The Cartan–Kähler theorem

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Contents

1 Partial differential equations: the embarrasment of taxonomy 2
1.1 Genera 3

2 Determined equations 3
2.1 Cartan's vision 3

3 Differential forms 3
3.1 Tableau of Laplace equation 4
3.2 Algebra versus analysis 4
3.3 Exterior differential systems 4

4 The Laplace equation 5

5 Prolongation 5
5.1 Cartan's lemma 5
5.2 The Laplace equation 5

6 Torsion 6

7 Cartan–Kähler theorem 6

8 Examples: surfaces in 3-dimensional Euclidean space 7
8.1 Setting up Darboux's structure equations 7
8.2 Structure equations 8
8.3 Adapting to a surface 8
8.4 Shape operator 8
8.5 Exterior differential system 9
8.6 Example: constant mean curvature 9
8.7 Example: surfaces with all points umbilic 9
8.8 Prolongation 10

9 Example: isometric embedding of surfaces **10**
9.1 Structure equations of surfaces 10
9.2 Exterior differential system 11
9.3 Cartan's count 11

10 For more information **12**

1 Partial differential equations: the embarrassment of taxonomy

PDE	*Zoology*
lots of "types"; taxonomy	lots of genera, species; taxonomy
$+, -, \times, /, \frac{\partial}{\partial x}$	4 amino acids
imagine general theory, existence, uniqueness, smoothness	imagine what viable animals could be made of all possible DNA, where they could live, what they could eat, symbiosis

can't do this, so settle for examining creatures that arise in nature

Typical article: *Existence of solutions of the initial value problem for the quasilinear hyperbolic second order equation . . .*	Typical article: *Study of bat guano . . .*

1.1 Genera

underdetermined
fewer PDEs than unknown functions
Gromov
"topological"
e.g. Legendre submanifolds of contact manifolds
e.g. curves in Carnot geometries

determined
as many PDEs as unknown functions
Cauchy
analysis
e.g. heat, waves, ..., physics

overdetermined
more PDEs than unknown functions
Cartan
"algebraic"
e.g. special solutions, supersymmetric backgrounds

2 Determined equations

No general theory of PDE, but in the analytic category, the study of local solutions is essentially algebra.

Theorem 1 (Cauchy–Kovalevskii) *A determined analytic PDE system admits a unique solution with any noncharacteristic initial data.*

Idea: expand data in Taylor series along the submanifold where it is defined. Use PDE system to differentiate in directions across that submanifold, giving all Taylor coefficients of the solution. "Noncharacteristic" means the PDEs tell you how to move across the submanifold.

2.1 Cartan's vision

Sequence of noncharacteristic determined problems. Solve each with Cauchy–Kovalevskii. Proceed dimension at a time. To avoid hitting any underdetermined problems along the way, restrict to a submanifold which just kills all of the underdeterminacy. This is how Cartan's proof works, but the statement of the theorem doesn't manifest any of this induction.

3 Differential forms

1. An ODE
$$\frac{dy}{dx} = f(x, y)$$

can be written as

$$dy = f(x,y)\,dx.$$

2. A PDE

$$\frac{\partial u}{\partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}\right)$$

can be written as

$$du = p\,dx + f\left(x, y, u, p\right)\,dy.$$

3. The Laplace equation

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

can be written $\vartheta^1 = \vartheta^2 = \vartheta^3 = 0$, where

$$\begin{aligned}
\vartheta^1 &= du - p\,dx - q\,dy \\
\vartheta^2 &= dp - r\,dx - s\,dy \\
\vartheta^3 &= dq - s\,dx + r\,dy.
\end{aligned}$$

3.1 Tableau of Laplace equation

Take exterior derivative:

$$d\begin{pmatrix}\vartheta^1 \\ \vartheta^2 \\ \vartheta^3\end{pmatrix} = -\begin{pmatrix}0 & 0 \\ \pi^1 & \pi^2 \\ \pi^2 & -\pi^1\end{pmatrix} \wedge \begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix},$$

modulo ϑ-terms, where

$$\begin{aligned}
\pi^1 &= dr, \pi^2 = ds, \\
\omega^1 &= dx, \omega^2 = dy.
\end{aligned}$$

This is called the *tableau*.

