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Differential equations uniquely determined by algebras of point symmetries [1]

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Abstract

In a recent paper [12], within the framework of the inverse Lie problem, the definitions of strongly as well as weakly Lie remarkable equations have been introduced. Strongly Lie remarkable equations are uniquely determined by their Lie point symmetries, whereas weakly Lie remarkable equations are equations which do not intersect other equations admitting the same symmetries. In this paper we start from some relevant algebras of vector fields on $\mathbb{R}^k$ (such as the isometric, affine, projective or conformal algebra), and characterize strongly Lie remarkable equations admitted by the considered Lie algebras.

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1 Introduction

One of the most successful achievements in the geometric theory of differential equations (DEs), either ordinary or partial, is the theory of symmetries [1, 2, 3, 4, 9, 10, 15, 16, 17]. Symmetries of DEs are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former, with the further property of sending solutions into solutions. The knowledge of the symmetries of a DE may lead

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to compute some of its solutions, or to transform to in a more convenient form; in the case of an ordinary differential equation (ODE) it may allow to reduce the order, determine integrating factors, etc. Among symmetries, there is a distinguished class, that of symmetries coming from a transformation of the independent and dependent variables: point symmetries. In this paper we focus our attention on this kind of symmetries.

The problem of finding the symmetries of a DE has associated a natural "inverse" problem, namely, the problem of finding the most general form of a DE admitting a given Lie algebra as subalgebra of infinitesimal point symmetries.

A way which leads to the solution of this problem is to classify all possible realizations of the given Lie algebra as algebra of vector fields on the base manifold. The second step is to find differential invariants I_i of the realization under consideration (see for instance [8, 18]). Then, it is well known (see [16]) that, under suitable hypotheses of regularity, the most general DE admitting a given Lie algebra as subalgebra of point symmetries is locally given by $F^\mu(I_1, I_2, \ldots, I_k) = 0$ where F^μ are arbitrary smooth functions.

In this paper we specialize the above problem in the following way: we would like to find the *unique* DE which admits a given Lie algebra as subalgebra of its point symmetries. An aspect of this problem has been considered in [12], where the authors, starting from a given DE, found necessary and sufficient conditions for it to be uniquely determined by its point symmetries. By following the terminology already used in [11, 12, 13, 14], we call such a DE *Lie remarkable*. A similar problem was also considered by Rosenhaus in [19, 20, 21]; in fact, among various results, the author proved that the equation of vanishing Gaussian curvature of surfaces in $\mathbb{R}^3$ is the unique second order equation admitting the projective algebra of $\mathbb{R}^3$ as point symmetries.

The plan of the paper is the following.

In section 2, we introduce a DE of order r as a submanifold of a suitable jet space (of order r), which is a manifold whose coordinate functions of a chart can be interpreted as "independent" and "dependent" variables, and by the derivatives of the latter with respect to the former up to the order r. Then we introduce two distinguished types of Lie remarkable equations: *strongly* and *weakly* Lie remarkable equations. Strongly Lie remarkable equations are uniquely determined by their point symmetries in the whole jet space; on the contrary, weakly Lie remarkable equations are equations which do not intersect other equations admitting the same symmetries. Then we recall the main results obtained in [11]. More precisely, we report necessary as well as sufficient conditions for an equation to be strongly or weakly Lie remarkable.

In section 3, we find strongly Lie remarkable equations associated with isometric, affine, projective and conformal algebra of $\mathbb{R}^k$, where $\mathbb{R}^k$ is provided with metrics of various signatures. Since we start from concrete algebras and not from abstract ones, we do not have the problem of realizing them as vector fields. In particular, with regard to the affine algebra in $\mathbb{R}^k$ we recover the homogeneous second order Monge-Ampère equations [5], and in $\mathbb{R}^3$ also a third order PDE that, to the author's knowledge has not been described heretofore in literature. Also in the case of conformal algebra in $\mathbb{R}^3$, an interesting second order PDE is recovered, *i.e.* the equation for a surface $u(x, y)$ having the square of the (scalar) mean curvature equal to the Gaussian curvature.

