, (5.13) the *sheaf-theoretic character* of the present account entails for the objects under consideration a varying status by the very nature of the sheaf theory employed. Thus, this is an indispensable issue concerning a "quantum field theory" perspective of looking at things, by a direct reference to the latter, without the intervention of any "coordinate space", in the classical sense of the term. By contrast, any resort to *coordinatization* refers to the "arithmetics" (alias, "laboratory"), that is, in any way implemented by us, of course!

Accordingly, all told,

(5.14) the employed framework of abstract differential geometry (ADG), as advocated by the present study, turns out to be virtually *eo ipso* a quantized one, either by referring to pairs ((free) elementary particles) of the form (5.3) or to those of (5.8.1); yet, in particular, as a spin-off of its *sheaf-theoretic flavor*, the same framework is a "second-quantized" (*viz.*, a relativistic) one.

The above, however, just, as it concerns, of course, the "vacuum" (!); notwithstanding, this might still be instructive, for other more involved situations, as, at least, to the real nature (role) of the particular issues, under consideration.

Thus, concerning the natural question (see I. Raptis [3]),

(5.15) "… whether … quantizing a classical theory is physically meaningful at all …"

one might respond, by remarking that

(5.16) (it always depends on the theory we apply!), this is, of course, always related with the type of the theory (device/mechanism) that one employs in describing the physical laws, via the corresponding equations.

Now, the above can certainly be construed as being in accord with the spirit of the epigraph to this section.

Thus, as an outcome of the foregoing, we can further remark that

(5.17) the device has to be innate, alias, "fully covariant" (both terms are actually due to I. Raptis, loc cit.), with the physical objects (for instance, elementary particles) that the device tries to describe.

Thus, it is very likely that the "abstract differential geometry" developed so far provides such a device, as in (5.17). Yet, another spin-off of the same technique is the possibility of describing physical objects by an appropriate exploitation of cohomological language, as was hinted at at the beginning of this section (see the maxims quoted therein, along with the subsequent comments thereon), as, for instance, characteristic classes and the like, a fact that one may still consider as another "advanced arithmetics" pertaining to the theory of elementary particles.

5.1 Morita Equivalence, as Applied to Second Quantization

Our purpose in the following discussion is to sustain the idea that

(5.18) the nomenclature "matrix disguise" that was applied earlier (see (5.8.2)) appears to be susceptible to acquiring a real meaning concerning the essential use of the second compound of that term.

Indeed, it seems that this can be attained by an appropriate exploitation of the (category) equivalence hinted at in the title of this section. (My debt here is due actually to I. Raptis for communicating to me a relevant allusion lately of M.B.P. Wright to *Morita equivalence,* which further reminded me of some remarks in the past pertaining to the "Morita equivalence," within the context of *topological algebras theory*, as appeared in A. Mallios [2]).

In this context, we further note that the "matrix framework" pointed out by the preceding is still present "locally," in the moduli space of $\mathcal{A}$-connections of a given $\mathcal{A}$-module $\mathcal{E}$ on X, as will become clear in Chapter III; see, for instance, (3.17) and (3.18).

Now, it is a basic inference of commutative algebra (thus, in effect, of algebraic geometry) that there is a category equivalence

$$\mathcal{E}^{f}_{\mathcal{A}}(X) \sim \mathcal{P}(\mathbb{A}) \tag{5.19}$$

between locally free $\mathcal{A}$-modules of finite rank and projective finitely generated $\mathbb{A}$-modules, with $\mathbb{A}$ a unital commutative $\mathbb{C}$-algebra whose (prime) spectrum (alias, "spectrum space") is

$$X \equiv Spec(\mathbb{A}), \tag{5.20}$$

and $\mathcal{A}$, as above, is the associated (to X) "structure sheaf" (in effect, a $\mathbb{C}$-algebra sheaf on X) so that one has, in particular, according to the very definitions, the relation

$$\mathcal{A}(X) = \mathbb{A}, \tag{5.21}$$

within an isomorphism of (unital commutative) $\mathbb{C}$-algebras. (See, for instance, A. Grothendieck–J.A. Diendonné. [1: p. 198, Definition 1.3.4, and p. 207, Corollary 1.4.4]).

On the other hand, according to K. Morita, one has the following equivalence of (module) categories ("Morita equivalence"):

$$\mathcal{M}od_{\mathbb{A}} \sim \mathcal{M}od_{M_n(\mathbb{A})} \tag{5.22}$$

for any $n \in \mathbb{N}$ (see, e.g., F.W. Anderson–K.R. Fuller [1: p. 265, Corollary 22.6]). Furthermore, a category equivalence respects finitely generated projective modules (see, e.g., P.M. Cohn [1: p. 1157]), so that one actually obtains

$$\mathcal{P}(\mathbb{A}) \sim \mathcal{P}(M_n(\mathbb{A})) \tag{5.23}$$

for any $n \in \mathbb{N}$.

We proceed by assuming, henceforth, that

(5.24) $\mathbb{A}$ is a unital topological Q-algebra (*viz.*, $\mathbb{A}^{\bullet}$ is a neighborhood of 1).

(See A. Mallios [3: p. 43, Definition 6.2], along with A. Mallios [4: p. 419, Section 5]). Then, the algebra

(5.25) $M_n(\mathbb{A})$ is still a unital topological Q-algebra.

See the same last quotation. Therefore, one can further consider the following:

(5.26) $M_n(\mathbb{A})$-vector bundles (of finite rank) over X, along with the corresponding sheaves of global continuous sections, the latter being, in particular, locally free $\mathcal{A}$-modules of finite rank over X, such that

(5.26.1) $$\mathcal{A} \equiv \mathcal{C}_X^{M_n(\mathbb{A})}.$$

See, for instance, loc. cit., along with p. 403, therein, Lemma 1.1. Accordingly, by employing the notation of (5.19), as above, one gets the following equivalence of the categories concerned:

(5.27) $$\mathcal{E}^f_{\mathcal{C}_X^{\mathbb{A}}}(X) \sim \mathcal{E}^f_{\mathcal{C}_X^{M_n(\mathbb{A})}}(X).$$

We apply here the Morita equivalence (see (5.23)) plus *generalized Serre-Swan*, ibid., p. 420, or even A. Mallios [1: p. 481, Theorem 4.2], as well as L.N. Vaserstein [1].

Yet, setting

(5.28) $$\mathcal{A} \equiv \mathcal{C}_X^{\mathbb{A}}$$

in (5.27), one then obtains, by the very definitions,

(5.29) $$\mathcal{C}_X^{M_n(\mathbb{A})} = M_n(\mathcal{C}_X^{\mathbb{A}}) \equiv M_n(\mathcal{A}),$$

so that we can still write (5.27) in the form

(5.30) $$\mathcal{E}^f_{\mathcal{A}}(X) \sim \mathcal{E}^f_{M_n(\mathcal{A})}(X),$$

within a category equivalence, for any $n \in \mathbb{N}$, with $\mathcal{A}$ given by (5.28). In other words, speaking in terms of Morita equivalence, one concludes that

(5.31) locally free $\mathcal{A}$-modules of finite rank (*viz.*, vector sheaves, with respect to $\mathcal{A}$ as well as those relative to $M_n(\mathcal{A})$, with $n \in \mathbb{N}$) are actually (modulo Morita equivalence) the same.

The above might be very likely related to Finkelstein's adage that (emphasis below is ours, of course!)

(5.32) *"noncommutativity does not necessarily imply* [(!)] *the quantum deep."*

(See, e.g., D.R. Finkelstein [2: p. 134, beginning of Section 4.4.3].)

3

Geometry of Yang–Mills $\mathcal{A}$-Connections

"...to understand what's what ... is a vital aspect of Mathematics."

S. Mac Lane in *Mathematics: Form and Function* (Springer-Verlag, New York, 1986). p. 288.

"...the emphasis is on generality and careful formulation rather than on the technique of solving problems."

G. W. Mackey in *The Mathematical Foundations of Quantum Mechanics* (W.A. Benjamin, New York, 1963). p. vii.

The geometry referred to in the title concerns an application of classical differential-geometric notions/methods in the study of structure properties (geometry) of the space that interests us here, which is the space of solutions (again!) of the so-called Yang–Mills equations; these solutions are, by definition, $\mathcal{A}$-connections in the sense of the present treatise, which thus appear on the stage through their corresponding curvature (field strength), which is actually involved in the equations at issue. (See also Chapter I, Section 4, for the precise terminology employed.) On the other hand, since, by virtue of their own nature, the objects concerned ($\mathcal{A}$-connections solutions) are not distinguished insofar as they are "gauge equivalent;" one is led to consider not the initial solution space, as above, but, in effect, an appropriate "quotient" of it—the so-called "moduli space" of the solutions ($\mathcal{A}$-connections) under consideration.

Of course, the spaces involved herewith are not finite-dimensional (vector) spaces. However, the things are finally not as bad see as they might look at first sight! Indeed, as we shall see (see Sections 2 and 3 in the subsequent discussion), this is essentially due to the important property of being, namely, the space of $\mathcal{A}$-connections an affine space. Thus, at the very end, one is led to consider $\mathbb{C}$-vector spaces (however, infinite-dimensional, anyway). Yet, with something making the things more familiar, the same spaces can, for an appropriate structure sheaf $\mathcal{A}$, become suitable topological vector spaces, so that classical differential-geometric notions pertaining, for instance, to the notion of a tangent vector or of a tangent space and the like can be formulated, as they indeed are.

Thus, with all these actually being abstract, even within the classical framework, we further point out that the same can still be treated in a more general context, showing the effectiveness of the abstract methods applied thus far, even in the present, more sophisticated case of the classical theory. Yet, this whole enterprise is not without a profit: indeed, as a result, one further gains, among other things, the possibility

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of employing the same abstract methods as advocated by this treatise, in more general situations that occur in the applications where the classical setup is no more accessible!

1 Abstract Differential-Geometric Jargon in the Moduli Space of $\mathcal{A}$-Connections

To start with, and also to be general enough while specializing occasionally, as the particular case at hand may demand,

(1.1) suppose that we are given the framework of Section 1 in Chapter II.

So we basically consider a vector sheaf

$$\mathcal{E}, \text{ with } rk\mathcal{E} = n \in \mathbb{N} \tag{1.2}$$

on a topological space X, a carrier of a given differential triad

$$(\mathcal{A}, \partial, \Omega^1) \tag{1.3}$$

(see also Chapt. II, Section 2), while we still assume that

$$\Omega^1 \tag{1.4}$$

is a vector sheaf on X as well. On the other hand,we further suppose that the given vector sheaf $\mathcal{E}$ admits an $\mathcal{A}$-connections namely, equivalently (see Chapt. II, (2.1)), we also assume that

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \neq \{0\}. \tag{1.5}$$

Therefore (see Volume I, Chapt. I, (5.7)), if D is an $\mathcal{A}$-connection of $\mathcal{E}$, one has

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) = D + \Omega^1(\mathcal{E}nd\mathcal{E})(X); \tag{1.6}$$

that is,

(1.7) $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ is an affine space, modeled after the $\mathcal{A}(X)$-module

$$\Omega^1(\mathcal{E}nd\mathcal{E})(X). \tag{1.7.1}$$

In fact, in view of our hypothesis for $\mathcal{E}$ and Ω^1, the above $\mathcal{A}(X)$-module is another vector sheaf on X as well.

In this connection, see also [VS: Chapt. VI, p. 32, Theorem 7.1]. We are going to make substantial use of the latter vector sheaf, as in (1.7.1), the "model" of the affine space of the $\mathcal{A}$-connections of $\mathcal{E}$, strictly speaking, of its "local structure" (see Section 2), when trying to identify, for instance, the "tangent space" at D of the various "slices" of the moduli space of $\mathcal{E}$ (see Chapt. II (2.9)), these slices being the orbits, $\mathcal{O}_D$, of the given $\mathcal{A}$-connection D of $\mathcal{E}$, as D varies through the space (1.5) (see also loc. cit. (2.50)).

Now, apart from $\mathcal{O}_D$, the orbit of the $\mathcal{A}$-connection D of $\mathcal{E}$, as above, defined under the action of $Aut\mathcal{E}$ (or even of $\mathcal{A}ut\mathcal{E}$) on (1.5) (loc. cit. (2.19), along with (2.48) therein), we shall also need in the sequel (see Section 2) the notion of the isotropy group of D.

Thus, by considering the aforementioned action of $Aut\mathcal{E}$ (group of gauge transformations of $\mathcal{E}$ (Chapt. I, (5.19)) on $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ (Chapt. II, (2.48)), along with an $\mathcal{A}$-connection D of $\mathcal{E}$—namely, an element

(1.8) $$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$$

—one defines the isotropy, or stability group of D, denoted by $\mathcal{O}(D)$, according to the relation

(1.9) $$\mathcal{O}(D) := \{\phi \in Aut\mathcal{E} : \phi \cdot D = D\},$$

hence, by the very definitions (ibid. (2.20) and (2.24)), a subgroup of $Aut\mathcal{E}$. On the other hand, by taking into account our previous remarks in Chapter II that follow (2.18), along with (2.31) as well as (2.32) therein, one can look at the isotropy group of D as that subgroup of $\mathcal{A}ut\mathcal{E}$ that is given, by analogy with (1.9), by the relation

(1.9′) $$\mathcal{O}(D) := \{\phi \in \mathcal{A}ut\mathcal{E} : \phi \cdot D = D\}.$$

In the sequel we shall follow, indifferently, either one of the two definitions (1.9) or (1.9′), as the case may be.

Thus, by still working with definition (1.9), for example, we can say that

(1.10) the isotropy group of D, as above (see (1.8) and (1.9)), $\mathcal{O}(D)$, consists of those $\phi \in Aut\mathcal{E}$ whose "adjoint representations" leave D invariant; that is, one has

(1.10.1) $$\mathcal{O}(D) = \{\phi \in Aut\mathcal{E} : Ad(\phi) \cdot D = D\} < Aut\mathcal{E} \equiv (\mathcal{A}ut\mathcal{E})(X).$$

Indeed, our assertion in (1.10) is just another way of writing the defining relation $\phi \cdot D = D$, as in (1.9); that is, one has (see also Chapt. II, (2.20))

(1.11) $$\phi \cdot D := (\phi \otimes 1) \circ D \circ \phi^{-1} \equiv Ad(\phi) \equiv \phi \cdot D \cdot \phi^{-1},$$

which is exactly our notation in (1.10.1). ■

Of course, relation (1.11) still entails, by definition (loc. cit), the group action of $Aut\mathcal{E}$ (or even of $\mathcal{A}ut\mathcal{E}$, as the case may be; see (1.9) or (1.9′), respectively) on (the affine space; see (1.7)) $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$.

On the other hand, by further looking at the defining relation of $\mathcal{O}(D)$,

(1.12) $$\phi \cdot D = D$$

with $\phi \in Aut\mathcal{E}$, or $\phi \in \mathcal{A}ut\mathcal{E}$, as above, while also taking (1.11) into account, one obtains

$$\phi \cdot D \equiv (\phi \otimes 1) \circ D \circ \phi^{-1} = D, \tag{1.13}$$

so that one has

$$(\phi \otimes 1) \circ D = D \circ \phi. \tag{1.14}$$

Hence, by further setting

$$(\phi \otimes 1) \circ D \equiv \phi \cdot D \ \text{ and } \ D \circ \phi \equiv D \cdot \phi, \tag{1.15}$$

one finally gets, in view of (1.14), the relation

$$\phi D = D\phi. \tag{1.16}$$

(In this connection, we still note that, for convenience, we employed above an obvious (and usual!) abuse of notation, pertaining to (1.11) and the first of (1.15), as before.) Therefore, by virtue of (1.16), one still obtains

$$[D, \phi] \equiv D\phi - \phi D = 0, \tag{1.17}$$

which thus can now be viewed as the defining relation of $\mathcal{O}(D)$ as well. In other words, one thus concludes that

(1.18)

$\mathcal{O}(D)$, the isotropy group of

$$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{1.18.1}$$

are those $\phi \in Aut\mathcal{E}$ (see, e.g., (1.9)) for which one has

$$[D, \phi] \equiv D\phi - \phi D = 0 \tag{1.18.2}$$

(*viz.*, those $\phi \in Aut\mathcal{E}$, that "commute" with D). Accordingly, one obtains

$$\begin{aligned} \mathcal{O}(D) &= \{\phi \in Aut\mathcal{E} : [D, \phi] \equiv D\phi - \phi D = 0\} \\ &= \{\phi \in Aut\mathcal{E} : D\phi = \phi D\}. \end{aligned} \tag{1.18.3}$$

Now, another expression of the same group $\mathcal{O}(D)$ as before that will also be of use below, can still be obtained through the "covariant differential"

$$D_{\mathcal{E}nd\mathcal{E}}; \tag{1.19}$$

that is, by means of the $\mathcal{A}$-connection of the vector sheaf

$$\mathcal{E}nd\mathcal{E} := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \tag{1.20}$$

induced on it by the given $\mathcal{A}$-connection D of $\mathcal{E}$. (See also Chapter I, (4.28), along with (1.18), as above.) That is, one has the following useful (as we shall see, by the ensuing discussion) definition of the group, under consideration:

$$\begin{aligned}\mathcal{O}(D) &= \{\phi \in Aut\mathcal{E} : D_{\mathcal{E}nd\mathcal{E}}(\phi) = 0\} \\ &= \ker(D_{\mathcal{E}nd\mathcal{E}|_{Aut\mathcal{E}}}) = (\ker D_{\mathcal{E}nd\mathcal{E}}) \cap Aut\mathcal{E}.\end{aligned} \tag{1.21}$$

In this connection, we recall that, according to the very definition of (1.19), one has the map

$$D_{\mathcal{E}nd\mathcal{E}} : \mathcal{E}nd\mathcal{E} \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E}) \tag{1.22}$$

as a $\mathbb{C}$-linear (sheaf) morphism of the $\mathcal{A}$-modules (in effect, vector sheaves; see (1.4)) involved in (1.22), which thus explains the notation employed in (1.21). So we come now to the following:

Proof of (1.21) Based on the definition of the covariant differential, as in (1.22) (see Chapt. I, (4.28), along with (1.15) and (1.17) in the preceding), one obtains

$$D_{\mathcal{E}nd\mathcal{E}}(\phi) = D \circ \phi - (\phi \otimes 1) \circ D \equiv D\phi - \phi D \equiv [D, \phi]; \tag{1.23}$$

that is, one actually has the relation

$$D_{\mathcal{E}nd\mathcal{E}}(\phi) = [D, \phi] \tag{1.24}$$

for any $\phi \in \mathcal{E}nd\mathcal{E}$ (see also (1.22), along with our comments in the foregoing concerning (1.9) and (1.9′)). Thus, our assertion in (1.21) in now an immediate consequence of (1.24), together with the previous definition of $\mathcal{O}(D)$, through (1.18.3). ■

Note 1.1 By looking at the second member of (1.24) as the *Lie derivative* of $\phi \in \mathcal{E}nd\mathcal{E}$, with respect to the given $\mathcal{A}$-connection D of $\mathcal{E}$ denoted by

$$\mathcal{L}_D, \tag{1.25}$$

one has, in view of (1.24), the relation

$$D_{\mathcal{E}nd\mathcal{E}} = \mathcal{L}_D. \tag{1.26}$$

The latter provides, at least (!), an easy way to recall the form of the first member of (1.26). In this regard, see also [VS: Chapt. VI, p. 21 (5.26)].

Thus, by applying the notation of (1.26) in (1.21), one further obtains

$$\mathcal{O}(D) = \ker(\mathcal{L}_D|_{Aut\mathcal{E}})) \equiv \{\phi \in Aut\mathcal{E} : \mathcal{L}_D(\phi) \equiv [D, \phi]) = 0\}. \tag{1.27}$$

We are going to apply the preceding notions throughout this chapter. On the other hand, by further employing *dual differentials* (see Chapt. II, Section 2), one succeeds in decomposing the “model” vector sheaf

(1.28) $$\Omega^1(\mathcal{E}nd\mathcal{E}),$$

in point of fact, locally, *viz.*, the corresponding $\mathcal{A}(U)$-modules,

(1.29) $$\Omega^1(\mathcal{E}nd\mathcal{E})(U),$$

for any given open $U \subseteq X$, in terms of the image of the operator

(1.30) $$D_{\mathcal{E}nd\mathcal{E}|_{Aut\mathcal{E}}} : Aut\mathcal{E} \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E})(X)$$

and its "orthogonal complement." As we shall see, the above two subspaces of $\Omega^1(\mathcal{E}nd\mathcal{E})(X)$ (however, always locally; see (1.29)) may finally be construed as the "tangent space" of $\mathcal{O}_D$, the orbit of D in (1.6) (see Chapt. II, (2.42)) at D, and the corresponding "normal subspace" (orthogonal complement), relative to a given $\mathcal{A}$-metric on $\mathcal{E}$, hence, on the space (1.28) as well, when assuming, as usual, the appropriate pair

(1.31) $$(X, \mathcal{A}).$$

Note 1.2 (Physical meaning of (1.26)) By further commenting on (1.26), while still employing the terminology of Chapter I, Section 4, we can say that

given an element

(1.32.1) $$\phi \in Aut\mathcal{E},$$

(1.32) the "flow" of ϕ relative to D (in point of fact, relative to the "Lie operator" defined by D) is that one of ϕ relative to $D_{\mathcal{E}nd\mathcal{E}}$; namely, one has the relation

(1.32.2) $$\mathcal{L}_D(\phi) = D_{\mathcal{E}nd\mathcal{E}}(\phi),\ \phi \in Aut\mathcal{E}.$$

Now, the above, in conjunction with the remarks in Chapter II, Section 5, suggests that

the description of the shift

$$\mathcal{E} \rightsquigarrow \mathcal{E}nd\mathcal{E},$$

(1.33) as akin to a *quantization process* (ibid.), is further supported by relation (1.26) through

$$D_{\mathcal{E}nd\mathcal{E}} = \mathcal{L}_D.$$

Thus, "differentiating with respect to $\mathcal{E}nd\mathcal{E}$" is reduced, in effect, to a quantization procedure (Lie derivative).

On the other hand, by using the terminology of Chapter I, (4.59), one concludes that

(1.34) (1.32.2) is tantamount to converting ϕ, the same already being an $\mathcal{A}$-automorphism of $\mathcal{E}$ in the category

(1.34.1) $$\mathcal{V}ect\mathcal{S}h_X$$

(vector sheaves on X), into an automorphism of $\mathcal{E}$ in the (Yang–Mills) category

(1.34.2) $$\mathcal{YM}_X.$$

Thus, by the very definitions, this is (i.e., the previous transition of φ is)

(1.35) still equivalent to ϕ being in the isotropy (stability) group of D, $\mathcal{O}(D)$.

2 Tangent Spaces

For the sake of generality, hence, of simplicity, we suppose that we have the same framework as in Section 1 (see also Chapt. II, Section 1), following the relevant terminology therein.

So we start by first giving the definition of what one may consider as the

(2.1) tangent space of $\mathcal{C}onn_{\mathcal{E}}(\mathcal{E})$ (see Chapt. II, Section 2) at

(2.1.1) $$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

which we denote, following classical notation (finite-dimensional differential geometry), by

(2.1.2) $$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D).$$

Now, for all the technical difficulties that at first sight appear to be inherent in our previous task, as presented by (2.1), note that the space of $\mathcal{A}$-connections of $\mathcal{E}$; that is, the set

(2.2) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$

nontrivial by our hypothesis, is not, of course, a usual manifold (!). Things are actually made much easier by just remarking that the space at issue is an affine space, modeled after $\Omega^1(\mathcal{E}nd\mathcal{E})(X)$; that is, one has

(2.3) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) = D + \Omega^1(\mathcal{E}nd\mathcal{E})(X)$$

for any given $\mathcal{A}$-connection D of $\mathcal{E}$; the last equality above holds true, within a bijection (see (1.6) in the preceding section), given by the map

(2.4) $$\tau_D : \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E})(X),$$

such that one sets

(2.5) $$\tau_D(D') := D' - D \equiv u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$$

for any $D' \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$. See also [VS: Chapt. VI, p. 30, (7.6)]. Yet, by still employing our previous notation in Chapter II, (2.31) and (2.47), one also obtains, concerning the same map (2.4) as above, (2.6) $$\tau_D(D') = D' - D = D^{(D',D)}_{\mathcal{E}nd\mathcal{E}}(1_{\mathcal{E}}) \equiv D_{\mathcal{E}nd\mathcal{E}}(1)$$

for any D, D' in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$.

Therefore, based on (2.3), one way look at the space $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ is through the bijection τ_D, as defined by (2.5), for any given fixed $\mathcal{A}$-connection D of $\mathcal{E}$. In other words, one may always employ herewith the bijection

(2.7) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \underset{\tau_D}{\cong} \Omega^1(\mathcal{E}nd\mathcal{E})(X),$$

the second member of (2.7) being an $\mathcal{A}(X)$-module, hence, *a fortiori*, a $\mathbb{C}$-vector space (of course, infinite-dimensional, in general!).

Thus, based on the preceding, it is now quite natural, by following standard patterns (see also Section 3), to come to the following definition by also meeting the desired question set forth already by (2.1). So we set

(2.8) $$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D) := \Omega^1(\mathcal{E}nd\mathcal{E})(X).$$

Now, we proceed to Section 3, to provide a "geometrical" meaning to definition (2.8) by suitably specializing on our structure sheaf $\mathcal{A}$ (also see Section 4).

Namely, the idea here is that one can consider an appropriate topological vector space as a suitable model to develop differential-geometric notions within an infinite-dimensional framework. Of course, this is already quite standard and old (!), being also, in effect, the essential meaning of the above identification (2.8), as will become clearer in Section 4.

3 Geometrical Meaning of $T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D)$

As the title of this section indicates, we give a "geometrical meaning" to any particular element, say

(3.1) $$u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X),$$

this being viewed, by virtue of (2.7) and (2.8) as a "tangent vector" of $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ at D, the given fixed (loc. cit.) $\mathcal{A}$-connection of $\mathcal{E}$. Thus, our claim is that

(3.2) the element $u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$, as in (3.1), may be viewed as a suitable tangent vector to a curve in $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, the latter space being further identified, in view of (2.7) above, with $\Omega^1(\mathcal{E}nd\mathcal{E})(X)$.

Thus, the above framework, as presented by (3.2), is now exactly what one understands, according to the standard usage of the term (classical differential geometry of smooth manifolds), by a "geometric" (*viz.*, "newtonian" (!)) description of the notion of a tangent vector.

Now, according to standard patterns, the notion of a "tangent vector to a curve" at some of its points is, in effect, locally determined. Therefore, one needs to have at one's disposal the local behavior of the space in which the curve in question arrives (see (3.2)). In other words, it is enough to look at the space of $\mathcal{A}$-connections of $\mathcal{E}$, $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$, only locally; yet, it suffices just to consider the space (see also (2.7) in the preceding section)

$$(3.3)\qquad \begin{aligned} \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U &\subseteq \mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U) \\ &\cong (\Omega^1|_U)(\mathcal{E}nd(\mathcal{E}|_U))(U) = \Omega^1(\mathcal{E}nd\mathcal{E})(U) \end{aligned}$$

(everything gets localized !), where U is any open set in X, the last member in (3.3) being a $\mathbb{C}$-vector space. (It is essentially an $\mathcal{A}(U)$-module; see also (3.8) below.) Now, in this connection, what one really wants is to have the latter space as a topological ($\mathbb{C}$-)vector space (see (3.24)), something that, as already mentioned, one can actually achieve by a suitable supplementary hypothesis for our structure sheaf $\mathcal{A}$ (see Section 4; we still note that this extra hypothesis for $\mathcal{A}$ is always satisfied in the classical case of smooth (*viz.*, $\mathcal{C}^{\infty}$-)manifolds, where one has, as we know, $\mathcal{A} \equiv \mathcal{C}^{\infty}_X$; see Volume I, Chapt. I, (1.15)).

However, before we come to that matter, more comments on our previous relations in (3.3) are still in order. Thus, by referring to the first member of (3.3), one actually considers the $\mathcal{A}$-connections of $\mathcal{E}$ restricted to (the open set) $U \subseteq X$; that is, in effect,

(3.4) $\mathcal{A}|_U$-connections of the vector sheaf $\mathcal{E}|_U$ (thus, in fact, an $\mathcal{A}|_U$-module) on U,

hence, the first "inclusion" relation in (3.3). Thus, one gets, in effect, the restriction of a given Yang–Mills field $(\mathcal{E}, D)$ on X to $U \subseteq X$; that is, one has

$$(3.5)\qquad (\mathcal{E}, D)|_U := (\mathcal{E}|_U, D|_U) \equiv i_U^*((\mathcal{E}, D)).$$

The last term in (3.5) denotes the pull-back of $(\mathcal{E}, D)$ under the canonical inclusion map

$$(3.6)\qquad U \underset{i_U}{\overset{\subset}{\longrightarrow}} X.$$

It is actually the last map which, by the pull-back map i_U^* associated with it, defines a "differential triad" on U that corresponds to the given one on X, as in (1.3). Thus, one sets

$$(3.7)\qquad i_U^*((\mathcal{A}, \partial, \Omega^1)) := (\mathcal{A}|_U, \partial|_U, \Omega^1|_U),$$

which also provides the "differential setting" for (3.3). So, concerning the second relation (bijection) therein, this follows, of course, straightforwardly from (2.7) by setting $X = U$ and taking still (3.7) into account. In this connection, see also [VS: Chapt. VI, p. 28, Section 6.1].

Finally, pertaining to the last relation in (3.3), one actually gets the following calculations:

$$
\begin{aligned}
\Omega^1(\mathcal{E}nd\mathcal{E})(U) &\equiv (\Omega^1 \otimes_{\mathcal{A}} \mathcal{E}nd\mathcal{E})(U) \\
&= \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^1(\mathcal{E}))(U) = Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \Omega^1(\mathcal{E})|_U) \\
&\equiv Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \Omega^1|_U \otimes_{\mathcal{A}|U} \mathcal{E}|_U) \qquad (3.8) \\
&= \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \Omega^1|_U \otimes_{\mathcal{A}|_U} \mathcal{E}|_U)(U) \\
&= (\Omega^1|_U)(\mathcal{E}nd(\mathcal{E}|_U))(U),
\end{aligned}
$$

which, of course, along with the preceding, completely justifies (3.3). ■

(In this regard, see also [VS: Chapt. I, p. 55, (11.40), and Chapt. II, p. 132, Lemma 5.1, along with p. 135, (6.8) and (6.9) as well as Chapt. IV, p. 304, Corollary 6.1].)

On the other hand, according to our hypothesis, $\mathcal{E}$ and Ω^1 are vector sheaves on X, so let

$$
rk\mathcal{E} = n \quad \text{and} \quad rk\Omega^1 = m. \qquad (3.9)
$$

Thus, assuming further that the above open set $U \subseteq X$ is, in particular, a common local gauge of both $\mathcal{E}$ and Ω^1 (something that we may always do, according to the very definitions), one obtains, by definition and (3.9), that

$$
\mathcal{E}|_U = \mathcal{A}^n|_U \quad \text{and} \quad \Omega^1|_U = \mathcal{A}^m|_U \qquad (3.10)
$$

within $\mathcal{A}|_U$-isomorphisms of the $\mathcal{A}|_U$-modules involved.

Accordingly, by looking at (3.8) and in view of (3.10), one has

$$
\begin{aligned}
\Omega^1(\mathcal{E}nd\mathcal{E})(U) &= (\Omega^1|_U)(\mathcal{E}nd(\mathcal{E}|_U))(U) \\
&= (\Omega^1|_U)(\mathcal{E}nd(\mathcal{A}^n|_U))(U) = \Omega^1(\mathcal{E}nd\mathcal{A}^n)(U) \qquad (3.11) \\
&= \Omega^1(M_n(\mathcal{A}))(U) \equiv (\Omega^1 \otimes_{\mathcal{A}} M_n(\mathcal{A}))(U) \\
&\equiv M_n(\Omega^1)(U) = M_n(\Omega^1(U)).
\end{aligned}
$$

Therefore, by virtue of (3.3) and (3.11), one finally obtains

$$
\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \subseteq \mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U) = M_n(\Omega^1(U)) = M_n(\Omega^1)(U). \qquad (3.12)
$$

Consequently,

(3.13)

> the space of $\mathcal{A}$-connections of $\mathcal{E}$,
>
> $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),$$
>
> when fastened at a certain particular element ($\mathcal{A}$-connection)
>
> $$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$$
>
> has, according to (2.7) and by further restriction to a local gauge U of $\mathcal{E}$—namely, the space
>
> $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \subseteq \mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U) \qquad (3.13.1)$$

—the form of the "sheaf of connection coefficients" of $\mathcal{E}$,

(3.13.2) $$M_n(\Omega^1) := \Omega^1(M_n(\mathcal{A})),$$

the latter being also similarly localized; that is, the space (3.13.1) is of the form

(3.13.2) $$\begin{aligned}(M_n(\Omega^1)|_U)(U) &= M_n(\Omega^1|_U)(U)\\ &= M_n(\Omega^1)(U) = M_n(\Omega^1(U))\end{aligned}$$

(see also (3.12)).

