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Linear Control Theory in Geometric Terms

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References

Wonham
Linear Multivariable Control – A Geometric Approach, Springer Verlag, 1974-1985.

Basile and Marro
Controlled and Conditioned Invariants in Linear System Theory, Prentice Hall, 1992

Trentelman, Stoorvogel and Hautus
Control Theory for Linear Systems, Springer Verlag, 2001

The first papers of the geometric approach

Basile and Marro
Controlled and Conditioned Invariant Subspaces in Linear System Theory, Journal of Optimization Theory and Applications, vol. 3, n. 5, 1969.
This paper first introduced the *controlled invariant* and the *conditioned invariant*, the main tools of the geometric approach.

Wonham and Morse
Decoupling and Pole Assignment in Linear Multivariable Systems: a Geometric Approach, SIAM Journal on Control, vol. 8, n. 1, 1970.
This paper first introduced the *controllability subspace*, but completely missed the controlled invariant and the conditioned invariant.

Other Early Contributions (1971-1985)
(see the above-mentioned books for complete references)

- Akashi
- Commault
- Francis
- Hautus
- Imai
- Malabre
- Molinari
- Pearson
- Schumacher
- Silverman
- Willems

The Wonham-Morse and Basile-Marro results were derived independently. However Wonham and Morse in their first contribution did not realize that their algorithm for solving the decoupling problem defined indeed a new important algebraic object, the basic tool of the geometric approach, already named "controlled invariant" by Basile and Marro.

However Wonham renamed it "(A,B)-invariant" *five years after* in his book.

The name (A,B)-invariant, although second-best and drawing forth some notational confusion, is the most popular today. An exception is the recent book by Trentelman, Hautus and Stoorvogel, in which both our and Wonham's name are used together.

This Particular Presentation

The material herein presented is our today interpretation of thirtythree years of contributions to the geometric approach by several authors. Its main features are:

- Preferential treatment of discrete-time systems, that facilitates understanding of some limitations inherent in the continuous-time case.

- Use of lattices of self-bounded controlled invariants and their duals: this reduces the solution of many problems to a very standard routine.

- Stress on duality: the original duality of controlled and conditioned invariants prods to set and investigate the dual for every problem, thus providing a simpler, unified theory.

- Separate investigation of structural and stabilizability properties: this gives a significant insight into the substance of the problems considered.

- A useful classification of the types of signal involved: disturbances, signals accessible for measurement (like reference), and signals known in advance or *previewed* (like reference in many cases).

- Standard computational algorithms: a our own "geometric approach toolbox" in Matlab enables easy transfer from theory to practice. Software can be freely downloaded from:

http://www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

1 - Introduction to Control Problems

The Reference Block Diagrams

Consider the following very standard block diagram that shows the feedback connection of a *controlled system* (plant) Σ and a *controller* Σ_r.

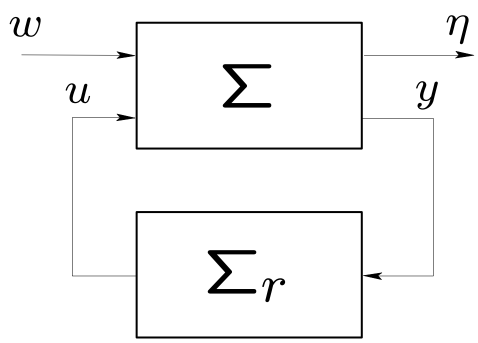


Fig. 1.1. A standard reference block diagram.

Fig.1.2 shows a possible detailed feedback regulation layout. This fits the above concise block diagram, with the assumptions $w := \{r, d_1, d_2, d_3\}$, $\eta := \{e, y_1\}$, $y := \{e, d_1, y_2\}$ (with feedthrough on d_1).

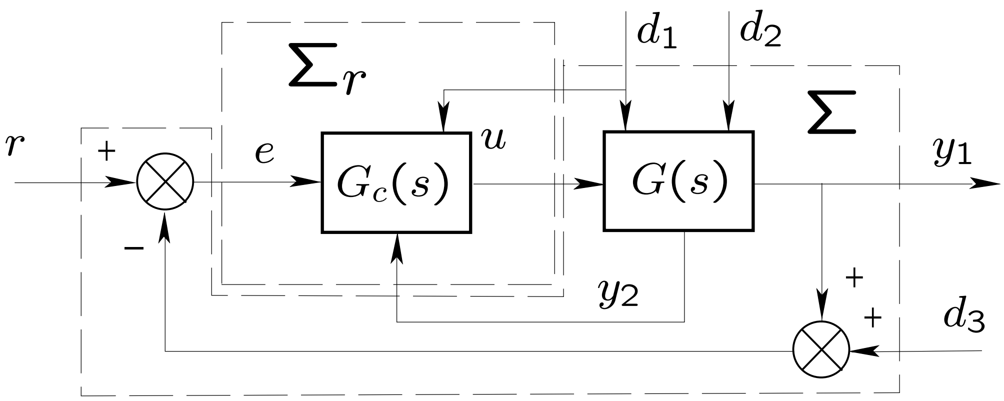


Fig. 1.2. A feedback regulation scheme.

Consider the Fig. 1.3 that includes a *controlled system* (plant) Σ and a *controller* Σ_r, with a feedback part Σ_c and a feedforward part Σ_f. This is more complete than the previous one and it is what we need for a correct exposition of regulation theory.

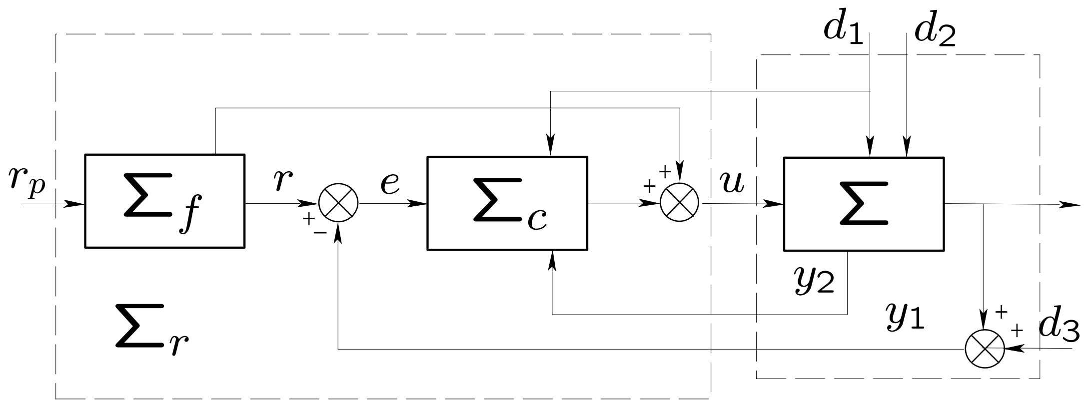


Fig. 1.3. A feedforward/feedback regulation scheme.

- r_p previewed reference
- r reference
- y_1 controlled output
- y_2 informative output
- e error variable
- u manipulated input
- d_1 measurable disturbance
- d_2 non-measurable disturbance
- d_3 non-measurable disturbance

The block diagram shown in Fig. 1.3 can be concisely represented by the one shown in Fig. 1.4, whose differences from that in Fig 1.1 are basic.

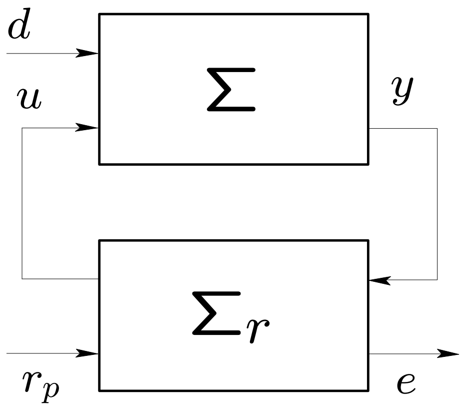


Fig. 1.4. A more comprehensive block diagram.

In the above figure $d := \{d_1, d_2, d_3\}$, $y := \{y_1, y_2, d_1\}$.

All the symbols in the figure denote *signals*, represented by real vectors varying in time.

Both the plant and the controller are assumed to be linear (zero state and superposition property).

The blocks represent *oriented systems* (inputs, outputs), that are assumed to be causal.

The plant Σ is given and the controller Σ_r is to be designed to (possibly) maintain $e(\cdot) = 0$.

This can be achieved at steady state according to the *internal model principle*.

State Space Models

- Continuous-time systems:

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}} \tag{1.1}$$

with the *state* $x \in \mathcal{X} = \mathbb{R}^n$, the *input* $u \in \mathcal{U} = \mathbb{R}^p$, the *output* $y \in \mathcal{Y} = \mathbb{R}^q$ and A, B, C, D real matrices of suitable dimensions. The system will be referred to as the *quadruple* (A, B, C, D) or, if $D = 0$, the *triple* (A, B, C). If $D \neq 0$ the system is said to be *non-purely dynamic* and the corresponding term in (1.1) is referred to as *feedthrough*. Most of the theory will be derived referring to triples since conditions and algorithms are simpler and subsequent extension to quadruples is straightforward.

- Discrete-time systems:

$$\boxed{\begin{aligned} x(k+1) &= A_d\,x(k) + B_d\,u(k) \\ y(k) &= C_d\,x(k) + D_d\,u(k) \end{aligned}} \tag{1.2}$$

Recall that a continuous-time system is internally asymptotically stable iff all the eigenvalues of A belong to $\mathbb{C}^-$ (the open left half plane of the complex plane) and a discrete-time system is internally asymptotically stable iff all the eigenvalues of A_d belong to $\mathbb{C}^{\odot}$ (the open unit disk of the complex plane).

- Connection with transfer matrix models

By taking the Laplace transform of (1.1) or the Z transform of (1.2) we obtain the *transfer matrix* representations

$$Y(s) = G(s)\,U(s) \quad \text{with}$$
$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and

$$Y(z) = G_d(z)\,U(z) \quad \text{with}$$
$$G_d(z) = C_d\,(zI - A_d)^{-1}\,B_d + D_d$$

The H_2 norm in the continuous-time case is

$$\|G\|_2 = \left(\frac{1}{2\pi}\,\mathrm{tr}\left(\int_{-\infty}^{\infty} G(j\omega)\,G^*(j\omega)\,d\omega\right)\right)^{1/2} \qquad (1.3)$$

$$= \left(\mathrm{tr}\left(\int_{0}^{\infty} g(t)\,g^\top(t)\,dt\right)\right)^{1/2} \qquad (1.4)$$

where $g(t)$ denotes the impulse response of the system (the inverse Laplace transform of $G(s)$), and in the discrete-time case it is

$$\|G_d\|_2 = \left(\frac{1}{2\pi}\,\mathrm{tr}\left(\int_{-\pi}^{\pi} G_d(e^{j\omega})\,G_d^*(e^{j\omega})\,d\omega\right)\right)^{1/2} \qquad (1.5)$$

$$= \left(\mathrm{tr}\left(\sum_{k=0}^{\infty} g_d(k)\,g_d^\top(k)\,dt\right)\right)^{1/2} \qquad (1.6)$$

where $G_d(e^{j\omega})$ denotes the frequency response of the discrete-time system for unit sampling time and $g_d(k)$ the impulse response of the system (the inverse Z transform of $G_d(z)$). Relations (1.3,1.4) and (1.5,1.6) express the *Parseval theorem*.

A Few Words on Notation

Let us consider the following multivariable system, with a *non-manipulable input* h, a *manipulable input* u, a *regulated output* e and an *informative output* y.

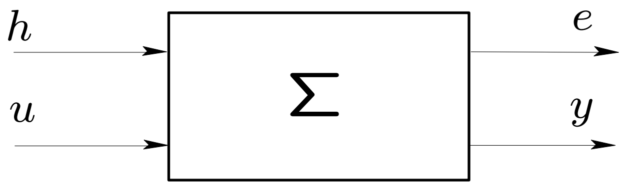


Fig. 1.5. A five-map system.

The system equation are

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + H\,h(t) + B\,u(t) \\ e(t) &= E\,x(t) \qquad\qquad\quad + D_1\,u(t) \\ y(t) &= C\,x(t) + D_2\,h(t) \end{aligned}$$

or

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c|cc} A & H & B \\ \hline E & 0 & D_1 \\ C & D_2 & 0 \end{array}\right] \begin{bmatrix} x(t) \\ h(t) \\ u(t) \end{bmatrix}$$

or else, by using a today very popular notation, we can denote the system *transfer matrix* with

$$G(s) = \left[\begin{array}{c|cc} A & H & B \\ \hline E & 0 & D_1 \\ C & D_2 & 0 \end{array}\right]$$

This has the advntage of pointing out very clearly the existence of feedthrough terms.

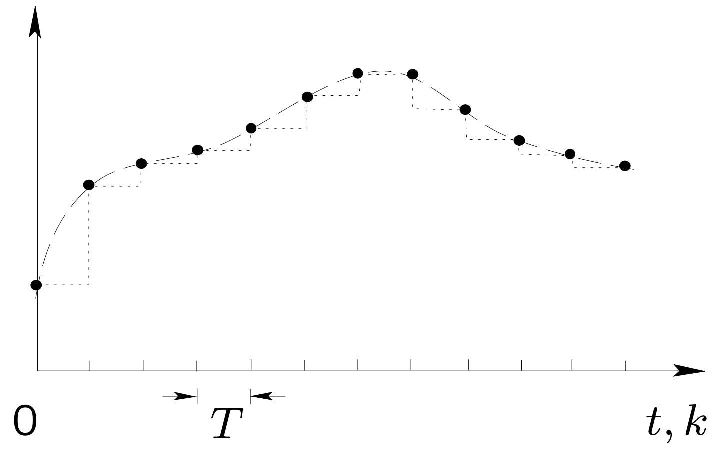


Fig. 1.6. Zero-order hold.

Theorem 1.1 (The sampling theorem) *Let* (A, B) *be controllable. The corresponding zero-order hold pair* (A_d, B_d) *is controllable if the spectrum of* A *does not contain eigenvalues whose imaginary part is multiple of* π/T*, where* T *is the sampling time.*

The regulator Σ_r shown in Fig. 1.4 may be a continuous-time or a discrete-time system ruled by

$$\boxed{\begin{aligned} \dot{z}(t) &= N\,z(t) + M\,y(t) \\ u(t) &= L\,z(t) + K\,y(t) \end{aligned}} \tag{1.7}$$

or

$$\boxed{\begin{aligned} z(k+1) &= N_d\,z(k) + M_d\,y(k) \\ u(k) &= L_d\,z(k) + K_d\,y(k) \end{aligned}} \tag{1.8}$$

but to solve some basic control theory problems like perfect or almost perfect decoupling, perfect or almost perfect tracking and their duals some other signal processing techniques are necessary. These are the *finite delay* and the *convolutor* or *FIR system*.

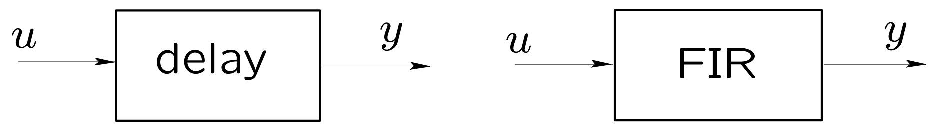


Fig. 1.7. Further components of regulators.

The mathematical models of the delay and of the convolutor or Finite Impulse Response (FIR) system are set as follows.

- Continuous-time:

$$\boxed{y(t) = u(t - t_0)} \tag{1.9}$$

$$\boxed{y(t) = \int_0^{t_f} W(\tau)\, u(t - \tau)\, d\tau} \tag{1.10}$$

where $W(\tau)$, $\tau \in [0, t_f]$, is a $q \times p$ real matrix of time functions, referred to as the *gain* of the FIR system, while $[0, t_f]$ is called the *window* of the FIR system.

- Discrete-time:

$$\boxed{y(k) = u(k - k_0)} \tag{1.11}$$

$$\boxed{y(k) = \sum_{l=0}^{k_f} W(l)\, u(k - l)} \tag{1.12}$$

where $W(k)$, $k \in [0, k_f]$, is a $q \times p$ real matrix of time functions, referred to as the *gain* of the FIR system, while $[0, k_f]$ is called the *window* of the FIR system.

Duality

System Σ : (A, B, C, D);
Dual system $\Sigma^\top$: $(A^\top, C^\top, B^\top, D^\top)$.

FIR system Σ : $W(\tau)$;
Dual FIR system $\Sigma^\top$: $W^\top(\tau)$.

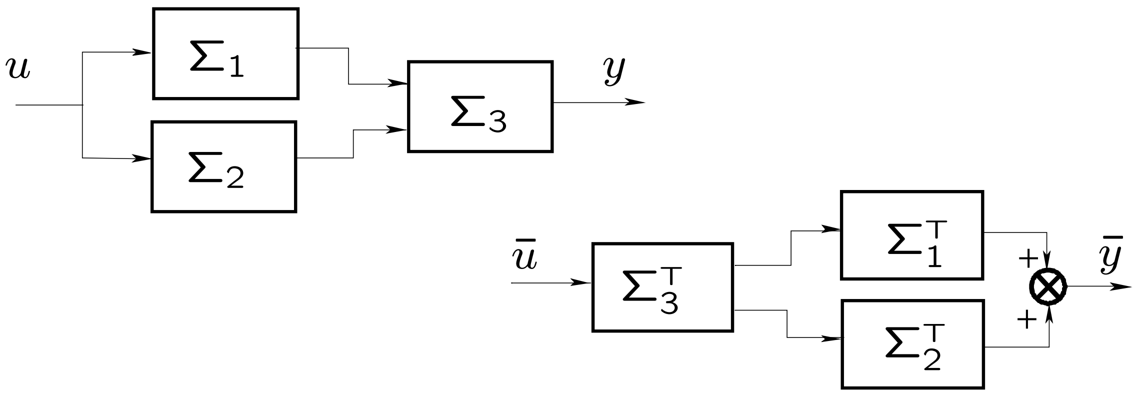


Fig. 1.8. The dual of an interconnection.

The overall dual system is obtained by reversing the order of serially connected systems and interchanging branching points with summing junctions and vice versa. In fact, referring to the above figure, we have:

$$G(s) = \left[\begin{array}{ccc|c} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & 0 & B_2 \\ B_{3,1}C_1 & B_{3,2}C_2 & A_3 & 0 \\ \hline 0 & 0 & C_3 & 0 \end{array}\right]$$

$$G^\top(s) = \left[\begin{array}{ccc|c} A_1^\top & 0 & C_1^\top B_{3,1}^\top & 0 \\ 0 & A_2^\top & C_2^\top B_{3,2}^\top & 0 \\ 0 & 0 & A_3^\top & C_3^\top \\ \hline B_1^\top & B_2^\top & 0 & 0 \end{array}\right]$$

The Seven Properties of Multivariable System:

- Controllability
- Observability
- Internal and External Stability
- Right Invertibility
- Left Invertibility
- Relative Degree
- Phase Minimality

These will be all expressed in geometric terms, i.e., in terms of invariants, controlled invariants and conditioned invariants.

The Problems Considered with Geometric Tools:

- Signal Decoupling and its dual
- Noninteraction
- Perfect Tracking and its dual
- Feedforward Model Matching and its dual
- Feedback Model Matching
- Disturbance Decoupling with output feedback
- Regulation with Internal Model

Furthermore, disturbance decoupling with state or output feedback, measurable signal decoupling, perfect tracking, model matching and their duals can also be geometrically solved in the optimal H_2 sense by referring to the corresponding Hamiltonian systems.

