What is involutivity?

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Involutivity is a geometric condition, that is, if it holds in one co-ordinate system, it holds in all. Being a DGB/characteristic set is not.

Example (G. Reid): The system

$$u_{xx} = 0, \qquad u_{yy} = 0$$

is a DGB. Change co-ordinates to $X = x + y$, $Y = x - y$ and the system becomes

$$u_{XX} + u_{YY} = 0, \qquad u_{XY} = 0.$$

which has an integrability condition, with $Y < X$,

$$u_{YYY} = 0$$

Think: Changes of co-ordinates introduce new orderings. . .

A primitive “completion” algorithm:

[A] differentiate everything to get all eqns the same order

[B] do linear algebra on the top order terms to get into echelon form

[C] if get a lower order eqn, differentiate it to be the same order as where you are right now, re-do the linear algebra

[D] differentiate all eqns once

[E] go to step B

Problem: When do we know we have “finished”? Can we achieve a co-ordinate free result? That is, if we change co-ordinates, will we finish at the same order?

Naive answer: when the top order terms “saturate”.

As we know, there is one huge impediment to such a naive approach, which is solved by using Gröbner basis or more generally "critical pair" type techniques, namely, finding systematically the integrability conditions of lower order than the order of the eqns required to find them.

One major, well known and much studied method of making co-ordinate free conditions algebraically testable is to construct a functor into the category of chain complexes.

$$X \in \mathcal{X} \mapsto \qquad \stackrel{\delta}{\leftarrow} \mathcal{C}^{n+1} \stackrel{\delta}{\leftarrow} \mathcal{C}^{n} \stackrel{\delta}{\leftarrow} \mathcal{C}^{n-1} \stackrel{\delta}{\leftarrow}, \qquad \delta^2 = 0$$

where the $\mathcal{C}^n$ are abelian groups (modules, vector spaces). Any function $f : X \to Y$ for $X, Y \in \mathcal{X}$ is mapped to a map of chain complexes, $\{f^{\#n} : \mathcal{C}^n(Y) \to \mathcal{C}^n(X) \,|\, n \in \mathbb{N}\}$ such that $(1_X)^{\#n} = 1_{\mathcal{C}^n}$ and $(f \circ g)^{\#n} = g^{\#n} \circ f^{\#n}$.
The cohomology groups are defined by

$$H^n(X) = \ker \delta|_{\mathcal{C}^n} / \operatorname{im} \delta|_{\mathcal{C}^{n-1}}$$

and are thought of as invariants or obstructions.

Assume the system $\Phi = \{\Phi_a\}$ is a characteristic set in a termordering with total degree of differentiation as a main criterion, and each element of Φ has order q.

Set

$$u^\alpha_K = \frac{\partial^{|K|} u^\alpha}{\partial x^K}, \qquad K = (K_1, \ldots, K_p).$$

Then the r^{th} symbol of Φ is given by

$$\sigma_r(\Phi) = \left\{ \sum_{\substack{\alpha \\ |K|=q}} \frac{\partial \Phi_a}{\partial u^\alpha_K} \mathbf{v}^\alpha_{K+L} \ : \ |L| = r, a \in \mathcal{A} \right\}$$

The symbol lies in a module over a polynomial ring . . . or in a symmetric tensor fibre bundle depending on your point of view.

Example calculations

Equation	The σ_0 Symbol element
$u_x^2 + u_y^2 - u^2$	$2u_x v_x + 2u_y v_y$
$u_x u_{xx} + u_{xy}^2 - 3u_{yy}$	$u_x v_{xx} + 2u_{xy} v_{xy} - 3v_{yy}$
$\frac{\partial}{\partial x^3}(u_x^2 + u_y^2)$	$2u_x v_{xxxx} + 2u_y v_{xxxy}$

By the product rule and the fact Φ is a characteristic set, so there are no integrability conditions of lower order than those required to calculate it, we have

$$\sigma_r(\{\frac{\partial}{\partial x^K}\Phi_a \ : \ a \in \mathcal{A}, |K| = s\}) = \sigma_{r+s}(\Phi).$$

Think: (1) the v_K^α form a basis of $S^{|K|}T^*$ ('S' is for symmetric; the index on the v's is symmetric).