Concerning the applied terminology above, see Volume I, Chapt. I, (2.45), along with [VS: Chapt. VI, p. 46, (9.10), and subsequent comments therein].

Yet, by still employing our previous argument in (3.11), one gets at the relation

(3.14) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \subseteq \mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U) = \mathcal{A}^{n^2 \cdot m}(U) \equiv \mathcal{A}^k(U),$$

where, of course, we set $k \equiv n^2 \cdot m \in \mathbb{N}$, as in (3.10). Thus, by complementing (3.13), as above, one finally concludes that

(3.15)

the space of $\mathcal{A}$-connections of a given Yang–Mills field

$$(\mathcal{E}, D)$$

(fastened at D, as in (3.13)), when taken locally on a given local gauge U of $\mathcal{E}$ (in effect, U can be taken, as a common local gauge of $\mathcal{E}$ and Ω^1, as well)—that is, the space (see (3.13.1)),

(3.15.1) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \subseteq \mathcal{C}onn_{\mathcal{A}|U}(\mathcal{E}|_U)$$

—has the form

(3.15.2) $$\mathcal{A}^k(U) = \mathcal{A}(U)^k,$$

such that

(3.15.3) $$k = n^2 \cdot m \in \mathbb{N}$$

with n and m in $\mathbb{N}$ as given by (3.10).

Now, our previous conclusion in (3.15), pertaining to the form of the space (3.15.1), hints already, through (3.15.2), at the desired property of being, namely,

(3.16) $$\Omega^1(\mathcal{E}nd\mathcal{E})$$

a *topological vector space sheaf* on X, when our structured sheaf $\mathcal{A}$ is further suitably enriched. This matter, being also crucially related with our initial claim in (3.2), is examined in Section 4.

However, we still remark, for later use, that as a byproduct of the preceding discussion, one can get at an analogous conclusion to (3.13), pertaining now, in effect, to the moduli space of $\mathcal{E}$ itself (see Chapt. II, Section 2, yet, in particular, (2.16) therein). Thus, one obtains that

(3.17) the moduli space of $\mathcal{E}$ (ibid. (2.50)), when similarly restricted on a local gauge of $\mathcal{E}$, say $U \subseteq X$, as above (see (3.13))—that is, the space (see also (3.15.1))

(3.17.1) $$(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/Aut\mathcal{E})|_U \subseteq \mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U)/Aut(\mathcal{E}|_U)$$

—is of the form

(3.17.2) $$M_n(\Omega^1(U))/GL(n, \mathcal{A}(U)) = M_n(\Omega^1)(U)/\mathcal{GL}(n, \mathcal{A})(U)$$

(see also (3.13.2), along with Chapt. II, (1.14.2)).

On the other hand, by performing on (3.17.2) analogous calculations to (3.14), or even to (3.15), one finally concludes that

(3.18) the moduli space of $\mathcal{E}$ is locally (*viz.*, when restricted to a common local gauge of $\mathcal{E}$ and Ω^1, say $U \subseteq X$; see (3.17.1)) of the form

(3.18.1) $$\mathcal{A}^k(U)/GL(n, \mathcal{A}(U))$$

with $k = m \cdot n^2 \in \mathbb{N}$, as in (3.10) (see also (3.15.3)), in such a manner that one still sets, concerning the last relation, as above,

(3.18.2) $$\mathcal{A}^k(U) \cong M_n(\mathcal{A}^m(U)).$$

Yet, within the same vein of ideas, we explain our argument in (3.17), hence in (3.18), by some of our relevant calculations in the foregoing. Indeed, by referring to (3.17.1), one can first consider the restriction of $\mathcal{A}ut\mathcal{E}$ on U, taking then, according to the comments in Chapter II, following (2.50) therein, its (global) sections over U, so that one thus obtains

(3.19) $$((\mathcal{A}ut\mathcal{E})|_U)(U) = (\mathcal{A}ut\mathcal{E})(U) = Isom_{\mathcal{A}|U}(\mathcal{E}|_U, \mathcal{E}|_U) \equiv Aut(\mathcal{E}|_U)$$

(see also [VS: Chapt. VI, p. 93, (17.23)]). Now, the last group can further be viewed as acting on the space

(3.20) $$\mathcal{C}onn_{\mathcal{A}|_U}(\mathcal{E}|_U),$$

according to (3.5) and (3.7) (see also Chapt. II, (2.55)), which thus explains the assertion in (3.17.1). In particular, supposing now that U is a local gauge of $\mathcal{E}$ (see (3.10)), one gets (see also (3.19))

(3.21) $$\begin{aligned} Aut(\mathcal{E}|_U) &= Aut(\mathcal{A}^n|_U) = (\mathcal{A}ut\mathcal{A}^n)(U) \\ &= M_n(\mathcal{A}^{\bullet})(U) = \mathcal{GL}(n, \mathcal{A})(U) = GL(n, \mathcal{A}(U)), \end{aligned}$$

which, in turn, justifies (3.17.2). On the other hand, as already said, (3.18) is simply an immediate consequence of our relevant argument in (3.15), as above. ■

Now, the preceding, in conjunction with our previous considerations in (3.13), or even in (3.15), lead us also to the following general remarks:

Scholium 3.1 (α) Concerning our previous conclusion, for instance, in (3.12), the fact that

when looking at the matter, locally, always (!),

(3.22)

(3.22.1) the space of $\mathcal{A}$-connections of a given vector sheaf $\mathcal{E}$ over X is essentially a space of "matrices of 1-forms,"

a consequence, in effect, of the very nature of $\mathcal{E}$ (locally free $\mathcal{A}$-module), comes short of information for the space under consideration. In fact, the preceding result is more direct, than that we already know, namely, that

an $\mathcal{A}$-connection D of $\mathcal{E}$ is locally represented by a family of matrices of 1-forms, this family parametrized, for instance, by the local gauges of a given local frame of $\mathcal{E}$, say

(3.23.1) $$\mathcal{U} = (U_\alpha)_{\alpha \in I};$$

that is, one has an identification (bijective correspondence)

(3.23) (3.23.2) $$D \longleftrightarrow \omega = (\omega^{(\alpha)}) \in C^0(\mathcal{U}, M_n(\Omega^1)),$$

so that one has

(3.23.3) $$\omega^{(\alpha)} \equiv (\omega_{ij}^{(\alpha)}) \in M_n(\Omega^1(U_\alpha)) = M_n(\Omega^1(U_\alpha))$$

for any $\alpha \in I$, as in (3.23.1). Of course, the above provide already the "inclusion" in (3.12), the range of (3.23.2) being otherwise specified by the so-called *transformation law of potentials*. (Thus, inclusion, as before, is actually proper (!)).

In this concern, see also Volume I, Chapter I, (2.50), (2.54), and (2.56.1); yet, see [VS: Chapt. VII, p. 119, Theorem 3.2].

However, what is actually deduced from the previous discussion is the fact that

by fixing a particular $\mathcal{A}$-connection of $\mathcal{E}$, the space of all $\mathcal{A}$-connections of $\mathcal{E}$, due to its affine nature, attains the structure of a ($\mathbb{C}$-)vector space. In particular, as already said in the foregoing,

(3.24)

(3.24.1) the structure of a topological ($\mathbb{C}$-)vector space, provided we can dispose of the pertinent "structure sheaf" $\mathcal{A}$, is something that one actually needs (see Sections 4 and 5), as we have explained at the beginning of this Section. Yet, our remarks in (3.24.1) are especially highlighted by the corresponding argument in (3.15) in the preceding; see (3.15.2) therein, exhibiting the

(3.24.2) type of space that is actually reduced to, locally (!), the space of $\mathcal{A}$-connections of $\mathcal{E}$, when fixed at some of its points

(see also (3.18.2); yet, see (2.7) as well as (3.2) in the foregoing).

(β) On the other hand, the same correspondence (3.23.2), as above, suggests and further justifies the (physical) aspect that

(3.25) a (gauge) field may be construed as an $\mathcal{A}$-connection.

The previous assertion can be further supported, since (i) an $\mathcal{A}$-connection can be viewed as an "observable" by means of the respective curvature (=field strength), to which property one can still attribute the "geometrical substance" of an ($\mathcal{A}$-)connection, which, occasionally, is usually associated with an $\mathcal{A}$-connection, the latter notion being, in effect, an analytic (algebraic) one when viewed as, as it actually is, a (global $\mathbb{C}$-) linear sheaf morphism, having further the *Leibniz property*, alias a Leibniz ($\mathbb{C}$-)linear sheaf morphism, or, briefly, just a Leibniz (sheaf) morphism of the $\mathcal{A}$-modules concerned (see Chapt. I, Definition 2.1). On the other hand, (ii) by looking at an $\mathcal{A}$-connection through correspondence (3.23.2), one understands it (locally (!)) as

(3.26) a set (family) of (functions) sections, defined (locally) on a topological space X (which, otherwise, may stand here for the classical notion of "space–time"), that are subject further to a certain particular transformation law (*viz.*, to the so-called *transformation law of potentials*) under the "structure group" of the theory. (Notice that the aforesaid family is usually parametrized by a local frame of $\mathcal{E}$; see (3.23.1).)

In this context, we also remark that the aforementioned group in (3.26) may be identified with the (internal) symmetry group considered in the foregoing, which still appears in disguised form (representation; see Chapt. II, Section 1.1) as the group sheaf $\mathcal{A}ut\mathcal{E}$, or its corresponding group of global sections,

$$(\mathcal{A}ut\mathcal{E})(X) \equiv Aut\mathcal{E}, \tag{3.27}$$

which again is reduced, locally, to the group sheaf

$$\mathcal{GL}(n, \mathcal{A}) \equiv M_n(\mathcal{A})^{\bullet} \tag{3.28}$$

with $n = rk\mathcal{E} \in \mathbb{N}$. The latter still appears, through its various sets (in effect, groups) of local sections, as

$$\mathcal{GL}(n, \mathcal{A})(U) = GL(n, \mathcal{A}(U)), \tag{3.29}$$

with U varying over the local gauges (*generalized coordinate systems*) of $\mathcal{E}$ (see, for instance, (3.17)).

The preceding, in conjunction with our claim in (3.25), may also be viewed as being still in accord with the ever-existing standard interpretation of a "field." See, for instance, R.F. Streater–A.S. Wightman [1: p. 96]. ■

On the other hand, the previous discussion points out once more the instrumental role of the structural sheaf (= our (generalized) "arithmetics") $\mathcal{A}$. Thus, in point of fact, we have several times realized, so far, that

(3.30) all our calculations are virtually performed, locally (!), in terms of (local) sections of $\mathcal{A}$.

Of course, the same holds true in the classical theory as well, concerning the sheaf of (germs of) smooth functions

$$\mathcal{A} \equiv \mathcal{C}_X^{\infty}. \tag{3.31}$$

However,

(3.32) in complete contrast to the classical case, where (3.31) is supplied by the type of particular smooth manifold X considered, here that is, within the present abstract setup (ADG),

(3.32.1) all this is transferred to $\mathcal{A}$ itself,

in the sense that the "structure sheaf" $\mathcal{A}$ is assumed to have *eo ipso* (*viz.*, axiomatically, not due to the particular space, at issue!) the necessary (abstract) differential-geometric mechanism that is appropriate to our purposes, without the intervening of any space, supplying the aforementioned mechanism (!) (*viz.*, in other words, (3.31), as above, either in the classical or in the "abstract" sense).

Yet, we may further look at the above attitude as the *Leibniz's point of view.* See, for instance, A. Mallios [9], along with N. Bourbaki [2: Chapt. I, Note hist., p. 161, footnote 1].

We close this section by assembling our main conclusions concerning the "tangent space"

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D), \tag{3.33}$$

the "geometrical meaning" of which is still to be explained, as this was stated at the beginning of the present section. (See Section 5.)

So, as an outcome of the foregoing, one gets the following relations when looking at a fixed $\mathcal{A}$-connection D of $\mathcal{E}$ (see (2.7), along with (3.3), (3.8), (3.11), (3.15.2), and (3.18.2)):

$$\begin{aligned} \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \underset{\longrightarrow}{\subset} \Omega^1(\mathcal{E}nd\mathcal{E})(U) &= (\Omega^1|_U)(\mathcal{E}nd(\mathcal{E}|_U))(U) \\ &\cong T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U, D) = M_n(\Omega^1(U)) \\ &= M_n(\mathcal{A}^m(U)) \equiv \mathcal{A}^k(U), \end{aligned} \tag{3.34}$$

the $\mathcal{A}$-connection D of $\mathcal{E}$ being actually understood as

(3.35) $$D \equiv D|_U := i_U^*(D),$$

in the sense of (3.6) and (3.7), while U stands here for a local gauge of both $\mathcal{E}$ and Ω^1. Now, as we shall see (Section 5), the above relations (3.34) will actually permit us to look at the space (3.33) at its "localization" (restriction) on U, as in (3.34), as a topological vector space for suitable sheaf $\mathcal{A}$. Of course, our task here is obviously completely facilitated, in view of the last part of (3.34), as above, and the relevant remarks at the end of Section 2, concerning relation (2.8), therein (see Section 4).

4 $\Omega^1(\mathcal{E}nd\ \mathcal{E})$, as a Topological ($\mathbb{C}$-)Vector Space Sheaf

As already noted in the foregoing, we consider in this section

(4.1) the case that our "structure sheaf" $\mathcal{A}$ supplies the local section $\mathcal{A}(U)$-modules, for the various open $U \subseteq X$, with the structure of a topological $\mathbb{C}$-vector space.

Thus, we assume henceforth, concerning the present section, that

(4.2) $\mathcal{A}$ is a topological ($\mathbb{C}$-)vector space sheaf on X (the latter being an arbitrary topological space, as usual, so far), in the sense that

(4.2.1) for every open set $U \subseteq X$, the local section $\mathcal{A}(U)$ yields a topological ($\mathbb{C}$-)vector space such that the corresponding transition map

$$\rho_V^U : A(U) \longrightarrow A(V)$$

for any open sets $V \subseteq U \subseteq X$ is a continuous linear map. Yet, we assume herewith (for convenience) that

(4.2.2) the preceding spaces have nontrivial topological duals—namely, we accept that

$$A(U)' \neq \{0\}$$

for any open $U \subseteq X$.

Examples (i) The classical case

(4.3) $$\mathcal{A} \equiv \mathcal{C}_X^\infty$$

(see (3.31)) is an important particular instance of the previous situation, as in (4.2). In point of fact, the respective sheaf

(4.4) $\mathcal{C}_X^\infty$ on a given smooth ($\mathcal{C}^\infty$-)manifold X is an example of a locally convex ($\mathbb{C}$-)vector space sheaf on X, for which (4.2.2) is in force.

The respective (locally convex) vector space topology on each ($\mathbb{C}$-)vector space,

(4.5) $$\mathcal{C}_X^\infty(U) \cong \mathcal{C}^\infty(U)$$

for any open $U \subseteq X$, is the so-called *Schwartz topology*—namely, that one of the "uniform convergence on compacta" of the ($\mathcal{C}^\infty$-)functions in (4.5) (we consider them here as $\mathbb{C}$-valued) and of all of their partial derivatives (see, for instance, A. Mallios [TA: p. 131]).

In point of fact, the same sheaf as in (4.3) is, in particular,

(4.6) a topological ($\mathbb{C}$-)algebra sheaf on X of the type locally m-convex (topological $\mathbb{C}$-)algebra, in the sense that each one of the sets, as in (4.5), is a $\mathbb{C}$-algebra of the previous sort, while the same (topological) algebra still has the important property that

(4.6.1) $$\mathfrak{M}(\mathcal{C}^\infty(U)) = U$$

within a homeomorphism of the topological spaces involved (TA: p. 227, Theorem 2.1, along with Scholium 2.1 therein). By the very definitions, (4.6.1) is already stronger than (4.2.2).

Topological algebra sheaves, in general, of the previous type, even of a milder one, concerning the above condition (4.6.1), have been considered already in the foregoing; see Chapter I, Section 7.1, in particular, Scholium 7.1. Yet, in Section 8, in particular, Note 8.1, where we looked at $\mathcal{A}$, as a topological ($\mathbb{C}$-)vector space sheaf on X (see (4.2) as above).

(ii) Topological algebra sheaves of the same type as in (4.6) are also the sheaf of (germs of) continuous ($\mathbb{C}$-valued) maps

(4.7) $$\mathcal{C}_X$$

on a completely regular (Hausdorff) space X, where one has

(4.8) $$\mathcal{C}_X(U) \cong \mathcal{C}_c(U)$$

for any open $U \subseteq X$. In the second member of (4.8), one considers the "compact-open topology" on U, such that one has

(4.9) $$\mathfrak{M}(\mathcal{C}_c(U)) = U$$

within a homeomorphism of the spaces concerned (see also [TA: Chapt. VII, p. 223, Theorem 1.2]).

On the other hand, another classical important topological algebra sheaf is

(4.10) $$\mathcal{O}_X$$

(*viz.*, the sheaf of (germs of) holomorphic functions on a complex (analytic) manifold X, in particular, when X is a Stein manifold (see loc. cit. pp. 228ff)). For such a manifold X (not necessarily Stein), one can choose a basis of its topology consisting of (open) *Stein submanifolds*, so that one has

$$\mathfrak{M}(\mathcal{O}(U)) = U \tag{4.11}$$

within a homeomorphism of the spaces involved, with U running over the aforesaid basis. (See TA: p. 230, Lemma 3.1 as well as L. Kaup-B. Kaup [1: p. 224, Corollary 51.5]).

4.1 Vector Sheaves, Locally Topological Modules

Suppose we are given a $\mathbb{C}$-algebraized space

$$(X, \mathcal{A}), \tag{4.12}$$

as usual, on a topological space X, such that the "structure sheaf" $\mathcal{A}$ is, in particular, a topological algebra sheaf on X (see Chapt. I, Section 7.1). On the other hand, suppose that $\mathcal{E}$ is a given vector sheaf on X, with

$$rk\mathcal{E} = n \in \mathbb{N}. \tag{4.13}$$

Then,

(4.14) $\mathcal{E}$ can be construed as a topological ($\mathbb{C}$-)vector space sheaf on X;

that is, one actually proves that

(4.15) for any open $U \subseteq X$, the corresponding local section set $\mathcal{E}(U)$ is a $\mathbb{C}$-vector space, such that the various stalks of $\mathcal{E}$,

$$\mathcal{E}_x, \ x \in X, \tag{4.15.1}$$

are topological $\mathbb{C}$-vector spaces.

For convenience, we recall that, by virtue of our hypothesis for $\mathcal{E}$,

(4.16) every point $x \in X$ has a fundamental system (basis) of given neighborhoods, consisting of local gauges of $\mathcal{E}$.

See [VS: Chapt. II, p. 125, (4.6)]. Thus, the assertion in (4.15) is now a consequence of (4.16), the relation (see also (4.13))

$$\mathcal{E}(U) = \mathcal{A}^n(U) = \mathcal{A}(U)^n, \tag{4.17}$$

valid, for any local gauge U of $\mathcal{E}$, within $\mathcal{A}(U)$-isomorphisms of the $\mathcal{A}(U)$-modules concerned, in conjunction with our argument in [VS: Chapt. XI, p. 301, Theorem 1.1], applied, in particular, on the stalks of $\mathcal{E}$, (4.18) $$\mathcal{E}_x = \varinjlim_{U \in \mathcal{B}(x)} \mathcal{A}^n(U), \ x \in X,$$

with $\mathcal{B}(x)$ the basis mentioned in (4.16). Yet, the same argument (loc. cit.) justifies our claim now in (4.14). ■

Note 4.1 By clarifying our previous argument in (4.15), as connected with (4.18) in the preceding, we still note, in that context, that one considers on $\mathcal{E}_x$, $x \in X$ (loc. cit.), the so-called *inductive limit vector space topology* defined on it, according to our hypothesis in (4.2.1). Now, in the case that the various $\mathcal{A}(U)$, with U open in X (ibid.) are, in particular, topological algebras, then the same topology, as before, makes $\mathcal{E}_x$, as in (4.18), a topological algebra too. In this regard, see also A. Mallios [TA: p. 115, Lemma 2.2, first part] or even A. Mallios [VS: Chapt. XI, p. 301, Theorem 1.1.].

Thus, the preceding leads now to the general conclusion that

(4.19) the topological-algebraic sheaf structure that may occasionally have our structure sheaf $\mathcal{A}$ is inherited, at least (see also Note 4.1), concerning the topological vector space structure, to every vector sheaf $\mathcal{E}$ on X (see (4.14)).

Now, the above remarks in (4.19), together with those in Note 4.1, explain also the precise meaning of the title of the present section.

On the other hand, by looking further, in particular, at the vector sheaf on X,

(4.20) $$\Omega^1(\mathcal{E}nd\mathcal{E}),$$

as in the preceding Section 3, one obtains

(4.21) $$\Omega^1(\mathcal{E}nd\mathcal{E})(U) = \mathcal{A}^k(U) = \mathcal{A}(U)^k,$$

within $\mathcal{A}(U)$-isomorphism of the $\mathcal{A}(U)$-modules involved, for every local gauge U of $\mathcal{E}$, which might also be viewed as a local gauge of (4.20); see (3.8), (3.10), (3.11), and (3.15.3). Therefore (see (4.19)),

(4.22) $\Omega^1(\mathcal{E}nd\mathcal{E})$ is a topological ($\mathbb{C}$)-vector space sheaf on X.

In this context, it should be remarked here that

(4.23) although (4.21) is valid, in general, only for a local gauge of $\mathcal{E}$ that may also be such for the vector sheaf (4.20) too, one concludes from (4.19), in conjunction with [VS: Chapt. XI, p. 302, (1.7)], that

(4.23.1) $$\mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$$

is, in effect, a topological ($\mathbb{C}$-)vector space for every open $U \subseteq X$ and for any vector sheaf $\mathcal{E}$ on X and, therefore, in particular, for the vector sheaf $\Omega^1(\mathcal{E}nd\mathcal{E})$ on X, as well. (Thus, relation (4.21) is not necessarily valid for any open $U \subseteq X$(!).)

The above conclusion is, indeed, what one actually needs to look at the "tangent space" $T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D)$ (see Section 3), as the totality of its "tangent vectors," in the classical sense of the latter notion. We consider this matter in Section 5.

5 Geometric Meaning of $T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D)$ (continued)

We prove here that, what we have called thus far, the "tangent space,"

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D), \tag{5.1}$$

of the space of $\mathcal{A}$-connections of $\mathcal{E}$,

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{5.2}$$

at a given $\mathcal{A}$-connection D of $\mathcal{E}$ can be construed as the totality of its "tangent vectors."

Now, as already remarked in the preceding, this actually constitutes the description of (5.1) in the classical sense, or even the "geometric meaning" of the same space (5.1) in the point of view of the classical differential geometry (CDG) of smooth (*viz.*, $\mathcal{C}^{\infty}$-)manifolds.

In other words, by employing herewith standard terminology, we virtually prove that

(5.3)

for every element

$$u \in T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D), \tag{5.3.1}$$

one obtains

$$u = \dot{\alpha}(0) \tag{5.3.2}$$

for some curve

$$\alpha : I \longrightarrow \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{5.3.3}$$

with I an open neighborhood of $0 \in \mathbb{R}$, such that

$$\alpha(0) = D. \tag{5.3.4}$$

Moreover, concerning the terminology applied in (5.3), one sets

$$\dot{\alpha}(0) := \lim_{t \to 0} \frac{1}{t}(\alpha(t) - \alpha(0)). \tag{5.4}$$

However, first, we have to comment a bit more on the previously applied notation. Thus, first, as already remarked in Section 2, we consider henceforth the space of $\mathcal{A}$-connections of $\mathcal{E}$, as pinched at the given $\mathcal{A}$-connection D of $\mathcal{E}$, so that one actually obtains (see (2.3))

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \cong \Omega^1(\mathcal{E}nd\mathcal{E})(X); \tag{5.5}$$

hence, one further defines (5.1) by the relation (see (2.8))

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D) \equiv \Omega^1(\mathcal{E}nd\mathcal{E})(X). \tag{5.6}$$

Therefore, our previous considerations in (5.3) are actually transferred, by definition, to the space

$$\Omega^1(\mathcal{E}nd\mathcal{E})(X) \tag{5.7}$$

(see also the comments at the beginning of Section 3). However,

(5.8) assuming, henceforth, the framework of Section 4, we know (ibid.) that

$$\Omega^1(\mathcal{E}nd\mathcal{E}) \tag{5.8.1}$$

is a topological ($\mathbb{C}$-)vector space sheaf on X (see (4.22)).

Accordingly, for any element (see (3.1) and (3.6))

$$u \in T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D) \cong \Omega^1(\mathcal{E}nd\mathcal{E})(X), \tag{5.9}$$

one can consider the curve

$$\alpha : \mathbb{R} \longrightarrow \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) : t \longmapsto \alpha(t) := D + t \cdot u \tag{5.10}$$

such that one has

$$\alpha(0) = D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \cong \Omega^1(\mathcal{E}nd\mathcal{E})(X). \tag{5.11}$$

On the other hand, the same curve as above can still be considered as a family

$$(\alpha_U)_{U \in \mathcal{U}}, \tag{5.12}$$

parametrized by the elements (local gauges) of a local frame of $\mathcal{E}$, so that one has

$$\alpha_U : I_U \longrightarrow \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U \cong \Omega^1(\mathcal{E}nd\mathcal{E})(U) \cong M_n(\Omega^1(U)) \cong \mathcal{A}^k(U), \tag{5.13}$$

where I_U is an open neighborhood of $0 \in \mathbb{R}$, such that (5.13) have a meaning; namely,

$$\alpha_U(I_U) \subseteq \Omega^1(\mathcal{E}nd\mathcal{E})(U) \cong \mathcal{A}^k(U) \cong \mathcal{A}(U)^k. \tag{5.14}$$

Thus, (5.4) and the desired condition (5.3.2) are now satisfied, according to the very definitions and the particular structure of the range of α_U, with U as in (5.12) (see also (4.22) and (4.23)).

The above explains completely our argument in (5.3), as well as the claimed structure of (5.1), through its various local expressions

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})|_U, D), \tag{5.15}$$

(see also (3.34) in the preceding) as a "tangent space" of $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$ at D in the classical sense, which was exactly formulated by (5.3). ■

6 Tangent Space of the Orbit of an $\mathcal{A}$-Connection, $T(\mathcal{O}_D, D)$

Given a vector sheaf $\mathcal{E}$ on X and an $\mathcal{A}$-connection of it, say

$$D \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{6.1}$$

we further consider the orbit of D in the space of $\mathcal{A}$-connections of $\mathcal{E}$, as above, under the action of the group sheaf of $\mathcal{A}$-automorphisms of $\mathcal{E}$, $\mathcal{A}ut\mathcal{E}$, or of the respective one of its global sections,

$$Aut\mathcal{E} := (\mathcal{A}ut\mathcal{E})(X). \tag{6.2}$$

Thus, one has, as already defined in the preceding (see Chapt. II, (2.37) and (2.42)), the set (orbit space of D)

$$\begin{aligned} \mathcal{O}_D &:= \{\phi \cdot D : \phi \in Aut\mathcal{E}\} = \{D - D_{\mathcal{E}nd\mathcal{E}}(\phi)\phi^{-1} : \phi \in Aut\mathcal{E}\} \\ &\subseteq \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \cong D + \Omega^1(\mathcal{E}nd\mathcal{E})(X) \end{aligned} \tag{6.3}$$

(see also Note 2.1 therein as well as (2.3) in this chapter, or even Chapter I, (5.7) of Volume I of this treatise, concerning the last bijection, as above).

Therefore, in view of (6.3), by looking at the relation

$$\mathcal{O}_D \subseteq \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \cong D + \Omega^1(\mathcal{E}nd\mathcal{E})(X), \tag{6.4}$$

as well as at the identification (see (2.8)),

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D) \equiv \Omega^1(\mathcal{E}nd\mathcal{E})(X), \tag{6.5}$$

one can further consider the relation

$$T(\mathcal{O}_D, D) \underset{\longrightarrow}{\subset} T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D), \tag{6.6}$$

which may still be justified by the previous relations and the (canonical) identification (see (2.71)),

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \underset{\tau_D}{\cong} \Omega^1(\mathcal{E}nd\mathcal{E})(X). \tag{6.7}$$

The same (6.6) will also be justified right away in the sequel, through the relation

$$T(\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), D) = T(\mathcal{O}_D, D) \oplus T(\mathcal{O}_D, D)^{\perp}, \tag{6.8}$$

where the direct command

$$T(\mathcal{O}_D, D)^{\perp}, \tag{6.9}$$

being, in effect, an "orthogonal complement" of the "tangent space" of $\mathcal{O}_D$ at D,

$$T(\mathcal{O}_D, D), \tag{6.10}$$

along with the last space, as well, will suitably be defined through the subsequent discussion:

Thus, based on the second relation in (6.3) (definition of $\mathcal{O}_D$), (6.4), and the identifications (6.7) and (6.5), one can still define

(6.11) $$T(\mathcal{O}_D, D) := im(D_{\mathcal{E}nd\mathcal{E}}|Aut\mathcal{E}).$$

Now, to explain (6.8), we employ in our argument the notion of an $\mathcal{A}$-metric, first, of course, on our "structure sheaf" $\mathcal{A}$; that is, assume, henceforth, that we are given

(6.12) the framework of Chapter I, Section 4, supposing thus also that X is a paracompact (Hausdorff) space. As a result (Chapt. I, (4.2)), every vector sheaf $\mathcal{E}$ on X becomes a Riemannian vector sheaf

(6.12.1) $$(\mathcal{E}, \rho),$$

while every vector sheaf on X admits an $\mathcal{A}$-connection (loc. cit. (4.3)) as well.

The above hypothesis implies, in effect, that

(6.13) in the sequel, when we refer to the group $Aut\mathcal{E}$ (see (6.2)), we actually mean its subgroup,

(6.13.1) $$(Aut\mathcal{E})_\rho,$$

as the latter was defined in Chapter I, (5.46.2); see also Chapter II, (3.3.2).

Now, based on (6.12), apart from the sheaf morphism ($\mathcal{A}$-connection)

(6.14) $$D_{\mathcal{E}nd\mathcal{E}} : \mathcal{E}nd\mathcal{E} \longrightarrow \Omega^1(\mathcal{E}nd\mathcal{E}),$$

one can further consider its "dual morphism"

(6.15) $$\delta^1_{\mathcal{E}nd\mathcal{E}} : \Omega^1(\mathcal{E}nd\mathcal{E}) \longrightarrow \mathcal{E}nd\mathcal{E}$$

(see Chapt. I, (2.39), (2.29) and (2.30)). Thus, by virtue of the relation, defining (6.15) (Chapt. I, (2.30)), one has

(6.16) $$\rho(D_{\mathcal{E}nd\mathcal{E}}(\phi), t) = \rho(\phi, \delta^1_{\mathcal{E}nd\mathcal{E}}(t))$$

for any

(6.17) $$\phi \in (\mathcal{E}nd\mathcal{E})(X) = End\mathcal{E} \quad \text{and} \quad t \in \Omega^1(\mathcal{E}nd\mathcal{E})(X).$$

Note that the $\mathcal{A}$-metric ρ in (6.16) is actually referred to the vector sheaves $\mathcal{E}nd\mathcal{E}$ and $\Omega^1(\mathcal{E}nd\mathcal{E})$ on X.

Accordingly, by virtue of (6.16), one gets

(6.18) $$im D_{\mathcal{E}nd\mathcal{E}} \subseteq (\ker \delta^1_{\mathcal{E}nd\mathcal{E}})^{\perp},$$

the second member of the previous relation denoting the "orthogonal space" of

(6.19) $$\ker \delta^1_{\mathcal{E}nd\mathcal{E}} \subseteq \Omega^1(\mathcal{E}nd\mathcal{E}),$$

in the obvious sense, with respect to the $\mathcal{A}$-metric, that is still induced on the domain of definition of (6.15) by the given one on $\mathcal{A}$ (see also the Note in (6.17)). Yet, due to the $\mathcal{A}$-bilinearity of ρ (see Chapt. I, (2.1); Condition i)), one easily concludes that

(6.20) $\ker \delta^1_{\mathcal{E}nd\mathcal{E}}$ is, in particular, an $\mathcal{A}$-module too, not just a $\mathbb{C}$-vector space sheaf on X, hence its orthogonal space,

(6.20.1) $$(\ker \delta^1_{\mathcal{E}nd\mathcal{E}})^{\perp},$$

as well.

Of course,

(6.21) the totality of the preceding relations are in force, locally, as well, relative to a local gauge, say $U \subseteq X$, of $\mathcal{E}$ which (due to our hypothesis for $\mathcal{E}$ and Ω^1) may also be considered as such, even for the rest of the vector sheaves involved in the preceding—namely, of $\mathcal{E}nd\mathcal{E}$, Ω^1, and $\Omega^1(\mathcal{E}nd\mathcal{E})$.

