

Lie point symmetries and first integrals: The Kowalevski top

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We show how the Lie group analysis method can be used in order to obtain first integrals of any system of ordinary differential equations. The method of reduction/increase of order developed by Nucci [J. Math. Phys. **37**, 1772–1775 (1996)] is essential. Noether's theorem is neither necessary nor considered. The most striking example we present is the relationship between Lie group analysis and the famous first integral of the Kowalevski top. © 2003 American Institute of Physics.
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I. INTRODUCTION

In January 2001, the first Whiteman prize for notable exposition on the history of mathematics was awarded to Thomas Hawkins by the American Mathematical Society. In the citation, published in the Notices of AMS **48**, 416–417 (2001), one reads that Thomas Hawkins "... has written extensively on the history of Lie groups. In particular, he has traced their origins to [Lie's] work in the 1870s on differential equations ... the *idée fixe* guiding Lie's work was the development of a Galois theory of differential equations ... [Hawkins's book¹⁰] highlights the fascinating interaction of geometry, analysis, mathematical physics, algebra, and topology ..."

In the Introduction of his book,³⁶ Olver wrote that "it is impossible to overestimate the importance of Lie's contribution to modern science and mathematics. Nevertheless, anyone who is already familiar with [it] ... is perhaps surprised to know that its original inspirational source was the field of differential equations."

Lie's monumental work on transformation groups,^{20–22} and in particular contact transformations,²³ led him to achieve his goal.²⁴ Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations. Any known integration technique can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, i.e., the Lie symmetry algebra. In particular, Bianchi's theorem^{2,36} states that if an admitted n -dimensional solvable Lie symmetry algebra is found, then the general solution of the corresponding n order system of ordinary differential equations can be obtained by quadratures. The admitted Lie symmetry algebra can be easily derived by a straightforward although lengthy procedure. As computer algebra softwares become widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is getting easier to carry out.

A major drawback of Lie's method is that it is useless when applied to systems of n first order equations, because they admit an infinite number of symmetries, and there is no systematic way to find even one-dimensional Lie symmetry algebra, apart from trivial groups like translations in time admitted by autonomous systems. One may try to derive an admitted n -dimensional solvable Lie symmetry algebra by making an ansatz on the form of its generators.

However, Nucci³⁰ has remarked that any system of n first order equations could be transformed into an equivalent system where at least one of the equations is of second order. Then, the

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admitted Lie symmetry algebra is no longer infinite dimensional, and nontrivial symmetries of the original system could be retrieved.³⁰ This idea has been successfully applied in several instances.^{30,43,5,40,32,33,35,31}

Here we show another striking application of such an idea. If we consider a system of first order equations, and by eliminating one of the dependent variables derive an equivalent system which has one equation of second order, then Lie group analysis applied to that equivalent system yields the first integral(s) of the original system which does not contain the eliminated dependent variable. Of course, if such first integrals exist. The procedure should be repeated on as many times as there are dependent variables in order to find all such first integrals. The first integrals correspond to the characteristic curves of determining equations of parabolic type⁴² which are constructed by the method of Lie group analysis.

We would like to remark that interactive (not automatic) programs for calculating Lie point symmetries such as Refs. 28 and 29 are most appropriate for performing this task.

It is well known that if one finds a transformation which leaves invariant a functional describing a variational problem, then Noether's theorem²⁷ provides a first integral of the corresponding Euler–Lagrange system. Unfortunately, a general method for finding such a transformation does not exist. In addition, many equations of physical interest (e.g., Lorenz system in meteorology²⁵) do not come from a variational problem. On the contrary, our method can be applied to any system of ordinary differential equations, even if they do not derive from a variational problem,³¹ and we do not make any a priori hypothesis on the form of the first integrals, apart missing one of the unknowns.

In the next section, we describe the method in detail, in Secs. III and IV, we present the classical example of the Kowalevski top, and in Sec. V the three-dimensional Kepler problem in Cartesian coordinates. The last section contains some final comments.

II. OUTLINE OF THE METHOD

Let us consider the following autonomous (which could also be nonautonomous) system of N first order ordinary differential equations:

$$\begin{aligned}\dot{w}_1 &= F_1(w_1, w_2, \dots, w_N), \\ \dot{w}_2 &= F_2(w_1, w_2, \dots, w_N), \\ &\dots, \\ \dot{w}_N &= F_N(w_1, w_2, \dots, w_N).\end{aligned}\tag{1}$$

Let

$$I = I(w_1, w_2, \dots, w_{s-1}, w_{s+1}, \dots, w_N),\tag{2}$$

be a first integral which does not depend on w_s , and

$$X = V(t, w_1, \dots, w_N) \partial_t + \sum_{k=1}^N G_k(t, w_1, \dots, w_N) \partial_{w_k}\tag{3}$$

be a generator of a Lie point symmetry group for (1). If we derive w_s from one of the equations (1), say the first, then we obtain a system of $N-2$ equations of first order in $w_2, \dots, w_{s-1}, w_{s+1}, \dots, w_N$ and one of second order in w_1 . We remark that the method does not depend on the equation we choose from (1) to derive w_s . After introducing the new notation u_j ($j=1, \dots, N-1$), we can write the system we obtain as

$$\begin{aligned} \ddot{u}_1 &= f_1(u_1, u_2, \dots, u_{N-1}, \dot{u}_1), \\ \dot{u}_2 &= f_2(u_1, u_2, \dots, u_{N-1}, \dot{u}_1), \\ &\dots, \\ \dot{u}_{N-1} &= f_{N-1}(u_1, u_2, \dots, u_{N-1}, \dot{u}_1). \end{aligned} \tag{4}$$

A generator of a Lie point symmetry group for (4) is

$$\bar{X} = \bar{V}(t, u_1, \dots, u_{N-1}) \partial_t + \sum_{j=1}^{N-1} \bar{G}_j(t, u_1, \dots, u_{N-1}) \partial_{u_j}. \tag{5}$$

If we apply Lie group analysis to system (4) using the interactive REDUCE programs developed by Nucci,^{28,29} then we obtain a determining equation of parabolic type for V . Its characteristic curves will yield $m < N - 1$ transformations, which eliminate \dot{u}_1 from all the first order equations in (4). Thus, we have obtained a system of $N - 2$ equations of first order and one equation of second order in the new dependent variables \bar{u}_j such that $u_1 = \bar{u}_1$ and each of the other variables \bar{u}_j are either the original u_j itself, if \dot{u}_1 did not appear in the j -equation of system (4), or the corresponding characteristic curve. If we apply Lie group analysis to this final system, then again a determining equation of parabolic type will be derived, and its characteristic curve, when rewritten in the original variables, will be exactly the first integral (2).

Now let us consider a system of M second order ordinary differential equations

$$\ddot{x}_i = H_i(x_1, \dots, x_M, \dot{x}_1, \dots, \dot{x}_M) \quad (i = 1, \dots, M). \tag{6}$$

A generator of a Lie point symmetry group for this system has the form

$$\Gamma = \tau(t, x_1, \dots, x_M) \partial_t + \sum_{i=1}^M \eta_i(t, x_1, \dots, x_M) \partial_{x_i}. \tag{7}$$

System (6) can be converted into the following autonomous system of $2M$ first order ordinary differential equations:

$$\begin{aligned} \dot{w}_i &= w_{M+i}, \\ \dot{w}_{M+i} &= H_i(w_1, \dots, w_M, w_{M+1}, \dots, w_{2M}). \end{aligned} \tag{8}$$

At this point, we could either proceed as indicated above or choose one of the dependent variables to be the new independent variable y in order to reduce the order of system (8) by one.³⁰ For example, we could take $x_M \equiv w_M = y$. Then, system (8) becomes the following nonautonomous system of $2M - 1$ first order ordinary differential equations with independent variable y :

$$\begin{aligned} \frac{d}{dy} w_h &= w_{M+h} / w_{2M}, \\ \frac{d}{dy} w_{M+h} &= H_h(w_1, \dots, w_{M-1}, y, w_{M+1}, \dots, w_{2M}) / w_{2M}, \end{aligned} \tag{9}$$

where $h = 1, \dots, M - 1$. Now, our method can be applied to this system as if it was system (1). The fact that system (9) is not autonomous does not effect the result, as we will show in the case of the three dimensional Kepler problem in Cartesian coordinates.

