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DIFFERENTIAL EQUATIONS UNIQUELY DETERMINED BY ALGEBRAS OF POINT SYMMETRIES

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We continue to investigate strongly and weakly Lie remarkable equations, which we defined in a recent paper. We consider some relevant algebras of vector fields on $\mathbb{R}^k$ (such as the isometric, affine, projective, or conformal algebras) and characterize strongly Lie remarkable equations admitted by the considered Lie algebras.

Keywords: Lie symmetries of differential equations, jet space

1. Introduction

One of the fundamental achievements in the geometric theory of differential equations (DEs), both ordinary (ODEs) and partial (PDEs), is the theory of symmetries [1]–[5]. Symmetries of DEs are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former with the further property that solutions are sent into solutions. Knowing the symmetries of a DE can allow computing some of its solutions or transforming it into a more convenient form; in the case of an ODE, it can allow reducing the order, determining the integrating factor, and so on. There is a distinguished class of symmetries, those coming from a transformation of the independent and dependent variables: point symmetries. In this paper, we focus our attention on them.

Associated with the problem of finding the symmetries of a DE is the natural "inverse" problem of finding the most general form of a DE admitting a given Lie algebra as a subalgebra of infinitesimal point symmetries. A way leading to the solution of this problem is to classify all possible realizations of the given Lie algebra as an algebra of vector fields on the base manifold. The second step is to find differential invariants I_i of the realization under consideration (see, e.g., [6]). Then, as is well known [5], under suitable regularity hypotheses, the most general DE admitting a given Lie algebra as a subalgebra of point symmetries is given locally by $F^\mu(I_1, I_2, \dots, I_k) = 0$, where the F^μ are arbitrary smooth functions.

In this paper, we specialize the above problem as follows: we wish to find a *unique* DE that admits a given Lie algebra as a subalgebra of its point symmetries. An aspect of this problem was considered in [7], where we started from a given DE and found necessary and sufficient conditions for it to be uniquely determined by its point symmetries. Following the terminology already used in [7]–[9], we say that such a DE is *Lie remarkable*. A similar problem was also considered by Rosenhaus [10]; in fact, among various results, he proved that the equation of vanishing Gaussian curvature of surfaces in $\mathbb{R}^3$ is the unique second-order equation admitting the projective algebra of $\mathbb{R}^3$ as point symmetries.

This paper is organized as follows. In Sec. 2, we introduce a DE of order r as a submanifold of a suitable jet space (of order r), which is a manifold whose coordinate functions of a chart can be interpreted

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as "independent" and "dependent" variables and the derivatives of the latter with respect to the former up to the order r. We then introduce two distinguished types of Lie remarkable equations: *strongly* and *weakly* Lie remarkable equations. Strongly Lie remarkable equations are uniquely determined by their point symmetries in the whole jet space; in contrast, weakly Lie remarkable equations are equations that do not intersect with other equations admitting the same symmetries. We then recall the main results obtained in [8]: we report necessary and sufficient conditions for an equation to be strongly or weakly Lie remarkable.

In Sec. 3, we find strongly Lie remarkable equations associated with isometric, affine, projective, and conformal algebras of $\mathbb{R}^k$, where $\mathbb{R}^k$ is provided with metrics of various signatures. Because we start from concrete, not abstract, algebras, we do not have the problem of realizing them as vector fields. In particular, with regard to the affine algebra in $\mathbb{R}^k$, we recover the homogeneous second-order Monge–Ampère equations [11], and in $\mathbb{R}^3$ also a third-order PDE that, to our knowledge, has not been previously described in the literature. Also, in the case of the conformal algebra in $\mathbb{R}^3$, we recover an interesting second-order PDE, i.e., the equation for a surface $u(x, y)$ with the square of the (scalar) mean curvature equal to the Gaussian curvature.

2. Theoretical framework

We recall some basic facts about jet spaces (see [3], [4], [12] for more details) and the basic theory of DEs determined by their Lie point symmetries [7]. All manifolds and maps are assumed to be C^∞. If E is a manifold, then we let $\chi(E)$ denote the Lie algebra of vector fields on E. Also, for the sake of simplicity, all submanifolds of E are *embedded* submanifolds.

Let E be an $(n+m)$-dimensional smooth manifold and L be an n-dimensional embedded submanifold of E. Let (V, y^A) be a local chart on E. The coordinates (y^A) can be divided into two sets, $(y^A) = (x^\lambda, u^i)$, $\lambda = 1, \dots, n$ and $i = 1, \dots, m$, such that the submanifold L is locally described as the graph of a vector function $u^i = f^i(x^1, \dots, x^n)$. In what follows, Greek indices range from 1 to n and Latin indices range from 1 to m unless otherwise specified.

