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An extension of Cartan's method of equivalence to immersions: I. Necessary conditions

Abdol-Reza Mansouri *

Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada

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ABSTRACT

In this paper, we show that the existence of immersions preserving given geometric structures between manifolds can be expressed in terms of a relation between suitably constructed moving coframes on the manifolds, and we show that the key steps in Cartan's method of equivalence can be extended to yield necessary conditions for the existence of such immersions.

1. Introduction

Consider the following problems:

(a) Let $M \subset \mathbb{R}^3$ open and $N \subset \mathbb{R}^4$ open; let $\mathcal{D}_1$ be a smooth rank 2 distribution on M, and $\mathcal{D}_2$ a smooth rank 2 distribution on N. We wish to investigate whether there exists an immersion $f : M \to N$ that maps $\mathcal{D}_1$ to $\mathcal{D}_2$, i.e. such that $\forall p \in M$, $\forall v \in T_pM$: $v \in \mathcal{D}_1(p) \Rightarrow T_p f(v) \in \mathcal{D}_2(f(p))$. Restricting M and N if necessary, let $\omega = (\omega^1, \omega^2, \omega^3)^T$ be a coframe on M such that $\mathcal{D}_1 = \{\omega^3 = 0\}$, and let $\Omega = (\Omega^1, \Omega^2, \Omega^3, \Omega^4)^T$ be a coframe on N such that $\mathcal{D}_2 = \{\Omega^3 = \Omega^4 = 0\}$. An immersion $f : M \to N$ will map $\mathcal{D}_1$ to $\mathcal{D}_2$ (in the precise sense defined above) if and only if

$$f^\star \begin{pmatrix} \Omega^3 \\ \Omega^4 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \omega^3$$

where c, d are smooth $\mathbb{R}$-valued functions on M with $c^2 + d^2 \neq 0$. Equivalently, we wish to investigate the existence of a smooth mapping $f : M \to N$ such that

$$f^\star \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix},$$

* Fax: +1 (613) 533 2964.
E-mail address: mansouri@mast.queensu.ca.

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with the matrix function

$$V = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix}$$

taking values in the set of real 4×3 rank 3 matrices. Note that the matrix function V can be written as $V = UJ$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and U is a matrix function on M with values in the subgroup

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A, C \in GL(2, \mathbb{R}) \right\}$$

of $GL(4, \mathbb{R})$.

(b) Let $M, N, \mathcal{D}_1, \mathcal{D}_2$ as in (a), and assume now given a Riemannian metric g_1 on $\mathcal{D}_1$ and a Riemannian metric g_2 on $\mathcal{D}_2$. We wish to investigate whether there exists an immersion $f : M \to N$ that maps $\mathcal{D}_1$ to $\mathcal{D}_2$ and g_1 to g_2, i.e. such that $\forall p \in M,\ \forall v \in T_pM\colon\ v \in \mathcal{D}_1(p) \Rightarrow T_pf(v) \in \mathcal{D}_2(f(p))$, and such that $\forall p \in M,\ \forall v, w \in T_pM\colon\ g_2(T_pf(v), T_pf(w)) = g_1(v, w)$. Restricting M and N if necessary, let $\omega = (\omega^1, \omega^2, \omega^3)^T$ be a coframe on M such that $\mathcal{D}_1 = \{\omega^3 = 0\}$ and such that $g_1 = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$, and let $\Omega = (\Omega^1, \Omega^2, \Omega^3, \Omega^4)^T$ be a coframe on N such that $\mathcal{D}_2 = \{\Omega^3 = \Omega^4 = 0\}$ and such that $g_2 = \Omega^1 \otimes \Omega^1 + \Omega^2 \otimes \Omega^2$. An immersion $f : M \to N$ will map $\mathcal{D}_1$ to $\mathcal{D}_2$ and g_1 to g_2 (in the precise sense defined above) if and only if

$$f^\star \begin{pmatrix} \Omega^3 \\ \Omega^4 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \omega^3$$

and

$$f^\star \begin{pmatrix} \Omega^1 \\ \Omega^2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \omega^3$$

where c, d are smooth $\mathbb{R}$-valued functions on M with $c^2 + d^2 \neq 0$ and $a_1, a_2, a_3, b_1, b_2, b_3$ smooth $\mathbb{R}$-valued functions on M with

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in O(2, \mathbb{R}).$$

Equivalently, we wish to investigate the existence of a smooth mapping $f : M \to N$ such that

$$f^\star \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix},$$

with the matrix function

$$V = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix}$$

taking values in the set of real 4×3 rank 3 matrices having upper-left 2×2 block in $O(2, \mathbb{R})$. In this problem as well V can be written as $V = UJ$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and U is a matrix function on M with values in the subgroup

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in O(2, \mathbb{R}), C \in GL(2, \mathbb{R}) \right\}$$

of $GL(4, \mathbb{R})$.

(c) Let (M, g_M) be a Riemannian manifold of dimension m, and (N, g_N) a Riemannian manifold of dimension n, with $m < n$; does there exist an isometric immersion $f : M \to N$, i.e. a C^∞ map f such that $f^\star g_N = g_M$? Re-expressing the problems in terms of suitable adapted coframes (and restricting M and N if necessary), and letting $\Omega = (\Omega^1, \dots, \Omega^n)^T$ be a coframe on N such that $g_N = \sum_i \Omega^i \otimes \Omega^i$, and $\omega = (\omega^1, \dots, \omega^m)^T$ a coframe on M such that $g_M = \sum_i \omega^i \otimes \omega^i$, it is easy to see that $f : M \to N$ is an isometric immersion if and only if there exists a matrix-valued function V on M satisfying $V^T V = I$ and such that $f^\star \Omega = V\omega$. Note that if m and n were equal, this problem would become the standard Riemannian isometry problem. Note also that any real $n \times m$ matrix V satisfying $V^T V = I$ can be written as $V = UJ$, where $U \in O(n)$ and $J = (e_1 \dots e_m)$, where e_k denotes the kth basis vector in the canonical basis of $\mathbb{R}^n$.

(d) With (M, g_M) and (N, g_N) as in (c), assume we now wish to investigate whether there exists a smooth mapping $f : M \to N$ such that $f^\star g_N = \alpha^2 g_M$, for some smooth function $\alpha : M \to \mathbb{R}^\star$. In other words, we are interested in conformal immersions of M into N. Letting Ω and ω be the same coframes as in (c), it is easy to see that $f : M \to N$ is a conformal immersion if and only if there exists a matrix-valued function V on M satisfying $V^T V = \alpha^2 I$ for some smooth function $\alpha : M \to \mathbb{R}^\star$ and such that $f^\star \Omega = V\omega$. Here again, any real $n \times m$ matrix V satisfying $V^T V = \alpha^2 I$ can be written as $V = UJ$, where $U \in CO(n) = \{\alpha \hat{U} \mid \alpha \in \mathbb{R}^\star, \hat{U} \in O(n)\}$ and $J = (e_1 \dots e_m)$, where e_k denotes the kth basis vector in the canonical basis of $\mathbb{R}^n$.

The problems described in (a), (b), (c), and (d) are all instances of the following general problem:

(P) Let G be a Lie subgroup of $GL(n)$, let M, N be open subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively with $m < n$; let W and V be real vector spaces of dimension m and n, respectively; let ω_M be a W-valued smooth one-form on M, and Ω_N a smooth V-valued one-form on N; assume $\omega_M(q)$ is onto W, $\forall q \in M$, and $\omega_N(q)$ onto V, $\forall q \in N$; let $J : W \to V$ be a monomorphism. Assume given a representation of G on V, and, viewing V as a G-module, denote the corresponding operation $G \times V \to V$ by $(g, v) \mapsto g \cdot v$. Does there exist an immersion $f : M \to N$ such that $f^\star \Omega_N = \lambda \cdot J\omega_M$, where $\lambda : M \to G$ is a C^∞ mapping on M with values in G?