The same method can be applied to a single ordinary differential equation of order N which can be easily transformed into a system of N equations of first order. It should be noticed that there

could be several different ways of transforming an equation of order N into a system of N equations of first order. Then, the just described method may give different results, videlicet (*viz.*) no first integrals with certain reductions, all the first integrals with different reductions.

III. FINDING THE KOWALEVSKI TOP

The motion of a heavy rigid point about a fixed point is one of the most famous problems of classical mechanics.⁷ In 1750, Euler⁶ derived the equations of motion which now bear his name, and described what is nowadays known as the Euler–Poinsoot case because of the geometrical description given by Poinsoot about 100 years later.³⁸ It was Jacobi¹³ who integrated this case by using the elliptic functions which he had developed (along with Legendre, Abel, and Gauss²⁶) and mastered¹⁴ (we have translated this fundamental text into Italian and extensively commented⁴¹). Another case was described by Lagrange,¹⁹ and it is known as the Lagrange–Poisson case, due to the extensive study done later by Poisson.³⁹ This case can also be integrated by using Jacobi elliptic functions.⁴⁴ At the time, it seemed that other cases could easily be found and similarly integrated. In 1855, the Prussian Academy of Science proposed this topic for a competition, but nobody applied.⁴ The problem was so elusive that the German mathematicians called it the mathematical mermaid (*die mathematische Nixe*).¹⁷ More than 30 years elapsed before the Bordin prize was awarded to Kowalevski for finding and reducing to hyperelliptic quadratures the third case¹⁶ which is since known as the Kowalevski top. She solved the problem by looking for solutions which are single-valued meromorphic functions in the entire complex plane of the variable t .⁷ Her method became what is now known as the Painlevé–Kowalevski (or just Painlevé) method.¹² Hawkins had established “the nature and extent of Jacobi’s influence upon Lie.”⁹ It is a remarkable coincidence that the mathematical mermaid can also be found by using Lie group analysis as we show in the following.

The Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point are¹⁶

$$\begin{aligned} \dot{p} &= ((B-C)rq + mg(\beta z_G - \gamma y_G))/A, \\ \dot{q} &= ((C-A)pr + mg(\gamma x_G - \alpha z_G))/B, \\ \dot{r} &= ((A-B)pq + mg(\alpha y_G - \beta x_G))/C, \\ \dot{\alpha} &= \beta r - \gamma q, \\ \dot{\beta} &= \gamma p - \alpha r, \\ \dot{\gamma} &= \alpha q - \beta p, \end{aligned} \tag{10}$$

with A , B , C the principal moments of inertia, $p(t)$, $q(t)$, $r(t)$ the components of the angular velocity, m the mass of the body, g the acceleration of gravity, x_G , y_G , z_G the coordinates of the center of mass, and $\alpha(t)$, $\beta(t)$, $\gamma(t)$ the component of the unit vertical vector. There are three first integrals for system (10): conservation of energy, i.e.,

$$I_1 = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + mg(x_G\alpha + y_G\beta + z_G\gamma) \tag{11}$$

conservation of the vertical component of the angular momentum, i.e.,

$$I_2 = Ap\alpha + Bq\beta + Cr\gamma \tag{12}$$

the length of the unit vertical vector, i.e.,

$$I_3 = \alpha^2 + \beta^2 + \gamma^2 (= 1). \tag{13}$$

If we apply our method to system (10), then we find only the first integral of the unit vertical vector which has p, q, r as missing variables. Kowalevski found that if one imposes the following conditions on the parameters:

- (1) $A = B = 2C$,
 - (2) $z_G = 0$, and either $x_G \neq 0$ or $y_G \neq 0$
- then there exists a fourth integral, i.e.,

$$I_4 = \left(p^2 - q^2 - mg \frac{x_G \alpha - y_G \beta}{C} \right)^2 + \left(2pq - mg \frac{x_G \beta + y_G \alpha}{C} \right)^2. \tag{14}$$

We notice that γ and r are missing in (14). Thanks to our method, we can find the Kowalevski top by searching for a first integral which does not contain γ . First we derive γ from the second equation of system (10), i.e.,

$$\gamma = \frac{B\dot{q} + (A - C)pr + mgz_G\alpha}{mgx_G}$$

which implies that x_G must be different from zero. We obtain the following system of four equations of first order, and one of second order:

$$\begin{aligned} \ddot{u}_1 = & \dot{u}_1(Au_1z_G + (A - C)u_3y_G)/Ax_G - (A - B)(A - C)u_1u_2^2/BC + (A - C)^2y_Gu_2u_3^2/ABx_G \\ & + (A - C)u_1u_2u_3z_G/Bx_G - (A - C)(B - C)u_1u_3^2/AB - (Au_2x_G - Cu_3z_G) \\ & \times (A - C)mgy_Gu_4/ABCx_G + (A(A - 2C)u_2x_G + C(C - 2A)u_3z_G)mgu_5/ABC \\ & + (x_G^2 + z_G^2)mgu_1u_4/Bx_G, \end{aligned} \tag{15}$$

$$\ddot{u}_2 = -\dot{u}_1By_G/Ax_G + u_3((C - A)y_Gu_2 + (B - C)x_Gu_1)/Ax_G + mgz_G(-u_4y_G + u_5x_G)/Ax_G, \tag{16}$$

$$\dot{u}_3 = ((A - B)u_1u_2 + mg(u_4y_G - u_5x_G))/C, \tag{17}$$

$$\dot{u}_4 = -\dot{u}_1Bu_1/mgx_G + (C - A)u_1u_2u_3/mgx_G + (u_3u_5x_G - u_1u_4z_G)/x_G, \tag{18}$$

$$\dot{u}_5 = \dot{u}_1Bu_2/mgx_G + (A - C)u_2^2u_3/mgx_G + z_Gu_2u_4/x_G - u_3u_4 \tag{19}$$

with

$$u_1 = q, \quad u_2 = p, \quad u_3 = r, \quad u_4 = \alpha, \quad u_5 = \beta. \tag{20}$$

Now we apply Lie group analysis to system (15)–(19). An operator Γ

$$\Gamma = V(t, u_1, u_2, u_3, u_4, u_5) \partial_t + \sum_{k=1}^5 G_k(t, u_1, u_2, u_3, u_4, u_5) \partial_{u_k} \tag{21}$$

is said to generate a Lie point symmetry group if its second prolongation

$$\Gamma_2 = \Gamma + \sum_{k=1}^5 \left(\frac{dG_k}{dt} - \dot{u}_k \frac{dV}{dt} \right) \partial_{\dot{u}_k} + \left(\frac{d}{dt} \left(\frac{dG_1}{dt} - \dot{u}_1 \frac{dV}{dt} \right) - \ddot{u}_1 \frac{dV}{dt} \right) \partial_{\ddot{u}_1}$$

applied to system (15)–(19), on their solutions, is identically equal to zero, i.e.,

$$\Gamma_2(15)|_{(15)-(19)} = 0,$$

$$\Gamma_2(16)|_{(15)-(19)} = 0,$$

$$\Gamma(17)|_{(15)-(19)} = 0, \tag{22}$$

$$\Gamma(18)|_{(15)-(19)} = 0,$$

$$\Gamma(19)|_{(15)-(19)} = 0.$$

The five determining equations (22) constitute an overdetermined system of linear partial differential equations in the unknowns $V, G_k (k=1,5)$. In fact, they are polynomials in \dot{u}_1 , each coefficient of which must become identically equal to zero. In particular, the first determining equation in (22) is a polynomial of degree three for \dot{u}_1 . The coefficient of highest degree yields an equation of parabolic type for V in four independent variables u_1, u_2, u_4, u_5 , i.e.,

$$\begin{aligned} &A^2 m g^2 x_G^2 \frac{\partial^2 V}{\partial u_1^2} - 2 A B m^2 g^2 x_G y_G \frac{\partial^2 V}{\partial u_1 u_2} - 2 A^2 B m g u_1 x_G \frac{\partial^2 V}{\partial u_1 u_4} + 2 A^2 B m g u_2 x_G \frac{\partial^2 V}{\partial u_1 u_5} \\ &+ B^2 m^2 g^2 y_G^2 \frac{\partial^2 V}{\partial u_2^2} + 2 A B^2 m g u_1 y_G \frac{\partial^2 V}{\partial u_2 u_4} - 2 A B^2 m g u_2 y_G \frac{\partial^2 V}{\partial u_2 u_5} + A^2 B^2 u_1^2 \frac{\partial^2 V}{\partial u_4^2} \\ &- 2 A^2 B^2 u_1 u_2 \frac{\partial^2 V}{\partial u_4 u_5} + A^2 B^2 u_2^2 \frac{\partial^2 V}{\partial u_5^2} - A^2 B m g x_G \frac{\partial V}{\partial u_4} - A B^2 m g y_G \frac{\partial V}{\partial u_5} = 0. \end{aligned} \tag{23}$$