3.2 Algebra versus analysis

Writing PDEs in differential forms makes coordinate changes very easy, analysis hard (because analysts like functions).

3.3 Exterior differential systems

Any PDE system can be written as $\vartheta^i = 0$, subject to ω^i being linearly independent. Moreover, taking d:

$$d\vartheta^i = -\varpi^i_j \wedge \omega^j \pmod{\vartheta}.$$

4 The Laplace equation

For the Laplace equation,

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\end{pmatrix} = -\begin{pmatrix}0 & 0\\ \pi_1 & \pi_2\\ \pi_2 & -\pi_1\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \pmod{\vartheta}.$$

Let

$$s_1 = \#\text{of independent } \pi\text{'s in 1st column,}$$
$$s_2 = \#\text{of independent } \pi\text{'s in 2nd column, mod those in 1st,}$$

etc.

These are the *Cartan characters*. Warning: they change if we change permute the ω's, so permute ω's to make s_1 as big as possible, s_2 as big as possible subject to value of s_1, etc. For Laplace equation, $s_1 = 2, s_2 = 0$.

5 Prolongation

Given equation, e.g. $\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial y}$, differentiate both sides to learn that

$$\frac{\partial^2 u}{\partial x^2} = 3\frac{\partial^2 u}{\partial x \partial y},$$
$$\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial^2 u}{\partial y^2}.$$

Call this *prolongation.* In diff'l forms?

5.1 Cartan's lemma

If $\sum_j \phi_j \wedge \omega^j = 0$, ω^j independent, then there are unique functions $a_{ij} = a_{ji}$ so that $\phi_i = a_{ij}\omega^j$.

5.2 The Laplace equation

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\end{pmatrix} = -\begin{pmatrix}0 & 0\\ \pi_1 & \pi_2\\ \pi_2 & -\pi_1\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}.$$

If $\vartheta^j = 0$, then $d\vartheta^j = 0$, so

$$0 = \begin{pmatrix}0 & 0\\ \pi_1 & \pi_2\\ \pi_2 & -\pi_1\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}.$$

Apply Cartan's lemma twice, find unique u, v functions with

$$\pi_1 = u\omega^1 + v\omega^2$$
$$\pi_2 = v\omega^1 - u\omega^2.$$

The prolongation is the system with the new 1-forms

$$\vartheta_3 = \pi_1 - u\omega^1 - v\omega^2,$$
$$\vartheta_4 = \pi_2 - v\omega^1 + u\omega^2.$$

Differentiate:

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\\ \vartheta_3\\ \vartheta_4\end{pmatrix} = -\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ \pi_3 & \pi_4\\ -\pi_4 & \pi_3\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},$$

mod ϑ's. Have 2 new parameters, u, v, so the number of new parameters is $S = 2$.

6 Torsion

Trouble: what if there are $\omega \wedge \omega$ terms in the tableau? Then setting $\vartheta = 0$ can give inconsistent conditions. We call the $\omega \wedge \omega$ terms in the tableau the *torsion*. For example,

$$\vartheta = dy - t^2\,dx - dt - z\,dz = 0$$

with $\omega^1 = dx, \omega^2 = dz$, has tableau

$$d\vartheta = 2t\omega^1 \wedge \omega^2,$$

so torsion $2t$. Thus only has solutions where $t = 0$.

7 Cartan–Kähler theorem

Theorem 2 *Suppose the tableau has no torsion, after suitable choice of ϑ, ω, π. Then Cartan characters satisfy $s_1 + 2s_2 + \cdots + ns_n \geq S$. Equality is called* involution*. Involution implies that there is a solution. In fact, there is a sequence of well-posed determined problems with initial data depending on s_k functions of k variables, s_k the last nonzero character.*

Example: Laplace equation has $s_1 = 2, s_2 = 0, S = 2$, no torsion, so $s_1 + 2s_2 = 2 = S$, so solutions depend on 2 functions of 1 variable.