On the other hand, as another consequence of (6.16), one obtains

(6.22) $$\ker \delta^1_{\mathcal{E}nd\mathcal{E}} = (im D_{\mathcal{E}nd\mathcal{E}})^{\perp} \subseteq \Omega^1(\mathcal{E}nd\mathcal{E}),$$

yielding thus, by analogy with (6.20), sub-$\mathcal{A}$-modules of the last space in (6.22): The assertion follows, indeed, from (6.16), in conjunction with (6.15), and the pertinent properties of ρ (see, for instance, Chapt. I, (2.6)–(2.8)). ■

Accordingly, by now employing a "local argument" (*viz.*, modulo a local gauge U of $\mathcal{E}$, as in (6.21)), one gets at the crucial relation

(6.23) $$\Omega^1(\mathcal{E}nd\mathcal{E})(U) = im D_{\mathcal{E}nd\mathcal{E}} \oplus \ker \delta^1_{\mathcal{E}nd\mathcal{E}},$$

so that we further set (see also (6.11))

(6.24) $$T(\mathcal{O}_D, D) := im(D_{\mathcal{E}nd\mathcal{E}}|_{Aut\mathcal{E}}) = (\ker(\delta^1_{\mathcal{E}nd\mathcal{E}|Aut\mathcal{E}}))^{\perp}.$$

Now, within the same vein of ideas, by setting

(6.25) $$\begin{aligned} \mathcal{S}_D &:= D + \ker \delta^1_{\mathcal{E}nd\mathcal{E}} = \{D + u \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) : \delta^1_{\mathcal{E}nd\mathcal{E}}(u) = 0\} \\ &\subseteq D + \Omega^1(\mathcal{E}nd\mathcal{E})(X) = \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \end{aligned}$$

(see also (6.22), along with the last relation in (6.23)), we still conclude, by virtue of (6.24), that

(6.26) $\mathcal{S}_D$ can be construed locally (see (6.21)) as the "orthogonal complement", with respect to $\Omega^1(\mathcal{E}nd\mathcal{E})$, of the tangent space of the orbit of D at D.

Thus, the essential point of view of the preceding discussion is, in effect, that

(6.27) we look at the "geometry" of the space of $\mathcal{A}$-connections of $\mathcal{E}$, respectively, of that one of the corresponding moduli space of $\mathcal{E}$, by reducing it to the "geometry" of the "model" of the aforesaid spaces.

(6.28) Thus, together with the partition of the space of $\mathcal{A}$-connections of $\mathcal{E}$, through the orbits of its elements, via the action of the group (sheaf) $\mathcal{A}ut\mathcal{E}$, there is (naturally) associated a similar partition of the model through the corresponding tangent spaces to the orbits of the $\mathcal{A}$-connections of $\mathcal{E}$; yet, by employing suitable $\mathcal{A}$-metrics on the various vector sheaves involved, one still gets at a partition of the model that is "normal" ("orthogonal") to the former one.

We are going to apply the above, right away, to Section 7, looking still within the present abstract setting, at the classical *Gribov's case* ("ambiguity").

Before we close the present section, we still look at certain consequences of the preceding discussion. Thus, our first remark is that by using the notation applied in Note 1.1 in the preceding, one concludes that

(6.29) the *Lie derivative-functor* with respect to a given $\mathcal{A}$-connection D of $\mathcal{E}$,

(6.29.1) $$\mathcal{L}_D \equiv D_{\mathcal{E}nd\mathcal{E}}$$

(loc. cit. (1.25)), preserves the "affine structure of the respective space of $\mathcal{A}$-connections of $\mathcal{E}$.

That is, one has

(6.30) $$\mathcal{L}_{D+u} = \mathcal{L}_D + \mathcal{L}_u$$

in the sense that

(6.31) $$(\mathcal{L}_{D+u})(\phi) = \mathcal{L}_D(\phi) + \mathcal{L}_u(\phi)$$

or even

(6.32) $$[D + u, \phi] = [D, \phi] + [u, \phi]$$

for any $\phi \in \mathcal{E}nd\mathcal{E}$. Indeed, one has (operating formally)

(6.33) $$\begin{aligned}[D + u, \phi] &= (D + u)\phi - \phi(D + u)\\ &= D\phi - \phi D + u\phi - \phi u\\ &\equiv [D, \phi] + [u, \phi]\end{aligned}$$

for any ϕ, as in (6.33)—that is, our assertion. ■

Therefore (see (6.29.1)), we can still write (6.30) (in effect, by definition) in the form

(6.34) $$(D + u)_{\mathcal{E}nd\mathcal{E}} = D_{\mathcal{E}nd\mathcal{E}} + u$$

for any $u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$.

Yet, by applying herewith our previous notation in (6.24), one obtains

$$\begin{aligned} T(\mathcal{O}_{D+u}, D+u) &= im(D+u)_{\mathcal{E}nd\mathcal{E}} = im(D_{\mathcal{E}nd\mathcal{E}} + u) \\ &= \{(D_{\mathcal{E}nd\mathcal{E}} + u)\phi : \phi \in Aut\mathcal{E}\} \\ &= \{D_{\mathcal{E}nd\mathcal{E}}(\phi) + u(\phi) : \phi \in Aut\mathcal{E}\} \\ &= im D_{\mathcal{E}nd\mathcal{E}} + im(u) = T(\mathcal{O}_D, D) + im(u) \end{aligned} \tag{6.35}$$

—in other words, what we may call

(6.36) the "tangent-space functor"—still preserves the "affine structure" of the space of $\mathcal{A}$-connections of $\mathcal{E}$.

Finally, by looking at the "operators" ($\mathbb{C}$-linear (sheaf) morphisms) considered in the foregoing, we can still make the following general remark:

(6.37) for any given vector sheaf $\mathcal{E}$ on X, the operator

$$\delta D (\equiv \delta^1 D^0) : \mathcal{E} \longrightarrow \mathcal{E} \tag{6.37.1}$$

is self-adjoint; that is, one has

$$\rho((\delta D)(s), t) = \rho(s, (\delta D)(t)) \tag{6.37.2}$$

for any $s, t \in \mathcal{E}(U)$.

Indeed, one obtains

$$\begin{aligned} \rho((\delta D)(s), t) &= \rho(t, \delta D s) = \rho(Dt, Ds) \\ &= \rho(Ds, Dt) = \rho(s, \delta D t), \end{aligned} \tag{6.38}$$

our claim. ■

Yet, an analogous conclusion holds true for the operator $D\delta$; that is one has

$$\rho((D\delta)(s), t) = \rho(s, (D\delta)(t)) \tag{6.39}$$

for any $s, t \in \mathcal{E}(U)$. More generally, concerning the rank of the operators considered, the previous conclusions are in force for any $p \in \mathbb{Z}_+$.

7 The Moduli Space of $\mathcal{A}$-Connections as an Affine Space. Gribov's Ambiguity (à la Singer)

Assuming the framework of Section 6, consider the space

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \tag{7.1}$$

of $\mathcal{A}$-connections of a given vector sheaf $\mathcal{E}$ on X, along with the corresponding moduli space of $\mathcal{E}$ (see also Chapt. II, (2.50))

$$M(\mathcal{E}) \equiv \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/Aut\mathcal{E}. \tag{7.2}$$

Now, our first remark here is that

(7.3) the action of $Aut\mathcal{E}$ on the space of $\mathcal{A}$-connections of $\mathcal{E}$ (Chapt. II, Note 2.1) preserves further the "affine structure" of the latter space.

Indeed, it is an easy consequence of the very definitions—the relation

(7.4) $$\phi \cdot (D + u) \equiv (\phi \otimes 1)D\phi^{-1} = \phi D + \phi \cdot u,$$

for any $\phi \in Aut\mathcal{E}$ and $u \in \Omega^1(\mathcal{E}nd\mathcal{E})(X)$—that actually proves the assertion (see also, for instance, (6.4)). ■

In other words, one infers that

(7.5) the group of $\mathcal{A}$-automorphisms of $\mathcal{E}$,

(7.5.1) $$Aut\mathcal{E} := (\mathcal{A}ut\mathcal{E})(X),$$

acts, in effect, on the affine space of $\mathcal{A}$-connections of $\mathcal{E}$, not just on the respective set. Yet, in what amounts to the same thing,

(7.5.2) $Aut\mathcal{E}$ can still be construed, in particular, as the affine group of the affine space of $\mathcal{A}$-connections of $\mathcal{E}$.

In point of fact, what we are actually going to consider herewith is the "Yang–Mills case," that is, in effect, the group

(7.6) $$(Aut\mathcal{E})_\rho := (\mathcal{A}ut\mathcal{E})_\rho(X)$$

(see Chapt. II, (3.3.2)) of metric-preserving $\mathcal{A}$-automorphisms of $\mathcal{E}$. Therefore, the corresponding moduli space of $\mathcal{E}$,

(7.7) $$\mathcal{M}(\mathcal{E})_\rho \equiv \mathcal{C}onn_{\mathcal{A}}(\mathcal{E})/(Aut\mathcal{E})_\rho$$

(loc. cit. (3.6)). Accordingly, by further specializing (7.5.2), one concludes that

(7.8) $(Aut\mathcal{E})_\rho$ can also be considered as an affine group of the Yang–Mills $\mathcal{A}$-connections of $\mathcal{E}$.

Thus, based on the preceding, we can look now at the model of the moduli space of the Yang–Mills $\mathcal{A}$-connections of $\mathcal{E}$, the latter space still being viewed as an affine space. Indeed, by restricting ourselves to the group (7.6) and also taking into account the preceding relations (6.16), (6.22) and (6.25), one obtains that

(7.9) the model of the affine space (7.7) is given by the relations

(7.9.1) $$\Omega^1(\mathcal{E}nd\mathcal{E})(X)/(Aut\mathcal{E})_\rho = (im(D_{\mathcal{E}nd\mathcal{E}|_{(Aut\mathcal{E})_\rho}}))^\perp = \mathcal{S}_D$$

within bijections (in effect, $\mathbb{C}$-vector space isomorphisms).

Yet, by virtue of what has been said thus far concerning "tangent spaces" within the present setting, we still conclude that

(7.10) the "tangent space" of the moduli space $\mathcal{M}(\mathcal{E})_\rho$, as in (7.7), at one of its elements, namely, at an orbit of a Yang–Mills $\mathcal{A}$-connection D of $\mathcal{E}$,

(7.10.1) $$\mathcal{O}_D \equiv [D]_\rho \in \mathcal{M}(\mathcal{E})_\rho,$$

is (isomorphic to) $\mathcal{S}_D$; that is, one has

(7.10.2) $$T(M(\mathcal{E})_\rho, \mathcal{O}_D) = \mathcal{S}_D,$$

within a bijection (see also (7.9.1)).

On the other hand,

(7.11) whether the intersection

(7.11.1) $$\mathcal{S}_D \cap T(\mathcal{O}_D, D)$$

within the space $\Omega^1(\mathcal{E}nd\mathcal{E})(X)$ is just the $\mathcal{A}$-connection D of $\mathcal{E}$ is classically viewed, following I.M. Singer [1], as the so-called *Gribov's ambiguity*.

Yet, according to the same classical terminology (loc. cit.), we still call $\mathcal{S}_D$, as above (see (6.22)), the abstract (generalized) *Coulomb gauge*, that is associated with the given $\mathcal{A}$-connection D of $\mathcal{E}$ (see also K.B. Marathe-G. Martucci [1: p. 161].

We terminate the present discussion by another aspect of Gribov's ambiguity within the above abstract setting, which, however, will require some extra conditions on our structure sheaf $\mathcal{A}$ (always!), that, of course, are fulfilled in the classical case of $\mathcal{C}^\infty$-manifolds (see (3.31)).

Thus, looking at the same framework as above, let us consider again the "gauge group" of a given vector sheaf $\mathcal{E}$ on X, $Aut\mathcal{E}$ (*viz.*, the group of $\mathcal{A}$-automorphisms of $\mathcal{E}$ (see (7.5.1))). On the other hand, given an element $x \in X$, consider the set

(7.12) $$(Aut\mathcal{E})_x := \{\phi \in Aut\mathcal{E} : \phi_x = (id_{\mathcal{E}})_x\} \equiv \ker(\delta_x) \subseteq Aut\mathcal{E},$$

where we further set

(7.13) $$\delta_x(\phi) := \phi(x) \equiv \phi_x,\ x \in X,$$

where the last map in (7.13) stands for the corresponding fiber component of $\phi \in Aut\mathcal{E}$ at $x \in X$. Now, it is immediate, by the same definitions, that

(7.14) (7.14.1) $$(Aut\mathcal{E})_x \leq Aut\mathcal{E}$$

for any $x \in X$; namely, (7.12) defines an abelian subgroup of $Aut\mathcal{E}$.

On the other hand, what one further realizes here, concerning the same group as above, is that

(7.15) for any $x \in X$ the (abelian) group $(Aut\mathcal{E})_x$ (see (7.12)) acts freely on the set $\mathcal{C}onn_{\mathcal{A}}(\mathcal{E})$; that is, as we also say,

(7.15.1) $$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \text{ is a } (Aut\mathcal{E})_x\text{-principal set, for every } x \in X.$$

Indeed, as already hinted at in the preceding, we verify the previous assertion, under a suitable supplementary hypothesis for our "differential setup" considered thus far; in point of fact, one proves, under the same extra hypothesis for the "differentials," a stronger result than (7.15). However, we have first to comment a bit more on the relevant terminology.

Thus, given a vector sheaf $\mathcal{E}$ on X, we have already remarked in the foregoing that

(7.16) $\mathcal{E}nd\mathcal{E}$ is an $\mathcal{A}$-algebra sheaf on X, having as the unit element the identity $\mathcal{A}$-automorphism of $\mathcal{E}$,

(7.16.1) $$id_{\mathcal{E}} \equiv \mathbf{1}_{\mathcal{E}}.$$

Furthermore, its group of units is given by the relation

(7.16.2) $$(\mathcal{E}nd\mathcal{E})^{\bullet} = \mathcal{A}ut\mathcal{E},$$

being thus the group sheaf of $\mathcal{A}$-automorphisms of $\mathcal{E}$, such that one has

(7.16.3) $$\begin{aligned}(\mathcal{E}nd\mathcal{E})^{\bullet}(X) &= ((\mathcal{E}nd\mathcal{E})(X))^{\bullet} \equiv (End\mathcal{E})^{\bullet} \\ &= (\mathcal{A}ut\mathcal{E})(X) = Aut\mathcal{E}.\end{aligned}$$

In this context, see also [VS: Chapt. II; p. 138, Definition 6.2, and p. 139, (6.30) and (6.31) as well as Chapt. V, p. 390, Scholium 8.2]. Now, according to our hypothesis, the "structure sheaf" $\mathcal{A}$ is a unital $\mathbb{C}$-algebra sheaf on X. Therefore, in view also of (7.16.1), one has

(7.17) $$\mathbb{C} \equiv \mathbb{C}_X \underset{\longrightarrow}{\subset} \mathcal{A} \underset{\longrightarrow}{\subset} \mathcal{E}nd\mathcal{E}$$

within $\mathbb{C}$-algebra sheaves isomorphisms (into), so that by further looking at the corresponding group sheaves of units, one obtains (see (7.16.2))

(7.18) $$\mathbb{C}^{\bullet} \equiv \mathbb{C}^{\bullet}{}_X \underset{\longrightarrow}{\subset} \mathcal{A}^{\bullet} \underset{\longrightarrow}{\subset} (\mathcal{E}nd\mathcal{E})^{\bullet} = \mathcal{A}ut\mathcal{E},$$

within group sheaves isomorphisms (into), or even by taking global sections (applying the global section functor $\Gamma_X \equiv \Gamma$ on (7.18)), one gets

(7.19) $$\mathbb{C}^{\bullet} \underset{\longrightarrow}{\subset} \mathcal{A}^{\bullet}(X) = \mathcal{A}(X)^{\bullet} \underset{\longrightarrow}{\subset} (\mathcal{E}nd\mathcal{E})^{\bullet}(X) = Aut\mathcal{E}$$

(see also (7.16.3)) within group isomorphisms (into), as indicated.

Thus, taking now (6.14) and (7.19) into account, for the notation employed just below, consider the condition

$$\ker(D_{\mathcal{E}nd\mathcal{E}|_{Aut\mathcal{E}}}) = \mathbb{C}^{\bullet} \cdot \mathbf{1}_{\mathcal{E}} \equiv \mathbb{C}^{\bullet}. \tag{7.20}$$

Indeed, since (see (1.21) in the preceding)

$$\ker(D_{\mathcal{E}nd\mathcal{E}|_{Aut\mathcal{E}}}) = \mathcal{O}(D), \tag{7.21}$$

the second member of (7.21) standing for the isotropy (alias, stability) group of the given $\mathcal{A}$-connection D of $\mathcal{E}$, one sees that (7.20) is equivalent to the condition

(7.22) the isotropy group of D is trivial; namely, one has

(7.22.1) $$\mathcal{O}(D) = \mathbb{C}^{\bullet} \equiv \mathbb{C}^{\bullet} \cdot \mathbf{1}_{\mathcal{E}} \leq Aut\mathcal{E}.$$

Thus, our assertion now is the following:

(7.23) Suppose we are given a Yang–Mills field

(7.23.1) $$(\mathcal{E}, D).$$

Moreover, assume that

(7.23.2) $$\phi \in \mathcal{O}(z) \cap \mathcal{O}(D) \subseteq Aut\mathcal{E}$$

holds true, such that

(7.23.3) $0 \neq z \in \mathcal{E}$ and D has trivial isotropy group (*viz.*, (7.22.1) is valid).

Then, one obtains

(7.23.4) $$\phi = \mathbf{1}_{\mathcal{E}}.$$

Indeed, our claim is an immediate consequence of (7.23.2) and (7.23.3) and the previous relation (7.21). [Concerning the notation employed in (7.23.2), we set $\mathcal{O}(z) = \{\phi \in Aut\mathcal{E} : \phi(z) = z\}$, the isotropy group of $z \in \mathcal{E}$, under the action of $Aut\mathcal{E}$ on $\mathcal{E}$]. ■

Of course, (7.15) is now a straightforward consequence of (7.23), according to the very definitions. ■

Accordingly, as an outcome of the preceding discussion, one concludes that

(7.24) supposing that condition (7.22.1) holds true for any given $\mathcal{A}$-connection D of $\mathcal{E}$, one further employs a condition, like (7.23.2), namely,

(7.24.1) $$\phi(z) = z,$$

for some $z \neq 0$ in $\mathcal{E}$ (varying, in general, with ϕ(!), one thus effectuates here a "continuous selection" of such an element z in $\mathcal{E}$), so that (7.23.4) be then in force.

Condition (7.24.1) is fulfilled, for instance, in (7.12); hence, our conclusion in (7.15). Thus, the latter guarantees the "local" (pointwise) freeness of the action of $Aut\mathcal{E}$ on the space (7.1), while the same choice being rendered ambiguous, in general, "globally"! (*Gribov's phenomenon*).

Concerning the classical counterpart of the previous account, apart from the work of I.M. Singer [1] that also was as already mentioned our main motivation to the above considerations in the present section, see also A.S. Schwarz [2: p. 274, §15.9] as well as C. Nash [1: p. 219] and P.K. Mitter–C.M. Viallet [1], or even K.B. Marathe–G. Martucci [1: p. 160, §6.5].

Part III

General Relativity

4

General Relativity, as a Gauge Theory. Singularities

> "...the evolution of the relativity of space time does not end with the paradox of the singularities."
>
> D. J. Raine and M. Heller in *The Science of Space–Time* (Pachart Publ. House, 1981). p. 230.

> "...the description of our own measurements of a quantum system must use classical, commutative ***c***-numbers ..."
>
> N. Bohr in *Quantum Theory and Measurement* (J.A. Wheeler–W.H. Zurek (Eds.), Princeton University Press, Princeton, 1983).

> "Sensible mathematics involves neglecting a quantity when it turns out to be small—not neglecting it just because it is infinitely great and you do not want it."
>
> P.A.M. Dirac in *Directions in Physics* (H. Hora–J.R. Shepanski (Eds.), J. Wiley, London, 1978). p. 36.

> "...the general theory of relativity can be conceived only as a field theory."
>
> A. Einstein in *The Meaning of Relativity* (5th edition) (Princeton University Press, Princeton, NJ, 1956). p. 140.

Nowadays we understand that it is much more geometrical to lay the "geometry" (and, in particular, the differential geometry) we apply on the functions rather, which are employed in its description, than on an *a priori* existed "space." Yet, the former seems to be more akin to the physicists point of view, according to which "geometry, mechanics, and physics form an inseparable theoretical whole". (See, for instance, S.Y. Auyang [1: p. 144]). Indeed, the functions we alluded to above are essentially the fields, in the physics terminology, hence (see Volume I, Chapt. II), "sections" of appropriate algebra sheaves and/or of their (sheaf) modules, in particular, of what we called throughout the present abstract (sheaf-theoretic) framework the vector sheaves (ibid. Definition 6.1).

Especially, our aim in this chapter, as the title indicates, is to treat the fundamentals of General Relativity, for instance, to obtain Einstein's equation (*in vacuo*) within the abstract framework, which has been advocated thus far, throughout the preceding. So, by looking at general relativity as a physical theory of the gravitational field, as it actually is, for that matter, we have here again a so-called "gauge (field) theory"; in other words, a physical theory, pertaining to the study of a particular

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gauge field (= $\mathcal{A}$-connection), indeed, of the whole space of such, or even of the corresponding "moduli space" (see Chapter II in the foregoing), which is carried by a "vector sheaf" that is here factuated by means (of the states (= sections) of the previous sheaf; see Vol. I, Chapt. II) of some elementary particle, which, for the case at hand, might be the so-called "graviton." (In this connection, see also J. Baez–J.P. Muniain [1: p. 402]).

Now, a nontrivial spin-off of such a treatment of general relativity, in terms, namely, of the abstract (differential-geometric) technique employed herewith, is a potential application in our calculations of functions (that is to say, of sections) that may have a large amount of singularities (!) in the classical sense of the term, as they actually are, for instance, the elements (sections) of the *Rosinger's algebra sheaf* (see Section 5 in the sequel); such a presence of singularities, at all, is, of course, so far, the most awkward obstacles for the classical theory, as it mainly concerns, among other things, the important issue of its relation to quantization. To quote A. Einstein himself "... we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions" (see A. Einstein [1: p. 165]). We do consider such a sort of results, along with further reflexions, in the last two sections of this chapter.

On the other hand, within the same vein of ideas, by referring to general relativity we still note that the gravitational interaction, which is, of course, the subject matter of the same theory, can also be viewed as (the study-object of) a gauge theory, in the previous sense, as well. See, for instance, T.W. Kibble [1], along with Volume I, Chapter II, Section 9.2 in the forgoing. Yet, to quote Einstein himself, as in the frontispiece of this chapter, "the general theory of relativity can be conceived only as a field theory." Now, "field theory" means, in effect, that physical–mathematical theory that actually concerns a Yang–Mills field, in general,

$$(\mathcal{E}, D), \tag{0.1}$$

as this term has been employed in the preceding (see Chapt. I, (4.13)); this can also be a Maxwell field

$$(\mathcal{L}, D) \tag{0.2}$$

(see also Volume I, Chapt. III, (1.3)), according to the particular case at issue, as is, for instance, the case for the electromagnetic field (loc. cit. Definition 1.1), while this latter situation, as in (0.2), refers to the gravitational field, as well (graviton; see Section 9 in the sequel). Hence, according to what has been said in the foregoing, one thus realizes that

(0.3) field theory means, in point of fact, a gauge theory;

therefore, our previous considerations, pertaining to general relativity, is viewed as a gauge theory.

Thus, one has first to establish the appropriate differential-geometric framework, in the sense of this treatise, within which one can further consider, according to the preceding, the general theory of relativity as an abstract gauge theory, a task that we take on straightforwardly in the next section.

1 Abstract Differential-Geometric Setup

To start with, we assume that we are given our usual abstract setting—that is, a differential triad

$$(1.1)\qquad (\mathcal{A}, \partial, \Omega^1)$$

on an arbitrary topological space X; on the other hand, these initial data will be further specialized, as we proceed, in accordance with the particular case at hand.

So our first specialization is to suppose that we actually have a curvature space X, or a curvature datum on X,

$$(1.2)\qquad (\mathcal{A}, \partial, \Omega^1, d^1 \equiv d, \Omega^2)$$

(see, for instance, Volume I, Chapt. I, (7.19)), while we also assume that we are given a vector sheaf $\mathcal{E}$ on X, such that

$$(1.3)\qquad rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} = n \in \mathbb{N}.$$

Now, what is further special here is our assumption about the dual (vector) sheaf $\mathcal{E}^*$ of $\mathcal{E}$; that is, we still accept that

$$(1.4)\qquad \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) = \Omega^1,$$

which yields, of course, that the given (see (1.1)) $\mathcal{A}$-module Ω^1 is, in particular, a vector sheaf on X as well (see [VS: Chapt. IV, p. 301, (5.19)]), so that by virtue of (1.3) and (1.4), along with the "reflexivity" of a given vector sheaf on X, as is $\mathcal{E}$ here (loc. cit. p. 299, Theorem 5.1), one further obtains

$$(1.5)\qquad \mathcal{E} = \mathcal{E}^{**} = (\Omega^1)^*,$$

within $\mathcal{A}$-isomorphisms of the $\mathcal{A}$-modules (in effect, of the vector sheaves) involved (ibid.), a fact that will be of essential use presently below (see for instance, (1.51) in the sequel). Yet, one has (loc. cit.)

$$(1.5')\qquad rk\Omega^1 = rk(\mathcal{E}^*) = rk\mathcal{E} = n \in \mathbb{N}.$$

Note 1.1 The conclusion from what has been said before is thus that in the particular case of the general theory of relativity, as this can be formulated, within our abstract setting, one considers the differential triad

$$\text{(i)}\qquad (\mathcal{A}, \partial, \Omega^1),$$

as in (1.1), where the given $\mathcal{A}$-module Ω^1 is now a vector sheaf with, say,

$$\text{(ii)}\qquad rk\Omega^1 = n \in \mathbb{N},$$

while the "variant" vector sheaf $\mathcal{E}$, that we consider on X, is still, by definition, the dual (vector sheaf) of Ω^1; namely, we set (see (1.5))

$$\text{(iii)} \qquad \mathcal{E} := (\Omega^1)^*.$$

The above obviously corresponds to the classical case, where the space–time is a suitable (4-dimensional) $\mathcal{C}^\infty$-manifold, whose tangent bundle $\mathcal{T}(X)$, along with the respective cotangent bundle $\mathcal{T}^*(X)$—the two bundles being, virtually, identified, through the Lorentz metric (see (1.13.2) in the sequel)—represent here, by means of their corresponding sheaves of sections, the previous scheme, as described by (i) and (iii).

On the other hand, for the sake of generality and without any extra cost for that matter, we assumed by (ii) that the (finite) rank of Ω^1, hence of $\mathcal{E} = (\Omega^1)^*$ as well (see (1.5′)), is, in general, $n \in \mathbb{N}$, not necessarily $n = 4$ (classical case). Thus, practically speaking, the framework adopted herewith contributes also to the presentation, within the abstract setting, advocated by this study, of more general aspects of general relativity, as are, for example, several *unified field theories*, as such the *Kaluza–Klein theory* (see, e.g., C. von Westenholz [1: pp. 504 and 554]). Relevant to this note is also Scholium 1.1 in the sequel.

Finally, and in anticipation, with respect to the subsequent discussion, concerning the above differential framework, as given by (i) and (iii), what will diversify it is, as we shall see, the particular metric that in each case will be associated with it. So, in this regard, the preceding setup might also be characterized, as an Einstein (differential) triad, a terminology that will further be justified by the ensuing discussion (see for instance, Definition 3.1 in the sequel).

Now, suppose that our vector sheaf $\mathcal{E}$ on X, as before (see (1.5) or even Note 1.1 (iii)), is further endowed with an $\mathcal{A}$-connection D; thus, by using the previously employed terminology (see Chapt. I), we actually assume, according to the preceding, that we are given a Yang–Mills field

$$\text{(1.6)} \qquad (\mathcal{E}, D)$$

on X. In this connection, we still note that by virtue of the general theory (see, e.g., [VS: Chapt. VI, p. 85, Theorem 16.1]), the situation just described can be achieved under suitable conditions for $\mathcal{A}$ and X, that, in practice, appear in several particular important cases, apart, of course, from the classical one (smooth case) that virtually fulfills the aforementioned conditions (loc. cit.).

Thus, by looking at the corresponding field strength of the field, at issue, that is, by definition, at the curvature of the given $\mathcal{A}$-connection D of $\mathcal{E}$, as in (1.6) (see also (1.2)), denoted, as usual, by

$$\text{(1.7)} \qquad R(D) \equiv R,$$

one has (see Volume I, Chapt. I, (7.22))

$$(1.7') \qquad \begin{aligned} R(D) \equiv R \in \Omega^2(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \Omega^2(\mathcal{E}nd\mathcal{E})) \\ \subseteq C^0(\mathcal{U}, \Omega^2(\mathcal{E}nd\mathcal{E})) = \prod_\alpha \Omega^2(\mathcal{E}nd\mathcal{E})(U_\alpha) = \prod_\alpha M_n(\Omega^2(U_\alpha)), \end{aligned}$$

so that one finally obtains that

$$(1.7'') \qquad R(D) \equiv R = (R^{(\alpha)}) \in \prod_\alpha M_n(\Omega^2(U_\alpha)).$$

Here, as usual,

$$(1.8) \qquad \mathcal{U} = (U_\alpha)_{\alpha \in I}$$

stands for a given local frame of (the vector sheaf) $\mathcal{E}$, such that on each local gauge U_α, $\alpha \in I$, of $\mathcal{E}$, one has (see (1.3))

$$(1.9) \qquad \mathcal{E}|_{U_\alpha} = \mathcal{A}^n|_{U_\alpha} = (\mathcal{A}|_{U_\alpha})^n$$

for any $\alpha \in I$, within $\mathcal{A}|_{U_\alpha}$-isomorphisms of the $\mathcal{A}|_{U_\alpha}$-modules involved. Therefore, in view of (1.7′) and by complete analogy with the classical case, one concludes that

(1.10) the field strength $R(D) \equiv R$ of the field $(\mathcal{E}, D)$, as in (1.6), is given by a 0-cocycle of (local) $n \times n$ matrices with entries (local sections) from

$$(1.10.1) \qquad \Omega^2 \equiv \Omega^1 \wedge \Omega^1$$

(*viz.*, (local) "2-forms" on X). Therefore, locally (see also (1.7″)) one obtains

$$(1.10.2) \qquad R(D)|_{U_\alpha} \equiv R^{(\alpha)} = (\omega_{ij}^{(\alpha)}) \in M_n(\Omega^2(U_\alpha)) = M_n(\Omega^2)(U_\alpha)$$

for any $\alpha \in I$, as in (1.8).

Now, in connection with the preceding and by making full use of our previous considerations in (1.4), (1.7′), and (1.10), we further assume, in the next subsection, that we are also given an appropriate $\mathcal{A}$-metric on $\mathcal{E}$, the source, in a nutshell, of what we may call *curvature operators* on $\mathcal{E}$, that will also be our main issues in all that follows.

1.1 Curvature Operators

We continue working, within the above abstract setting: thus, to fix the terminology applied, we recall that we have assumed thus far that

(1.11) we are given a curvature datum

(i) $$(\mathcal{A}, \partial, \Omega^1, d^1, \Omega^2) \equiv (\mathcal{A}, \partial, \Omega^1, d, \Omega^2) \equiv (\partial, d)$$

(*viz.*, we set $d^0 \equiv \partial$ and $d^1 \equiv d$) on an arbitrary topological space X, where we still suppose that

(ii) $$\mathcal{E} := (\Omega^1)^*$$

(i.e., the dual $\mathcal{A}$-module of Ω^1) is a vector sheaf on X, such that

(iii) $$rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} = n \in \mathbb{N}.$$

Therefore, the same, as for $\mathcal{E}$, holds true for Ω^1 as well, the latter being thus a vector sheaf on X too (see, for instance, (1.4), (1.5), along with (1.5′)).

Now, to proceed, we further suppose the existence of a "metric," in our abstract (sheaf-theoretic) sense (see below); this is, in effect, the most essential as well as convenient assumption we make for all that follows: Indeed,

(1.12) gravity is a metric, after all; mathematically, however, roughly speaking, which is, also in the same jargon, the source, alias the cause, of every thing, that refers to general relativity,

the latter being, of course, our subject matter in the present chapter, viewed, as an (abstract) gauge theory. Thus, precisely speaking and within always our abstract setting, we next assume that

(1.13) we are given an $\mathcal{A}$-metric on $\mathcal{E}$ (see (1.11) (ii)), namely, a sheaf morphism, say

(1.13.1) $$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A},$$

being also, by definition (see also Chapt. I, Section 2), an $\mathcal{A}$-valued scalar product, in the sense that the following three conditions are further satisfied:

(i) ρ is $\mathcal{A}$-bilinear, with respect to the $\mathcal{A}$-modules involved in (1.13.1).
(ii) ρ is symmetric (see Chapt. VI, (2.2)).
(iii) ρ is strongly nondegenerate; that is, we suppose that one has

(1.13.2) $$\mathcal{E} \underset{\tilde{\rho}}{\cong} \mathcal{E}^*,$$

within an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules involved.