The set of equivalence classes $[L]^r_p$ of submanifolds L having a contact of order r at $p \in E$ is called the *r-jet of n-dimensional submanifolds of E* (also known as extended bundles [4]) and is denoted by $J^r(E, n)$. If E is endowed with a fibration $\pi\colon E \to M$, where $\dim M = n$, then the rth-order jet $J^r\pi$ of local sections of π is an open dense subset of $J^r(E, n)$. We have the natural maps $j_r L\colon L \to J^r(E, n)$, $p \mapsto [L]^r_p$ and $\pi_{k,h}\colon J^k(E, n) \to J^h(E, n)$, $[L]^k_p \mapsto [L]^h_p$, $k \geq h$.

The set $J^r(E, n)$ is a smooth manifold whose dimension is

$$\dim J^r(E, n) = n + m \sum_{h=0}^{r} \binom{n+h-1}{n-1} = n + m \binom{n+r}{r} \tag{1}$$

and whose charts are $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $u^i_{\boldsymbol{\sigma}} \circ j_r L = \partial^{|\boldsymbol{\sigma}|} f^i / \partial x^{\boldsymbol{\sigma}}$, $0 \leq |\boldsymbol{\sigma}| \leq r$. On $J^r(E, n)$, there is a distribution (the contact distribution) generated by the vectors

$$D_\lambda \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}}, \qquad \frac{\partial}{\partial u^j_{\boldsymbol{\tau}}},$$

where $0 \leq |\boldsymbol{\sigma}| \leq r-1$, $|\boldsymbol{\tau}| = r$, and $\boldsymbol{\sigma}\lambda$ denotes the multi-index $(\sigma_1, \dots, \sigma_{r-1}, \lambda)$. The vector fields D_λ are the (truncated) total derivatives. Any vector field $\Xi \in \chi(E)$ can be lifted to a vector field $\Xi^{(r)} \in \chi\bigl(J^r(E, n)\bigr)$ that preserves the contact distribution. In coordinates, if $\Xi = \Xi^\lambda \partial/\partial x^\lambda + \Xi^i \partial/\partial u^i$ is a vector field on E, then its k-lift $\Xi^{(k)}$ has the coordinate expression

$$\Xi^{(k)} = \Xi^\lambda \frac{\partial}{\partial x^\lambda} + \Xi^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \tag{2}$$

where $\Xi^j_{\boldsymbol{\tau},\lambda} = D_\lambda(\Xi^j_{\boldsymbol{\tau}}) - u^j_{\boldsymbol{\tau},\beta} D_\lambda(\Xi^\beta)$ with $|\boldsymbol{\tau}| < k$.

A *differential equation* $\mathcal{E}$ *of order* r *on* n*-dimensional submanifolds of a manifold* E is a submanifold of $J^r(E,n)$. The manifold $J^r(E,n)$ is called the *trivial equation*. An *infinitesimal point symmetry* of $\mathcal{E}$ is a vector field of the type $\Xi^{(r)}$ that is tangent to $\mathcal{E}$. Let $\mathcal{E}$ be locally described by $\{F^i = 0\}$, $i = 1 \dots k$, with $k < \dim J^r(E,n)$. Then finding point symmetries amounts to solving the system

$$\Xi^{(r)}(F^i) = 0$$

whenever $F^i = 0$ for some $\Xi \in \chi(E)$.

We let $\mathrm{sym}(\mathcal{E})$ denote the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$. By an *rth-order differential invariant* of a Lie subalgebra $\mathfrak{s}$ of $\chi(E)$, we mean a smooth function $I\colon J^r(E,n) \to \mathbb{R}$ such that for all $\Xi \in \mathfrak{s}$, we have $\Xi^{(r)}(I) = 0$. The problem of determining the Lie algebra $\mathrm{sym}(\mathcal{E})$ is called the *direct Lie problem*. Conversely, given a Lie subalgebra $\mathfrak{s} \subset \chi(E)$, we consider the *inverse Lie problem* of characterizing the equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathfrak{s} \subset \mathrm{sym}(\mathcal{E})$ [2], [13].

In what follows, we analyze the inverse Lie problem. We start with the definition and main properties, contained in [7], of DEs that are uniquely determined by their point symmetries, which we call *Lie remarkable* DEs.