Our objective in this paper is to show that Cartan's method of equivalence [1,3,5] can be extended, with suitable modifications, to address this class of problems, yielding necessary conditions for a solution to exist. We will treat the sufficient conditions for the existence of a solution to this problem in a future paper. Note that Problem (P) with $m = n$ is exactly the problem addressed by Cartan's method of equivalence. Also, in order to facilitate the comparison of our proposed extension to Cartan's method of equivalence, we have tried to keep our notation as close as possible to that used in the standard Refs. [3,5] on Cartan's method of equivalence.

2. Extending Cartan's equivalence method

2.1. Principal bundles, coframes, and structure equations

As with Cartan's equivalence method, the first major simplification is afforded by lifting the problem to the principal bundle of coframes. Consider then the principal G-bundles $M \times G$ and $N \times G$ with left G-action given by $g_1 \cdot (q, g_2) = (q, g_1 g_2)$, and canonical surjections $\pi_M : M \times G \to M$, $p_M : M \times G \to G$ and $\pi_N : N \times G \to N$, $p_N : N \times G \to G$. Let $(\hat{e}_i)_{i=1}^m$ be a basis of W, and let $(e_i)_{i=1}^n$ be a basis of V such that $e_i = J\hat{e}_i$, $\forall i \in \{1, \dots, m\}$; denote by $(f^j)_{j=1}^n$ the basis of $V^\star$ dual to $(e_i)_{i=1}^n$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $V^\star$ and V. On $M \times G$ we define the V-valued one-form ω by

$$\omega = p_M \cdot J\pi_M^\star \omega_M,$$

and, on $N \times G$, the V-valued one-form Ω by

$$\Omega = p_N \cdot \pi_N^\star \Omega_N.$$

We define the real-valued smooth one-forms $(\omega^i)_{i=1}^n$ on $M \times G$ and $(\Omega^i)_{i=1}^n$ on $N \times G$ by

$$\omega = \sum_{i=1}^n \omega^i e_i, \qquad \Omega = \sum_{i=1}^n \Omega^i e_i.$$

The following result is the key result on which the proposed extension rests:

Theorem 1. *There exists an immersion $f : M \to N$ and a C^∞ map $\lambda : M \to G$ such that $f^\star \Omega_N = \lambda \cdot J\omega_M$ if and only if there exists a left-equivariant C^∞ map $\hat{f} : M \times G \to N \times G$ such that $\hat{f}^\star \Omega = \omega$.*

Proof. Assume first that there exists a left-equivariant C^∞ map $\hat{f} : M \times G \to N \times G$ such that $\hat{f}^\star \Omega = \omega$. Note that it follows from left-equivariance of $\hat{f}$ that $\pi_N \circ \hat{f} = f \circ \pi_M$ for some C^∞ map $f : M \to N$. We have:

$$
\begin{aligned}
\hat{f}^\star\Omega = \omega \quad &\Leftrightarrow \quad \hat{f}^\star\Omega|_x = \omega|_x, \quad \forall x \in M \times G, \\
&\Leftrightarrow \quad \big(\hat{f}^\star(p_N \cdot \pi_N^\star \Omega_N)\big)\big|_x = \big(p_M \cdot J\pi_M^\star(\omega_M)\big)\big|_x, \\
&\Leftrightarrow \quad \big[(p_N \circ \hat{f}) \cdot (\pi_N \circ \hat{f})^\star \Omega_N\big]\big|_x = \big(p_M \cdot J\pi_M^\star(\omega_M)\big)\big|_x, \\
&\Leftrightarrow \quad \big[(f \circ \pi_M)^\star \Omega_N\big]\big|_x = \big((p_N \circ \hat{f})^{-1} \cdot p_M \cdot J\pi_M^\star(\omega_M)\big)\big|_x, \\
&\Leftrightarrow \quad \pi_M^\star(f^\star\Omega_N)|_x = \big((p_N \circ \hat{f})(x)\big)^{-1} \cdot p_M(x) \cdot J\pi_M^\star(\omega_M)|_x, \quad \forall x \in M \times G.
\end{aligned}
$$

By left-equivariance of $\hat{f}$, it follows that $\forall x \in M \times G$, $\forall g \in G$:

$$
\begin{aligned}
\big((p_N \circ \hat{f})(g \cdot x)\big)^{-1} \cdot p_M(g \cdot x) &= \big((g \cdot p_N \circ \hat{f})(x)\big)^{-1} \cdot g \cdot p_M(x) \\
&= \big((p_N \circ \hat{f})(x)\big)^{-1} \cdot p_M(x),
\end{aligned}
$$

in other words, the G-valued mapping $x \mapsto ((p_N \circ \hat{f})(x))^{-1} \cdot p_M(x)$ on $M \times G$ factors through the projection π_M. There therefore exists a C^∞ map $\lambda : M \to G$ such that $((p_N \circ \hat{f})(x))^{-1} \cdot p_M(x) = \lambda \circ \pi_M(x)$, $\forall x \in M \times G$. Hence:

$$
\begin{aligned}
\hat{f}^\star\Omega = \omega \quad &\Rightarrow \quad \pi_M^\star(f^\star\Omega_N) = (\lambda \circ \pi_M) \cdot J\pi_M^\star\omega_M \\
&\Leftrightarrow \quad \pi_M^\star(f^\star\Omega_N) = \pi_M^\star(\lambda \cdot J\omega_M) \\
&\Leftrightarrow \quad f^\star\Omega_N = \lambda \cdot J\omega_M,
\end{aligned}
$$

by virtue of π_M being a submersion. Conversely, assume there exists a C^∞ map $f : M \to N$ and a C^∞ map $\lambda : M \to G$ such that $f^\star\Omega_N = \lambda J\omega_M$. Defining $\hat{f} : M \times G \to N \times G$ by $\hat{f}(q, g) = (f(q), g \cdot (\lambda(q))^{-1})$, we obtain:

$$
\begin{aligned}
\hat{f}^\star\Omega|_{(q,g)} &= \hat{f}^\star(p_N \cdot \pi_N^\star\Omega_N)|_{(q,g)} \\
&= (p_N \circ \hat{f}) \cdot (\pi_N \circ \hat{f})^\star \Omega_N|_{(q,g)} \\
&= (p_N \circ \hat{f})(q, g) \cdot (f \circ \pi_M)^\star \Omega_N|_{(q,g)} \\
&= g \cdot \big(\lambda(q)\big)^{-1} \cdot \pi_M^\star(\lambda \cdot J\omega_M)|_{(q,g)} \\
&= g \cdot \big(\lambda(q)\big)^{-1} \cdot \lambda(q) \cdot J\pi_M^\star(\omega_M)|_{(q,g)} \\
&= g \cdot J\pi_M^\star(\omega_M)|_{(q,g)} \\
&= \omega|_{(q,g)}, \quad \forall (q, g) \in M \times G,
\end{aligned}
$$

and left-equivariance of $\hat{f}$ is immediately verified. □

The key simplification afforded by Theorem 1 lies in the fact that the dependence on the group-valued function is removed; instead, the group itself is introduced through the product spaces. At this point, the problem is reduced to pulling back a family of one-forms to a given family of one-forms. It is important to note that since $m < n$, the family of linear forms $(\omega^i(q))_{i=1}^n$ is linearly dependent $\forall q \in M \times G$, whereas the family of linear forms $(\Omega^i(q))_{i=1}^n$ is linearly independent $\forall q \in N \times G$. Furthermore, $\forall q \in M \times G$, exactly m of the linear forms $(\omega^i(q))_{i=1}^n$ form a linearly independent family in $T_q(M \times G)$. We can exploit this redundancy in the ω^i by effecting a first reduction of the structure group G. For every $(n - m)$-tuple of indices $(i_1, \ldots, i_{n-m})$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_{n-m} \leqslant n$, and for every $q \in M$, we define the subset $G_{q,(i_1,\ldots,i_{n-m})}$ of G as follows:

$$
G_{q,(i_1,\ldots,i_{n-m})} = \big\{g \in G \mid \omega^{i_1}(q, g) = \cdots \omega^{i_{n-m}}(q, g) = 0\big\}.
$$