2 Theoretical framework

Here we recall some basic facts regarding jet spaces (for more details, see [4, 15, 22]) and the basic theory on DEs determined by their Lie point symmetries [12].

All manifolds and maps are supposed to be C^∞. If E is a manifold then we denote by $\chi(E)$ the Lie algebra of vector fields on E. Also, for the sake of simplicity, all submanifolds of E are *embedded* submanifolds.

Let E be an $(n+m)$-dimensional smooth manifold and L an n-dimensional embedded submanifold of E. Let (V, y^A) be a local chart on E. The coordinates (y^A) can be divided in two sets, $(y^A) = (x^\lambda, u^i)$, $\lambda = 1 \dots n$ and $i = 1 \dots m$, such that the submanifold L is locally described as the graph of a vector function $u^i = f^i(x^1, \dots, x^n)$. In what follows, Greek indices run from 1 to n and Latin indices run from 1 to m unless otherwise specified.

The set of equivalence classes $[L]^r_p$ of submanifolds L having at $p \in E$ a contact of order r is said to be the *r-jet of n-dimensional submanifolds of E* (also known as extended bundles [15]), and is denoted by $J^r(E, n)$. If E is endowed with a fibring $\pi : E \to M$ where $\dim M = n$, then the r-th order jet $J^r\pi$ of local sections of π is an open dense subset of $J^r(E, n)$. We have the natural maps $j_rL \colon L \to J^r(E, n)$, $p \mapsto [L]^r_p$, and $\pi_{k,h} \colon J^k(E, n) \to J^h(E, n)$, $[L]^k_p \mapsto [L]^h_p$, $k \geq h$.

The set $J^r(E, n)$ is a smooth manifold whose dimension is

$$(1) \qquad \dim J^r(E, n) = n + m \sum_{h=0}^{r} \binom{n+h-1}{n-1} = n + m \binom{n+r}{r},$$

whose charts are $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $u^i_{\boldsymbol{\sigma}} \circ j_rL = \partial^{|\boldsymbol{\sigma}|} f^i / \partial x^{\boldsymbol{\sigma}}$, where $0 \leq |\boldsymbol{\sigma}| \leq r$. On $J^r(E, n)$ there is a distribution, the contact distribution, which is generated by the vectors

$$D_\lambda \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \quad \text{and} \quad \frac{\partial}{\partial u^j_{\boldsymbol{\tau}}},$$

where $0 \leq |\boldsymbol{\sigma}| \leq r-1$, $|\boldsymbol{\tau}| = r$ and $\boldsymbol{\sigma}\lambda$ denotes the multi-index $(\sigma_1, \dots, \sigma_{r-1}, \lambda)$. The vector fields D_λ are the (truncated) total derivatives. Any vector field

$X \in \chi(E)$ can be lifted to a vector field $X^{(r)} \in \chi(J^r(E,n))$ which preserves the contact distribution. In coordinates, if $\Xi = \Xi^\lambda \partial/\partial x^\lambda + \Xi^i \partial/\partial u^i$ is a vector field on E, then its k-lift $\Xi^{(k)}$ has the coordinate expression

$$\Xi^{(k)} = \Xi^\lambda \frac{\partial}{\partial x^\lambda} + \Xi^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \tag{2}$$

where $\Xi^j_{\boldsymbol{\tau},\lambda} = D_\lambda(\Xi^j_{\boldsymbol{\tau}}) - u^j_{\boldsymbol{\tau},\beta} D_\lambda(\Xi^\beta)$ with $|\boldsymbol{\tau}| < k$.

A *differential equation* $\mathcal{E}$ *of order* r *on* n*-dimensional submanifolds of a manifold* E is a submanifold of $J^r(E,n)$. The manifold $J^r(E,n)$ is called the *trivial equation*. An *infinitesimal point symmetry* of $\mathcal{E}$ is a vector field of the type $X^{(r)}$ which is tangent to $\mathcal{E}$.

