Covering Spaces
and
the Fundamental Group

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Overview

Homotopy theory reveals certain topological invariants of a path connected space M through investigation of continuous maps from $\mathbb{S}^n$ into M. These maps are naturally partitioned and form a group. This in turn partitions the category of path connected spaces into equivalence classes by identifying spaces that share identical group structures. This lecture will introduce the powerful idea of covering maps and use them to discover some elementary results about π_1, the "fundamental group" of a space, consisting of continuous maps $\mathbb{S}^1 \to M$. We will also apply results to the special cases where M is a group and where M is a Lie group.

Definitions

Let M, N be topological spaces. A **path** in M is a continuous function from $I = [0, 1]$ into M. M is **path connected** if there exists a path connecting any two points of M. Let $f, g : M \to N$. A **homotopy** Φ from f to g is a continuous transformation of f to g. Specifically we need

$$\Phi : I \times M \to N \qquad \text{where} \qquad \Phi(0, x) = f(x),\ \Phi(1, x) = g(x)$$

and Φ is continuous, of course. (Equivalently, we can define a homotopy as a path in the function space given with the compact-open topology). Throughout this lecture let M be a path connected topological space. A **loop** in M **at** x_0 is a path c with

$$c(0) = c(1) = x_0.$$

(Equivalently a continuous function $\mathbb{S}^1 \to M$, $1 \mapsto x_0$). When we speak of **path homotopies** and **loop homotopies** we mean these as per the definitions above with the added constraint that the end points remain fixed (see exercise 3). We will now investigate loop homotopies.

Group Structure Emerges

Let $\Omega(M, x)$ be the loop space at x. Define an operation $*$ on $\Omega(M, x)$

called **concatenation** as follows.

$$f * g : I \longrightarrow M \qquad \text{by} \qquad t \mapsto \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} < t \leq 1 \end{cases}$$

Now consider the relation $\sim$ on $\Omega(M, x)$ such that $f \sim g$ when there exists a homotopy from f to g (we say f is **homotopic** to g). $\sim$ is an equivalence relation. Then $\Omega(M, x)/\sim$ is well defined and equivalent to the path components in $\Omega(M, x)$.

Theorem 1. *$*$ induces a group operation $\cdot$ on $G = \Omega(M, x)/\sim$.*

Proof. Let f_0, f_1 belong to some class $[f]$, and g_0, g_1 to $[g]$, and let Φ_f,Φ_g be respective loop homotopies. * induces an operation on loop homotopies as follows.

$$\Phi_f * \Phi_g : I \times I \longrightarrow M \qquad \text{by} \qquad (s,t) \mapsto \begin{cases} \Phi_f(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ \Phi_g(s, 2t-1) & \frac{1}{2} < t \leq 1 \end{cases}$$

Which gives the desired homotopy

$$\Phi_f * \Phi_g(0,t) = f_0 * g_0(t), \qquad \Phi_f * \Phi_g(1,t) = f_1 * g_1(t)$$

and shows * is independent of choice of class representative.
With a moments reflection the group operations should be evident.⊣

***The* Fundamental Group** Let $\pi_1(M, x)$ be the group $\Omega(M, x)/\sim$ with operation $\cdot$ induced by $*$.

Theorem 2. *$\pi_1(M, x)$ is isomorphic to $\pi_1(M, y)$, for all $x, y \in M$.*

Proof. We show that any path p from x to y determines a homomorphism $\Psi_p : \pi_1(M, x) \longrightarrow \pi_1(M, y)$ with inverse. Let $\Psi_p([f]) = [p^{-1}][f][p]$ where $[p]$ is the homotopy equivalence class of the *path* p. Then

$$\Psi_p([f][g]) = [p^{-1}][f][g][p] = [p^{-1}][f][p][p^{-1}][g][p] = \Psi_p([f])\Psi_p([g])$$

and we have a similarly defined homorphism $\Upsilon_{p^{-1}}$ going back from $\pi_1(M, y)$ to $\pi_1(M, x)$ along $[p^{-1}]$, and

$$\Upsilon_{p^{-1}} \circ \Psi_p([f]) = [p][p^{-1}][f][p][p^{-1}] = [f]$$

so

$$\Upsilon_{p^{-1}} \circ \Psi_p = Id_{\pi_1(X,x)},$$

Since Ψ_p has a left inverse it is injective. A similar argument shows

$$\Psi_p \circ \Upsilon_{p^{-1}} = Id_{\pi_1(X,y)}$$

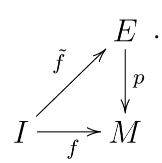
and hence Ψ_p is an isomorphism. $\dashv$

Quiz:
What is $\pi_1(\mathbb{S}^2)$?
Proof?
Maybe there is a little surprise here.