Proposition 1. *$\forall q \in M$, $G_{q,(m+1,\ldots,n)}$ is a Lie subgroup of G independent of $q \in M$, and denoting it by G_0, for every $(n - m)$-tuple of indices $(i_1, \ldots, i_{n-m})$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_{n-m} \leqslant n$, $G_{q,(i_1,\ldots,i_{n-m})}$ is independent of $q \in M$ and is either empty or a right coset of G_0.*

Proof. By Cartan's theorem [4], it will be sufficient to show that $G_{q,(m+1,\ldots,n)}$ is a closed subgroup of G. The fact that $G_{q,(m+1,\ldots,n)}$ is a closed subset of G follows immediately from continuity of the one-forms $\omega^{m+1}, \ldots, \omega^n$ on $M \times G$. Furthermore:

$$
\begin{aligned}
\omega &= \sum_{i=1}^m (\pi_M^\star \omega_M^i) p_M \cdot e_i \\
&= \sum_{j=1}^n \sum_{i=1}^m (\pi_M^\star \omega_M^i) \langle f^j, p_M \cdot e \rangle e_j,
\end{aligned}
$$

and therefore, since $(\omega_M^i)_{i=1}^m$ is a coframe on M and π_M is a submersion, we obtain, $\forall q \in M$:

$$G_{q,(m+1,\dots,n)} = \{g \in G \mid \langle f^j, g \cdot e_i\rangle = 0,\ \forall i = 1,\dots,m,\ \forall j = m+1,\dots,n\}.$$

Note that it follows from the above characterization of $G_{q,(m+1,\dots,n)}$ that:

$$g \in G_{q,(m+1,\dots,n)} \quad \Leftrightarrow \quad g \cdot e_i \in \operatorname{span}(e_1,\dots,e_m), \quad \forall i = 1,\dots,m.$$

Furthermore:

1. $\forall g_1, g_2 \in G_{q,(m+1,\dots,n)}$, $\forall i = 1,\dots,m$, $\forall j = m+1,\dots,n$, $\langle f^j, g_1 g_2 \cdot e_i\rangle = \langle f^j, g_1 \cdot (g_2 \cdot e_i)\rangle = 0$, as follows from the above characterization, which implies $g_1 g_2 \in G_{q,(m+1,\dots,n)}$;
2. $1 \in G_{q,(m+1,\dots,n)}$ since $\langle f^j, e_i\rangle = 0$, $\forall i = 1,\dots,m$, $\forall j = m+1,\dots,n$;
3.
$$\begin{aligned} g \in G_{q,(m+1,\dots,n)} \quad &\Leftrightarrow \quad g \cdot e_i \in \operatorname{span}(e_1,\dots,e_m), \quad \forall i = 1,\dots,m, \\ &\Leftrightarrow \quad e_i \in \operatorname{span}(g^{-1} \cdot e_1,\dots,g^{-1} \cdot e_m), \quad \forall i = 1,\dots,m, \\ &\Leftrightarrow \quad \operatorname{span}(g^{-1} \cdot e_1,\dots,g^{-1} \cdot e_m) = \operatorname{span}(e_1,\dots,e_m) \\ &\Leftrightarrow \quad g^{-1} \in G_{q,(m+1,\dots,n)}, \end{aligned}$$

   as follows immediately from the above characterization of $G_{q,(m+1,\dots,n)}$.

This shows that $G_0 = G_{q,(m+1,\dots,n)}$ is a Lie subgroup of G and is independent of $q \in M$. Consider now $G_{q,(i_1,\dots,i_{n-m})}$; either it is the empty set or there is an element $g_{(i_1,\dots,i_m)}$ of G acting on V through an automorphism permuting e_{i_1} and e_{m+1}, e_{i_2} and e_{m+2}, …, $e_{i_{n-m}}$ and e_n, and leaving all other e_i fixed; in the latter case,every $g \in G_{q,(i_1,\dots,i_{n-m})}$ can be written as $g = g_{(i_1,\dots,i_m)} g_0$ where $g_0 \in G_0$. Hence, $G_{q,(i_1,\dots,i_{n-m})}$ is either empty or a right coset of G_0. □

The key idea at this point is to reduce the original problem to one involving only the subgroup G_0; by reducing the structure group of the bundles to G_0, the problem does become simpler. The next result, the proof of which follows directly from the definitions, shows that the existence of a solution to the original problem, involving G, implies the solution of a similar problem, involving G_0. Note that there is a natural left action of G_0 on $N \times G$, inherited from the left action of G on $N \times G$, and that for each $(n-m)$-tuple of indices $(i_1,\dots,i_{n-m})$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_{n-m} \leqslant n$ and for which $G_{(i_1,\dots,i_{n-m})}$ is non-empty, $M \times G_{(i_1,\dots,i_{n-m})}$ is a (trivial) principal G_0-bundle over M.

Remark. In all that follows, we denote by ρ_0 the dimension $\dim(\mathcal{G}_0)$ of $\mathcal{G}_0$.

Proposition 2. *Assume $\hat{f} : M \times G \to N \times G$ is G-left-equivariant and satisfies $\hat{f}^\star \Omega = \omega$. Then, for each $(n-m)$-tuple of indices $(i_1,\dots,i_{n-m})$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_{n-m} \leqslant n$ and for which $G_{(i_1,\dots,i_{n-m})}$ is non-empty, the restriction of $\hat{f}$ to $M \times G_{(i_1,\dots,i_{n-m})}$ (denoted by the same symbol) is G_0-left-equivariant and satisfies $\hat{f}^\star \Omega = \omega|_{M \times G_{(i_1,\dots,i_{n-m})}}$.*

By virtue of this proposition, we can reduce the original problem to one on $M \times G_{(i_1,\dots,i_{n-m})}$. By virtue of the fact that the pullback operation and exterior derivative commute, it follows from Theorem 1 and Proposition 2 that a necessary condition for the existence of a C^∞ map $f : M \to N$ and a C^∞ map $\lambda : M \to G$ such that $f^\star \Omega_N = \lambda J \omega_M$ is that there exist, for every $(n-m)$-tuple of indices $(i_1,\dots,i_{n-m})$ such that $1 \leqslant i_1 < i_2 < \cdots < i_{n-m} \leqslant n$ and for which $G_{(i_1,\dots,i_{n-m})}$ is non-empty, a G_0-left-equivariant C^∞ map $\hat{f} : M \times G_{(i_1,\dots,i_{n-m})} \to N \times G$ such that $\hat{f}^\star d\Omega = d\omega|_{M \times G_{(i_1,\dots,i_{n-m})}}$. We shall therefore investigate the structure equations of the quasi-coframes ω and Ω, on $M \times G_{(i_1,\dots,i_{n-m})}$ and $N \times G$, respectively. We have:

$$\begin{aligned} d\omega &= d(p_M \cdot J \cdot \pi_M^\star \omega_M) \\ &= dp_M \wedge J\pi_M^\star \omega_M + p_M \cdot J\pi_M^\star\, d\omega_M \\ &= dp_M \cdot (p_M)^{-1} \wedge \omega + p_M \cdot J\pi_M^\star\, d\omega_M. \end{aligned}$$