Let $\mathcal{E}$ be locally described by $\{F^i = 0\}$, $i = 1 \ldots k$ with $k < \dim J^r(E,n)$. Then finding point symmetries amounts to solve the system

$$\Xi^{(r)}\left(F^i\right) = 0 \quad \text{whenever} \quad F^i = 0$$

for some $\Xi \in \chi(E)$.

We denote by $\operatorname{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$.

By an *r-th order differential invariant* of a Lie subalgebra $\mathfrak{s}$ of $\chi(E)$ we mean a smooth function $I\colon J^r(E,n) \to \mathbb{R}$ such that for all $\Xi \in \mathfrak{s}$ we have $\Xi^{(r)}(I) = 0$.

The problem of determining the Lie algebra $\operatorname{sym}(\mathcal{E})$ is said to be the *direct Lie problem*. Conversely, given a Lie subalgebra $\mathfrak{s} \subset \chi(E)$, we consider the *inverse Lie problem*, *i.e.*, the problem of characterizing the equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathfrak{s} \subset \operatorname{sym}(\mathcal{E})$ [2, 7].

In what follows, we will devote ourselves to the analysis of the inverse Lie problem. We start by the definition and main properties, contained in [12], of DEs which are uniquely determined by their point symmetries, that we call *Lie remarkable* DEs.

1 Definition. Let E be a manifold, $\dim E = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An l-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be

1. *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ passing at any $\theta \in \mathcal{E}$ admitting $\operatorname{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries;

2. *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ admitting $\operatorname{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Of course, a strongly Lie remarkable equation is also weakly Lie remarkable. Some direct consequences of our definitions are due. For each $\theta \in$

$J^r(E,n)$ denote by $S_\theta(\mathcal{E}) \subset T_\theta J^r(E,n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. Let us set $S(\mathcal{E}) \stackrel{\text{def}}{=} \bigcup_{\theta\in J^r(E,n)} S_\theta(\mathcal{E})$. In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E,n)$. The following inequality holds:

$$\dim \operatorname{sym}(\mathcal{E}) \geq S_\theta(\mathcal{E}),\ \forall\, \theta \in J^r(E,n), \tag{3}$$

where $\dim \operatorname{sym}(\mathcal{E})$ is the dimension, as real vector space, of the Lie algebra of infinitesimal point symmetries $\operatorname{sym}(\mathcal{E})$ of $\mathcal{E}$. If the rank of $S(\mathcal{E})$ at each $\theta \in J^r(E,n)$ is the same, then $S(\mathcal{E})$ is an involutive (smooth) distribution.

A submanifold N of $J^r(E,n)$ is an *integral submanifold* of $S(\mathcal{E})$ if $T_\theta N = S_\theta(\mathcal{E})$ for each $\theta \in N$. Of course, an integral submanifold of $S(\mathcal{E})$ is an equation in $J^r(E,n)$ which admits all elements in $\operatorname{sym}(\mathcal{E})$ as infinitesimal point symmetries. The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$ [12]. It follows that $\mathcal{E}$ can not coincide with the set of points of maximal rank of $S(\mathcal{E})$. The following theorems [12] can be proved.

2 Theorem.

1. *A necessary condition for the differential equation $\mathcal{E}$ to be strongly Lie remarkable is that*
$$\dim \operatorname{sym}(\mathcal{E}) > \dim \mathcal{E}.$$
2. *A necessary condition for the differential equation $\mathcal{E}$ to be weakly Lie remarkable is that*
$$\dim \operatorname{sym}(\mathcal{E}) \geq \dim \mathcal{E}.$$

In [12] also sufficient conditions have been established, that reveal useful when computing examples and applications.

3 Theorem.

1. *If $S(\mathcal{E})|_{\mathcal{E}}$ is an l-dimensional distribution on $\mathcal{E} \subset J^r(E,n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.*
2. *Let $S(\mathcal{E})$ be such that for any $\theta \notin \mathcal{E}$ we have $\dim S_\theta(\mathcal{E}) > l$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.*

The next theorem [12] gives the relationship between Lie remarkability and differential invariants.