Covering Spaces

Given a map F between topological spaces, we say that a set U in the range is **evenly covered** by F if U is open and U's fiber $F^{-1}U$ is a disjoint number of homeomorphic copies of U, and $F|_U$ is such a homeomorphism. A **covering map** is a surjective map $p : E \to M$ between topological spaces such that every point in M has a neighborhood evenly covered by p. In such a case we call E a **covering space**.

Lemma. *Fixing a point $e_0 \in p^{-1}(x)$, and given a loop f in M we can construct a unique path $\tilde{f}$ in E beginning at e_0, such that the following diagram commutes.*



We refer to $\tilde{f}$ as the **lift** *or* **lifted path** *of f, by p.*

Rough sketch. Cover the image of f by evenly covered sets $\{U\}$. By compactness reduce to a finite subcover $\{U_i\}$. (Order these *locally* along the image of f using the loop parameter.) Now, by our local homeomorphism from U_1 to a neighborhood about e_0 we have a unique *begining* of a path. Using the fact that $U_i \cap U_{i+1}$ contains a point evenly covered by both neighborhoods, glue the lifted path together in this unique bf way. $\dashv$

Lemma. *Loop homotopies in M lift to path homotopies in E, and path homotopies on paths between $e_0, e_1 \in \pi^{-1}(x_0)$ descend via p to loop homotopies on loops at x_0.*

Rough sketch. Let Ψ be a loop homotopy from f to g, both loops at x_0. By the local homeomorphisms, given some $j \in (0,1)$ there is an epsilon ϵ such that $\Psi(j,t)$ and $\Psi(j \pm \epsilon, t)$ are all within the same cover of evenly covered sets, and so have the same end point. Let $A \subset I$ be defined by

$$A = \{s \in I \mid \Psi(s,1) = f(1)\}.$$

Then by the above argument A is open in I. But by continuity of the local homeomorphisms and Ψ, A is closed. Hence $A = I$. As for homotopies on E descending to M, this is just a consequence of the continuity of p. $\dashv$

Caveat Lector Munkres book *Topology* contains generously detailed proofs of these lemmas, for example making use of the *Lebesgue number lemma*. I don't see a need for this, so there's probably hidden subtlety here.

Conclusion

Consider the implication of Lemma 1 and 2 when E is simply connected. For every point $e \in \pi^{-1}x_0$ we have exactly one homotopy class of paths connecting e_0 to e, which descends to a distinct loop homotopy on M. Thus the fundamental group has the cardinality of a typical fiber, and in fact in the case of topological groups there is much more structure here.

Theorem 3. *Let (G) be a topological group. Define an operation $\cdot$ on $\Omega(G,e)$ by*

$$(f \cdot g)(t) = f(t)g(t).$$

a) $\cdot$ is well defined on $\Omega(G,e)/\sim$, the homotopy classes of loops at e.
b) Concatenation, $*$, and path product, $\cdot$, are identical on $\Omega(G,e)/\sim$.
c) The fundamental group of G is abelian.

Proof. a) Let $f_0, f_1 \in [f]$ and $g_0, g_1 \in [g]$ with Φ_f, Φ_g the respective homotopies. We must show that $f_0 \cdot g_0 \sim f_1 \cdot g_1$. Let $\Phi(s,t) = \Phi_f(s,t) \cdot \Phi_g(s,t)$.
b) by a) we can write

$$[f] \cdot [g] = [f * e] \cdot [e * g]$$

where e is denoting the constant loop. But this is clearly equal to [f*g]=[f][g].
c) We must find a homotopy from $f(t) \cdot g(t)$ to $g(t) \cdot f(t)$. Try

$$\Phi(s,t) = f^{-1}(s,t) \cdot f(t) \cdot g(t) \cdot g^{-1}(s,t) \qquad \dashv$$

In a more extensive lecture we would see that a covering space for a topological group has a unique (up to some symmetry) group structure that turns the covering map into a homomorphism, that the fiber over e is contained in the center of the covering group, and that the action of the covering group on itself that leaves the fiber invariant is a group isomorphic to the fiber subgroup. In this direction, we would want to include that a simply connected covering space E of some space X is unique up to homeomorphism and that whenever Y covers X there is a covering map $E \to Y$ such that it composes to give our covering $E \to X$. I want to discuss the existence of simply connected covering spaces. Such a covering space is call a **universal covering space**.