Letting $\mathcal{G}_0$ denote the Lie subalgebra of $\mathcal{G}$ associated with the Lie subgroup G_0 of G, $dp_M \cdot (p_M)^{-1}$ is a $\mathcal{G}_0$-valued C^∞ one-form on $M \times G_0$, invariant under the right action of G_0 on $M \times G_0$. Let $(a_\rho)_\rho$ be a basis of the Lie algebra $\mathcal{G}$ of G such that the first ρ_0 elements form a basis of the Lie subalgebra $\mathcal{G}_0$. Then there exist right-invariant Maurer–Cartan forms $(\mu^\rho)_\rho$ on G_0 such that

$$dp_M \cdot p_M^{-1} = \sum_{1 \leqslant \rho \leqslant \rho_0} a_\rho p_M^\star(\mu^\rho).$$

The representation of G_0 on V induces a representation of the Lie algebra $\mathcal{G}_0$ on V, i.e. a Lie algebra homomorphism $\mathcal{G}_0 \to Hom(V, V)$. We therefore obtain:

$$d\omega = \sum_{1 \leqslant \rho \leqslant \rho_0} a_\rho p_M^\star(\mu^\rho) \wedge \omega + p_M \cdot J\pi_M^\star d\omega_M.$$

Since $\omega_M = \sum_{i=1}^m \omega_M^i \hat{e}_i$ is such that $\omega_M(q)$ is onto W for all $q \in M$, and M is m-dimensional, there exist uniquely defined C^∞ functions c^i_{jk} on M such that

$$d\omega_M^i = \sum_{1 \leqslant j<k \leqslant m} c^i_{jk} \omega_M^j \wedge \omega_M^k, \quad 1 \leqslant i \leqslant m,$$

and therefore

$$\begin{aligned} J\pi_M^\star d\omega_M &= \sum_{\substack{1 \leqslant j<k \leqslant m \\ 1 \leqslant i \leqslant m}} \big[(c^i_{jk} \circ \pi_M)\pi_M^\star \omega_M^j \wedge \pi_M^\star \omega_M^k\big] J\hat{e}_i \\ &= \sum_{\substack{1 \leqslant j<k \leqslant m \\ 1 \leqslant i \leqslant m}} \big[(c^i_{jk} \circ \pi_M)\pi_M^\star \omega_M^j \wedge \pi_M^\star \omega_M^k\big] e_i. \end{aligned}$$

We now express $\pi_M^\star \omega_M^j, \pi_M^\star \omega_M^k$ in terms of $\omega = \sum_{i=1}^n \omega^i \; e_i$. Note that

$$\begin{aligned} \omega = p_M \cdot J\pi_M^\star \omega_M \quad &\Leftrightarrow \quad J\pi_M^\star \omega_M = p_M^{-1} \cdot \omega \\ &\Leftrightarrow \quad \sum_{i=1}^m (\pi_M^\star \omega_M^i) e_i = \sum_{i=1}^n \omega^i \; p_M^{-1} \cdot e_i. \end{aligned}$$

Hence, $\forall 1 \leqslant j \leqslant m$:

$$\begin{aligned} \pi_M^\star \omega_M^j &= \left\langle f^j, \sum_{i=1}^m (\pi_M^\star \omega_M^i) \; e_i \right\rangle \\ &= \langle f^j, J\pi_M^\star \omega_M \rangle \\ &= \langle p_M \cdot f^j, \omega \rangle. \end{aligned}$$

It follows therefore that

$$\begin{aligned} p_M \cdot J\pi_M^\star d\omega_M &= \sum_{\substack{1 \leqslant j<k \leqslant m \\ 1 \leqslant i \leqslant m}} (c^i_{jk} \circ \pi_M)\big(\langle p_M \cdot f^j, \omega\rangle \wedge \langle p_M \cdot f^k, \omega\rangle\big) p_M \cdot e_i \\ &= \sum_{\substack{1 \leqslant r<s \leqslant m \\ 1 \leqslant t \leqslant m}} \gamma^t_{rs} \langle f^r, \omega\rangle \wedge \langle f^s, \omega\rangle e_t \\ &= \sum_{\substack{1 \leqslant r<s \leqslant m \\ 1 \leqslant t \leqslant m}} \gamma^t_{rs} \omega^r \wedge \omega^s e_t \end{aligned}$$

where the functions γ^t_{rs}, called torsion coefficients, are given $\forall 1 \leqslant t \leqslant m, 1 \leqslant r < s \leqslant m$ by:

$$\gamma^t_{rs} = \sum_{\substack{1 \leqslant j<k \leqslant m \\ 1 \leqslant i \leqslant m}} (c^i_{jk} \circ \pi_M)\langle f^j, p_M^{-1} \cdot e_r\rangle \langle f^k, p_M^{-1} \cdot e_s\rangle \langle p_M^{-1} \cdot f^t, e_i\rangle. \tag{1}$$

We now consider the group action on $V \otimes \bigwedge^2 V^\star$ induced from the group action on V. The following result is a direct consequence of the definition of torsion coefficients given in Eq. (1), and it highlights a desirable property of the restriction of the torsion coefficients to $M \times G_0$.

Proposition 3. *Consider the smooth function $\gamma = \sum \gamma^t_{rs} e_t \otimes f^r \wedge f^s$ defined on $M \times G_0$ with values in $V \otimes \bigwedge^2 V^\star$; we have, $\forall q \in M$, $\forall g, \hat{g} \in G_0$:*

$$g \cdot \gamma(q, g') = \gamma(q, gg').$$

Proof. We have, $\forall q \in M$, $\forall g, \hat{g} \in G_0$:

$$\begin{aligned}
g \cdot \gamma(q, \hat{g}) &= \sum \gamma_{rs}^t(q, \hat{g}) g \cdot e_t \otimes g \cdot f^r \wedge g \cdot f^s \\
&= g \cdot \Big(\sum c_{jk}^i(q) \langle f^j, \hat{g}^{-1} \cdot e_r \rangle \langle f^k, \hat{g}^{-1} \cdot e_s \rangle \langle \hat{g}^{-1} \cdot f^t, e_i \rangle e_t \otimes f^r \wedge f^s \Big) \\
&= g \cdot \Big(\sum c_{jk}^i(q) \langle \hat{g} \cdot f^j, e_r \rangle \langle \hat{g} \cdot f^k, e_s \rangle \langle f^t, \hat{g} \cdot e_i \rangle e_t \otimes f^r \wedge f^s \Big) \\
&= g \cdot \Big(\sum c_{jk}^i(q) \big(\langle f^t, \hat{g} \cdot e_i \rangle \big) e_t \otimes \big(\langle \hat{g} \cdot f^j, e_r \rangle \big) f^r \wedge \big(\langle \hat{g} \cdot f^k, e_s \rangle \big) f^s \Big) \\
&= g \cdot \Big(\sum c_{jk}^i(q) \hat{g} \cdot e_i \otimes \hat{g} \cdot f^j \wedge \hat{g} \cdot f^k \Big) \\
&= g\hat{g} \cdot \gamma(q, e),
\end{aligned}$$

where e denotes the identity element of G_0. Using once more the same chain of equalities with different arguments yields $g\hat{g} \cdot \gamma(q, e) = \gamma(q, g\hat{g})$, proving the desired result. $\square$

The structure equations for Ω are given by:

$$\begin{aligned}
d\Omega &= d(p_N \cdot \pi_N^\star \Omega_N) \\
&= dp_N \cdot p_N^{-1} \wedge \Omega + p_N \cdot \pi_N^\star \, d\Omega_N;
\end{aligned}$$

since $dp_N \cdot p_N^{-1}$ is a $\mathcal{G}$-valued C^∞ one-form on $N \times G$, invariant under the right action of G on $N \times G$, there exist right-invariant Maurer–Cartan forms $(\Pi^\rho)_\rho$ on G such that

$$dp_N \cdot p_N^{-1} = \sum_{1 \leqslant \rho \leqslant \dim(\mathcal{G})} a_\rho p_N^\star(\Pi^\rho),$$

and since $\Omega_N = \sum_{i=1}^n \Omega_N^i e_i$ is such that $\Omega_N(q)$ is onto V, $\forall q \in N$, and N is n-dimensional, there exist uniquely defined C^∞ functions C_{jk}^i on N such that

$$d\Omega_N^i = \sum_{1 \leqslant j < k \leqslant n} C_{jk}^i \Omega_N^j \wedge \Omega_N^k, \quad 1 \leqslant i \leqslant n,$$

and since $\Omega = p_N \cdot \pi_N^\star \Omega_N$, we have $\pi_N^\star \Omega_N = p_N^{-1} \cdot \Omega$, and the structure equations for Ω can be rewritten as:

$$\begin{aligned}
d\Omega &= \sum a_\rho p_N^\star(\Pi^\rho) \wedge \Omega + \sum \Gamma_{rs}^t \langle f^r, \Omega \rangle \wedge \langle f^s, \Omega \rangle e_t \\
&= \sum a_\rho p_N^\star(\Pi^\rho) \wedge \Omega + \sum \Gamma_{rs}^t \Omega^r \wedge \Omega^s e_t
\end{aligned}$$

where the torsion coefficients Γ_{rs}^t are given, $\forall 1 \leqslant t \leqslant n,\ 1 \leqslant r < s \leqslant n$, by

$$\Gamma_{rs}^t = \sum_{\substack{1 \leqslant j < k \leqslant n \\ 1 \leqslant i \leqslant n}} (C_{jk}^i \circ \pi_M) \langle f^j, p_M^{-1} \cdot e_r \rangle \langle f^k, p_M^{-1} \cdot e_s \rangle \langle p_M^{-1} \cdot f^t, e_i \rangle. \tag{2}$$

Let now $\hat{f} : M \times G_0 \to N \times G$ be a G_0-left equivariant map such that $\hat{f}^\star \Omega = \omega$; then:

$$\omega = \hat{f}^\star \Omega \quad \Rightarrow \quad d\omega = \hat{f}^\star d\Omega$$

Hence, letting $a_\rho \cdot e_j = \sum_{i=1}^n a_{j\rho}^i e_i$, we can identify $a_\rho \in \mathcal{G}$ with $\sum_{i,j=1}^n a_{j\rho}^i e_i \otimes f^j \in V \otimes V^\star \simeq Hom(V, V)$. We therefore obtain:

$$\begin{aligned}
\omega = \hat{f}^\star \Omega \quad \Rightarrow \quad d\omega^i &= \sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \rho_0}} a_{j\rho}^i p_M^\star(\mu^\rho) \wedge \omega^j + \sum_{1 \leqslant j < k \leqslant m} \gamma_{jk}^i \omega^j \wedge \omega^k \\
&= \sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \rho_0}} a_{j\rho}^i (p_N \circ \hat{f})^\star(\Pi^\rho) \wedge \omega^j + \sum_{1 \leqslant j < k \leqslant m} (\Gamma_{jk}^i \circ \hat{f}) \omega^j \wedge \omega^k, \quad i = 1, \ldots, m;
\end{aligned}$$

by Cartan's lemma, this in turn implies

$$\sum_{1 \leqslant \rho \leqslant \rho_0} a_{k\rho}^i \big[(p_N \circ \hat{f})^\star(\Pi^\rho) - p_M^\star(\mu^\rho) \big] + \sum_{1 \leqslant j \leqslant m} \big[(\Gamma_{jk}^i \circ \hat{f}) - \gamma_{jk}^i \big] \omega^j = \sum_{1 \leqslant j \leqslant m} b_{jk}^i \omega^j, \quad i, k = 1, \ldots, m.$$

with $b_{kj}^i = b_{jk}^i$.

Remark. Since $(a_\rho)_{\rho=1}^{\dim(\mathcal{G})}$ is assumed to be basis of the Lie algebra $\mathcal{G}$ such that $(a_\rho)_{\rho=1}^{\rho_0}$ be a basis of the Lie subalgebra $\mathcal{G}_0$, we have that $\forall \rho \in \{1,\dots,\rho_0\}$, $a^i_{j\rho} = 0$ whenever $i > m$ and $j \leqslant m$; furthermore, it follows from the characterization of the subgroup G_0 of G that we can assume without loss of generality that $\forall \rho \in \{\rho_0+1,\dots,\dim(\mathcal{G})\}$, $a^i_{j\rho} = 0$ for $0 \leqslant i, j \leqslant m$.

2.2. Torsion absorption

We have, for any family $(\nu^\rho_k)_{\substack{1\leqslant k\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}}$ of smooth $\mathbb{R}$-valued functions on $M \times G_0$:

$$\sum_{\substack{1\leqslant j\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho}\left(p^\star_M(\mu^\rho) + \sum_{1\leqslant k\leqslant m} \nu^\rho_k \omega^k\right)\wedge\omega^j = \sum_{\substack{1\leqslant j\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho} p^\star_M(\mu^\rho)\wedge\omega^j - \sum_{\substack{1\leqslant j<k\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} \left(a^i_{j\rho}\nu^\rho_k - a^i_{k\rho}\nu^\rho_j\right)\omega^j\wedge\omega^k.$$

Let V_m denote the vector subspace of V spanned by $e_1,\dots,e_m$; we identify its dual $V^\star_m$ with the vector subspace of $V^\star$ spanned by $f^1,\dots,f^m$. Define the vector space homomorphism

$$\mathbf{L} : \mathcal{G}_0 \otimes V^\star_m \to V_m \otimes \bigwedge^2 V^\star_m$$

by:

$$\mathbf{L}\left(\sum_{\substack{1\leqslant k\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} \nu^\rho_k a_\rho \otimes f^k\right) = -\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j<k\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} \left(a^i_{j\rho}\nu^\rho_k - a^i_{k\rho}\nu^\rho_j\right) e_i \otimes f^j \wedge f^k.$$

Denoting $-\sum_{1\leqslant\rho\leqslant\rho_0}(a^i_{j\rho}\nu^\rho_k - a^i_{k\rho}\nu^\rho_j)$ by $(\mathbf{L}(\nu))^i_{jk}$, we therefore obtain:

$$d\omega^i = \sum_{\substack{1\leqslant j\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho}\left(p^\star_M(\mu^\rho) + \sum_{1\leqslant k\leqslant m} \nu^\rho_k\omega^k\right)\wedge\omega^j + \sum_{1\leqslant j<k\leqslant m}\left[\gamma^i_{jk} - \left(\mathbf{L}(\nu)\right)^i_{jk}\right]\omega^j\wedge\omega^k, \quad i = 1,\dots,m.$$

For torsion absorption, we need to solve as many of the equations

$$\gamma^i_{jk} - \left(\mathbf{L}(\nu)\right)^i_{jk} = 0, \quad i = 1,\dots,m,\ 1 \leqslant j < k \leqslant m, \tag{3}$$

on $M \times G_0$ as possible. Similarly, we have, for any family $(\hat\nu^\rho_k)_{\substack{1\leqslant k\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}}$ of smooth $\mathbb{R}$-valued functions on $N \times G$:

$$\begin{aligned} d\Omega^i = {} & \sum_{\substack{1\leqslant j\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho}\left(p^\star_N(\Pi^\rho) + \sum_{1\leqslant k\leqslant m}\hat\nu^\rho_k\Omega^k\right)\wedge\Omega^j + \sum_{1\leqslant j<k\leqslant m}\left[\Gamma^i_{jk} - \left(\mathbf{L}(\hat\nu)\right)^i_{jk}\right]\Omega^j\wedge\Omega^k \\ & + \sum_{\substack{m+1\leqslant j\leqslant n\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho} p^\star_N(\Pi^\rho)\wedge\Omega^j + \sum_{\substack{1\leqslant j\leqslant n\\ \rho_0+1\leqslant\rho\leqslant\dim(\mathcal{G})}} a^i_{j\rho} p^\star_N(\Pi^\rho)\wedge\Omega^j \\ & + \sum_{1\leqslant j\leqslant m<k\leqslant n}\Gamma^i_{jk}\Omega^j\wedge\Omega^k + \sum_{m+1\leqslant j<k\leqslant n}\Gamma^i_{jk}\Omega^j\wedge\Omega^k, \quad i = 1,\dots,n. \end{aligned}$$

In this case we need to solve as many of the equations

$$\Gamma^i_{jk} - \left(\mathbf{L}(\hat\nu)\right)^i_{jk} = 0, \quad i = 1,\dots,m,\ 1 \leqslant j < k \leqslant m, \tag{4}$$

on $N \times G$ as possible, that is, we need to absorb the torsion coefficients Γ^i_{jk} with coefficients i, j, k ranging from 1 to m only. In the special case that all the torsion coefficients in Eqs. (3) and (4) can be absorbed, we have the following result:

Theorem 2. *Assume there exists a unique family (ν^ρ_k) of smooth $\mathbb{R}$-valued functions on $M \times G_0$ such that all of the equalities in (3) are satisfied, and a family $(\hat\nu^\rho_k)$ of smooth $\mathbb{R}$-valued functions on $N \times G$ such that all of the equalities in (4) are satisfied; if, after torsion absorption for the $d\omega$- and $d\Omega$-structure equations, we have the structure equations*

$$d\omega^i = \sum_{\substack{1\leqslant j\leqslant m\\ 1\leqslant\rho\leqslant\rho_0}} a^i_{j\rho}\theta^\rho\wedge\omega^j, \quad i = 1,\dots,m,$$

$$d\Omega^i = \sum_{\substack{1\leqslant j\leqslant n\\ 1\leqslant\rho\leqslant\dim(\mathcal{G})}} a^i_{j\rho}\Theta^\rho\wedge\Omega^j + \sum_{1\leqslant j\leqslant m<k\leqslant n}\Gamma^i_{jk}\Omega^j\wedge\Omega^k + \sum_{m+1\leqslant j<k\leqslant n}\Gamma^i_{jk}\Omega^j\wedge\Omega^k, \quad i = 1,\dots,n,$$

where, for each $\rho \in \{1,\dots,\rho_0\}$, θ^ρ is a smooth one-form on $M \times G_0$, and for each $\rho \in \{1,\dots,\dim(\mathcal{G})\}$, Θ^ρ a smooth one-form on $N \times G$, and if the left-equivariant C^∞ map $\hat{f}: M \times G \to N \times G$ satisfies $\hat{f}^\star \Omega = \omega$, then:

$$\hat{f}^\star \Theta^\rho = \theta^\rho, \quad \forall \rho \in \{1,\dots,\rho_0\}.$$

Proof. By Proposition 2, left G-equivariance of $\hat{f}$ implies left G_0-equivariance of the restriction of $\hat{f}$ to $M \times G_0$ (which we denote by the same symbol); it follows therefore that:

$$\begin{aligned}\hat{f}^\star \Omega = \omega \quad &\Rightarrow \quad \sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \rho_0}} a^i_{j\rho} \theta^\rho \wedge \omega^j = \sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \dim(\mathcal{G})}} a^i_{j\rho} \hat{f}^\star(\Theta^\rho) \wedge \omega^j, \quad i = 1,\dots,m, \\ &\Leftrightarrow \quad \sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \rho_0}} a^i_{j\rho} \big(\theta^\rho - \hat{f}^\star(\Theta^\rho)\big) \wedge \omega^j - \sum_{\substack{1 \leqslant j \leqslant m \\ \rho_0+1 \leqslant \rho \leqslant \dim(\mathcal{G})}} a^i_{j\rho} \hat{f}^\star(\Theta^\rho) \wedge \omega^j = 0, \quad i = 1,\dots,m,\end{aligned}$$

and since $a^i_{j\rho} = 0$ for $\rho \geqslant \rho_0 + 1$ and $1 \leqslant i, j \leqslant m$, we obtain

$$\sum_{\substack{1 \leqslant j \leqslant m \\ 1 \leqslant \rho \leqslant \rho_0}} a^i_{j\rho} \big(\theta^\rho - \hat{f}^\star(\Theta^\rho)\big) \wedge \omega^j = 0, \quad i = 1,\dots,m,$$

from which it follows that there exists a family (α^ρ_k) of smooth $\mathbb{R}$-valued functions on $M \times G_0$ such that

$$\theta^\rho - \hat{f}^\star(\Theta^\rho) = \sum_{k=1}^m \alpha^\rho_k \omega^k, \quad \rho = 1,\dots,\rho_0,$$

and by the assumption that there is a unique family (ν^ρ_k) that satisfies all the equalities in (3), we must have $\alpha^\rho_k = 0$ for all k, ρ, which gives the desired conclusion.

Remark. Theorem 2 is the counterpart of the "unique torsion absorption" theorem in Cartan's method of equivalence and its importance lies in the fact that the necessary condition $\hat{f}^\star \Theta^\rho = \theta^\rho$ leads in turn to the necessary condition $\hat{f}^\star d\Theta^\rho = d\theta^\rho$ which can be exploited through the integrability conditions $dd\omega = 0$ and $dd\Omega = 0$, exactly along the lines of what Gardner [3] calls an "abstract computation."

2.3. Structure group reduction

We now study the case when not all of the torsion coefficients can be absorbed. As with Cartan's equivalence method, the goal is to normalize the torsion and reduce the structure group to the isotropy subgroup of the normalized value (compare with [3], Chapter 4).

Consider the G_0-action on $\mathcal{G}_0$ induced from the G_0-action on V_m under the identification $a_\rho \mapsto \sum a^i_{j\rho} e_i \otimes f^j$. If, for $g \in G_0$ and $a_\rho \in \mathcal{G}_0$, we let $g \cdot a_\rho = \sum \alpha^\tau_\rho a_\tau$, $g \cdot e_i = \sum A^r_i e_r$, and $g \cdot f^j = \sum B^j_k f^k$, then, under the identification of $g \cdot a_\rho$ with $g \cdot (\sum a^i_{j\rho} e_i \otimes f^j)$, we have:

$$g \cdot \Big(\sum a^i_{j\rho} e_i \otimes f^j\Big) = \sum a^i_{j\rho} A^r_i B^j_k e_r \otimes f^k = \sum \alpha^\tau_\rho a^r_{k\tau} e_r \otimes f^k,$$

which yields the equality

$$\sum \alpha^\tau_\rho a^r_{k\tau} = \sum a^i_{j\rho} A^r_i B^j_k.$$

Proposition 4. *The mapping $\mathbf{L}: \mathcal{G}_0 \otimes V^\star_m \to V_m \otimes \bigwedge^2 V^\star_m$ is a homomorphism of G_0-modules.*

Proof. We have, $\forall \{\nu^\rho_k\} \subset \mathbb{R}$, $\forall g \in G_0$:

$$\begin{aligned}\mathbf{L}\Big(g \cdot \Big(\sum \nu^\rho_k a_\rho \otimes f^k\Big)\Big) &= \mathbf{L}\Big(\sum \nu^\rho_k g \cdot a_\rho \otimes g \cdot f^k\Big) \\ &= \sum (a^i_{j\rho} A^r_i B^j_s B^k_t \nu^\rho_k - a^i_{k\rho} A^r_i B^k_s B^j_t \nu^\rho_j) e_r \otimes f^s \wedge f^t \\ &= \sum (a^i_{j\rho} \nu^\rho_k - a^i_{k\rho} \nu^\rho_j) g \cdot (e_i \otimes f^j \wedge f^k) = g \cdot \mathbf{L}\Big(\sum \nu^\rho_k a_\rho \otimes f^k\Big).\end{aligned}$$

which proves the desired result.

It follows that the quotient mapping $\xi: V_m \otimes \bigwedge^2 V^\star_m \to V_m \otimes \bigwedge^2 V^\star_m / Im(\mathbf{L})$ is also a homomorphism of G_0-modules. The following result is a direct consequence of this observation and of Proposition 3:

Proposition 5. *Restricted to each fiber of $M \times G_0$, the $V_m \otimes \bigwedge^2 V_m^\star / Im(\mathbf{L})$-valued function $\xi \circ \gamma$, where $\gamma = \sum \gamma_{rs}^t e_t \otimes f^r \wedge f^s$ with γ_{rs}^t defined in (1), is a homomorphism of G_0-modules. Hence, restricted to each fiber of $M \times G_0$, $\xi \circ \gamma$ takes values in an orbit of the action of G_0 on $V_m \otimes \bigwedge^2 V_m^\star / Im(\mathbf{L})$.*

Consider now the projection $p_m : V \otimes \bigwedge^2 V^\star \to V_m \otimes \bigwedge^2 V_m^\star$ defined by

$$p_m\Bigg(\sum_{\substack{1 \leqslant j<k \leqslant n \\ 1 \leqslant i \leqslant n}} b_{jk}^i e_i \otimes f^j \wedge f^k\Bigg) = \sum_{\substack{1 \leqslant j<k \leqslant m \\ 1 \leqslant i \leqslant m}} b_{jk}^i e_i \otimes f^j \wedge f^k.$$

Note that p_m is not necessarily G_0-equivariant; this leads us to define the subgroup $G_0^\star$ of G_0 by:

$$G_0^\star = \big\{ g \in G_0 \mid \langle f^i, g \cdot e_j \rangle = 0, \ \forall\ 1 \leqslant i \leqslant m, \ m+1 \leqslant j \leqslant n \big\}.$$

By $G_0^\star$-equivariance of p_m, it follows that, restricted to each fiber of $N \times G$, the $V_m \otimes \bigwedge^2 V_m^\star / Im(\mathbf{L})$-valued function $\xi \circ p_m \circ \Gamma$, where $\Gamma = \sum \Gamma_{rs}^t \Omega^r \wedge \Omega^s e_t$ with Γ_{rs}^t defined in (2), is also a homomorphism of $G_0^\star$-modules.

Definition 1.

1. The $V_m \otimes \bigwedge^2 V_m^\star / Im(\mathbf{L})$-valued function $\tau = \xi \circ \gamma$ on $M \times G_0$ is called the intrinsic torsion of the quasi-coframe ω.
2. The $V_m \otimes \bigwedge^2 V_m^\star / Im(\mathbf{L})$-valued function $\Upsilon = \xi \circ p_m \circ \Gamma$ on $N \times G$ is called the intrinsic torsion of the quasi-coframe Ω.
3. Problem (P) is called regular if $\forall \tau_0 \in \tau(M \times G_0^\star) \cap \Upsilon(N \times G)$, $\tau^{-1}(\tau_0)$ is a submanifold of $M \times G_0^\star$ submersing onto M, and $\Upsilon^{-1}(\tau_0)$ a submanifold of $N \times G$ submersing onto N.

Remark. Let $G_0^{\tau_0}$ denote the isotropy subgroup of τ_0 in $G_0^\star$; Proposition 5 implies that $\tau^{-1}(\tau_0)$ and $\Upsilon^{-1}(\tau_0)$ inherit a $G_0^{\tau_0}$-action from the $G_0^\star$-action on $M \times G_0^\star$ and $N \times G$, respectively.

Theorem 3.

(a) *If $\tau(M \times G_0) \cap \Upsilon(N \times G) = \emptyset$, then there is no solution to Problem (P),*

(b) *Assume Problem (P) is regular; if $\tau(M \times G_0^\star) \cap \Upsilon(N \times G) \neq \emptyset$ and $\tau_0 \in \tau(M \times G_0^\star) \cap \Upsilon(N \times G)$, and if there exists a smooth G-left-equivariant map $\hat{f} : M \times G \to N \times G$ that satisfies $\hat{f}^\star \Omega = \omega$, then the restriction $\hat{f}|_{\tau^{-1}(\tau_0)} : \tau^{-1}(\tau_0) \to \Upsilon^{-1}(\tau_0)$ of $\hat{f}$ to $\tau^{-1}(\tau_0)$ is $G_0^{\tau_0}$-left-equivariant and satisfies $\hat{f}|^\star_{\tau^{-1}(\tau_0)} \Omega|_{\Upsilon^{-1}(\tau_0)} = \omega|_{\tau^{-1}(\tau_0)}$.*

Proof. Assume $\tau(M \times G_0) \cap \Upsilon(N \times G) = \emptyset$ and assume that there exists a smooth G-left-equivariant map $\hat{f} : M \times G \to N \times G$ that satisfies $\hat{f}^\star \Omega = \omega$. We must then have $\hat{f}^\star d\Omega = d\omega$, which then implies $\xi \circ \gamma = \xi \circ p_m \circ \Gamma \circ \hat{f}$, which implies $\tau(M \times G_0) \cap \Upsilon(N \times G) \neq \emptyset$, which contradicts the assumption; this proves (a). To prove (b), note first that it follows from the proof of (a) that $\forall \tau_0 \in \tau(M \times G_0^\star) \cap \Upsilon(N \times G)$, $\hat{f}|_{\tau^{-1}(\tau_0)}(\tau^{-1}(\tau_0)) \subset \Upsilon^{-1}(\tau_0)$. The desired result follows trivially by restriction of the forms on both sides of the equality $\hat{f}^\star \Omega = \omega$ to $\tau^{-1}(\tau_0)$.

Remark. Under the condition of (b) of Theorem 3, Problem (P) leads to Problem (P_{τ_0}): Find a $G_0^{\tau_0}$-left-equivariant map $\hat{g} : \tau^{-1}(\tau_0) \to \Upsilon^{-1}(\tau_0)$ such that $\hat{g}^\star \Omega|_{\Upsilon^{-1}(\tau_0)} = \omega|_{\tau^{-1}(\tau_0)}$. As with Cartan's method of equivalence, the goal is to reduce the structure group as much as possible. When the structure group is reduced to the identity, no group parameters are involved anymore and no Lie algebra-compatible torsion absorption is feasible; the intrinsic torsion of the quasi-coframes is then the same as their torsion. In the simplest case, that of constant torsion, equality of corresponding torsions of the quasi-coframes provides the desired necessary conditions. When the torsions are not constant, the problem is to identify the functional dependencies between them. This is done exactly as in Cartan's method of equivalence, as detailed in [3,5].

3. Examples

We illustrate the proposed extension to Cartan's equivalence method on two examples:

(a) Let $M \subset \mathbb{R}^3$ an open subset of $\mathbb{R}^3$ with coordinates (x, y, z), and let $N \subset \mathbb{R}^4$ an open subset of $\mathbb{R}^4$ with coordinates (X, Y, Z, W). Consider the rank 2 distribution $\mathcal{D}_1$ on M defined by $dz - (x\,dy - y\,dx) = 0$, and the rank 2 distribution $\mathcal{D}_2$ on N defined by $dZ = dW = 0$. Does there exist an immersion $f : M \to N$ that maps $\mathcal{D}_1$ to $\mathcal{D}_2$? In order to express this problem in terms of suitable adapted coframes, we define the one-forms

$$\omega_M^1 = dx, \quad \omega_M^2 = dy, \quad \omega_M^3 = dz - (x\,dy - y\,dx),$$

on M, and the one-forms

$$\Omega_N^1 = dX, \quad \Omega_N^2 = dY, \quad \Omega_N^3 = dZ, \quad \Omega_N^4 = dW,$$

on N. As seen in Problem (a) of Section 1, we need to consider the subgroup

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A, C \in GL(2, \mathbb{R}) \right\}$$

of $GL(4, \mathbb{R})$. Proceeding as detailed in Section 2, we have a structure equation of the form

$$d\omega = \begin{pmatrix} d\omega^1 \\ d\omega^2 \\ d\omega^3 \\ d\omega^4 \end{pmatrix} = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pi_M^\star \begin{pmatrix} d\omega_M^1 \\ d\omega_M^2 \\ d\omega_M^3 \end{pmatrix}$$
$$= \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \pi_M^\star(2\,dx \wedge dy) \\ 0 \end{pmatrix}$$

for ω on $M \times G$, and a structure equation of the form

$$d\Omega = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix}$$

for Ω on $N \times G$. The subgroup G_0 of G fixing ω^4 to 0 is given by

$$G_0 = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A, C \in GL(2, \mathbb{R}),\ C \text{ upper triangular} \right\}.$$

It follows from the characterization of G_0 that there exists a smooth non-vanishing function a on $M \times G_0$ such that $\pi_M^\star(2dx \wedge dy) = a\omega^1 \wedge \omega^2$; hence, the structure equation for ω on $M \times G_0$ becomes (omitting ω^4):

$$d\omega = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ a\omega^1 \wedge \omega^2 \end{pmatrix},$$

from which it follows that the ω-structure equation has a non-vanishing torsion term ($a\omega^1 \wedge \omega^2$) which cannot be absorbed in the (truncated) Maurer–Cartan matrix due to the presence of the zero block; hence, that torsion is intrinsic. On the other hand, the Ω-structure equation has zero intrinsic torsion. Applying Theorem 3, we are led to a contradiction. Hence, there exists no immersion $f : M \to N$ mapping distribution $\mathcal{D}_1$ to distribution $\mathcal{D}_2$.

(b) Let $M \subset \mathbb{R}^2$ an open subset of $\mathbb{R}^2$ with coordinates (x, y), and let $N \subset \mathbb{R}^3$ an open subset of $\mathbb{R}^3$ with coordinates (X, Y, Z). Consider the rank 2 distribution $\mathcal{D}_1$ on M defined by TM itself, and the rank 2 distribution $\mathcal{D}_2$ on N defined by $dZ - Z\,dX = 0$. Consider the Riemannian metric g_1 on $\mathcal{D}_1$ defined by $g_1 = dx^2 + dy^2$, and the (sub-)Riemannian metric g_2 on $\mathcal{D}_2$ defined by $g_2 = (1 + Y^2)^2\,dX^2 + dY^2$. Does there exist an immersion $f : M \to N$ that maps $\mathcal{D}_1$ to $\mathcal{D}_2$ and such that the Riemannian metrics are preserved, i.e. such that $f^\star g_2 = g_1$? In order to express this problem in terms of suitable adapted coframes, we define the one-forms

$$\omega_M^1 = dx, \quad \omega_M^2 = dy,$$

on M, and the one-forms

$$\Omega_N^1 = (1 + Y^2)\,dX, \quad \Omega_N^2 = dY, \quad \Omega_N^3 = dZ - Z\,dX,$$

on N. As seen in Problem (b) of Section 1, we need to consider the subgroup

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} \mid A \in O(2, \mathbb{R}), c \in \mathbb{R}^\star \right\}$$

of $GL(3, \mathbb{R})$. Note that the subgroup G_0 of G fixing ω^3 to 0 is G itself. Proceeding as detailed in Section 2, we have a structure equation of the form

$$d\omega = \begin{pmatrix} d\omega^1 \\ d\omega^2 \\ d\omega^3 \end{pmatrix} = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \pi_M^\star \begin{pmatrix} d\omega_M^1 \\ d\omega_M^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \alpha & \star \\ -\alpha & 0 & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

for ω on $M \times G$, with α (the pullback, via p_M, of) a Maurer–Cartan form on $O(2;\mathbb{R})$, and a structure equation of the form

$$d\Omega = \begin{pmatrix} 0 & \mu & \star \\ -\mu & 0 & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \end{pmatrix} + \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} \begin{pmatrix} -2Y\,dX \wedge dY \\ 0 \\ 0 \end{pmatrix}$$

for Ω on $N \times G$, with μ the (pullback, via p_N, of) a Maurer–Cartan form on $O(2;\mathbb{R})$. We have:

$$dX \wedge dY \equiv \frac{1}{1+Y^2} \Omega^1 \wedge \Omega^2 \mod \Omega^3,$$

and, with the parametrization

$$\begin{pmatrix} A & B \\ 0 & c \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & b_1 \\ \sin t & \cos t & b_2 \\ 0 & 0 & c \end{pmatrix},$$

we can write the Ω-structure equation as:

$$\begin{aligned} d\Omega &= \begin{pmatrix} 0 & \mu & \star \\ -\mu & 0 & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \end{pmatrix} + \begin{pmatrix} -\frac{2Y}{1+Y^2}\cos t\,\Omega^1 \wedge \Omega^2 + (0 \bmod \Omega^3) \\ -\frac{2Y}{1+Y^2}\sin t\,\Omega^1 \wedge \Omega^2 + (0 \bmod \Omega^3) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \lambda & \star \\ -\lambda & 0 & \star \\ 0 & 0 & \star \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \end{pmatrix} + \begin{pmatrix} (0 \bmod \Omega^3) \\ (0 \bmod \Omega^3) \\ 0 \end{pmatrix}, \end{aligned}$$

where

$$\lambda = \mu - \left(\frac{2Y}{1+Y^2}\cos t\right)\Omega^1 - \left(\frac{2Y}{1+Y^2}\sin t\right)\Omega^2.$$

Since the one-form α is uniquely determined, it follows from Theorem 2 that if there did exist an immersion with the desired properties, λ would have to pull back to α under a left-equivariant map. However, we have:

$$d\alpha = d(dt) = 0,$$

$$d\lambda \equiv \frac{2}{1+Y^2}\Omega^1 \wedge \Omega^2 \mod \Omega^3,$$

i.e. α is closed whereas λ is not. We conclude that there cannot exist an immersion $f: M \to N$ that maps $\mathcal{D}_1$ to $\mathcal{D}_2$ and such that the Riemannian metrics are preserved.

4. Conclusion

In this paper, we have proposed an extension of Cartan's method of equivalence to immersions; more specifically, we have shown how the basic steps in Cartan's method, starting with the encoding of the desired geometric structures in terms of suitable moving coframes, can be carried out in order to find obstructions to the existence of immersions between given geometric structures. It would be interesting to relate the results on Riemannian immersions that could be obtained with the method proposed here to known results on Riemannian immersions [2].

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