Definition. Let E be a manifold, $\dim E = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An l-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ for any $\theta \in \mathcal{E}$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries and is said to be *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Of course, a strongly Lie remarkable equation is also weakly Lie remarkable. There are some direct consequences of this definition. For each $\theta \in J^r(E,n)$, let $S_\theta(\mathcal{E}) \subset T_\theta J^r(E,n)$ denote the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. We set $S(\mathcal{E}) \stackrel{\mathrm{def}}{=} \bigcup_{\theta \in J^r(E,n)} S_\theta(\mathcal{E})$. In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E,n)$. The inequality

$$\dim \mathrm{sym}(\mathcal{E}) \geq S_\theta(\mathcal{E}) \tag{3}$$

holds for all $\theta \in J^r(E,n)$, where $\dim \mathrm{sym}(\mathcal{E})$ is the (real-vector-space) dimension of the Lie algebra of infinitesimal point symmetries $\mathrm{sym}(\mathcal{E})$ of $\mathcal{E}$. If the rank of $S(\mathcal{E})$ is the same at each $\theta \in J^r(E,n)$, then $S(\mathcal{E})$ is an involutive (smooth) distribution.

A submanifold N of $J^r(E,n)$ is an *integral submanifold* of $S(\mathcal{E})$ if $T_\theta N = S_\theta(\mathcal{E})$ for each $\theta \in N$. Of course, an integral submanifold of $S(\mathcal{E})$ is an equation in $J^r(E,n)$ that admits all elements in $\mathrm{sym}(\mathcal{E})$ as infinitesimal point symmetries. The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$ [7]. It follows that $\mathcal{E}$ cannot coincide with the set of points of maximal rank of $S(\mathcal{E})$. The following theorems [7] can be proved.

Theorem 1. 1. *A necessary condition for the differential equation* $\mathcal{E}$ *to be strongly Lie remarkable is that*

$$\dim \mathrm{sym}(\mathcal{E}) > \dim \mathcal{E}.$$

2. *A necessary condition for the differential equation* $\mathcal{E}$ *to be weakly Lie remarkable is that*

$$\dim \mathrm{sym}(\mathcal{E}) \geq \dim \mathcal{E}.$$

Sufficient conditions, which prove useful in computing examples and applications, were also established in [7].

Theorem 2. *1. If $S(\mathcal{E})|_{\mathcal{E}}$ is an l-dimensional distribution on $\mathcal{E} \subset J^r(E,n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.*

2. If $S(\mathcal{E})$ is such that $\dim S_\theta(\mathcal{E}) > l$ for any $\theta \notin \mathcal{E}$, then $\mathcal{E}$ is a strongly Lie remarkable equation.

The next theorem [7] gives the relation between Lie remarkability and differential invariants.

Theorem 3. *Let $\mathfrak{s}$ be a Lie subalgebra of $\chi\big(J^r(E,n)\big)$. If the r-prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E,n)$ and the set of rth-order functionally independent differential invariants of $\mathfrak{s}$ reduces to a unique element $I \in C^\infty\big(J^r(E,n)\big)$, then the submanifold of $J^r(E,n)$ described by $\Delta(I) = 0$ (in particular, $I = k$ for any $k \in \mathbb{R}$) with Δ being an arbitrary smooth function is a weakly Lie remarkable equation.*

Several examples of strongly and weakly Lie remarkable equations were provided in [7]. We recall some of them.

1. The equation of minimal surfaces in $\mathbb{R}^4$ or $\mathbb{R}^5$ is neither strongly nor weakly Lie remarkable, but it is weakly Lie remarkable in $\mathbb{R}^3$ and $\mathbb{R}^6$ if we remove singular equations.

2. The equation of an unparameterized geodesic on a complete simply connected Riemannian two-dimensional manifold E is strongly Lie remarkable if and only if E has a constant Gaussian curvature.

3. The equation $u_{xx}u_{yy} - u_{xy}^2 = \kappa$ is weakly Lie remarkable if $\kappa \neq 0$, but it is strongly Lie remarkable if $\kappa = 0$.