2 - Geometric Approach (GA)

Geometric Approach: is a control theory for multivariable linear systems based on:

- vector spaces and subspaces
- linear tranformations

The geometric approach consists of

- an algebraic part (theoretical)
- an algorithmic part (computational)

Most of the mathematical support is developed in coordinate-free form: this choice leads to simpler and more elegant results, which facilitate insight into the actual meaning of statements and procedures. The computational aspects are considered independently of the theory and handled by means of the standard methods of matrix algebra, once a suitable coordinate system is defined.

Basic Operations

Let $A : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear map between the vector spaces $\mathcal{V}$ and $\mathcal{W}$.
Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be subspaces of $\mathcal{V}$.
Let $\mathcal{H}$ be a subspace of $\mathcal{W}$.

We define:

- *sum:* $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$
- *linear transformation:* $\mathcal{H} = A\,\mathcal{X}$
- *orthogonal complementation:* $\mathcal{Y} = \mathcal{X}^\perp$
- *intersection:* $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}$
- *inverse linear transformation:* $\mathcal{X} = A^{-1}\,\mathcal{H}$

Basic Relations

Let $A : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear map between the vector spaces $\mathcal{V}$ and $\mathcal{W}$.
Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be subspaces of $\mathcal{V}$.
Let $\mathcal{H}$ be a subspace of $\mathcal{W}$.

$$
\begin{array}{rcl}
\mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) & \supseteq & (\mathcal{X} \cap \mathcal{Y}) + (\mathcal{X} \cap \mathcal{Z}) \\
\mathcal{X} + (\mathcal{Y} \cap \mathcal{Z}) & \subseteq & (\mathcal{X} + \mathcal{Y}) \cap (\mathcal{X} + \mathcal{Z}) \\
(\mathcal{X}^{\perp})^{\perp} & = & \mathcal{X} \\
(\mathcal{X} + \mathcal{Y})^{\perp} & = & \mathcal{X}^{\perp} \cap \mathcal{Y}^{\perp} \\
(\mathcal{X} \cap \mathcal{Y})^{\perp} & = & \mathcal{X}^{\perp} + \mathcal{Y}^{\perp} \\
A\,(\mathcal{X} \cap \mathcal{Y}) & \subseteq & A\,\mathcal{X} \cap A\,\mathcal{Y} \\
A\,(\mathcal{X} + \mathcal{Y}) & = & A\,\mathcal{X} + A\,\mathcal{Y} \\
A^{-1}\,(\mathcal{X} \cap \mathcal{Y}) & = & A^{-1}\,\mathcal{X} \cap A^{-1}\,\mathcal{Y} \\
A^{-1}\,(\mathcal{X} + \mathcal{Y}) & \supseteq & A^{-1}\,\mathcal{X} + A^{-1}\,\mathcal{Y} \\
(A^{-1}\,\mathcal{H})^{\perp} & = & A^{T}\mathcal{H}^{\perp} \\
A\,\mathcal{X} \subseteq \mathcal{H} & \Leftrightarrow & A^{T}\mathcal{H}^{\perp} \subseteq \mathcal{X}^{\perp}
\end{array}
$$

Modular Rule: the first two relations hold with the equality sign if one of the involved subspaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ is contained in any of the others.

Grassmann's Rule:

$$\dim(\mathcal{X} + \mathcal{Y}) + \dim(\mathcal{X} \cap \mathcal{Y}) = \dim\mathcal{X} + \dim\mathcal{Y}$$

Grassmann's manifold

Definition 2.1 *Let $\mathcal{X}$ be a vector space.*

$$\mathfrak{Gr}(\mathcal{X}) = \{\mathcal{W} \subseteq \mathcal{X} \mid \mathcal{W} \text{is a subspace of } \mathcal{X}\}$$

Theorem 2.1 $(\mathfrak{Gr}(\mathcal{X}), +, \cap; \subseteq)$ *is a non-distributive modular lattice, whose universal bounds are $\mathcal{X}$ and $\{0\}$.*

This means that, for every pair $\mathcal{X}_1, \mathcal{X}_2$,

1. $\mathcal{X}_1 + \mathcal{X}_2$ is the smallest subspace of $\mathcal{X}$ containing both $\mathcal{X}_1$ and $\mathcal{X}_2$
2. $\mathcal{X}_1 \cap \mathcal{X}_2$ is the largest subspace of $\mathcal{X}$ contained in both $\mathcal{X}_1$ and $\mathcal{X}_2$

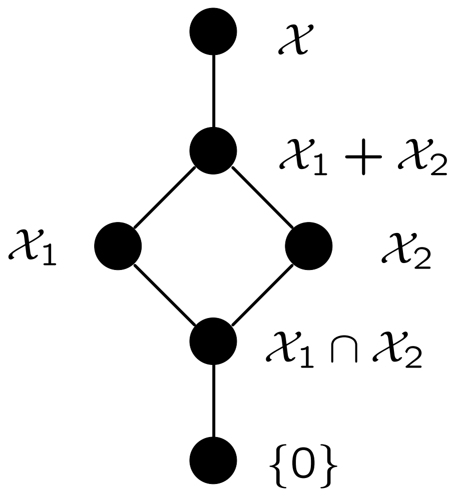


Fig. 2.1. Hasse diagram of the lattice $(\mathfrak{Gr}(\mathcal{X}), +, \cap; \subseteq)$

Invariant Subspaces

Definition 2.2 (Invariant Subspace)
Given a linear map $A : \mathcal{X} \to \mathcal{X}$, *a subspace* $\mathcal{J} \subseteq \mathcal{X}$ *is an* A*-invariant if*

$$\boxed{A\,\mathcal{J} \subseteq \mathcal{J}} \tag{2.1}$$

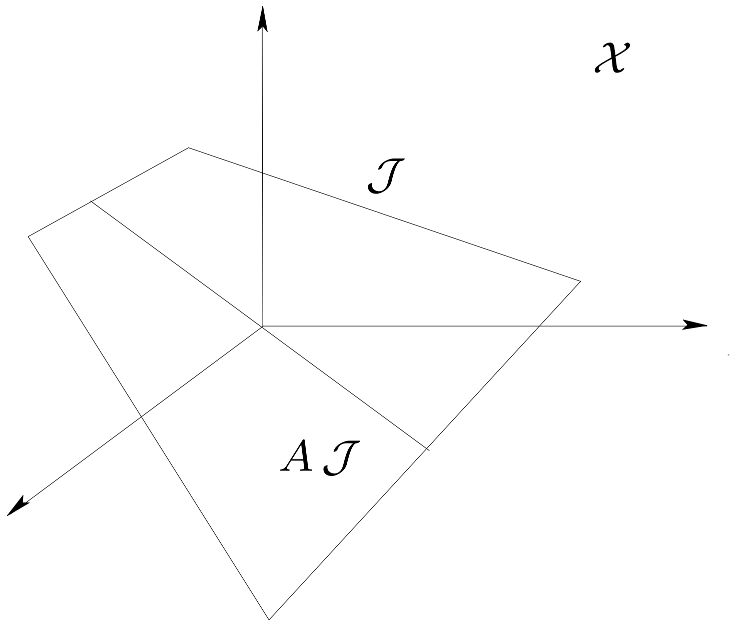


Fig. 2.2. $\mathcal{J}$ is an A-invariant subspace of $\mathcal{X}$

Let J be a basis matrix of the subspace $\mathcal{J} \subseteq \mathcal{X}$; the following statements are equivalent:

- $\mathcal{J}$ is A-invariant
- a matrix X exists such that $A\,J = J\,X$
- $\mathcal{J}$ is a locus of state trajectories of the free system

$$\dot{x}(t) = A\,x(t), \qquad x(0) = x_0 \in \mathcal{J} \tag{2.2}$$

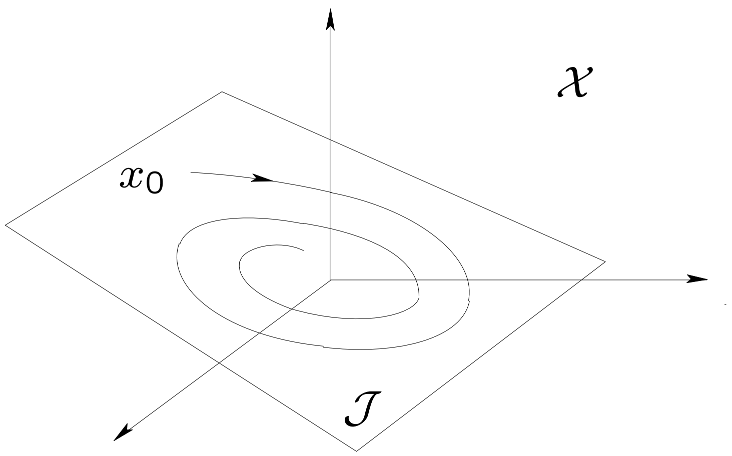


Fig. 2.3. $\mathcal{J}$ as a locus of free trajectories

Theorem 2.2 *Let $\mathcal{J}$ be a h-dimensional A-invariant. Let $\{e_1, \ldots, e_h, \ldots, e_n\}$ be a basis for $\mathcal{X}$ adapted to $\mathcal{J}$. The matrix A, with respect to this basis, has the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$

Let $\mathcal{W}$ be a subspace of $\mathcal{X}$. We define the quotient space $\mathcal{X}/\mathcal{W}$ as the set of all the linear varieties parallel to $\mathcal{W}$, and we denote the canonical projection on the quotient space $\mathcal{X}/\mathcal{W}$ by

$$\pi : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{W}, \quad x \rightarrow \{x\} + \mathcal{W}$$

Consider a partition of the state vector according to the basis defined in Theorem 2.2. We thus obtain the following system:

$$\begin{aligned} \dot{x}_1(t) &= A_{11}\, x_1(t) + A_{12}\, x_2(t) & \qquad x_1(0) &= x_{10} \\ \dot{x}_2(t) &= A_{22}\, x_2(t) & \qquad x_2(0) &= x_{20} \end{aligned}$$

Definition 2.3 (Internal Stability of an Invariant)
The A-invariant $\mathcal{J}$ is internally stable if every state-trajectory that originates on it lies completely on it and converges to the origin as t approaches infinity.

According to the basis defined in Theorem 2.2, if $x_{20} = 0$ ($x(0) \in \mathcal{J}$), then $x_2(t) = 0 \ \forall t$: the motion on $\mathcal{J}$ is described only by A_{11}:

$$\dot{x}_1(t) = A_{11}\, x_1(t), \quad x_1(0) = x_{10}$$

Submatrix A_{11} represents $A|_{\mathcal{J}}$, i.e. the restriction of A to the subspace $\mathcal{J}$.

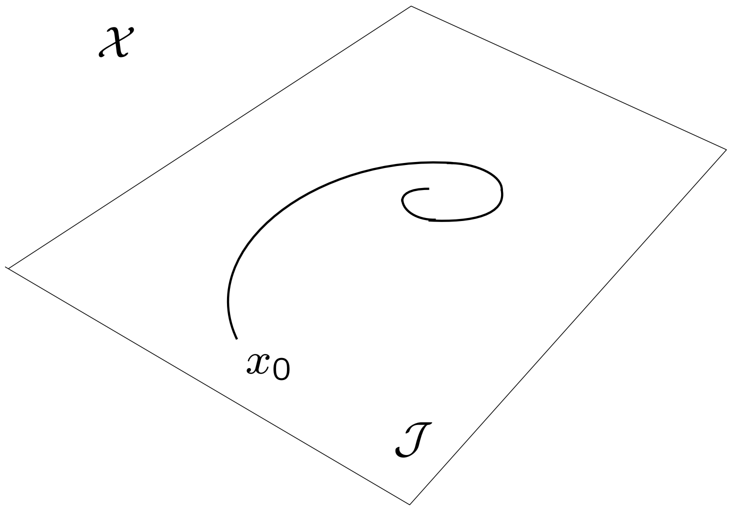


Fig. 2.4. Internal Stability of an Invariant

Since the motion on the invariant $\mathcal{J}$ is completely described by the eigenvalues of $A|_{\mathcal{J}}$, $\mathcal{J}$ is internally stable if and only if $A|_{\mathcal{J}}$ is stable.

Definition 2.4 (External Stability of an Invariant)
The A-invariant $\mathcal{J}$ is externally stable if every state-trajectory that originates out of it converges to $\mathcal{J}$ as t approaches infinity.

According to the basis defined in Theorem 2.2, if $x_{20} \neq 0$ ($x(0) \notin \mathcal{J}$), the state trajectory converges to $\mathcal{J}$ if and only if submatrix A_{22} is stable:

$$\dot{x}_2(t) = A_{22}\, x_2(t), \qquad x_2(0) = x_{20} \neq 0$$

Submatrix A_{22} represents the map induced by A on the quotient space $\mathcal{X}/\mathcal{J}$.

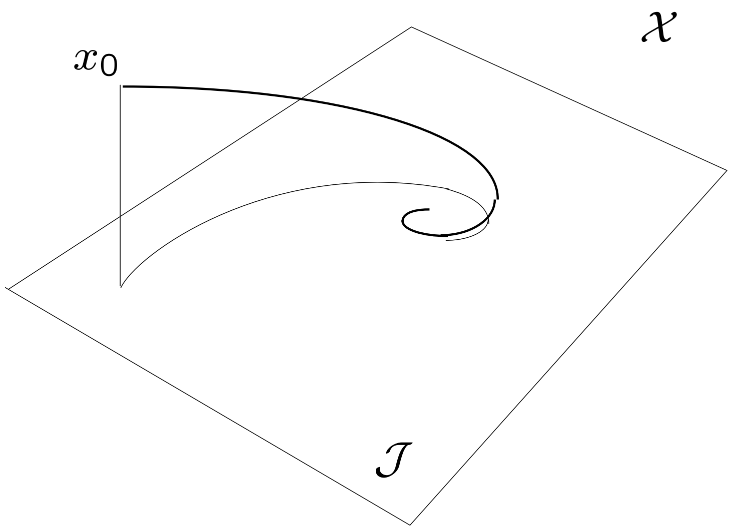


Fig. 2.5. External Stability of an Invariant

Since the dynamics of the second component of the state only depend from the eigenvalues of $A|_{\mathcal{X}/\mathcal{J}}$, $\mathcal{J}$ is externally stable if and only if $A|_{\mathcal{X}/\mathcal{J}}$ is stable.

Definition 2.5 *Let $\mathcal{E}$ be a subspace of $\mathcal{X}$;*

$$\mathcal{J}(A,\mathcal{E}) = \{\mathcal{J} \in \mathfrak{Gr}(\mathcal{X}) \mid A\,\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{J} \subseteq \mathcal{E}\}$$

Proposition 2.1 $(\mathcal{J}(A,\mathcal{E}),+,\cap;\subseteq)$ *is a non-distributive modular lattice, whose minimum is* $\{0\}$ *and whose maximum is*

$$\max \mathcal{J}(\mathsf{A},\mathcal{E}) = \sum_{\mathcal{J}\in\mathcal{J}(\mathsf{A},\mathcal{E})} \mathcal{J}$$

Algorithm 2.1 *(Computation of* $\max\mathcal{J}(A,\mathcal{E})$*)*

$$\boxed{\begin{array}{ll} \mathcal{Z}_1 = \mathcal{E} & \\ \mathcal{Z}_i = \mathcal{E} \cap A^{-1}\,\mathcal{Z}_{i-1} & i = 2,3,\ldots \end{array}} \tag{2.3}$$

$\max\mathcal{J}(A,\mathcal{E}) = \mathcal{E} \cap A^{-1}\max\mathcal{J}(A,\mathcal{E})$ *is obtained when the sequence stops.*

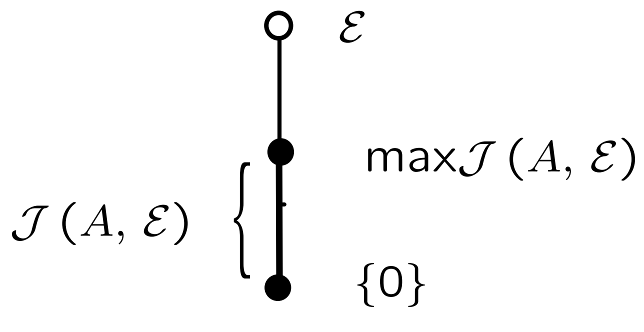


Fig. 2.6. The lattice $\mathcal{J}(A,\mathcal{E})$

Definition 2.6 *Let $\mathcal{D}$ be a subspace of $\mathcal{X}$;*

$$J\,(A,\mathcal{D}) = \{\mathcal{J} \in \mathfrak{Gr}(\mathcal{X}) \;\mid\; A\,\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{J} \supseteq \mathcal{D}\}$$

Proposition 2.2 $(\mathcal{J}\,(A,\mathcal{E}),+,\cap;\subseteq)$ *is a non-distributive modular lattice, whose maximum is $\mathcal{X}$ and whose maximum is*

$$\min \mathcal{J}\,(\mathrm{A},\mathcal{D}) = \bigcap_{\mathcal{J}\in\mathcal{J}\,(\mathrm{A},\mathcal{D})} \mathcal{J}$$

Algorithm 2.2 *(Computation of* $\min\mathcal{J}(A,\mathcal{D})$*)*

$$\boxed{\begin{array}{ll} \mathcal{Z}_1 = \mathcal{D} & \\ \mathcal{Z}_i = \mathcal{D} + A\,\mathcal{Z}_{i-1} & i = 2, 3, \ldots \end{array}} \qquad (2.4)$$

$\min\mathcal{J}(A,\mathcal{D}) = \mathcal{D} + A\,\min\mathcal{J}(A,\mathcal{D})$ *is obtained when the sequence stops.*

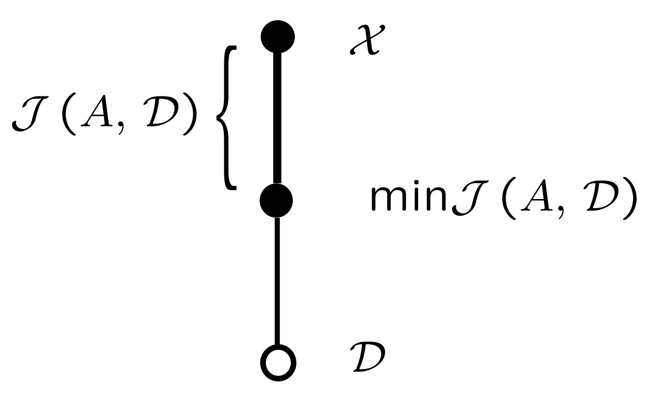


Fig. 2.7. The lattice $\mathcal{J}\,(A,\,\mathcal{D})$

Definition 2.7 *Let $\mathcal{E}$ and $\mathcal{D}$ be subspaces of $\mathcal{X}$; we define*

$$\Theta\,(\mathcal{D},\mathcal{E}) = \{\mathcal{J} \in \mathfrak{Gr}(\mathcal{X}) \mid A\,\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{D} \subseteq \mathcal{J} \subseteq \mathcal{E}\}$$

Proposition 2.3 $(\Theta(\mathcal{D},\mathcal{E}),+,\cap;\subseteq)$ *is a non-distributive modular lattice.*

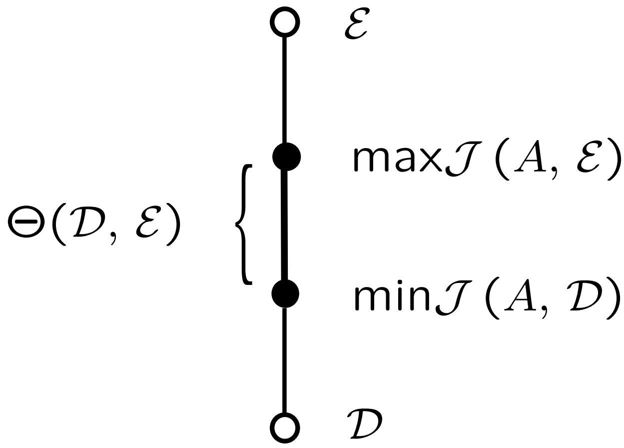


Fig. 2.8. The lattice $(\Theta(\mathcal{D},\,\mathcal{E}),+,\cap;\subseteq)$

The above lattice is non-empty if and only if $\mathcal{D} \subseteq \max\mathcal{J}\,(A,\,\mathcal{E})$ or $\min\mathcal{J}\,(A,\,\mathcal{D}) \subseteq \mathcal{E}$.