Its three characteristic curves yield the following transformations:

$$\begin{aligned} u_2 &= s_2 - \frac{B u_1 y_G}{A x_G} \quad \text{viz.} \quad p = s_2 - \frac{B q y_G}{A x_G}, \\ u_4 &= s_4 - \frac{B u_1^2}{2 m g x_G} \quad \text{viz.} \quad \alpha = s_4 - \frac{B q^2}{2 m g x_G}, \\ u_5 &= s_5 + B u_1 \frac{B y_G u_1 + 2 A x_G u_2}{2 A m g x_G^2} \quad \text{viz.} \quad \beta = s_5 + B q \frac{B y_G q + 2 A x_G p}{2 A m g x_G^2} \end{aligned} \tag{24}$$

with s_2, s_4 , and s_5 new unknown functions of t .

As outlined in Sec. II, transformations (24) eliminate \dot{u}_1 from all the first order equations in system (15)–(19).

In fact, system (15)–(19) becomes

$$\begin{aligned} \ddot{u}_1 &= (-6 A^2 B C u_1 \tilde{u}_2^2 x_G^2 - 4 A^2 B C u_1 x_G \tilde{u}_2 u_3 z_G - 2 A^2 B m g u_1 x_G^2 y_G \tilde{u}_5 - 2 A^2 C^2 u_3 x_G u_1 \tilde{u}_2 z_G \\ &+ 2 A^2 B m g u_1 x_G y_G^2 \tilde{u}_4 - 3 A^2 B C u_1^2 y_G u_3 z_G + 2 A^2 B C u_1 u_3 x_G y_G - 5 A^2 B C u_1^2 y_G \tilde{u}_2 x_G \\ &+ 3 A B^3 u_1^3 y_G^2 - 2 A^2 B C u_1 u_3^2 y_G^2 - A^2 B C u_1^3 x_G^2 - 2 A^2 B C u_1 u_3^2 x_G^2 - 2 A C^3 u_3^2 x_G^2 u_1 \\ &+ 2 A^2 C m g x_G u_1 \tilde{u}_4 z_G^2 + 2 A^2 C m g x_G^3 u_1 \tilde{u}_4 - 4 A^2 C m g x_G^2 u_3 z_G \tilde{u}_5 - 4 A^2 C m g x_G^3 \tilde{u}_2 \tilde{u}_5 \\ &+ 2 A^2 C m g x_G u_3 z_G \tilde{u}_4 y_G - 7 A^2 B^2 y_G u_1^2 \tilde{u}_2 x_G + 2 A^2 C m g x_G^2 \tilde{u}_2 \tilde{u}_4 y_G - 4 A^2 C^2 u_3^2 x_G \tilde{u}_2 y_G \\ &- 3 A^2 B^2 y_G^2 u_1^3 - B^2 C^2 y_G u_1^2 u_3 z_G - 2 u_1 A^4 \tilde{u}_2^2 x_G^2 + 2 A^3 \tilde{u}_2 x_G C u_3^2 y_G + 2 A^3 \tilde{u}_2 x_G C u_1 u_3 z_G \\ &+ 2 A^3 \tilde{u}_2^2 x_G^2 C u_1 - 2 u_1 B C^3 y_G^2 u_3^2 + 5 A^3 \tilde{u}_2 x_G B u_1^2 y_G - 2 A^3 \tilde{u}_2 x_G^2 m g \tilde{u}_4 y_G + 4 A^3 \tilde{u}_2^2 x_G^2 B u_1 \\ &+ 4 A B C m g u_1 x_G^2 y_G \tilde{u}_5 - 2 A B C m g u_1 x_G y_G^2 \tilde{u}_4 - 2 A B C^2 \dot{u}_1 u_3 x_G y_G + 2 A B C^2 u_1 x_G \tilde{u}_2 u_3 z_G \\ &+ 3 A B C^2 u_1^2 y_G u_3 z_G + 4 A B C^2 u_1 u_3^2 y_G^2 + 2 A B C^2 u_1 u_3^2 x_G^2 + 2 A B^2 C u_1^2 y_G u_3 z_G \\ &+ 10 A B^2 C u_1^2 y_G \tilde{u}_2 x_G + 2 A C^3 u_3^2 x_G \tilde{u}_2 y_G + 2 A C^2 m g u_3 x_G^2 z_G \tilde{u}_5 - 4 B^3 C y_G^2 u_1^3 \end{aligned}$$

$$\begin{aligned}
 & -2AC^2mgu_3x_Gz_G\tilde{u}_4y_G+2A^2C^2u_3^2x_G^2u_1-A^2BCu_1^3z_G^2+3AB^2Cu_1^3y_G^2 \\
 & +2A^2BCu_1x_G\dot{u}_1z_G+2A^3\tilde{u}_2x_G^3mg\tilde{u}_5)/2A^2BCx_G^2, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 \ddot{u}_2 = & (-2A^2\tilde{u}_2u_3x_Gy_G+ABu_1^2y_Gz_G+2ABu_1\tilde{u}_2x_Gz_G+2ABu_1u_3x_G^2+2ABu_1u_3y_G^2-2ACu_1u_3x_G^2 \\
 & +2AC\tilde{u}_2u_3x_Gy_G-2Amg\tilde{u}_4x_Gy_Gz_G+2Amg\tilde{u}_5x_G^2z_G-2BCu_1u_3y_G^2)/2A^2x_G^2, \tag{26}
 \end{aligned}$$

$$\begin{aligned}
 \dot{u}_3 = & (2A^2u_1\tilde{u}_2x_G-3ABu_1^2y_G-4Bu_1\tilde{u}_2Ax_G+2Amg\tilde{u}_4x_Gy_G-2\tilde{u}_5x_G^2Amg+3B^2y_Gu_1^2)/2ACx_G, \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 \ddot{u}_4 = & (-2A^2u_1\tilde{u}_2u_3x_G+ABu_1^3z_G+2ABu_1^2u_3y_G+2ABu_1\tilde{u}_2u_3x_G+2ACu_1\tilde{u}_2u_3x_G \\
 & -2Amgu_1\tilde{u}_4x_Gz_G+2Amgu_3\tilde{u}_5x_G^2-B^2u_1^2u_3y_G-2BCu_1^2u_3y_G)/2Amgx_G^2, \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 \ddot{u}_5 = & (2A^2\tilde{u}_2^2u_3x_G-ABu_1^2\tilde{u}_2z_G+ABu_1^2u_3x_G-2ABu_1\tilde{u}_2u_3y_G-2AC\tilde{u}_2^2u_3x_G+2Amg\tilde{u}_2\tilde{u}_4x_Gz_G \\
 & -2Amgu_3\tilde{u}_4x_G^2-2B^2u_1^2\tilde{u}_2z_G-2B^2u_1^2u_3x_G+2BCu_1^2u_3x_G+2BCu_1\tilde{u}_2u_3y_G \\
 & -2Bmgu_1\tilde{u}_5x_Gz_G)/2Amgx_G^2, \tag{29}
 \end{aligned}$$

with

$$\tilde{u}_2 = s_2, \quad \tilde{u}_4 = s_4, \quad \tilde{u}_5 = s_5. \tag{30}$$