4. Some higher-order Monge–Ampère equations [14] are weakly Lie remarkable if we remove singular subsets.

3. Strongly Lie remarkable equations determined by Lie algebras of vector fields on $\mathbb{R}^k$

In what follows, we consider only scalar PDEs, i.e., according to our notation, we restrict our attention to the case $m = 1$. We let $\mathcal{I}(\mathbb{R}^{n+1})$, $\mathcal{A}(\mathbb{R}^{n+1})$, $\mathcal{P}(\mathbb{R}^{n+1})$, and $\mathcal{C}(\mathbb{R}^{n+1})$ denote the respective isometric, affine, projective, and conformal algebras of $\mathbb{R}^{n+1}$ with the metric $g = \sum_{i=1}^n k_i\, dx^i \otimes dx^i + du \otimes du$, where k_i, $i = 1,\dots,n$, are nonvanishing real constants (to simplify the notation in what follows, we use the respective variables x and y instead of x^1 and x^2 in the case where $n = 2$). Even if the algebras $\mathcal{I}(\mathbb{R}^{n+1})$ and $\mathcal{C}(\mathbb{R}^{n+1})$ depend on k_i, we still let the same symbols denote them. For instance, $\mathcal{I}(\mathbb{R}^4)$ can represent both the Euclidean and the Poincaré algebra for suitable values of k_i. For each of the previous algebras, we determine strongly Lie remarkable PDEs of various orders associated with them.

We first study the case of the algebra $\mathcal{I}(\mathbb{R}^{n+1})$. To explain how the methods presented in Sec. 2 work, we calculate in the case of $\mathbb{R}^3$ endowed with the metric $k_1\, dx \otimes dx + k_2\, dy \otimes dy + du \otimes du$. For the remaining algebras, we only give the results with some useful comments.

3.1. The case of $\mathcal{I}(\mathbb{R}^{n+1})$. The algebra $\mathcal{I}(\mathbb{R}^{n+1})$ has the dimension $(n+1)(n+2)/2$. By Theorem 1, strongly Lie remarkable equations can then be of the first order. By statement 2 in Theorem 2, to obtain them, we must construct the matrix $M = \big(\Xi_i^{(1)j}\big)$ of the one-prolongations of the isometries characterized by the vector fields

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x}, && \Xi_2 = \frac{\partial}{\partial y}, && \Xi_3 = \frac{\partial}{\partial u}, \\
&\Xi_4 = k_2 y\frac{\partial}{\partial x} - k_1 x\frac{\partial}{\partial y}, && \Xi_5 = u\frac{\partial}{\partial x} - k_1 x\frac{\partial}{\partial u}, && \Xi_6 = u\frac{\partial}{\partial y} - k_2 y\frac{\partial}{\partial u}.
\end{aligned} \tag{4}$$

We must then see whether the rank of this matrix is five on $J^1(\mathbb{R}^3, 2)$ except for a four-dimensional submanifold where the rank decreases. Such a submanifold is then the sought strongly Lie remarkable equation. More precisely, the matrix M is given by

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
y & -\dfrac{k_1}{K_2}x & 0 & \dfrac{k_1}{k_2}u_y & -u_x \\
u & 0 & -k_1 x & -k_1 - u_x^2 & -u_x u_y \\
0 & u & -k_2 y & -u_x u_y & -k_2 - u_y^2
\end{pmatrix}.$$

The rank of this matrix is five except on the submanifold

$$1 + \frac{u_x^2}{k_1} + \frac{u_y^2}{k_2} = 0, \tag{5}$$

which describes the vanishing of the infinitesimal area element, where the rank decreases. The generalization to arbitrary n is straightforward. The strongly Lie remarkable equation in this case is

$$1 + \sum_{i=1}^{n} \frac{u_{x^i}^2}{k_i} = 0. \tag{6}$$

Of course, for Eqs. (5) and (6) to be nonempty, we must require that not all k_i are positive.

3.2. The case of $\boldsymbol{\mathcal{A}}(\mathbb{R}^{n+1})$. The algebra $\mathcal{A}(\mathbb{R}^{n+1})$ has the dimension n^2+3n+2. By Theorem 1, we see that a strongly Lie remarkable equation can be of the second or third order if $n = 2$ and of the second order if $n \geq 3$. The infinitesimal generators of this algebra are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b} \tag{7}$$

for all $a, b \in \{x^1, x^2, \dots, x^n, u\}$. We see that the strongly Lie remarkable second-order equation in $J^2(\mathbb{R}^3, 2)$ is the homogeneous Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{8}$$

which describes surfaces of $\mathbb{R}^3$ with the vanishing Gaussian curvature.