4 Theorem. *Let $\mathfrak{s}$ be a Lie subalgebra of $\chi(J^r(E,n))$. Let us suppose that the r-prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E,n)$ and that the set of r-th order functionally independent differential invariants of $\mathfrak{s}$ reduces to a unique element $I \in C^\infty(J^r(E,n))$. Then the submanifold of $J^r(E,n)$ described by $\Delta(I) = 0$ (in particular $I = k$ for any $k \in \mathbb{R}$), with Δ an arbitrary smooth function, is a weakly Lie remarkable equation.*

Several examples of strongly and weakly Lie remarkable equations are provided in [12]. We recall some of them.

1. The equation of minimal surfaces in $\mathbb{R}^4$ or $\mathbb{R}^5$ is nor strongly neither weakly Lie remarkable, whereas it is weakly Lie remarkable in $\mathbb{R}^3$ and $\mathbb{R}^6$, provided we remove singular equations.

2. The equation of unparametrized geodesic on a complete simply connected Riemannian 2–dimensional manifold E is strongly Lie remarkable if and only if E has constant Gaussian curvature.

3. The equation $u_{xx}u_{yy} - u_{xy}^2 = \kappa$ is weakly Lie remarkable if $\kappa \neq 0$, whereas it is strongly Lie remarkable if $\kappa = 0$.

4. Some higher order Monge-Ampère equations [6] are weakly Lie remarkable, provided we remove singular subsets.

3 Strongly Lie remarkable equations determined by Lie algebras of vector fields on $\mathbb{R}^k$

In what follows we shall consider only scalar partial differential equations (PDEs), that is, according to our notations, we restrict our attention to the case $m = 1$. We denote by $\mathcal{I}(\mathbb{R}^{n+1})$, $\mathcal{A}(\mathbb{R}^{n+1})$, $\mathcal{P}(\mathbb{R}^{n+1})$ and $\mathcal{C}(\mathbb{R}^{n+1})$, respectively, the isometric, affine, projective and conformal algebra of $\mathbb{R}^{n+1}$ with respect to the metric $g = \sum_{i=1}^{n} k_i dx^i \otimes dx^i + du \otimes du$, where k_i ($i = 1, \ldots, n$) are non-vanishing real constants (in the following, in the case when $n = 2$, to light the notation, we use the variables x and y instead of x^1 and x^2, respectively). Even if the algebras $\mathcal{I}(\mathbb{R}^{n+1})$ and $\mathcal{C}(\mathbb{R}^{n+1})$ depend on k_i, we shall continue to denote them by the same symbol. For instance $\mathcal{I}(\mathbb{R}^4)$ can represent both the Euclidean and Poincaré algebra for suitable values of k_i. For each of the previous algebras we shall determine strongly PDEs of various order associated with them.

We first study the case of the algebra $\mathcal{I}(\mathbb{R}^{n+1})$. To explain how the methods exposed in section 2 works, in the next section we make computations in the case of $\mathbb{R}^3$ provided with the metric $k_1 dx \otimes dx + k_2 dy \otimes dy + du \otimes du$. For the remaining algebras, we will give just the results together with some useful comments.

3.1 The case of $\mathcal{I}(\mathbb{R}^{n+1})$

The algebra $\mathcal{I}(\mathbb{R}^{n+1})$ has dimension equal to $(n+1)(n+2)/2$. Then, in view of theorem 2, strongly Lie remarkable equations can be of order 1. In view of 2) of theorem 3, in order to get them, we have to construct the matrix

$M = \left(\Xi_i^{(1)\,j}\right)$ of the 1-prolongations of the isometries, characterized by the following vector fields:

$$(4)\quad \begin{aligned} &\Xi_1 = \frac{\partial}{\partial x}, && \Xi_2 = \frac{\partial}{\partial y}, && \Xi_3 = \frac{\partial}{\partial u}, \\ &\Xi_4 = k_2 y\frac{\partial}{\partial x} - k_1 x\frac{\partial}{\partial y}, && \Xi_5 = u\frac{\partial}{\partial x} - k_1 x\frac{\partial}{\partial u}, && \Xi_6 = u\frac{\partial}{\partial y} - k_2 y\frac{\partial}{\partial u}. \end{aligned}$$