Property 2.1 *Dualities*

$$\max\mathcal{J}(A,\mathcal{C}) = \min\mathcal{J}(A^T,\mathcal{C}^\perp)^\perp$$

$$\min\mathcal{J}(A,\mathcal{B}) = \max\mathcal{J}(A^T,\mathcal{B}^\perp)^\perp$$

Controllability and Observability

Consider a triple (A, B, C), i.e., refer to

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

The reachability subspace of (A, B), i.e., the set of all the states that can be reached from the origin in any finite time by means of control actions, is $\mathcal{R} = \min \mathcal{J}(A, \operatorname{im} B)$.

- if $\mathcal{R} = \mathcal{X}$, the pair (A, B) is said to be *completely controllable*
- if $\mathcal{R} \neq \mathcal{X}$ but $\mathcal{R}$ is externally stabilizable, the pair (A, B) is said to be *stabilizable*

The unobservability subspace of (A, C), i.e., the set of all the initial states that cannot be recognized from the output function, is $\mathcal{Q} = \max \mathcal{J}(A, \ker C)$.

- if $\mathcal{Q} = \{0\}$, the pair (A, C) is said to be *completely observable*
- if $\mathcal{Q} \neq \{0\}$ but $\mathcal{Q}$ is internally stabilizable, the pair (A, C) is said to be *detectable*

Pole Assignment

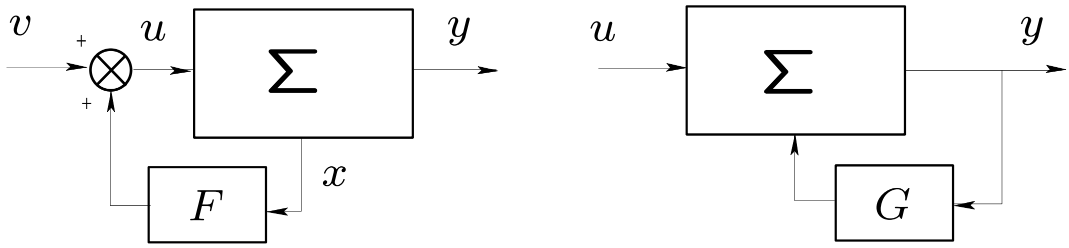


Fig. 2.9. State feedback and output injection

State feedback

$$\begin{aligned} \dot{x}(t) &= (A+BF)\,x(t) + B\,v(t) \\ y(t) &= C\,x(t) \end{aligned} \tag{2.5}$$

Output injection

$$\begin{aligned} \dot{x}(t) &= (A+GC)\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \tag{2.6}$$

The eigenvalues of $A+BF$ are arbitrarily assignable by a suitable choiche of F if and only if the system is completely controllable and those of $A+GC$ are arbitrarily assignable by a suitable choice of G if and only if the system is completely observable.

A more general dual setting is the following

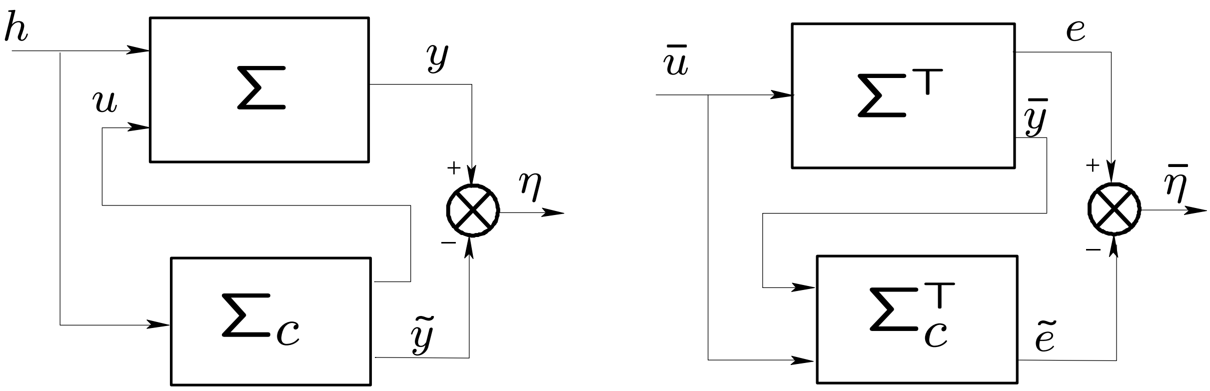


Fig. 2.10. Dual observer and observer.

The overall system and its dual are, in this case,

$$G(s) = \left[\begin{array}{cc|c} A & BF & H \\ 0 & A+BF & H \\ \hline C & -C & 0 \end{array}\right]$$

$$G^{\top}(s) = \left[\begin{array}{cc|c} A^{\top} & 0 & C^{\top} \\ F^{\top}B^{\top} & A^{\top}+F^{\top}B^{\top} & -C^{\top} \\ \hline H^{\top} & H^{\top} & 0 \end{array}\right]$$

$$\equiv \left[\begin{array}{cc|c} A^{\top} & 0 & C^{\top} \\ -F^{\top}B^{\top} & A^{\top}+F^{\top}B^{\top} & C^{\top} \\ \hline H^{\top} & -H^{\top} & 0 \end{array}\right]$$

Note that the sign of both the input and output of the observer have been changed with respect to the strict dual to obtain the input-output equivalent layout shown in the figure at right.

Controlled and Conditioned Invariants

Definition 2.8 *Given a linear map* $A : \mathcal{X} \to \mathcal{X}$ *and a subspace* $\mathcal{B} \subseteq \mathcal{X}$*, a subspace* $\mathcal{V} \subseteq \mathcal{X}$ *is an* $(A, \mathcal{B})$*-controlled invariant if*

$$A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} \tag{2.7}$$

Let B be a basis matrix of $\mathcal{B}$: the following statements are equivalent:

- $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant
- a matrix F exists such that $(A + BF)\,\mathcal{V} \subseteq \mathcal{V}$
- matrices X and U exist such that $A\,V = V\,X + B\,U$
- $\mathcal{V}$ is a locus of trajectories of the pair (A, B)

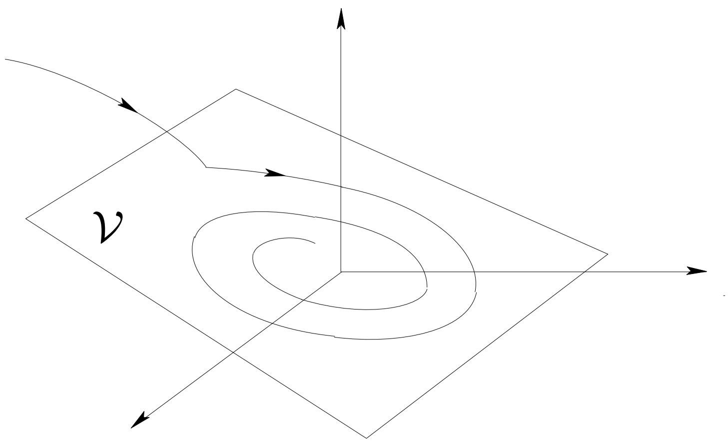


Fig. 2.11. The controlled invariant as a locus of trajectories.

The sum of any two controlled invariants is a controlled invariant, while the intersection is not.

We denote by $\mathcal{V}(A, \mathcal{B}, \mathcal{E})$ the set of all the $(A, \mathcal{B})$-controlled invariants contained in the subspace $\mathcal{E} \subseteq \mathcal{X}$.

Property 2.2 $(\mathcal{V}(A, \mathcal{B}, \mathcal{E}), +; \subseteq)$ *is a non-distributive modular upper semilattice.*

We will denote by $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} = \max \mathcal{V}(A, \mathcal{B}, \mathcal{E})$ the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{E}$.

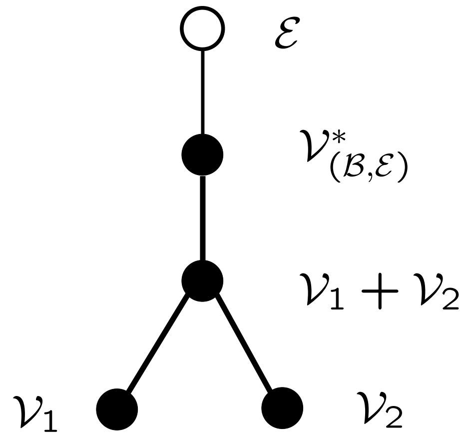


Fig. 2.12. The upper semilattice $(\mathcal{V}(A, \mathcal{B}, \mathcal{E}), +; \subseteq)$

<u>Meaning of $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$</u>

$\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ is the maximum locus of trajectories contained in $\mathcal{E}$: this means that a suitable control action can mantain the trajectory on $\mathcal{E}$ if and only if $x_0 \in \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$.

We will use the symbol $\mathcal{V}^*$ for $\max \mathcal{V}(A, \operatorname{im} B, \ker C)$: it is the maximal locus of controlled state trajectories such that the output is identically zero.

Definition 2.9 *Given a linear map $A : \mathcal{X} \to \mathcal{X}$ and a subspace $\mathcal{C} \subseteq \mathcal{X}$, a subspace $\mathcal{S} \subseteq \mathcal{X}$ is an $(A,\mathcal{C})$-conditioned invariant if*

$$A\,(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S} \tag{2.8}$$

Let C be a basis matrix of $\mathcal{C}$. The following statements are equivalent:

- $\mathcal{S}$ is an $(A,\,\mathcal{C})$-conditioned invariant
- a matrix G exists such that $(A + GC)\,\mathcal{S} \subseteq \mathcal{S}$

The intersection of any two conditioned invariants is a conditioned invariant, while the sum is not.
We denote by $\mathcal{S}\,(A,\,\mathcal{C},\,\mathcal{D})$ the set of all the $(A,\,\mathcal{C})$-conditioned invariants containing the subspace $\mathcal{D} \subseteq \mathcal{X}$.

Property 2.3 $(\mathcal{S}\,(A,\,\mathcal{C},\,\mathcal{D}), \cap; \subseteq)$ *is a non-distributive modular lower semilattice.*

We denote by $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} = \min \mathcal{S}\,(A,\,\mathcal{C},\,\mathcal{D})$ the minimal $(A,\,\mathcal{C})$-conditioned invariants containing the subspace $\mathcal{D} \subseteq \mathcal{X}$ and by $\mathcal{S}^*$ the subspace $\min\mathcal{S}(A, \ker C, \mathrm{im} B)$, the most important conditioned invariant concerning the triple (A, B, C).

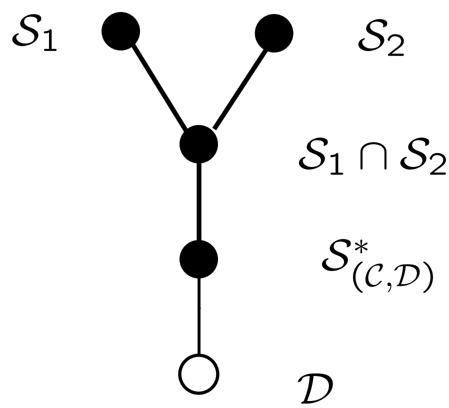


Fig. 2.13. The upper semilattice $(\mathcal{S}\,(A,\,\mathcal{C},\,\mathcal{D}), \cap; \subseteq)$

Algorithms

Algorithm 2.3 *(Computation of $\mathcal{S}^*_{(\mathcal{C},\mathcal{B})}$)*

$$\boxed{\begin{array}{ll}\mathcal{S}_0 = \mathcal{B} & \\ \mathcal{S}_i = \mathcal{B} + A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) & i = 2, 3, \ldots\end{array}} \tag{2.9}$$

$\mathcal{S}^*_{(\mathcal{C},\mathcal{B})} = \mathcal{B} + A\,(\mathcal{S}^*_{(\mathcal{C},\mathcal{B})} \cap \mathcal{C})$ *is obtained when the sequence stops.*

Algorithm 2.4 *(Computation of $\mathcal{V}^*_{(\mathcal{B},\mathcal{C})}$)*

$$\boxed{\begin{array}{ll}\mathcal{V}_0 = \mathcal{C} & \\ \mathcal{V}_i = \mathcal{C} \cap A^{-1}\,(\mathcal{V}_{i-1} + \mathcal{B}) & i = 2, 3, \ldots\end{array}} \tag{2.10}$$

$\mathcal{V}^*_{(\mathcal{B},\mathcal{C})} = \mathcal{C} \cap A^{-1}\,(\mathcal{V}^*_{(\mathcal{B},\mathcal{C})} + \mathcal{B})$ *is obtained when the sequence stops.*

Meaning of $\mathcal{S}^*$

Refer to a discrete-time triple (A, B, C): Algorithm 2.3 at the generic i-th step provides the set of all states reachable from the origin through trajectories having all states but the last one belonging to Ker C, hence invisible at the output. Thus $\mathcal{S}^*$ is the maximum subspace of $\mathcal{X}$ reachable from the origin through this type of trajectories in ρ steps, being ρ the number of iterations required for the sequence $\mathcal{S}_i$ to converge to $\mathcal{S}^*$.

Duality

Controlled and conditioned invariant are dual to each other.

Property 2.4 *Given a subspace $\mathcal{L} \subseteq \mathcal{X}$, the orthogonal complement of an $(A, \mathcal{L})$-controlled invariant is an $(A^T, \mathcal{L}^\perp)$-conditioned invariant (and viceversa).*

As a consequence of the previous property the following relations hold:

$$\max \mathcal{V}(A, \mathcal{B}, \mathcal{C}) = [\min \mathcal{S}(A^T, \mathcal{B}^\perp, \mathcal{C}^\perp)]^\perp \qquad (2.11)$$

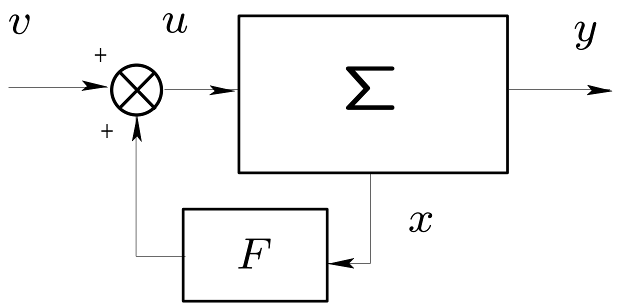
$$\min \mathcal{S}(A, \mathcal{C}, \mathcal{B}) = [\max \mathcal{V}(A^T, \mathcal{C}^\perp, \mathcal{B}^\perp)]^\perp \qquad (2.12)$$

Reachable subspace on a controlled invariant

Given the triple (A,B,C), the $(A,\,\mathrm{im}\,B)$-controlled invariant $\mathcal{V}$ and the input function $u(t) = F\,x(t) + v(t)$:

$$\dot{x}(t) = (A + B\,F)\,x(t) + B\,v(t)\,, \quad x_0 \in \mathcal{V}$$
$$y(t) = C\,x(t)$$

the state trajectories belong to $\mathcal{V}$ if and only if $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ and $v(t) \in B^{-1}\,\mathcal{V}\;\forall t$:



We denote by $R_{\mathcal{V}}$ the reachable subspace from the origin by state trajectories constrained to belong to $\mathcal{V}$:

$$\mathcal{R}_{\mathcal{V}} = \min \mathcal{J}\,(A + B\,F,\, \mathcal{V} \cap \mathrm{im}\,B)$$

Being $(A + B\,F)$-invariant, $R_{\mathcal{V}}$ is an $(A,\,\mathrm{im}\,B)$ controlled invariant itself.

Theorem 2.3 *The following equality holds:*

$$\mathcal{R}_{\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}} = \min \mathcal{J}\,(A + B\,F,\, \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathrm{im}\,B) = \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B})}$$

Self-bounded Controlled Invariants

Definition 2.10 *Let $\mathcal{B}, \mathcal{E}$ be subspaces of $\mathcal{X}$. Let $\mathcal{V}$ be an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{E}$; $\mathcal{V}$ is said to be self-bounded with respect to $\mathcal{E}$ if*

$$\mathcal{V} \supseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{B} \tag{2.13}$$

We define

$$\Phi(\mathcal{B}, \mathcal{E}) = \{\mathcal{V} \in \mathcal{V}\,(A, \mathcal{B}, \mathcal{E}) \mid \; \mathcal{V} \supseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{B}\}$$

If $\mathcal{V} \in \Phi(\mathcal{B}, \mathcal{E})$, then $\mathcal{V}$ cannot be exited by means of any trajectory on $\mathcal{E}$.