Moreover, there also exists a strongly Lie remarkable equation of third order in $J^3(\mathbb{R}^3, 2)$ with the local expression

$$\begin{aligned}
&u_{xx}^3 u_{yyy}^2 + u_{xxx}^2 u_{yy}^3 + 6u_{xx}u_{xxx}u_{xy}u_{yy}u_{yyy} - 6u_{xxx}u_{xxy}u_{xy}u_{yy}^2 - \\
&\quad - 6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 6u_{xx}^2 u_{xy}u_{xyy}u_{yyy} - 6u_{xx}^2 u_{xxy}u_{yy}u_{yyy} - \\
&\quad - 8u_{xxx}u_{xy}^3 u_{yyy} + 9u_{xx}u_{xxy}^2 u_{yy}^2 + 9u_{xx}^2 u_{xyy}^2 u_{yy} + \\
&\quad + 12u_{xxx}u_{xy}^2 u_{xyy}u_{yy} + 12u_{xx}u_{xxy}u_{xy}^2 u_{yyy} - 18u_{xx}u_{xxy}u_{xy}u_{xyy}u_{yy} = 0.
\end{aligned} \tag{9}$$

To our knowledge, Eq. (9) has not been previously described in the literature; nevertheless, we are currently unable to interpret it geometrically or physically. Further investigations are in progress.

It now remains to find the equations in $J^2(\mathbb{R}^{n+1}, n)$ with $n \geq 3$. In this case, we still have a strongly Lie remarkable equation for each n, namely,

$$\det\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j}\right) = 0, \tag{10}$$

i.e., the second-order homogeneous Monge–Ampère equation in n variables [11].

3.3. The case of $\mathcal{P}(\mathbb{R}^{n+1})$. The algebra $\mathcal{P}(\mathbb{R}^{n+1})$ has the dimension $n^2 + 4n + 3$. Its infinitesimal generators are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + u\frac{\partial}{\partial u}\right) \tag{11}$$

for all $a, b \in \{x^1, x^2, \dots, x^n, u\}$. We now realize that we must seek strongly Lie remarkable equations in the same jet spaces that we considered in the preceding subsection. Furthermore, we must also discuss the case of an equation in $J^3(\mathbb{R}^4, 3)$. We realize that Eqs. (8)–(10) *are also strongly Lie remarkable in this case* and that *there are no strongly Lie remarkable equations in* $J^3(\mathbb{R}^4, 3)$. In fact, the prolongations of vector fields (11) at each point of $J^3(\mathbb{R}^4, 3)$ span a subspace of at most dimension 20. Because the dimension of the sought equation is 22, by Theorem 1, we have no chance to find a strongly Lie remarkable equation.

3.4. The case of $\mathcal{C}(\mathbb{R}^{n+1})$. The algebra $\mathcal{C}(\mathbb{R}^{n+1})$ has the dimension $(n+2)(n+3)/2$. We must seek second-order strongly Lie remarkable equations. We start with $n = 2$. The infinitesimal generators of this algebra are the generators in (4) together with

$$\begin{aligned}
&\frac{1}{2}\frac{k_1x^2 - k_2y^2 - u^2}{k_1}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xu\frac{\partial}{\partial u},\\
&xy\frac{\partial}{\partial x} + \frac{1}{2}\frac{k_2y^2 - k_1x^2 - u^2}{k_1}\frac{\partial}{\partial y} + yu\frac{\partial}{\partial u},\\
&xu\frac{\partial}{\partial x} + yu\frac{\partial}{\partial y} + \frac{1}{2}(u^2 - k_1x^2 - k_2y^2)\frac{\partial}{\partial u},\\
&x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.
\end{aligned} \tag{12}$$

Before computing strongly Lie remarkable equations, we recall some basic notions of the theory of surfaces in $\mathbb{R}^3$. We consider the metric $k_1\,dx \otimes dx + k_2\,dy \otimes dy + du \otimes du$ on $\mathbb{R}^3$, where the k_i are nonzero real constants. Then the (scalar) mean curvature H of a general surface $u = u(x, y)$ is

$$H = \frac{1}{2}\frac{(k_2 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (k_1 + u_x^2)u_{yy}}{(k_1k_2 + k_2u_x^2 + k_1u_y^2)^{3/2}}, \tag{13}$$

and the Gaussian curvature G is

$$G = \frac{u_{xx}u_{yy} - u_{xy}^2}{(k_1k_2 + k_2u_x^2 + k_1u_y^2)^2}. \tag{14}$$

Analyzing the rank of the matrix of two-prolongations of vector fields (4) and (12), we then realize that the unique second-order equation that is strongly Lie remarkable with respect to the conformal algebra is

$$G = H^2. \tag{15}$$

Remark. By a direct computation, we realize that a unique second-order scalar differential invariant I of the algebra formed by $\mathcal{I}(\mathbb{R}^3)$ with the addition of homotheties of $\mathbb{R}^3$ is $I = H^2/G$. Then $I = k$, where k is a constant, is a weakly Lie remarkable equation by Theorem 3. Therefore, we could seek strongly Lie remarkable equations among the equations $I = k$. In fact, from the above discussion, we realize that $I = 1$ is just the sought strongly Lie remarkable equation.