We now apply Lie group analysis to system (25)–(29). An operator $\tilde{\Gamma}$

$$\begin{aligned}
 \tilde{\Gamma} = & \tilde{V}(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_t + \tilde{G}_1(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_{u_1} + \tilde{G}_2(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_{\tilde{u}_2} \\
 & + \tilde{G}_3(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_{u_3} + \tilde{G}_4(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_{\tilde{u}_4} + \tilde{G}_5(t, u_1, \tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)\partial_{\tilde{u}_5} \tag{31}
 \end{aligned}$$

is said to generate a Lie point symmetry group if its second prolongation $\tilde{\Gamma}_2$ applied to system (25)–(29), on their solutions, is identically equal to zero, i.e.,

$$\begin{aligned}
 \tilde{\Gamma}_2(25)|_{(25)-(29)} &= 0, \\
 \tilde{\Gamma}_2(26)|_{(25)-(29)} &= 0, \\
 \tilde{\Gamma}_2(27)|_{(25)-(29)} &= 0, \tag{32} \\
 \tilde{\Gamma}_2(28)|_{(25)-(29)} &= 0, \\
 \tilde{\Gamma}_2(29)|_{(25)-(29)} &= 0.
 \end{aligned}$$

The five determining equations (32) constitute an overdetermined system of linear partial differential equations in the unknowns $\tilde{V}, \tilde{G}_k(k=1,5)$. In fact, they are polynomials in \dot{u}_1 , each coefficient of which must become identically equal to zero. In particular, the fifth determining equation in (32) is a polynomial of degree one for \dot{u}_1 . We call its two coefficients $c5k1$ and $c5k0$. For the sake of simplicity, we assume $\tilde{G}_k=0, \partial\tilde{V}/\partial t=0$. Then, the coefficient of degree one, i.e., $c5k1$, yields

$$\frac{\partial \tilde{V}}{\partial u_1} = 0.$$

Now, $c5k0$ is a polynomial of degree five in u_1 . Therefore, its coefficients, call them $c5m5, c5m4, c5m3, c5m2, c5m1, c5m0$, must become identically equal to zero. The coefficient of degree five in u_1 , i.e., $c5m5$, yields

$$\frac{\partial \tilde{V}}{\partial \tilde{u}_4} A^2 B^2 C z_G (- (A + 2B) \tilde{u}_2 z_G + (A - 2B + 2C) u_3 x_G) = 0 \tag{33}$$

which gives the condition on the parameter

$$z_G = 0. \tag{34}$$

Then, the coefficient of degree four in u_1 , i.e., $c5m4$, yields

$$\begin{aligned} & - \left(3(A - B) \frac{\partial \tilde{V}}{\partial u_3} m g y_G x_G - (A - 2B + 2C) \frac{\partial \tilde{V}}{\partial \tilde{u}_5} C u_3 x_G - (2A - B - 2C) \frac{\partial \tilde{V}}{\partial \tilde{u}_4} C u_3 y_G \right) \\ & \times (A - 2B + 2C) A B^2 u_3 x_G = 0 \end{aligned} \tag{35}$$

which gives the condition on the parameters

$$A = 2B - 2C. \tag{36}$$

Then, the coefficient of degree three in u_1 , i.e., $c5m3$, becomes

$$12 \left(\frac{\partial \tilde{V}}{\partial u_3} m g x_G - \frac{\partial \tilde{V}}{\partial \tilde{u}_4} C u_3 \right) (2B - 3C)(B - C)(B - 2C) B^2 \tilde{u}_2 u_3 y_G^2 = 0 \tag{37}$$

which gives the further condition on the parameters

$$B = 2C. \tag{38}$$

Thus, we have found the Kowalevski top. We also notice that either condition $2B = 3C$ or $B = C$ leads to the Lagrange top. Finally, we are left with two linear first order partial differential equations in $\tilde{V} = \tilde{V}(\tilde{u}_2, u_3, \tilde{u}_4, \tilde{u}_5)$, the coefficient of degree two in u_1 , i.e., $c5m2$,

$$2 \left(\frac{\partial \tilde{V}}{\partial \tilde{u}_4} x_G - \frac{\partial \tilde{V}}{\partial \tilde{u}_5} y_G \right) C u_3 \tilde{u}_2 - 4 \frac{\partial \tilde{V}}{\partial u_3} m g x_G^2 \tilde{u}_2 + (x_G^2 + y_G^2) \frac{\partial \tilde{V}}{\partial \tilde{u}_2} m g u_3 = 0 \tag{39}$$

and the coefficient of degree one in u_1 , i.e., $c5m1$

$$\begin{aligned} & 2C(C\tilde{u}_2^2 - m g \tilde{u}_4 x_G - m g \tilde{u}_5 y_G) \frac{\partial \tilde{V}}{\partial \tilde{u}_4} x_G u_3 \tilde{u}_2 \\ & - 4C(C\tilde{u}_2^2 - m g \tilde{u}_4 x_G) \frac{\partial \tilde{V}}{\partial \tilde{u}_5} y_G u_3 \tilde{u}_2 \\ & - 2(2C\tilde{u}_2^2 x_G - 2m g \tilde{u}_4 x_G^2 + m g \tilde{u}_4 y_G^2 - m g \tilde{u}_5 x_G y_G) \frac{\partial \tilde{V}}{\partial u_3} m g x_G \tilde{u}_2 \end{aligned} \tag{40}$$

$$+(C\tilde{u}_2^2x_G^2+2C\tilde{u}_2^2y_G^2-mg\tilde{u}_4x_G^3-mg\tilde{u}_4x_Gy_G^2)\frac{\partial\tilde{V}}{\partial\tilde{u}_2}mg u_3=0.$$

If \tilde{V} satisfies equations (39) and (40), then it is easy to prove that the determining equations (32) are identically satisfied by considering conditions (34), (36), (38) as well.

From (39) it is easy to obtain that $\tilde{V}=\tilde{V}(\eta_1, \eta_2, \eta_3)$ with

$$\eta_1=u_3^2+\frac{4x_G^2\tilde{u}_2^2}{x_G^2+y_G^2}, \quad \eta_2=\tilde{u}_4-\frac{Cx_G\tilde{u}_2^2}{mg(x_G^2+y_G^2)}, \quad \eta_3=\tilde{u}_5+\frac{Cy_G\tilde{u}_2^2}{mg(x_G^2+y_G^2)}. \tag{41}$$

Then, (40) becomes

$$2mg(y_G\eta_2-x_G\eta_3)\frac{\partial\tilde{V}}{\partial\eta_1}+C\frac{\partial\tilde{V}}{\partial\eta_2}\eta_3-C\frac{\partial\tilde{V}}{\partial\eta_3}\eta_2=0. \tag{42}$$

Its characteristic curves are

$$\xi_1=\eta_1+\frac{2mg}{C}(y_G\eta_3+x_G\eta_2), \quad \xi_2=\eta_2^2+\eta_3^2. \tag{43}$$

Finally, we have that $\tilde{V}=\Psi(\xi_1, \xi_2)$ with Ψ an arbitrary function of ξ_1, ξ_2 , and consequently operator

$$\tilde{\Gamma}=\Psi(\xi_1, \xi_2)\partial_t \tag{44}$$

is a generator of a Lie point symmetry for system (25)–(29). Transforming (43) into the original unknown functions by using (41), (30), (24), and (20) yields

$$\xi_1=\frac{2}{C}\left(\frac{C}{2}(2p^2+2q^2+r^2)+mg(x_G\alpha+y_G\beta)\right),$$

$$\xi_2=C^2\frac{\left(p^2-q^2-mg\frac{x_G\alpha-y_G\beta}{C}\right)^2+\left(2pq-mg\frac{x_G\beta+y_G\alpha}{C}\right)^2}{m^2g^2(x_G^2+y_G^2)}$$

which correspond to the first integral of conservation of energy (11), and that derived by Kowalevski (14), respectively.