Theorem 2.4 $(\Phi(\mathcal{B}, \mathcal{E}), +, \cap; \subseteq)$ *is a non-distributive modular lattice, whose maximum is $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ and whose minimum is $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B})}$.*

Being $\mathcal{D}$ a subspace of $\mathcal{X}$, if $\mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ it can be proven that $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} = \mathcal{V}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}$, so that

$$\Phi(\mathcal{B}+\mathcal{D}, \mathcal{E}) = \{\mathcal{V} \in \mathcal{V}\,(A, \mathcal{B}+\mathcal{D}, \mathcal{E}) \mid \; \mathcal{V} \supseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap (\mathcal{B}+\mathcal{D})\}$$

is the lattice of $(A, \mathcal{B}+\mathcal{D})$-controlled invariants with forcing action in $\mathcal{B}+\mathcal{D}$ self-bounded with respect to $\mathcal{E}$, whose maximum is $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ and whose minimum is

$$\mathcal{V}_m = \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{D})}$$

Self-Hidden Conditioned Invariants

Definition 2.11 *Let $\mathcal{C}, \mathcal{D}$ be subspaces of $\mathcal{X}$. Let $\mathcal{S}$ be an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{D}$; $\mathcal{S}$ is said to be self-hidden with respect to $\mathcal{D}$ if*

$$\mathcal{S} \subseteq \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} + \mathcal{E} \tag{2.14}$$

We define

$$\Psi(\mathcal{C}, \mathcal{D}) = \{\mathcal{S} \in \mathcal{S}\,(A, \mathcal{C}, \mathcal{D}) \mid\ \mathcal{S} \subseteq \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} + \mathcal{E}\}$$

Theorem 2.5 $(\Psi(\mathcal{C}, \mathcal{D}), +, \cap; \subseteq)$ *is a non-distributive modular lattice, whose maximum is $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} + \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$ and whose minimum is $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})}$.*

Being $\mathcal{E}$ a subspace of $\mathcal{X}$, if $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subseteq \mathcal{E}$ it can be proven that $\mathcal{S}^*_{(\mathcal{D},\mathcal{C}\cap\mathcal{E})} = \mathcal{S}^*_{(\mathcal{D},\mathcal{C}\cap\mathcal{E})}$, so that

$$\Psi(\mathcal{D}, \mathcal{C} \cap \mathcal{E}) = \{\mathcal{S} \in \mathcal{S}\,(A, \mathcal{C} \cap \mathcal{E}, \mathcal{D}) \mid\ \mathcal{S} \subseteq \mathcal{S}^*_{(\mathcal{D},\mathcal{E})} + (\mathcal{C} \cap \mathcal{E})\}$$

is the lattice of $(A, \mathcal{C} \cap \mathcal{E})$-conditioned invariants self-hidden with respect to $\mathcal{D}$, whose minimum is $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})}$ and whose maximum is

$$\mathcal{S}_M = \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} + \mathcal{V}^*_{(\mathcal{D},\mathcal{C}\cap\mathcal{E})}$$

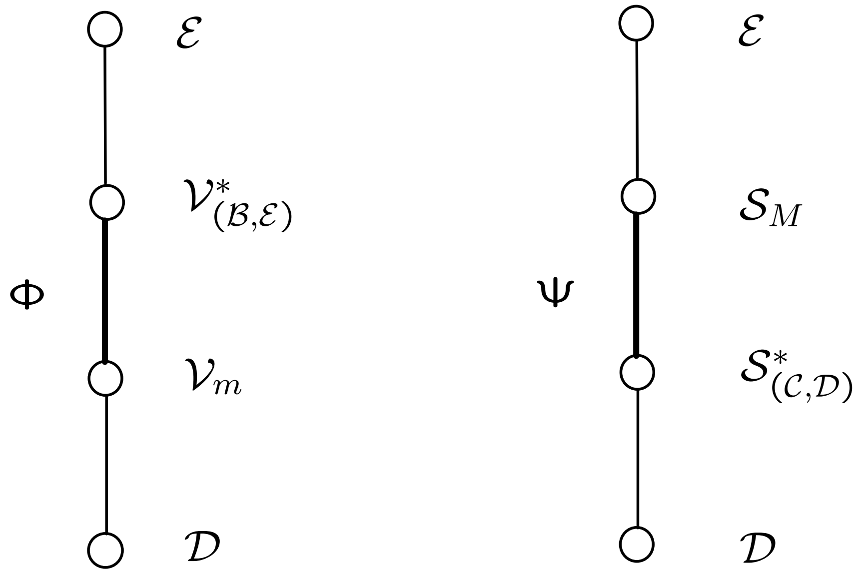


Fig. 2.14. The lattices Φ and Ψ.

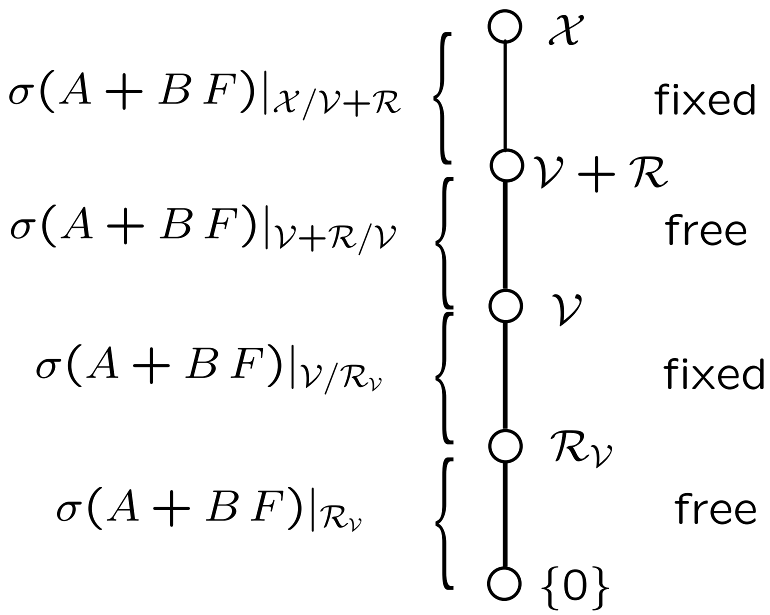
The main theorem and its dual

Theorem 2.6 *Let $\mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$. There exists at least one internally stabilizable $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ such that $\mathcal{D} \subseteq \mathcal{V} \subseteq \mathcal{E}$ if and only if $\mathcal{V}_m$ is internally stabilizable.*

Theorem 2.7 *Let $\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subseteq \mathcal{E}$. There exists at least one externally stabilizable $(A, \mathcal{C})$-conditioned invariant $\mathcal{S}$ such that $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{E}$ if and only if $\mathcal{S}_M$ is internally stabilizable.*

Internal and External Eigenvalues

Let $\mathcal{V}$ be an $(A, \mathcal{B})$-controlled invariant and F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$. Being $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$, we can partition the spectrum of $(A + B\,F)$ as in figure.



$\mathcal{V}$ is said to be

- internally stablizable if $\forall x_0 \in \mathcal{V}$, the trajectory can be mantained on $\mathcal{V}$ converging to the origin by a suitable control action; this happens if and only if $(A + B\,F)|_{\mathcal{V}}$ is stable, i.e. if and only if $(A + B\,F)|_{\mathcal{V}/R_\mathcal{V}}$ is stable

- externally stabilizable if $\forall x_0 \notin \mathcal{V}$, the trajectory converge to $\mathcal{V}$ by a suitable control action; this happens if and only if $(A + B\,F)|_{\mathcal{X}/\mathcal{V}}$ is stable, i.e. if and only if $(A + B\,F)|_{\mathcal{X}/\mathcal{V}+\mathcal{R}}$ is stable

Computation of matrix F

Let V be a basis matrix of the $(A, \mathrm{im}\, B)$-controlled invariant $\mathcal{V}$. From equation $A\,V = V\,X + B\,U$ it is possible to derive a matrix F such that $(A{+}B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ as follows:

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \end{bmatrix}^{+} A\,V + \Gamma\,\alpha$$

where the symbol $^{+}$ denotes the pseudoinverse Γ a basis matrix of $\ker \begin{bmatrix} V & B \end{bmatrix}$ and α is an arbitrary vector. Then compute

$$F = -U\,(V^T\,V)^{-1}\,V^T$$

The degree of freedom α allows to assign the internal assignable eigenstructure of $\mathcal{V}$. If the system is left-invertible, $\Gamma = 0$ and no internal eigenvalue can be assigned: in fact $\mathcal{R}_{\mathcal{V}*} = \{0\}$.

Algorithmic Procedures

A subspace $\mathcal{X}$ is given through a basis matrix of maximum rank X such that $\mathcal{X} = \text{im}X$.

The operations on subspaces are all performed through an orthonormalization process (*subroutine ima.m* in Matlab) that computes an orthonormal basis of a set of vectors in $\mathbb{R}^n$ by using methods of the *Gauss–Jordan* or *Gram–Schmidt* type.

Computational support with Matlab

Q = ima(A,p) Orthonormalization.

Q = ortco(A) Complementary orthogonalization.

Q = sums(A,B) Sum of subspaces.

Q = ints(A,B) Intersection of subspaces.

Q = invt(A,X) Inverse transform of a subspace.

Q = ker(A) Kernel of a matrix.

Q = mininv(A) Min A-invariant containing imB.

Q = maxinv(A) Max A-invariant contained in imC.

Q = mainco(A,B,X) Maximal $(A, \text{im}B)$-controlled invariant contained in im$X$

Q = miinco(A,C,X) Minimal $(A, \text{im}C)$-conditioned invariant containing im$X$

In program *ima* the flag *p* allows for permutations of the input column vectors.

[P,Q] = stabi(A,X) Matrices for the internal and external stability of the A-invariant im$X$

[P,Q] = stabv(A,B,X) Matrices for the internal and external stability of the $(A, \text{im}B)$-controlled invariant im$X$

F = effe(A,B,X) State feedback matrix such that $(A+BF)\,\text{im}X \subseteq \text{im}X$

F = effest(A,B,X,ei,ee) Stabilizing feedback matrix setting the assignable eigenvalues as ei and the assignable external eigenvalues as ee

Internal and External Stability of an Invariant

Algorithm 2.5 *Matrices P and Q representing $A|_{\mathcal{J}}$ and $A|_{\mathcal{X}/\mathcal{J}}$ up to an isomorphism, are derived as follows. Let us consider the similarity transformation $T := [J \; T_2]$, with* im$J = \mathcal{J}$ *(J is a basis matrix of $\mathcal{J}$) and T_2 such that T is nonsingular. In the new basis the linear transformation A is expressed by*

$$A' = T^{-1} A \, T = \begin{bmatrix} A'_{11} & A'_{12} \\ O & A'_{22} \end{bmatrix} \tag{2.15}$$

The requested matrices are defined as $P := A'_{11}$, $Q := A'_{22}$.

Complementability of an Invariant

An A-invariant $\mathcal{J} \subseteq \mathcal{X}$ is said to be *complementable* if an A-invariant $\mathcal{J}_c$ exixts such that $\mathcal{J} \oplus \mathcal{J}_c = \mathcal{X}$; if so, $\mathcal{J}_c$ is called a *complement* of $\mathcal{J}$.

Algorithm 2.6 *Let us consider again the change of basis introduced in Algorithm 2.5. $\mathcal{J}$ is complementable if and only if the Sylvester equation*

$$A'_{11} \, X - X \, A'_{22} = -A'_{12} \tag{2.16}$$

admits a solution. If so, a basis matrix of $\mathcal{J}_c$ is given by $J_c := J \, X + T_2$.

Algorithm 2.7 *Computation of matrix F such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$. Let V be a basis matrix of the $(A,\mathcal{B})$-controlled invariant $\mathcal{V}$. First, compute*

$$\begin{bmatrix} X \\ U \end{bmatrix} = [V B]^{+} A\, V$$

where the symbol $^{+}$ denotes the pseudoinverse. Then, compute

$$F := -U\,(V^T V)^{-1}\, V^T$$

Algorithm 2.8 *Computation of the internal unassignable eigenstructure of an $(A,\mathcal{B})$-controlled invariant. A matrix P representing the map $(A+BF)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}}$ up to an isomorphism, is derived as follows. Let us consider the similarity transformation $T := [T_1\; T_2\; T_3]$, with $\mathrm{im}T_1 = \mathcal{R}_{\mathcal{V}}$, $\mathrm{im}T_2 = \mathcal{V}$ and T_3 such that T is nonsingular. In the new basis matrix $A+BF$ is expressed by*

$$(A+BF)' = T^{-1}(A+BF)\,T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}$$

The requested matrix is $P := A'_{22}$.

3 - System Properties and Basic Problems

Invariant Zeros

Roughly speaking, an invariant zero corresponds to a mode that, if suitably injected at the input of a dynamic system, can be nulled at the output by a suitable choice of the initial state.

Definition 3.1 *The* <u>*invariant zeros*</u> *of* (A, B, C) *are the internal unassignable eigenvalues of* $\mathcal{V}^*$. *The* <u>*invariant zero structure*</u> *of* (A, B, C) *is the internal unassignable eigenstructure of* $\mathcal{V}^*$.

Recall that $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$. The invariant zeros are the eigenvalues of the map $(A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$, where F denotes any matrix such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$.

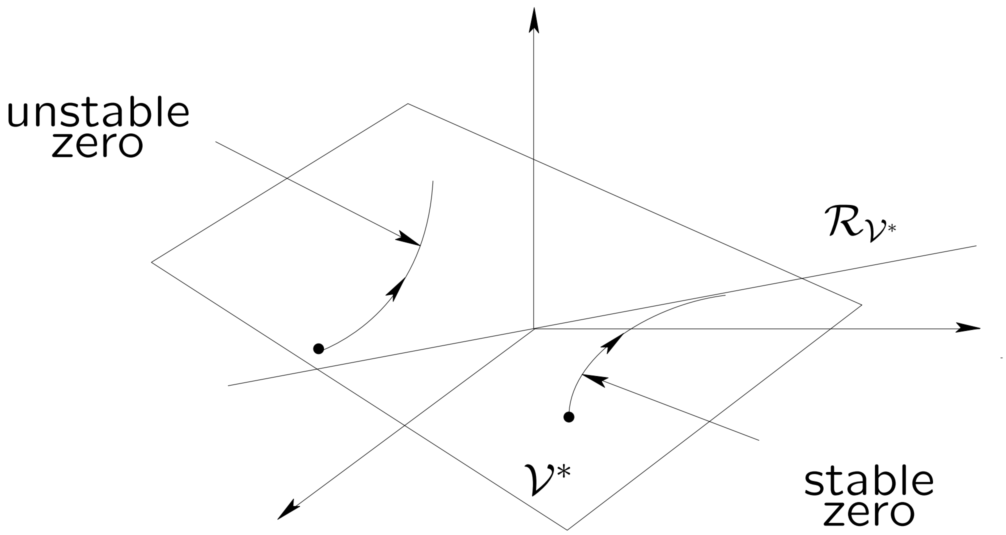


Fig. 3.1. Decomposition of the map $(A+BF)|_{\mathcal{V}^*}$

 [InvariantZeros]

Consider the change of basis defined by transformation $T = [T_1, T_2, T_3]$ with $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im} T_2 = \mathcal{V}^*$, then

$$T^{-1}(A + BF)T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & 0 & A'_{33} \end{bmatrix}.$$

Moreover, being $\mathcal{B} \in \mathcal{S}^*$ and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$,

$$T^{-1}B = \begin{bmatrix} B'_{11} & 0 & B'_{13} \\ 0 & 0 & 0 \\ 0 & B'_{23} & B'_{33} \end{bmatrix}.$$

The invariant zeros of (A, B, C) are the eigenvalues of matrix A'_{22}.

Computational support with Matlab

z = gazero(A,B,C,[D])	Invariant zeros of (A, B, C) or (A, B, C, D)

[exzeros.m]

Property 3.1 *Let W be a real $m \times m$ matrix having the invariant zero structure of (A, B, C) as eigenstructure. A real $p \times m$ matrix L and a real $n \times m$ matrix X exist, with (W, X) observable, such that by applying to (A, B, C) the input function*

$$u(t) = L\, e^{Wt}\, v_0 \tag{3.1}$$

where $v_0 \in \mathbb{R}^m$ denotes an arbitrary column vector, and starting from the initial state $x_0 = X\, v_0$, the output $y(\cdot)$ is identically zero, while the state evolution (on ker$C$*) is described by*

$$x(t) = X\, e^{Wt}\, v_0 \tag{3.2}$$

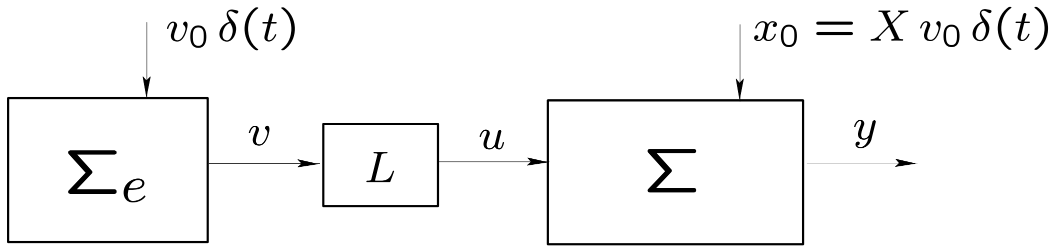


Fig. 3.2. The meaning of Property 3.1

Remark. In the discrete-time case equations (3.1) and (3.2) are replaced by $u(k) = LW^k\, v_0$ and $x(k) = XW^k\, v_0$, respectively.

Left and Right Invertibility

Consider the standard continuous-time system – triple (A, B, C)

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned} \tag{3.3}$$

or the standard discrete-time system – triple (A_d, B_d, C_d)

$$\begin{aligned} x(k+1) &= A_d\,x(k) + B_d\,u(k) \\ y(k) &= C_d\,x(k) \end{aligned} \tag{3.4}$$

We consider triples since they provide a better insight and extension to quadruples is straightforward – obtainable with a suitable state extension.

Systems (3.3) and (3.4) with $x(0)=0$ define linear maps $\mathcal{T}_f : \mathcal{U}_f \rightarrow \mathcal{Y}_f$ from the space $\mathcal{U}_f$ of the admissible input functions to the functional space $\mathcal{Y}_f$ of the zero-state responses. These maps are defined by the convolution integral and the convolution summation

$$y(t) = C \int_0^t e^{A\,(t-\tau)}\,B\,u(\tau)\,d\tau \tag{3.5}$$

$$y(k) = C_d \sum_{h=0}^{k-1} A_d^{(k-h-1)}\,B_d\,u(h) \tag{3.6}$$

The admissible input functions are:
- piecewise continuous and bounded functions of time t for (3.5);
- bounded functions of the discrete time k for (3.6).

Definition 3.2 *Assume that B has maximal rank. System (3.3) is said to be* <u>*invertible*</u> *(left-invertible) if, given any output function $y(t)$, $t \in [0, t_1]$ $t_1 > 0$ belonging to* $\text{im}\mathcal{T}_f$*, there exists a unique input function $u(t)$, $t \in [0, t_1)$, such that (3.5) holds.*

Definition 3.3 *Assume that B_d has maximal rank. System (3.4) is said to be* <u>*invertible*</u> *(left-invertible) if, given any output function $y(k)$, $k \in [0, k_1]$, $k_1 \geq n$ belonging to* $\text{im}\mathcal{T}_f$ *there exists a unique input function $u(k)$, $k \in [0, k_1 - 1]$ such that (3.6) holds.*

?! $[k-1]$

Definition 3.4 *Assume that C has maximal rank. System (3.3) is said to be* <u>*functionally controllable*</u> *(right-invertible) if there exists an integer $\rho \geq 1$ such that, given any output function $y(t)$, $t \in [0, t_1]$, $t_1 > 0$ with ρ-th derivative piecewise continuous and such that $y(0) = 0, \ldots y^{(\rho)}(0) = 0$, there exists at least one input function $u(t)$, $t \in [0, t_1)$ such that (3.5) holds. The minimum value of ρ satisfying the above statement is called the relative degree of the system.*

Definition 3.5 *Assume that C_d has maximal rank. System (3.4) is said to be* <u>*functionally controllable*</u> *(right-invertible) if there exists an integer $\rho \geq 1$ such that, given an output function $y(k)$, $k \in [0, k_1]$, $k_1 \geq \rho$ such that $y(k) = 0$, $k \in [0, \rho - 1]$, there exists at least one input function $u(k)$, $k \in [0, k_1 - 1]$ such that (3.6) holds. The minimum value of ρ satisfying the above statement is called the relative degree of the system.*

?! [controllability vs. functional controllability.]

Theorem 3.1 *System (3.3) or (3.4) is invertible if and only if*

$$\mathcal{V}^* \cap \mathcal{S}^* = \{0\} \qquad (3.7)$$

[LINK to pag. 29] Recall that $\mathcal{B} \subseteq \mathcal{S}^*$, then

$$\mathcal{V}^* \cap \mathcal{S}^* = \{0\} \quad \Rightarrow \quad \mathcal{V}^* \cap \mathcal{B} = \{0\}$$

There are no inputs that do not influence the output.