8 Examples: surfaces in 3-dimensional Euclidean space

8.1 Setting up Darboux's structure equations

Write inner products in $\mathbb{R}^3$ as $x \cdot y = x_i y_i$. Let G be the group of rigid motions of Euclidean space $\mathbb{R}^3$. Identify G with the set of matrices

$$\begin{pmatrix} e & x \\ 0 & 1 \end{pmatrix}$$

where e is an orthogonal matrix, and $x \in \mathbb{R}^3$ a vector, by defining

$$\begin{pmatrix} e & x \\ 0 & 1 \end{pmatrix} y = ey + x,$$

for $y \in \mathbb{R}^3$. Write $e = (e_{ij})\,, x = (x_i)\,.$ Define the Maurer–Cartan 1-form on G as

$$\omega = g^{-1}\,dg,$$

for

$$g = \begin{pmatrix} e & x \\ 0 & 1 \end{pmatrix}.$$

Writing it out:

$$\omega = \begin{pmatrix} e^{-1}\,de & e^{-1}\,dx \\ 0 & 0 \end{pmatrix}.$$

Since e orthogonal, $e^{-1} = e^t$, so can write

$$\begin{aligned} e^{-1}\,de &= \left(e^{-1}_{ik}\,de_{kj}\right) \\ &= \left(e_{ki}\,de_{kj}\right) \\ &= \left(e_i \cdot de_j\right), \end{aligned}$$

where

$$e = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix},$$

i.e. the e_i are the column vectors e_{ki}. Similarly, $e^{-1}\,dx = (e_i \cdot dx)\,.$ Here e_i form any orthonormal basis for $\mathbb{R}^3$, not necessarily the usual one. We let $\omega_{ij} = e_i \cdot de_j$ and $\omega_i = e_i \cdot dx$, so that

$$\omega = \begin{pmatrix} \omega_{ij} & \omega_i \end{pmatrix}.$$

Exercise: $\omega_{ij} = -\omega_{ji}$.

8.2 Structure equations

Calculate: since $\omega = g^{-1}\, dg$, must have $g\omega = dg$ so take exterior derivative:

$$dg \wedge \omega + g d\omega = 0,$$

so

$$d\omega = -g^{-1}\, dg \wedge \omega,$$

giving Cartan's structure equation:

$$d\omega = -\omega \wedge \omega.$$

Computing this out in components gives

$$\begin{aligned} d\omega_i &= -\omega_{ij} \wedge \omega_j \\ d\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj}. \end{aligned}$$

8.3 Adapting to a surface

Let $\Sigma \subset \mathbb{R}^3$ be any smooth surface, oriented. An *adapted frame* at a point $x \in \Sigma$ is a choice of orthonormal basis e_1, e_2, e_3 for which e_1, e_2 are tangent to Σ at x, a positively oriented basis, and e_3 is normal. Let $F\Sigma \subset G$ be the set of elements

$$\begin{pmatrix} e & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \Sigma$ and e an adapted frames of Σ at x. The map $\pi : F\Sigma \to \Sigma$ given by

$$\pi \begin{pmatrix} e & x \\ 0 & 1 \end{pmatrix} = x,$$

has fibers circles (rotate e_1, e_2, fix e_3). By definition, e_3 is perpendicular to the tangent space to Σ, so $e_3 \cdot dx = 0$ (i.e. directions dx in which you can move are perpendicular to e_3). So $\omega_3 = 0$. Hence $F\Sigma \subset G$ is a solution of the differential equations $\omega_3 = 0$. Because $\omega_3 = 0$ on $F\Sigma$, that implies that

$$\begin{aligned} 0 &= d\omega_3 \\ &= -\omega_{31} \wedge \omega_1 - \omega_{32} \wedge \omega_2 \\ &= \omega_{13} \wedge \omega_1 + \omega_{23} \wedge \omega_2. \end{aligned}$$

By Cartan's lemma,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

for some functions $a_{ij} : F\Sigma \to \mathbb{R}$, with $a_{ij} = a_{ji}$.