Proposition 1. *For M a* reasonably[1] *connected topological space there exists a universal covering space.*

Crux. We will construct the covering space. Let $\Lambda(M,x)$ be the space of paths begining at x. Let $\sim$ be the usual path homotopy relation.We will show that $\Lambda(M,x)/\sim$ can be given the topology of M, locally, and that $\Lambda(M,x)/\sim$ is simply connected. Surjectivity will follow from path connectivity of M. Now, let $f \in \Lambda(M,x)$ be a path to $y \in M$ and let $\hat{U}$ be a neighborhood in $\Lambda(M,x)$ about f. There exists a neighborhood about y with the property that for any $z \in U$ there is a path $g \in \hat{U}$ connecting x to z,and a path in $\Lambda(M,x)$ connecting f to g. This gives us a local surjection from $\hat{U}$ onto U. By local simple connectedness, when we quotient out $\Lambda(M,x)$ by $\sim$ we get injectivity, so have a local bijection with U around the point $[f] \in \Lambda(M,x)/\sim$. Next, we need that the induced topology on $\Lambda(M,x)/\sim$ is equivalent to M's topology. This is left to the reader. Finally, that $\Lambda(M,x)/\sim$ is simply connected follows from this construction. When we form a loop in $\Lambda(M,x)\sim$, we are defining the boundary of a D^2 disc, in

[1]semi-locally simply connected should be enough, at any rate anything resembling a manifold or variety is okay. For sake of simplicity say M is path connected.

M. The contraction of D^2 in M corresponds to the contraction of our loop to a point in $\Lambda(M,x)/\sim$. I will attempt a convincing illustration.

A Few Exercises on Loop Quotients

Exercise 1. Let $\Omega(\mathbb{R}^2, 0)$ be the loop space at 0 of the plane, endowed with the concatenation product $*$.
a) Show that there exists no identity element.
b) Show that $f * g = g * f \implies f = g$.
c) Prove that for two loops f, g, not both equal to the trivial loop, we cannot have $f * g = f$ ... however,
d) allowing f to be discontinuous at a single point this can be achieved, and such that neither of f, g is the trivial loop.

Since L fails to be a group, we might ask ourselves *what is the weakest quotienting or modifying we need to do to get a group structure out of $\Omega(\mathbb{R}^2, 0)$.*

Exercise 2. Let $\sim$ be the relation on $\Omega(\mathbb{R}^2, 0)$ of loop homotopy *within the images of the loops*. That is, let $f \in \Omega(\mathbb{R}^2, 0)$ and let $\bar{f}$ denote the image of f in $\mathbb{R}^2$. Let $\hat{f}$ be the set of loops with image in $\bar{f}$ and homotopic to f within this subspace. Then $f \sim g$ whenever $\hat{f} \cap \hat{g} \neq \varnothing$. Show that $*$ is a well defined group operation on $\Omega(\mathbb{R}^2, 0)/\sim$.

Exercise 3. Let X be a path connected topological space with some distinguished point x_0. Define a (courser than normal homotopy) relation $\sim$ where the point $f(0) = f(1)$ is itself allowed to loop away from x_0 and then back. More precisely, Let $\Upsilon(X) = \{f : \mathbb{S}^1 \to X)$ and define the inclusion $\iota : \Omega(X, x_0) \hookrightarrow \Upsilon(X)$ according to the usual correspondence

$$I \ni x \mapsto e^{2\pi i x} \in \mathbb{S}^1 \subset \mathbb{C}^1$$

Now for $f, g \in \Omega(X, x_0)$ let $f \sim g$ if $\iota(f)$ and $\iota(g)$ are connected by a path in $\Upsilon(X)$. Show that $\sim$ is well defined on $\pi_1(X, x_0)$ (this is *trivial*). Provide an example where the quotienting is not trivial, (i.e., is not equivalent to normal loop homotopy). Can the quotient in such a case be a group? Provide an example where the quotient is a group or prove it cannot be one.