We now analyze the case $n = 3$. We must now seek strongly Lie remarkable equations in the jet space $J^2(\mathbb{R}^4, 3)$. In this case, we see that we have no strongly Lie remarkable equations. In fact, the prolongations of vector fields forming the algebra $\mathcal{C}(\mathbb{R}^4)$ at each point of $J^2(\mathbb{R}^4, 3)$ span a subspace of at most dimension 12. Because the dimension of the sought equation is 12, by Theorem 1, we have no chance to find a strongly Lie remarkable equation. We obtain the same negative result for the algebra $\mathcal{C}(\mathbb{R}^{n+1})$ with $n > 3$.

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REFERENCES

1. G. Baumann, *Symmetry Analysis of Differential Equations with Mathematica*, Springer, New York (2000); G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Appl. Math. Sci., Vol. 81), Springer, New York (1989); N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics* (Math. Appl., Vol. 3), Reidel, Dordrecht (1985); N. H. Ibragimov, ed., *Handbook of Lie Group Analysis of Differential Equations*, Vols. 1–3, CRC, Boca Raton, Fl. (1994–1996); L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Acad. Press, New York (1982).
2. G. W. Bluman and J. D. Cole, *Similarity Methods of Differential Equations* (Appl. Math. Sci., Vol. 13), Springer, New York (1974).
3. A. M. Vinogradov and I. S. Krasil'shchik, eds., *Symmetries and Conservation Laws of Equations of Mathematical Physics* [in Russian], Faktorial, Moscow (1997); English transl.: I. S. Krasil'shchik and A. M. Vinogradov, eds. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Transl. Math. Monogr., Vol. 182), Amer. Math. Soc., Providence, R. I. (1999).
4. P. J. Olver, *Applications of Lie Groups to Differential Equations* (2nd ed.), Springer, New York (1993).
5. P. J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge Univ. Press, New York (1995).
6. W. I. Fushchych and I. Yehorchenko, *Acta Appl. Math.*, **28**, 69–92 (1992); G. Rideau and P. Winternitz, *J. Math. Phys.*, **31**, 1095–1105 (1990).
7. G. Manno, F. Oliveri, and R. Vitolo, “On differential equations characterized by their Lie point symmetries,” *J. Math. Anal. Appl.* (to appear).
8. G. Manno, F. Oliveri, and R. Vitolo, “On an inverse problem in group analysis of PDEs: Lie-remarkable equations,” in: *Waves and Stability in Continuous Media* (Proc. 13th Conf. Wascom, Catania, Italy, 2005, R. Monaco, G. Mulone, S. Rionero, and T. Ruggeri, eds.), World Scientific, Hackensack, N. Y. (2006), p. 420–432.
9. F. Oliveri, *Note Mat.*, **23**, No. 2, 195–216 (2004/2005); “Sur une propriété remarquable des équations de Monge-Ampère,” in: *Non Linear Hyperbolic Fields and Waves: A Tribute to Guy Boillat* (Rend. Circ. Mat. Palermo (2) Suppl., No. 78) (2006), p. 243–257.

10. V. Rosenhaus, "On one-to-one correspondence between the equation and its group: The Monge–Ampère equation," Preprint F. 18, Acad. Sci. Estonian SSR, Tartu (1982); *Algebras Groups Geom.*, **3**, 148–166 (1986); **5**, 137–150 (1988).
11. G. Boillat, *C. R. Acad. Sci. Paris*, **313**, 805–808 (1991).
12. D. J. Saunders, *The Geometry of Jet Bundles* (London Math. Soc. Lect. Note Ser., Vol. 142), Cambridge Univ. Press, Cambridge (1989).
13. W. I. Fushchych, *Scientific Works*, Inst. Math., Nat. Acad. Sci. Ukraine, Kiev (2000); http://www.imath.kiev.ua/symfushchych/papers/Volume2.pdf.
14. G. Boillat, *C. R. Acad. Sci. Paris*, **315**, 1211–1214 (1992).