8.4 Shape operator

Let $A = (a_{ij})$, called the shape operator. The *Gauss curvature* is $G = \det A$, while the *mean curvature* is $H = \operatorname{tr} A$. Exercise: show that G, H are actually functions on Σ.

8.5 Exterior differential system

Set

$$\begin{aligned} \vartheta_0 &= \omega_3 \\ \vartheta_1 &= \omega_{13} - a_{11}\omega_1 - a_{12}\omega_2 \\ \vartheta_2 &= \omega_{23} - a_{21}\omega_1 - a_{22}\omega_2. \end{aligned}$$

Then

$$d\begin{pmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

modulo ϑ's, where

$$\pi_{ij} = \pi_{ji} = da_{ij} + \omega_{ik}a_{kj} - a_{ik}\omega_{kj}.$$

This is an exterior differential system (i.e. a set of equations expressed in differential forms) on the manifold $G \times \mathbb{R}^3_{a_{11},a_{12},a_{22}}$. Count: $s_1 = 2, s_2 = 1, S = 4$, so Cartan's test: $s_1 + 2s_2 = S$ is satisfied and we find that surfaces in $\mathbb{R}^3$ depend on 1 function of 2 variables. They are locally graphs of functions, so this is right.

8.6 Example: constant mean curvature

Now what if we ask for surfaces of constant mean curvature?

$$\begin{aligned} a_{11} + a_{22} &= \text{constant} \\ da_{11} + da_{22} &= 0 \\ \pi_{11} + \pi_{22} &= 0. \end{aligned}$$

This kills off a 1-form from the second column, so now $s_1 = 2, s_2 = 0$ and check that $S = 2$, so Cartan's test $s_1 + 2s_2 = S$ is satisfied, and surfaces of constant mean curvature depend on 2 functions of 1 variable.

Same idea if we ask for any one equation on G, H.

8.7 Example: surfaces with all points umbilic

Umbilic points are points where $A = (a_{ij})$ is a multiple of the identity matrix,

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Thus

$$\begin{aligned} a_{12} &= 0, a_{11} = a_{22}, \\ da_{12} &= 0, da_{11} = da_{22}, \\ \pi_{12} &= 0, \pi_{11} = \pi_{22}. \end{aligned}$$

Now the tableau is

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\end{pmatrix} = -\begin{pmatrix}0 & 0\\ \pi_{11} & 0\\ 0 & -\pi_{11}\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

modulo ϑ's. Cartan's test fails: $s_1 = 1, s_2 = 0$ while by Cartan's lemma $\pi_{11} = 0$ on solutions, so $S = 0$, so $s_1 + 2s_2 = 1 \neq S = 0$.

8.8 Prolongation

Now what? Cartan's lemma told us that $\pi_{11} = 0$, so throw that in to the equations, say $\vartheta_3 = \pi_{11}$. Now you find tableau

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\\ \vartheta_3\end{pmatrix} = -\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

modulo ϑ's.

Theorem 3 (Frobenius) *Define s_0 to be the number of independent ϑ's. Then Cartan's test extends to the case of vanishing tableau: for vanishing tableau, the solutions depend on s_0 functions of no variables, i.e. s_0 constants.*

So totally umbilic surfaces in $\mathbb{R}^3$ depend on 4 constants. In fact, they turn out to be just the spheres (3 parameters for center, 1 for radius).

9 Example: isometric embedding of surfaces

Given an oriented surface Σ with a Riemannian metric, let $F\Sigma$ be the set of pairs (x, E_1, E_2) where $x \in \Sigma$ and E_1, E_2 an orthonormal basis of $T_x\Sigma$. Suppose that we have an isometric embedding $\phi : \Sigma \to \mathbb{R}^3$. Define a map $\Phi : F\Sigma \to G$ by

$$\Phi(x, E_1, E_2) = \begin{pmatrix}e & \phi(x)\\ 0 & 1\end{pmatrix},$$

where

$$e = \begin{pmatrix}e_1 & e_2 & e_3\end{pmatrix},$$

with $e_j = \phi'(x)E_j$ and e_3 is the unit normal vector. The graph of Φ is a submanifold of $F\Sigma \times G$.