Then we have to see if the rank of this matrix is 5 on $J^1(\mathbb{R}^3, 2)$ except for a 4-dimensional submanifold where the rank decreases. Such a submanifold will be the strongly Lie remarkable equation which we were looking for. More precisely, the matrix M is the following:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ y & -\frac{k_1}{K_2}x & 0 & \frac{k_1}{k_2}u_y & -u_x \\ u & 0 & -k_1 x & -k_1 - u_x^2 & -u_x u_y \\ 0 & u & -k_2 y & -u_x u_y & -k_2 - u_y^2 \end{pmatrix}$$

The rank of the previous matrix is 5 except on the submanifold

$$(5)\quad 1 + \frac{u_x^2}{k_1} + \frac{u_y^2}{k_2} = 0,$$

which describes the vanishing of the infinitesimal area element, where the rank decreases. The generalization to arbitrary n is straightforward. The strongly Lie remarkable equation in this case is

$$(6)\quad 1 + \sum_{i=1}^{n} \frac{u_{x^i}^2}{k_i} = 0$$

Of course, in order that equations (5) and (6) to be nonempty, we have to require that not all k_i are positive.

3.2 The case of $\mathcal{A}(\mathbb{R}^{n+1})$

The algebra $\mathcal{A}(\mathbb{R}^{n+1})$ has dimension $n^2 + 3n + 2$. By using theorem 2, we see that a strongly Lie remarkable equation can be of order 2 or 3 in the case $n = 2$, and of order 2 if $n \geq 3$.

The infinitesimal generators of this algebra are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b} \tag{7}$$

$\forall\, a, b \in \{x^1, x^2, \ldots, x^n, u\}$.

We see that the strongly Lie remarkable second order equation in $J^2(\mathbb{R}^3, 2)$ is the homogeneous Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{8}$$

which describes surfaces of $\mathbb{R}^3$ with vanishing Gaussian curvature.

Moreover, there exists also a strongly Lie remarkable equation of third order in $J^3(\mathbb{R}^3, 2)$ which has the following local expression:

$$\begin{aligned} & u_{xx}^3 u_{yyy}^2 + u_{xxx}^2 u_{yy}^3 + 6u_{xx}u_{xxx}u_{xy}u_{yy}u_{yyy} - 6u_{xxx}u_{xxy}u_{xy}u_{yy}^2 \\ & -6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 6u_{xx}^2 u_{xy}u_{xyy}u_{yyy} - 6u_{xx}^2 u_{xxy}u_{yy}u_{yyy} \\ & -8u_{xxx}u_{xy}^3 u_{yyy} + 9u_{xx}u_{xxy}^2 u_{yy}^2 + 9u_{xx}^2 u_{xyy}^2 u_{yy} \\ & +12u_{xxx}u_{xy}^2 u_{xyy}u_{yy} + 12u_{xx}u_{xxy}u_{xy}^2 u_{yyy} - 18u_{xx}u_{xxy}u_{xy}u_{xyy}u_{yy} = 0. \end{aligned} \tag{9}$$

To the authors' knowledge, equation (9) has not been heretofore described in literature; nevertheless, at the present we are not able to give for it a geometrical or a physical interpretation. Further investigations are in progress.

Now it remains to find the equations which live in $J^2(\mathbb{R}^{n+1}, n)$ with $n \geq 3$. In this case, for each n, we still have a strongly Lie remarkable equation, namely

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 0, \tag{10}$$

that is, the second order homogeneous Monge-Ampère equation in n variables [5].

3.3 The case of $\mathcal{P}(\mathbb{R}^{n+1})$

The algebra $\mathcal{P}(\mathbb{R}^{n+1})$ has dimension $n^2 + 4n + 3$. Its infinitesimal generators are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + u\frac{\partial}{\partial u}\right), \tag{11}$$

for all $a, b \in \{x^1, x^2, \ldots, x^n, u\}$.