Yet, as a result of (1.13.2) and (1.4) (see also (1.5)), one still obtains that

(1.14) $$\Omega^1 = \mathcal{E}^* \underset{\tilde{\rho}^{-1}}{\cong} \mathcal{E},$$

while, for convenience, we recall (see Chapt. I, (2.6) and (2.7)) that the $\mathcal{A}$-morphism (in effect, $\mathcal{A}$-isomorphism, in view of our hypothesis in (1.13.2)) $\tilde{\rho}$ is defined by the relation

$$\tilde{\rho}(s)(t) \equiv \rho_s(t) := \rho(s,t) \tag{1.15}$$

for any s, t in $\mathcal{E}(U)$ and any open $U \subseteq X$.

Based on (1.4) and (1.13.2), one can further define the subsequent curvature operator, or *curvature endomorphism*, which is associated with the given Yang–Mills field $(\mathcal{E}, D)$ as in (1.6) under conditions (1.4) and (1.13.2). So, by looking at the corresponding curvature $R(D) \equiv R$ of the given $\mathcal{A}$-connection D of $\mathcal{E}$ locally (i.e., in terms of local sections of the $\mathcal{A}$-modules (in effect, vector sheaves) involved in (1.7′)), one obtains the following relation for any open $U \subseteq X$, being a local gauge of (the vector sheaf) $\mathcal{E}$ as well (see (1.9)):

$$\begin{aligned}(R|_U)(\cdot, s)(t) &\equiv R(\cdot, s)(t) \equiv R(\cdot, s)t \\ &\in (\mathcal{E}nd\mathcal{E})(U) \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(U) \\ &= Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)\end{aligned} \tag{1.16}$$

with s, t in $\mathcal{E}(U)$, where we have also taken into account (1.14) in conjunction with (1.7′) (in the particular case that $U_\alpha = U$).

Concerning (1.16), our argument is based, first, on the following general result:

Lemma 1.1 Given two $\mathcal{A}$-modules $\mathcal{E}, \mathcal{F}$ on X and an open $U \subseteq X$, local gauge of any one of them (not necessarily of finite rank), one has

$$(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})(U) = \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U), \tag{1.17}$$

within an $\mathcal{A}(U)$-isomorphism of the $\mathcal{A}(U)$-modules concerned.

Proof See [VS: Chapt. VII, p. 100, (1.10)]. ■

On the other hand, within the same vein of ideas, as in the previous lemma, and assuming that $\mathcal{E}$ is a vector sheaf on X, one has, if $\mathcal{E} = \mathcal{E}^*$ (see, e.g., (1.13.2)),

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} = \mathcal{E}^{**} \otimes_{\mathcal{A}} \mathcal{F} = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}^*, \mathcal{F}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}), \tag{1.18}$$

within $\mathcal{A}$-isomorphisms (see, [VS: Chapt. IV; p. 302, Theorem 6.1]). Furthermore, for any given $\mathcal{A}$-module $\mathcal{E}$ on X, one obtains, by definition,

$$\wedge^n \mathcal{E}^* = (\wedge^n \mathcal{E})^*, \quad n \in \mathbb{N}, \tag{1.19}$$

up to an $\mathcal{A}$-isomorphism of $\mathcal{A}$-modules.

Furthermore, by virtue of (1.7′) and the preceding, a local constituent of the curvature of R takes the form

$$\begin{aligned}R(D)|_U \equiv R|_U &\in \Omega^2(\mathcal{E}nd\mathcal{E})(U) \equiv (\Omega^2 \otimes_{\mathcal{A}} \mathcal{E}nd\mathcal{E})(U) \\ &= \Omega^2(U) \otimes_{\mathcal{A}(U)} (\mathcal{E}nd\mathcal{E})(U) = \mathcal{H}om_{\mathcal{A}}((\Omega^2)^*, \mathcal{E}nd\mathcal{E})(U) \\ &= \mathcal{H}om_{\mathcal{A}}(\Omega^2, \mathcal{E}nd\mathcal{E})(U) = \mathcal{H}om_{\mathcal{A}}(\wedge^2\mathcal{E}, \mathcal{E}nd\mathcal{E})(U) \\ &= Hom_{\mathcal{A}|_U}(\wedge^2(\mathcal{E}|_U), (\mathcal{E}nd\mathcal{E})|_U),\end{aligned} \tag{1.20}$$

so that the local sections s and t that appear in (1.16) are by virtue of (1.10.1), (1.19), and (1.14), first of all elements of $\Omega^1(U)$, but finally of $\mathcal{E}(U)$, exactly as asserted in (1.16), which thus completely justifies the relation. ■

On the other hand, a compilation of (1.16) with (1.9) yields the following:

$$(1.21)\qquad \begin{aligned}(R|_U)(\cdot, s)(t) &\equiv R(\cdot, s)t \in Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)\\ &= Hom_{\mathcal{A}|_U}(\mathcal{A}^n|_U, \mathcal{A}^n|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A}^n)(U)\\ &\equiv M_n(\mathcal{A})(U) = M_n(\mathcal{A}(U))\end{aligned}$$

(see also [VS: Chapt. IV, p. 294, (3.24)], concerning the notation applied in (1.21)).

In conclusion, based on (1.21) and by applying an abbreviated notation, we get the following relation, which justifies the term "curvature endomorphism" employed at the beginning; that is, one has (see also (1.27.1))

$$(1.22)\qquad R(\cdot, s)t \in (\mathcal{E}nd\mathcal{E})(U) = M_n(\mathcal{A}(U)) = M_n(\mathcal{A})(U)$$

with s and t in $\mathcal{E}(U)$ and U open in X, as in (1.9).

To fix our terminology, we further single out the following basic notion for the subsequent discussion.

Definition 1.1 Suppose we have a curvature space X (see (1.2)) endowed with an $\mathcal{A}$-valued scalar product (see (1.13)), and let $(\mathcal{E}, D)$ be a given Yang–Mills field on X satisfying the relation

$$(1.23)\qquad \mathcal{E} = (\Omega^1)^*.$$

Then, for any open $U \subseteq X$, a local gauge of $\mathcal{E}$ (see (1.9)), and for any s and t in $\mathcal{E}(U)$, one defines the *Ricci operator* of $\mathcal{E}$, denoted by

$$(1.24)\qquad \mathcal{R}ic(\mathcal{E}) \equiv R(s, t),$$

according to the following relation:

$$(1.25)\qquad \mathcal{R}ic(\mathcal{E}) \equiv R(s, t) := tr(R(\cdot, s)t);$$

that is, as the trace of the matrix appearing in (1.22).

In other words, and by virtue of (1.22) and (1.25), one gets a map

$$(1.26)\qquad R : \mathcal{E}(U) \times \mathcal{E}(U) \longrightarrow \mathcal{A}(U) : (s, t) \longmapsto R(s, t) := tr(R(\cdot, s)t),$$

being thus identified, according to the preceding, as the Ricci operator of $\mathcal{E}$, yielding by definition a local map for any local gauge $U \subseteq X$ of $\mathcal{E}$, as in (1.9). (Warning (notational)! An obvious abuse of notation has been applied in (1.26), pertaining to the usual symbol for the curvature R, being easily spotted from the context.)

To summarize the preceding, we thus conclude that

(1.27) working within the previous framework, for any two (local) sections, say s and t of $\mathcal{E}$ (over a local gauge U of $\mathcal{E}$), one obtains through the curvature R of $\mathcal{E}$ a local endomorphism of $\mathcal{E}$ (over U; *viz.*, what we called above the curvature endomorphism, or curvature operator of $\mathcal{E}$), via the map (see (1.21) and (1.22))

$$\begin{aligned} R(\cdot, s)(t) &\equiv R(\cdot, s)t \in Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \\ &= \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(U) \equiv (\mathcal{E}nd\mathcal{E})(U) \\ &\equiv End(\mathcal{E}|_U) = M_n(\mathcal{A}(U)). \end{aligned} \tag{1.27.1}$$

Thus, by virtue of the last relation a consequence, of course, of our hypothesis for $\mathcal{E}$ and U (see (1.9)), the trace of the corresponding matrix from the last member of (1.27.1), being an element of $\mathcal{A}(U)$, yields now, by definition (see (1.25)), the Ricci operator of $\mathcal{E}$ as noted by (1.26).

To fix the terminology employed, we still note that, as follows from (1.27.1),

(1.28) $R(\cdot, s)t$ (*viz.*, the curvature operator, or curvature endomorphism, of $\mathcal{E}$), is an $\mathcal{A}|_U$-morphism of $\mathcal{E}|_U$ into itself, or an $\mathcal{A}|_U$-endomorphism of (the $\mathcal{A}|_U$-module) $\mathcal{E}|_U$ for any s, t, and U, as above.

The preceding naturally leads to the definition of the following map of a sheaf morphism; namely, one has (applying, for convenience, an obvious abuse of notation, again with respect to (1.24))

$$\mathcal{R}ic(\mathcal{E}) : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}, \tag{1.29}$$

so that locally (*viz.*, for any local gauge U of $\mathcal{E}$), one has

$$\begin{aligned} \mathcal{R}ic(\mathcal{E})_U(s, t) \equiv \mathcal{R}ic(\mathcal{E})(s, t) &:= \mathcal{R}ic(\mathcal{E}) \equiv R(s, t) \\ &:= tr(R(\cdot, s)t) \in \mathcal{A}(U) \end{aligned} \tag{1.30}$$

for any s and t in $\mathcal{E}(U)$.

Consequently, based on the preceding, one sets, by definition,

$$\mathcal{R}ic(\mathcal{E}) \equiv (\mathcal{R}ic(\mathcal{E})_U), \tag{1.31}$$

with U varying over a given local frame of $\mathcal{E}$ (see (1.8) and (1.9) in the foregoing), the latter family being a basis of the topology of X; therefore, (1.31), hence (1.29), is well defined as a sheaf morphism, which was exactly our assertion concerning the map (1.29) as defined by (1.30).

We call (1.29), or equivalently (1.31), the global, or even generalized, Ricci operator of $\mathcal{E}$. Yet, for brevity's sake, we also name it simply the Ricci morphism of $\mathcal{E}$. In this connection, we also note in anticipation that we shall use the previous map, even in its local form, as given by (1.25), to formulate in our case *Einstein's equation* (*in vacuo*; see Definition 3.1 in the sequel).

On the other hand, the same map (1.29), as given by (1.31), is by definitions (see (1.20)) $\mathcal{A}$-bilinear and also symmetric (see (1.44) in the sequel). By further extending to our abstract case the classical situation, an Einstein ($\mathcal{A}$-)metric on $\mathcal{E}$ is defined as a "scalar multiple" of $\mathcal{R}ic(\mathcal{E})$; namely, we set

(1.32) $$\rho_{Ein} \equiv \rho := \alpha \cdot \mathcal{R}ic(\mathcal{E}),$$

such that

(1.32′) $$\alpha \in \mathcal{A}(X) = Z^0(\mathcal{U}, \mathcal{A}) \subseteq C^0(\mathcal{U}, \mathcal{A}) = \prod_U \mathcal{A}(U),$$

while we still assume that the map ρ, as in (1.32), is strongly nondegenerate as well. In other words, we thus conclude that

(1.33) the map ρ, as in (1.32), is an $\mathcal{A}$-valued scalar product on $\mathcal{E}$ (see (1.13)), being, in particular, a "scalar multiple," in the previous sense, of $\mathcal{R}ic(\mathcal{E})$.

Now, specializing further on our structure sheaf $\mathcal{A}$, we also consider in the next subsection the abstract version in our case of the standard notion of the *scalar curvature* for a given $\mathcal{A}$-connection D of $\mathcal{E}$, as in (1.6).

1.2 Scalar Curvature

As already stated, to deal with the notion in the title of this subsection, we need some extra structure in $\mathcal{A}$, providing us with the appropriate abstract framework, analogous to that in the standard case. What we really need here is to have at our disposal an

(1.34) enriched ordered algebraized space

(1.34.1) $$(X, \mathcal{A})$$

(see, for instance, Chapt. I, (7.1), along with (5.1) and (5.2)), which is also a curvature space (see (1.2)). Furthermore, we also assume that X is a paracompact (Hausdorff) space, while we still accept that our structure sheaf $\mathcal{A}$ is a strictly positive fine sheaf on X (see [VS: Chapt. IV, p. 327, Definition 8.5]).

Finally, we assume that we are given an $\mathcal{A}$-metric ρ on $\mathcal{A}$, so that

(1.34.2) $$(\mathcal{A}, \rho)$$

is a Riemannian $\mathcal{A}$-module on X in the sense that ρ is an $\mathcal{A}$-valued inner product on $\mathcal{A}$ ($\mathcal{A}$-bilinear, symmetric, positive definite, and strictly nondegenerate sheaf morphism; [loc. cit. p. 318, Definition 8.2]).

Note 1.2 (i) Our hypothesis, as in (1.34), that the structure sheaf $\mathcal{A}$ is a fine sheaf on a paracompact (Hausdorff) space X (see also [VS: Chapt. IV, p. 327; (8.47)) entails, among other things, the inheritance of the given "metric" structure of $\mathcal{A}$, as in (1.34.2), to any vector sheaf on X as well, according to a standard argument, pertaining to the existence of "partitions of unity" for $\mathcal{A}$: See, for example, [VS: Chapt. IV, p. 327, Definition 8.5 as well as p. 328, Theorem 8.3].

As consequence of the preceding, one can consider here two notions pertaining to "orthonormality" that refer to vector sheaves: Thus, we first remark that given a vector sheaf $\mathcal{E}$ on X, one can consider it, by virtue of what has been said in Note 1.2, still as a Riemannian vector sheaf

(1.35) $$(\mathcal{E}, \rho).$$

Moreover, if $U \subseteq X$ is a local gauge of $\mathcal{E}$ (see (1.9)), one can assume that this is an orthonormal one. Indeed, one proves that

(1.36) given a vector sheaf $(\mathcal{E}, \rho)$, as in (1.35), there exists a basis of the topology of X consisting of orthonormal local gauges of $\mathcal{E}$. Otherwise stated, there is a basis of the topology of X yielding an orthonormal local frame of $\mathcal{E}$.

In this connection, we recall, for convenience, that a local gauge of $\mathcal{E}$,

(1.37) $$e^U \equiv (U; e_1, \ldots, e_n),$$

is said to be orthonormal whenever one has the relation (see also (1.35))

(1.38) $$\rho(e_i, e_j) = \delta_{ij}, \qquad 1 \leqq i, j \leqq n = rk\mathcal{E}.$$

Of course, the analogous property to (1.9) is still valid for U by definition, while

(1.39) $$(e_1, \ldots, e_n) \subseteq \mathcal{E}(U)^n = \mathcal{E}^n(U)$$

is an (orthonormal) basis of the $\mathcal{A}(U)$-module $\mathcal{E}(U)$, such that

(1.40) $$| e_i |= 1, \qquad 1 \leqq i \leqq n$$

(see also (VS: Chapt. IV, p. 336, (10.8) and (10.9)).

The preceding argument in (1.36) is actually based on an extension to the case of vector sheaves of the usual *Gram–Schmidt orthogonalization* procedure, valid already, as we know, for free $\mathcal{A}$-modules of finite rank (ibid. p. 340, Theorem 10.1). This extension is accomplished on the basis of the defining property of a local gauge of a given vector sheaf, as, for example, in (1.9), in conjunction with our previous result. (In this connection, see also Scholium 1.2 in the sequel for a possible further extension of the same result as above pertaining to "indefinite" or *semi-Riemannian metrics* on vector sheaves, according to the standard argument.) By further referring to (1.36), one can actually prove that

(1.41) for a vector sheaf $\mathcal{E}$ on X as above, the set of locally finite (in the obvious sense) orthonormal local frames of $\mathcal{E}$, as in (1.36), provides a cofinal subset of the set of all (proper) open coverings of X.

Equivalently, one thus concludes, always within the previous setting, that

(1.42) given a vector sheaf $\mathcal{E}$ on X as before, for any open covering of X, there always exists an open (locally finite) refinement of it, consisting of local orthonormal gauges of $\mathcal{E}$ (thus, providing an orthonormal local frame of $\mathcal{E}$ as well).

Note 1.3 Thinking of the above results pertaining to "orthonormality," we remark that these are valid without any "differentiability" assumptions on X. Thus, they still hold true with X being an

(1.43) enriched ordered algebraized space that is also paracompact (Hausdorff), while the corresponding structure sheaf $\mathcal{A}$ is assumed to be a strictly positive fine sheaf on X that is further endowed with a Riemannian $\mathcal{A}$-metric.

Thus, the above constitute (see also Note 1.2) a very convenient framework to formulate within our abstract (sheaf-theoretic) context the standard theory of "inner product spaces."

Finally, as another consequence of the above, we obtain the following result, which has been used in the preceding (true in a generalized form; however, see Scholium 1.1 in the sequel). Thus, one concludes that

(1.44) the Ricci operator of $\mathcal{E}$, as given by (1.25) (see also (1.30) and (1.31)), is symmetric; that is, one has, as concerns the scalar curvature,

(1.44.1) $$R(s,t) = R(t,s)$$

for any s and t in $\mathcal{E}(U)$, as in (1.30). (Here the local gauge U of $\mathcal{E}$ involved may be taken to be an orthonormal one; see (1.36). Yet, the vector sheaf $\mathcal{E}$ as in (1.11) and (1.43) may be of the form (1.6) and (1.35) (*viz.*, a Riemannian Yang–Mills field).)

Our assertion in (1.44) is, as already said, an outcome of our previous considerations on "orthonormality" concepts within our abstract setup in conjunction with the relevant classical argument; see W.A. Poor [1: p. 130, Proposition 3.44]. ■

Having the previous abstract (differential-geometric) context as given by (1.43) above along with (1.2) and (1.4), suppose again that

(1.45) $$(\mathcal{E}, D)$$

is a given Yang–Mills field on X. Let us further consider the sheaf morphism

(1.46) $$\mathcal{R} : \mathcal{E} \longrightarrow \mathcal{A}$$

given (sectionwise) by the relation (see also (1.25))

$$\mathcal{R}(s) := R(s, s)/\|s\|^2, \tag{1.47}$$

where $s \in \mathcal{E}(U)$, with $s \neq 0$, while U is a local gauge of $\mathcal{E}$, which according to the preceding (see (1.36)) may still be considered as an orthonormal one. Concerning the notation employed in (1.47), see also Chapter I, (5.5.2).

The map (1.46), as given by (1.47), is called the Ricci curvature of $(\mathcal{E}, D)$.

Note 1.4 By referring to our hypothesis on the given vector sheaf $\mathcal{E}$ on X, as in (1.45), we further remark here that according to our assumptions on the $\mathbb{C}$-algebraized space $(X, \mathcal{A})$ set forth in (1.2), (1.4), and (1.43), one concludes that

(1.48) given a vector sheaf $\mathcal{E}$ on X, this may always be viewed as a Riemannian Yang–Mills field on X; namely, it is of the form

$$(\mathcal{E}, D; \rho), \tag{1.48.1}$$

with D and ρ as in (1.6) and (1.35), respectively. Moreover, D is an $\mathcal{A}$-connection on $\mathcal{E}$ compatible with the $\mathcal{A}$-metric ρ, or a Riemannian $\mathcal{A}$-connection on $\mathcal{E}$, emanating from the given $\mathcal{A}$-metric ρ. Concerning the given vector sheaf $\mathcal{E}$ on X, we may still think in terms of an orthonormal local frame of $\mathcal{E}$, which can also be chosen to be a basis of the topology of X (see (1.36)).

We come now to the definition of the sort of curvature that is indicated by the title of the present subsection: having the above framework, as declared by Note 1.4, and according also to the classical pattern (see, e.g., W.A. Poor [1: p. 131, Definition 3.45]), one now defines the

(1.49) scalar curvature (of the Ricci operator; see (1.24) or (1.31)) of $\mathcal{E}$ (strictly speaking, of $(\mathcal{A}, \rho)$, as in (1.34.2)) by the relation

$$\sigma(\mathcal{E}) := \sum_{i,j} \rho(R(e_i, e_j)e_j, e_i), \tag{1.49.1}$$

with $1 \leqq i, j \leqq n = rk\mathcal{E}$. See also (1.22) concerning the previously applied notation.

In other words, one considers

the trace (value) of the Ricci operator as defined by (1.21) (see also (1.22)) with respect to the given $\mathcal{A}$-metric ρ (see (1.34.2)) and an orthonormal local gauge of $\mathcal{E}$ (see (1.37)–(1.40) as well as (1.36)).

Scholium 1.1 Concerning the framework that has been employed so far, we remark that

(1.50) this was just the initially given differential triad

(1.50.1) $$(\mathcal{A}, \partial, \Omega^1)$$

(see (1.1)), that is simply the basic differential

(1.50.2) $$\partial (\equiv d^0 : \Omega^0 \equiv \mathcal{A} \longrightarrow \Omega^1)$$

(see Chapter I, (1.3′), (1.4), and (1.5′) concerning the notation applied in (1.50.2)), which has been further supplemented through the addition of one more differential [*viz.*, of d^1 (see (1.2), as well as Section 1)], while we also assumed that

(1.50.3) $$\Omega^1 \text{ is a vector sheaf on } X \text{ with } rk\Omega^1 = n \in \mathbb{N}$$

(see also (1.4)–(1.5′)) such that

(1.50.4) $$(\Omega^1)^* \equiv \mathcal{E}.$$

The above particular vector sheaf $\mathcal{E}$ on X (i.e., the dual (vector sheaf) of Ω^1), was the vector sheaf under consideration throughout the foregoing. Therefore, our relevant framework was essentially that of (1.50.1) within the particular assumptions as above, these being further appropriately supplemented, as we shall discuss in the sequel. Thus, we can say that

(1.51) equivalently (with respect to (1.50)) we are first given a $\mathbb{C}$-algebraized space

(1.51.1) $$(X, \mathcal{A}),$$

while we further consider a vector sheaf $\mathcal{E}$ on X; it is good to bear in mind that we actually have, through $\mathcal{E}$, pieces of some (finite) power of $\mathcal{A}$ (see, e.g., (1.9)).

The next crucial assumption pertaining to our differential setting is that we assume the existence of a differential triad of the form

(1.51.2) $$(\mathcal{A}, \partial, \Omega^1),$$

where (by definition) we have set

(1.51.3) $$\Omega^1 := \mathcal{E}^* \equiv \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}).$$

In other words, we suppose the existence of a map like the one in (1.50.2) having the appropriate properties (Leibniz map, or (flat) $\mathcal{A}$-connection of $\mathcal{A}$) with Ω^1 as in (1.51.3). [Thus, the situation here concerning the existence of the basic differential ∂ is reminiscent of the Kähler theory, according to which, roughly speaking, "every (unital commutative) algebra has its own differential" (with an appropriately constructed Ω^1, module of Pfaffians, by means of the algebra concerned). For a relevant account in the case of a suitable topological algebra sheaf, viewed as the corresponding structure (algebra) sheaf, see [VS: Chapt. XI, Section 5], along with Section 5 in the subsequent discussion.]

Consequently, one concludes through our previous discussion in (1.50) and (1.51) that one is actually working within the framework of (1.50.1) or, equivalently, (1.51.2), depending on what one starts with—that is, (1.50.4) or (1.51.3). This framework is further specialized according to the following fundamental assumption pertaining to the $\mathcal{A}$-metric involved.

However, before we come to the matter of the $\mathcal{A}$-metric, we first comment a bit on our assumptions concerning the topological space X and the structure sheaf $\mathcal{A}$, which, as we shall presently see, fit well with our subsequent hypothesis for the $\mathcal{A}$-metric. Thus, we have supposed that X is a paracompact (Hausdorff) space, a fact that has important cohomological implications, in conjunction with differential-geometric ones, in the case that $\mathcal{A}$ is also a fine sheaf on X, which we assume as well. (In this connection, see also [VS: Chapt. III, as well as Chapt. VI; existence of $\mathcal{A}$-connections and the like].) As a result, one concludes that every vector sheaf $\mathcal{E}$ on X admits an $\mathcal{A}$-connection, so that one may look at $\mathcal{E}$ as a Yang–Mills field on X,

$$(\mathcal{E}, D) \tag{1.52}$$

(see also loc. cit. Chapt. VI, p. 85, Theorem 16.1). In particular, this holds for Ω^1 (a vector sheaf on X too, by virtue of our hypothesis, as in (1.50.3), hence for its dual sheaf $\mathcal{E}$ on X (see (1.50.4)), as well as [VS: Chapt. VI, p. 22, Section 5.4]). Consequently one can finally say that

(1.53) the fundamental assumption here, by taking also into account those on X and $\mathcal{A}$, is our hypothesis that

(1.53.1) Ω^1 is a locally free sheaf of finite rank (*viz.*, a vector sheaf on X).

In conclusion, the existence of an $\mathcal{A}$-connection for Ω^1, hence for $\mathcal{E}$ too, was a consequence of our particular hypothesis for X and $\mathcal{A}$, along with the existence of ∂, already an $\mathcal{A}$-connection of $\mathcal{A}$, the latter being guaranteed by the definition of (1.51.2). (In this connection, see also [VS: Chapt. VII, p. 101, Theorem 1.1, or p. 103, (1.24)].)

We come now to our final assumption in the preceding, which was concerned with the

(1.54) existence of the pair

(1.54.1) $$(\mathcal{A}, \rho)$$

—in other words, with that of a (Riemannian) $\mathcal{A}$-metric on $\mathcal{A}$.

See, for instance, (1.13) along with (1.34.2) and (1.48); thus, one has to distinguish here between Riemannian $\mathcal{A}$-metrics (or $\mathcal{A}$-valued inner products) on $\mathcal{A}$, which are by definition positive definite (see, e.g., Chapt. I, (2.5), or Definition 2.1) and by an extension of the classical terminology, semi-Riemannian $\mathcal{A}$-metrics (alias, $\mathcal{A}$-valued scalar products) on $\mathcal{A}$, which are thus by definition semidefinite (positive or negative) or indefinite. We return to this latter case in Section 3 and also through Section 2 by considering an important particular type of the latter metrics (*viz.*, the Lorentz $\mathcal{A}$-metrics on $\mathcal{A}$).

Certain additional technical assumptions pertaining to an appropriate order structure on $\mathcal{A}$ are to be made here, as we did in the preceding discussion (see, for example, (1.43)), this order structure on $\mathcal{A}$ being naturally involved according to the general theory (see [VS: Chapt. IV, Section 8]) in the definition of the concept of an $\mathcal{A}$-metric. Thus, for convenience of reference, we recapitulate the corresponding items of the framework we have employed so far; that is, one considers

(1.55)

an enriched ordered $\mathbb{C}$-algebraized space

$$(X, \mathcal{A}) \tag{1.55.1}$$

with X a paracompact (Hausdorff) space and $\mathcal{A}$ a strictly positive fine sheaf on X endowed with a Riemannian $\mathcal{A}$-metric ρ (see (1.54.1)). We assume that X is a curvature space; namely, we are given

$$(\mathcal{A}, \partial, \Omega^1, d^1, \Omega^2) \tag{1.55.2}$$

as in (1.2), while Ω^1 is a vector sheaf on X.

Concerning the terminology applied above, see also [VS: Chapt. IV, p. 336, Definition 10.1, and p. 327, Definition 8.5, as well as (8.47)].

A crucial fact that results from (1.55) is that

(1.56) every vector sheaf $\mathcal{E}$ on a topological space X, as in (1.55) (an abbreviation of saying that we assume that the framework of (1.55)) admits an $\mathcal{A}$-metric ρ as well, along with an $\mathcal{A}$-connection D compatible with ρ (see Chapt. I, Section 9 or [VS: Chapt. VII, p. 155, Definition 8.1, along with p. 168, Definition 9.1, and the comments preceding the latter]). Indeed, D is determined through the $\mathcal{A}$-metric ρ itself and the differential structure of X (see (1.51.2) or (1.55.2)) in view of the *Levi–Civita identity* (Volume I, Chapt. I, (9.7)).

The preceding is a consequence of what we may call in our case the fundamental lemma of Riemannian vector sheaves, along with the Levi–Civita identity, the latter classical relation being extended to the present abstract setting; see [VS: Chapt. VII, p. 168, Theorem 9.1, as well as p. 166, (8.80) and p. 160, (8.46)].

The previous discussion pointed out our aim, the particular significance and contribution throughout the foregoing of the structure sheaf $\mathcal{A}$, especially of the differential triad $(\mathcal{A}, \partial, \Omega^1)$, under the additional assumption (1.53.1), as well as of the pair

$(\mathcal{A}, \rho)$, with the special hypothesis, however, very useful (!), for many purposes, for $\mathcal{A}$ and X (see (1.55)); in particular, we assumed that $\mathcal{A}$ was a (strictly positive) fine sheaf on the paracompact (Hausdorff) space X.

The preceding have, of course, a special bearing on the classical case, pertaining in particular to Riemannian manifolds; more specifically, the above abstract framework can be employed, classically, in the study of Einstein (-Riemannian) manifolds, [*viz.*, according to the standard definition, Riemannian manifolds of constant Ricci curvature (see for instance, A.L. Besse [1: p. 4, 0.14])]. By the term *Ricci curvature*, one means here what we called in the preceding the Ricci operator of a given vector sheaf $\mathcal{E}$ (see Definition 1.3). In the classical case, $\mathcal{E}$ stands for the sheaf of germs of sections (*viz.*, vector fields) of the tangent bundle $\mathcal{T}(X)$ of the Riemannian ($\mathcal{C}^{\infty}$-)manifold X under consideration; in this connection, see also Chapter I, Section 2.1 for the terminology).

On the other hand, the preceding can also be applied in many abstract instances, a fact that is of our concern here. Thus, the above framework is particularly in force in the case that our structure sheaf $\mathcal{A}$ is the so-called *Rosinger's algebra sheaf*, which will be considered in the subsequent discussion (see Section 5 of this chapter). The base space of the same sheaf is by definition a paracompact (Hausdorff) space. Our interest here lies, as we shall see (see Sections 6 and 8 in the sequel), in the fact that through an application of the previous sheaf, as a sheaf of coefficients, we are able to absorb "singularities," in the classical sense of the latter term, throughout our argument, as if these peculiarities as well as disadvantages of the classical theory were not there at all (!), in the sense that their presence does not entirely affect our argument, hence our reckoning as well, thanks to the abstract form of the differential geometry employed. The same framework seems to acquire a potential application as well pertaining to the problem of quantization of gravity (see Section 9).

1.3 Semi-Riemannian $\mathcal{A}$-Modules

We have already noted (see Scholium 1.1, comments following (1.54)) the chance of considering, through the following discussion, more general $\mathcal{A}$-metrics than those employed so far (Riemannian $\mathcal{A}$-metrics; see (1.34.2)). These are, by extending the standard terminology, the so-called semi-Riemannian $\mathcal{A}$-metrics, as the title of this subsection indicates. We are going to employ such types of $\mathcal{A}$-metrics throughout Section 2 (Lorentz $\mathcal{A}$-metrics). The crucial difference between the latter and the Riemannian $\mathcal{A}$-metrics considered in the foregoing is that the semi-Riemannian $\mathcal{A}$-metrics are $\mathcal{A}$-metrics on a given $\mathcal{A}$-module $\mathcal{E}$ on X—that is, sheaf morphisms

$$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A} \tag{1.57}$$

(see, for instance, (1.13.1)), which are $\mathcal{A}$-bilinear, symmetric and strongly nondegenerate (see (1.13)), being otherwise not necessarily positive definite; see, for instance, Chapter VI, (2.5). Based on these properties of ρ, as in (1.57), and by further extending the relevant classical terminology (see for instance, B. O'Neill [1: p. 47, Definition 20]), we may call such a ρ an $\mathcal{A}$-valued scalar product of the $\mathcal{A}$-module $\mathcal{E}$, as above.

Thus, by applying the notation of Chapter I, (2.5), one concludes for an $\mathcal{A}$-metric, as before, that

(1.58) $\rho(s,s) \in \mathcal{A}(U)$ is not necessarily an element of $\mathcal{P}(U) \subseteq \mathcal{A}(U)$, as well (*viz.*, "positive") for given $s \in \mathcal{E}(U)$ and open $U \subseteq X$.

Therefore, by defining the "norm" of (a local section) $s \in \mathcal{E}(U)$, one sets

$$\|s\| := \sqrt{|\rho(s,s)|}. \tag{1.59}$$

Of course, we have to assume here, concerning (1.59), that

(1.60) we are still given an enriched ordered $\mathbb{C}$-algebraized space

(1.60.1) $$(X, \mathcal{A}),$$

alias an ordered $\mathbb{C}$-algebraized space with square root.