9.1 Structure equations of surfaces

Let $\pi : (x, E_1, E_2) \in F\Sigma \mapsto x \in \Sigma$. Define 1-forms η_1, η_2 on $F\Sigma$ by

$$v \lrcorner \eta_j = e_j \cdot \pi' v$$

for $v \in T_{(x,E_1,E_2)}F\Sigma$.

Theorem 4 (Levi–Civita) *There is a unique 1-form $\eta_{12} = -\eta_{21}$ on $F\Sigma$ for which*

$$d\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = -\begin{pmatrix}0 & \eta_{12}\\ \eta_{21} & 0\end{pmatrix} \wedge \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Taking exterior derivative gives

$$d\eta_{12} = G\eta_1 \wedge \eta_2,$$

and G is the Gauss curvature.

9.2 Exterior differential system

Let

$$\begin{aligned}
\vartheta_0 &= \omega_3\\
\vartheta_1 &= \omega_1 - \eta_1\\
\vartheta_2 &= \omega_2 - \eta_2\\
\vartheta_3 &= \omega_{12} - \eta_{12}\\
\vartheta_4 &= \omega_{13} - a_{11}\,\omega_1 - a_{12}\,\omega_2\\
\vartheta_5 &= \omega_{23} - a_{21}\,\omega_1 - a_{22}\,\omega_2
\end{aligned}$$

and

$$\pi_{ij} = da_{ij} + \eta_{ik}a_{kj} + a_{ik}\eta_{kj}.$$

Calculate

$$d\begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\\ \vartheta_3\end{pmatrix} = -\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ \pi_{11} & \pi_{12} & 0\\ \pi_{21} & \pi_{22} & 0\end{pmatrix} \wedge \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_{12}\end{pmatrix},$$

and $a_{11}a_{22} - a_{12}^2 = G$, giving

$$\pi_{11}a_{22} + a_{11}\pi_{22} - 2a_{12}\pi_{12} = dG.$$

9.3 Cartan's count

$dG = G_1\,\eta_1 + G_2\,\eta_2$, some functions $G_1, G_2 : F\Sigma \to \mathbb{R}$, so this is one equation on these π's, modulo ϑ, η. Calculate $s_1 = 2, s_2 = 0$, and $S = 2$, so involution with general isometric embedding (local) depending on 2 functions of 1 variable, as long as we don't run into trouble with these a_{ij} variables. We really need the equation on the π's to be nontrivial, so need one of the $a_{ij} \neq 0$. Therefore the general isometric embedding with nonzero second fundamental form depends on 2 functions of 1 variable.

Similar calculations work for any real analytic n-manifold M, isometrically embedding locally in $\mathbb{R}^N$, $N = n(n+1)/2$.

In C^∞ category, every surface with $G > 0$ or $G < 0$ is locally isometrically embeddable in $\mathbb{R}^3$ (for $G > 0$, elliptic PDE; for $G < 0$ hyperbolic PDE). But there are C^∞ surfaces with a single point of $G = 0$, for which no neighborhood of that point isometrically embeds in $\mathbb{R}^3$. In any C^k or Sobolev space, $k \geq 2$, every surface is arbitrarily closely approximated by surfaces which locally isometrically embed. Every surface isometrically embeds into $\mathbb{R}^5$ (Gromov).

10 For more information

Nothing in this document is true, not even this sentence. For a more lengthy introduction, and more accurate statements, see [2, 3] (especially the exercises in the second half of [2]). For a comprehensive, accurate and definitive treatment see [1].

References

[1] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems.* Springer-Verlag, New York, 1991.

[2] Élie Cartan. *Les systèmes différentiels extérieurs et leurs applications géométriques.* Actualités Sci. Ind., no. 994. Hermann et Cie., Paris, 1945.

[3] Thomas A. Ivey and J. M. Landsberg. *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, volume 61 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2003.