Can other cases of integrability (viz. integration by quadrature) be obtained by using our method? We leave the answer to a future paper. In Ref. 43, the application of our method led to an integrable case for a nonlinear system of three ordinary differential equations which does not possess the Painlevé property.

IV. FIRST INTEGRALS OF THE KOWALEVSKI TOP

We apply our method to the Kowalevski top itself which corresponds to the following conditions on the parameters:

- (1) $A=B=2C$,
- (2) $y_G=z_G=0, \quad x_G>0$.

The condition on y_G can be added without loss of generality. Then, system (10) become

$$\begin{aligned}
 \dot{p} &= rq/2, \\
 \dot{q} &= -pr/2 + mgx_G\gamma/2C, \\
 \dot{r} &= -mgx_G\beta/C, \\
 \dot{\alpha} &= \beta r - \gamma q, \\
 \dot{\beta} &= \gamma p - \alpha r, \\
 \dot{\gamma} &= \alpha q - \beta p.
 \end{aligned} \tag{45}$$

The first integrals for the Kowalewski top are

- (1) conservation of energy, i.e.,

$$I_1 = \frac{C}{2}(2p^2 + 2q^2 + r^2) + mgx_G\alpha; \tag{46}$$

- (2) conservation of the vertical component of the angular momentum, i.e.,

$$I_2 = C(2p\alpha + 2q\beta + r\gamma); \tag{47}$$

- (3) the length of the unit vertical vector, i.e.,

$$I_3 = \alpha^2 + \beta^2 + \gamma^2 (=1); \tag{48}$$

- (4) the first integral derived by Kowalevski, i.e.,

$$I_4 = \left(p^2 - q^2 - \frac{x_G\alpha mg}{C} \right)^2 + \left(2pq - \frac{x_G\beta mg}{C} \right)^2. \tag{49}$$

If our method is applied to (45), then all the first integrals can be obtained, apart from (47) which has all the unknown variables $p, q, r, \alpha, \beta, \gamma$ appearing in its expression. Let us observe that

β does not appear in I_1 ,

γ does not appear in both I_1 and I_4 ,

p does not appear in I_3 .

In the following, we eliminate α, β, γ, p from system (45) one at a time.

A. Eliminating α

First we show a negative result: no first integral obtained. Let us assume that we do not know any of the first integrals. Therefore, we do not know *a priori* that none of the first integrals can be obtained by deriving α . We derive α from the fifth equation of system (45), i.e.,

$$\alpha = \frac{p\gamma - \dot{\beta}}{r}$$

and obtain the following system of four equations of first order, and one of second order:

$$\begin{aligned}
 \ddot{u}_1 &= (-2Cu_1u_3^3 - 2Cu_1u_3u_4^2 - 2C\dot{u}_1u_2u_4 + 3Cu_2u_3^2u_5 \\
 &\quad + 2Cu_2u_4^2u_5 - 2mgu_1\dot{u}_1x_G + 2mgu_1u_4u_5x_G)/2Cu_3, \\
 \dot{u}_2 &= (-Cu_3u_4 + mgu_5x_G)/2C, \\
 \dot{u}_3 &= (-mgu_1x_G)/C, \\
 \dot{u}_4 &= (u_2u_3)/2, \\
 \dot{u}_5 &= (-u_1u_3u_4 - \dot{u}_1u_2 + u_2u_4u_5)/u_3
 \end{aligned} \tag{50}$$

with

$$u_1 = \beta, \quad u_2 = q, \quad u_3 = r, \quad u_4 = p, \quad u_5 = \gamma. \tag{51}$$

If we apply Lie group analysis to system (50), then we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve is

$$u_5 u_3 + u_1 u_2$$

which yields the following transformation:

$$u_5 = \frac{s_5 - u_2 u_1}{u_3} \quad \text{viz.} \quad \gamma = \frac{s_5 - q\beta}{r} \tag{52}$$

with s_5 a new unknown function of t . Then, system (50) transforms into

$$\begin{aligned} \ddot{u}_1 = & (-3Cu_1 u_2^2 u_3^2 - 2Cu_1 u_2^2 u_4^2 - 2Cu_1 u_3^4 - 2Cu_1 u_3^2 u_4^2 - 2C\dot{u}_1 u_2 u_3 u_4 + 3Cu_2 u_3^2 \tilde{u}_5 \\ & + 2Cu_2 u_4^2 \tilde{u}_5 - 2mgu_1^2 u_2 u_4 x_G - 2mgu_1 \dot{u}_1 u_3 x_G + 2mgu_1 u_4 \tilde{u}_5 x_G) / 2Cu_3^2, \\ \dot{u}_2 = & (-Cu_3^2 u_4 - mgu_1 u_2 x_G + mg\tilde{u}_5 x_G) / 2Cu_3, \\ \dot{u}_3 = & (-mgu_1 x_G) / C, \\ \dot{u}_4 = & u_2 u_3 / 2, \\ \ddot{u}_5 = & (-2Cu_1 u_2^2 u_4 - 3Cu_1 u_3^2 u_4 + 2Cu_2 u_4 \tilde{u}_5 + mgu_1^2 u_2 x_G - mgu_1 \tilde{u}_5 x_G) / 2Cu_3, \end{aligned} \tag{53}$$

with

$$\tilde{u}_5 = s_5. \tag{54}$$

If we apply Lie group analysis to system (53), then we obtain a two dimensional Lie symmetry algebra generated by the following two operators:

$$\Gamma_1 = \frac{\partial}{\partial t}, \tag{55}$$

$$\Gamma_2 = -t \frac{\partial}{\partial t} + 2u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3} + u_4 \frac{\partial}{\partial u_4} + 3\tilde{u}_5 \frac{\partial}{\partial \tilde{u}_5} \tag{56}$$

which in the original unknown functions correspond to

$$\Gamma_1 = \frac{\partial}{\partial t}, \tag{57}$$

$$\Gamma_2 = -t \frac{\partial}{\partial t} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} + r \frac{\partial}{\partial r} + 2\alpha \frac{\partial}{\partial \alpha} + 2\beta \frac{\partial}{\partial \beta} + 2\gamma \frac{\partial}{\partial \gamma}. \tag{58}$$

This is a trivial finding.

B. Eliminating β

We derive β from the third equation of system (45), i.e.,

$$\beta = -\frac{C\dot{r}}{mgx_G}$$

and obtain the following system of four equations of first order, and one of second order:

$$\begin{aligned}\ddot{u}_1 &= (mgx_G(u_1u_4 - u_3u_5))/C, \\ \ddot{u}_2 &= (-Cu_1u_3 + mgx_Gu_5)/2C, \\ \dot{u}_3 &= u_1u_2/2, \\ \dot{u}_4 &= (-Cu_1\dot{u}_1 - mgx_Gu_2u_5)/mgx_G, \\ \dot{u}_5 &= (C\dot{u}_1u_3 + mgx_Gu_2u_4)/mgx_G\end{aligned}\tag{59}$$

with

$$u_1 = r, \quad u_2 = q, \quad u_3 = p, \quad u_4 = \alpha, \quad u_5 = \gamma.\tag{60}$$