Then we realize that we have to look for strongly Lie remarkable equations in the same jet spaces we considered in the previous section. Furthermore, we have to discuss also the case of an equation which lives in

$J^3(\mathbb{R}^4,3)$. We realize that *equations* (8), (9), (10) *are strongly Lie remarkable also in this case, and that there are no strongly Lie remarkable equations in* $J^3(\mathbb{R}^4,3)$. In fact, the prolongations of vector fields (11) span, at each point of $J^3(\mathbb{R}^4,3)$, a subspace of dimension at most 20. Since the dimension of the equation we are looking for is 22, in view of theorem 2, we have no chances to find a strongly Lie remarkable equation.

3.4 The case of $\mathcal{C}(\mathbb{R}^{n+1})$

The algebra $\mathcal{C}(\mathbb{R}^{n+1})$ has dimension equal to $(n+2)(n+3)/2$. We have to look for second order strongly Lie remarkable equations.

We start with $n=2$. The infinitesimal generators of this algebra are the generators (4) together with

$$(12)\qquad \begin{cases} \dfrac{1}{2}\dfrac{k_1x^2-k_2y^2-u^2}{k_1}\dfrac{\partial}{\partial x}+xy\dfrac{\partial}{\partial y}+xu\dfrac{\partial}{\partial u} \\[2ex] xy\dfrac{\partial}{\partial x}+\dfrac{1}{2}\dfrac{k_2y^2-k_1x^2-u^2}{k_1}\dfrac{\partial}{\partial y}+yu\dfrac{\partial}{\partial u} \\[2ex] xu\dfrac{\partial}{\partial x}+yu\dfrac{\partial}{\partial y}+\dfrac{1}{2}(u^2-k_1x^2-k_2y^2)\dfrac{\partial}{\partial u} \\[2ex] x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}+u\dfrac{\partial}{\partial u} \end{cases}$$

Before computing strongly Lie remarkable equations, let us recall some basic notions of the theory of surfaces in $\mathbb{R}^3$.

Let us consider the metric $k_1dx\otimes dx+k_2dy\otimes dy+du\otimes du$ on $\mathbb{R}^3$, with k_i non-vanishing real constants. Then the (scalar) mean curvature H of a generic surface $u=u(x,y)$ is

$$(13)\qquad H=\frac{1}{2}\frac{(k_2+u_y^2)u_{xx}-2u_xu_yu_{xy}+(k_1+u_x^2)u_{yy}}{(k_1k_2+k_2u_x^2+k_1u_y^2)^{\frac{3}{2}}},$$

and the Gaussian curvature G is

$$(14)\qquad G=\frac{u_{xx}u_{yy}-u_{xy}^2}{(k_1k_2+k_2u_x^2+k_1u_y^2)^2}.$$

Then, by analyzing the rank of the matrix of 2-prolongations of the vector fields (4) and (12), we realize the the unique second order equation which is strongly Lie remarkable with respect to the conformal algebra is

$$(15)\qquad G=H^2.$$

5 Remark. By a direct computation, we realize that the unique second order scalar differential invariant I of the algebra formed by $\mathcal{I}(\mathbb{R}^3)$ with the addition of homotheties of $\mathbb{R}^3$ is $I = H^2/G$. Then $I = k$, with k constant, is a weakly Lie remarkable equation in view of theorem 4. Therefore we could look for strongly Lie remarkable equations among the equations $I = k$. In fact, from the above discussion, we realize that $I = 1$ is the strongly Lie remarkable equation we were looking for.

Now we analyze the case $n = 3$. Then we have to search strongly Lie remarkable equations in the jet space $J^2(\mathbb{R}^4, 3)$. In this case, we realize that we have not strongly Lie remarkable equations. In fact, the prolongations of vector fields forming the algebra $\mathcal{C}(\mathbb{R}^4)$, at each point of $J^2(\mathbb{R}^4, 3)$, span a subspace of dimension at most 12. Since the dimension of the equation we are looking for is 12, in view of theorem 2, we have no chances to find a strongly Lie remarkable equation. The same negative result is achieved by considering the algebra $\mathcal{C}(\mathbb{R}^{n+1})$, where $n > 3$.

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