(2) the r^{th} symbol of the system Φ defines a linear map $\tau_{q+r} : S^{q+r}T^* \to E$

Example •

$$\Phi : \quad \begin{cases} f_1 = u_{yz} - u_{xx} \\ f_2 = u_{zz} - u_{xz} \end{cases}$$

For this example, the image space G_2 of τ_2 and the kernel g_2 of τ_2 are given by:

$$\begin{aligned} G_2 &= \langle v_{yz} - v_{xx}, v_{zz} - v_{xz} \rangle \\ g_2 &= \langle v_{yz} + v_{xx}, v_{zz} + v_{xz}, v_{xy}, v_{yy} \rangle \end{aligned}$$

Spencer's definition of involutivity

We tensor the symmetric tensor space ST^* with the exterior algebra ΛT^*. For

$$\omega \in \Lambda^\ell T^* \otimes S^{k+1}T^*, \qquad \omega = \sum_{|\mu|=\ell, |\nu|=k+1} \omega_{\mu,\nu} dx^\mu \otimes v_\nu$$

define

$$\delta(\omega) = \frac{1}{k} \sum_{i,\nu_i \neq 0} \omega_{\mu,\nu}\, dx^\mu \wedge dx^i \otimes v_{\nu - 1_i}$$

Example

$$\delta(dz \otimes (v_{yz} - v_{xx})) = dzdy \otimes v_z - dzdx \otimes v_x$$

The sequence

$$0 \to S^r T^* \xrightarrow{\delta} T^* \otimes S^{r-1} T^* \xrightarrow{\delta} \Lambda^2 T^* \otimes S^{r-2} T^*$$
$$\xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda^n T^* \otimes S^{r-n} T^* \to 0$$

where $n = \#x$'s and $r \geq n$, is exact.

That is, not only $\delta^2 = 0$ but $\operatorname{im} \delta = \ker \delta$.

The map δ restricts to the kernels g_{q+k} of the symbol maps τ_{q+r}, and so the sequence

$$0 \to g_{q+r} \xrightarrow{\delta} T^* \otimes g_{q+r-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes g_{q+r-2} \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} \Lambda^n T^* \otimes g_{q+r-n} \to 0$$

is well-defined.

Denote by

$$H\left(\Lambda^p T^* \otimes g_{q+r-p}\right)$$

the quotient of $\ker \delta / \operatorname{im} \delta$. We say that the symbol of Φ is involutive if

$$H\left(\Lambda^p T^* \otimes g_{q+r-p}\right) = 0, \text{ for all } 0 \leq p \leq n, r \geq p$$

Recall the Example • $\Phi: \begin{cases} f_1 = u_{yz} - u_{xx} \\ f_2 = u_{zz} - u_{xz} \end{cases}$

The first two kernel spaces are

$$g_2 = \langle v_{yz} + v_{xx},\ v_{zz} + v_{xz},\ v_{xy}, v_{yy} \rangle$$
$$g_3 = \langle v_{yyz} + v_{xxy},\ v_{xyy},\ v_{yyy},\ v_{xxx} + v_{xyz} + v_{yzz} + v_{xxz} + v_{xzz} + v_{zzz} \rangle$$

Then

$$1 = \dim H(\Lambda^2 T^* \otimes g_2) = \dim \ker \delta|_{\Lambda^2 T^* \otimes g_2} - \dim\ \operatorname{im}\ \delta|_{T^* \otimes g_3}$$

and a representative element is

$$dydz \otimes (v_{zz} + v_{xz}) + dxdz \otimes (v_{yz} + v_{xx})$$

Note that "new" conditions for this system that appear at order 3 include $u_{yzz} - u_{zzz}$ if $x > z > y$ and $u_{xxz} - u_{xxx}$ if $z > x > y$

To investigate the map δ, and what the homology spaces might be detecting, I needed a map $\mathcal{S}$, which "goes the same way as differentiation", that is, the indices on the v's are added to, not subtracted.

The map $\mathcal{S}$ is defined to be the adjoint of δ with respect to the simplest inner product,

$$\langle dx^I \otimes v_K, dx^\Gamma \otimes v_L \rangle = \begin{cases} \pm 1 & K = L, dx^I = \pm dx^\Gamma \\ 0 & \text{else} \end{cases}$$

Thus $\mathcal{S}$ is defined by

$$\langle \delta\omega, \rho \rangle = \langle \omega, \mathcal{S}\rho \rangle$$

We have that $\mathcal{S}$ is zero on 0-forms ($|I| = 0$) and otherwise

$$\mathcal{S}(dx^I \otimes v_K) = \sum_i \text{sign}(I, i) dx^{I-i} \otimes v_{K+1_i}$$

where

$$\text{sign}(I, i) dx_i \wedge dx^{I-i} = dx^I$$

Recall δ was defined by

$$\delta(dx^\mu \otimes v_\nu) = \frac{1}{k} \sum_{i, \nu_i \neq 0} dx^\mu \wedge dx^i \otimes v_{\nu - 1_i}$$