If we apply Lie group analysis to system (59), then we obtain a determining equation of parabolic type for V in three independent variables. Its two characteristic curves yield the following transformations:

$$\begin{aligned}u_4 &= \frac{s_4 - Cu_1^2}{2mgx_G} \quad \text{viz.} \quad \alpha = \frac{s_4 - Cr^2}{2mgx_G}, \\ u_5 &= \frac{Cu_3u_1 + s_6}{mgx_G} \quad \text{viz.} \quad \gamma = \frac{Cpr + s_6}{mgx_G}\end{aligned}\tag{61}$$

with s_4 and s_6 new unknown functions of t . Then, system (50) transforms into

$$\begin{aligned}\ddot{u}_1 &= (-Cu_1^3 - 2Cu_1u_3^2 + u_1\tilde{u}_4 - 2u_3\tilde{u}_5)/2C, \\ \dot{u}_2 &= \tilde{u}_5/2C, \\ \dot{u}_3 &= u_1u_2/2, \\ \dot{u}_4 &= -2u_2(Cu_1u_3 + \tilde{u}_5), \\ \dot{u}_5 &= u_2(-2Cu_1^2 + \tilde{u}_4)/2\end{aligned}\tag{62}$$

with

$$\tilde{u}_4 = s_4, \quad \tilde{u}_5 = s_6.\tag{63}$$

If we apply Lie group analysis to system (62), then we obtain two first order partial differential equations for V :

$$\frac{\partial V}{\partial u_3} - 4Cu_3 \frac{\partial V}{\partial \tilde{u}_4} = 0,\tag{64}$$

$$\frac{\partial V}{\partial u_2} - 4Cu_2 \frac{\partial V}{\partial \tilde{u}_4} = 0, \tag{65}$$

with $V \equiv V(u_2, u_3, \tilde{u}_4)$. From (64) it is easy to obtain that $V \equiv V(\eta, u_2)$ with

$$\eta = 2Cu_3^2 + \tilde{u}_4. \tag{66}$$

Then, (65) becomes

$$\frac{\partial V}{\partial u_2} - 4Cu_2 \frac{\partial V}{\partial \eta} = 0. \tag{67}$$

Its characteristic curve is

$$\xi_1 = 2Cu_2^2 + \eta. \tag{68}$$

Finally, we have that $V = \psi(\xi_1)$ with ψ an arbitrary function of ξ , and consequently operator

$$\Gamma_1 = \psi(\xi_1) \partial_t \tag{69}$$

is a generator of a Lie point symmetry for system (62). Transforming (68) into the original unknown functions by using (66), (63), (61), (60) yields

$$\xi_1 = \frac{C}{2}(2p^2 + 2q^2 + r^2) + mg x_G \alpha$$

which is exactly the first integral of conservation of energy (46). In addition, we have algorithmically derived that (69) is a generator of a Lie point symmetry for system (45).

C. Eliminating γ

We derive γ from the second equation of system (45), i.e.,

$$\gamma = \frac{C(2\dot{q} + pr)}{mgx_G}$$

and obtain the following system of four equations of first order, and one of second order:

$$\begin{aligned} \ddot{u}_1 &= u_1(-Cu_3^2 + 2mgu_4x_G)/4C, \\ \dot{u}_2 &= u_1u_3/2, \\ \dot{u}_3 &= -mgu_5x_G/C, \\ \dot{u}_4 &= (-2Cu_1\dot{u}_1 - Cu_1u_2u_3 + mgu_3u_5x_G)/mgx_G, \\ \dot{u}_5 &= (2C\dot{u}_1u_2 + Cu_2^2u_3 - mgu_3u_4x_G)/mgx_G \end{aligned} \tag{70}$$

with

$$u_1 = q, \quad u_2 = p, \quad u_3 = r, \quad u_4 = \alpha, \quad u_5 = \beta. \tag{71}$$

If we apply Lie group analysis to system (70), then we obtain a determining equation of parabolic type for V in three independent variables. Its two characteristic curves yield the following transformations:

$$\begin{aligned}
 u_4 &= \frac{s_4 - C u_1^2}{m g x_G} \quad \text{viz.} \quad \alpha = \frac{s_4 - C q^2}{m g x_G}, \\
 u_5 &= \frac{2 C u_1 u_2 + s_5}{m g x_G} \quad \text{viz.} \quad \beta = \frac{2 C p q + s_5}{m g x_G}
 \end{aligned}
 \tag{72}$$

with s_4 and s_5 new unknown functions of t . Then, system (70) transforms into

$$\begin{aligned}
 \ddot{u}_1 &= [u_1(-2 C u_1^2 - C u_3^2 + 2 \tilde{u}_4)]/4C, \\
 \dot{u}_2 &= u_1 u_3/2, \\
 \dot{u}_3 &= (-2 C u_1 u_2 - \tilde{u}_5)/C, \\
 \dot{\tilde{u}}_4 &= u_3(C u_1 u_2 + \tilde{u}_5), \\
 \dot{\tilde{u}}_5 &= u_3(C u_2^2 - \tilde{u}_4)
 \end{aligned}
 \tag{73}$$

with

$$\tilde{u}_4 = s_4, \quad \tilde{u}_5 = s_5. \tag{74}$$

If we apply Lie group analysis to system (73), then we obtain two first order partial differential equations for V :

$$u_3 \frac{\partial V}{\partial u_2} - 4 u_2 \frac{\partial V}{\partial u_3} + 4 C u_2 u_3 \frac{\partial V}{\partial \tilde{u}_4} = 0, \tag{75}$$

$$8 C u_2 \tilde{u}_5 \frac{\partial V}{\partial u_3} - C u_3 \tilde{u}_5 \frac{\partial V}{\partial \tilde{u}_4} + C u_3 (\tilde{u}_4 - C u_2^2) \frac{\partial V}{\partial \tilde{u}_5} = 0 \tag{76}$$

with $V \equiv V(u_2, u_3, \tilde{u}_4, \tilde{u}_5)$. From (75) it is easy to obtain that $V \equiv V(\eta_1, \eta_2, \tilde{u}_5)$ with

$$\eta_1 = 4 u_2^2 + u_3^2, \quad \eta_2 = C u_2^2 - \tilde{u}_4. \tag{77}$$

Then, Eq. (76) becomes

$$2 C u_3 \tilde{u}_5 \frac{\partial V}{\partial \eta_1} + C u_3 \tilde{u}_5 \frac{\partial V}{\partial \eta_2} - C u_3 (u_4 - C u_2^2) \frac{\partial V}{\partial \tilde{u}_5} = 0. \tag{78}$$

Its characteristic curves are

$$\xi_1 = C \eta_1 - 2 \eta_2, \quad \xi_2 = \eta_2^2 + \tilde{u}_5^2. \tag{79}$$

Finally, we have that $V = \Psi(\xi_1, \xi_2)$ with Ψ an arbitrary function of ξ_1 , ξ_2 , and consequently operator

$$\Gamma_1 = \Psi(\xi_1, \xi_2) \partial_t \tag{80}$$

is a generator of a Lie point symmetry for system (73). Transforming (79) into the original unknown functions by using (77), (74), (72), (71) yields

$$\xi_1 = \frac{C}{2} (2 p^2 + 2 q^2 + r^2) + m g x_G \alpha,$$

$$\xi_2 = \left(p^2 - q^2 - \frac{x_G \alpha m g}{C} \right)^2 + \left(2pq - \frac{x_G \beta m g}{C} \right)^2$$

which are exactly the first integral of conservation of energy (46), and that derived by Kowalevski (49), respectively. In addition, we have algorithmically derived that (80) is a generator of a Lie point symmetry for system (45).

D. Eliminating p

We derive p from the second equation of system (45), i.e.,

$$p = \frac{mg \gamma x_G - 2C \dot{q}}{Cr}$$

and obtain the following system of four equations of first order, and one of second order:

$$\begin{aligned} \ddot{u}_1 &= u_1(-Cu_3^2 + 2mgu_4x_G)/4C, \\ \dot{u}_2 &= (Cu_1u_3u_4 + 2C\dot{u}_1u_5 - mgu_2u_5x_G)/Cu_3, \\ \dot{u}_3 &= -mgu_5x_G/C, \\ \dot{u}_4 &= -u_1u_2 + u_3u_5, \\ \dot{u}_5 &= (-2C\dot{u}_1u_2 - Cu_3^2u_4 + mgu_2^2x_G)/Cu_3 \end{aligned} \tag{81}$$

with

$$u_1 = q, \quad u_2 = \gamma, \quad u_3 = r, \quad u_4 = \alpha, \quad u_5 = \beta. \tag{82}$$

If we apply Lie group analysis to system (81), then we obtain a determining equation of parabolic type for V in three independent variables. Its two characteristic curves yield the following transformations:

$$\begin{aligned} u_2 &= \sqrt{s_6} \cos(2s_5 - 2u_3) \quad \text{viz.} \quad \gamma = \sqrt{s_6} \cos(2s_5 - 2r), \\ u_5 &= \sqrt{s_6} \sin(2s_5 - 2u_3) \quad \text{viz.} \quad \beta = \sqrt{s_6} \sin(2s_5 - 2r) \end{aligned} \tag{83}$$

with s_6 and s_5 new unknown functions of t . Then, system (81) transforms into

$$\begin{aligned} \ddot{u}_1 &= u_1(-Cu_3^2 + 2mgu_4x_G)/4C, \\ \ddot{u}_2 &= 2u_4\sqrt{\tilde{u}_2} \cos(2u_3 - 2\tilde{u}_5)[\tan(2u_3 - 2\tilde{u}_5)u_3 + u_1], \\ \dot{u}_3 &= \sqrt{\tilde{u}_2} \sin(2u_3 - 2\tilde{u}_5)mgx_G/C, \\ \dot{u}_4 &= -\sqrt{\tilde{u}_2} \cos(2u_3 - 2\tilde{u}_5)[\tan(2u_3 - 2\tilde{u}_5)u_3 + u_1], \\ \dot{u}_5 &= \sqrt{\tilde{u}_2} \cos(2u_3 - 2\tilde{u}_5)[-(Cu_3^2u_4 - mg\tilde{u}_2x_G) - 2C\dot{u}_1\tilde{u}_2 \\ &\quad + (Cu_1u_4 + 2mg\tilde{u}_2x_G)\tan(2u_3 - 2\tilde{u}_5)u_3]/2C\tilde{u}_2u_3 \end{aligned} \tag{84}$$

with

$$\tilde{u}_2 = s_6, \quad \tilde{u}_5 = s_5. \tag{85}$$

If we apply Lie group analysis to system (84), then we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve yields the following transformation:

$$\tilde{u}_5 = \frac{ss_5 - u_1}{u_3} \quad \text{viz.} \quad \begin{aligned} \gamma &= \sqrt{s_6} \cos\left(2 \frac{ss_5 - q - r^2}{r}\right) \\ \beta &= \sqrt{s_6} \sin\left(2 \frac{ss_5 - q - r^2}{r}\right), \end{aligned} \quad (86)$$

with ss_5 a new unknown function of t . Then, system (84) transforms into

$$\begin{aligned} \dot{u}_1 &= u_1(-Cu_3^2 + 2mgu_4x_G)/4C, \\ \dot{\tilde{u}}_2 &= 2u_4\sqrt{\tilde{u}_2} \cos((2u_1 + 2u_3^2 - 2\hat{u}_5)/u_3)[u_1 + \tan((2u_1 + 2u_3^2 - 2\hat{u}_5)/u_3)u_3], \\ \dot{u}_3 &= \sqrt{\tilde{u}_2} \sin((2u_1 + 2u_3^2 - 2\hat{u}_5)/u_3)mgx_G/C, \\ \dot{u}_4 &= -\sqrt{\tilde{u}_2} \cos((2u_1 + 2u_3^2 - 2\hat{u}_5)/u_3)[u_1 + \tan((2u_1 + 2u_3^2 - 2\hat{u}_5)/u_3)u_3], \\ \dot{\tilde{u}}_5 &= \sqrt{\tilde{u}_2} \cos((2u_3^2 - 2\hat{u}_5 + 2u_1)/u_3)[[(2u_3^2 + 2\hat{u}_5 - 2u_1)mg\tilde{u}_2x_G + Cu_1u_3^2u_4] \\ &\quad \times \tan((2u_3^2 - 2\hat{u}_5 + 2u_1)/u_3) + (mg\tilde{u}_2x_G - cu_3^2u_4)u_3]/2c\tilde{u}_2u_3 \end{aligned} \quad (87)$$

with

$$\hat{u}_5 = ss_5. \quad (88)$$

If we apply Lie group analysis to system (87), then we obtain one first order partial differential equation for V :

$$\frac{\partial V}{\partial u_4} - 2u_4 \frac{\partial V}{\partial \tilde{u}_2} = 0 \quad (89)$$

with $V \equiv V(\tilde{u}_2, u_4)$. Its characteristic curve is

$$\xi_1 = \tilde{u}_2 + u_4^2. \quad (90)$$

Finally, we have that $V = \psi(\xi_1)$ with ψ an arbitrary function of ξ_1 , and consequently operator

$$\Gamma_1 = \psi(\xi_1) \partial_t \quad (91)$$

is a generator of a Lie point symmetry for system (87). Transforming (90) into the original unknown functions by using (88), (86), (85), (83), (82) yields

$$\xi_1 = \alpha^2 + \beta^2 + \gamma^2$$

which is exactly the first integral of the length of the unit vertical vector (48). In addition, we have algorithmically derived that (91) is a generator of a Lie point symmetry for system (45).

V. KEPLER PROBLEM

In Ref. 33, Nucci's method³⁰ was used to find symmetries additional to those reported by Krause¹⁸ in his study of the complete symmetry group of the Kepler problem. A consequence of the application of Nucci's method was the demonstration of the group theoretical relationship between the simple harmonic oscillator and the Kepler problem. In Ref. 33, polar coordinates were

used, and Nucci's method was not applied to the three-dimensional case with the purpose of finding first integrals. We do it here by considering Cartesian coordinates.

The equations of motion of the Kepler problem are given by the following well-known three equations of second order:

$$\begin{aligned} \ddot{x}_1 &= -\mu x_1 / ((x_1^2 + x_2^2 + x_3^2)^{3/2}), \\ \ddot{x}_2 &= -\mu x_2 / ((x_1^2 + x_2^2 + x_3^2)^{3/2}), \\ \ddot{x}_3 &= -\mu x_3 / ((x_1^2 + x_2^2 + x_3^2)^{3/2}). \end{aligned} \tag{92}$$

The first integrals for the Kepler problem are conservation of energy E , conservation of angular momentum \mathbf{K} , the Laplace–Runge–Lenz vector \mathbf{L} . None of the unknowns $x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3$ are missing in the expression of E and the components of \mathbf{L} . This is not true for the three components of \mathbf{K} , i.e.,

$$K_1 = x_3 \dot{x}_2 - \dot{x}_3 x_2, \tag{93}$$

$$K_2 = x_3 \dot{x}_1 - \dot{x}_3 x_1, \tag{94}$$

$$K_3 = x_1 \dot{x}_2 - \dot{x}_1 x_2. \tag{95}$$

Therefore, we can only obtain the three components of \mathbf{K} using our method. However, neither E nor \mathbf{L} are needed to reduce system (92) to a linear oscillator, as we show in the following. Let us transform system (92) into a system of six equations of first order

$$\begin{aligned} \dot{w}_1 &= w_4, \\ \dot{w}_2 &= w_5, \\ \dot{w}_3 &= w_6, \\ \dot{w}_4 &= -\mu w_1 / ((w_1^2 + w_2^2 + w_3^2)^{3/2}), \\ \dot{w}_5 &= -\mu w_2 / ((w_1^2 + w_2^2 + w_3^2)^{3/2}), \\ \dot{w}_6 &= -\mu w_3 / ((w_1^2 + w_2^2 + w_3^2)^{3/2}) \end{aligned} \tag{96}$$

with

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3, \quad w_4 = \dot{x}_1, \quad w_5 = \dot{x}_2, \quad w_6 = \dot{x}_3. \tag{97}$$

Consequently, the components of the angular momentum become

$$K_1 = w_3 w_5 - w_6 w_2, \tag{98}$$

$$K_2 = w_3 w_4 - w_1 w_6, \tag{99}$$

$$K_3 = w_1 w_5 - w_4 w_2. \tag{100}$$

We choose one of the dependent variables to be the new independent variable y in order to reduce the order of system (96) by one.³⁰ We take $w_3 = y$. Then, system (96) becomes the following nonautonomous system of five first order ordinary differential equations:

$$w_1' = w_4 / w_6,$$

$$\begin{aligned}
 w_2' &= w_5/w_6, \\
 w_4' &= -\mu w_1/(w_6(w_1^2 + w_2^2 + y^2)^{3/2}), \\
 w_5' &= -\mu w_2/(w_6(w_1^2 + w_2^2 + y^2)^{3/2}), \\
 w_6' &= -\mu y/(w_6(w_1^2 + w_2^2 + y^2)^{3/2})
 \end{aligned} \tag{101}$$

with prime denoting differentiation with respect to y . Let us observe that

w_4 does not appear in K_1 ,

w_5 does not appear in K_2 ,

w_6 does not appear in K_3 ,

We should remark that other variables are missing too. For example, w_1 is also missing in K_1 . However, our method will yield the result whatever the choice of a missing variable. In the following, we eliminate w_4, w_5, w_6 from system (101) one at a time.

A. Eliminating w_4

We derive w_4 from the first equation of system (101), i.e.,

$$w_4 = w_1' w_6$$

and obtain the following nonautonomous system of three equations of first order, and one of second order:

$$\begin{aligned}
 u_4'' &= \mu(u_4' y - u_4)/(u_3^2(u_2^2 + u_4^2 + y^2)^{3/2}), \\
 u_3' &= -\mu y/(u_3(u_2^2 + u_4^2 + y^2)^{3/2}), \\
 u_2' &= u_1/u_3, \\
 u_1' &= -\mu u_2/(u_3(u_2^2 + u_4^2 + y^2)^{3/2})
 \end{aligned} \tag{102}$$

with

$$u_4 = w_1, \quad u_2 = w_2, \quad u_3 = w_6, \quad u_1 = w_5. \tag{103}$$

If we apply Lie group analysis to system (102), then after several reductions we obtain one first order partial differential equations for G_3 ,

$$u_1 \frac{\partial G_3}{\partial u_2} + u_3 \frac{\partial G_3}{\partial y} = 0 \tag{104}$$

with $G_3 \equiv G_3(u_1, u_2, u_3, y)$. Its solution is $G_3 = \psi(\xi_1)$ with ψ an arbitrary function of

$$\xi_1 = u_3 u_2 - y u_1. \tag{105}$$

Transforming (105) into the original unknown functions by using (97), (103) yields

$$\xi_1 = \dot{x}_3 x_2 - \dot{x}_2 x_3$$

which is exactly the first component of the angular momentum (93).

B. Eliminating w_5

We derive w_5 from the second equation of system (101), i.e.,

$$w_5 = w_2' w_6$$

and obtain the following nonautonomous system of three equations of first order, and one of second order:

$$\begin{aligned} u_4'' &= \mu(-u_4 + u_4'y)/(u_2(u_1^2 + u_4^2 + y^2)^{3/2}), \\ u_3' &= -\mu u_1/(u_2(u_1^2 + u_4^2 + y^2)^{3/2}), \\ u_2' &= -\mu y/((u_1^2 + u_4^2 + y^2)^{3/2}), \\ u_1' &= u_3/u_2 \end{aligned} \tag{106}$$

with

$$u_4 = w_2, \quad u_2 = w_6, \quad u_3 = w_4, \quad u_1 = w_1. \tag{107}$$

If we apply Lie group analysis to system (106), then after several reductions we obtain one first order partial differential equation for G_2 ,

$$u_3 \frac{\partial G_2}{\partial u_1} + u_2 \frac{\partial G_2}{\partial y} = 0 \tag{108}$$

with $G_2 \equiv G_2(u_1, u_2, u_3, y)$. Its solution is $G_2 = \phi(\xi_2)$ with ϕ an arbitrary function of

$$\xi_2 = y u_3 - u_1 u_2. \tag{109}$$

Transforming (109) into the original unknown functions by using (97), (107) yields

$$\xi_2 = x_3 \dot{x}_1 - \dot{x}_3 x_1$$

which is exactly the second component of the angular momentum (94).

C. Eliminating w_6

We derive w_6 from the first equation of system (101), i.e.,

$$w_6 = \frac{w_4}{w_1'}$$

and obtain the following nonautonomous system of three equations of first order, and one of second order:

$$\begin{aligned} u_4'' &= \mu u_4'^2(-u_4 + u_4'y)/(u_1^2(u_2^2 + u_4^2 + y^2)^{3/2}), \\ u_3' &= -\mu u_2 u_4'/(u_1(u_2^2 + u_4^2 + y^2)^{3/2}), \\ u_2' &= u_3 u_4'/u_1, \\ u_1' &= -\mu u_4 u_4'/(u_1(u_2^2 + u_4^2 + y^2)^{3/2}) \end{aligned} \tag{110}$$

with

$$u_4 = w_1, \quad u_2 = w_2, \quad u_3 = w_5, \quad u_1 = w_4. \quad (111)$$

If we apply Lie group analysis to system (110), then after several reductions we obtain one first order partial differential equation for G_3 ,

$$u_3 \frac{\partial G_3}{\partial u_2} + u_1 \frac{\partial G_3}{\partial u_1} = 0 \quad (112)$$

with $G_3 \equiv G_3(u_1, u_2, u_3, u_4)$ and with $G_3 \equiv G_3(u_1, u_2, u_3, y)$. Its solution is $G_3 = \varphi(\xi_3)$ with φ an arbitrary function of

$$\xi_3 = u_1 u_2 - u_4 u_3. \quad (113)$$

Transforming (113) into the original unknown functions by using (97), (111) yields

$$\xi_3 = \dot{x}_2 x_1 - \dot{x}_1 x_2$$

which is exactly the third component of the angular momentum (94).

Now let us derive w_5 , w_4 , and w_2 from (98), (99) and (100), i.e.,

$$w_5 = \frac{-\xi_3 w_6 y + \xi_2 \xi_1 + \xi_1 w_1 w_6}{\xi_2 y}, \quad (114)$$

$$w_4 = \frac{\xi_2 + w_1 w_6}{y}, \quad (115)$$

$$w_2 = \frac{-\xi_3 y + \xi_1 w_1}{\xi_2} \quad (116)$$

with ξ_1 , ξ_2 , ξ_3 new unknown functions of y . Substituting (114), (115), (116) into (101), and deriving w_6 from the first equation yields the following system of three equations of first order, and one of second order:

$$\begin{aligned} u_4'' &= (\mu u_2 (-u_4^3 + 3u_4^2 u_4' y - 3u_4 u_4'^2 y^2 + u_4'^3 y^3)) / ((u_1^2 y^2 - 2u_3 u_1 u_4 y + u_2^2 u_4^2 + u_2^2 y^2 + u_3^2 u_4^2)^{3/2}), \\ u_3' &= 0, \\ u_2' &= 0, \\ u_1' &= 0 \end{aligned} \quad (117)$$

with

$$u_4 = w_1, \quad u_3 = \xi_3, \quad u_2 = \xi_2, \quad u_1 = \xi_1.$$

It is easy to show that system (117) admits an eleven-dimensional Lie symmetry algebra. In fact, the first equation of (117) itself admits a Lie symmetry algebra of dimension eight, which means that it is linearizable through a point transformation.²⁴ Thus, we have reduced the equations of motion of the Kepler problem to the harmonic oscillator^{33,35} by using Lie group analysis.

VI. FINAL COMMENTS

We have found that Lie group analysis yields the first integrals admitted by any system of ordinary differential equations if the method developed by Nucci³⁰ is applied, the only limitation being the absence of at least one of the unknowns in each first integral.

Is it possible to obtain all of the first integrals by means of Lie group analysis? Also, what is the link between Painlevé method and Lie group analysis?³⁴ In addition, can Lax pairs be found by Lie group analysis? So far these are open questions that we hope to address in future work.

Let us conclude by underlining that the application of Nucci's method to the Kowalevski top have led us to understand how first integrals can be found by using Lie group analysis. In 1984, Cooke⁴ wrote "Kowalevskaya's work is an ingenious application of mathematics to a system of equations of great mathematical interest ... but since the case to which it applies is rather special, the details of her arguments are no longer worth troubling about." About the same time, a revival of interest into integrable problems of mechanics has led to numerous papers on the Kowalevski top. Just to cite a few, see Refs. 37, 11, 8, 1, 3, 15 and the entire No. 11 issue of J. Phys. A **34** (2001).

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