Theorem 3.2 *System (3.3) or (3.4) is functionally controllable if and only if*

$$\mathcal{V}^* + \mathcal{S}^* = \mathcal{X} \qquad (3.8)$$

[LINK to pag. 29] Recall that $\mathcal{C}\,(:= \ker C) \supseteq \mathcal{V}^*$, then

$$\mathcal{V}^* + \mathcal{S}^* = \mathcal{X} \quad \Rightarrow \quad \mathcal{C} + \mathcal{S}^* = \mathcal{X} \quad \Leftrightarrow \quad C\mathcal{S}^* = \operatorname{im} C$$

The output space is completely spanned by trajectories in $\mathcal{S}^*$.

Note the duality: if system (A, B, C) or (A_d, B_d, C_d) is invertible (functionally controllable), the dual system $(A^\top, C^\top, B^\top)$ or $(A_d^\top, C_d^\top, B_d^\top)$ is functionally controllable (invertible).

[LINK to pag. 30 and 13] Recall that $\mathcal{V}^{*\perp} = \mathcal{S}_d^*$ and $\mathcal{S}^{*\perp} = \mathcal{V}_d^*$.

Relative Degree

Property 3.2 *Assume that (3.8) holds and consider the conditioned invariant computational sequence* $\mathcal{S}_i$ $(i = 1, 2, \ldots)$. *The relative degree is the least integer* ρ *such that*

$$\mathcal{V}^* + \mathcal{S}_\rho = \mathcal{X}$$

In discrete time, one has to wait ρ steps before controlling the output along any discrete time output function.

 [RelDeg]

Computational support with Matlab

r = reldeg(A,B,C,[D])	Relative degree of (A, B, C) or (A, B, C, D)

A geometric insight on how to track a given trajectory.

A very recent interpretation of $\mathcal{S}^*$ the minimal conditioned invariant $(A_d, \mathcal{C}_d)$-**conditioned invariant** containing $\mathcal{B}_d$.

For the sake of simplicity discrete time system are considered.

$$\begin{aligned} \mathcal{S}_0 &:= \mathcal{B}_d \\ \mathcal{S}_i &:= \mathcal{B}_d + A_d(\mathcal{S}_{i-1} \cap \mathcal{C}_d). \end{aligned}$$

Subspace $\mathcal{S}_p$ corresponds to the set of states reachable in p ($p \geq 0$) steps from $x_0 = 0$, with the state trajectory constrained to evolve on $\mathcal{C}_d$ in the preceding p-steps interval $[0, p-1]$.


[how to track]

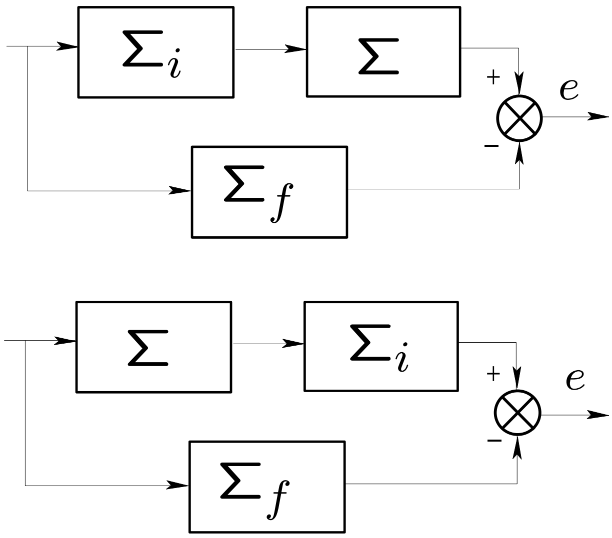


Fig. 3.3. Connections for right and left inversion

In Fig. 3.3 Σ_f denotes a suitable relative-degree filter in the continuous-time case or a relative degree delay in the discrete-time case. The inverse system Σ_i is to be designed to null the error e.

If the system is nonminimum phase, i.e, has some unstable zeros, the inverse system is internally unstable, so that the time interval considered for the system inversion must be finite.

Extension to Quadruples

How to reduce a quadruple to an equivalent triple.

Extension to quadruples of the above definitions and properties can be obtained through a simple contrivance.

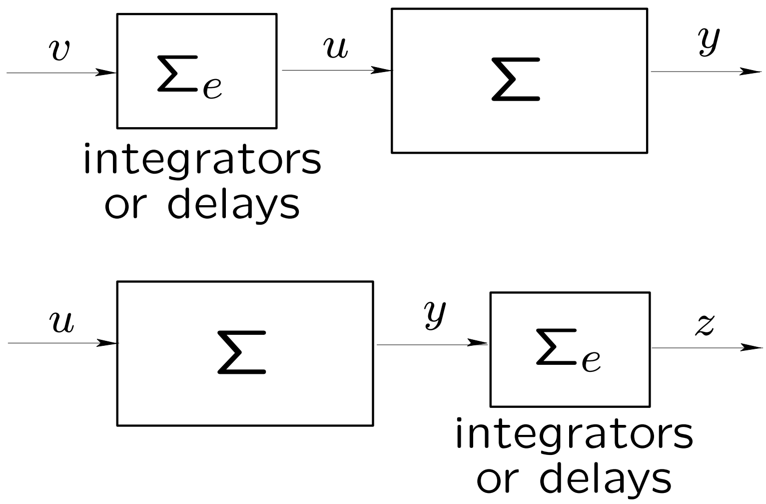


Fig. 3.4. Artifices to reduce a quadruple to a triple

Refer to the first figure: system Σ_d is modeled by

$$\dot{u}(t) = v(t)$$

and the overall system by

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\,\hat{x}(t) + \hat{B}\,v(t) \\ y(t) &= \hat{C}\,\hat{x}(t) \end{aligned}$$

with

$$\hat{x} := \begin{bmatrix} x \\ u \end{bmatrix} \qquad \hat{A} := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

$$\hat{B} := \begin{bmatrix} 0 \\ I_p \end{bmatrix} \qquad \hat{C} := \begin{bmatrix} C & D \end{bmatrix}$$

The addition of integrators at inputs or outputs does not affect the system right and left invertibility, while the relative degree of $(\hat{A}, \hat{B}, \hat{C})$ must be simply reduced by 1 to be referred to (A, B, C, D).

In the discrete-time case Σ_d is described by

$$u(k+1) = v(k)$$

and the overall system by

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}_d\,\hat{x}(k) + \hat{B}_b\,v(k) \\ y(k) &= \hat{C}_d\,\hat{y}(t) \end{aligned}$$

with the extended matrices $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$ defined like in the continuous-time case in terms of A_d, B_d, C_d, D_d.

This contrivance can also be used in most of the synthesis problems considered in the sequel.

Disturbance Decoupling

The *disturbance decoupling problem* is one of the earliest, [Basile Marro, 69] and [Wonham Morse, 70], applications of the geometric approach.

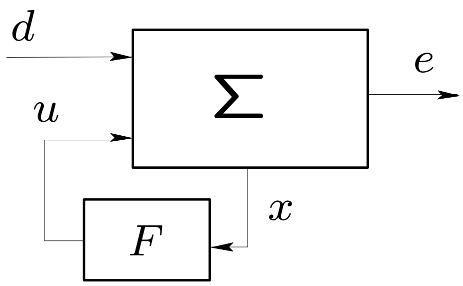


Fig. 3.5. Disturbance decoupling with state feedback

Let us consider the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + D\,d(t) \\ e(t) &= E\,x(t) \end{aligned} \qquad (3.9)$$

where u denotes the manipulable input, d the disturbance input. Let $\mathcal{B} := \text{im}B$, $\mathcal{D} := \text{im}D$, $\mathcal{E} := \ker E$.

<u>Unaccessible disturbance decoupling problem:</u> determine, if possible, a state feedback matrix F such that disturbance d has no influence on output e.

The system with state feedback is described by

$$\begin{aligned} \dot{x}(t) &= (A + B\,F)\,x(t) + D\,d(t) \\ e(t) &= E\,x(t) \end{aligned} \qquad (3.10)$$

It behaves as requested if and only if its reachable set by d, i.e., the minimum $(A + BF)$-invariant containing $\mathcal{D}$, is contained in $\mathcal{E}$.

Let $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} := \max \mathcal{V}(A,\mathcal{B},\mathcal{E})$. Since any $(A+BF)$-invariant is an $(A,\mathcal{B})$-controlled invariant, the unaccessible disturbance decoupling problem has a solution if and only if

$$\mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \tag{3.11}$$

Equation (3.11) is a structural condition and does not ensure internal stability. If stability is requested, we have the *disturbance decoupling problem with stability*. Stability is easily handled by using self-bounded controlled invariants. Assume that (A,B) is stabilizable (i.e., that $\mathcal{R} = \min \mathcal{J}(A,B)$ is externally stable) and let

$$\mathcal{V}_m := \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{D})} \tag{3.12}$$

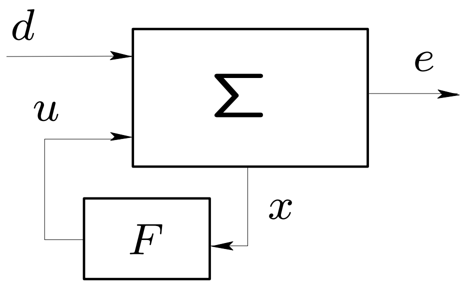
This subspace has already been defined in Property 2.3. The following result, providing both the structural and the stability condition, is a direct consequence of Theorem 2.6.

[LINK to pag. 34]
Theorem 2.6 Let $\mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$. There exists at least one internally stabilizable $(A,\mathcal{B})$-controlled invariant $\mathcal{V}$ such that $\mathcal{D} \subseteq \mathcal{V} \subseteq \mathcal{E}$ if and only if $\mathcal{V}_m$ is internally stabilizable.

Corollary 3.1 *The disturbance decoupling problem with stability admits a solution if and only if*

$$\boxed{\begin{array}{l} \mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \\ \mathcal{V}_m \textit{ is internally stabilizable} \end{array}} \qquad (3.13)$$

If conditions (3.13) are satisfied, a solution is provided by a state feedback matrix such that $(A+BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and $\sigma(A+BF)$ is stable.



If the state is not accessible, disturbance decoupling may be achieved through a dynamic unit similar to a state observer. This is called *disturbance decoupling problem with dynamic measurement feedback*, and will be considered later.

Feedforward

Decoupling of Measurable Signals

Consider now the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + H\,h(t) \\ e(t) &= E\,x(t) \end{aligned} \tag{3.14}$$

The triple (A, B, E) is assumed to be stable. This is similar to (3.9), but with a different symbol for the non-manipulable input, to denote that it is accessible for measurement. Let $\mathcal{H} := \mathrm{im} H$.

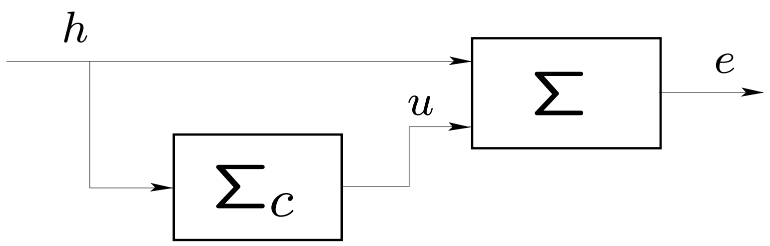


Fig. 3.6. Measurable signal decoupling

The *measurable signal decoupling problem* is: determine, if possible, a feedforward compensator Σ_c such that the input h has no inflence on the output e. Conditions for this problem to be solvable with stability are similar to those of disturbance decoupling problem.

?! [feedforwad is now possible.]

Structural Condition

Recall that the structural condition for the unaccessible disturbance decoupling problem is

$$\mathcal{H} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}.$$

The *structural condition* for the *measurable signal decoupling* problem (h is accessible for measurement) relax to

$$\mathcal{H} \subseteq \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} + \mathcal{B}.$$

The decoupling control cancels, through input $u(t)$, the part of disturbance $Hh(t)$ not in $\mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})}$

$$u(t) = -B^+Hh(t)$$

and then the postaction acts to localize the other part

$$Hh(t) - BB^+Hh(t) = (I - BB^+)Hh(t)$$

in the nullspace of the output matrix as for the unaccessible disturbance decoupling problem.

 [proof sketch of the stability condition.]

Solvability conditions with stability

Let $\mathcal{V}_m := \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{H})}$.

The corollary is a consequence of Theorem 2.6.

Corollary 3.2 *The measurable signal decoupling problem with stability admits a solution if and only if*

$$\boxed{\begin{array}{l} \mathcal{H} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} + \mathcal{B} \\ \mathcal{V}_m \text{ is internally stabilizable} \end{array}} \qquad (3.15)$$

The feedforward unit Σ_c has state dimension equal to the dimension of $\mathcal{V}_m$ and includes a state feedback matrix F such that $(A+BF)|_{\mathcal{V}_m}$ is stable. It is not necessary to reproduce $(A+BF)|_{\mathcal{X}/\mathcal{V}_m}$ in Σ_c since it is not influenced by input h.

The assumption that Σ is stable is not restrictive. It can be relaxed to Σ being stabilizable and detectable.

Note that internal stabilizability of $\mathcal{V}_m$ is ensured if the plant is minimum phase (with all the invariant zeros stable), since the internal unassignable eigenvalues of $\mathcal{V}_m$ are a part of those of $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$, that are invariant zeros of the plant.

It is possible to include feedthrough terms in (3.14) by using the extensions to quadruples previously described. In this case addition of a dynamic unit with relative degree one at the output achieves our aim.

The Dual Problem: Unknown-Input Observation

Consider the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + D\,d(t) \\ y(t) &= C\,x(t) \\ e(t) &= E\,x(t) \end{aligned} \qquad (3.16)$$

Triple (A, D, C) is assumed to be stable. Output e denotes a linear function of the state to be estimated (possible the whole state).

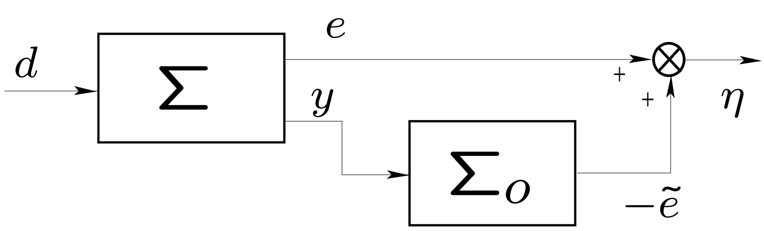
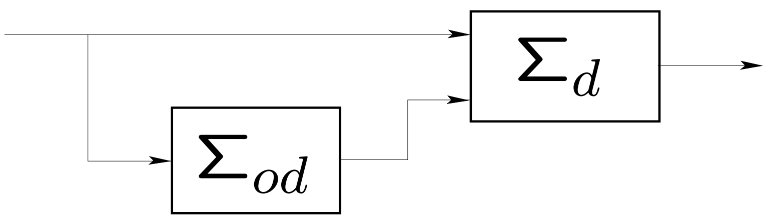


Fig. 3.7. Unknown-input observation

The *unknown-input observation problem* is: determine, if possible, an observer Σ_o such that the input d has no effect on the output η. Conditions for this problem to be solvable with stability are dual to those of the measurable signal decoupling problem.




[Duality]

The problem can be solved by duality. Define

$$\mathcal{S}_M = \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} + \mathcal{V}^*_{(\mathcal{D},\mathcal{C}\cap\mathcal{E})} \tag{3.17}$$

like in (2.15). The solvability conditions are consequence of Theorem 2.7.

Corollary 3.3 *The unknown-input observation problem with stability admits a solution if and only if*

$$\boxed{\begin{array}{l} \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \cap \mathcal{C} \subseteq \mathcal{E} \\ \mathcal{S}_M \text{ is externally stabilizable} \end{array}} \tag{3.18}$$

Previewed Signal Decoupling (Discrete-Time)

The role of controlled and conditioned invariants is very clearly pointed out by the previewed signal decoupling problem in the discrete-time case.

Refer to the discrete time system

$$\begin{aligned} x(k+1) &= A_d\, x(k) + B_d\, u(k) + H_d\, h(k) \\ e(k) &= E_d\, x(k) \end{aligned}$$

and suppose that signal $h(k)$, to be decoupled, is previewed, i.e. known in advance.

p-preview: the k-th sample of the signal to be decoupled $h(k)$ is available to the controller p steps before.

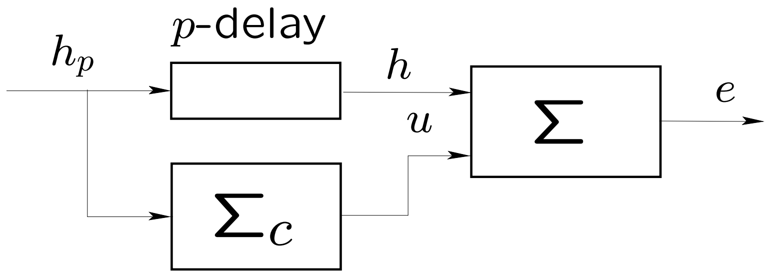


Fig. 3.8. Previewed signal decoupling

Note that the $p = 0$ case addresses the *measurable signal decopling* problem.

?! [Preview in continuos time.]

Previewed signal decoupling: Assume that signal $h(k)$ is previewed by p instants of time, $p \geq 0$. Determine a control law $u(k)$ which, using this preview, is able to maintain the output $e(k)$ identically zero (for zero initial condition).

Differently from the unaccessible disturbance decoupling problem, in this case the preview on $h(k)$ is used to "prepare" the system dynamics to localize signal $h(k)$ on $\ker(E_d)$.

Notes on the literature

Basile and Marro (1981) derived conditions for the dual problem: observation of the state with unknown inputs and postknowledge.

Willems (1982) derived, in the continuous time domain, conditions for PSDP with pole placement, see also [Bonilla and Malabre, 99]. Note that distributions are not practically implementable.

Imai and Shinozuka (1983) proposed conditions for the PSDP with stability in both discrete and continuous time cases.

Previous results do not care about dimensionality of the resolving controlled invariant subspace. Minimal dimension yields to reduce the order of the controller units.

Dimension optimization - new solution for *previewed signal decoupling problem* with stability based on a subspace with minimal dimension (through self-bounded controlled invariants) [F. Barbagli, G. Marro, D. Prattichizzo, JOTA 2001].

Unifying conditions for unaccessible, measurable and previewed signal decoupling problem with stability - unique necessary and sufficient condition for signal decoupling problems with stability independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed. [F. Barbagli, G. Marro, D. Prattichizzo, CDC 2000].

The control system exploits the p-preview by preparing the system state dynamics to cancel the previewed signal $H_d h(k)$ when it will occur at step k. Such preaction should evolve in the nullspace of output matrix E_d.