Example

$$\mathcal{S}(dxdy \otimes v_{yy} - 2dxdz \otimes x^2 u_{yy} v_{xz})$$
$$= dy \otimes v_{xyy} - dx \otimes v_{yyy}$$
$$-2dz \otimes x^2 u_{yy} v_{xxz}$$
$$+2dx \otimes x^2 u_{yy} v_{xzz}$$

$\mathcal{S}$ yields a symbolic differentiation

$$\mathcal{S}(dx_i \otimes v_K) = x_i * v_K = v_{K+1_i}$$

x_i* treats everything except the v_K as constants and

$$\sigma_{q+1}\left(\frac{\partial f}{\partial x_i}\right) = x_i * \sigma_q(f).$$

We have $\mathcal{S}^2 = 0$, and

$$\stackrel{\mathcal{S}}{\to} \Lambda^{r+1} T^* \otimes S^k T^* \stackrel{\mathcal{S}}{\to} \Lambda^r T^* \otimes S^{k+1} T^* \stackrel{\mathcal{S}}{\to} \cdots$$

$$\stackrel{\mathcal{S}}{\to} \Lambda^1 \otimes S^{k+r} T^* \stackrel{\mathcal{S}}{\to} S^{k+r+1} T^* \to 0$$

is exact. Moreover

$$\mathcal{S} : \Lambda^k T^* \otimes G_{q+r} \to \Lambda^{k-1} T^* \otimes G_{q+r+1}$$

where the G_{q+r} are the r^{th} symbol equations.

Theorem

$$H_\delta(\Lambda^k T^* \otimes g_{q+r}) \approx H_{\mathcal{S}}(\Lambda^{k-1} T^* \otimes G_{q+r+1})$$

In other words, the information the two sets of homology groups encode is the same!!

Recall the Example • $\Phi : \begin{cases} f_1 = u_{yz} - u_{xx} \\ f_2 = u_{zz} - u_{xz} \end{cases}$

so that $G_2 = \langle v_{yz} - v_{xx}, v_{zz} - v_{xz} \rangle$. Since

$$\frac{\partial}{\partial x} f_2 = u_{xzz} - u_{xxz}, \quad \frac{\partial}{\partial z} f_1 - \frac{\partial}{\partial x} f_1 - \frac{\partial}{\partial y} f_2 = u_{xxx} - u_{xxz}$$

we have $v_{xzz} - v_{xxz}, \quad v_{xxx} - v_{xxz} \in G_3$. So

$$\omega = dz \otimes (v_{xxx} - v_{xxz}) - dx \otimes (v_{xxz} - v_{xzz}) \in \Lambda^1 T^* \otimes G_3$$

Can check $\mathcal{S}(\omega) = 0$ but $\omega \notin \mathcal{S}(\Lambda^2 T^* \otimes G_2)$ and thus

$$[\omega] \in H_{\mathcal{S}}(\Lambda^1 T^* \otimes G_3)$$

and is a representative generator.

BUT LOOK!!

$$\omega = dz \otimes [z * \sigma(f_1) - x * \sigma(f_1) - y * \sigma(f_2)] + dx \otimes x * \sigma(f_2)$$

so that

$$\mathcal{S}(\omega) = 0$$

is the equation

$$(z^2 - xz) * \sigma(f_1) + (x^2 - zy) * \sigma(f_2) = 0$$

which is nothing other than

the SYZYGY or compatibility condition of the symbol equations

Thus, we can try to understand involutivity

in terms of syzygies,

which is much more familiar territory

(at least to me!)

Let $s = (s_1, s_2, \dots s_r)$ be a syzygy of the symbol equations $\sigma(f_1), \sigma(f_2), \dots \sigma(f_r)$ that is,

$$\sum s_i * \sigma(f_i) = 0.$$

Let $\mathcal{H}$ be $\mathcal{H}$ homogeneous polynomials in the x_i and let $\omega_i \in \Lambda^1 T^* \otimes \mathcal{H}$ be such that

$$\mathcal{S}(\omega_i) = s_i$$

Define $\omega_s = \sum_i \omega_i \otimes \sigma(f_i)$ Then

$$\omega_s \in \Lambda^1 T^* \otimes G_{q+deg(s)-1}$$

is defined up to an element in im $\mathcal{S}$.