See Chapter I, (5.1) and subsequent comments therein or [VS: Chapt. IV, p. 336, (10.8)] for more details on the above notion. As already mentioned, we are going to consider $\mathcal{A}$-metrics of the previous type in Section 2, with $\mathcal{A}$ satisfying (1.60); see, for instance, Note 2.1 in.

On the other hand, semi-Riemannian $\mathcal{A}$-metrics were already considered in Section 9 of Chapter I, in Volume I of this treatise, through the so-called Einstein $\mathcal{A}$-metrics; see (9.61) and (9.62) therein.

2 Lorentz $\mathcal{A}$-Metrics

In the ensuing discussion we specialize in a certain particular type of $\mathcal{A}$-metric, as the title of this section indicates, whose classical counterpart is of fundamental importance in relativity theory. Our aim here is to establish the appropriate abstract setup, within which one can formulate in our case the analogous concept to the classical notion, as suggested by the title of the present section.

To start with, suppose that we are given a $\mathbb{C}$-algebraized space

$$(X, \mathcal{A}) \tag{2.1}$$

on a topological space X (see Volume I, Chapt. I, (1.4)) and let $\mathcal{E}$ be an $\mathcal{A}$-module on X, for which we further assume that one has a local gauge of the form

$$e^U \equiv \{U \subseteq X, \text{ open}: \ (e_i)_{0 \leqq i \leqq n} \subseteq \mathcal{E}(U)\}. \tag{2.2}$$

By definition, this means that we are given the above (finite) family of local sections of $\mathcal{E}$ over the open set $U \subseteq X$ as in (2.2); namely, one has

$$e_i \in \mathcal{E}(U), \qquad 0 \leqq i \leqq n, \tag{2.3}$$

which also supplies a basis of the corresponding $\mathcal{A}(U)$-module $\mathcal{E}(U)$, while we still assume that

(2.4) $$\mathcal{E}|_U = \mathcal{A}^{n+1}|_U = (\mathcal{A}|_U)^{n+1},$$

within an $\mathcal{A}|_U$-isomorphism of the $\mathcal{A}|_U$-modules concerned. (Here our hypothesis concerns only the first of the previous relations, while the rest is proved!)

The preceding is just the local aspect of the important notion of a vector sheaf, in particular, of the type that will mainly concern us in the sequel. After the above preliminary material, we come now to the following basic notion.

Definition 2.1 Given an $\mathcal{A}$-module $\mathcal{E}$ on X as above, a Lorentz $\mathcal{A}$-metric on $\mathcal{E}$ is a sheaf morphism

(2.5) $$\rho : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$$

such that the following conditions are in force:

(i) ρ is an $\mathcal{A}$-bilinear morphism of the $\mathcal{A}$-modules involved in (2.5).
(ii) ρ is symmetric (see (1.13), condition (ii)).
(iii) ρ is strongly nondegenerate; that is, one has the relation

(2.6) $$\mathcal{E} \underset{\tilde{\rho}}{\cong} \mathcal{E}^*,$$

within an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned (see also (1.4) for the relevant notation).
(iv) ρ satisfies the following *Lorentz condition*; that is, one has (see also (2.2)–(2.4))

(2.7) $$\rho(e_i, e_j) \equiv \eta_{ij} \begin{cases} 0, & i \neq j, \\ -1, & i = j = 0, \\ 1, & i = j \neq 0, \end{cases} \qquad 0 \leqq i, j \leq n.$$

(We still speak here of (2.2) as a Lorentz local gauge of $\mathcal{E}$.)

On the other hand, concerning (2.7) and by analogy with the Riemannian case with regard to the Kronecker δ, we also speak about the Lorentz η, pertaining, according to (2.7), to the $\mathcal{A}(X)$-valued matrix

(2.8) $$\eta \equiv (\eta_{ij}) \in M_{n+1}(\mathcal{A}(X)) = M_{n+1}(\mathcal{A})(X),$$

such that one has

(2.9) $$\eta_{ij} = 0, \pm 1 \in \mathcal{A}(X), \qquad \text{with } 0 \leqq i, j \leqq n.$$

In particular, by referring to (2.7), the same matrix is restricted to the given local gauge $U \subseteq X$ of $\mathcal{E}$, so that by looking at (2.9), one actually has

(2.10) $$\eta_{ij} \in \mathcal{A}(U), \qquad 0 \leqq i, j \leqq n.$$

What amounts to the same thing, one has the $(n+1) \times (n+1)$ (Lorentz) matrix

$$(2.11) \qquad \eta \equiv \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & \ddots_{\ n \text{ times}} & \\ 0 & & & +1 \end{pmatrix} \in M_{n+1}(\mathcal{A}(X)),$$

restricted to $U \subseteq X$, as above, or the respective (Lorentz) diagonal matrix, having signature

$$(2.12) \qquad sign(\eta) = (-1, \underbrace{+1, \ldots, +1}_{n \text{ times}}).$$

The above are certainly justified in view of our hypothesis for the $\mathcal{A}$-metric ρ, as in (2.5), along with our assumption for the structure sheaf $\mathcal{A}$ itself; indeed, one has the following $\mathbb{C}$-isomorphisms of ($\mathbb{C}$-vector) sheaves:

$$(2.13) \qquad \mathbb{R}_X \equiv \mathbb{R} \underset{\rightarrow}{\subset} \mathbb{C} \equiv \mathbb{C}_X \underset{\rightarrow_\varepsilon}{\subset} \mathcal{A},$$

with $\mathbb{R}_X$ and $\mathbb{C}_X$ denoting the respective constant (numbers, as indicated) sheaves on X (see Volume I, Chapt. I, (1.5)). Hence, constant numbers are essentially considered as constant sections of the relevant sheaves, as before.

Thus, having fixed the terminology that we are going to employ throughout the subsequent discussion, we further consider in the next subsection the $\mathcal{A}$-modules endowed with Lorentz $\mathcal{A}$-metrics (see Definition 2.1), in analogy with the similar situation that one encounters in the Riemannian case (see [VS: Chapt. IV, Section 8]).

2.1 Lorentz $\mathcal{A}$-Modules

Having the framework of (2.1), we call a Lorentz vector sheaf on X a pair

$$(2.14) \qquad (\mathcal{E}, \rho)$$

consisting of a vector sheaf $\mathcal{E}$ on X and a Lorentz $\mathcal{A}$-metric ρ on it (see Definition 2.1). So we assume that

(2.15) we are given a local frame of $\mathcal{E}$, say

$$(2.15.1) \qquad \mathcal{U} = (U),$$

such that the Lorentz condition (see (2.7)) holds for every local gauge U of $\mathcal{E}$ belonging to $\mathcal{U}$, as in (2.15.1).

For convenience, we also assume that

$$(2.16) \qquad rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} = n+1, \ n \in \mathbb{N}.$$

(However, see also, e.g., (10.27).)

On the other hand, by mimicking the corresponding situation, one has in the Riemannian case (see [VS: Chapt. IV, p. 322, Theorem 8.1]) that a Lorentz $\mathcal{A}$-metric ρ on a given vector sheaf $\mathcal{E}$ on X is provided by means of a Lorentz local frame of $\mathcal{E}$; that is, we assume that we are given a local frame, say

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}, \tag{2.17}$$

of $\mathcal{E}$ along with a 0-cocycle

$$\begin{aligned} \rho_{\mathcal{U}} &= (\rho_\alpha) \in Z^0(\mathcal{U}, \mathcal{GL}(n+1, \mathcal{A})) = \mathcal{GL}(n+1, \mathcal{A})(X) \\ &= GL(n+1, \mathcal{A}(X)) = M_{n+1}(\mathcal{A}(X))^{\bullet} \subseteq M_{n+1}(\mathcal{A}(X)) \\ &\subseteq \prod_\alpha M_{n+1}(\mathcal{A}(U_\alpha)), \end{aligned} \tag{2.18}$$

in such a manner that one has, in view of (2.7) (see also (2.12)),

$$sign(\rho_\alpha) = (-1, \underbrace{+1, \dots, +1}_{n \text{ times}}), \ \alpha \in I. \tag{2.19}$$

Thus, based now on (2.19), one further obtains that

$$^t\rho_{\mathcal{U}} = \rho_{\mathcal{U}} \tag{2.20}$$

(symmetry of $\rho \equiv \rho_{\mathcal{U}}$) or, equivalently,

$$^t\rho_\alpha = \rho_\alpha, \ \alpha \in I \tag{2.21}$$

(see also [VS: Chapt. IV, p. 320ff]). In this connection, we can speak of a Lorentz $\mathcal{A}$-metric on a given vector sheaf $\mathcal{E}$ on X as a (global) section matrix

$$\eta = (\eta_{ij}) \in M_{n+1}(\mathcal{A}(X)) = (M_{n+1}(\mathcal{A}))(X), \tag{2.22}$$

as in (2.11) (see also (2.8)), which, according to (2.7), corresponds to a given local frame $\mathcal{U}$ of $\mathcal{E}$ (see (2.2) for a particular local gauge of $\mathcal{E}$ in $\mathcal{U}$).

To sum up, we can say that

(2.23)

> we have a Lorentz vector sheaf
>
> $$(\mathcal{E}, \rho) \tag{2.23.1}$$
>
> on X whenever we are given a vector sheaf $\mathcal{E}$ on X along with a Lorentz local frame of $\mathcal{E}$
>
> $$\{\mathcal{U} = (U_\alpha); \rho = (\rho_\alpha)\} \tag{2.23.1}$$
>
> in such a manner that (2.18) and (2.19) hold. In this connection, we also refer to ρ as a Lorentz $\mathcal{A}$-metric associated with (or corresponding to) a given local frame $\mathcal{U}$ of $\mathcal{E}$.

In particular, the preceding specializes to our structure sheaf $\mathcal{A}$, or rather to any (finite) power (direct sum) of it, $\mathcal{A}^n$, $n \in \mathbb{N}$. In this connection, we note that according to (2.16), we consider here the (free) $\mathcal{A}$-modules

$$\mathcal{A}^n, \qquad \text{with } n \geqq 2. \tag{2.24}$$

(In this regard, see also Section 9 in the sequel.) Our next objective, in agreement with our previous remarks in Scholium 1.1, is to conclude the above situation as in (2.23) for any given vector sheaf $\mathcal{E}$ on X, from a similar assumption on $\mathcal{A}$ (*viz.*, again, reduce, in a sense, everything to $\mathcal{A}$). Indeed, as we shall see, this can be achieved under appropriate further assumptions for $\mathcal{A}$ and X (see (2.27) and/or (2.33) in the sequel). To repeat, the extra assumptions alluded to before are fulfilled in certain important particular instances that interest us, having to do with potential applications of the abstract differential-geometric setup in problems related to singularities and *relativistic quantization* (see Sections 5 and 9 in the sequel along with Sections 6 and 8).

Taking the preceding comments into account, assume now that

$$(\mathcal{A}^{n+1}, \rho) \tag{2.25}$$

is a Lorentz (free) $\mathcal{A}$-module. So our aim here is to conclude the assumption in (2.25) for any locally free $\mathcal{A}$-module $\mathcal{E}$ on X with

$$rk\mathcal{E} = n + 1 \tag{2.26}$$

(i.e., for a vector sheaf $\mathcal{E}$ on X of the previous rank). Indeed, by analogy with the Riemannian case (see [VS: Chapt. IV, Section 8]), we assume henceforth that

(2.27) we are given, in particular, an ordered algebraized space

$$(X, \mathcal{A}), \tag{2.27.1}$$

where X is a (Hausdorff) paracompact space and $\mathcal{A}$ is a strictly positive fine $\mathcal{A}$-module on X.

We refer to the above cited work for the terminology applied in (2.27). The crucial ingredient here is that as a consequence of our previous hypothesis, one can employ a convenient partition of unity of $\mathcal{A}$, say

$$(\varphi_\alpha)_{\alpha \in I} \subseteq End\mathcal{A} = \mathcal{A}(X), \tag{2.28}$$

in fact, a strictly positive one (loc. cit., p. 326, Definition 8.4), in such a manner that the relation

$$\bar{\rho} := \sum_\alpha \varphi_\alpha \cdot \rho_\alpha \tag{2.29}$$

yields a Lorentz $\mathcal{A}$-metric for the vector sheaf $\mathcal{E}$ as above (see also (2.26)). In this connection, one defines

$$\rho_\alpha : (\mathcal{E} \oplus \mathcal{E})|_{U_\alpha} \longrightarrow \mathcal{A}|_{U_\alpha} \tag{2.30}$$

as the Lorentz $\mathcal{A}$-metric on

(2.31) $$\mathcal{E}|_{U_\alpha} = \mathcal{A}^{n+1}|_{U_\alpha},$$

which is supplied already in view of our hypothesis for $\mathcal{E}$ and (2.25). Furthermore,

(2.32) $$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

stands for a given local frame of $\mathcal{E}$, which by virtue of the hypothesis for X can be viewed as being locally finite ibid. [p. 325, (8.42)], the partition of unity (2.28) being subordinated to it. (See also [loc. cit. p. 325, proof of Theorem 8.2].)

As a consequence, one now gets the following basic result:

(2.33) suppose we are given an ordered algebraized space

(2.33.1) $$(X, \mathcal{A}),$$

with X a paracompact (Hausdorff) space and $\mathcal{A}$ a strictly positive fine $\mathcal{A}$-module on X, such that

(2.33.2) $$(\mathcal{A}^{n+1}, \rho)$$

is a Lorentz $\mathcal{A}$-module (for some given (hence for any) $n \in \mathbb{N}$). Then every vector sheaf $\mathcal{E}$ on X (of rank $n+1$) admits a Lorentz $\mathcal{A}$-metric as well; namely, $\mathcal{E}$ is a Lorentz vector sheaf too.

Note 2.1 Concerning the recursive procedure of defining a Lorentz $\mathcal{A}$-metric (*viz.*, via Whitney sums and pull-back) as alluded to above, we further remark that one can consider the analogue of the Gram–Schmidt orthogonalization process by assuming that the pair

(2.34) $$(X, \mathcal{A}),$$

as in (2.33.1), is still an enriched ordered algebraized space. (In this connection, we refer to [VS: Chapt. IV, Section 10] for relevant motivating thoughts, which can easily be adapted to the present framework as well.)

We consider next Lorentz $\mathcal{A}$-modules that are further endowed with appropriate $\mathcal{A}$-connections (metric invariant ones), something that will be of particular significance in the sequel.

2.2 Lorentz Yang–Mills Fields

To begin with, suppose that we are given the framework as in (2.33). Moreover, suppose that we have a vector sheaf $\mathcal{E}$ on X with

(2.35) $$rk\mathcal{E} = n + 1,$$

so that by virtue of our hypothesis and of (2.33), one may look at a Lorentz vector sheaf on X,

$$(\mathcal{E}, \rho). \tag{2.36}$$

(For convenience, we applied here an obvious abuse of notation in connection with (2.33.2).)

On the other hand, by further assuming that we are given a differential triad

$$(\mathcal{A}, \partial, \Omega^1), \tag{2.37}$$

one concludes, in view of our hypothesis for X and $\mathcal{A}$ as in (2.33), that the given vector sheaf $\mathcal{E}$ admits an $\mathcal{A}$-connection D as well (see [VS: Chapt. VI, p. 85, Theorem 16.1, in conjunction with Chapt. III, p. 247, (8.56)]). Accordingly, one obtains the Yang–Mills field

$$(\mathcal{E}, D) \tag{2.38}$$

on X too (see also Chapt. I, (4.12)).

In view of the preceding, we come now to the following basic notion, the main issue of this particular subsection. As already stated in the preceding, we consider here, for convenience, only vector sheaves of rank at least 2; see (2.26).

Definition 2.2 Suppose we have a differential triad, as in (2.37), on a topological space X (no extra assumptions on X and $\mathcal{A}$ are to be assumed here *a priori*) and let $\mathcal{E}$ be a given vector sheaf on X with

$$rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} = n + 1. \tag{2.39}$$

We say that we have a *Lorentz Yang–Mills field* on X, denoted by

$$(\mathcal{E}, \rho; D), \tag{2.40}$$

whenever the pair

$$(\mathcal{E}, \rho) \tag{2.41}$$

provides a Lorentz vector sheaf on X (see (2.14) or (2.23)), while the corresponding pair

$$(\mathcal{E}, D) \tag{2.42}$$

entails a Yang–Mills field on X (see Chapt. I, (4.12) of the present volume) and in such a manner that the following relation holds:

$$D_{\mathcal{H}om(\mathcal{E}, \mathcal{E}^*)}(\tilde{\rho}) = 0. \tag{2.43}$$

Concerning the notation applied in (2.43), see (2.6) as well as Scholium 2.1, where the same relation is further commented on from a physical perspective.

Scholium 2.1 (Terminological) Having the framework of Definition 2.2, the $\mathcal{A}$-isomorphism $\tilde{\rho}$ appearing in (2.43) is the one defined by (2.6), so that (2.43) means by definition that

(2.44) the $\mathcal{A}$-isomorphism $\tilde{\rho}$ as in (2.6) is horizontal with respect to the canonical $\mathcal{A}$-connection

(2.44.1) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}^*)}$$

defined on the vector sheaf

(2.44.2) $$\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}^*) = \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E}^* = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})^*$$

through (2.42). (See also [VS: Chapt. VI, (6.16)].)

If (2.43) is valid, we say that the given $\mathcal{A}$-connection D, as in (2.42), is compatible with the (Lorentz) $\mathcal{A}$-metric ρ (see (2.40) or (2.41)). On the other hand, we can express (2.43) by saying that

(2.45) (2.43) entails a gauge equivalence of the $\mathcal{A}$-connections of $\mathcal{E}$ and $\mathcal{E}^*$ (dual of $\mathcal{E}$), which is provided by the $\mathcal{A}$-isomorphism $\tilde{\rho}$ (see (2.6)).

Based on Chapt. II, (1.11) and Chapt. I, (4.23) as well as on [VS: Chapt. VII, Section 8, in particular, p. 165, proof of Theorem 8.2], one sees that (2.43) is equivalent to the following relation (see also Chapt. I, (4.42) and (4.43)):

(2.46) $$D^* \equiv (\tilde{\rho} \otimes 1) \circ D \circ \tilde{\rho}^{-1} \equiv \tilde{\rho} \circ D \circ \tilde{\rho}^{-1} \equiv Ad(\tilde{\rho}) \cdot D \equiv \tilde{\rho}_*(D),$$

where we put

(2.46′) $$D^* \equiv D_{\mathcal{E}^*}.$$

The dual $\mathcal{A}$-connection of D (*viz.*, the $\mathcal{A}$-connection of $\mathcal{E}^*$) induced on the latter vector sheaf (dual of $\mathcal{E}$) by the given $\mathcal{A}$-connection of $\mathcal{E}$, $D \equiv D_{\mathcal{E}}$ (see (2.42). See also [VS: Chapt. VI, p. 22, Section 5.4]).

On the other hand, by employing a usual (and obvious as well) abuse of notation, we can further put (2.46) into the form

(2.47) $$D^* \circ \tilde{\rho} \equiv D^* \cdot \tilde{\rho} = \tilde{\rho} \cdot D \equiv \tilde{\rho} \circ D$$

(see Scholium 2.2 for the physical significance of the middle equality in the above relation) or in the form

(2.48) $$D^* \underset{\tilde{\rho}}{\sim} D,$$

which by definition entails an equivalent expression for (2.45); namely, one has that

(2.48′) D^* (the $\mathcal{A}$-connection of $\mathcal{E}^*$) is gauge equivalent to D (the $\mathcal{A}$-connection of $\mathcal{E}$) via $\tilde{\rho}$ the $\mathcal{A}$-isomorphism as given by (2.6).

Accordingly, (2.6), which has been assumed for the (Lorentz; see also Chapt. I, (2.9)) $\mathcal{A}$-metric ρ, according to Definition 2.1, is further transferred, through (2.48), to a similar relation for the corresponding $\mathcal{A}$-connections of the vector sheaves involved in (2.6), a condition that is still required by Definition 2.2 (compatibility of D, with respect to ρ; see (2.43)). So

(2.49) the required identification ($\mathcal{A}$-isomorphism $\tilde{\rho}$) of the carriers of the fields involved in (2.6) is further decreed to be in force as a similar identification (gauge equivalence) of the corresponding $\mathcal{A}$-connections of the fields concerned.

In other words, one has, through Definition 2.2,

(2.50) an equivalence in the Yang–Mills category, $\mathcal{YM}_X$ (see Chapt. I, (4.19)) of the fields under consideration (*viz.*, of $(\mathcal{E}, D)$ and its dual $(\mathcal{E}^*, D^*)$), and not simply in the category of vector sheaves on X, $\mathcal{V}ect\mathcal{S}h_X$. One has by definition the relation

(2.50.1) $$\mathcal{YM}_X \subseteq \mathcal{V}ect\mathcal{S}h_X$$

concerning the categories at issue. (In this connection, see also Volume I, Chapter II, (6.29) and (9.40) and Chapter I, Sections 4.2 and 4.3.)

On the other hand, (2.43) is the abstract version, in our case, of the familiar condition in general relativity, that

(2.51) the Lorentz metric satisfies the relation

(2.51.1) $$\nabla \rho = 0.$$

(See also Scholium 2.2 for a physical interpretation of the above relation.) So it is now a consequence of the general theory that (2.51), hence equivalently, in the abstract case considered, (2.43), implies that

(2.52) the (Levi–Civita) $\mathcal{A}$-connection D of $\mathcal{E}$ (see (2.42)) induced on it by the Lorentz $\mathcal{A}$-metric ρ as in (2.41) is torsion free.

In this regard, see also [VS: Chapt. VIII, (10.9)], pertaining to the notion of the torsion of a given $\mathcal{A}$-connection; see, for example, Volume I, Chapter I, Section 8.2 of the present treatise. On the other hand, we further note that by analogy with the classical case, one proves that

(2.53) the Levi–Civita $\mathcal{A}$-connection D of $\mathcal{E}$ (see (2.42)) is characterized by (2.51.1) (*viz.*, (2.43) (metric compatibility)) and the vanishing of the corresponding torsion.

See also [VS: Chapt. VIII, Section 10, along with Chapt. VII, p. 128; (5.34) and (5.35) and p. 131, (5.52)]; concerning the classical counterpart of the previous assertion, see M. Nakahara [1: p. 247, §7.8.4]. ■

To facilitate further the terminology applied, we set the following definition.

Definition 2.3 We call a topological space X for which the hypothesis of (2.33) is in force a *Lorentz space of order* $n + 1$ ($n \in \mathbb{N}$).

As a consequence of the above definition and our conclusion in (2.33), one now obtains the following, certainly more succinct, restatement of that result; that is, one has (see also Scholium 2.2)

(2.54) Given a Lorentz space X of order $n+1$ ($n \in \mathbb{N}$), every vector sheaf $\mathcal{E}$ on X of rank $n+1$ becomes a Lorentz Yang–Mills field (see Definition 2.2).

However, what one is really confronted with in the applications (see Section 3) is the framework that is depicted in the following result:

(2.55) Suppose we have the hypothesis of (2.33), apart from (2.33.2), and let

(2.55.1) $$(\mathcal{E}, \rho)$$

be a Lorentz vector sheaf on X of rank $n + 1$ (see (2.14)). Then $\mathcal{E}$ admits an $\mathcal{A}$-connection D as well, compatible with ρ (see (2.43), along with the comments following (2.44)); that is, $\mathcal{E}$ becomes a Lorentz Yang–Mills field on X,

(2.55.2) $$(\mathcal{E}, \rho; D)$$

(see Definition 2.2).

Indeed, except for (2.43), the assertion in (2.55) follows from our previous discussion, before Definition 2.2, based on the pertinent part of our hypothesis in (2.33). On the other hand, one obtains (2.43) by applying a similar argument to that used in the Riemannian case (see [VS: Chapt. VII, p. 168, Theorem 9.1]). ■

As a matter of fact, the preceding describes the framework that one has in the case of Einstein's equations (*in vacuo*), always within the present abstract setting, which will be our objective in Section 3.

Scholium 2.2 (Physical significance) Referring to (2.43),

(2.56) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0$$

(Levi–Civita identity), we remark that one may still look at it as an algebraic expression of the fact that (the given Lorentz $\mathcal{A}$-metric) ρ, hence $\tilde{\rho}$ (see (2.6)) as well, is a tensor (i.e., geometric objects), so that they are characterized by a flow. However, the latter is causally stationary (see Chapter I, Scholium 4.1) with respect (concerning the causality at issue) to $D_{\mathcal{E}} \equiv D$ and $D_{\mathcal{E}^*} \equiv D^*$ (see (2.50)); that is, relative to the $\mathcal{A}$-connection

(2.57) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}^*)}.$$

Thus, one finally obtains (2.56). Accordingly, to recapitulate, one concludes that

(2.58) $\tilde{\rho}$ is "stationary" or even "constant" relative to the (covariant) derivative defined by D and D^*—in other words, with respect to the $\mathcal{A}$-connection (2.57).

Furthermore, according to (2.47), the physical–geometric significance of (2.56) can be expressed through the

(2.59) commutativity of the flow (*viz.*, of $\tilde{\rho}$) relative to the causality (as determined by D and D^*). Taking into account the terminology of Chapter I, (4.30), we can also say that (2.56) is equivalent to the relation

(2.59.1) $$\tilde{\rho} \in \mathcal{YM}_X - Isom_{\mathcal{A}}((\mathcal{E}, D), (\mathcal{E}^*, D^*))$$

(see loc. cit. (4.54.1)), which also recalls the aforesaid commutativity, according to the definitions.

On the other hand, within the same vein of ideas, one can say that

(2.60) the Levi–Civita identity as in (2.43) provides another form of the principle of equivalence, or the principle of least action (alias *principle of Maupertuis* or *Fermat's principle*).

In this connection, see M. Nakahara [1: p. 28, along with pp. 257ff].

3 Einstein Field Equations

Our aim in the following discussion is to present Einstein's field equations *in vacuo*—that is, to quote R. Penrose [1: p. 23], "...in the absence of all physical fields except gravity ..." [emphasis here is ours], in other words, except for the gravitational field (graviton)—within our abstract framework that is advocated in the present treatise. As we shall see in conjunction with the last few sections of this chapter, that this gives us the possibility, concerning the classical aspect of the subject, to consider solutions of the equations at issue, including singularities, something unattainable, thus far by referring to the standard theory. This newly existing situation, as alluded to above, may have (see also Section 9) significant consequences pertaining to the quantum theory of gravity.

So to start with, we first have to fix the terminology we are going to employ in the subsequent discussion. Generally speaking, the framework within which we shall work is that of a Lorentz space of order $n+1$, in the sense of Section 2 (see (2.53), along with (2.33)).

Suppose that

(3.1) $$(\mathcal{E}, \rho)$$

is a Lorentz vector sheaf of order $n + 1$ (see (2.23), along with Definition 2.2). Furthermore, concerning the given differential triad

$$(\mathcal{A}, \partial, \Omega^1) \tag{3.2}$$

on X (we also remark here that no extra hypothesis is imposed on X so far), we assume that we have

$$\Omega^1 = \mathcal{E}^*, \tag{3.3}$$

so that one obtains (see [VS: Chapt. IV, p. 299, Theorem 5.1])

$$\mathcal{E} = (\Omega^1)^*. \tag{3.4}$$

Equivalently, we can say that

(3.5) we are given a differential triad

$$(\mathcal{A}, \partial, \Omega^1) \tag{3.5.1}$$

on a topological space X such that

$$\Omega^1$$

is a Lorentz vector sheaf on X of rank $n + 1$.

In this connection, see also (3.3), along with (2.6) and (2.23). ■

On the other hand, we also remark that

(3.6) (3.5) is in force whenever we assume that X is a Lorentz space of order $n + 1$ while Ω^1 is a vector sheaf on X of rank $n + 1$.

The previous assertion in (3.6) is a straightforward consequence of (2.33). ■

One concludes that

(3.7) by assuming that X is a Lorentz space (of order $n + 1$), along with the same hypothesis for Ω^1 as in (3.6), one obtains that (see also (2.55))

$$(\Omega^1, \rho; D) \tag{3.7.1}$$

is a Lorentz Yang–Mills field on X (see Definition 2.2), so that the same holds for

$$\mathcal{E} = (\Omega^1)^*. \tag{3.7.2}$$

See (3.4) along with (2.54). ■

We are now in a position to formulate, within the present abstract setting, hence availing ourselves of the merits of this formulation (e.g., incorporation of singularities, as already mentioned), the equations in the heading of this section. In fact, the same equations are contained in the following basic definition.

Definition 3.1 Suppose we are given a differential triad

(3.8) $$(\mathcal{A}, \partial, \Omega^1)$$

on a topological space X, along with a Lorentz vector sheaf on X of rank $n+1$, say

(3.9) $$(\mathcal{E}, \rho).$$

Moreover, we assume that (see also (3.3) and (3.4))

(3.10) $$\mathcal{E} = (\Omega^1)^*.$$

In this context, we then speak of X as an abstract Einstein space. In particular, we shall say that (the Lorentz vector sheaf) $\mathcal{E}$ satisfies the *Einstein equation in vacuo* whenever one has

(3.11) $$\mathcal{R}ic(\mathcal{E}) = 0.$$

(See also (1.24) along with Scholium 3.1.) To make the contribution of $\mathcal{E}$ in (3.11) more explicit, we can rewrite the same relation in the form

(3.11′) $$\mathcal{R}ic(\mathcal{E}) = tr(R(\cdot, s)t) \equiv tr(R(D_{\mathcal{E}})(\cdot, s)t) = 0$$

for any s and t in $\mathcal{E}(U)$, with U a local gauge of $\mathcal{E}$. (See also (1.16), (122), (1.25), and (1.27.1).) Indeed, as we shall see (see Section 4), one concludes, implicitly, that

(3.11″) the Einstein equations can be construed as Yang–Mills equations for the curvature tensor.

The claim is based on what we can consider within the present abstract setting as the Einstein–Yang–Mills action principle (Section 4, (4.20.1)). ■

In this context, see also W. Drechsler–M.E. Mayer [1: p. 2], as well as D. Bleecker [1: p. 134, Theorem 9.3.3].

Scholium 3.1 (Terminological) According to the previous definition, an (abstract) Einstein space X (precisely speaking, we should say of order $n+1$) is the underlying space of a differential triad, as in (3.8), on which there is given a Lorentz vector sheaf of order $n+1$,

(3.12) $$(\mathcal{E}, \rho),$$

such that (3.10) holds.

In particular, the above are in force whenever one has the framework of (3.7); in that case we also speak of X as an (abstract) Lorentz–Einstein space (of order $n+1$). It is actually this particular case that usually occurrs in the classical theory, as indicated by (3.17).

On the other hand, concerning Einstein's (vacuum) equation (3.11), we further remark that (3.11) represents the global (generalized) aspect of Einstein's equation within the present abstract setting; namely, let

(3.13) $$\mathcal{U} = (U)$$

be a local frame of $\mathcal{E}$ as in (2.15)—in other words, a Lorentz local frame of $\mathcal{E}$ (see (2.23)). Therefore, one has, concerning the first part of (3.11), the relation

$$\mathcal{R}ic(\mathcal{E}) = (\mathcal{R}ic(\mathcal{E})_U)_{U \in \mathcal{U}}, \tag{3.14}$$

such that one sets

$$\mathcal{R}ic(\mathcal{E})_U \equiv R : \mathcal{E}(U) \times \mathcal{E}(U) \longrightarrow \mathcal{A}(U), \tag{3.15}$$

as given by (1.21), for any given $U \in \mathcal{U}$, as in (3.13). Thus, one obtains

$$\mathcal{R}ic(\mathcal{E}) \in \mathcal{M}or_{\mathcal{S}h_X}(\mathcal{E} \oplus \mathcal{E}, \mathcal{A}). \tag{3.16}$$

See also (1.24) as well as [VS: Chapt. I, p. 75, (13.19), along with App., p. 405, (1.1)].

Finally, one further remark should be made here in connection with (3.11): As already said, the equation at issue refers, as is also the case classically (see Section 3.1), to an empty space (vacuum). Thus, for the more general case that some other forces (i.e., fields), apart from the gravitational field, are present (as, for instance, the so-called in the classical theory stress-energy tensor), (3.11) can be appropriately supplemented by some tensor—that is, by a suitable $\mathcal{A}$-morphism of the $\mathcal{A}$-modules involved, as the particular case at hand may demand. (Of course, one always takes here into account (3.10) and (2.6).) The resulting equation (3.11), being supplemented as alluded to above, is called the extended Einstein equation.

3.1 The Classical Counterpart

Equation (3.11) stands here, of course, for Einstein's vacuum field equations of the classical theory of general relativity. Thus, within that standard setup, one takes as the structure sheaf (*viz.*, our arithmetics in the terminology applied so far)

$$\mathcal{A} \equiv {}^{\mathbb{R}}\mathcal{C}^{\infty}_X; \tag{3.17}$$

that is, one considers the $\mathbb{R}$-algebra sheaf of germs of $\mathbb{R}$-valued smooth (*viz.*, $\mathcal{C}^{\infty}$) functions on X, the latter space being a 4-dimensional space–time manifold—thus, by definition, a Lorentz manifold (space) of order 4 ($= 3 + 1$, in the previous terminology). See also Volume I, Chapter I or [VS: Chapt. X, §1, and Chapt. VI, §2], pertaining to the rest of the standard terminology, with regard to the present abstract setting; to put the above into perspective with the classical theory, in particular as it concerns equation (3.11), see also (3.15), in conjunction with (1.22), (1.25), and (1.27.1).

In this connection, let us remark once more concerning the above setting of the classical theory as presented in (3.17) and the abstract framework adopted here ((3.11)), that the impact here of the previous discussion lies exactly in the possibility of choosing for $\mathcal{A}$, hence for what we may also call Einstein vector sheaves, *viz.*, solutions of Einstein's equation (3.11), an algebra sheaf that can now contain a large number of singularities, in point of fact, the largest so far! (see also Sections 6 and 8), a situation that certainly was infeasible thus far, according to the standard theory. However, this now is possible, simply as a result of the present abstract setting, in conjunction with the appropriate choice of $\mathcal{A}$ (*viz.*, of a new arithmetic, instead, in comparison with (3.17)). See Section 5.

3.2 Einstein Algebra Sheaves

Suppose we are given the framework of (3.5), while we also assume that (3.4) is still in force (see, for instance, (3.6)). Suppose, moreover, that the vector sheaf

(3.18) $$\mathcal{E} = (\Omega^1)^*$$

also satisfies Einstein's equation (*in vacuo*) (3.11); that is, one has in addition the relation

(3.19) $$\mathcal{R}ic(\mathcal{E}) = 0.$$

Then one speaks (succinctly, pertaining in particular to our structure sheaf $\mathcal{A}$, of an *Einstein algebra sheaf*, or a *differential Einstein algebra sheaf*) simply to point out the intervention here of the differential triad (3.8) (see also (3.20)).

In this connection, by further extending the classical terminology on the subject, one can say that

(3.20) an Einstein algebra sheaf (*viz.*, a given differential triad

(3.20.1) $$(\mathcal{A}, \partial, \Omega^1)$$

on a topological space X having the above properties) determines a (generalized) space–time vacuum geometry, the topological space X, as before, being the solution space of Einstein's equation (3.11), or a (generalized) Einstein universe, by extending the standard terminology.

Note 3.1 Taking into account that our general treatise is to appropriately associate any property we want with the structure sheaf $\mathcal{A}$ itself, not with the carrier space X involved, the common base space of the sheaves concerned, we have thus to explain the phrase

(3.20′) "solution space of Einstein's equation."

According to the aforementioned perspective, one means here the $\mathcal{A}$-modules (Yang–Mills fields of the pertinent type, Lorentz vector sheaves as above) that might be solutions of Einstein's equation. Under suitable hypotheses for $\mathcal{A}$, the (topological) space X itself might be supplied by the same ($\mathbb{C}$-algebra) sheaf $\mathcal{A}$; see topological algebra theory and/or (topological) algebraic geometry. Notice that this is the case in the classical framework as well.

Furthermore, as we shall presently see (see Section 4), the same equation (3.11) can be provided by a variation of the analogous Lagrangian (density), the latter being suitably formulated, within the present abstract setting or, as we also say, by that of the Einstein–Hilbert action.

On the other hand, Einstein algebras have been considered in the past, either as appropriate abstract algebras (linear associative unital commutative algebras over the reals)—R. Geroch [1] in the early 1970s—or, more recently, in the early 1990s in a sheaf-theoretic context in terms of differential spaces by suitably functionalizing (via a Gel'fand transform) abstract algebras: See M. Heller [1, 2] as well as M. Heller–W. Sasin [1]; in this context, see also A. Mallios [10], along with D.G. Northcott [1: pp. 17ff].

It is worth mentioning at this point that in the latter sheaf-theoretic perspective of Einstein algebras, its connection and potential application in problems related to the so-called singularities in the quantum régime that concern general relativity have been pointed out. In that context see also our relevant comments, within the present abstract framework (ADG), in Chapter IV, Section 5 in the sequel.

3.3 Einstein–Riemannian Algebra Sheaves

As the title of this subsection indicates, we are concerned here with the abstract analogue within the present framework of the standard counterpart pertaining to the classically so-called Einstein manifolds—namely, Riemannian manifolds ($\mathcal{C}^{\infty}$-manifolds endowed with a positive definite symmetric bilinear smooth form, alias (0, 2)-tensor field, which is also positive definite and symmetric) such that the Ricci (curvature) tensor (see (1.24) above for the abstract counterpart of this notion) is a constant multiple of the given metric (see Volume I, Chapter I, (9.62) for the abstract analogue of this notion).

Note 3.2 (Terminological) Referring to the previously applied terminology, for the classical case see A.L. Besse [1: p. 3]; we further remark that by the same term "Einstein manifold," one also refers, classically, to a "semi-Riemannian manifold" (see, for instance, B. O'Neil [1: p. 54, Definition 2 as well as p. 96, Ex. 21]), the rest of the conditions remaining the same as before. So one assumes that the ($\mathcal{A}$-)metric concerned is (strongly) nondegenerate, in place of positive definite, as one assumes in the Riemannian case. It is this latter case that we suppose here (see Volume I, Chapter I, Section 9.4). In this connection, see also Sections 1.2 and 1.3.

By analogy with Section 3.1, we assume in the sequel that

(3.21)

we are given a differential triad

(3.21.1) $$(\mathcal{A}, \partial, \Omega^1)$$

on a topological space X such that

(3.21.2) $$\Omega^1$$

is a vector sheaf on X of rank $n \in \mathbb{N}$, while

(3.21.3) $$(\Omega^1)^* = \mathcal{E},$$

where

(3.21.4) $$(\mathcal{E}, \rho)$$

is a semi-Riemannian (or even pseudo-Riemannian) vector sheaf on X (of rank $n \in \mathbb{N}$ as well, in view of (3.21.3)), in an obvious sense of the latter term, by virtue of what has been stated in Note 3.2 (see also Volume I, Chapt. I, Section 9.4).

In other words,

(3.22) ρ in (3.21.4) is by definition an $\mathcal{A}$-metric on $\mathcal{E}$ that is also symmetric and strongly nondegenerate.

By employing analogous terminology to that of the previous subsection, one can speak of the structure sheaf $\mathcal{A}$ involved in (3.21.1) as an Einstein–Riemannian algebra sheaf on X or a differential Einstein–Riemannian algebra sheaf by referring in particular to the differential triad (3.21.1) whenever the pair $(\mathcal{E}, \rho)$ as in (3.21.4) satisfies Einstein's equation in *vacuo*; that is,

(3.23) $$\mathcal{R}ic(\mathcal{E}) = 0.$$

Of course, the vector sheaf $\mathcal{E}$, as in (3.21.4) and (3.23), is here supposed to be a pseudo-Riemannian Yang–Mills field

(3.24) $$(\mathcal{E}, D),$$

in the obvious sense, pertaining to the compatibility of D with ρ (see Chapt. I, (9.7)); therefore, one understands the presense of $\mathcal{E}$ in (3.23) via its curvature $R(D) \equiv R$ (see (1.22) and (1.25)). In other words,

(3.25) the relevant exposition in Section 1 holds for the pseudo-Riemannian case as well (by analogy with what has been said, the $\mathcal{A}$-metric is thus of importance in that context).

In this connection, we remark that the preceding can be formulated in terms of an appropriate pair

(3.26) $$(\mathcal{A}, \rho)$$

(see (1.34.2)) in conjunction with a suitable topological space X, the base space of the sheaves considered. See (1.43) as well as Note 3.1.

4 Einstein–Hilbert Functional and Its First Variation

Our purpose now is to derive Einstein's equation (*in vacuo*) as in (3.11) from a variation of the Lagrangian density or from that of an Einstein–Hilbert action, alias functional, all this within our abstract framework of Section 3. Following the analogous classical argument (see J. Baez–J.P. Muniain [1: pp. 397ff]), we start with first fixing the relevant notions and terminology that will be of use in the sequel.

Assume that we have the framework of Definition 3.1 and let

$$(\mathcal{E}, \rho) \tag{4.1}$$

be a given Lorentz vector sheaf on X of rank $n+1$. In others words, we suppose that we are given an Einstein space X of order $n+1$. For convenience, we recall that one has the previous framework whenever one is given an (abstract) Lorentz–Einstein space of order $n+1$, a situation that has a direct bearing on the classical case as well (for $n = 3$ [loc. cit. (3.17)]).

By considering the previous (Lorentz) vector sheaf $\mathcal{E}$ on X, as in (4.1), the corresponding Einstein–Hilbert functional is by definition the map

$$\mathcal{EH} : \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}) \longrightarrow \mathcal{A}(X) : D \longmapsto \mathcal{EH}(D) := tr R \equiv \mathcal{R}ic(\mathcal{E}) \tag{4.2}$$

(see also (3.11)), whose domain of definition is thus the (affine) space of $\mathcal{A}$-connections of $\mathcal{E}$, its values being taken from $\mathcal{A}(X)$ (see (1.29) and (1.30) in the preceding).

By following the classical counterpart, our next objective is to show the following.

Theorem 4.1 The critical points of the Einstein–Hilbert functional, as in (4.2), that corresponds to a given Lorentz vector sheaf $\mathcal{E}$ on X are exactly the solutions of the Einstein field equations (*in vacuo*; see (3.11)).

That is, one has to find, within the present abstract setting, the analogous first variational formula of $\mathcal{EH}$ (*viz.*, of the previous map $\mathcal{EH}$), as in (4.2).

4.1 First Variational Formula of the Einstein–Hilbert Functional

Applying a similar argument to that in the case of the Yang–Mills functional (see Chapt. I, Sections 6 and 8), the functional considered here being a spin-off of the latter one (see Chapt. I), one first considers an $\mathcal{A}$-connection curve in the (affine) space of $\mathcal{A}$-connections of $\mathcal{E}$,

$$\mathcal{C}onn_{\mathcal{A}}(\mathcal{E}), \tag{4.3}$$

along with the corresponding curvature curve. Thus, if

$$D_t \equiv D + t\tilde{\omega} \in \mathcal{C}onn_{\mathcal{A}}(\mathcal{E}),\ t \in \mathbb{R}, \tag{4.4}$$

is an $\mathcal{A}$-connection curve of $\mathcal{E}$ in the space (4.3), where

$$\tilde{\omega} \in \Omega^1(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})), \tag{4.5}$$

then (see Chapt. VI, Lemma 6.1) the curvature curve of $\mathcal{E}$ associated with (4.4) is given by the relation

$$R_t \equiv R(D_t) = R + t \cdot D^1_{\mathcal{E}nd\mathcal{E}}(\tilde{\omega}) + t^2(\tilde{\omega} \wedge \tilde{\omega}), \tag{4.6}$$

with $t \in \mathbb{R}$ and $R \equiv R(D_t)$, so that one has (see also (6.2))

$$R_t \equiv R(D_t) \in \Omega^2(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \Omega^2(\mathcal{E}nd\mathcal{E})), \tag{4.7}$$

where

$$\mathcal{U} = (U_\alpha) \tag{4.8}$$

is a given local frame of $\mathcal{E}$ ((6.1.4)). Hence, in particular one obtains, pertaining to (4.5) and (4.6),

$$\tilde{\omega} \equiv (\tilde{\omega}^{(\alpha)}) \in Z^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) \subseteq C^0(\mathcal{U}, \Omega^1(\mathcal{E}nd\mathcal{E})) = C^0(\mathcal{U}, M_k(\Omega^1)), \tag{4.9}$$

where we have set

$$k \equiv n + 1 = rk_{\mathcal{A}}(\mathcal{E}) \equiv rk\mathcal{E} \tag{4.10}$$

(see (3.1) along with (2.39)), taking also into account (3.4).

To proceed, we adopt that

(4.11) in addition to our previous assumptions concerning the present framework (see Definition 3.1), we are also given the pertinent setup to formulate the notion of a volume element, as in Chapter I, Section 7.

Therefore, still following the classical pattern, one can further define

(4.12) the Einstein–Hilbert action according to the relation

$$S(\mathcal{E}) = \int_X \mathcal{R}ic(\mathcal{E}) \cdot vol. \tag{4.12.1}$$

On the other hand, by virtue of Chapter I, Section 7, one has the following relation pertaining to the volume element associated with the (Lorentz) metric ρ, as in (4.1):

$$vol \equiv \omega = \sqrt{|\ \rho\ |} \cdot \varepsilon_1 \wedge \cdots \wedge \varepsilon_k \in \mathcal{A}(X) \tag{4.13}$$

(see also (4.10)) as well as [VS: Chapt. IV, (11.3) and (11.4)]), such that

$$|\ \rho\ | := |\ \det(\rho(\varepsilon_i, \varepsilon_j))\ | \tag{4.14}$$

with $1 \leqq i, j \leqq k$ and

$$\varepsilon \equiv (\varepsilon_i)_{1 \leqq i \leqq k} \subseteq \mathcal{A}^k(X) = \mathcal{A}(X)^k, \tag{4.15}$$

the Kronecker gauge of $\mathcal{A}^k$ (Chapt. I, (7.4); see also (7.3) and (7.5)). In connection with (4.14), one further obtains, by virtue of the Lorentzian character of ρ (see (2.7)),

$$| \det \rho(\varepsilon_i, \varepsilon_j) | \equiv | \det \rho | = -\det \rho. \tag{4.16}$$

Scholium 4.1 (The integral in (4.12.1)) The integral sign that appears in (4.12.1) corresponds to an $\mathcal{A}(X)$-valued integral (see [VS: Chapt. IV, (11.4)], along with (3.14) or (3.15)). Moreover,

(4.17) we assume that the sections involved in (4.12.1) have compact support (they vanish outside a compact subset of their domain of definition).

The latter hypothesis enables one to apply the usual integration procedure associated with Radon-like measures, provided some suitable topological algebra structure is given on the local section algebras of our structure sheaf $\mathcal{A}$. In this connection, see also the relevant comments in Chapter I, (7.24), as well as Section 5 below.

On the other hand, concerning the same matter as above, see also the relevant comments in Chapter I, Scholium 6.1, which we also employ presently below. In other words, the need to integrate, in the previous sense, is actually circumvented, as we shall see presently; that is integration is needed only fiberwise (ibid., (6.38)), which is usually much easier, without resorting to the whole framework of Scholium 4.1.

Based on the preceding, we next define the first variational formula of the Einstein–Hilbert functional (see (4.2)), or of the Einstein–Hilbert action, by the relation

$$\begin{aligned} \delta S(\mathcal{E}) &\equiv \frac{d}{dt}\Big|_{t=0} S(\mathcal{E}) \equiv \frac{d}{dt}\Big|_{t=0} \left(\int_X \mathcal{R}(\mathcal{E}) \cdot vol \right) \\ &= \int_X \frac{d}{dt}\Big|_{t=0} (\mathcal{R}ic(\mathcal{E}) \cdot vol). \end{aligned} \tag{4.18}$$

Here, the previously applied commutation

$$\left(\frac{d}{dt}\Big|_{t=0} \right) \circ \int_X = \int_X \circ \left(\frac{d}{dt}\Big|_{t=0} \right) \tag{4.19}$$

is based on a standard appropriate use of the topological duals of the topological-algebraic structures involved in (4.18) (see also Scholium 4.1) in conjunction with the limit operation denoted therein, yet fiberwise. Now,

(4.20) by definition, the critical points of the Einstein–Hilbert functional, or the Einstein–Hilbert action, are those (Lorentz) vector sheaves $\mathcal{E}$, as in (4.1), for which one has

$$\delta S(\mathcal{E}) = 0; \tag{4.20.1}$$

hence, equivalently, by virtue of our previous comments concerning the meaning of the integral in (4.12.1), hence in (4.18) too, fiberwise (see the remarks following (4.19)), one gets

$$(4.20.2) \qquad \frac{d}{dt}\Big|_{t=0}(\mathcal{R}ic(\mathcal{E})\cdot vol) = 0.$$

Consequently, in view of (1.29) and (4.13), one obtains (see also (4.6) in conjunction with (1.18) and (1.20), modulo the easily spotted abuse of notation)

$$(4.21) \qquad \frac{d}{dt}\Big|_{t=0}(\mathcal{R}ic(\mathcal{E})\cdot vol) \equiv \overset{\cdot}{\overbrace{\mathcal{R}ic(\mathcal{E})\cdot vol}}(0) \in \mathcal{A}(X).$$

Thus, by further following, within the present abstract setting, the relevant classical argument, one finally obtains

$$(4.22) \qquad \delta S(\mathcal{E}) = \mathcal{R}ic(\mathcal{E})\delta\rho,$$

where

$$(4.23) \qquad \rho \longrightarrow \rho + \delta\rho$$

stands for the variation of the metric employed here. (In this connection, concerning the classical counterpart see M. Nakahara [1: pp. 258ff]; cf. also N. Bourbaki [1: Chap. 3, p. 70, Corollaire and p. 50, Section 4]. Yet, see also A. Mallios [13] [15: Section 4, Applications]). So, the foregoing provides the proof of Theorem 4.1. ■

Stating the preceding result otherwise, we have thus arrived, within the present abstract setting, at the classical conclusion that (see also D. Bleecker [1: p. 120])

(4.24) the Einstein field equation *in vacuo*—namely,

$$(4.24.1) \qquad \mathcal{R}ic(\mathcal{E}) = 0$$

(see (3.19)), arises from setting the first variation of the integral of the scalar curvature equal to zero.

5 Rosinger's Algebra Sheaf

> "It does not seem reasonable ...to introduce into a continuum theory [field theory] points (or lines etc.) for which the field equations do not hold."
>
> A. Einstein in *The Meaning of Relativity* (Princeton University Press, Princeton NJ, 1989). p. 164.

Our ambitious aim in the subsequent discussion is to show that the (algebra) sheaf, as in the title of this section, can actually be employed, as our sheaf of coefficients (alias our arithmetics within the present framework of abstract differential geometry (ADG)), so that one could finally formulate, according to what has been already said in the preceding chapters of this treatise, such types of differential equations as, for instance, Einstein's equation (in vacuum; see this chapter, Section 3) or Yang–Mills equations (see Chapt. I, Section 4.4, Definition 4.2). The profit thereof lies in the fact that the aforesaid equations might then engulf, or, what virtually amounts to the same thing, could coexist with, so many singularities, in the classical sense of this term, as the functions (in point of fact, sections) of the same sheaf, as above, can actually contain, by virtue, as we shall see, of its very definition. Thus, the final outgrowth of it might be, as we shall explain in the sequel, a type of a unified field theory, expressing by means of such equations the classical part of reality as it was the case, so far, nevertheless simultaneously, the "singular", *viz.*, the "discrete" (!) part of it, as well, that, exactly, might be construed, in that context, as a new potential issue (!) of the present study, as a whole.

5.1 Basic Definitions

We start with fixing the relevant terminology, thus giving fundamental definitions and also explaining necessary rudimentary concepts. For further details we refer to the work of E.E. Rosinger [1] as well as to A. Mallios–E.E. Rosinger [1].

We first describe what we may call a Rosinger presheaf on a topological space X. For convenience, we take the space X, the base of the (pre)sheaves involved, as a (nonvoid) open subset of $\mathbb{R}^k$; in fact, what we are going to say is valid any time this happens just locally; that is, in place of $\mathbb{R}^k$, one might also consider a (smooth; *viz.*, $\mathcal{C}^\infty$-)manifold (see M. Kunzinger [1]). On the other hand, the aforesaid presheaf consists, as we shall see, of commutative unital quotients of complex(-valued) function algebras (although real-valued functions can still be considered).

Taking the space

$$X \text{ as an open subset of } \mathbb{R}^k \tag{5.1}$$

along with an

$$\text{open } U \subseteq X, \tag{5.2}$$

we further set

$$A(U) \equiv \mathcal{C}^\infty(U)^{\mathbb{N}}, \tag{5.3}$$

thus getting a commutative unital $\mathbb{C}$-algebra whose elements are by definition sequences of $\mathbb{C}$-valued smooth ($\mathcal{C}^\infty$-)functions on $U \subseteq X$. Hence, as U varies over the open subsets of X, one gets, through (5.3) and the obvious restriction maps, a presheaf of $\mathbb{C}$-algebras on X; that is,

$$A \equiv \{(A(U); \rho_V^U\}. \tag{5.4}$$

The above is a complete presheaf (Leray), i.e., a sheaf (Lazard); in other words, the Cartesian product sheaf,

$$({}^{\mathbb{C}}\mathcal{C}_X^\infty)^{\mathbb{N}} \equiv (\mathcal{C}_X^\infty)^{\mathbb{N}}, \tag{5.5}$$

(see also, e.g., R. Godement [1: p. 117, no 1.10]) or the sheaf of germs of sequences of ($\mathbb{C}$-valued) $\mathcal{C}^\infty$-functions on X, with X, as in (5.1). In fact, Rosinger's (algebra) sheaf is a pertinent quotient of the above sheaf (5.5). We define this through the corresponding presheaf that generates it.

The crux of the definition is what we may call here Rosinger's ideal, by definition a (2-sided) ideal of each of the algebras appearing in (5.3) which for convenience we define below for the global section algebra of the sheaf (5.5)—namely, for the commutative unital $\mathbb{C}$-algebra,

(5.6) $$\mathcal{C}^\infty_{\mathbb{C}}(X) \equiv \mathcal{C}^\infty(X) \cong \Gamma(X, (\mathcal{C}^\infty_X)^{\mathbb{N}}) \equiv (\mathcal{C}^\infty_X)^{\mathbb{N}}(X).$$

Definition 5.1 Take X an open subset of $\mathbb{R}^k$ and the algebra $\mathcal{C}^\infty(X)$ (see (5.6)). Then we define the Rosinger ideal $I(X)$ of $\mathcal{C}^\infty(X)$ as those sequences of ($\mathbb{C}$-valued) $\mathcal{C}^\infty$-functions on X,

(5.7) $$t \equiv (t_n) \in (\mathcal{C}^\infty(X))^{\mathbb{N}},$$

for which the following asymptotic vanishing condition is satisfied:

(5.8)

there exists a closed, nowhere dense set $\Gamma \subseteq X$,

(5.8.1) namely, we assume that $\overset{\circ}{\Gamma} = \emptyset$, hence, $\overline{\complement\Gamma} = X$,

such that for every relatively compact $K \subseteq \complement\Gamma$, there exists $m(K) \equiv m \in \mathbb{N}$ such that

(5.8.2) $$t_n|_K \equiv 0$$

for any $n \geqq m$.

It is clear by definition that

(5.9) the subset of $\mathcal{C}^\infty(X)^{\mathbb{N}}$ defined by (5.8) is a (2-sided) ideal of the aforesaid algebra.

We call (5.8), Rosinger's asymptotic vanishing condition. We further define Rosinger's (algebra) presheaf

(5.10) $$A/I \equiv \{A(U)/I(U); \sigma^U_V\},$$

so that, by definition,

(5.11) Rosinger's algebra sheaf $\mathcal{A}_{nd} \equiv \mathcal{A}$ is the $\mathbb{C}$-algebra sheaf on X generated by the presheaf (5.10).

Now one proves that

(5.12) Rosinger's presheaf, as given by (5.10), is, in effect, a complete presheaf, in the sense of J. Leray.

See [VS: Chapt. I, p. 46, Section 11] concerning the terminology applied in (5.12). Thus, what we actually prove is that

(5.13) Rosinger's presheaf is a monopresheaf, while the same is already localized as well.

See also [loc. cit. p. 51, Proposition 11.1] along with A. Mallios [10], concerning, in particular, the term *localized*, which we shall explain in the sequel. We first remark that (5.4) is a monopresheaf on X, being, by definition, a functional one (see also [VS: p. 50, (11.14), and p. 54, (11.36′)]), while

(5.14) Rosinger's (algebra) presheaf is a monopresheaf according to the definitions, as follows (5.10).

See also [VS: p. 32, (7.22)]. Now, the point is that

(5.15) Rosinger's (algebra) presheaf is already a localized one, as well; that is, any (local) function

(5.15.1) $\alpha : U \to \mathcal{A}$ that locally belongs to (5.10) is of the same type.

In this context, the meaning of (5.15.1), concerning the given function a, is that

(5.16) for any point $x \in U$, there exists an open neighborhood V of x with

(5.16.1) $$x \in V \subseteq U$$

and an element

$$t \in A(V)/I(V)$$

such that

$$\alpha|_V = t.$$

The verification of our assertion in (5.13) and (5.15) is a consequence of the argument used in the second part of the proof of Lemma 2 in A. Mallios–R. Rosinger [1: p. 246; Section 5, Appendix]. In this connection, see also A. Mallios [10: (2.16)]. ■

Furthermore, referring to the inner structure of Rosinger's algebra sheaf

(5.17) $$\mathcal{A}_{nd} \equiv \mathcal{A},$$

as given by (5.11), one has the fundamental property

(5.18) $$\mathfrak{D}'_X \subseteq \mathcal{A},$$

where the first member of (5.18) denotes the sheaf of germs of the (Schwartz) distributions on X, considered as a ($\mathbb{C}$-)vector space sheaf on X. See E.E. Rosinger [1: p. IX]. Indeed, one has the fuller relation

(5.19) $$\mathcal{C}^\infty_X \subsetneqq \mathfrak{D}'_X \subseteq \mathcal{A},$$

the first member denoting the sheaf of (germs of) $\mathbb{C}$-valued $\mathcal{C}^\infty$-functions on X.

Note 5.1 (Historical) By referring to (5.18), it is to be noticed that the same relation was in fact the instrumental motivating impulse both for J.F. Colombeau [1] and E.E. Rosinger [1] concerning their innovation in defining an appropriate algebra containing (Schwartz) distributions, after the remark of L. Schwartz himself [1] that the space of distributions is not always an algebra, an annoying fact referring to important applications of the theory—for instance, to PDEs (linear or not)! For further remarks on this, along with comments pertaining to the relation among the various aspects on the matter of the so-called generalized functions in relation to (5.18), we refer to the pertinent remarks of E.E. Rosinger [1], the same author being actually among the founders of one of the major classes of this type of (sheaves of) functions, which we also consider here.

5.2 The Differential Triad, Based on Rosinger's Algebra Sheaf

As the title of this subsection indicates, our aim is to show that

(5.20) Rosinger's algebra sheaf

(5.20.1) $$\mathcal{A}_{nd} \equiv \mathcal{A},$$

as defined by (5.11), can be employed as the sheaf of coefficients, or what amounts to the same thing, as our (generalized) arithmetics in the instrumental sense that these terms have been applied throughout the present treatise.

Our first objective in the following discussion is to show the existence of the fundamental

(5.21) basic differential operator

(5.21.1) $$\partial : \mathcal{A} \longrightarrow \Omega^1,$$

according to the general principles of abstract differential geometry; see Volume I, Chapter I, Section 1 or [VS: Chapt. VI, Section 1].

The subsequent discussion is essentially based on A. Mallios–E.E. Rosinger [1]: Indeed, the idea of defining the sequence of differential operators needed in the abstract differential geometric mechanism that is employed throughout the present treatise (see, for instance, Chapter I, Section 1) is rooted in the classical counterpart of the notion, due to the presence of the sheaf $\mathcal{C}^\infty_X$ in (5.19) as well as to the invariance of the latter (standard) operators, by taking (Rosinger) quotients as in (5.10). Thus, taking coordinatewise the aforementioned classical differential operators, in view of (5.5) or (5.6), and then passing to the corresponding quotients, one defines

(5.22) $$d^p([x]) := [d^p(x)] := [(d^p x_n)_{n\in\mathbb{N}}]$$

for every

(5.23) $$x \equiv (x_n)_{n\in\mathbb{N}} \in (\mathcal{C}^\infty(U))^{\mathbb{N}}$$

and U open in X (see (5.2)); that is, denoting by

$$A_{nd} \equiv \{A_{nd}(U) \equiv A(U)/I(U); \sigma_V^U\} \tag{5.24}$$

the Rosinger's (algebra) presheaf, as given by (5.10), and in view of (5.12), we actually defined through (5.22) the $\mathbb{C}$-linear operators, indeed sheaf morphisms (derivations),

$$d_U^p \equiv d^p : \mathcal{A}_{nd}(U) \longrightarrow \Omega^p(U) \tag{5.25}$$

(see also (5.32) in the sequel) for any open $U \subseteq X$ and $p \in \mathbb{Z}_+$ such that

$$d_U^0 \equiv d^0 \equiv \partial : \mathcal{A}_{nd}(U) \longrightarrow \Omega^1(U), \tag{5.26}$$

where by definition, following classical patterns, we assume that

(5.27) Ω^1 is the free $\mathcal{A}$-module of rank $k \in \mathbb{N}$ (see also (5.1), along with (5.11) or (5.17)), (freely) generated by

$$dx_1, \ldots, dx_k. \tag{5.27.1}$$

Therefore, a generic element of $\Omega^1(U)$, with U open in X, is of the form

$$t \equiv \sum_{i=1}^{k} \partial_i([t])dx_i \tag{5.28}$$

(see also (5.7)), such that one sets, according to (5.22) (see also (5.24)), for $p = 0$,

$$\partial_i([t]) := [(\partial_i(t_n))_n], \ 1 \leqq i \leqq k, \tag{5.29}$$

where we further set, following standard notation,

$$\partial_i(t_n) := \left(\frac{\partial}{\partial x_i}\right)(t_n), \text{ with } t_n \in \mathcal{C}^\infty(U) \text{ and } n \in \mathbb{N}, \tag{5.30}$$

while $1 \leqq i \leqq k$. Occasionally, we also define

$$\psi(t) := (t, t, \ldots, t, \ldots) \in \mathcal{C}^\infty(U)^{\mathbb{N}} \tag{5.31}$$

for any $t \in \mathcal{C}^\infty(U)$ (*viz.*, when taking in particular the diagonal of the Cartesian product, as in (5.7); especially, this is actually the case, by effectuating, otherwise, according to classical patterns, the "free generators" dx_i, as in (5.27.1)).

Applying the standard procedure of exterior algebra in terms of sheaf theory (see [VS: Chapt. IV, Section 7]), we can further define the corresponding exterior algebra in terms of Rosinger's generalized functions—that is, by means of the sheaf $\mathcal{A}_{nd} \equiv \mathcal{A}$; that is, as usual, one defines

$$\Omega^p := \bigwedge^p \Omega^1, \ p \in \mathbb{Z}_+, \tag{5.32}$$

where we set

$$\Omega^0 := \mathcal{A}_{nd} \equiv \mathcal{A}. \tag{5.33}$$

We also define the corresponding (Leibniz) differentials (i.e., is, sheaf morphisms),

$$(d^p)_{p \in \mathbb{Z}_+}, \tag{5.34}$$

proving that

$$d^{p+1} \circ d^p = 0, \ \ p \in \mathbb{Z}_+ \tag{5.35}$$

(see also (5.26). See A. Mallios-Rosinger [1], along with Chapter I, Section 1 of the present treatise). What is of importance here is the fact that

(5.36) the differential triad

$$(\mathcal{A}_{nd} \equiv \mathcal{A}, \partial, \Omega^1), \tag{5.36.1}$$

as defined in the preceding discussion, can be used as such to develop the classical differential geometry on smooth ($\mathcal{C}^\infty$-) manifolds, within the abstract (spaceless) setup, as given by [VS] and subsequent relevant work, this being subsumed in the present treatise. However, see also A. Mallios [9, 11], as well as Section 7 in the sequel concerning potential physical applications in terms of a differential triad, like (5.36.1), and the concomitant differential-geometric machinery, which can be defined according to the point of view that has been elaborated thus far.

In this context, a major issue is the exactness of the corresponding de Rham complex (see also Chapter I, (1.10) in the preceding)—namely, of the complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\ \varepsilon\ } \mathcal{A}_{nd} \equiv \mathcal{A} \xrightarrow{\ \partial\ } \Omega^1 \xrightarrow{\ d^1\ } \Omega^2 \xrightarrow{\ d^2\ } \cdots, \tag{5.37}$$

which reduces to the validity of the relation

$$\ker d^{p+1} = \mathrm{im} d^p, \ \ p \in \mathbb{Z}_+ \tag{5.38}$$

—that is, to the validity in our case of the famous *Poincaré lemma*, a result of an extremely special character, yet, local, notwithstanding. Thus, within the present context, the proof of (5.38) is reduced to that of its classical counterpart, due to the definition of $\mathcal{A}_{nd}$ and (5.22), along with the fact that exactness is checked fiberwise (hence locally) and to the force of (5.38) as well. (See also A. Mallios–E.E. Rosinger [1: p. 243, Theorem 1].) ∎

Scholium 5.1 The fiberwise checking of the exactness of a given sequence of $\mathcal{A}$-modules, thus for the case considered here of the validity of (5.38), is a classical matter concerning the more general situation pertaining to sheaf morphisms in general; so we refer, for instance, to [VS: Chapt. I, Section 12] for pertinent comments and further details. In this context, we still refer to [loc. cit. p. 10, (2.10)–(2.13) as well as p. 12, Section 3] concerning the local character of the above, as applied in the preceding, this being the result of the general motto "a sheaf is its sections" or even that "a sheaf is [made out of] the germs of its sections" [loc. cit., p. 10].