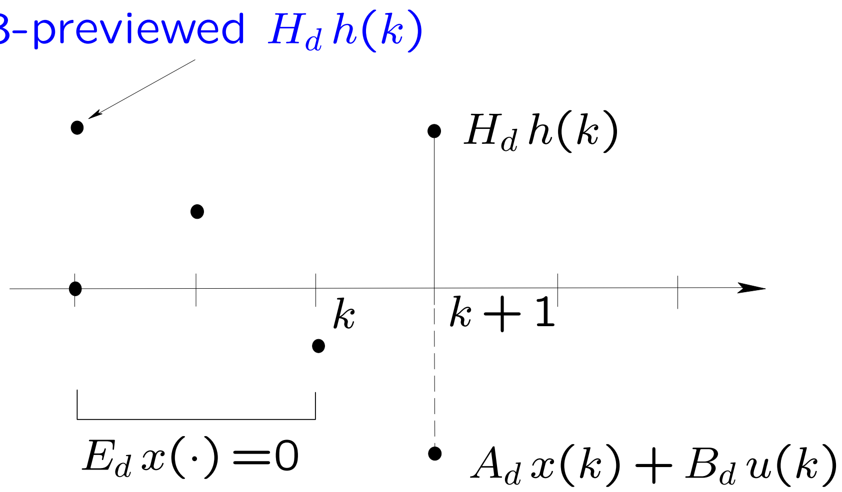


Fig. 3.9. Preaction on $\mathcal{S}_p$.

Refer to $x(k+1) = A_d x(k) + B_d u(k) + H_d h(k)$. It is possible to (silently) cancel $H_d h(k)$ or a part of it (with a p-preview) if $H_d h(k) \in \mathcal{S}_p$.

Subspace $\mathcal{S}_p$ corresponds to the set of states reachable in p ($p \geq 0$) steps from $x_0 = 0$, with the state trajectory constrained to evolve on $\mathcal{E}_d$ in the preceding p-steps interval $[0, p-1]$.

The algorithm for $\mathcal{S}^* := \min S(A_d, \mathcal{E}_d, \mathcal{B}_d)$, the minimal $(A_d, \mathcal{E}_d)$-conditioned invariant containing $\mathcal{B}_d$ is [Basile and Marro, 69]

$$\begin{aligned} \mathcal{S}_0 &:= \mathcal{B}_d \\ \mathcal{S}_i &:= \mathcal{B}_d + A_d(\mathcal{S}_{i-1} \cap \mathcal{E}_d). \end{aligned}$$

Theorem 3.3 *Necessary and sufficient condition for previewed signal decoupling problem to be solved is that* $\mathcal{H}_d \subseteq \mathcal{V}^*_{(\mathcal{B}_d,\mathcal{E}_d)} + \mathcal{S}_p$.

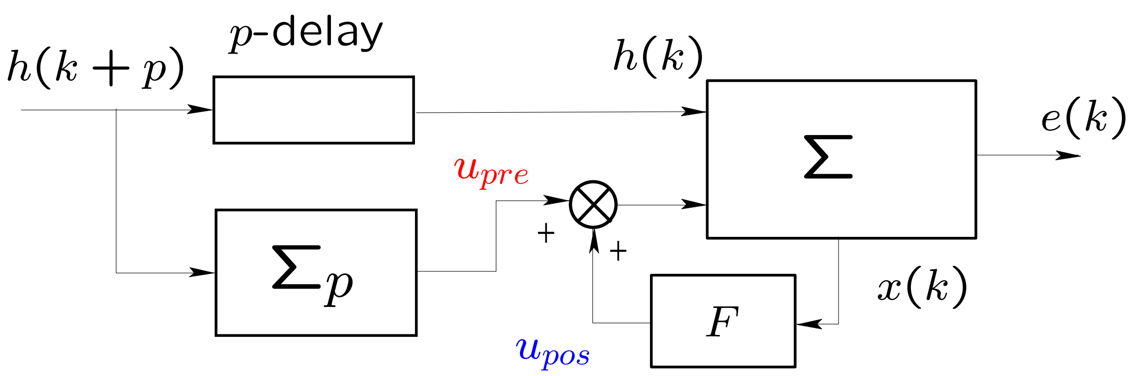


Fig. 3.10. Preaction and postaction.

Postaction $u_{\text{pos}}(k)$ is synthesized through a state feedback matrix F s.t. $\mathcal{V}^*_{(\mathcal{B}_d,\mathcal{E}_d)}$ is $(A_d + B_d F)$–invariant or by an equivalent feedforward unit with reduced dimension.

The preaction unit $(u_{\text{pre}}(k))$ is synthesized by controlling the state trajectory on the special reachability subspace $\mathcal{S}_p$.

The disturbance components to be canceled with pre or post action are chosen as follows

$$H_d = H_V + H_S \quad \text{s.t.} \quad \begin{array}{rcl} \mathcal{H}_V & := & \text{im}(H_V) \subseteq \mathcal{V}^*_{(\mathcal{B}_d,\mathcal{E}_d)} \\ \mathcal{H}_S & := & \text{im}(H_S) \subseteq \mathcal{S}_p. \end{array}$$

?! [Structure of the controller Σ_p.]

FIR Structure of the controller Σ_p

Postaction: $x(k+1) = (A + BF)x(k) + Bu_{\text{pre}}(k) + H_s h(k) + H_v h(k)$.

The **preaction** unit is a p-step FIR system $u_{\text{pre}}(k) = \sum_{l=0}^{p} \Phi(l) h(k+l)$ which, previewing the signal $h(k)$ p-steps in advance, is able to prepare system dynamics to cancel component $H_S h(k)$ when it occurs (at the time instant k).

Ω_0, Ω_1 ... Ω_p exist s.t. $\begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_p \end{bmatrix} = M^{\#} \begin{bmatrix} -H_s \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$M = \begin{bmatrix} B & A_F B & A_F^2 B & \cdots & A_F^p B \\ 0 & CB & CA_F B & \cdots & CA_F^{p-1} B \\ 0 & 0 & CB & \cdots & CA_F^{p-2} B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & CB \end{bmatrix}.$$

Consistency is guaranteed by $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_p$.

Recall that subspace $\mathcal{S}_p$ is the set of states reachable in p ($p \geq 0$) steps from $x_0 = 0$, with state trajectory constrained to evolve onto $\mathcal{C}$ in the preceding interval $[0, p-1]$.

Stability condition

Problem 3.1 *Assume that the dynamic system is stabilizable and that input h is previewed by p instants of time, $p \geq 0$. Determine a control law which, using the preview, is able to maintain the output $e(k)$ identically zero and the state trajectory bounded.*

The solution is based on:

- Lattices of self-bounded controlled invariants [Basile and Marro. 82] [Schumacher, 83].
- A special attention is devoted to the dimension of the resolving controlled invariant.

Theorem 3.4 *The signal decoupling problem with stability for the p-previewed signal $h(k)$ is solvable if and only if the structural condition $\mathcal{H}_d \subseteq \mathcal{V}^*_{(\mathcal{B}_d,\mathcal{E}_d)} + \mathcal{S}_p$ is satisfied and the self bounded controlled invariant*

$$\mathcal{V}_{m2} = \mathcal{V}^*{}_{(\mathcal{B}_d,\mathcal{E}_d)} \cap \mathcal{S}^*_{(\mathcal{E}_d,\mathcal{H}_d+\mathcal{S}_p)}$$

is internally stabilizable.
[Barbagli, Prattichizzo and Marro, JOTA 2001]

Unifying signal decoupling conditions

Theorem (unaccessible signal) The unaccessible disturbance localization problem with stability has a solution iff

1. $\mathcal{H} \subseteq \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})}$;
2. $\mathcal{V}_m := \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{H})}$ is I.S.

I.S.: internally stabilizable

Theorem (measurable signal) The measurable disturbance (signal) localization problem with stability has a solution iff

1. $\mathcal{H} \subseteq \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} + B$;
2. $\mathcal{V}_m := \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{H})}$ is I.S.

Theorem 3.5 *(previewed signal)* The p-previewed disturbance (signal) localization problem with stability has a solution iff

1. $\mathcal{H}_d \subseteq \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} + \mathcal{S}_p$;
2. $\mathcal{V}_m := \mathcal{V}^*{}_{(\mathcal{B},\mathcal{E})} \cap \mathcal{S}^*_{(\mathcal{E},\mathcal{B}+\mathcal{H})}$ is I.S.

[Barbagli, Prattichizzo and Marro, 2000 — *CDC*]

Decoupling conditions only differ by the structural condition.

An illustrative example

$$A = 0.1 \begin{bmatrix} 1 & 2 & 1 & -1 & -2 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 3 & 1 & -1 & -1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathcal{V}^* = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}^* = \mathcal{S}_1 = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix}.$$

$$\mathcal{H} \not\subseteq \mathcal{V}^* + \mathcal{B} \quad \text{but} \quad \mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_1 = \mathcal{V}^* + \mathcal{S}^*.$$

$p = 1$ is the minimum number of previewed steps necessary to solve the decoupling problem.

$$H_V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

$$\mathcal{V}_m = \mathcal{V}_{m1} = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>postaction</u>

$$F = \begin{bmatrix} -0.1 & 0 & 0 & 0.1 & 0 \\ -0.1 & 0 & 0 & -0.2 & 0 \end{bmatrix};$$

<u>preaction</u>

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -5.7735 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Subspace $\mathcal{V}_g^*$ proposed in [Willems, 82] has a dimension which is double of $\mathcal{V}_m$.

The postaction can be implemented through a feedforward unit Σ_c having state dimension equal to the dimension of $\mathcal{V}_m$ and with a state feedback matrix F such that $(A+BF)|_{\mathcal{V}_m}$ is stable. It is not necessary to reproduce $(A+BF)|_{\mathcal{X}/\mathcal{V}_m}$ in Σ_c since it is not influenced by input h.

Note that internal stabilizability of $\mathcal{V}_m$ is ensured if the plant is minimum phase (with all the invariant zeros stable), since the internal unassignable eigenvalues of $\mathcal{V}_m$ are a part of those of $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}$, that are invariant zeros of the plant.

It is possible to include feedthrough terms in (3.14) by using the extensions to quadruples previously described. In this case addition of a dynamic unit with relative degree one at the output achieves our aim.

[hud.m]

Large Preview (Non-Minimum Phase Systems)

A large preview time enables to overcome the stability condition, thus making it possible to obtain signal decoupling also in the nonminimum-phase case. “Large” means significantly greater than the time constant of the unstable zero closest to the unit circle.

Property 3.3 *The “largely” previewed signal decoupling problem with stability admits a solution if and only if*

$$\boxed{\mathcal{H} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} + \mathcal{S}^*_{(\mathcal{E},\mathcal{B})}} \qquad (3.19)$$

Suppose that an impulse is scheduled at input h at time ρ. It can be decoupled with an input signal u of the type shown in the following figure with preaction concerning unstable zeros and postaction stable zeros.

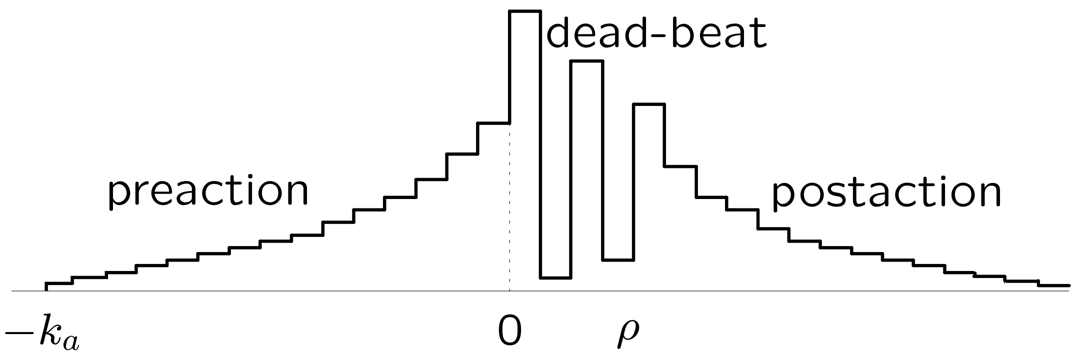


Fig. 3.11. Input sequence for decoupling an impulse at time ρ.

Localization of a previewed generic signal $h(\cdot)$ is achievable through a FIR system having such type of functions as gain.

Two different strategies are outlined according to whether condition 2 in Theorem 3.5 is satisfied or not. The basic idea is synthesized as follows.

Denote by ρ the least integer such that $\mathcal{H} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} + \mathcal{S}_\rho$. Let us recall that $\mathcal{V}_m$ is a locus of initial states in $\mathcal{E}$ corresponding to trajectories controllable indefinitely in $\mathcal{E}$, while $(\mathcal{S}_\rho)$ is the maximum set of states that can be reached from the origin in ρ steps with all the states in $\mathcal{E}$ except the last one. Suppose that an impulse is applied at input h at the time instant ρ, producing an initial state $x_h \in \mathcal{H}$, decomposable as $x_h = x_{h,s} + x_{h,v}$, with $x_{h,s} \in \mathcal{S}_\rho$ and $x_{h,v} \in \mathcal{V}_m$. Let us apply the control sequence that drives the state from the origin to $-x_{h,s}$ along a trajectory in $\mathcal{S}_\rho$, thus nulling the first component. The second component can be maintained on $\mathcal{V}_m$ by a suitable control action in the time interval $\rho \leq k < \infty$ while avoiding divergence of the state if all the internal unassignale modes of $\mathcal{V}_m$ are stable or stabilizable. If not, it can be further decomposed as $x_{h,v} = x'_{h,v} + x''_{h,v}$, with $x'_{h,v}$ belonging to the subspace of the stable or stabilizable internal modes of $\mathcal{V}_m$ and $x''_{h,v}$ to that of the unstable modes. The former component can be maintained on $\mathcal{V}_m$ as before, while the latter can be nulled by reaching $-x''_{h,v}$ with a control action in the time interval $-\infty < k \leq \rho - 1$ corresponding to a trajectory in $\mathcal{V}_m$ from the origin.

 [Plots]

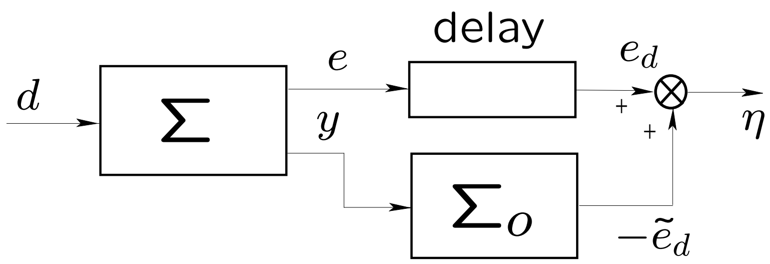


Fig. 3.12. Unknown-input delayed observation

The dual problem is the unknown-input observation of a linear function of the state with relative degree delay if Σ is minimum phase or "large" delay if not. The duals of Theorem 3.5 and Property 3.3 are stated as follows.

Corollary 3.4 *The unknown-input observation problem of a linear function of the state with relative degree delay and stability admits a solution if and only if*

$$\boxed{\begin{array}{l} \mathcal{V}^*_{(\mathcal{D},\mathcal{C})} \cap \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subseteq \mathcal{E} \\ \mathcal{S}_M \text{ is externally stabilizable} \end{array}} \tag{3.20}$$

where $\mathcal{S}_M$ is defined again by (3.17).

Note that the unknown-input observation of any linear function of the state (possibly the whole state) with relative degree delay is achievable if Σ is left-invertible and minimum phase.

Property 3.4 *The unknown-input observation problem of a linear function of the state with "large" delay and stability admits a solution if and only if*

$$\boxed{\mathcal{V}^*_{(\mathcal{D},\mathcal{C})} \cap \mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subseteq \mathcal{E}} \tag{3.21}$$

Feedforward Model Following

The feedforward model following problem reduces to decoupling of measured signals, as the following figure shows.

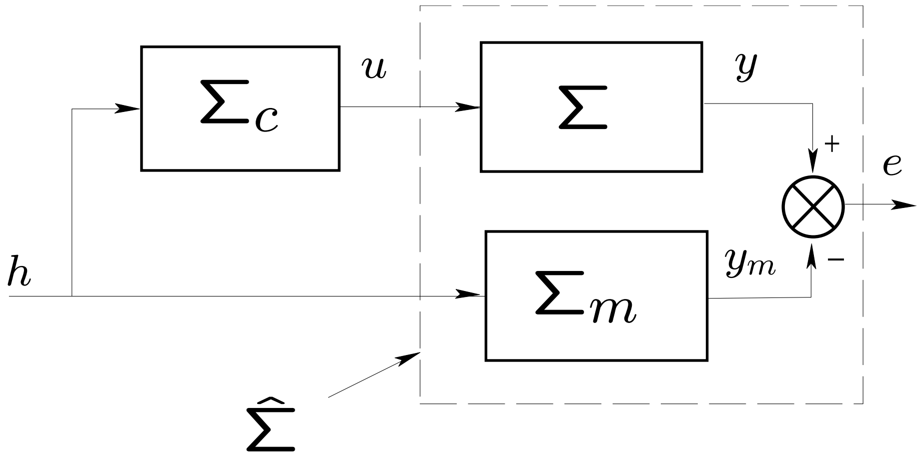


Fig. 3.13. Feedforward model following

Assume that system Σ is described by the triple (A, B, C) and model Σ_m by the triple (A_m, B_m, C_m). The overall sistem $\hat{\Sigma}$ is described by

$$\hat{A} := \begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix} \qquad \hat{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$\hat{H} := \begin{bmatrix} 0 \\ B_m \end{bmatrix} \qquad \hat{E} := \begin{bmatrix} C & -C_m \end{bmatrix} \tag{3.22}$$

Both system and model are assumed to be stable, square, left and right invertible. The structural condition expressed by the former of (3.15) is satisfied if and only if the relative degree of Σ_m is at least equal to that of Σ.

Let us assume that Σ and Σ_m have no equal invariant zeros: it can be shown that the internal eigenvalues of $\widehat{\mathcal{V}}_m$ are the union of the invariant zeros of Σ and the eigenvalues of A_m, so that in general model following with stability is not achievable if Σ is nonminimum-phase. If, on the other hand, the model Σ_m consists of q independent single-input single-output systems all having as zeros some invariant zeros of Σ, these are canceled as internal eigenvalues of $\widehat{\mathcal{V}}_m$. This makes it possible to achieve both input-output decoupling and internal stability, but restricts the model choice.

Note that the right inversion layout shown in Fig. 3.3 is achievable with a model consisting of q independent relative-degree filters in the continuous-time case or q independent relative-degree delays in the discrete-time case.

The dual problem of model following is model following by output feedforward correction, that reduces to the left inversion layout shown in Fig. 3.1 if a model consisting of p independent relative-degree filters in the continuous-time case or p independent relative-degree delays in the discrete-time case is adopted.

4 - Feedback

Disturbance Decoupling by Dynamic Output Feedback

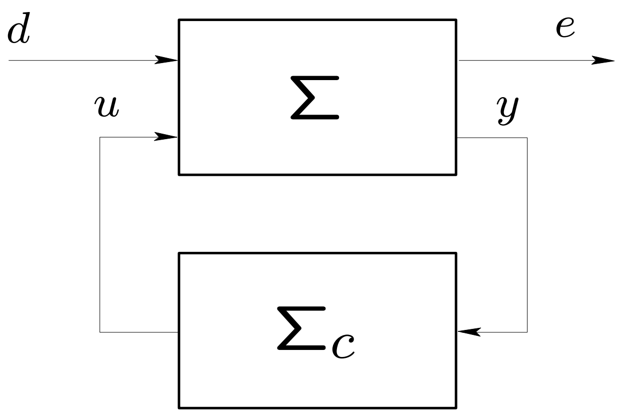


Fig. 4.1. Disturbance decoupling by dynamic output feedback

Model of Σ:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + D\,d(t) \\ y(t) &= C\,x(t) \\ e(t) &= E\,x(t) \end{aligned} \qquad (4.1)$$

The inputs u and d are the *manipulable input* and the *disturbance input*, respectively, while outputs y and e are the *measured output* and the *controlled output*, respectively.