Theorem

Let s be a syzygy of the symbolic system $G_q = \{\sigma_0(f_i) : i\}$ of degree greater than one.

Suppose s is *minimal*, that is,

$$s \neq \sum_j h_\gamma t^\gamma, \qquad t^\gamma \text{ a syzygy, } h_\gamma \in \mathcal{H}$$

with deg $t^\gamma <$ deg s.

Then

$$0 \neq [\omega_s] \in H_{\mathcal{S}}(\Lambda^1 T^* \otimes G_{q+\deg(s)-1})$$

Moreover:

all elements of

$$H_{\mathcal{S}}(\Lambda^1 T^* \otimes G_{q+\deg(s)-1})$$

are of the form

$$[\omega_s]$$

for some syzygy s of G_q.

Overall result

Assume the system is a DGB with respect to a total degree termordering.

As you prolong the system, the degree of the minimal syzygies of the symbol goes down monotonically to 1, and then stays at 1 forever. When the syzygies have degree 1, the $\mathcal{S}$ homology groups are all zero, and then the system is involutive.

In fact all the groups $H_{\mathcal{S}}(\Lambda^k T^* \otimes G_{q+r})$ can be obtained in terms of k^{th} order syzygies, that is, the syzygies of the syzygies of the syzygies of the . . . .

So, is the $\mathcal{S}$ sequence the same as the Koszul sequence for G_q?

If $I = \langle f_1, \ldots f_\rho \rangle \subset R$ is a polynomial ideal, one way to define the Koszul sequence is

$$0 \longrightarrow \Lambda^\rho \xrightarrow{\lrcorner f} \Lambda^{\rho-1} \xrightarrow{\lrcorner f} \cdots$$

$$\cdots \xrightarrow{\lrcorner f} \Lambda^2 \xrightarrow{\lrcorner f} \Lambda^1 \xrightarrow{\lrcorner f} R \longrightarrow 0$$

where Λ is the exterior algebra on ρ symbols $\{e_i\}$ over R, and the map $\lrcorner f$ is linear and

$$\lrcorner_f(e_{i_1} e_{i_2} \cdots e_{i_r}) = \sum_j (-1)^j f_{i_j} e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_r}.$$

For example, $\lrcorner_f(\sum g_i e_i) = \sum f_i g_i$. The sequence $f_1, \ldots, f_\rho$ is said to be regular if the homology groups of $\lrcorner f$ are zero.

Example: involutive $\not\Rightarrow$ regular

Let $\Phi = \{u_{xx}, u_{xy}, u_{yy}\}$. The symbol, written in the form of polynomials, are the prolongations of

$$\sigma_0(\Phi) = \{t_1^2,\ t_1 t_2,\ t_2^2\}.$$

We have $R = \mathbb{R}[t_1, t_2]$ and $f = t_1^2,\ t_1 t_2,\ t_2^2$. Then $\ker \lrcorner_f|_{\Lambda^1} = \langle s_1, s_2 \rangle$ where

$$s_1 = t_2 e_1 - t_1 e_2, \qquad s_2 = t_2 e_2 - t_1 e_3.$$

Since the minimal syzgies of σ_0 have order 1, the symbol is involutive. Now, the image of $\lrcorner_f|_{\Lambda^2}$ is given by

$$\text{im } \lrcorner_f = \langle t_1 s_1, t_2 s_2, t_2 s_1 + t_1 s_2 \rangle$$

and thus $H^1_{\text{Koszul}} \approx \mathbb{R}^3$.

Name	Map	Space
Spencer	δ	$\wedge^* \otimes q_{q+*}$
	$\mathcal{S}$	$\wedge^* \otimes G_{q+*}$
Koszul	$\lrcorner f$	$\wedge^*$

For Spencer, the map is universal and the space particular. For Koszul, the map is particular and the space is universal.

Conclusions

Involutivity is a geometric, that is, co-ordinate free form of being a characteristic set.

A characteristic set is involutive when the generating syzygies of its zeroth order symbol have degree 1. This can always be achieved by prolongation.

There are at least two other sequences in terms of the syzygies of a system, the Koszul sequence and the Janet sequence (the analogue of the Hilbert resolution). The relationship between these sequences and their applications is a point for further study.