The above is also in accord with the local character of the definition of Rosinger's generalized functions—that is, of the respective presheaf as in (5.10), a fact that is also at the basis of the completeness of the same presheaf, as already mentioned (see also A. Mallios [10: (1.21)]).

The previous situation will also be of a particular significance for applications of the aforesaid (complete pre-)sheaf in the sequel.

Another very useful spin-off of the preceding is the validity of the so-called short exact exponential (sheaf-)sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon} \mathcal{A}_{nd} \equiv \mathcal{A} \xrightarrow{\exp} \mathcal{A}^{\bullet} \longrightarrow 1, \tag{5.39}$$

where $\mathbb{Z}$ stands for the constant sheaf of integers over X and

$$\mathcal{A}^{\bullet} \equiv \mathcal{A}^{\bullet}_{nd} \tag{5.40}$$

denotes the group sheaf of units (invertible elements) of the algebra sheaf on X, $\mathcal{A}_{nd} \equiv \mathcal{A}$, as before (see (5.11)). Concerning our notation in (5.39), we also define

$$\exp([t]) := [(\exp t_n)_{n\in\mathbb{N}}] \tag{5.41}$$

for any

$$t \equiv (t_n) \in \mathcal{C}^{\infty}(U)^{\mathbb{N}}, \tag{5.42}$$

with U open in X.

On the other hand, the group sheaf of units $\mathcal{A}^{\bullet}$ appearing in (5.40) can be defined as follows (see also A. Mallios–E.E. Rosinger [1: p. 237, Lemma 3], to which we refer for further details). By considering an open set U in X, the persheaf of groups of units of Rosinger's algebra presheaf, as defined by (5.10)—namely, that whose individual members (groups) are given by

$$\begin{aligned} A^{\bullet}(U) = \mathcal{A}^{\bullet}_{nd}(U) &\equiv \mathcal{A}^{\bullet}(U) = \mathcal{A}(U)^{\bullet} \\ = A_{nd}(U)^{\bullet} &\equiv A(U)^{\bullet} \equiv (A(U)/I(U))^{\bullet} \end{aligned} \tag{5.43}$$

(see also (5.12), along with [VS: Chapt. IV, p. 282, Lemma 1.1, in particular, (1.15), p. 283])—is identified according to the following relation:

$$\begin{aligned} A^{\bullet}_{nd}(U) = \{[t] \in \mathcal{A}_{nd}(U) :\ &\exists\, \Gamma \subseteq U, \text{ closed nowhere dense, such that} \\ &\text{for any relatively compact } K \subseteq (\complement\Gamma) \cap U, \exists\, m \in \mathbb{N}, \text{ with } t_n \neq 0|_K, \text{ for} \\ &\text{any } n \geqq m\} \end{aligned} \tag{5.44}$$

(see also (5.42)), while the identity element of the latter algebra is given based on (5.31). Our assertion concerning the exactness of (5.39) is supplied by the relevant argument in A. Mallios–E.E. Rosinger [1: p. 241, in particular Lemma 5]. ■

Note 5.2 Both of the preceding two fundamental conditions, which for the general theory of abstract differential geometry, are the exactness of the sequences (5.37) and (5.39), the first being the corresponding Poincaré lemma, have been often employed in the previous chapters of this study. See, for instance, Volume I of the present treatise, Chapter III, Section 3: *Weil scheme* or *Weil space,* used in the formulation/proof of *Weil's integrality theorem.*

In this context, we note that

(5.45) the base space X of Rosinger's algebra sheaf, being by definition (see (5.1)) an (open) subset of $\mathbb{R}^k$, hence metrizable, is a paracompact (Hausdorff) space.

Furthermore, it has been already remarked, see A. Mallios–E.E. Rosinger [1: p. 237, Lemma 2, along with pp. 248f] that

(5.46) Rosinger's algebra sheaf

(5.46.1) $$\mathcal{A}_{nd} \equiv \mathcal{A}$$

is a fine and flabby sheaf on X.

The fineness of the structure sheaf $\mathcal{A}$ on a paracompact (Hausdorff) space X, as is the case for the sheaf $\mathcal{A}_{nd}$, is of a particular significance for at least cohomological issues, another fundamental item within the present abstract treatment. Flabbiness of the same sheaf (see (5.46)) will also be of use and importance too, as we shall presently see (see Section 7).

5.3 $\mathcal{A}_{nd}$-Metrics

Another important issue throughout the preceding account has been the notion of an $\mathcal{A}$-metric on a given $\mathcal{A}$-module $\mathcal{E}$, hence on $\mathcal{A}$ as well, $\mathcal{A}$ being the sheaf of coefficients in the particular case under consideration—here, specifically, Rosinger's algebra sheaf: Concerning the relevant abstract theory, we refer for details to Volume I of this treatise, Chapter I, Section 9 as well as to [VS: Chapt. V, Section 8] for a fuller account; see Chapter I in the preceding, Section 2.

Thus, based on what has been said above pertaining to the latter sheaf, which concerns us in this section, we simply want to point out here that

(5.47) Rosinger's algebra sheaf can now be construed as a Riemannian vector sheaf on X (of rank 1, in effect, a free such sheaf), which can further be employed in all the corresponding particular cases that this notion ($\mathcal{A}$-metric) has been used in the preceding sections within the general framework of the abstract theory.

In this connection, see also, in particular, Chapter I, (9.37) and (9.40), in Volume I of the present study, which can be related to the definition of an $\mathcal{A}_{nd}$-metric on $\mathcal{A}_{nd} \equiv \mathcal{A}$, as before. Similar remarks to (5.47) are still in force pertaining, in particular, to Einstein and/or Lorentz $\mathcal{A}$-metrics (Volume I, Chapt. I, Sections 9.4 and 9.5, as well as Chapt. I, Section 7; in this regard, see also our previous comments in (5.45) and (5.46) of this section).

5.4 $\mathcal{A}_{nd}$ as a Topological Algebra Sheaf. Radon-Like Measures

By treating the so-called Yang–Mills equations, within the present abstract setting, (see Chapt. I), one is naturally led to consider, following classical patterns, the corresponding Yang–Mills functional (Section 5), as well as the associated variation formulas (Sections 6 and 8). Thus, what one actually needs here (see Chapt. I, Scholium 7.1) is an appropriate topological vector space structure on the fibers of the structural sheaf (alias generalized arithmetic) under consideration providing a nontrivial (topological) dual space.

So our main objective in this subsection is to show that

(5.48) by employing Rosinger's algebra sheaf (see (5.46)) as our generalized arithmetic, one secures the previously described situation, depicted, by Scholium 7.1 of Chapter I.

Thus, following standard reasoning in topological algebra theory (see, for instance, A. Mallios [TA]), we first remark that

(5.49) each of the algebras

$$A(U) \equiv \mathcal{C}^{\infty}(U)^{\mathbb{N}},$$

with U open in X (see (5.1)), yields a Fréchet locally m-convex algebra, hence, in particular, a Fréchet locally convex (topological vector) space.

See [TA: p. 130, (4.12), along with p. 82, Lemma 1.1 and subsequent comments therein; see also p. 9, Definition 1.5].

On the other hand, by considering the corresponding Rosinger's quotient algebras

$$(A/I))U) \equiv A(U)/I(U) \tag{5.50}$$

(see (5.10)), one gets again locally m-convex algebras [TA: p. 71, (7.9)]; the latter are still Fréchet, provided $I(U)$ is a closed (2-sided) ideal of $A(U)$, which, however, may not be always the case (E.E. Rosinger: oral communication). Accordingly, we consider the so-called Hausdorff completion of (5.50), thus being again a Fréchet locally m-convex algebra; see, for instance, N. Bourbaki [3: Chapt. II, p. 23, Définition 4], along with [TA: p. 22, Lemma 4.1]. We denote it by

$$A(\widehat{U)/I}(U), \text{ with } U \text{ open in } X. \tag{5.51}$$

Consequently, employing the notation of (5.10), one gets by definition a presheaf morphism

$$A/I \xrightarrow{\ i\ } \widehat{A/I}, \tag{5.52}$$

with the obvious meaning of the target of i according to (5.50) and (5.51), so that by taking further (topological) duals, one thus obtains the precosheaf morphism

$$(5.53) \qquad {}^t i : (\widehat{A/I})' \longrightarrow (A/I)'$$

(see also G.E. Bredon [1: p. 281]—namely, a natural transformation of the respective covariant functors (from the (category of) open sets of X to the category of locally convex spaces) defining the domain and range of ${}^t i$, as above (see also [loc. cit., p. 8, Section 2]). This also sustains, for the case at hand, a argument similar to that in Chapter I, Section 7.1.

6 Rosinger's Algebra Sheaf (continued): Multifoam Algebra Sheaves

"We do not possess any method at all to derive systematically solutions that are free of singularities."

A. Einstein in *The Meaning of Relativity* (Princeton University Press, Princeton, NJ, 1989). p. 165.

We continue in the present section the preceding discussion, referring to Rosinger's algebra sheaf, by extending the corresponding framework, as was exhibited in Section 5, to a much more general situation, pertaining now to a replacement of the structural sheaf $\mathcal{A} \equiv \mathcal{A}_{nd}$ considered thus far (see (5.11)) by what we are going to define below as a multifoam algebra sheaf (see definition (6.20.2)). Thus, the aforesaid generalization concerns a particularly significant enlargement of the size of the sets of singularities that can be considered, this being the largest one, so far, in the literature (see A. Mallios–E.E. Rosinger [2]). Indeed, as we shall presently see, the

(6.1) "singularities" might be arbitrary sets, under the only proviso that their complements are dense in X (residual sets; see J. Dugundji [1: p. 92]).

Thus, in the case that one takes for the space X the real line $\mathbb{R}$, then the singularities can be all the irrational points, while the nonsingular points can be reduced to the rational ones (see [loc. cit., p. 61]).

6.1 Basic Definitions

As in Section 5, we take (for convenience) as

(6.2) base space of the sheaves involved, an open subset X of $\mathbb{R}^k$.

Of course, a (smooth) manifold X could also be considered, as already hinted earlier.

The so-called singularity sets that have been considered in Section 5, being by definition closed nowhere dense sets (see Definition 5.1; in particular, (5.8.1)), hence, the notation nd, are here replaced by arbitrary sets under the only restriction that their complements are dense subsets of X, with X as in (6.2). In the sequel, we follow, in principle, the terminology of A. Mallios and E.E. Rosinger [2], to which we refer for further details.

We start with the definition of the family of singularity sets, the largest one thus far. Indeed, as already mentioned, consider the following family of subsets of X—namely, we define

$$\mathcal{S}_{\mathcal{D}}(X) := \{\Sigma \subseteq X : \overline{\complement\Sigma} = X\}. \tag{6.3}$$

The point is that each of the previous sets might be the set of singularities of a given generalized function. It is clear that the family, say

$$\mathcal{S}_{nd}(X), \tag{6.4}$$

of closed nowhere dense subsets of X is contained in (6.3) (see (5.8.1)); that is, one has the relation

$$\mathcal{S}_{nd}(X) \subseteq \mathcal{S}_{\mathcal{D}}(X). \tag{6.5}$$

In this context, we refer to the aforementioned work concerning the term "generalized function" in the sense of E.E. Rosinger; thus, one considers an element of the algebra

$$\mathcal{A}_{nd}(X) \equiv \mathcal{A}(X) \tag{6.6}$$

(see (5.17), along with (5.12)) when the said function is globally defined, or of the algebra

$$\mathcal{A}_{nd}(U) \equiv \mathcal{A}(U), \text{ with } U \text{ open in } X, \tag{6.7}$$

for locally defined such functions.

On the other hand, by analogy with the concept of Rosinger's ideal, as given by Definition 5.1, we further define the analogous pertinent notion within the present, more general, framework. Thus, consider an (upward) directed set Λ (see, for instance, J.L. Kelley [1: p. 65]), along with the (Cartesian product) algebra

$$\mathcal{C}^{\infty}(X)^{\Lambda} \tag{6.8}$$

(see also (5.3)), as well as an element

$$\Sigma \in \mathcal{S}_{\mathcal{D}}(X). \tag{6.9}$$

Then, one defines the following ideal of the algebra (6.8), denoted by

$$\mathcal{J}_{\Lambda,\Sigma}(X), \tag{6.10}$$

consisting precisely of those elements

$$t \equiv (t_{\lambda}) \in \mathcal{C}^{\infty}(X)^{\Lambda} \tag{6.11}$$

that satisfy the following (Rosinger's) asymptotic vanishing condition:

for every $x \in \complement\Sigma$, there exists $\lambda \in \Lambda$ such that

(6.12) $$(D^p t_\mu)(x) = 0$$

for any $\mu \geqq \lambda$ and $p \in \mathbb{N}^k$.

See also (5.22) along with A. Mallios-E.E. Rosinger [2: p. 64, Section 2.2].

One now defines the next quotient algebra,

(6.13) $$B_{\Lambda,\Sigma}(X) := (\mathcal{C}^\infty(X)^\Lambda)/\mathcal{J}_{\Lambda,\Sigma}(X),$$

which we call a foam algebra over X, that is associated with the (residual) singularity set Σ, as in (6.9).

On the other hand, based on the definitions, one further concludes that, apart from (6.10), the set

(6.14) $$\mathcal{J}_{\Lambda,\mathcal{S}}(X) := \bigcup_{\Sigma\in\mathcal{S}} \mathcal{J}_{\Lambda,\Sigma}(X)$$

is an ideal of the algebra (6.8) as well, where we also set

(6.15) $$\mathcal{S} \subseteq \mathcal{S}_{\mathcal{D}}(X);$$

by definition, $\mathcal{S}$ is a family of residual subsets of X (see (6.1)), in such a manner that we assume that

(6.16) for any Σ, Σ' in $\mathcal{S}$, there exists $\Sigma'' \in \mathcal{S}$ with

(6.16.1) $$\Sigma \cup \Sigma' \subseteq \Sigma''.$$

Thus, equivalently, by looking at the family $\mathcal{S}$ as a partially ordered set (poset), by means of set inclusion we assume that

(6.17) $\mathcal{S}$ is an (upward) directed family, through set-inclusion, of residual subsets of X; the same are considered, as singularity sets in X.

Accordingly, by analogy now with (6.13), we can further define the next (quotient) algebra

(6.18) $$B_{\Lambda,\mathcal{S}}(X) := (\mathcal{C}^\infty(X)^\Lambda)/\mathcal{S}_{\Lambda,\mathcal{S}}(X),$$

which we call the multifoam algebra of X associated with the given set of singularities $\mathcal{S}$, as in (6.17)—equivalently, a subset of the power set of X, satisfying (6.15) and (6.16).

In this regard, and in conjunction with (6.5), we also note that

(6.19) the family $\mathcal{S}_{nd}(X)$, which already satisfies by definition (see (5.8.1)), (6.15) it does (6.16) as well.

In this context, see also J. Dugundji [1: p. 92; Problem 21.(3)].

On the other hand, by looking at (6.13) and/or (6.18), locally (i.e., in terms of the open sets of X), one obtains a corresponding presheaf of foam, respectively of multifoam, algebras on X. In fact, one proves here that

(6.20) the presheaf of foam, or of multifoam, algebras on X is a complete one, hence (J. Leray) a sheaf on X. Furthermore, the same is a fine, as well as a flabby, sheaf on X. We denote the aforesaid (complete pre)sheaves by

(6.20.1)
$$\mathcal{B}_{\Lambda,\Sigma},$$

respectively by

(6.20.2)
$$\mathcal{B}_{\Lambda,\mathcal{S}}.$$

As a consequence of (6.20) and our previous notation in (6.13) and (6.18), one obtains, within an isomorphism of algebras, the following relations:

(6.21)
$$B_{\Lambda,\Sigma}(X) = \mathcal{B}_{\Lambda,\Sigma}(X),$$

respectively

(6.22)
$$B_{\Lambda,\mathcal{S}}(X) = \mathcal{B}_{\Lambda,\mathcal{S}}(X).$$

Concerning the proof of (6.20), we refer to the treatise of A. Mallios and E.E. Rosinger [2]. ■

Note 6.1 Referring to the proof of the assertion in (6.20) that the presheaves appeared therein are complete—that is (J. Leray), sheaves on X, one can employ an analogous argument to that used in (5.13), (5.14), and (5.15) of Section 5, appropriately adjusted, of course, to the present (more general) setting.

6.2 A Differential Triad Related to Rosinger's Multifoam Algebra Sheaf

By analogy with our previous conclusion in (5.20) of Section 5, here we want to point out that one can obtain a differential triad of the form

(6.23)
$$(\mathcal{B}_{\Lambda,\mathcal{S}}, \partial, \Omega^1),$$

by taking as corresponding sheaf of coefficients, alias generalized arithmetic, the previously defined (Rosinger's) multifoam algebra sheaf on X (see (6.20.2)). The procedure is entirely parallel to that one, pertaining to the nowhere dense algebra sheaf, as has been exhibited in Section 5 (see (5.17), (5.22), along with (5.27) and (5.28)); thus, we refer to that section for further technical details, as well as to the above cited work. *In toto*, one thus concludes that

(6.24) the whole theory pertaining to potential applications of standard differential geometric techniques within the framework of abstract differential geometry, using now as sheaf of coefficients the multifoam algebra sheaf (see (6.20.2)), remains in force, as exhibited in Section 5, for the particular case of the sheaf (5.17).

On the other hand, we further remark that

(6.25) an analogous argument to that in (6.24) is also in effect, referring in particular to the corresponding behavior of the sheaf (6.20.2) as concerns our considerations in Sections 5.3 and 5.4. Indeed, the assertion here can straightforwardly be rooted on easily detected formal analogies, as well as on the pertinent definitions pertaining to the preceding discussions.

Scholium 6.1 It is a general outcome of the entire perspective of abstract differential geometry, as has been employed throughout the present treatise, that

(6.26) by applying in each particular case an appropriate algebra (sheaf) of coefficients, one can succeed in overcoming standard problems of the classical theory, mainly due to the entanglement of the notion of a (space–time) smooth manifold, a characteristic example being the so-called singularities that deplore the traditional differential geometric treatment of relativistic physics and also of quantum field theory.

In this connection, see also the discussion in the sections below. On the other hand, an extremely important example in conjunction with our previous comments in (6.26) appears now to be the so-called generalized functions—in particular, those presented above that were initially invented by the relevant work of E.E. Rosinger (see E.E. Rosinger [1]), being devised in order to cope with the classical problem of multiplication of (Schwartz) distributions. Thus, as already explained by the preceding discussion as well as by that in Section 5, one realizes that the corresponding

(6.27) Rosinger's algebra sheaves *viz.*, what we may describe, briefly, as the nowhere dense (see (5.17)) along with the much more general multi-foam algebra sheaves) can be used for the aforementioned purpose, as in (6.26), being, as we have seen, very well suited to the (algebraic, in fact character of the) machinery of abstract differential geometry. (Of course, no space–time manifold is needed.)

There is a special interest concerning the previous situation in the sense that the above perspective, hinted at by (6.26) and (6.27), seems very likely to be applicable in problems connected with the important theme of the quantization of the gravitational field, pertaining in particular to the pathologies (singularities) that already traditionally plague the subject. In this regard, see also the ensuing few sections.

7 Singularities

"…Laestrygonians and Cyclopes, angry Poseidon,
Such obstacles you will never encounter in your way,
As long as you do not carry them in your soul,
As long as your soul does not raise them before you…"

Constantinos Cavafis [1]

Our main objective here is to support the idea that the so-called

(7.1) *singularities*

that appear when we try to apply techniques of classical differential geometry (CDG)—that is, differential geometry on smooth ($\mathcal{C}^{\infty}$-)manifolds, in problems pertaining to theoretical physics such as in general relativity or in quantum field theory—

(7.2) are simply due to the way we understand the techniques (i.e., as a spin-off of the underlying $\mathcal{C}^{\infty}$-manifold).

—hence, the epigraph of this section with the verses of Cavafis.

Thus, it is the instrumental outcome of the whole point of view of abstract differential geometry (ADG), as this technique has been employed throughout the present study (see also [VS]), that

(7.3) the mechanism of CDG is an inherent (innate) algebraic (relational, or functorial) technique that can be referred directly to the (geometrical) objects at issue, irrespective of any underlying smooth (or otherwise) manifold.

In this connection, we can further remark that the above are in complete accord with what Leibniz, already in his time, was asking for—namely,

(7.4) to concoct a geometrical calculus (mechanism) acting directly on the geometrical objects without any intervention of coordinates.

See also N. Bourbaki [2: Chapt. 1; Note historique, p. 161, footnote 1], as well as A. Mallios [9: (8.9) and (8.10)].

On the other hand, it is the crucial conclusion of our previous discussion in Sections 5 and 6 that the new perspective that is advocated here is that one can actually

(7.5) reduce the singularities that were attributed thus far to an ill (peculiar) behavior of space, although they were simply the result of using an insufficient class of functions on that space, in the sense of the latter class being very restrictive, to a better (*viz.*, more efficient (larger)), class of functions (see, for instance, (5.19) or (6.5)). Now, this new class could, at the very best, contain many, if not all, types of singularities. Furthermore (and this is the important issue!), the standard differential geometric apparatus could still be in force within the new larger class of functions, as well, something, of course, that it was thus far at least in the simple way, that ADG provides, unattainable (!), due, simply, to the type of functions and the associated (differential) geometry that was employed before.

In this regard, see also A. Mallios [6: p. 174, concluding remarks] or A. Mallios [8: p. 96, Section 6, and p. 98, Scholium].

Thus, the following standard association, which has been advocated so far, especially by general relativity—that is,

(7.6)

(7.6.1) space–time $\longleftrightarrow$ carrier of the physical fields (a substratum of the physical world),

where the source of the above association is modeled on an appropriate type of a smooth ($\mathcal{C}^\infty$) manifold—appears nowadays to be of a disputable substance.

In fact, this disagreement has been present from the early days of general relativity. Thus, even A. Einstein himself, as early as 1916, led by problems pertaining to the quantum world (hence with a unified field theory), seems to strongly doubt the naturalness of the previous correspondence as in (7.6.1), yet confessing, on the other hand, the lack, at his time, of any relevant replacement for the same correspondence. Indeed, we can quote from J. Stachel [1: p. 280] the following passage from a letter of A. Einstein, written in 1916:

(7.7) ... continuum space–time ... should be banned from the theory as a supplementary construction not justified by the essence of the problem, ... which corresponds to nothing "real." But we still lack the mathematical structure ... How much have I already plagued myself in this way [of the manifold] !

The emphasis above is ours, as well as the adjunction of "[...]" to increase comprehension.

On the other hand, as already noticed several times in the foregoing, it is just at this point that the central message of ADG comes to the fore, indeed, in an instrumental manner, in the sense that

(7.8) the drawback (or even pathogeny) of the space–time model as a (smooth) manifold (continuum) is the only culprit of the so-called singularities (see also (7.10)).

Indeed, what we actually conclude from the whole perspective of ADG is that

(7.9) there is no notion of manifold in the classical sense of this term that intrudes itself into our calculations as concerns the whole setup of ADG, this being in complete accord with Einstein's inquiry/suggestion as in (7.6).

Of course, concerning the previous considerations, one understands the essential identification

(7.10) $$X \longleftrightarrow \mathcal{C}^\infty(X) \cong \mathcal{C}^\infty_X(X) \equiv \Gamma(X, \mathcal{C}^\infty_X),$$

the first association, hinted at above, being effectuated by means of the spectrum (Gel'fand space) of the (topological) algebra $\mathcal{C}^\infty(X)$ and the latter being identified with the global section algebra of the sheaf $\mathcal{C}_X^\infty$, with X a given smooth ($\mathcal{C}^\infty$-) manifold whose appropriate particular case models, as usual, space–time. In this regard, see also A. Mallios [TA: p. 227, Theorem 2.1] or [VS: Chapt. XI, p. 313, Theorem 3.1, along with p. 315, (3.11)].

Yet, within the same vein of ideas, we further remark that the preceding identification, as in (7.10), is essentially what in practice we do, since we usually prefer to deal with functions that come from (or live on) a given space, rather than with the space itself. On the other hand, when we speak of geometry, we usually understand, or even confound, it with our own model that we devise for what one can conceive as physical (natural) geometry, a notion that might be identified with what we understand as space. However, the latter (i.e.,

(7.11) the "physical space") is in fact what fills it up; that is, the physical objects (*viz.*, the constituents of it), or just the geometrical elements à la Leibniz (see (7.4)).

Thus, in view of the preceding,

(7.12) the space model is equivalently expressed through the functions, the smooth ($\mathcal{C}^\infty$) ones, that live on it, to which the so-called singularities actually refer as well. See also (7.5) as well as (7.3).

Thus, to paraphrase P.G. Bergmann [1], the problem now arises as to whether one expresses oneself in terms of physical geometry or within the context of geometrical physics, something that is directly connected with the technical identification

$$\textit{Euclidean} \longleftrightarrow \textit{Cartesian}. \tag{7.13}$$

Here the left-hand member of (7.13) refers by definition to the physical geometry—that is, to the relational one, in the sense of the ancient Greeks, for example, in drawing conclusions by comparison (of the geometric (physical) objects). This same point of view might also be conceived as akin to the Leibnizian perspective (see, for instance, (7.4) and (7.11) above) or even the Machian one, yet in a more technical language, to the Kleinian, or Kählerian point of view.

On the other hand, the Cartesian (one might call it the Cartesian–Newtonian) perspective hints at the classical differential-geometric point of view—that is, at that of differential geometry on smooth ($\mathcal{C}^\infty$-) manifolds (CDG), which, according to what has been said, leads us directly to the core of the problem of the so-called singularities. Now, this same issue is erroneously presented (physically associated) thus far, due essentially to the previous (implicitly made) identification, as in (7.13), of two drastically different perspectives of what we usually understand as geometry.

Thus, according to what has been stated in the preceding,

(7.14) ADG seems to correspond to what has been described, in the previous discussion, as physical geometry (*viz.*, not *a priori* given space, direct study/description of what we understand as "geometrical" (physical) objects/structures), while this latter aspect (of geometry) appears to be, in fact, the only real one.

On the other hand, since gauge theories of current physics are considered as physical theories of a geometrical character (M.F. Atiyah), therefore (J.A. Wheeler) as dynamical (*viz.*, relational ones), it naturally follows that to look at the aforesaid theories from the point of view of ADG (which is also the title of the whole treatise) might be of a significantly particular interest, at least, as well as something of a new applicability pertaining to potential physical interpretations. In this connection, see also (9.3) in the subsequent discussion, along with relevant comments.

8 Eddington–Finkelstein Coordinates

As another consequence of the preceding discussion, we can say that

(8.1) we have to find a mechanism through which we could interpret the singularities of the classical theory (CDG) simply as another aspect of the reality incorporating them within the usual laws of nature. Thus, quantum (field) theory, for instance, was once of the same character.

Accordingly, one is led to the conclusion that we should find

(8.2) equations that can incorporate singularities as if the latter did not exist at all, something that is in complete accord with the point of view that

(8.3) *nature has no singularities*.

In this regard, see also A. Mallios [9: remarks following (0.6) and (4.4)]. Now, such a program would also be in agreement with our own looking at the physis (participation in it) in a unified way, a fact still in accordance with (8.3).

Thus, a characteristic example of the above line of ideas is certainly the case of the so-called, in the standard literature, *Eddington–Finkelstein coordinates*, as in the title of the present section; see also C.W. Misner et al. [1: p. 828, Box 31.2]. For technical details related to the ensuing discussion, we refer to the aforementioned work, as well as to the original article of D.R. Finkelstein [1]. Thus, as already stated, the case under consideration shows exactly that to

(8.4) change from a given structure sheaf $\mathcal{A}$, in the sense of the terminology applied here, to another similar more appropriate one $\mathcal{A}'$, pertaining to the particular problem at issue—here to the classical *Schwarzschild singularity* [loc. cit.]—enables one to resolve, indeed, to engulf the “singularity” appeared, yet, without this to affect the differential-geometric “calculus” employed, which, for that matter, was, in effect, independent of the particular “spatial anomaly” (singularity) occurred.

On the other hand, the special interest here is exactly Finkelstein’s own interpretation of this phenomenal singularity (ibid.), which actually corresponds to (being, in fact, a forerunner and a convincing vindication of) what we actually demand by our statement in (9.3) below.

In this connection, we also refer to A. Mallios [9], as well as to a forthcoming article by A. Mallios–I. Raptis [4] concerning another similar perspective of the above, still from the point of view of ADG. Thus, to say it once again,

(8.5) smooth singularities in no way indicate a breakdown of differentiability;

in complete contrast to what CDG has maintained thus far (see also [ibid. Section 5]). Of course, by differentiability here, as in (8.5), one understands the innate (Leibnizian) mechanism, which is entailed as an upshot of the way classical differential geometry works—that is, as derived from a given smooth ($\mathcal{C}^{\infty}$-) manifold, thus from the local structure of classical Euclidean space.

9 Singularities (continued)

Paraphrasing A. Einstein [1: p. 165], referring to his well-known remarks on the characteristics of a field theory, we can further state, in conjunction with what has been said in the preceding, that

(9.1) any time we have a method to derive systematically solutions that are free of singularities, we might then have a field theory permitting us to understand the atomistic and quantum structure of reality.

On the other hand, it is the central message of ADG, according to our previous discussion, that an analogous situation to that as hinted at by (9.1) arrives within the pertinent framework of ADG (see, for instance, Sections 5 and 6).

Furthermore, having our arithmetic (structure sheaf $\mathcal{A}$ throughout the present treatise) capable of coping simultaneously with regular and singular points is the analogue, we can say, in terms of ADG of currently well-known supermathematics. Consequently, we have, in a sense, here too a means of confronting with the desire of understanding the "atomistic and quantum structure of reality" in accordance with A. Einstein, as quoted above, and this, at the same time, together with the rest function of our field theory pertaining to the regular (*viz.*, standard, alias smooth) situation (hence, again, a type of a unified field theory).

On the other hand, as already noticed occasionally in the foregoing, we are not compelled here, as usual (concerning the classical theory), to avoid singularities, looking for solutions free of singularities because such

(9.2) singularities (*viz.*, in point of fact, inconveniences of the standard (smooth) theory) can now be incorporated within our arithmetic (i.e., into our structure sheaf $\mathcal{A}$), while this does not affect the effectiveness of the corresponding (abstract) differential mechanism.

Therefore, the task now arises to find

(9.3) the appropriate physical interpretation of this, newly (concerning the present abstract setting) appearing potential eventuality.

Within the same context, it is important to make it clear once more that

(9.4) by excluding singularities from our calculations, we actually lose in information, in view, at least, of (8.3), while still quoting A. Einstein [loc. cit.],

(9.4.1) "... we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions."

On the other hand, by looking at the relevant phrase "manifold of solutions" as in (9.4.1) and also taking into account (3.11) (Einstein's equation, in vacuum), along with Note 3.1, we can further remark that very likely,

(9.5) Einstein did not, presumably, come to the eventuality that his equation, for instance, might have the same form (hold true) over singularities.

In connection with our comments in (9.3), we can refer, for example, to the preceding account in Section 8 concerning the Eddington–Finkelstein coordinates. Furthermore, within the same vein of ideas, we also mention the work of M. Heller [3], who makes analogous remarks pertaining to a potential production of particles, due to a similar situation, connected with (9.3), while he refers [(ibid.)] to a relevant work of T. Dray et al. [1].

9.1 "Singularities" of the Metric

According to the classical point of view, we consider gravity (the gravitational field) as a gauge theory—namely,

(9.6) (the behavior of) a field whose quantum is the metric.

The particular "is" above hints at the fact that the metric cannot, by definition, be a physical issue, being thus only our model of the corresponding physical notion (quantum of the gravitational field).

Note 9.1 The previous comments, related to (2.6), standing in complete accord with the general viewpoint of the present treatise (ADG), correspond to a relevant criticism of R.P. Feynman [1: p. xxxii, or even p. 113, §8.4] pertaining to the nature of the guilty object of gravitation (its quantum) in connection with our usual attitude toward the metric (geometrical interpretation of gravity). In this regard, see also our previous remarks with respect to (7.4).

On the other hand, we have to bear in mind here that for the case in point,

(9.7) this quantum (metric) extends over all the underlying space, the carrier of the field.

Indeed, the metric ρ refers to our arithmetics (*viz.*, to the structure sheaf $\mathcal{A}$) not to the base space X (see for instance, Chapt. I, (4.1.3), along with (4.2)). Otherwise, we would have a flat space, which is not, of course, the case; as a matter of fact, we do not virtually think through the space, but via the (geometric) objects that live on that space. The same do not actually curve the space (there is no space at all in the usual sense of the term), but they just curve (their paths) according to the laws of nature! This recalls A. Machado's verse (see A. Machado [1: The Road]),

(9.8) "Traveller there are no paths; paths are made by walking."

Thus, the role of our space X, base space of the sheaves ($\mathcal{A}$-modules) involved, is only a catalytic one in the sense that it is simply the carrier space of the objects (sheaves, alias les éléments géométriques à la Leibniz; see N. Bourbaki [2]), not having any other particular geometric structure than the topological one, hence not contributing to any specific geometric substance of the whole stage, the latter being thus exclusively rendered to $\mathcal{A}$ and the relevant $\mathcal{A}$-modules involved.

On the other hand, technically speaking, we know that the $\mathcal{A}$-metric ρ on a given vector sheaf $\mathcal{E}$ on X is just a 0-cocycle of $\mathcal{GL}(n, \mathcal{A})$, with $n = rk\mathcal{E}$ (what we may call the Gram cocycle of ρ; see [VS: Chapt. IV; p. 324]); namely one has

$$\rho \equiv (\rho_\alpha) \in Z^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})) \cong \mathcal{GL}(n, \mathcal{A})(X) \cong GL(n, \mathcal{A}(X)), \tag{9.9}$$

such that

$$\mathcal{U} = (U_\alpha)_{\alpha \in I} \tag{9.10}$$

is a local frame of $\mathcal{E}$ (see Volume I, Chapter I, (9.40) or [VS: Chapt. IV; p. 322, Theorem 8.1]). Hence, occasional singularities of ρ actually refer to similar ones of the elements (*viz.*, in effect, of sections) of $\mathcal{A}$, therefore appropriately reducible, (i.e., suitably absorbable, indeed, into a new "$\mathcal{A}$") by virtue of our considerations in Section 5 and 6.

Thus, the preceding provides a potential application of the present abstract technique in problems connected with occasional singularities of the metric, appearing thus far within the framework of the classical theory (CDG) such as those hinted at already in the preceding, while one should look at the present new situation in the perspective of our comments connected with (9.3).

10 Quantum Gravity

> "reality cannot at all be represented by a continuous field."
>
> A. Einstein in *The Meaning of Relativity* (Princeton Science Library, Princeton, NJ, 1988). p. 160.

As already hinted at in the preceding, our aim in this section is to discuss certain reflections pertaining to the theme in the title of the present section that are connected with a potential application of our previous considerations, in what, in particular, concerns an appropriate entanglement of an algebra sheaf containing singularities in the classical sense of the term, referred here, hence, of a not "continuous" object, yet, in the standard point of view, as a "structure sheaf", or else "sheaf of coefficients", for a differential-geometric framework, that one is willing to employ in confronting with the problem at issue (quantization of gravity). Of course, one is justified in the aforementioned trial through the generalized structure sheaf, in view of our experience thus far, emanating from a similar, very successful application of the relevant standard differential-geometric machinery in the classical smooth regime (continuum). In that manner, one hopes to have at one's disposal a semicontinuous, or quasi-continuous, so to speak, field (theory), by means of which one might be able to "understand the atomistic and quantum structure of reality" as well, a fact that A. Einstein himself seemed also to demand from a field theory (see A. Einstein [1: p. 165]). In this connection, he also emphasized, clearly, the fact that the difficulty here lies in that "we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions" [loc. cit.]; that is, by employing, technically speaking now, an algebra sheaf, as a sheaf of coefficients that cannot cope with singularities, in the classical sense of the term, we simply reduce the information we can get out of the field equations; thus, it would be very desirable if we could see beyond the singularities, although always, of course, the same (system of) field equations, since otherwise (see for instance, A. Einstein [loc. cit. p. 164]), "it does not seem reasonable ... to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold."

Note 10.1 By taking into account Einstein's phrase in the epigraph of this section, combined with his above remarks, we see that "continuum" here might be rather construed as the world ("cosmos," reality) we try to describe, so that it would fit rather well when looking at it in the generalized sense of a semicontinuum as alluded to above; thus, what amounts, by definition, to the same, one has to consider here an extended (or even generalized) continuum—that is, a continuum that can contain singularities in the classical sense (*viz.*, relative to a given class of functions). Yet, the field equations, being extended, are, as we shall presently see, still in force on the whole of the previous continuum.

Note 9.1 hints at the possibility of treating at the same time a generalized continuum, in the preceding sense, even an ordinary one, by employing extended field equations, a situation we further explain in the sequel. This possibility has exactly to be pointed out at this place, as opposed to further relevant remarks of A. Einstein [1: p. 165], who thus was noting, at the time, wishing presumably to have a kind of extended field as before, that "We do not possess any method at all to derive systematically solutions that are free of singularities."

However, it appears that the point here is simply that

(10.1) we do now possess methods that render us able to derive solutions of the field equations that are (may, indeed, be viewed as) free of singularities;

that is,

(10.2) by incorporating the singularities in the classical sense (*viz.*, with respect to the functions concerned) into the solutions themselves, while not affecting at all the (abstract) differential-geometric mechanism employed—that is, equivalently (incorporating the singularities), while the field equations concerned being still in force.

Thus, by considering, for instance, Einstein's equation (*in vacuo*), as in (3.11), we have an example of a field equation (the gravitational field) whose solutions (those vector sheaves satisfying (3.11)) are free of singularities, at least from those that can be encapsulated in our structure sheaf $\mathcal{A}$ (see (3.2)); now, the latter might be, for example, Rosinger's algebra sheaf (see Section 5), a sheaf that contains, as already mentioned, the largest number of singularities possible thus far (Rosinger's theory, space–time foam algebras). So (see Section 3) the abstract differential-geometric setup that is essentially needed to formulate the respective part of general relativity (see also Sections 1 and 2 above) is here in force concerning the previous sort of algebra sheaves.

Accordingly, to summarize the situation, we can single out the following lines of thought:

(10.3) by looking at the field equation of a given field, we first distinguish the corresponding algebra (sheaf) coefficients, with respect to which the field equation at issue is formulated. Then, one must try to incorporate into the initial algebra sheaf, as before, any singularities that may occur, getting thus at a new (larger) algebra (sheaf) of coefficients.

Now, in the case that this new algebra (sheaf) can be construed as the structure (algebra) sheaf, in the sense of the preceding discussion, providing thus a differential triad—having also reasonable properties for an abstract differential-geometric treatment of the situation then according to the preceding—one arrives at a field equation of the field concerned, having now solutions that are free of singularities (the latter being thus viewed now as ordinary solutions with respect to the enlarged (algebra) sheaf of coefficients).

In this connection, and this is certainly of importance here, one must

(10.4) find the appropriate physical meaning of the above new solutions/singularities (the last term being understood in the standard sense), behaving now as usual ones within the preceding extended context. This, indeed, deserves special attention, in view of the fact that such an appropriate physical interpretation might lead to entirely new information, vis-à-vis our knowledge (see Section 9).

Referring, for instance, to Einstein himself, we can assert that

(10.5) "the general theory of relativity can be conceived only as a field theory."

See A. Einstein [1: p. 140]. ■

On the other hand,

(10.6) a field theory is a gauge theory—that is, one referred to a particular Yang–Mills field

(10.6.1) $$(\mathcal{E}, D),$$

according to the terminology employed so far (see the relevant comments in the introduction to this chapter, (0.3)). By applying (10.5), in conjunction with (10.6), we can say that

(10.7) the general theory of relativity is the gauge theory of the gravitational field, hence, as already mentioned, one of a pair, as in (10.6.1).

However, in the case of the gravity, that is, equivalently, of the gravitational field, one has the gauge that corresponds to a boson, namely, to the 2-spin (massless) graviton; hence, according to our terminology (see Volume I, Chapt. I, (6.22), along with Chapt. II, Definition 1.1 therein),

(10.8) the gravitational field can be associated with a Maxwell field

(10.8.1) $$(\mathcal{L}, D).$$

In this connection, see also J. Baez–J.P. Muniain [1: p. 402], as well as R.P. Feynman [1: pp. 11 and 28].

Now, for obvious historical reasons and to distinguish (10.8.1) from other Maxwell fields, in the previously applied terminology,

(10.9) the Maxwell field

(10.9.1) $$(\mathcal{L}, D)$$

that represents the gravitational field in (10.8.1) will be named henceforward the Einstein field.

Thus, otherwise formulated

(10.10) the Einstein field is, by definition, the gravitational field that, being a boson (elementary particle of integral spin; see Chapt. I, (2.2)), can be expressed through a Maxwell field

(10.10.1) $$(\mathcal{L}, D).$$

See also Volume I, Chapt. II, (1.4). Here, as usual, $\mathcal{L}$ stands for the carrier of the field at issue (i.e., for the (hypothesized) graviton), while the $\mathcal{A}$-connection D as in (10.10.1) represents the field itself, or "the guilty object" (see R.P. Feynman et al. [1: p. 91]) of the situation concerned. The same is not, of course, a geometric object in the technical sense of the term, but a dynamical/algebraic one—that is, the source/cause of the whole matter, as exhibited by the pair in (10.10.1) (Einstein field). It certainly entails its geometry/flow, or what amounts to the same thing, it is further realized/conceived by means of its strength, that is, by the corresponding

(10.11) curvature $R(D) \equiv R$ of the $\mathcal{A}$-connection D, as in (10.10.1), or the field strength of D, in effect of $(\mathcal{L}, D)$ equivalently, by the

(10.11.1) gravity $\equiv R(D) \equiv R$ of the Einstein field $(\mathcal{L}, D)$, as before, namely, by definition (see (10.10)), by the gravity (result) of the gravitational field.

Scholium 10.1 It is a crucial matter to point out at this point that by virtue of our axiomatic thus far (see thus, Volume I, Chapter I in this treatise), one considers here as carrier of the field at issue not, as usual, the underlying space, that is, the topological space X, but the vector sheaf itself (here, of course, the line sheaf $\mathcal{L}$, as in (10.10.1)), whose sections (a sheaf is its sections, for that matter, see [VS: Chapt. I; Section 3]) represent, by our hypothesis (see Volume I, Chapt. II, (6.29)), the states of the field under consideration. We thus deliberately turn here our attention concerning the carrier of the field from the underlying topological space X to the field itself, the latter being, for that matter,

(10.12) "...an independent, not further reducible fundamental concept,"

according also to the current point of view; see A. Einstein [1: p. 140] for the previous quotation.

The same argument as before contributes to the contemporary aspect that there is no absolute space that is virtually akin to the (general-)relativistic perspective of the issue: Indeed, see A. Einstein [1: p. 140]:

(10.13) "...space as such is assigned a role in the system of physics that distinguishes it from all other elements of physical description. It plays a determining role in all processes, without in its turn being influenced by them. Though such a theory is logically possible, it is on the other hand rather unsatisfactory."

In that sense too, a sheaf-theoretic description of matters in terms of vector sheaves and the like, as in the preceding, seems thus to be more contiguous to the point of view of the general theory of relativity, while, at the same time, offers us the possibility of treating the (ever existing; see (10.19)) quantal aspect of gravity. Accordingly, one can say here, in a sense, that

(10.14) space is carried (determined) by matter (the field itself), not conversely.

Scholium 10.2 (Ubiquity of $\mathcal{A}$) The previous perspective, as expressed within the present sheaf-theoretic framework, focuses finally our attention on the structure (algebra) sheaf $\mathcal{A}$ again, which is also natural, given that any vector sheaf is, after all, locally finite many replicas of $\mathcal{A}$ (see (1.9)). So we are getting nearer to the aspect that

(10.15) anything we try to describe should be made as it actually happens entirely in terms of our domain of coefficients $\mathcal{A}$

[*viz.*, of our arithmetic, or structure sheaf, or sheaf of coefficients, or whatever other suggestive name might be assigned to the above algebra sheaf $\mathcal{A}$ (see (1.1))]. Yet, and this is of particular significance, even

(10.16) the space itself (*viz.*, in our case the topological space X) is to be described by means of $\mathcal{A}$.

The above point of view seems to be more natural, pertaining to our approach in describing reality, hence more economical. The same is also in accordance with the plethora of efforts in recent times to find an algebra-theoretic manner of describing the real world, hence, too, with Einstein's relevant suggestion that we should

(10.17) "... find a purely algebraic theory for the description of reality."

See A. Einstein [1: p. 166]. In this connection, we further note that by employing, for instance, suitable topological algebra sheaves $\mathcal{A}$, one can get (10.16); see [VS: Chapt. XI].

Scholium 10.3 Still referring to the advantages that one can obtain when applying the above abstract aspect of differential geometry to the general theory of relativity, we have already noticed that one can eventually cope, in that manner, with the atomistic and quantum structure of reality (see the relevant comments at the beginning of this section), in view of the possibility of incorporating, within the structure sheaf $\mathcal{A}$ itself, a large number of singularities (à la Rosinger; see A. Mallios–E.E. Rosinger [2])—indeed, the largest thus far!

On the other hand, coming now to our hypothesis that the structure sheaf $\mathcal{A}$ employed hitherto is a commutative algebra sheaf, we can further remark that this is certainly in agreement with the fact that $\mathcal{A}$ is essentially our arithmetic—that is, that domain within which we actually perform all our calculations, hence, in accord with Bohr's correspondence principle, in that

(10.18) "... the description of our own measurements of a quantum system must use classical commutative c-numbers ..."

(see the epigraphs of the present chapter). Thus (commutative) c-numbers for the so-called first quantization, while appropriate (unital) commutative algebras (*viz.*, extended numbers), in our case, in sheaf form, for the second quantization (field quantization; see also Volume I, Chapter V, Section 5).

Consequently, the moral here is that it would be extremely useful, for obvious reasons, to have a mechanism supplying the above commutative framework, even in noncommutative situations (e.g., quantum effects), while still working classically (i.e., as in the commutative case); yet we desire to have the same mechanism, effective even in the presence of singularities (e.g., quantum), in the classical sense of the term. In that sense, we are thus not compelled to have to change our structure sheaf $\mathcal{A}$ to a noncommutative one, as is traditionally usual, in order to be able to cope with the quantum. Such a program would still be in accordance with the point of view of Einstein, for instance, who, as already discussed, was asking at the time whether one could conceive the "atomistic and quantum structure of reality" by means of a field theory (*viz.*, in principle (see the previous discussion), by a continuum or a commutative theory), ascribing thus the obstacles appearing here to the presence of singularities—a fact, however, that can now be overcome, through the above abstract perspective of (the classical) differential geometry, when, in the particular case considered, the structure sheaf $\mathcal{A}$ contains the largest thus far possible number of singularities ("foam algebra sheaf" à la Rosinger; see E.E. Rosinger [3], along with A. Mallios–E.E. Rosinger [2]; see also Section 5 in the preceding).

On the other hand, by finally referring to the quantum of gravity (i.e., by definition, to the quantum (causality) of the relevant field; *viz.*, for the case at hand, of the gravitational field (see (10.11.1) above)), one concludes, by virtue of Volume I, Chapter IV, (5.37) and/or (5.127), in conjunction with (10.8) and (10.10), that the

(10.19) graviton is (pre)quantizable!

The above certainly contributes to Finkelstein's apostrophe that

(10.20) "... all is quantum."

(I am indebted to I. Raptis for letting me know the previous utterance of D. Finkelstein, an excerpt of a private communication. See D.R. Finkelstein [2: p. 477]). In this connection, one is rather tempted to say that

(10.21) everything is light,

a point of view that certainly needs further elaboration; see also A. Mallios [13: (6.21)–(6.24)]. However, to this we hope to return on some other occasion.

General remarks The following remarks are in perspective with the present abstract (algebraic) setting of differential geometry, in sharp contrast to the classical (spatial) aspect of the subject, while they also clarify several arguments in the preceding discussion.

Thus, we are of the (wrong) impression, as usual, that the applicable differential geometry we use depends each time on the underlying manifold (*viz.*, on the surrounding space, a fact *eo ipso* illusory), given that

(10.22) all our information about it is a consequence of what we call (algebra of) differentiable functions, which in our case corresponds to the algebra sheaf $\mathcal{A}$—that is, the structure sheaf of the (abstract) differential geometry at issue.

Accordingly,

(10.23) the differential-geometric mechanism we employ proves to be of a more innate (relational) character (*viz.*, of a more algebraic one than we thought).

Thus, speaking mathematically/physically, we can say that

(10.24) geometry (flow) is the result of a subsisting algebra (causality), *au fond*, not conversely.

Now, from this standpoint, we can further assert that

(10.25) a field theory is a continuum theory.

Indeed, by looking at a field theory from a differential-geometric point of view, we simply consider the respective flow (geometry; see (10.24) above) or the curvature of the field. In other words, one looks at the result, not at the "guilty object" as one should, to quote Feynman—that is, from the standpoint of this treatise, at the corresponding ($\mathcal{A}$-)connection (causality), the substance of the field itself. So, precisely speaking, by employing our previous terminology, one has to look here at the pair

$$(10.26) \qquad (\mathcal{E}, D)$$

(*viz.*, at the corresponding Yang–Mills field), or in particular, for the case in point, at the Einstein field, or gravitational field,

$$(10.27) \qquad (\mathcal{L}, D),$$

as above (see (10.10)). In that sense, the essence of the matter is that

(10.28) the quantum is the pair (10.27) that is in our case the graviton, and not whatever particular way one may use to describe it, as happens, for instance, in the case of the (Lorentz $\mathcal{A}$-)metric; see also Scholium 10.4.

Consequently, we thus arrive at a more algebraic perception of the matter, hence (Einstein) preferable to our classical (geometric) point of view. Yet, within the same vein of ideas, we can refer to a recent account on the subject by I. Raptis, where he considers, for instance, a quantum causal set in place of a quantum spacetime (see I. Raptis [1, 2]).

Referring to (10.25), our compensation in the preceding to confront reality algebraically (Einstein), while still employing a continuum theory—that is, (our structure sheaf) $\mathcal{A}$—is that now

(10.29) $\mathcal{A}$ contains holes, that is, singularities, as many as we want (thus far).

(See our previous comments in Scholium 10.3.) As already mentioned, we employ an extended (or generalized) continuum, to be able to cope with reality, quantal and/or not, simultaneously! From a physical standpoint this seems to be of particular significance, however, mathematically speaking as well.

Scholium 10.4 ("Physical geometry") By considering a Yang–Mills field

(10.30) $$(\mathcal{E}, D),$$

or, in particular, an Einstein field

(10.31) $$(\mathcal{L}, D),$$

as is the case here (see (10.10.1)) one remarks that

(10.32) even if the curvature $R(D) \equiv R$ of the given $\mathcal{A}$-connection D, as above, is zero, the cause that entails the curvature, in other words, the $\mathcal{A}$-connection D itself, is not necessarily zero.

A fundamental example is our basic differential triad

(10.33) $$(\mathcal{A}, \partial, \Omega^1),$$

for which one has, by definition (see Volume I, Chapt. I, (7.16)),

(10.34) $$R(\partial) \equiv d^1 \circ \partial = 0,$$

while, as already said (see also (1.50.2)),

(10.35) ∂ ($\equiv d^0$, the "basic differential") is the starting point of all(!),

as concerns the terminology employed here. Therefore,

(10.36) the field ($\mathcal{A}$-connection) itself may exist even if its strength (curvature) is zero.

On the other hand, physically speaking, we thus remark that

(10.36′) causality ($\mathcal{A}$-connection) may exist even if we cannot detect it!

Consequently, the preceding remarks show that within the framework that is advocated by the present treatise,

(10.37) the fundamental issue is the field $(\mathcal{E}, D)$ itself, or, as the case may be, $(\mathcal{L}, D)$, and not the underlying space X.

Of course, the physical laws are actually concerned with Yang–Mills fields in general (*viz.*, with pairs $(\mathcal{E}, D)$, as above (the Maxwell fields being comprised herein, for $\mathcal{E} \equiv \mathcal{L}$)). Therefore, physical geometry is referred not to the geometry of the underlying space X, but to that of the fields themselves with which we are concerned.

Now, the previous argument, pertaining to the geometry of $(\mathcal{E}, D)$'s, might also be construed as another expression of the principle of minimal coupling, or of the (strong) principle of equivalence. Accordingly, one can assert that

(10.38) physical geometry is the geometry of ($\mathcal{A}$-)connections, or the geometry of Yang–Mills fields (see Chapt. II); see also A. Mallios [12].

Equivalently, we can say that

(10.39) *physical geometry* is a (nonlinear partial) differential equation, whose the corresponding "*solution set*" expresses the (domain of) validity of a physical law, under consideration; cf., for instance, *Einstein's equation* (11.1) below. (At this point, we note that the latter set may contain, according to the present abstract setup, as many singularities in the classical sense of the term as we want; see (10.29).)

In this connection, we can say, therefore, that one employs here an algebraic point of view when taking the present perspective into account, in the sense that we focus our attention on the fields (equations, or laws of nature, physically speaking), hence on the $\mathcal{A}$-connections, or the "guilty objects" (à la Feynman) themselves, by referring, as before, to the pairs $(\mathcal{E}, D)$ (see (10.30)) or the relevant equations.

Scholium 10.5 ($\mathcal{A}$-Metric, again) In referring to the above objects $(\mathcal{E}, D)$ (see (10.30)), it is quite natural to consider among them those in particular that behave well with respect to the fundamental structural issue, that is, relative to the pair

(10.40) $$(\mathcal{A}^{n+1}, \rho), \ n \in \mathbb{N},$$

in other words, with respect to a Lorentz structure sheaf $\mathcal{A}$ (see Section 2); that is, technically speaking, one considers

gauge invariant $\mathcal{A}$-connections—namely, those D in (10.30) that obey the relation

(10.41) (10.41.1) $$D_{\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*)}(\tilde{\rho}) = 0$$

(see Definition 2.2 for the notation applied above, along with (the terminological) Scholium 2.1 in the same Section 2).

Accordingly, such types of $\mathcal{A}$-connections are compatible with the particular structure imposed on $\mathcal{A}$ by ρ (*viz.*, with the so-called (Lorentz) $\mathcal{A}$-metric on $\mathcal{A}$ (loc. cit.), in point of fact, on the (free $\mathcal{A}$-module) $\mathcal{A}^{n+1}$, $n \in \mathbb{N}$ (see also (2.25) above)). In this connection, we also recall that by further assuming that X, as in (10.33), is a paracompact (Hausdorff) space, while $\mathcal{A}$ is a strictly positive fine $\mathcal{A}$-module on X (see (2.27) in the preceding), one concludes that the (Lorentz) $\mathcal{A}$-metric is actually inherited on any vector sheaf $\mathcal{E}$ on X (of the appropriate rank; see (10.40), along with (2.33)), which is virtually also the content of (10.41.1).

As a result of the preceding, we further note that, again, the emphasis here is on the notion of a field and not on the space X! Indeed,

(10.42) the $\mathcal{A}$-metric ρ refers to our arithmetic (*viz.*, to the structure sheaf $\mathcal{A}$ itself (see also, however, (10.40)), not to the base space X(!)), on which it is not necessarily inherited, as usually happens in the classical case (inheritance of the metric on the underlying manifold).

In other words,

(10.43) the ($\mathcal{A}$-)metric comes from (or else, one "feels" the metric through) $\mathcal{A}$, not through the space X!

Even in the classical case (see Gauss–Riemann (!)), the metric is actually inherited on the underlying manifold X, through the tangent (vector) bundle, or, by employing our terminology, through the tangent (vector) sheaf on which it is defined, therefore, by means again of the differentiable functions involved, [i.e., in terms of $\mathcal{A} \equiv \mathcal{C}^{\infty}_{X}$ (see (3.17))]. Thus, as a further consequence of the present abstraction of the classical case, we can say that things are put in this manner even more within the pertinent perspective.

On the other hand, it is still crucial to note at this place that

(10.44) what one essentially employs here is $\mathcal{A}$-connections D that are ρ-invariant in the sense of (10.41.1). (See also (1.56).)

The above might also be viewed as a compensation of the classical way of looking at gravitation, namely, through the so-called (*differential-*) geometric point of view, in terms of the $\mathcal{A}$-metric involved, and that through field theory (R.P. Feynman [1] for instance). On the basis of our comments in (10.42), we further notice at this point that

(10.45) the topological space X does not (in point of fact, it is not necessary to) have any other geometric structure than the topological one!

The above is in accord, as already mentioned, with recent tendencies pertaining to the notion of vacuum as pure space; see P.J. Braam [1: p. 279].

On the other hand, as a further consequence of (10.45), in conjunction with the abstract differential-geometric setup that is advocated here and as concerns its applications in gauge theories of current physics, we remark that

(10.46) even a discrete (topological) space might be considered within the present (abstract) framework, provided, of course, that one has secured the appropriate differential triad, the particular topological space itself being always paracompact (Hausdorff).

11 Final Remark

We wish to close with some final remarks pertaining to the meaning of the framework that can be established through the previous treatise (see Sections 5 and 6) and the corresponding type of equations afforded, as, for example, by Einstein's equation (*in vacuo*; see (3.11))

(11.1) $$\mathcal{R}ic(\mathcal{E}) = 0.$$

According to what has been already said (see (8.5), along with (9.2), (9.3), or (10.29)),

(11.2) (11.1) might be full of singularities in the classical sense of the latter term, while still being in force (!),

to the extent, of course, that this is afforded by our structure sheaf $\mathcal{A}$ (see Sections 5 and 6). In this connection, it is remarkable to recall here what A.S. Eddington [1] said in 1920, in that (emphasis below is ours)

(11.3) "... the laws of motion of the singularities must be contained in the field-equations."

He remarks that [ibid.]

(11.4) "a particle of matter is a singularity of the gravitational field [(!)],"

postanticipating, albeit, more accurately, M. Faraday, as the latter is quoted by H. Weyl [1: p. 169]; in this regard, see also A. Mallios [12: (1.2)].

In employing the usual parlance of the present treatise, one can say, following Eddington, that

(11.5) the dynamics of the singularities must be implemented by those of the field itself (see also (11.6), along with (11.8) below).

By and large, based on what has been said, we can assert that

(11.6) to the extent that differential analysis, in the classical sense of this notion, is concerned, one can employ any algebra (sheaf) of generalized functions containing, as the case might be, the standard smooth functions; take, for instance, Rosinger's algebra sheaf (see (5.11) and (5.19), along with (6.18) and (6.27)). If, moreover, classical differential geometry is needed, then we can further employ, instead, abstract differential geometry (ADG), by having suitably chosen our (sheaf of) algebras (sheaf of coefficients).

In toto, the preceding perigram of thinking represents the central flavor (quintessence) of the whole rationale, that has been advocated throughout all of the previous discussion, while affording at the same time a variously potential applicability to relevant pestilential problems of the classical theory!

On the other hand, to paraphrase D.R. Finkelstein [2] in that context, one can look at the situation we meet in the quantum régime by considering a

(11.7) relativistic (*viz.*, covariant) dynamicalization, pertaining here to the quantum.

Therefore, what one actually concludes, based on the foregoing, is that in order to describe the preceding, one then has to concoct

(11.8) a "differential" equation, within the context of ADG!

Accordingly, we thus arrive again at a point of view, analogous to that which is already advocated by (11.6)!

11.1 On Einstein's Equation (continued)

By further looking at Einstein's equation *in vacuo*, as in (11.1),

(11.9) $$\mathcal{R}ic(\mathcal{E}) = 0$$

(see also (3.11)), we note that the description ("interpretation") of the graviton (i.e., of the quantum of the gravitational field) involved is accomplished

(11.10) on the basis of the classical theory (CDG), that is, by means of an

(11.10.1) $(n + 1) \times (n + 1)$ (Lorentz–Einstein) matrix,

with entries local sections of $\mathcal{A}$ when the whole matter is locally considered.

Therefore, one further concludes that following the classical pattern, we have thus concocted, within the context of ADG, an appropriate environment in order to catch/encircle the graviton, the latter particle being, as we have already assumed (this chapter), a boson (!), hence, the terminology applied in the sequel, in that context, by referring to it as an Einstein (–Maxwell) field: see (10.9), in conjunction with (10.28) in the ensuing discussion, pertaining to that particular (massless 2-)boson (i.e., to a Maxwell field), according to our terminology employed throughout the present treatise (see Volume I, Chapt. III, (1.4)). Consequently, the

(11.11) $(n + 1) \times (n + 1)$ Lorentz–Einstein matrix as in (11.10.1), see also Chapt. IV; (2.11), as well as the corresponding Yang–Mills field

(11.11.1) $$(\mathcal{E}, D),$$

as it appeared in (3.9) in the form of a solution space of Einstein's equation (see also Chapt. IV, (3.20′)), is but our own

(11.11.2) experimental cell within which the graviton (boson) lives, or, what amounts to the same thing, within which we can detect it through its field strength (curvature).

In this context, it is to be noticed here that

(11.12) the metric we use is defined in terms of $\mathcal{A}$ [in accordance with the general perspective of ADG, which is advocated throughout the present treatise, see (10.15) or (10.22)], thus without being at all of any spatial-temporal provenance, as is exactly the case in the classical theory (GR via CDG), hence along with all the concomitant inconveniences! (See also A. Mallios [11: (1.8), in conjunction with (1.9)], as well as (10.29) above.)

On the other hand, by considering Einstein's equation (*in vacuo*), as in (11.9), that is, within the framework of ADG, we also point out that

(11.13) we still retain the classical form of the said equation, however, concerning this aspect only (!), and not actually the (innate) structure of the equation, since we occasionally (this depending on the presence of singularities) have changed $\mathcal{A}$, the latter sheaf being now not the initially chosen one (see also (3.17)).

Nevertheless, we do not know so far, or at least not always (see Section 8), the physical content of the latter issue, along with the implemented consequences! (See also, for example, Sections 8 and 9.) Yet, in this connection, the quite recent tendency in the literature of quantum physics to consider the so-called background independent quantum gravity (see, for instance, A. Ashtekar–J. Lewandowski [2] and references therein) falls, in point of fact, half of the way in postanticipating ADG (see also, for example, (10.45), along with A. Mallios–I. Raptis [4: in particular, Sections 3 and 6]). Thus, one employs in that context the notion of manifold, but without any metric (!); therefore, one may say that we actually realize here what we may call the (ancient) Greek manner of doing geometry (no metric at all (!), just comparison, hence relational, thus physical geometry). On the other hand, as already mentioned, the manifold concept is kept throughout the aforesaid framework (within the so-called semiclassical theory) just to provide at least the differential-geometric environment, according to the standard pattern (CDG) a fundamental issue, in any case, to the extent, of course, that we intend to apply CDG in the whole story, a fact that, exactly at this point, and in a glaring contradistinction to the classical theory,

(11.14) ADG assures us that one can have the (classical) differential-geometric mechanism without any space at all !

See also, for instance, in that context, A. Mallios [12: (2.9), (2.14), as well as (3.1)]. Thus, in one word,

(11.15) modern/abstract differential geometry means to perform (differential) geometry without any underlying space at all by simply basing/employing a suitable algebra (of functions, true, of sections of an appropriate sheaf of algebras).

In other words, it is a fundamental upshot of the whole theory so far that

(11.16) the only reason to resort, classically speaking, to space–time is to afford calculus, hence the framework of classical differential geometry (CDG) too, something that is not necessarily the case, in order to have available the same inner mechanism of the latter theory, provided we dispose the appropriate (differential/ADG) algebra (see (11.15)) that can do the (same) job!

Thus, we can conclude here, by pointing out once more the utmost moral by far of ADG (abstract ($\equiv$modern) differential geometry), that

(11.17) it is quite irrelevant to the manner a Newtonian spark (see A. Mallios [12: (4.3) or (5.3)] for the terminology applied) comes from, provided it supplies an efficient differential-geometric mechanism (dgm)!

Furthermore, something that is not less important, this

(11.18) differential-geometric mechanism, hence, the entailed dynamics thereby, is quite indifferent (therefore, functorial) relative to any such choice as above, consequently natural(!) too.

What amounts to the same thing,

(11.19) our differential equations are thus functorial (hence natural) as well with respect to the dynamics (dgm) we choose!

In this context, we also remark that

(11.20) differential equation = (physically speaking) a relation describing a field $\longleftrightarrow$ physical law,

as this (field $\leftrightarrow$ law) arrives to us through its (field) strength, in fact, the only constituent of the field,

$$(\mathcal{E}, D), \tag{11.21}$$

that is functorial (covariant, natural, geometrical). So in that respect, we have always to bear in mind that

(11.22) what we call space–time is just the description/result of our arithmetic (*viz.*, of $\mathcal{A}$, in the terminology of ADG) since the time of Descartes, and even that was also the case with Newton, who, through the same description, was actually led to the discovery of calculus as a means to be able to write down the (expression/description of the) physical law he was concerned with! Furthermore, the same was also the case with Einstein himself, modulo, in his turn, the fact that the arithmetic (CDG) he applied was not the appropriate one(!), having now to do (to the extent, of course, we want it) with the quantum domain as well!

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