Model of Σ_c:

$$\begin{aligned} \dot{z}(t) &= N\,z(t) + M\,y(t) \\ u(t) &= L\,z(t) + K\,y(t) \end{aligned} \qquad (4.2)$$

The *disturbance decoupling problem by dynamic output feedback* is stated as follows: determine, if possible, a dynamic compensator (N, M, L, K) such that the disturbance d has no influence on the regulated output e and the overall system is internally stable.

It has been shown that output dynamic feedback of the type shown in Fig. 4.1 enables stabilization of the overall system provided that (A, B) is stabilizable and (A, C) detectable. Since overall system stability is required, these conditions on (A, B) and (A, C) are still necessary.

The overall system is described by

$$\begin{aligned} \dot{\widehat{x}}(t) &= \widehat{A}\,\widehat{x}(t) + \widehat{D}\,d(t) \\ e &= \widehat{E}\,\widehat{x}(t) \end{aligned} \qquad (4.3)$$

with

$$\begin{aligned} \widehat{x} := \begin{bmatrix} x \\ z \end{bmatrix} \quad & \widehat{A} := \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix} \\ \widehat{D} := \begin{bmatrix} D \\ 0 \end{bmatrix} \quad & \widehat{E} := \begin{bmatrix} E & 0 \end{bmatrix} \end{aligned} \qquad (4.4)$$

i.e., it can de described by a unique triple $(\widehat{A}, \widehat{D}, \widehat{E})$.

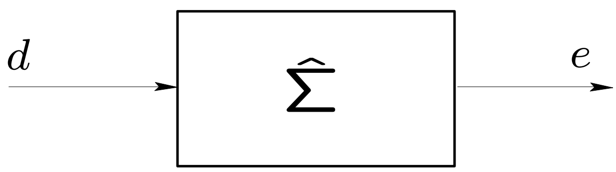


Fig. 4.2. The overall system

Output e is decoupled from input d if and only if $\min\mathcal{J}(\widehat{A}, \text{im}\widehat{D})$ (the reachable subpace of the pair $(\widehat{A}, \widehat{D})$) is contained in $\ker\widehat{E}$ or, equivalently, $\text{im}\widehat{D}$ is contained in $\max\mathcal{J}(\widehat{A}, \ker\widehat{E})$. Furthermore, in order the stability requirement to be satisfied, $\widehat{A}$ must be a stable matrix or $\min\mathcal{J}(\widehat{A}, \text{im}\widehat{D})$ and $\max\mathcal{J}(\widehat{A}, \ker\widehat{E})$ must be both internally and externally stable.

Stated in very simple terms, disturbance decoupling is achieved if and only if the overall system $(\widehat{A}, \widehat{D}, \widehat{E})$ exibits at least one $\widehat{A}$-invariant $\widehat{\mathcal{W}}$ such that

$$\boxed{\begin{array}{l} \widehat{\mathcal{D}} \subseteq \widehat{\mathcal{W}} \subseteq \widehat{\mathcal{E}} \\ \widehat{\mathcal{W}} \text{ is internally and externally stable} \end{array}} \qquad (4.5)$$

Necessary and sufficient conditions for solvability of our problem are stated in the following theorem.

Theorem 4.1 *The dynamic measurement feedback disturbance decoupling problem with stability admits at least one solution if and only if there exist an* $(A, \mathcal{B})$*-controlled invariant* $\mathcal{V}$ *and an* $(A, \mathcal{C})$*-conditioned invariant* $\mathcal{S}$ *such that:*

$$\boxed{\begin{array}{l} \mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{E} \\ \mathcal{S} \textit{ is externally stabilizable} \\ \mathcal{V} \textit{ is internally stabilizable} \end{array}} \qquad (4.6)$$

A short outline of the "only if" part of the proof. Define the following operations on subspaces of the extended state space $\widehat{x}$:

projection:

$$P(\widehat{\mathcal{W}}) = \left\{ x : \begin{bmatrix} x \\ z \end{bmatrix} \in \widehat{\mathcal{W}} \right\} \qquad (4.7)$$

intersection:

$$I(\widehat{\mathcal{W}}) = \left\{ x : \begin{bmatrix} x \\ 0 \end{bmatrix} \in \widehat{\mathcal{W}} \right\} \qquad (4.8)$$

Clearly, $I(\widehat{\mathcal{W}}) \subseteq P(\widehat{\mathcal{W}})$, $\mathcal{D} = I(\widehat{\mathcal{D}}) = P(\widehat{\mathcal{D}})$, $\mathcal{E} = P(\widehat{\mathcal{E}}) = I(\widehat{\mathcal{E}})$. The "only if" part of the proof of Theorem 4.1 follows from (4.5) and the following lemmas.

Lemma 4.1 *Subspace $\widehat{\mathcal{W}}$ is an internally and/or externally stable $\hat{A}$-invariant only if $P(\widehat{\mathcal{W}})$ is an internally and/or externally stabilizable $(A,\mathcal{B})$-controlled invariant.*

Lemma 4.2 *Subspace $\widehat{\mathcal{W}}$ is an internally and/or externally stable $\hat{A}$-invariant only if $I(\widehat{\mathcal{W}})$ is an internally and/or externally stabilizable $(A,\mathcal{C})$-conditioned invariant.*

The "if" part of the proof is constructive, i.e., if a resolvent pair $(\mathcal{S},\mathcal{V})$ is given, directly provides a compensator (N,M,L,K) satisfying all the requirements in the statement of the problem. This consists of a special type of state observer fed by the measured output y plus a special feedback connection from the observer state to the manipulable input u.

A more constructive set of necessary and sufficient conditions, based on the dual lattice structures af self-bounded controlled invariants and their duals, providing a convenient set of resolvent pair, is stated in the following theorem.

Theorem 4.2 *Consider the subspaces $\mathcal{V}_m$ and $\mathcal{S}_M$ defined in (2.14) and (2.15). The dynamic measurement feedback disturbance decoupling problem with stability admits at least one solution if and only if*

$$\boxed{\begin{array}{l} \mathcal{S}^*_{(C,D)} \subseteq \mathcal{V}^*_{(B,E)} \\ \mathcal{S}_M \text{ is externally stabilizable} \\ \mathcal{V}_M := \mathcal{V}_m + \mathcal{S}_M \text{ is internally stabilizable} \end{array}} \tag{4.9}$$

If Theorem 4.2 holds, $(\mathcal{S}_M, \mathcal{V}_M)$ is a convenient resolvent pair. Similarly, define $\mathcal{S}_m := \mathcal{V}_m \cap \mathcal{S}_M$. It can easily be proven that $(\mathcal{S}_m, \mathcal{V}_m)$ is also a convenient resolvent pair.

Note that conditions (4.9) consist of a *structural condition* ensuring feasibility of disturbance decoupling without internal stability and two *stabilizability conditions* ensuring internal stability of the overall system.

The layout of the possible resolvent pairs in the dual lattice structure is shown in the following figure, that also points out the correspondences between any self-bounded controlled invariant belonging to the first lattice and an element of the second and viceversa. This enables to derive other resolvent pairs satisfying Theorem 4.1.

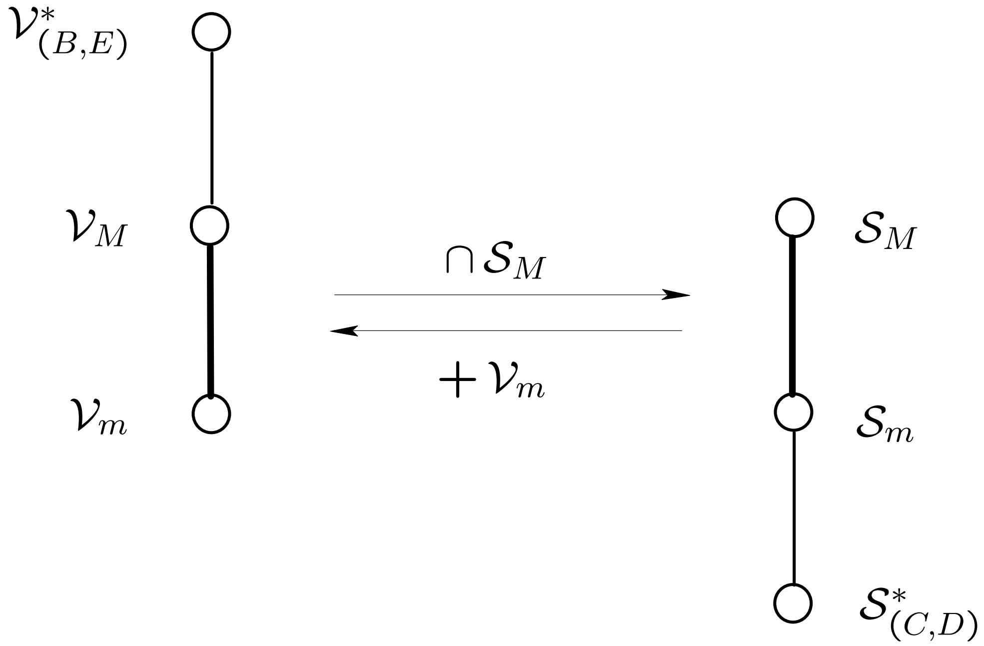


Fig. 4.3. The resolvents with minimum fixed poles

The Autonomous Regulator Problem

Consider the block diagram shown in the following figure.

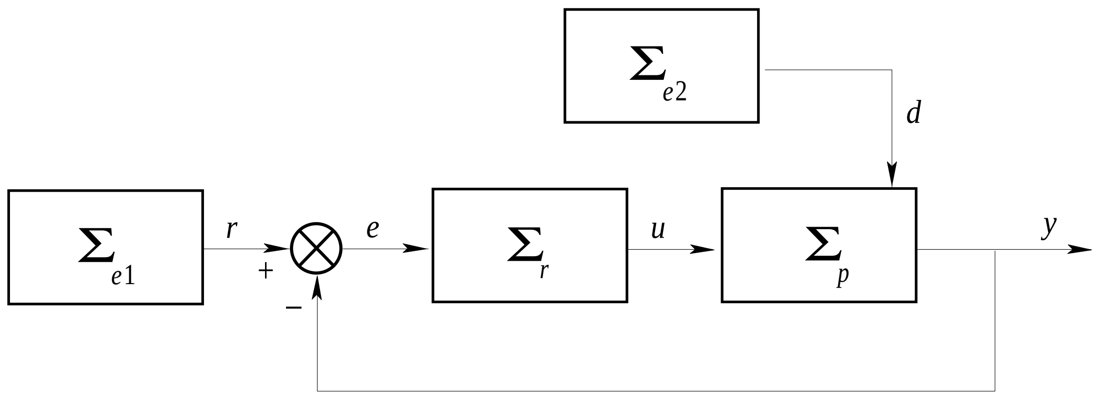


Fig. 4.4. The closed-loop control scheme.

The regulator Σ_r achieves:

(i) closed-loop asymptotic stability or, more generally, pole assignability;

(ii) asymptotic (robust) tracking of reference r and asymptotic (robust) rejection of disturbance d.

Both the reference and disturbance inputs are steps, ramps, sinusoids, that can be generated by the *exosystems* Σ_{e1} and Σ_{e2}. The eigenvalues of the exosystems are assumed to belong to the closed rigth half-place of the complex plane.

The overall system considered, included the exosystems, is described by a linear homogeneous set of differential equations, whose initial state is the only variable affecting evolution in time.

The plant and the exosystems are modelled as a unique *regulated system* which is not completely controllable or stabilizable (the exosystem is not controllable). The corresponding equations are

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ e(t) &= E\,x(t) \end{aligned} \tag{4.10}$$

with

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A := \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}$$

$$B := \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad E := \begin{bmatrix} E_1 & E_2 \end{bmatrix}$$

In (4.10) the plant corresponds to the triple (A_1, B_1, E_1). Note that the exosystem state x_2 influences both the plant through matrix A_3 and the error e through matrix E_2. (A_1, B_1) is assumed to be stabilizable and (A, E) detectable.

The regulator is modelled like in the disturbance decouplig problem by measurement feedback, i.e.

$$\begin{aligned} \dot{z}(t) &= N\,z(t) + M\,e(t) \\ u(t) &= L\,z(t) + K\,e(t) \end{aligned} \tag{4.11}$$

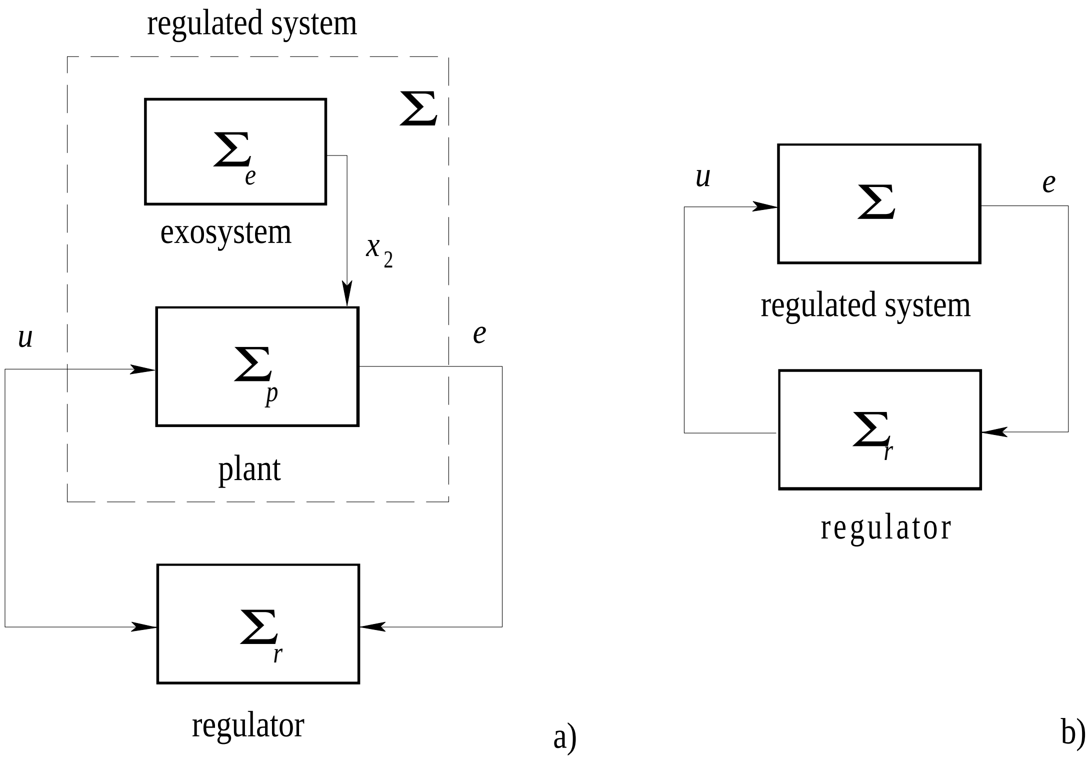


Fig. 4.5. Regulated system and regulator connection

The overall system is referred to as the autonomous *extended system*

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\,\hat{x}(t) \\ e(t) &= \hat{E}\,\hat{x}(t) \end{aligned} \qquad (4.12)$$

with

$$\hat{x} := \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$$

$$\hat{A} := \begin{bmatrix} A_1 + B_1KE_1 & A_3 + B_1KE_2 & B_1L \\ O & A_2 & O \\ ME_1 & ME_2 & N \end{bmatrix}$$

$$\hat{E} := \begin{bmatrix} E_1 & E_2 & O \end{bmatrix}$$

Let $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $z \in \mathbb{R}^m$. If the internal model principle is used to design the regulator, the autonomous extended system is characterized by an unobservability subspace containing these modes, that are all not strictly stable by assumption. In geometric terms, an $\hat{A}$-invariant $\hat{\mathcal{W}} \subseteq \ker \hat{E}$ having dimension n_2 exists, that is internally not strictly stable.

Since the eigenvalues of $\hat{A}$ are clearly those of A_2 plus those of the regulation loop, that are strictly stable, $\hat{\mathcal{W}}$ is externally strictly stable. Hence $\hat{A}|_{\hat{\mathcal{W}}}$ has the eigenstructure of A_2 (n_2 eigenvalues) and $\hat{A}_{\hat{\mathcal{X}}/\hat{\mathcal{W}}}$ that of the control loop ($n_1 + n_2$ eigenvalues).

The existence of this $\hat{A}$-invariant $\hat{\mathcal{W}} \subseteq \ker \hat{E}$ is preserved under parameter changes.

The *autonumous regulator problem* is stated as follows: derive, if possible, a regulator (N, M, L, K) such that the closed-loop system with the exosystem disconnected is stable and $\lim_{t \to \infty} e(t) = 0$ for all the initial states of the autonomous extended system.

In geometric terms it is stated as follows: refer to the extended system $(\hat{A}, \hat{E})$ and let $\hat{\mathcal{E}} := \ker \hat{E}$. Given the mathematical model of the plant and the exosystem, determine, if possible, a regulator (N, M, L, K) such that an $\hat{A}$-invariant $\hat{\mathcal{W}}$ exists satisfying

$$\begin{aligned} &\hat{\mathcal{W}} \subseteq \hat{\mathcal{E}} \\ &\sigma(\hat{A}|_{\hat{\mathcal{X}}/\hat{\mathcal{L}}}) \subseteq \mathbb{C}_- \end{aligned} \tag{4.13}$$

In the extended state space $\hat{\mathcal{X}}$ with dimension n_1+n_2+m, define the $\hat{A}$-invariant *extended plant* $\hat{\mathcal{P}}$ as

$$\hat{\mathcal{P}} := \{\, \hat{x} \,:\, x_2 = 0\} = \text{im} \begin{bmatrix} I_{n_1} & O \\ O & O \\ O & I_m \end{bmatrix} \qquad (4.14)$$

By a dimensionality argument, the $\hat{A}$ invariant $\hat{\mathcal{W}}$, besides (4.13), must satisfy

$$\hat{\mathcal{W}} \oplus \hat{\mathcal{P}} = \hat{\mathcal{X}} \qquad (4.15)$$

The main theorem on asymptotic regulation simply translates the extended state space conditions (4.13) and (4.15) into the plant plus exosystem state space where matrices A, B and E are defined. Define the A-invariant *plant* $\mathcal{P}$ through

$$\mathcal{P} := \{\, x \,:\, x_2 = 0\} = \text{im} \begin{bmatrix} I_{n_1} \\ O \end{bmatrix} \qquad (4.16)$$

Theorem 4.3 *Let $\mathcal{E} := \ker E$. The autonomous regulator problem admits a solution if and only if an $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ exists such that*

$$\boxed{\begin{aligned} &\mathcal{V} \subseteq \mathcal{E} \\ &\mathcal{V} \oplus \mathcal{P} = \mathcal{X} \end{aligned}} \qquad (4.17)$$

The "only if" part of the proof derives from (4.13) and (4.15), while the "if" part provides a quadruple (N, M, L, K) that solves the problem.

Unfortunately the necessary and sufficient conditions stated in Theorem 4.3 are nonconstructive. The following theorem provides constructive sufficient and almost necessary* conditions in terms of the invariant zeros of the plant.

Theorem 4.4 *Let us define* $\mathcal{V}^* := \max\mathcal{V}(A, \mathcal{B}, \mathcal{E})$. *The autonomous regulator problem admits a solution if*

$$\boxed{\begin{aligned} &\mathcal{V}^* + \mathcal{P} = \mathcal{X} \\ &\mathcal{Z}(A_1, B_1, E_1) \cap \sigma(A_2) = \emptyset \end{aligned}} \tag{4.18}$$

Remark:
We have again a structural condition and a stability condition in terms of invariant zeros. However, the stability condition is very mild in this case since it is only required that the plant has no invariant zeros equal to eigenvalues of the exosystem. Hence the autonomous regulator problem may be also solvable if the plant is nonminimum phase. In other words, minimality of phase is only required for perfect tracking, non for asymptotic tracking.

Corollary 4.1 (Uniqueness of the resolvent) *If the plant is invertible and conditions (4.18) are satisfied, a unique* $(A, \mathcal{B})$*-controlled invariant* $\mathcal{V}$ *satisfying conditions (4.17) exists.*

*The conditions become necessary if the boundedness of the control variable u is required. This is possible also when the output y is unbounded if a part of the internal model is contained in the plant.

Proof of Theorem 4.4:

Let F be a matrix such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$. Introduce the similarity transformation $T := [T_1 \; T_2 \; T_3]$, with im$T_1 = \mathcal{V}^* \cap \mathcal{P}$, im$[T_1 \; T_2] = \mathcal{V}^*$ and T_3 such that im$[T_1 \; T_3] = \mathcal{P}$.

In the new basis the linear transformation $A+BF$ has the structure

$$A' = T^{-1}(A+BF)\,T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} & O \\ O & O & A'_{33} \end{bmatrix} \tag{4.19}$$

Recall that $\mathcal{P}$ is an A-invariant and note that, owing to the particular structure of B, it is also an $(A+BF)$-invariant for any F.

By a dimensionality argument the eigenvalues of the exosystem are those of A'_{22}, while the invariant zeros of (A_1, B_1, E_1) are a subset of $\sigma(A'_{11})$ since $\mathcal{R}_{\mathcal{V}^*}$ is contained in $\mathcal{V}^* \cap \mathcal{P}$. All the other elements of $\sigma(A'_{11})$ are arbitrarily assignable with F. Hence, owing to (4.18), the Sylvester equation

$$A'_{11}\,X - X\,A'_{22} = -A'_{12} \tag{4.20}$$

admits a unique solution.

The matrix

$$V := T_1\,X + T_2$$

is a basis matrix of an $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ satisfying the solvability conditions (4.17).

Remarks:

- The proof of Theorem 4.4 provides the computational framework to derive a resolvent when the sufficient conditions stated (that are also necessary if the boundedness of the plant input is required) are satisfied.

- Relations (4.18) are respectively a *structural condition* and a *spectral condition*; they are easily checkable by means of the algorithms previously described.

- When a resolvent has been determined by means of the computational procedure described in the proof of Theorem 4.4, it can be used to derive a regulator with the procedure outlined in the "if" part of the proof of Theorem 4.3.

- The order of the obtained regulator is n (that of the plant plus that of the exosystem) with the corresponding $2n_1 + n_2$ closed-loop eigenvalues completely assignable under the assumption that (A_1, B_1) is controllable and (E, A) observable.

- The internal model principle is satisfied since the from the proof of the "if" part of Theorem 4.3 it follows that the eigenstructure of the regulator system matrix N contains that of A_2.

- It is necessary to repeat an exosystem for every regulated output to achieve independent steady-state regulation (different internal models are obtained in the regulator).

Feedback Model Following

The reference block diagram for feedback model following is shown in Fig. 4.6. Like in the feedforward case, both Σ and Σ_m are assumed to be stable and Σ_m to have at least the same relative degree as Σ.

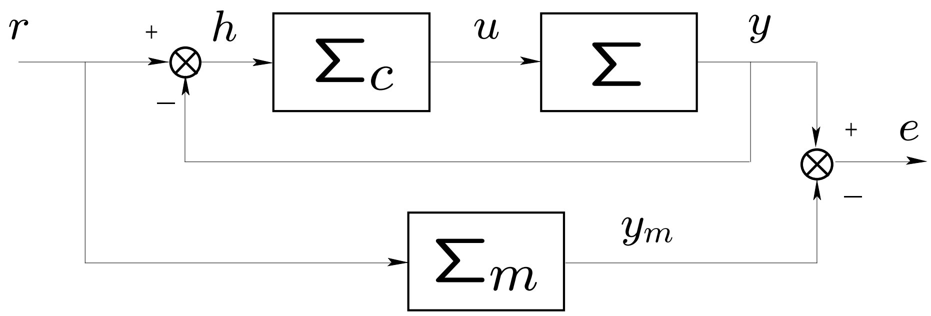


Fig. 4.6. Feedback model following

Replacing the feedback connection with that shown in Fig. 4.7 does not affect the structural properties of the system. However, it may affect stability. The new block diagram represents a feedforward model following problem.

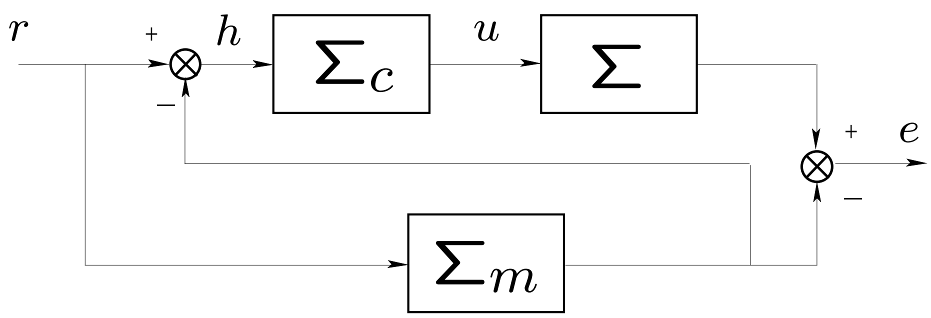


Fig. 4.7. A structurally equivalent connection

In fact, note that h is obtained as the difference of r (applied to the input of the model) and y_m (the output of the model). This corresponds to the parallel connection of Σ_m and the opposite of the identity matrix, that is invertible, having zero relative degree. Its inverse is Σ_m with a feedback connection through the identity matrix, as shown in Fig. 4.8.

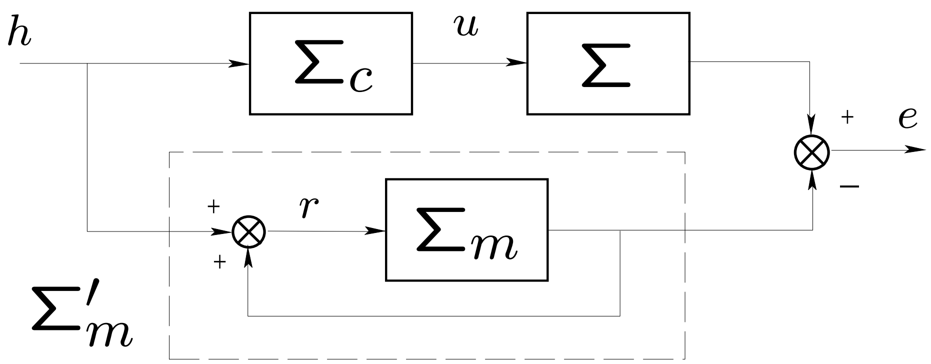


Fig. 4.8. A structurally equivalent block diagram

Let the model consist of q independent single-input single-output systems all having as zeros the unstable invariant zeros of Σ. Since the invariant zeros of a system are preserved under any feedback connection, a feedforward model following compensator designed with reference to the block diagram in Fig. 4.8 does not include them as poles.

It is also possible to include multiple internal models in the feedback connection shown in the figure (this is well known in the single input/output case), that are repeated in the compensator, so that both Σ'_m and the compensator may be unstable systems. In fact, zero output in the modified system may be obtained as the difference of diverging signals. However, stability is recovered when going back to the original feedback connection represented in Fig. 4.6.

[Ac,Bc,Cc,Dc]=hud(A,B,C,H,[D],[G])

Synthesis of a decoupling compensator (see Fig. 3.6) or, by duality, of an unknown-input observer (see Fig. 3.7). If the system is not left-invertible, the free poles on $\mathcal{R}_{\mathcal{V}^*}$ are assigned in interactive mode. It can be used both for continuous and discrete time systems.

test_hud

Enables testing of *hud* with numerical examples loaded in interactive mode as files of the type **.mat*.

modmtc

Uses *hud* to design a model following feedforward compensator (see Fig. 3.11) or a model following feedback regulator (see Fig. 4.6). It can be used both for continuous and discrete time systems.

modmtch2

Provides the H_2-optimal design of a model following feedforward compensator (see Fig. 3.11) or a model following feedback regulator (see Fig. 4.6). It works simply referring to the Hamiltonian system as explained in next Section. It can be used both for continuous and discrete time systems.

Remark: By duality, programs *modmtc* and *modmtch2* can be used for filtered left inversion (fault detection) exact and H_2-optimal respectively.

5 - Geometric Approach to LQR Problems

Consider again the disturbance decoupling problem by state feedback, corresponding to the state equations

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + D\,d(t) \\ e(t) &= E\,x(t) \end{aligned} \tag{5.1}$$

in the continuous-time case and to the equations

$$\begin{aligned} x(k+1) &= A_d\,x(k) + B_d\,u(k) + D_d\,d(k) \\ e(k) &= E_d\,x(k) \end{aligned} \tag{5.2}$$

in the discrete-time case. The corresponding block diagram is represented in Fig. 5.1.

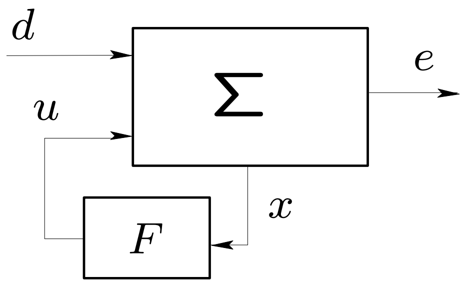


Fig. 5.1. Disturbance decoupling by state feedback

Assume that the necessary and sufficient conditions for its solvability with internal stability

$$\begin{aligned} &\mathcal{D} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{E})} \\ &\mathcal{V}_m \text{ is internally stabilizable} \end{aligned} \tag{5.3}$$

are not satisfied. In this case a convenient resort is to minimize the H_2 norm of the matrix transfer function from input d to output e, defined by equation (1.3) or (1.4) in the continuous-time case and equation (1.5) or (1.6) in the discrete-time case.

The continuous-time case

Consider the following problem:

Problem 5.1 *Referring to system (5.1), determine a state feedback matrix F such that $A+BF$ is stable and the corresponding state trajectory for any initial state $x(0)$ minimizes the performance index*

$$J = \int_0^\infty e(t)^\top e(t)\, dt = \int_0^\infty x(t)^\top E^\top E\, x(t)\, dt \qquad (5.4)$$

This problem is the so-called "cheap version" of the classical Kalman regulator problem or Linear Quadratic Regulator (LQR) problem. In the Kalman problem the performance index is

$$\begin{aligned} J &= \int_0^\infty x(t)^\top Q\, x(t) + u(t)^\top R\, u(t)\, dt \\ &= \int_0^\infty x(t)^\top C^\top C\, x(t) + u(t)^\top D^\top D\, u(t)\, dt \end{aligned}$$

where matrices Q and R are symmetric positive semidefinite and positive definite respectively, hence factorizable as shown. It can be proven that the cheap version is the more general, since the input to output feedthrough term $u(t)^\top D^\top D\, u(t)$ can be accounted for with a suitable state extension.

Problem 5.1 is solvable with the geometric tools. According to the classical optimal control approach, consider the Hamiltonian function

$$H(t) := x(t)^\top E^\top E\, x(t) + p(t)^\top (A\, x(t) + B\, u(t))$$

and derive the state, costate equations and stationary condition as

$$\begin{aligned}
\dot{x}(t) &= \left(\frac{\partial H(t)}{\partial p(t)}\right)^\top = A\, x(t) + B\, u(t) \\
\dot{p}(t) &= \left(\frac{\partial H(t)}{\partial x(t)}\right)^\top = -2\, E^\top E\, x(t) - A^\top p(t) \\
0 &= \left(\frac{\partial H(t)}{\partial u(t)}\right)^\top = B^\top p(t)
\end{aligned}$$

This overall Hamiltonian system can also be written as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \hat{A}\, \hat{x}(t) + \hat{B}\, u(t) \\
0 &= \hat{E}\, \hat{x}(t)
\end{aligned} \qquad (5.5)$$

with

$$\begin{aligned}
\hat{x} &= \begin{bmatrix} x \\ p \end{bmatrix} \qquad \hat{A} = \begin{bmatrix} A & 0 \\ -2E^\top E & -A^\top \end{bmatrix} \\
\hat{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \hat{E} = \begin{bmatrix} 0 & B^\top \end{bmatrix}
\end{aligned} \qquad (5.6)$$

Problem 5.1 admits a solution if and only if there exixts an internally stable $(\hat{A}, \hat{\mathcal{B}})$-controlled invariant of the overall Hamiltonian system contained in $\hat{\mathcal{E}}$ whose projection on the state space of system (5.1), defined as in (4.7), contains the initial state $x(0)$.

It can be proven that the internal unassignable eigenvalues of $\widehat{\mathcal{V}}^* := \max\mathcal{V}(\widehat{A},\widehat{\mathcal{B}},\widehat{\mathcal{E}})$ are stable-unstable by pairs. Hence a solution of Problem 5.1 is obtained as follows:

1. compute $\widehat{\mathcal{V}}^*$;
2. compute a matrix $\widehat{F}$ such that $(\hat{A}+\widehat{B}\widehat{F})\widehat{\mathcal{V}}^* \subseteq \widehat{\mathcal{V}}^*$ and the assignable eigenvalues (those internal to $\mathcal{R}_{\widehat{\mathcal{V}}^*}$) are stable;
3. compute $\widehat{\mathcal{V}}_s$, the maximum internally stable $(\widehat{A}+\widehat{B}\widehat{F})$-invariant contained in $\widehat{\mathcal{V}}^*$;
4. if $x(0) \in P(\widehat{\mathcal{V}}_s)$ the problem admits a solution F, that is easily computable as a function of $\widehat{\mathcal{V}}_s$ and $\widehat{F}$; if not, the problem has no solution.

Refer to Fig. 5.1. The above procedure also provides a state feedback matrix F corresponding to the minimum H_2 norm from d to e. This immediately follows from expression (1.4) of the H_2 norm in terms of the impulse response. In fact, the impulse response corresponds to the set of initial states defined by the column vectors of matrix D. Thus, the problem of minimizing the H_2 norm from d to e has a solution if and only if

$$\mathcal{D} \subseteq P(\widehat{\mathcal{V}}_s)$$

Thus, the minimum H_2 norm disturbance almost decoupling problem has no solution if the above condition is not satisfied. The discrete-time case is particularly interesting since a solution always exist. The reason for this will be pointed out below.

The discrete-time case

The discrete-time cheap LQR problem is stated as follows.

Problem 5.2 *Referring to system (5.2), determine a state feedback matrix F_d such that $A_d + B_d F_d$ is stable and the corresponding state trajectory for any initial state $x(0)$ minimizes the performance index*

$$J = \sum_{k=0}^{\infty} e(k)^\top e(k) = \sum_{k=0}^{\infty} x(k)^\top E_d^\top E_d\, x(k) \qquad (5.7)$$

In this case the Hamiltonian function is

$$H(k) := x(k)^\top E_d^\top E_d\, x(k) + p(k)^\top (A_d\, x(k) + B_d\, u(k))$$

and the state, costate equations and stationary condition are

$$\begin{aligned} x(k+1) &= \left(\frac{\partial H(k)}{\partial p(k+1)}\right)^\top = A_d\, x(k) + B_d\, u(k) \\ p(k) &= \left(\frac{\partial H(k)}{\partial x(k)}\right)^\top = 2\, E_d^\top E_d\, x(k) + A_d^\top p(k+1) \\ 0 &= \left(\frac{\partial H(k)}{\partial u(k)}\right)^\top = B^\top p(k+1) \end{aligned}$$

Like in the continuous-time case, it is convenient to state the overall Hamiltonian system in compact form:

$$\begin{aligned} \widehat{x}(k+1) &= \widehat{A}_d\,\widehat{x}(k) + \widehat{B}_d\,u(k) \\ 0 &= \widehat{E}_d\,\widehat{x}(k) \end{aligned} \tag{5.8}$$

with

$$\widehat{x} = \begin{bmatrix} x \\ p \end{bmatrix} \quad \widehat{A}_d = \begin{bmatrix} A_d & 0 \\ -2\,A_d^{-\top} E_d^\top E_d & -A_d^{-\top} \end{bmatrix}$$
$$\widehat{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix} \quad \widehat{E}_d = \begin{bmatrix} -2\,B_d^\top A_d^{-\top} E_d^\top E_d & B_d^\top A_d^{-\top} \end{bmatrix} \tag{5.9}$$

A solution of Problem 5.2 is obtained again with a geometric procedure, but, unlike the continuous-time case, in this case a dead-beat like motion is also feasible and $P(\widehat{\mathcal{V}}_s)$ covers the whole state space of system (5.2). Hence both Problem 5.2 and the problem of minimizing the H_2 norm from d to e are always solvable in the discrete-time case.

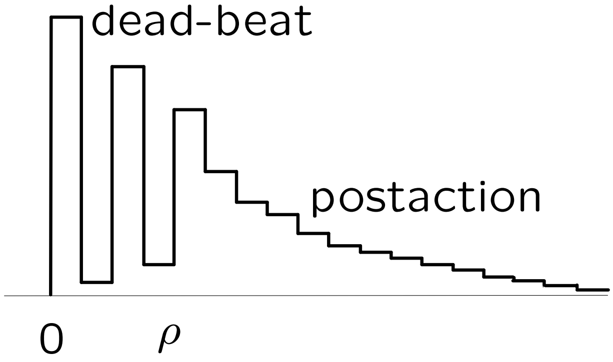


Fig. 5.2. Cheap H_2 optimal control.

A typical control sequence is shown in Fig. 5.2: as the sampling time approaches zero, the dead beat segment tend to a distribution, which is not obtainable with state feedback. For this reason solvability of the H_2 optimal decoupling problem is more restricted in the continuous-time case.

If the signal to be optimally decoupled is measurable and the system considered is stable, state feedback can be used in an auxiliary feedforward unit of the type shown in Fig. 3.6, while the dual layout shown in Fig 3.7 realizes the H_2-optimal observation of a linear function of the state or possibly of the whole state (Kalman filter).

However, if the signal is not measurable and state is not accessible, the problem of H_2-optimal decoupling with dynamic output feedback can be stated and solvability conditions derived by using geometric techniques again.

Conclusions

The three types of input signals:

- disturbance (eliminable only with feedback)
- measurable
- previewed

The seven characterizing properties of systems:

- (internal) stability
- controllability
- observability
- invertibility
- functional controllability
- relative degree
- minimality of phase

In general, the necessary and sufficient conditions for solvability of control problems consist of

- a structural condition
- s stability condition

When a tracking or disturbance rejection problem is not perfectly solvable with internal stability, it is possible to resort to H_2 optimal solutions that can also be obtained through the standard geometric tools and algorithms.

http://www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm