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Cartan's method of equivalence

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Preface

This book explains Cartan's method of equivalence, with some involved examples, and discussions of the more technical points. The manuscript began as my 8 pages of notes of Robert Bryant's 1996 lectures at the Institute for Advanced Study on Élie Cartan's method of equivalence.[1] Most of the added material can be found elsewhere in the literature. There is some new material here, but I ask credit principally for the mistakes.

[1] Those lectures became appendix 1 of chapter 2 of Bryant & Griffiths [18].

Contents

1

Introduction

1.1 The purpose of the method

The method of equivalence generates differential invariants of a wide variety of geometric structures on manifolds. It is more a recipe than an algorithm, so there are no criteria for its having succeeded or failed. It organizes calculations, without requiring local coordinates. Its greatest strengths are in calculating examples with large symmetry groups and in bringing to light hidden geometric features such as characteristic classes, foliations or complex structures. As examples (don't be concerned if you are not familiar with them), the Godbillion-Vey invariant of foliations of a 3 dimensional manifold by surfaces, the characteristic classes of web geometries, the Chern–Moser theory of CR-manifolds, and the existence of Riemannian manifolds of holonomy G_2 and Spin (7) were arrived at using essentially this point of view. Again: the reader need not be familiar with any of these topics to benefit from this book. The method works most effectively when applied to geometric objects with strong local structure, although we will apply it to some flabby objects as well. We will concentrate in this book on finding homogeneous examples of different types of geometry.

Question 1. Take a look over the hints and make sure that they read clearly.

Question 2. To relate the abstract computation of the Lie algebra of a homogeneous guy (using roots, for instance) to a specific matrix representation, allowing us a Cartan connection for the inhomogeneous guys, we will want to look at the splitting

$$\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \dots$$

where $\mathfrak{h}_{-1} = V$, $\mathfrak{h}_0 = \mathfrak{g}$ and $\mathfrak{h}_k \subset \mathfrak{g}^{(k)}$, I guess.

Question 3. I would like to include discussion of the tractor calculus and BGG somewhere.

1.2 Geometries

Many geometric structures on manifolds determine special bases for tangent spaces, and are determined by these bases, and any two such bases are matched up by some unique element of a particular subgroup of the general linear group. For examples, see table 1.1. We may think of these bases as choices of special coordinates, adapted to the geometry, but only defined as Taylor expansions up to first order. In the symplectic geometry example, if Ω is a symplectic form, a symplectic basis is a basis of vectors $u_1, \dots, u_n, v^1, \dots, v^n$ so that:

$$\Omega\left(u_i, u_j\right) = \Omega\left(v^i, v^j\right) = \Omega\left(u_i, v^j\right) - \delta_i^j = 0$$

Geometry	*Distinguished bases*	*Structure group*
Riemannian	orthonormal	$\mathrm{SO}(n)$
Symplectic	symplectic	$\mathrm{Sp}\left(2n, \mathbb{R}\right)$
Complex	holomorphic	$\mathrm{GL}\left(n, \mathbb{C}\right)$
Kähler	unitary holomorphic	$U(n)$

Table 1.1. Examples of geometric structures

1.3 Prerequisites

In these notes, I will assume that the reader is familiar with differential geometry at the level of a first graduate course, for example Spivak [78]; occasionally I will make some comments which assume familiarity with elementary representation theory of Lie algebras, but those comments can always be skipped. Far more than just the relevant Lie algebra theory is beautifully developed by Fulton & Harris [39] or Bröcker & tom Dieck [10]. Occasionally, we will employ the Cartan–Kähler theorem, a very general theorem for proving existence of solutions of systems of partial differential equations, although the reader will find that we often discover some way around using it; the theorem is explained more clearly than anywhere else by Ivey & Landsberg [47] and in the solved problems in the second half of Cartan's book [25], and proven in complete detail and generality by Bryant et. al. [12]. Ivey & Landsberg also provide a clear and short introduction to the equivalence method.

Part I

Ingredients

2

G-structures, frame bundles and torsion

2.1 The frame bundle

Let M be a manifold, and V a vector space of the same dimension as M. Define FM, the *frame bundle* of M, to be the bundle over M whose points $u \in F_x M$ are the linear isomorphisms

$$u : T_x M \to V.$$

When we want to be specific, we will call such an isomorphism a V *valued coframe*.[1] Let $\pi : FM \to M$ be the bundle map $\pi(u) = x$ for $u : T_x M \to V$. The frame bundle is a principal $\mathrm{GL}(V)$ bundle, with right action $u \mapsto r_g(u)$ defined by $r_g(u) = g^{-1}u$. (This will prove a superior choice to the obvious left action.)

Example 1. If V is a vector space, then $FV = V \times \mathrm{GL}(V)$.

Exercise 2.1 The tangent bundle is the quotient

$$TM = (FM \times V) / \mathrm{GL}(V).$$

Figure out what $\mathrm{GL}(V)$ action is needed here to make this work, and prove that it works. How can a similar construction produce the cotangent bundle?

Definition 1. *Suppose that $\phi : M \to N$ is a smooth map, which is locally a diffeomorphism. Define its* prolongation to frames $F\phi$ *as*

$$F\phi : FM \to FN \qquad F\phi(u) = u \circ \phi'(x)^{-1}$$

for $u \in F_x M$.

[1] In this book, we use the term *coframe* for an isomorphism of one tangent space with a vector space, and the term *coframing* for a smooth family of such isomorphisms, defined at each point of a manifold.

Exercise 2.2 Unwind definitions to show that $r_g F\phi = F\phi r_g$, i.e. the structure group $\mathrm{GL}(V)$ commutes with diffeomorphisms.

On FM there is a canonically defined *soldering form* ω defined by

$$\omega \in \Omega^1(FM) \otimes V \quad \omega(v) = u(\pi_* v) \text{ for all } v \in T_u FM.$$

Exercise 2.3

$$r_g^* \omega = g^{-1} \omega.$$

Exercise 2.4 The soldering form is invariant under prolongation to frames of a diffeomorphism $\phi : M \to M$:

$$F\phi^* \omega = \omega.$$

Exercise 2.5 The soldering form has the *reproducing property*: if η is a section of the frame bundle, i.e. a V-valued 1-form which identifies the tangent spaces of M with V,

$$\eta_x : T_x M \to V$$

then

$$\eta^* \omega = \eta$$

where on the left-hand side η^* means, thinking of $\eta : M \to FM$ as a map, pulling back ω, while η on the right-hand side means η as a V-valued 1-form.

Example 2. On a vector space V, the frame bundle is $FV = V \times \mathrm{GL}(V)$ and if $x \in V$ and $u \in \mathrm{GL}(V)$, then $\omega = u\, dx$. The right action is

$$r_g(x, u) = \left(x, g^{-1} u\right).$$

2.2 *G*-structures

Definition 2. *Suppose that G is a Lie group and V is a representation of G. Take M a manifold with $\dim M = \dim V$ and let FM be the V valued frame bundle of M. A G-*structure *is a right principal G bundle $B \to M$ together with a G equivariant bundle map $B \to FM$.*

2.2.1 Examples

Example 3. Traditionally in this subject, the group with 1 element, $\{1\}$, is denoted by e. An e-structure is a choice of section $M \to FM$ of the frame bundle $FM \to M$, i.e. a choice of coframing.

Example 4. Take a Lie group G and finite dimensional G representation V. Write the representation as $\rho : G \to \mathrm{GL}(V)$. The *standard flat G-structure* is the trivial bundle $B = V \times G \to M = V$, with right action

$$r_g(x_0, g_0) = \left(x_0, g^{-1} g_0\right)$$

which maps into $FV = V \times \mathrm{GL}(V)$ by the map

$$(x_0, g_0) \mapsto (x_0, \rho(g_0)).$$

Exercise 2.6 Prove that a foliation of a manifold by submanifolds determines and is determined by a G-structure.

Example 5 (Lie groups). Take $M = H$ a Lie group, $V = \mathfrak{h}$ the Lie algebra, and $B = H$ the same Lie group with map $B \to FM$ given by

$$h \in H = B \mapsto u \in FM$$

where

$$u = \left(r_h^{-1}\right)' : T_h H \to T_1 H = \mathfrak{h}.$$

This type of e-structure is particularly important, and will reappear frequently.

Example 6 (Pullback). If $\phi : M_0 \to M_1$ is a local diffeomorphism, and $B_1 \to M_1$ is any G-structure, then we can define the *pullback* G-structure $\phi^* B_1 \to M_0$ to be the pullback bundle as a principal G-bundle, turned into a G-structure via the map

$$u_1 \in \phi^* B_1 \mapsto u_1 \phi'(x) \in FM_0.$$

Exercise 2.7 (Pushforward) Suppose that $M_1 = \Gamma \backslash M_0$ is a quotient by a discrete group Γ, and that $B_0 \to M_0$ is a G-structure, and that Γ acts on B_0 as symmetries of the G-structure. Show that the *pushforward* $B_0/\Gamma \to M_1$ is a smooth principal right G-bundle. Define a map $B_0/\Gamma \to FM_1$ to create a G-structure.

Example 7 (Local picture). Every manifold is locally an open subset $M \subset V$ inside a vector space, and every right principal G bundle is locally a product $B = M \times G$, for instance with right action

$$r_g(x_0, g_0) = \left(x_0, g^{-1} g_0\right).$$

Such a trivialization of B is uniquely determined up to a gauge transformation $g : M \to G$ replacing

$$(x_0, g_0) \mapsto (x_0, g_0 g(x_0)).$$

Every G-structure is locally a map

$$(x_0, g_0) \in B \mapsto (x_0, \rho(g_0) f(x_0)) \in FM = M \times \mathrm{GL}(V)$$

for some function $f : M \to \mathrm{GL}(V)$. The choice of trivialization of B alters f by gauge transformation $f(x_0) \mapsto \rho(g(x_0))^{-1} f(x_0)$. Therefore the map

$$f/G : M \to \mathrm{GL}(V)/G$$

is well defined, and determines the G-structure. Locally, all G-structures are determined in this manner, but globally the story is more complicated.

Exercise 2.8 Show that a G-structure B on a manifold M determines and is determined by a section of the fiber bundle $FM/G \to M$.

Example 8. Take a homogeneous space $M = H/G$ and pick any frame $u_0 \in FM$. The action of an element $h \in H$ on M prolongs to an action Fh on FM as in definition 1 on page 5. The H-orbit of that frame is a G-structure:

$$\begin{array}{ccc} h \in H & \longmapsto & Fhu_0 \in FM \\ & \searrow \quad \swarrow & \\ & hm_0 \in M. & \end{array}$$

Most of our examples will be constructed in this way.

Example 9. Consider another point of view on the same construction. Let $M = H/G$ be a homogeneous space. Write the map $H \to M$ as $\pi : H \to M$. Take $h^{-1}\,dh$ to be the left invariant Maurer–Cartan 1-form, defined at a point $h_0 \in H$ as

$$h^{-1}\,dh = \left(L_{h_0}^{-1}\right)',$$

where L_{h_0} means left translation by h_0. Let $V = \mathfrak{h}/\mathfrak{g}$. Define a 1-form $\omega \in \Omega^1(H) \otimes V$ by $\omega = h^{-1}\,dh \pmod{\mathfrak{g}}$.

Lemma 1. *The 1-form ω vanishes on the fibers of $H \to M = H/G$.*

Proof. These fibers are hG translates, so on them $h^{-1}\,dh$ is valued in $\mathfrak{g}$.

So ω is semibasic[2] for $H \to M$, and for each point $h \in H$ we can define a 1-form at the corresponding point of M, written $u = u(h) \in T_{hG}M \otimes V$, to be the one which pulls back to ω:

$$u(h)\pi'(h) = \omega.$$

Map $H \to FM$ by $h \in H \mapsto u(h) \in FM$ to make an H-structure.

[2] A differential form η on a manifold X is called *semibasic* for a map $\phi : X \to Y$ if at each point $x \in X$, there is some $\xi \in \Lambda^*\left(T^*_{\phi(x)}Y\right)$ for which $\eta_x = \phi^*\xi$. For example, $y\,dx$ is semibasic for $(x, y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$.

Example 10. A variation on the previous example: take a G-structure $\pi : B \to M$ and suppose that a Lie group H of symmetries of the G-structure acts transitively on M. Then take any element $u \in B$, say $\pi(u) = m \in M$ and consider the maps $h \in H \mapsto Fh(u) \in B$, and $h \in H \mapsto h(m) \in M$. These make H an H_m-structure on M, where H_m is the stabilizer in H of $m \in M$. The structure group H_m acts on the right on H by usual right multiplication, while H acts on itself as symmetries of the H_m-structure, by usual left multiplication.

Exercise 2.9 Explain how the constructions in examples 8,9 and 10 are related.

2.2.2 Embedded G-structures

Returning to the general theory of G-structures, pull back the soldering 1-form ω to B so that we obtain another V valued 1-form which we call by the same name, satisfying the same equation:

$$r_g^* \omega = g^{-1} \omega$$

for $g \in G$.

Definition 3. *A G-structure is called an* embedded G-structure *if the map $B \to FM$ is injective.*

In fact, embeddedness depends only on the representation V of G: a G-structure $B \to M$ is embedded just when $G \to \mathrm{GL}(V)$ is an embedding, i.e. a closed subgroup. Define a subgroup N of G by

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathrm{GL}(V).$$

If G has image in $\mathrm{GL}(V)$ a closed subgroup (which happens in every case of interest), then B/N is an embedded G/N-structure. However, there can be more than one G-structure $B \to FM$ with the same image, i.e. the same embedded B/N structure.

Example 11. A *spin manifold* is a manifold with G-structure where $G = \mathrm{Spin}(n)$ acts on the vector space $V = \mathbb{R}^n$ via the usual irreducible representation. Recall the exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n) \to 1$$

and that $\mathrm{Spin}(n)$ is the only connected group that fits into this sequence. We get $G = \mathrm{Spin}(n)$ to act on $V = \mathbb{R}^n$ via the map to $SO(n)$. There can be more than one spin structure with the same $SO(n)$-structure (i.e. Riemannian metric). See Lawson & Michelsohn [56] for examples and a beautiful exposition of spin structures.

Example 12 (Hyperbolic and spherical geometry). Lets see an embedded G-structure explicitly in coordinates. On $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$) we can take coordinates x^i and consider the $SO(n)$-bundle whose elements are (x, u) with $u : T_x\mathbb{R}^n = \mathbb{R}^n \to \mathbb{R}^n$ given by

$$u^i_j = \frac{g^i_j}{a + \frac{\kappa |x|^2}{4a}}$$

where $|x|$ is the usual Euclidean distance from the origin, a and κ are arbitrary constants, $a \neq 0$, and $g = \left(g^i_j\right) \in SO(n)$. This $SO(n)$-structure is the Riemannian geometry of a sphere or hyperbolic space, depending on the sign of κ, as we will see later on, and κ will be the sectional curvature. (Picking $a = 1$ is convenient to show that it looks like the standard flat $SO(n)$-structure to first order. On the other hand, for $\kappa < 0$ it is more common to scale things so that it is defined precisely in the disk $r < 1$, i.e. to set $a = \sqrt{|\kappa| 2}$.)

Exercise 2.10 Recall example 8 on page 8: $M = H/G$, pick $u \in FM$ and map $H \to FM$ via $h \mapsto Fh(u)$. Show that such a G-structure is embedded precisely when G acts freely on $\mathfrak{h}/\mathfrak{g}$ via the adjoint action. Most of our examples will be constructed in this way.

2.2.3 Peripheral remarks on topology and embedded G-structures

The embedded G-structures on a manifold M are identified with sections of $FM/G \to M$. The existence of an embedded G-structure is therefore a problem of topology: a manifold M which is homotopy equivalent to a finite cell complex carries a characteristic class

$$c(m) \in \bigoplus_k H^{k+1}\left(M, \pi_k\left(F_m M/G\right)\right)$$

belonging to the cohomology with coefficients in the local system of homotopy groups of the fibers. This class vanishes when an embedded G-structure exists. A similar story describes whether two embedded G-structures can be deformed into one another. See Steenrod [83] for details.

Question 4. Try to give page references for citations wherever possible.

The one result on topology of principal bundles which we will make use of:

Proposition 1. *If $K \subset G$ is a maximal compact subgroup of a Lie group, then every principal G-bundle $B \to M$ contains a principal K-subbundle $B' \subset B$.*

Steenrod [83] gives the proof.

Definition 4. *For $G_0 \subset G$ a closed subgroup of a Lie group, a principal G_0-subbundle of a G-structure is called a G_0*-reduction.

Corollary 1. *If $K \subset G$ is a maximal compact subgroup of a Lie group G, then every G-structure admits a K-reduction.*

2.3 Equivalence

Definition 5. *Two G-structures $B \to M$ and $B' \to M'$ are called* equivalent *if there is some diffeomorphism $\phi : M \to M'$ and a right principal G bundle isomorphism $\Phi : B \to B'$ so that*

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ \downarrow & & \downarrow \\ FM & \xrightarrow{F\phi} & FM' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & M' \end{array}$$

(Recall that $F\phi$ means the prolongation to frames.) The pair (ϕ, Φ) is called an equivalence*. An autoequivalence is usually called a* symmetry*. Clearly the map Φ determines the map ϕ.*

Exercise 2.11 The symmetry group commutes with the structure group G.

Exercise 2.12 Writing $\mathrm{Aut}\,(B)$ for the symmetry group of a G-structure, show that the group $G \times \mathrm{Aut}\,(B)$ (with direct product group structure) acts on B via $(g, \Phi)b = r_g^{-1}\Phi(b)$. Show that the subgroup of $G \times \mathrm{Aut}\,(B)$ which acts trivially on B is the set of pairs (g, Φ) with $g \in N \cap Z(G)$ and $\Phi = r_g$. (Recall that N is the subgroup of G acting trivially on the representation V; see the remarks following definition 3 on page 9.) In particular, if the G-structure is embedded ($N = 1$), then an equivalence (ϕ, Φ) is completely determined by ϕ.

Example 13. Translations

$$\phi\,(x_0) = x_0 + \tau, \ \Phi\,(x_0, g_0) = (x_0 + \tau, g_0)$$

and the maps

$$\phi(x) = \rho(g)x_0, \ \Phi\,(x_0, g_0) = \left(\rho(g)x_0, g_0 g^{-1}\right)$$

are equivalences of the standard flat G-structure. We can put them together into an action of the semidirect product $G \ltimes V$ (in which V is the normal subgroup):

$$(g, \tau)x_0 = \rho(g)x_0 + \tau, \ (g, \tau)\,(x_0, g_0) = \left(\rho(g)x_0 + \tau, g_0 g^{-1}\right).$$

Recall that the semidirect product has multiplication

$$(g, \tau)\,(g', \tau') = (gg', \tau + \rho\,(g)\,\tau').$$

This symmetry group acts transitively on the bundle B, so the standard flat G-structure is homogeneous.

Example 14. The symmetry group of the standard flat G-structure could be much larger than $G \ltimes V$. For example, if $G = GL(2, \mathbb{C})$, and $V = \mathbb{C}^2$, then the standard flat G-structure has an infinite dimensional symmetry group, given precisely by the biholomorphisms of $\mathbb{C}^2$. Among them one finds

$$\phi(z_1, z_2) = (z_1, z_2 + f(z_1))$$

where $f(z_1)$ is any entire holomorphic function. These maps, sometimes called *shears*, are symmetries of the standard flat $SL(2, \mathbb{C})$-structure, and even of the standard flat G-structure, where G is the group of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

with $a \in \mathbb{C}$.

In general, the symmetry group of the standard flat G-structure (when G is embedded) consists in the maps $\phi : V \to V$ with $\phi'(x) \in G$ for all $x \in V$.

Exercise 2.13 The symmetry group of a flat G-structure might not be as large as the symmetry group of the standard flat G-structure, or may be larger. Calculate the symmetry group of the quotient of the standard flat G-structure on your favourite flat torus. However the standard flat G-structure does not always have the largest group of global symmetries; see the next example.

Example 15 (Conformal structures on Riemann surfaces). The standard flat structure is not always the most symmetrical. A *Riemann surface* is a surface with $\mathrm{GL}(1, \mathbb{C})$-structure; the representation is the obvious $V = \mathbb{C}$. (N.B. $\mathrm{GL}(1, \mathbb{C})$ is the identity component of $CO(2)$, so these are conformal structures.) There are three connected and simply connected Riemann surfaces (up to equivalence): the disk Δ, the complex affine line $\mathbb{C}$, and the complex projective line $\mathbb{CP}^1$ (also called the Riemann sphere); see Forster [38] for proof. Each is obviously included in the next. The complex affine line is the standard flat $\mathrm{GL}(1, \mathbb{C})$-structure.

Riemann surface	*Symmetry group Γ*		$\dim \Gamma$
Δ	$\mathbb{P}SL(2, \mathbb{R}) \cong \mathbb{P}\mathrm{SU}(1,1)$	hyperbolic isometries	3
$\mathbb{C}$	$\mathbb{C}^{\times} \rtimes \mathbb{C}$	complex affine maps	4
$\mathbb{CP}^1$	$\mathbb{P}SL(2, \mathbb{C})$	Möbius group	6

Table 2.1. Symmetry groups of simply connected Riemann surfaces

Example 16 (Pullback). The *pullback* of a G-structure (see example 6 on page 7) is a local equivalence.

Lemma 2 (Homogeneity). *Suppose that* $B_1 \to M_1$ *is a* G*-structure with transitive symmetry group. Take* $M_0 \to M_1$ *any covering space, and let* $B_0 \to M_0$ *be the pullback (see example 6 on page 7). The* G*-structure* $B_0 \to M_0$ *has locally transitive symmetry group (transitive, if it is a normal covering or if* M_0 *is connected).*

Proof. Each infinitesimal symmetry is a vector field, say X_1 on B_1, which projects to a vector field on M_1 as well, which we will call by the same name. The map $B_0 \to B_1$ is a local diffeomorphism, so determines a vector field X_0 on B_0, and one on M_0, which again we call by the same name. Because the maps $B_0 \to B_1$ and $M_0 \to M_1$ are covering maps, the integral curves are mapped locally diffeomorphically, and in particular, completeness of X_1 implies completeness of X_0. Therefore the Lie algebra of complete infinitesimal symmetries lifts to B_0, giving a group action of a connected covering Lie group by Palais' corollary 6 on page 73. The covering space must split into orbits, each of which is a homogeneous space and an open subset of M_0. Clearly these orbits cover M_0, and so each must be a path component.

Recall that a covering map $M_0 \to M_1$ is called *normal* (or *regular*) if $\pi_0(M_0) \to \pi_0(M_1)$ is an isomorphism, and $\pi_1(M_0) \to \pi_1(M_1)$ is a normal subgroup. This is the case precisely when the group of deck transformations is transitive on the fibers. Clearly the deck transformations are equivalences.

Exercise 2.14 Give an example of a G-structure whose symmetry group is transitive, for which the symmetry group of the pullback to some abnormal covering is not transitive.

Exercise 2.15 Fix a group $G \subset \mathrm{GL}(V)$. Let Γ be the group of linear transformations of V that intertwine the representation, i.e. Γ is the group of $T \in \mathrm{GL}(V)$ so that $TG = GT$. For any G-structure $\alpha : B \to FM$, show that $r_T\alpha : B \to FM$ is also a G-structure. Give an example in which these are inequivalent. Examples of G-structures with nontrivial Γ groups arise in some geometry problems (for instance, in hyperkähler geometry, or web geometry, see section 4.1).

Exercise 2.16 Continuing the previous exercise, let Z be the elements of $\mathrm{GL}(V)$ which commute with each element of G. Show that $W = \Gamma/Z$ acts on the category of G-structures, and that the symmetry groups and relations of equivalence and local equivalence are invariant under the W action.

Definition 6. *A* G*-structure is called* flat *if it is locally equivalent to the standard flat* G*-structure.*

Proposition 2. *An embedded* G *structure* $B \subset FM$ *is flat precisely if each point of* M *has a neighborhood, say* U*, where there is some* V *valued 1-form* $\eta \in \Omega^1(U) \otimes V$ *so that (1)* $d\eta = 0$ *and (2)* $\eta_x \in B_x$ *for each* $x \in U$*.*

Proof. Locally η is exact, $\eta = d\phi$. This ϕ is the map that identifies an open set of M with an open set of V, matching up G-structures. Conversely, given such a map ϕ, take $\eta = d\phi$.

Exercise 2.17 What does equivalence mean for G-structures of the type given in example 5 on page 7?

Determining whether two G-structures are equivalent has two parts: (1) ask if B/N and B'/N are equivalent as embedded G/N-structures and (2) ask if $B \to B/N$ and $B' \to B'/N$ are isomorphic as principal right N bundles. The second part of the problem belongs entirely to differential topology, and can be solved using the methods of Steenrod [83]. Therefore, our focus is entirely on the first part of the problem, in which we will employ the local geometry of an embedded G-structure.

Henceforth all G-structures are assumed embedded.

2.4 The significance of the soldering form

Recall the soldering form on FM, ω defined at a point $u \in FM$ by $\omega_u = u\pi'$, where $\pi : FM \to M$ is the obvious bundle map. Let B be a G-structure on M. We pull back ω to B via the bundle map $B \to FM$. Note that the 1-form ω is surjective at each point:

$$T_u B \xrightarrow{\omega_u} V \longrightarrow 0,$$

and its kernel consists of precisely the vertical vectors, i.e. the vectors tangent to the fibers of $B \to M$.

Proposition 3. *If $\phi : M_0 \to M_1$ is an equivalence between G-structures $B_0 \to M_0$ and $B_1 \to M_1$, then the prolongation to frames of ϕ is a diffeomorphism $F\phi : B_0 \to B_1$ matching up the soldering forms*

$$F\phi^* \omega_1 = \omega_0$$

(where ω_0 is the soldering form on B_0, and ω_1 the soldering form on B_1). Conversely if $U_0 \subset B_0$ is an open set, with connected fibers, and $\psi : U_0 \to B_1$ is any smooth map satisfying

$$\psi^* \omega_1 = \omega_0$$

then there is a local diffeomorphism

$$\bar{\psi} : \pi(U_0) \to M_1$$

so that $\psi = F\bar{\psi}$ and $\bar{\psi}$ is a local equivalence of B_0 and B_1, i.e. on small enough open sets U (forming a cover of $\pi(U_0)$), $\bar{\psi}$ restricts to be an equivalence of the G-structure $\pi_0^{-1}U \to U$ with the G-structure $\pi_1^{-1}\bar{\psi}(U) \to \bar{\psi}(U)$.

Proof. Because $\psi^*\omega_1 = \omega_0$, the fibers $\omega_0 = 0$ of $U_0 \to M_0$ are mapped into the fibers $\omega_1 = 0$ of $B_1 \to M_1$. Define $\bar{\psi}$ by $\psi(m_0) = m_1$ if ψ takes the fiber of m_0 into that of m_1.

$$\begin{aligned}
(\omega_0)_{u_0} &= u_0 \pi_0'(u_0) \\
&= (\psi^*\omega_1)_{u_0} \\
&= (\omega_1)_{u_1} \psi'(u_0) \\
&= u_1 \pi_1'(u_1)\, \psi'(u_0) \\
&= u_1 (\pi_1 \psi)'(u_0) \\
&= u_1 \left(\bar{\psi}\pi_0\right)'(u_0) \\
&= u_1 \bar{\psi}'(m_0)\, \pi_0'(u_0).
\end{aligned}$$

Therefore $u_0 = u_1 \bar{\psi}'(m_0)$. So

$$\begin{aligned}
u_1 &= u_0 \left(\bar{\psi}'(m_0)\right)^{-1} \\
&= F\bar{\psi}(u_0).
\end{aligned}$$

Therefore the local equivalence problem consists in the study of integral manifolds (in the sense of exterior differential systems) of the field of planes $(\omega_0 = \omega_1)$ in tangent spaces of $B_0 \times B_1$.

Exercise 2.18 If we have bundle maps

$$\begin{array}{ccc}
B_0 & \longrightarrow & B_1 \\
\downarrow & & \downarrow \\
M_0 & \longrightarrow & M_1
\end{array}$$

matching up soldering forms, then $M_0 \to M_1$ is an equivalence, and $B_0 \to B_1$ is its prolongation to frames.

Example 17. If M is an open subset of V, and $B = M \times G$ is trivial (locally, this is always the case) then we can write the immersion $B \to FM$ as

$$(x, g) \to (x, g f(x))$$

for some $f : M \to \mathrm{GL}(V)$. We can parameterize FM via writing down points (x, u) with $x \in M$ and $u \in \mathrm{GL}(V)$. The soldering form on FM is

$$\omega = u\, dx.$$

It pulls back to B to give

$$\omega = g f\, dx.$$

The exterior derivative is

$$\begin{aligned} d\omega =& dg \wedge f\,dx + g f'(x) dx \wedge dx \\ =& -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega \end{aligned}$$

where

$$\gamma = -dg\, g^{-1}$$

and T is a function valued in $\Lambda^2 (V^*) \otimes V$. This function T will be our source for invariants of the G-structure. It is not itself an invariant, as it depends on the choice of local trivialization, so we have to figure out how we can find build invariants out of it.

Example 18 (Hyperbolic and spherical geometry). Returning to example 12 on page 10, the soldering form has components

$$\omega^i = g^i_j \frac{dx^j}{a + \frac{\kappa r^2}{4a}}.$$

Exercise 2.19 (Nonembedded G-structures) Recall the exact sequence for nonembedded structures $1 \to N \to G \to \mathrm{GL}(V)$. Even if $B_0 \to M_0$ and $B_1 \to M_1$ are not embedded G-structures, if there is a map $\psi : U$ open $\subset B_0/N \to B_1/N$ with $\psi^* \omega_1 = \omega_0$, prove that there is a *local equivalence* between B_0 and B_1 (and define the term *local equivalence*).

2.5 Connections

We will refer to the fiber of $B \to M$ over $x \in M$ as $B_x \subset B$, and its tangent spaces $T_u(B_x)$ as the spaces of *vertical* vectors.

Definition 7. *A* connection H *on a G-structure is a G invariant choice of complementary subspace to the tangent space to the fiber at each point:*

$$H_u \oplus T_u(B_x) = T_u B, r_g^* H_u = H_{r_g u}$$

where B_x is the fiber of B above $x \in M$, and $u \in B_x$.

With this we can associate to any velocity vector v at $x \in M$ and coframe $u \in B_x$ a corresponding velocity vector in B: the one $\hat{v} \in H_u$ so that

$$\pi'(u)\hat{v} = v.$$

We can lift paths from M to paths upstairs in B, by asking that the lifted path be one which projects to the original path, and has velocity belonging to H, a process called *parallel transport.*

Example 19. On the standard flat G-structure $B = V \times G \to V$, take

$$H = TV \oplus 0 \subset TB$$

This is the *standard flat connection.*

How can we work with differential forms? On a Lie group G we have a canonical left invariant $\mathfrak{g}$ valued 1-form: we translate a vector from any point of G up to identity, by left translation. It gives a universal measure of velocity, allowing comparisons of velocity vectors at any points. We can imitate this on any right principal bundle P: we have a space of vertical vectors: $T_p(P_x) \subset T_pP$ in each tangent space, the tangent spaces of the fibers. These are spanned by the vectors that represent the infinitesimal G action. Write them, for $A \in \mathfrak{g}$, as

$$\vec{A}(p) = \left.\frac{d}{dt}\right|_{t=0} r_{e^{-tA}} p$$

so that

$$\left[\vec{A}_1, \vec{A}_2\right] = \overrightarrow{[A_1, A_2]}.$$

On these vectors we have a map $T_p(P_x) \to \mathfrak{g}$, $\vec{A} \mapsto A$, measuring a kind of velocity of motion up the fibers. Nothing this simple will give us a 1-form, because we only know how to measure vertical motion, so want can't apply it to arbitrary tangent vectors. But with a choice of connection H, we can define the connection 1-form

$$\gamma = \gamma_H \in \Omega^1(B) \otimes \mathfrak{g}$$

by

$$\begin{aligned} \gamma(v) &= 0 \text{ if } v \in H \\ \gamma\left(\vec{A}\right) &= A \end{aligned}$$

Example 20. For the standard flat connection, we can write for $x \in V, g \in G$

$$\begin{aligned} \omega &= g\,dx \\ \gamma &= -dg\,g^{-1} \end{aligned}$$

Alternatively, we could start with a suitable 1-form γ:

Definition 8. *A* connection form *(or sometimes, abusing terminology, a* connection*) for a G-structure $B \to FM$ is a $\mathfrak{g}$ valued 1-form $\gamma \in \Omega^1(B) \otimes \mathfrak{g}$ that satisfies*

$$\begin{aligned} r_g^* \gamma &= \operatorname{Ad}_{g^{-1}} \gamma \\ \gamma\left(\vec{A}\right) &= A \end{aligned}$$

The associated connection is $H = \ker \gamma$.

Exercise 2.20 On the standard flat G-structure $B = V \times G$ any connection can be written

$$\gamma = -dg\,g^{-1} + \operatorname{Ad}_g\left(\Gamma(x)\,dx\right)$$

where Γ is any function

$$\Gamma : V \to V^* \otimes \mathfrak{g}.$$

Conversely, any function Γ can be used to construct a connection γ.

Exercise 2.21 Any G-structure admits a connection.

Example 21. $H = 0$ is a connection on any e-structure.

Example 22 (The flat spin structure on Euclidean 4-space). Let $\mathbb{H}$ be the algebra of quaternions, i.e. the associative real algebra with identity generated by three elements i, j and k with relations

$$-1 = ijk = i^2 = j^2 = k^2.$$

The elements $1, i, j, k$ form a basis, with which we identify $\mathbb{H}$ with $\mathbb{R}^4$. The unit length quaternions form a group called $\operatorname{Sp}(1)$.

Let G be the group $G = \operatorname{Sp}(1)_+ \times \operatorname{Sp}(1)_-$, two copies of $\operatorname{Sp}(1)$. This group G is usually called $\operatorname{Spin}(4)$. Let G act on $\mathbb{H}$ by $(g_+, g_-)\, x = g_+ x \bar{g}_-$. Consider the flat G-structure: $M = \mathbb{H}$, $B = M \times G$, and soldering 1-form $\omega = g\, dx$. But $g \in G$ so $g = (g_+, g_-)$, allowing us to write

$$\omega = g_+\, dx\, \bar{g}_-.$$

The flat connection $\gamma = \gamma_+ \oplus \gamma_-$ is

$$\begin{aligned} \gamma_+ &= -dg_+\, \bar{g}_+ \\ \gamma_- &= -dg_-\, \bar{g}_- \end{aligned}$$

so that the structure equations are

$$\begin{aligned} d\omega &= -\gamma \wedge \omega \\ &= -\gamma_+ \wedge \omega - \omega \wedge \gamma_- \\ d\gamma_+ &= -\gamma_+ \wedge \gamma_+ \\ d\gamma_- &= -\gamma_- \wedge \gamma_- \end{aligned}$$

with ω a 1-form valued in $\mathbb{H}$ and γ_+ and γ_- 1-forms valued in $\operatorname{Im}\mathbb{H}$. This shows us how a peculiar representation of a group G imposes itself on the structure equations of the G-structure. We will return to these equations later when we briefly look at twistor theory in chapter 11.

2.5.1 The canonical connection

We will see below (example 27 on page 25, section 7.3 on page 106) that there is a canonical connection, called the *Levi-Civita connection*, on any $O(n)$-structure (where we think of $O(n)$ as acting on $\mathbb{R}^n$ in the irreducible representation). The Levi-Civita connection is the unique torsion-free connection. The same idea works for $O(p, q)$-structures: they have canonical torsion-free connections. In the next chapter, we will try to see how close we can come to picking a torsion-free connection for other G-structures, imitating Riemannian geometry. If we have a G-structure $B \to M$ for which the underlying representation V of G bears an invariant definite quadratic form, so that $G \subset O(p,q)$, then we can "fatten up" G to $O(p,q)$, and B to what we will call $B(O(p,q))$ (see section 6.2 on page 59). The resulting connection determines a connection on B, called the *canonical connection*, and determined by the requirement that the connection 1-form γ is the orthogonal projection to $\mathfrak{g} \subset \mathfrak{so}(p,q)$ of the Levi-Civita connection 1-form (via the adjoint invariant inner product on $\mathfrak{so}(p,q)$). This idea will become clearer later on. For example, this determines the canonical connection of a Kähler manifold, or in general any $U(p,q)$-structure. Any compact group G preserves a positive definite inner product on each of its representations, giving every G-structure a canonical connection. More on canonical connections in example 28 on page 25.

Remark 1 (Connections on bundles). We have only introduced connections on G-structures; there are more general definitions of connections on principal bundles, but for the most part we won't need these.

Example 23 (Symmetries of the standard flat connection). The reader might be curious to know why connections on the frame bundle of a manifold are called *affine connections*. The reason is:

Theorem 1. *The automorphism group of the standard flat connection is the semidirect product $G \rtimes V$ where G acts on V by the representation, and V acts on itself by translation.*

Proof. The automorphisms must be maps $\phi : V \to V$. Such a map has prolongation to frames

$$F\phi(x,g) = \left(\phi(x), g\phi'(x)^{-1}\right).$$

Exercise 2.22 Compute that

$$F\phi^*\gamma = \gamma + \mathrm{Ad}_g \left(\phi'(x)^{-1}\phi''(x)\right)$$

Thus ϕ is a local equivalence just when $\phi'' = 0$, i.e. ϕ is an affine map. But we also need $\phi' \in G$ to have an equivalence, so the symmetry group is the group $G \rtimes V$. Indeed every local symmetry has this form too.

This holds true even when the underlying standard flat G-structure has infinite dimensional symmetry group, e.g. $G = \mathrm{GL}(n,\mathbb{C})$ or $G = \mathrm{GL}(n,\mathbb{R})$. For such groups G, no connection is preserved by the symmetry group. However if G is compact (and in some other cases as well, as we shall see) then the canonical connection on the standard flat G-structure is the standard flat connection, and is invariant under symmetries, and therefore the symmetry group of the standard flat G-structure is $G \rtimes V$.

Example 24 (Hyperbolic and spherical geometry). Returning to example 18 on page 16, there is a natural connection, the Levi-Civita connection (we will learn more about it later), for which the connection form has components

$$\gamma^i_j = -dg^i_k \left(g^{-1}\right)^k_j + \frac{\kappa}{2a^2 + \frac{\kappa r^2}{2}} g^i_k \left(x^k \, dx^m - x^m \, dx^k\right) \left(g^{-1}\right)^m_j .$$

2.6 Pseudoconnections

It is not always possible to construct invariantly defined choices of connections on our G-structures. In such cases there are sometimes invariantly defined *pseudoconnections*. We will consider the origin of this problem in detail shortly.

Definition 9. *A* pseudoconnection *is a smooth choice of complement H_u to the vertical:*

$$H_u \oplus T_u\left(B_x\right) = T_u B$$

A pseudoconnection form *(or, again abusing terminology, a* pseudoconnection*) γ for a G-structure B is a $\mathfrak{g}$ valued 1-form on B so that*

$$\gamma\left(\vec{A}\right) = A \text{ for } A \in \mathfrak{g}$$

The associated pseudoconnection is $H = \ker \gamma$.

It follows that the map

$$(\omega, \gamma) : T_u B \to V \oplus \mathfrak{g}$$

is a linear isomorphism.

Exercise 2.23 Prove that any pseudoconnection on the standard flat G-structure $V \times G$ can be written

$$\gamma = -dg \, g^{-1} + \mathrm{Ad}_g \left(\Gamma(x, g) \, dx\right)$$

where Γ is any function

$$\Gamma : V \times G \to V^* \otimes \mathfrak{g}.$$

Conversely, any function Γ can be used to construct a pseudoconnection γ.

Parallel transport is still defined as before: the curves in B on which $\gamma = 0$ are the parallel transports of curves from M. Each curve in M has a unique parallel transport through each point of B which lies over it. But there is a crucial difference: for a connection, G invariance ensures that the parallel transport of a frame along a curve is induced by a single linear map from the tangent space in M at the initial point of the curve to the tangent space in M at the final point. Therefore, parallel transport can be thought of on M as defining linear identifications of tangent spaces along a curve. For a pseudoconnection, such a family of linear identifications still exists, but is generally dependent on the choice of initial point u from B_x, i.e. initial frame on $T_x M$, because a pseudoconnection is not G invariant (unless it is a connection).

A connection on a G-structure always imposes a connection on the frame bundle, determined uniquely by $\mathrm{GL}(V)$ invariance. However, a pseudoconnection on a G-structure does not generally have a canonical extension to a pseudoconnection on the frame bundle (see section 6.2 on page 59).

Proposition 4 (Structure equations). *Let γ be a pseudoconnection. There exists a unique map $T : B \to V \otimes \Lambda^2(V^*)$ called the* torsion *of γ, so that*

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega.$$

This equation is called the structure equation *of the G-structure. To emphasis that the V-valued 1-form ω has components ω^i in a basis for V, we can also call this the* structure equations *of the G-structure.*

Proof. Before I prove the result, I need to explain the notation. The object $\omega \wedge \omega$ is a 2-form with values in $\Lambda^2(V)$ defined by

$$\omega \wedge \omega(v, w) = 2\omega(v) \wedge \omega(w)$$

for $v, w \in T_u B$. The object $T(u)$ for any $u \in B$ will eat such a thing and give out a vector in V.

$$\begin{aligned} \mathcal{L}_{\vec{A}}\omega &= \left.\frac{d}{dt}\right|_{t=0} r^*_{e^{tA}}\omega \\ &= \left.\frac{d}{dt}\right|_{t=0} e^{-tA}\omega \\ &= -A\omega. \end{aligned}$$

By the Cartan family formula

$$\mathcal{L}_{\vec{A}}\omega = d\left(\vec{A} \lrcorner\, \omega\right) + \vec{A} \lrcorner\, d\omega$$

and since $\vec{A}$ is vertical, $\vec{A} \lrcorner\, \omega = 0$. This gives us

$$\vec{A} \lrcorner\, d\omega = -A\omega$$

or, if v, w are vectors tangent to B, and $v = \vec{A}$ is vertical, then

$$\begin{aligned} d\omega(v, w) &= -\gamma(v)\omega(w) \\ &= -\gamma \wedge \omega(v, w) \end{aligned}$$

(using the fact that v is vertical, so $\omega(v) = 0$). We have only to find a correction to this expression to make it work for arbitrary v and w: the torsion.

2.7 Developing and the geometric meaning of torsion

Surfaces with Riemannian metric provide some of the simplest examples of G-structures for which we have both rich geometric phenomena and intuition. Imagine rolling one surface on another, along a curve drawn on one of them. Their points of contact will roll out a curve on the other one. Call this process *developing along a curve.*

Take two G-structures, $B_0 \subset FM_0$ and $B_1 \subset FM_1$, whose soldering forms are ω_0 and ω_1 respectively, and suppose that they are equipped with pseudoconnections γ_0 and γ_1 respectively. Pick points $x_0 \in M_0$ and $x_1 \in M_1$. We can take any curve, say $C_0 \subset M_0$ passing through x_0, and take a coframe $u_j \in B_j$ for the tangent space at x_j. Now we form the parallel transport of C_0 through u_0: $\tilde{C}_0 \subset B_0$. On $\tilde{C}_0 \times B_1$, we solve the system of equations

$$\begin{aligned} \omega_1 - \omega_0 &= 0 \\ \gamma_1 &= 0 \end{aligned}$$

which will draw out a unique curve on $\tilde{C}_0 \times B_1$, projecting to a curve $\tilde{C}_1$ on B_1 through u_1, together with a map $\tilde{C}_0 \to \tilde{C}_1$. Call this the *development* of C_0 from u_0 to u_1.

To understand torsion, look at development of a G-structure B_0 on a manifold M_0 with a chosen pseudoconnection against the flat G-structure $G \times V \to V$. In the flat structure, we can draw a parallelogram, and develop along it.

Exercise 2.24 Take B a G-structure on a manifold M. The torsion of a pseudoconnection measures the failure at first order of the development of a small parallelogram from the standard flat G-structure V to close up in M.

The failure to close up in the bundle B is measured by the higher torsion (see section 7.1 on page 93). The straight lines of the standard flat G-structure (equipped with the standard flat connection) develop a family of curves in B called *geodesics*. The geodesics of a pseudoconnection are defined in the G-structure bundle, B, but their images on M are not in general determined by any system of differential equations. However, the geodesics of a connection always project to the solutions of a second order system of ordinary differential equations on the base M.

Exercise 2.25 Expressing a connection in local coordinates, show that its geodesics are precisely the solutions to a second order system of ordinary differential equations:

$$\frac{d^2x}{dt^2} + \Gamma(x)\frac{dx}{dt}\frac{dx}{dt} = 0.$$

From the same expression, show that there is no such second order system describing the geodesics of any pseudoconnection, unless the pseudoconnection is a connection.

3

Torsion controllable and uncontrollable

3.1 Controlling torsion

Suppose that γ' is another pseudoconnection on the same G-structure. How much can we affect the torsion by change of pseudoconnection? Suppose

$$d\omega = -\gamma' \wedge \omega + \frac{1}{2} T' \omega \wedge \omega$$

Then

$$(\gamma - \gamma') \wedge \omega = \frac{1}{2}(T - T')\omega \wedge \omega$$

But $\gamma - \gamma'$ is a multiple of ω because it vanishes on the vertical. Define

$$\delta : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 (V^*) \qquad \delta\eta(v, w) = \eta(v, w) - \eta(w, v)$$

Proposition 5. *There is some $Q : B \to \mathfrak{g} \otimes V^*$ so that*

$$T - T' = \delta Q$$

Proof. We know that $\gamma - \gamma'$ is a multiple of ω, say $Q\omega$, with $Q : B \to \mathfrak{g} \otimes V^*$. Thus $(\gamma - \gamma') \wedge \omega = (Q\omega) \wedge \omega$. But

$$\begin{aligned} (Q\omega) \wedge \omega(v, w) &= (Q\omega(v))\, \omega(w) - (Q\omega(w))\, \omega(v) \\ &= (\delta Q)\,(v, w). \end{aligned}$$

Exercise 3.1 Conversely, given any function $Q : B \to \mathfrak{g} \otimes V^*$, prove that $\gamma' = \gamma + Q\omega$ is a pseudoconnection for the G-structure B.

3.2 About the Lie algebra

Definition 10. *Let $\mathfrak{g}$ be a Lie algebra contained in $\mathfrak{gl}(V)$. Define $\mathfrak{g}^{(1)}$ to be the kernel of the map δ defined above:*

$$\delta : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 (V^*) \qquad \delta\eta(v,w) = \eta(v,w) - \eta(w,v)$$

Let $H^{0,2}(\mathfrak{g})$ be the cokernel of δ in $V \otimes \Lambda^2(V^*)$. We call $\mathfrak{g}^{(1)}$ the first prolongation *of $\mathfrak{g}$, and $H^{0,2}(\mathfrak{g})$ the $(0,2)$ Spencer cohomology group.(For an explanation of the Spencer cohomology groups, see the appendix A on page 283.)*

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 (V^*) \xrightarrow{[\,]} H^{0,2}(\mathfrak{g}) \longrightarrow 0$$

The space $\mathfrak{g}^{(1)}$ represents freedom to change pseudoconnection without affecting torsion, and $H^{0,2}(\mathfrak{g})$ represents the part of the torsion we can't change. Larger groups have larger prolongation and smaller Spencer cohomology.

Definition 11. *The* intrinsic torsion *of a G-structure $B \to M$, often [unambiguously!] referred to as the torsion of the G-structure, is the map*

$$[T] : B \to H^{0,2}(\mathfrak{g})$$

Proposition 6. *Consider that G acts on the right on the bundle B, and on the left on $H^{0,2}(\mathfrak{g})$ by the representation given by quotienting the representation of G on $V \otimes \Lambda^2(V^*)$:*

$$r_g t(u,v) = g t(g^{-1}u, g^{-1}v)$$

for $t \in V \otimes \Lambda^2(V^*)$. The intrinsic torsion is equivariant:

$$[T](r_g u) = g^{-1}[T](u)$$

Proof. Pick γ any pseudoconnection. Taking exterior derivative of both sides of $r_g^*\omega = g^{-1}\omega$, we calculate

$$0 = \left(\mathrm{Ad}_g^{-1}\gamma - r_g^*\gamma\right) \wedge \omega + \frac{1}{2}\left(r_g^*T - g^{-1}T\right) \wedge g^{-1}\omega.$$

Check that

$$0 = \vec{A} \lrcorner \left(\mathrm{Ad}_g^{-1}\gamma - r_g^*\gamma\right)$$

for $A \in \mathfrak{g}$. Therefore

$$\mathrm{Ad}_g^{-1}\gamma - r_g^*\gamma = Qg^{-1}\omega$$

for some $Q \in \mathfrak{g} \otimes V^*$. Therefore

$$r_g^*T = g^{-1}T + \delta Q.$$

Reduce to intrinsic torsion:

$$r_g^*[T] = g^{-1}[T].$$

Corollary 2. *The map (of sets)*

$$[T]/G : M \to H^{0,2}(\mathfrak{g})/G$$

is well defined. It is called the first order structure function*. For flat G-structures, the first order structure function vanishes. We will never employ the first order structure function, or refer to it again.*

Example 25. An e-structure has $\mathfrak{g} = 0$ so $H^{0,2}(\mathfrak{g}) = H^{0,2}(0) = V \otimes \Lambda^2(V^*)$. If the e-structure, thought of as a coframing, is $\eta : M \to FM$, then the reproducing property $\eta = \eta^*\omega$ implies that $d\eta = d\eta^*\omega$, so that $d\eta = \frac{1}{2}T\eta \wedge \eta$, with T our torsion.

Example 26 (Almost symplectic geometry). If $G = \mathrm{Sp}(2n, \mathbb{R})$, in its usual representation on $\mathbb{R}^{2n}$, then $H^{0,2}(\mathfrak{g}) \cong \Lambda^3(V^*)$ and $[T]$ is equal to the coefficients of $d\Omega$ in each given coframe, where Ω is the nondegenerate 2-form on M giving this structure. The prolongation is $\mathfrak{g}^{(1)} \cong \mathrm{Sym}^3(V^*)$. Recall that $\mathfrak{g} \cong \mathrm{Sym}^2(V^*)$. A $\mathrm{Sp}(2n, \mathbb{R})$-structure is called an *almost symplectic structure*. By the Darboux theorem (see [7]), every torsion-free $\mathrm{Sp}(2n, \mathbb{R})$-structure is flat.

Example 27 (The Levi-Civita connection). If $G = SO(p, q)$ in the usual $\mathbb{R}^{p+q}$ representation, then δ is an isomorphism, and there is a unique torsion-free pseudoconnection on B, the *Levi-Civita connection*. Note that $\mathfrak{g} \cong \Lambda^2(V^*)$. The intrinsic torsion always vanishes, because $H^{0,2}(\mathfrak{g}) = 0$, but nonetheless $SO(p, q)$-structures are not generally flat. We will soon look to second order to see the curvature. The Levi-Civita pseudoconnection is actually a connection, because its uniqueness forces it to be G-invariant.

Example 28 (Reductive groups with vanishing prolongation). If the group G is reductive (in the sense of the theory of Lie groups), then the G-equivariant map $[] : V \otimes \Lambda^2(V^*) \to H^{0,2}(\mathfrak{g})$ splits, giving a G-equivariant section $\sigma : H^{0,2}(\mathfrak{g}) \to V \otimes \Lambda^2(V^*)$. If it also turns out that $\mathfrak{g}^{(1)} = 0$ then we can specify a pseudoconnection γ by asking that the torsion of γ be $T = \sigma([T])$. This is a canonical choice of pseudoconnection, and therefore must be G-equivariant, i.e. a connection.

3.3 The torsion bundle

A useful view of torsion is via the *torsion bundle*

$$\mathrm{Tor}(B) := \left(B \times H^{0,2}(\mathfrak{g})\right)/G$$

which is a vector bundle over M. The intrinsic torsion of the G-structure $[T]$ can be interpreted as a section of this bundle.

Example 29 (Almost symplectic geometry). Returning to the case $G = \mathrm{Sp}(2n, \mathbb{R})$, the torsion bundle is canonically identified with 3-forms

$$\mathrm{Tor}(B) = \Omega^3(M)$$

and the torsion with $d\Omega$ where Ω is the 2-form on M giving (and given by) the G-structure.

Note, however, that the torsion bundle depends on the choice of G-structure B, i.e. there is no canonical identification of the torsion bundles of different G-structures on the same manifold. This is different from the torsion of a connection, which is always a section of

$$\Lambda^2(T^*M) \otimes TM.$$

The torsion bundle is canonically identified with a tensor bundle (independent of the choice of G structure, i.e. the torsion will be a tensor) precisely when there is an injective morphism of G representations

$$H^{0,2}(\mathfrak{g}) \to W$$

with W a $\mathrm{GL}(V)$ representation. Otherwise, the intrinsic torsion of different G-structures on the same manifold M live in different torsion bundles over M. A noninjective morphism will give rise to a tensor component of the torsion.

3.4 Pseudoconnections versus connections

The reason for our use of pseudoconnections rather than connections: if we take a linear section of the linear map

$$V \otimes \Lambda^2(V^*) \to H^{0,2}(\mathfrak{g}),$$

say $\sigma : H^{0,2}(\mathfrak{g}) \to V \otimes \Lambda^2(V^*)$, then we can define our pseudoconnection γ, up to choice of an element of $\mathfrak{g}^{(1)}$, at each point, by requiring that the torsion satisfy $T = \sigma([T])$, where $[T]$ is the intrinsic torsion, independent of γ. For example, if $\mathfrak{g}^{(1)} = 0$, that nails down a choice of pseudoconnection. But is it a connection? Only if it is G equivariant:

$$r_g^*\gamma = \mathrm{Ad}_g^{-1}\gamma.$$

But that can't happen unless our section σ is G invariant, i.e. a morphism of G representations. There may not exist a G invariant section; see section 4.3 on page 31 for an example. If the structure group G is reductive (for example semisimple), then all of its representations are sums of irreducible representations, and therefore there is always a G invariant section.

Exercise 3.2 Existence of a G invariant section $\sigma : V \otimes \Lambda^2(V^*) \to H^{0,2}(\mathfrak{g})$ occurs if and only if there is a connection γ with torsion T satisfying $T = \sigma([T])$, not just a pseudoconnection.

4

Four examples

4.1 Example: web geometry

Take a surface M with three foliations by curves, the curves from each foliation being nowhere tangent to those of either of the other foliations. Let $B \to M$ be the bundle of coframes $u : T_x M \to \mathbb{R}^2$ which identify the tangent lines of the leaves of the three foliations with three fixed lines drawn on $\mathbb{R}^2$, say the two axes and the diagonal. By some simple linear algebra, such coframes exist and are uniquely determined up to the action of the group G of linear transformations fixing those three lines in the plane. This group, the reader can easily calculate, consists just of the matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

where a is any nonzero real number. Conversely, any G-structure on a surface M will determine three foliations of M, with no curve from any of the three tangent to any curve from either of the other two. The structure equations with an arbitrary choice of pseudoconnection are

$$\begin{aligned} d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} &= - \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} t_1 \omega^1 \wedge \omega^2 \\ t_2 \omega^1 \wedge \omega^2 \end{pmatrix} \\ &= - \begin{pmatrix} \gamma + t_1 \omega^2 & 0 \\ 0 & \gamma - t_2 \omega^1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \\ &= - \begin{pmatrix} \gamma - t_2 \omega^1 + t_1 \omega^2 & 0 \\ 0 & \gamma - t_2 \omega^1 + t_1 \omega^2 \end{pmatrix}. \end{aligned}$$

Adding any multiples we like of ω^1, ω^2 to γ, it will still be a pseudoconnection, and we can arrange $t_1 = t_2 = 0$, and

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}.$$

Now there is no freedom in the choice of γ : our pseudoconnection is uniquely determined, because if two 1-forms existed satisfying this equation, say γ_1 and γ_2, the reader can check that their difference $\gamma_1 - \gamma_2$ would satisfy $(\gamma_1 - \gamma_2) \wedge \omega^j = 0$ for $j = 1, 2$, and so would vanish by Cartan's lemma.

This actually forces γ to be a connection, since the condition that determined γ, vanishing of torsion, is equivariant under the structure group G, so therefore γ is also invariant under the structure group (since $r_g^*\gamma$ satisfies the same equations that determined γ). The curvature κ of the connection is given by

$$d\gamma = \kappa\omega^1 \wedge \omega^2.$$

The function κ, analogous to Gauss curvature, is not actually defined on the surface M itself, but only up on B. However,

Exercise 4.1 $d\gamma$ is defined on M, although γ is not.

Being the curvature of the connection γ on the bundle B, the curvature form $d\gamma$ is a characteristic class,i.e. $\int_M d\gamma$ is a topological invariant, if M is compact.

4.1.1 The soldering form as chameleon

We can already see an important general idea emerging: the soldering 1-form ω is only defined on the bundle B of the G-structure. But it is semibasic, and so it "looks as if" it were a 1-form on the underlying manifold M. The reproducing property tells us that ω is something like a universal choice of 1-form on M adapted to the G-structure. Nonetheless, to be rigorous we must remember that ω is not actually defined on M at all.

4.1.2 Flat web geometries

Let us consider first what the web geometry looks like if the curvature vanishes. Then the structure equations are $d\omega = -\gamma \wedge \omega$ and $d\gamma = 0$. These same structure equations hold for any web geometry with vanishing curvature.

Exercise 4.2 Use the exponential map to prove that all web geometries with vanishing curvature are flat, locally equivalent to the web geometry obtained in the plane by taking 3 lines, no two being parallel, and foliating the plane thrice with their parallels.

Such webs exist on tori as well, by quotienting out any lattice.

Exercise 4.3 Show that tori are the only compact surfaces allowing a web geometry.

4.2 Example: conformal geometry

Let V be a finite dimensional real vector space and $\langle , \rangle$ a nondegenerate quadratic form of signature p, q. For $x \in V$, we write $x^* \in V^*$ for the element

$$x^*(y) = \langle x, y \rangle$$

and similarly for $\lambda \in V^*$ we write $\lambda^* \in V$ for the element

$$\langle \lambda^*, y \rangle = \lambda(y)$$

The conformal group of $\langle , \rangle$, written $CO\,(p, q)$, is the group of linear maps of V preserving the quadratic form up to a factor, i.e. the group of $g \in \mathrm{GL}\,(V)$ so that is some $a \in \mathbb{R}^{\times}$ with

$$\langle gx, gy \rangle = a \, \langle x, y \rangle$$

for any $x, y \in V$.

Exercise 4.4

$$a = \begin{cases} (\det g)^{2/n} & \text{if } n \text{ odd or } \langle , \rangle \text{ definite} \\ \pm (\det g)^{2/n} & \text{if } n \text{ even and } \langle , \rangle \text{ indefinite} \end{cases}$$

The Lie algebra of $CO\,(p, q)$ is $\mathfrak{co}\,(p, q)$, which is the Lie algebra of linear maps $A \in \mathfrak{gl}(V)$ so that

$$\langle Ax, y \rangle + \langle x, Ay \rangle = \frac{2}{n} \left(\operatorname{tr} A\right) \langle x, y \rangle$$

for any $x, y \in V$.

Proposition 7. *The prolongation of* $\mathfrak{co}\,(p, q)$ *is*

$$\mathfrak{co}\,(p, q)^{(1)} \cong V^*$$

where the identification is given by

$$\lambda \in V^* \mapsto \lambda' \in \mathfrak{co}\,(p, q)^{(1)} \subset V^* \otimes \mathfrak{co}\,(p, q)$$

where

$$\lambda'(x)y = \lambda(x)y + x\lambda(y) - \lambda^* \langle x, y \rangle$$

Proof. First, given a choice of λ we need the see that λ' belongs to the prolongation.

$$\delta\lambda'(x, y) = \lambda'(x)y - \lambda'(y)x$$

But $\lambda'(x)y$ is clearly symmetric in x, y, so $\delta\lambda' = 0$.

We need to see that every element α of the prolongation has the form $\alpha = \lambda'$ for some $\lambda \in V^*$. For α to belong to the prolongation,

$$0 = \delta\alpha(x, y) = \alpha(x)y - \alpha(y)x$$

Therefore

$$\langle \alpha(x)y, z \rangle = \langle \alpha(y)x, z \rangle \tag{4.1}$$

By definition of the Lie algebra $\mathfrak{co}\,(p, q)$, we also have

$$\langle \alpha(x)y, z \rangle + \langle \alpha(x)z, y \rangle = \frac{2}{n} \operatorname{tr} \alpha(x) \langle y, z \rangle$$

We can write this

$$\langle \alpha(x)y, z \rangle = \frac{2}{n} \operatorname{tr} \alpha(x) \langle y, z \rangle - \langle \alpha(x)z, y \rangle \tag{4.2}$$

We apply the two identities 4.1 and 4.2 one after the other to the expression $\langle \alpha(x)y, z \rangle$, and then repeat this two more times. The result is

$$\langle \alpha(x)y, z \rangle = \frac{1}{n} \left(\operatorname{tr} \alpha(x) \langle y, z \rangle + \operatorname{tr} \alpha(y) \langle x, z \rangle - \langle x, y \rangle \operatorname{tr} \alpha(z) \right)$$

or

$$\alpha(x) = \frac{1}{n} \left(\operatorname{tr} \alpha(x) I + x \otimes \operatorname{tr} \alpha(x) - (\operatorname{tr} \alpha)^* \otimes x^* \right)$$

so

$$\alpha = \frac{1}{n} (\operatorname{tr} \alpha)'$$

It remains to see that $\lambda \mapsto \lambda'$ is injective. If $\lambda' = 0$,

$$\lambda(x) \langle y, z \rangle + \lambda(y) \langle x, z \rangle = \lambda(z) \langle x, y \rangle$$

for any $x, y, z \in V$. Taking $x = y = z$ gives

$$2\lambda(x) \langle x, x \rangle = \lambda(x) \langle x, x \rangle$$

so that $\lambda(x) = 0$ or $\langle x, x \rangle = 0$. So λ vanishes on all nonnull vectors, and nonnull vectors span V.

Corollary 3.

$$H^{0,2}\left(\mathfrak{co}\,(p, q)\right) = 0$$

Proof. Suppose that $\dim V = n$. Then

$$\dim \mathfrak{co}\,(p, q) = \frac{n(n-1)}{2} + 1$$

and our exact sequence is

$$0 \to \mathfrak{co}\,(p, q)^{(1)} = V^* \to V^* \otimes \mathfrak{co}\,(p, q) \to \Lambda^2 (V^*) \otimes V \to H^{0,2}\left(\mathfrak{co}\,(p, q)\right) \to 0$$

We count dimensions:

$$\begin{aligned} \dim \mathfrak{co}\,(p,q)^{(1)} &= \dim V^* = n \\ \dim V^* \otimes \mathfrak{co}\,(p,q) &= n \left(\frac{n(n-1)}{2} + 1 \right) \\ \dim \Lambda^2\,(V^*) \otimes V &= \frac{n^2(n-1)}{2} \end{aligned}$$

and consequently

$$\dim H^{0,2}\,(\mathfrak{co}\,(p,q)) = 0$$

Corollary 4. *Every $CO\,(p,q)$-structure on a manifold admits a torsion free pseudoconnection, although it is never unique. Any two torsion free pseudoconnections differ by a map $B \to V^*$.*

See section 12.4 on page 259 for more on conformal connections.

4.3 Example: flag geometry

Consider a manifold equipped with a smoothly varying flag in its tangent spaces. This imposes a G-structure, where G is the group of invertible matrices of the form

$$\begin{pmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & * \end{pmatrix}$$

i.e. the Borel subgroup of $\mathrm{GL}\,(V)$. Let us ignore the G-structure for the moment, and learn more about the group. It is the group of linear maps fixing a flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

of subspaces, $\dim V_k = k$. The elements of the first prolongation $\mathfrak{g}^{(1)}$ are

$$\alpha = \left(\alpha^i_j \right)$$

1-forms valued in $\mathfrak{g}$. So each α^i_j is a 1-form, expressible in linear coordinates x^i on V by

$$\alpha^i_j = \alpha^i_{jk} dx^k$$

and to satisfy $\delta\alpha = 0$ it must satisfy

$$\alpha^i_{jk} = \alpha^i_{kj}.$$

To be valued in $\mathfrak{g}$, we need

$$\alpha^i_j = 0 \quad \text{for } i > j.$$

We see that α^i_{jk} is arbitrary for $i \le j \le k$, and hence

$$\begin{aligned} \dim \mathfrak{g}^{(1)} &= \dim \operatorname{Sym}^3(V) \\ &= \binom{n+2}{3} \end{aligned}$$

In particular, using $\alpha^i_{jk} - \alpha^i_{kj}$ terms to act on torsion, we can wipe out T^i_{jk} for $i \le j$. So we are left with T^i_{jk} terms with $i > j > k$. Writing out matrices makes it pretty easy to see how the torsion gets wiped out; consider the 3 dimensional case:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^3_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} T^1_{jk}\omega^j \wedge \omega^k \\ T^2_{jk}\omega^j \wedge \omega^k \\ T^3_{jk}\omega^j \wedge \omega^k \end{pmatrix}.$$

We can add any multiples of the $\omega^1, \omega^2, \omega^3$ to the γ^i_j, so we can wipe out the torsion from the first line, and the second line. On the third line there is no way to add anything to γ^3_3 which will affect the $\omega^1 \wedge \omega^2$ term in the torsion, since γ^3_3 only appears wedged with ω^3. So after absorbing:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^3_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ T^3_{12}\omega^1 \wedge \omega^2 \end{pmatrix}.$$

In fact, as G representations

$$\begin{aligned} \mathfrak{g}^{(1)} &= \bigoplus_{i=1}^{n} V_i \otimes \operatorname{Sym}^2(V/V_i)^* \\ H^{0,2}(\mathfrak{g}) &= \bigoplus_{i=2}^{n-1} (V/V_i) \otimes \Lambda^2(V_i^*) \end{aligned}$$

(It is easy to see that $\mathfrak{g}^{(1)}$ contains this space, and the equality follows by dimension count.) The weights of this representation are not those of any $\operatorname{GL}(V)$ representation, since the weights of a $\operatorname{GL}(V)$ representation are permutation invariant. Thus the torsion of such a G-structure is not a tensor, and the torsion bundle is not a tensor bundle.

Exercise 4.5 Calculate the weights.

Under the action of the structure group G the torsion transforms via:

$$r_g^* T^i_{jk} = \left(g^{-1}\right)^i_{i'} T^{i'}_{j'k'} g^{j'}_j g^{k'}_k .$$

In particular, in the 3 dimensional case, the T^3_{12} component gets added to the other $\omega^1 \wedge \omega^2$ components, with arbitrary multiples. Therefore there is no 1-dimensional linear subspace of T values which is G invariant and parameterized by T^3_{12}, no G invariant section of

$$V \otimes \Lambda^2 \left(V^*\right) \to H^{0,2} \left(\mathfrak{g}\right).$$

Consequently, if $T^3_{12} \neq 0$ then we are stuck using pseudoconnections.

4.4 Example: three line fields on a 3-manifold

Let

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \; abc \neq 0 \right\}.$$

A G-structure is thus a choice of 3 transverse line fields on a 3 manifold. Our structure equations with any choice of pseudoconnection γ are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & 0 & 0 \\ 0 & \gamma^2_2 & 0 \\ 0 & 0 & \gamma^3_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} T^1_{ij} \omega^i \wedge \omega^j \\ T^2_{ij} \omega^i \wedge \omega^j \\ T^3_{ij} \omega^i \wedge \omega^j \end{pmatrix}$$

It is clear that if I redefine the choice of γ^i_i by adding an arbitrary multiple of ω^i to it, then the equations remain valid, and γ remains a pseudoconnection. (Indeed the structure equations state that it is a pseudoconnection.) In this way, by adding (for example) $-T^1_{21}\omega^2$ to γ^1_1 we can arrange that $T^1_{21} = 0$ for some new pseudoconnection, γ'. Continuing in this way, we eventually get a pseudoconnection γ for which all of the torsion coefficients vanish except

$$T^1_{23}, T^2_{31}, T^3_{12}.$$

These constitute the intrinsic torsion. When an element of G acts on this intrinsic torsion,

$$T \mapsto g^{-1} T(g, g)$$

we find

$$\begin{pmatrix} T^1_{23} \omega^2 \wedge \omega^3 \\ T^2_{31} \omega^3 \wedge \omega^1 \\ T^3_{12} \omega^1 \wedge \omega^2 \end{pmatrix} \mapsto \begin{pmatrix} a^{-1} T^1_{23} bc \omega^2 \wedge \omega^3 \\ b^{-1} T^2_{31} ca \omega^3 \wedge \omega^1 \\ c^{-1} T^3_{12} ab \omega^1 \wedge \omega^2 \end{pmatrix}$$

where

$$g = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

We want to find the orbits of this group action. Note that we can not change the signs of products of any two of our torsion coefficients: the sign of $T^1_{23}T^2_{31}$ is invariant. Also, we can not make any torsion coefficient vanish by manipulating a, b, c. In case, for example, all of the torsion coefficients are positive, we can scale them to be all equal to 1. This reduces our group structure to a structure B_1 for the group

$$G_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a^2 = b^2 = c^2 = 1, abc = 1 \right\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The G_1-structure B_1 is the set of points of B at which the torsion coefficients are all equal to 1.

Part II

Recipes

5

Using torsion to reduce a G-structure

5.1 Reduction of G-structures

This will be the most difficult section of this book. We are going to use assumptions on the nature of the torsion to effect a reduction procedure, as in the last example.

Definition 12. *Let $H \subset G$ be a closed subgroup, and let $H^{0,2}(\mathfrak{g})_H$ be the elements of $H^{0,2}(\mathfrak{g})$ whose isotropy group in G is H. We say that a G-structure has* type H *if the intrinsic torsion satisfies*

$$[T](u) \in G \cdot H^{0,2}(\mathfrak{g})_H$$

for any $u \in B$, i.e. every $u \in B$ is mapped to a point of $H^{0,2}(\mathfrak{g})$ which has stabilizer subgroup conjugate to H.

Of course, H type is the same as K type if H and K are conjugate, so type depends only on conjugacy classes of closed subgroups of G. Not every G-structure need have a type: the stabilizers could be nonconjugate at various points of B. These are called *variable type* G-structures. A word on these a little later. Our reduction procedure will attempt (not always succeed) to cut an H subbundle out of a G-structure of type H.

If we have a two G equivariant maps $X \to Y$ and $Z \to Y$, we would like to lift to a map $Z \to X$, covering the given one, or perhaps a map from something transverse to the G orbits in Z to something transverse to those in X. This might then still be equivariant under a subgroup of G. We will ultimately apply this to the torsion: $Y = H^{0,2}(\mathfrak{g})$ and $X = \Lambda^2(V^*) \otimes V$ and $Z = B \to M$ a G-structure, so that for each value of intrinsic torsion in Y we will be looking for a value of extrinsic torsion in X which maps to it.

Definition 13. *Suppose that $\phi : X \to Y$ is a smooth G equivariant map of manifolds, for a Lie group G acting on the right on X and Y. Define a* section *of ϕ to be an immersed submanifold $\iota : S \hookrightarrow X$ so that*

1. *S is transverse to the stalks of ϕ, i.e.*

$$T_s S \cap \ker \phi'(x) = 0$$

and

2. *the tangent space of Y at each $y = \phi(s)$ for $s \in S$ is the direct sum of two transverse subspaces*

$$\phi'(s) \cdot T_s S \oplus T_y(yG) = T_y Y$$

(where yG is the G orbit through y) and

3. *for each $s \in S$, the stabilizer of $\phi(s) \in Y$ in G is the same subgroup $G_S \subset G$.*

See figure 5.1. Clearly

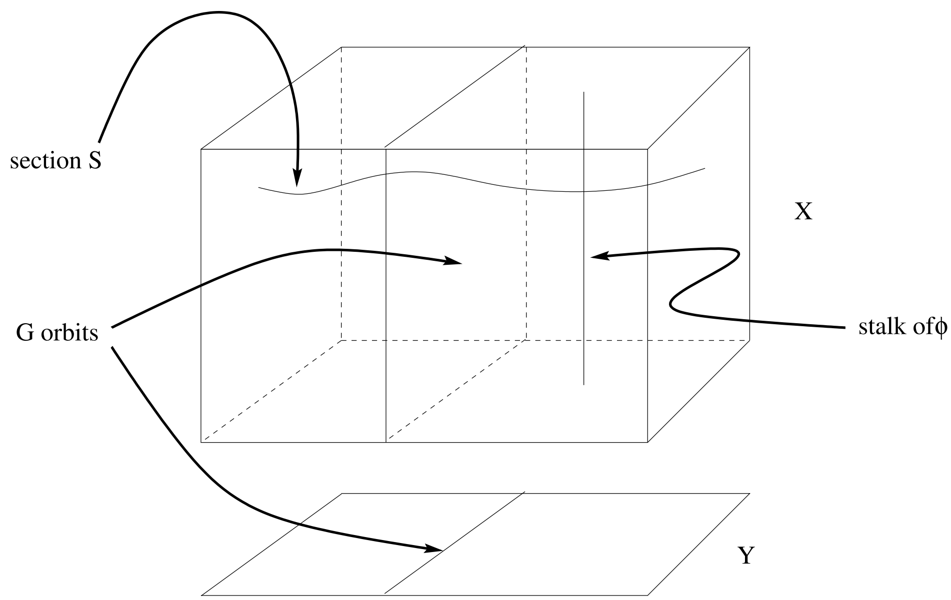


Fig. 5.1. A section of a G equivariant map

$$\dim S = \dim Y - \dim G + \dim G_S.$$

Given S, define an equivalence relation on S by setting two points $s_1, s_2 \in S$ equivalent if $\phi(s_1)$ and $\phi(s_2)$ lie in the same G orbit. The quotient by this equivalence relation

$$S \to S/\sim$$

is a surjective local diffeomorphism (not necessarily a covering map). Call S simple *if $S \subset X$ is embedded and $S/\sim = S$. Every section is locally simple. We* *call such a thing a section because it is essentially a multivalued local section of $X_{G_S}/N \to Y_{G_S}/N$, where Y_{G_S} is the set of points of Y whose G stabilizer is G_S, and $X_{G_S} = \phi^{-1} Y_{G_S}$, and N is the normalizer of G_S in G.*

Now fix a choice of group G, G equivariant map $\phi : X \to Y$, and section $S \subset X$. Take any right principal G bundle $B \to M$ with equivariant map $T : B \to Y$, so that for every $b \in B$ there is some $g \in G$ so that $T(bg) \in \phi(S)$.

Define the reduction of B by S to be the pushout

$$B_S = \{\, (b, s) \in B \times S \mid T(b) = \phi(s)\}$$

We get G_S to act on B_S by having it act on $B \times S$ acting trivially on S, and with the right action on B. Define the reduced map

$$\begin{aligned} T_S : B_S &\to S \\ (b, s) &\mapsto s. \end{aligned}$$

and the reduced base space to be

$$M_S = B_S / G_S \subset (B/G_S) \times S$$

with the obvious map $B_S \to M_S$. Note that the reduced map T_S is defined on M_S.

The idea is that the torsion T belongs to $\phi(S)$ precisely on a principal G_S-subbundle of B. But note that the reduced map is actually mapping to S. Locally, it looks like a map to X (globally, if S is embedded in X).

Proposition 8. *$B_S \to M_S$ is a principal G_S bundle on M_S, and $M_S \to M$ is a surjective local diffeomorphism. If $B \subset FM$ is a G-structure, then $B_S \subset FM_S$ is a G_S-structure.*

Proof. We write $\Delta \subset Y \times Y$ for the diagonal. Then $B_S = (T, \phi)^{-1} \Delta$. We have to check transversality, i.e. that

$$T_b B \oplus T_s S \xrightarrow{T'(b) \oplus \phi'(s)} T_y Y \oplus T_y Y \longrightarrow \frac{T_y Y \oplus T_y Y}{T_{(y,y)} \Delta}$$

is onto, where $y = T(b) = \phi(s)$. Obviously

$$\begin{aligned} \frac{T_y Y \times T_y Y}{T_{(y,y)} \Delta} &\cong T_y Y \\ (u, v) &\mapsto u - v \end{aligned}$$

so we have to show that

$$T'(b) - \phi'(s) : T_b B \oplus T_s S \to T_y Y$$

is onto. Since T is equivariant, $T'(b)$ is onto the tangent space of the G orbit through $y \in Y$. Also, $\phi'(s)$ is onto the transverse directions, by definition, so $T'(b) - \phi'(s)$ is onto $T_y Y$. Therefore $B_S \subset B \times S$ is a smooth embedded submanifold of dimension

$$\begin{aligned} \dim B_S &= \dim B + \dim S - \dim Y \\ &= \dim B + \dim G_S - \dim G. \end{aligned}$$

Clearly $B_S \subset B \times S$ is invariant under the action of G_S on B. If $v \in T_{(b,s)} B_S$ vanishes under the projection

$$T_{(b,s)} B_S \to T_b B \oplus T_s S \to T_b B \to T_m M$$

then write $v = \left(\dot{b}, \dot{s}\right)$, and we find

$$T'(b)\dot{b} = \phi'(s)\dot{s}$$

in order that v be tangent to B_S. But also we have $\dot{b}$ a vertical vector for $B \to M$, so that $T'(b)\dot{b}$ is tangent to the G orbit in Y. But $\phi'(s)\dot{s}$ can not be tangent to the G orbit unless $\dot{s} = 0$ and so $T'(b)\dot{b} = 0$. Therefore, $\dot{b}$ represents the action of an element of the Lie algebra of G_S, and we have

$$\ker\left(T_{(b,s)} B_S \to T_m M\right) = \left\{ (\vec{A}, 0) \,\middle|\, A \in \mathfrak{g}_S \right\}.$$

In particular, this ensures that $B_S \to M$ is a submersion, with the components of its fibers diffeomorphic to the maximal connected subgroup of G_S.

It is obvious that G_S acts freely on B_S, since $B_S \subset B \times S$ and G_S acts freely on B. We need to see that $B_S \to M_S$ is locally trivial. If we take an open subset $\Sigma \subset S$, we have $B_\Sigma \subset B_S$ an open subset. We can assume that $\Sigma \subset S$ is a simple section. We take the open subset $\Omega \subset M$ over which the torsion of B belongs to the G orbit of an element of $\phi(\Sigma)$. Thus $B_\Sigma \to \Omega$ is a submersion. If we have two points of B_Σ mapping to the same point of M, we can easily check that these two points differ by an element of G_S, i.e. $\Omega = B_\Sigma / G_S$. Consequently, our result is true for simple sections. If we take two simple sections, Σ_1, Σ_2 overlapping in a third simple section $\Sigma_3 = \Sigma_1 \cap \Sigma_2$, then it is easy to see that B_{Σ_1} and B_{Σ_2} glue together smoothly, forming an G_S bundle. This also shows that $M_S \to M$ is a surjective local diffeomorphism.

To see that B_S defines an G_S-structure on M_S, we need to show how to map it into the frame bundle FM_S. Using the map

$$B_S \subset B \times S \to B$$

we determine an element of $B \subset FM$ for each point of B_S. Using the surjective local diffeomorphism $M_S \to M$, and the map $B_S \to M_S$, this determines an element of FM_S. Clearly $B_S \to FM_S$ is an immersion, since $B_S \to B$ is an immersion. Also, since $B_S \to FM_S \subset FM \times S$ is an injection, we see that $B_S \to FM_S$ is injective.

Remark 2. Note the strange fact that B_S is not sitting inside B unless S is simple. Simple sections are the most common in practice.

Remark 3. To apply this process to a G-structure, we can take $X = V \otimes \Lambda^2(V^*)$ and $Y = H^{0,2}(\mathfrak{g})$. The process is *useless* unless the torsion of B sits inside S at least over some open subset of M. In general, we will want the torsion to belong to S up to G-action over every point of M.

Example 30. If $H^{0,2}(\mathfrak{g}) = 0$ then we can take $S = \{0\} \subset \Lambda^2(V^*) \otimes V$ as a section. For example, this happens with $G = SO(p,q)$ in its irreducible $V = \mathbb{R}^{p+q}$ representation.

Example 31. In our example of three line fields on a 3-manifold (see section 4.4 on page 33), we had structure group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, abc \neq 0 \right\}.$$

The intrinsic torsion had three components, acted on by the group according to

$$\begin{pmatrix} T^1_{23} \\ T^2_{31} \\ T^3_{12} \end{pmatrix} \mapsto \begin{pmatrix} a^{-1} T^1_{23} bc \\ b^{-1} T^2_{31} ca \\ c^{-1} T^3_{12} ab \end{pmatrix}.$$

The stabilizer of any torsion element will be

$$G_S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a^2 = b^2 = c^2 = 1, abc = 1 \right\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

as long as none of $T^1_{23}, T^2_{13}, T^3_{12}$ vanish. We find that the two points

$$\left(T^1_{23}, T^2_{13}, T^3_{12}\right) = (1,1,1) \text{ or } (1,1,-1)$$

(with all other $T^i_{jk} = 0$) form a section S.

Proposition 9. *If*

$$\begin{array}{ccc} B & \xrightarrow{F\phi} & B' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\phi} & M' \end{array}$$

is an equivalence of G-structures, then

$$F\phi^* [T'] = [T]$$

and consequently, the reduction commutes:

$$\begin{array}{ccc} B_S & \xrightarrow{F\phi} & B'_S \\ \pi \downarrow & & \downarrow \pi' \\ M_S & \xrightarrow{\phi} & M'_S \end{array}$$

i.e. equivalence obtains before reduction exactly if it obtains after reduction. (We are using the same section $S \subset V \otimes \Lambda^2 (V^*)$ of the map

$$[\,] : V \otimes \Lambda^2 (V^*) \to H^{0,2} (\mathfrak{g})$$

for both reductions.)

In the example of the 3 line fields, when all the torsion coefficients are positive, we can arrange that they become all equal to 1, and reduce to a G_S-structure, where G_S is finite. This says that we have specially distinguished coframes, up to 4 choices, and that these coframes are determined intrinsically by the choice of the three line fields. Note that we used a simple section in that example.

The two problems with reduction are (1) that we do not know how to write down sections, and (2) might not have torsion with conjugate stabilizer at every point, i.e. might not have constant type.

Remark 4 (Variable type). To study variable type structures, one can try to use a quotient of the torsion representation, to produce a quotient bundle, and take the image of the torsion in that bundle. Then one can impose constant type hypotheses on that object. Or one could even apply a G equivariant map $H^{0,2} (\mathfrak{g}) \to Y$ to any manifold Y with G action, and impose a constant type assumption on the image of the torsion inside Y. More generally, we could look only at the G-structures whose torsion belongs to some G-invariant subset of $H^{0,2} (\mathfrak{g})$, and define a map on that subset to some Y with a G action. This is the only way that one can reduce the structure group and thereby produce an invariantly defined reduction of each G-structure with given torsion behaviour.

Example 32 (Almost Kähler manifolds). An almost Kähler manifold, i.e. a $U(n)$-structure, might fail to have constant type, but the associated almost complex (i.e. $\mathrm{GL}(n, \mathbb{C})$) structure and the almost symplectic (i.e $\mathrm{Sp}(2n, \mathbb{R})$) structure might both have constant type; think of the maps

$$H^{0,2} (\mathfrak{u}(n)) \to H^{0,2} (\mathfrak{gl}(n, \mathbb{C}))$$
$$H^{0,2} (\mathfrak{u}(n)) \to H^{0,2} (\mathfrak{sp}(2n, \mathbb{R})) .$$

Remark 5. In carrying out reduction, it is essential to understand the geometric meaning of the constant type hypothesis one is making. Without knowing what our hypothesis means, we are probably unable to interpret the meaning of the resulting reduced structure, and we will be engulfed in a mess of structure equations.

Remark 6. Little is known about how to write down algebraic sections for various types. Perhaps in equivalence problems arising in algebraic geometry we might hope to understand sections of $X//N$ and $Y//N$ in geometric invariant theory.

5.2 Example: curved web geometries

Recall web geometries from section 4.1 on page 27. The action of the structure group is

$$r_g^* \omega^j = a^{-1} \omega^j$$
$$r_g^* \gamma = \gamma$$
$$r_g^* \kappa \omega^1 \wedge \omega^2 = \kappa \omega^1 \wedge \omega^2$$

for

$$g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

so that

$$r_g^* \kappa = a^2 \kappa.$$

Suppose that the curvature of our web geometry is nowhere zero. We can arrange $\kappa = \pm 1$ on a subbundle $B_1 \subset B$ with structure group $G_1 = \pm 1$. We now find that $\gamma = s_j \omega^j$ is semibasic, since the new structure group has trivial Lie algebra, and γ is always valued in the Lie algebra of the structure group. Our bundle $B_1 \to M$ is a 2-1 cover.

Proposition 10. *The symmetry group of a nowhere flat web geometry has dimension at most one, and consequently does not act transitively.*

Proof. If the symmetry group acts transitively on M, then it must have dimension at least two, and must act locally transitively on B_1. Later on (see section 8 on page 169), we will show that the symmetry group must embed into B_1, so that the symmetry group has at most two dimensions, and if it has two, then it acts locally transitively on B_1. So these s_j must be locally constant. But

$$d\gamma = ds_j \wedge \omega^j = \kappa \omega^1 \wedge \omega^2$$

so that the s_j cannot both be constant.

Proposition 11. *A compact surface with a web geometry must be a torus.*

Proof. Since G consists of oriented linear transformations, a G-structure imposes an orientation, so M is an oriented surface. Every G-structure B admits a reduction of structure group to any maximal compact subgroup of G, in this case $\pm 1 \subset G$; let $B_1 \subset B$ be any such reduction. The bundle $B_1 \to M$ is 2-1 cover, and B_1 has trivial tangent bundle (with global sections ω^j of the cotangent bundle), so B_1 is a torus, and therefore M is a torus.

Proposition 12. *On a compact surface, no web geometry can have nowhere vanishing curvature.*

Proof. On our 2-1 cover, we find

$$\begin{aligned} 0 &= \int_{B_1} d\gamma \\ &= 2 \int_M d\gamma \end{aligned}$$

since the 2-1 cover is orientation preserving. If there were negatively curved bits, they would have to be compensated for by positively curved bits.

The geometric meaning of web geometry curvature is not clear to the author. This is typical of the method of equivalence: it uncovers differential invariants, but does not always provide an interpretation of them.

5.2.1 Local coordinates

One can explicitly calculate the curvature of a given web geometry in local coordinates as follows: pick local coordinates x, y on the surface M, and suppose that the three foliations are each given by $dy = f_j \, dx$, where $j = 1, 2, 3$. Then the coframes that identify these with the usual foliations in $\mathbb{R}^2$ must look like

$$u = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}$$

where each η^j is a 1-form at a point of M. But then these 1-forms must look like

$$\eta^j = u^j \left(dy - f_j \, dx \right)$$

(no summation intended), to arrange that η^j lines up the tangent line to the j-th foliation with the appropriate line in $\mathbb{R}^2$, and to ensure that η^3 lines up the third foliation with the diagonal line, we need (as the reader can check)

$$\begin{aligned} u^3 &= u^2 - u^1 \\ u^2 &= u^1 \frac{f_3 - f_1}{f_3 - f_2}. \end{aligned}$$

The soldering 1-forms are

$$\omega^j = u^j \left(dy - f_j \, dx \right).$$

This is a general phenomenon: expressing the general coframe $u \in B$ in our bundle as a column of 1-forms η^j, we find that the soldering 1-forms are expressed by the same expressions, but now the parameters appearing (here

u^j) are local coordinate functions on the bundle B giving coordinates to the fibers of $B \to M$. So B has coordinates x, y, u^1,

We can calculate the 1-form γ on B by taking exterior derivatives of the soldering forms and applying Cartan's lemma. We find complicated expressions, along the lines of

$$\gamma = -\frac{du^1}{u^1} - \frac{\partial f_1}{\partial y}\, dx + h\,\omega^1$$

with h a very complicated function, which can be easily computed in a computer algebra program. Taking $d\gamma$, our computer algebra program spits out a huge mess giving an explicit expression for the curvature of the web.

Exercise 5.1 A pair of foliations by curves on a surface, with no curve from the first foliation tangent anywhere to any curve from the second, is always locally equivalent to the flat example: lines parallel to the x and y axes in the plane.

So lets suppose that we arrange the first and second of our foliations to have equations $f_1 = 1$ and $f_2 = -1$. Then we can calculate

$$\gamma = -\frac{du^1}{u^1} + \frac{\frac{\partial f_3}{\partial x} - \frac{\partial f_3}{\partial y}}{f_3^2 - 1}\,(dy - dx)$$

$$d\gamma = \frac{\left(f_3^2 - 1\right)\left(\frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} - 2f_3\left(\left(\frac{\partial f_3}{\partial x}\right)^2 - \left(\frac{\partial f_3}{\partial y}\right)^2\right)\right)}{\left(f_3^2 - 1\right)^2}\, dx \wedge dy.$$

Since the torsion vanishes, our results on osculation (see section 12.7 on page 264) show that one can always change coordinates to arrange that $f_3 = df_3 = 0$ at the origin, so that at the origin of coordinates, the curvature becomes

$$d\gamma = \left(\frac{\partial^2 f_3}{\partial y^2} - \frac{\partial^2 f_3}{\partial x^2}\right) dx \wedge dy$$

so that we can calculate curvature of some examples quite easily.

5.2.2 Web geometries with symmetry

Suppose that a web geometry has a one dimensional symmetry group whose generic orbits on M are curves. Replacing M by an open subset, we can suppose that all symmetry orbits are curves, and that the curvature vanishes nowhere. In tangent directions to these curves, we will have a relation between ω^1 and ω^2, since ω^1 and ω^2 are semibasic, say $\omega^2 = u\omega^1$ or $\omega^1 = v\omega^2$. If we swap the foliations, we can arrange that the relation is $\omega^2 = u\omega^1$, at least locally. This is a general technique in the method of equivalence: see exercise 2.16 on page 13. Check that this equation is invariant under structure group action, so that $u : M \to \mathbb{R}$ is defined on the underlying surface. All scalar invariants are constant along symmetry group orbits, and so

$$du = u_1 \left(\omega^2 - u\omega^1\right)$$

for some function $u_1 : B \to \mathbb{R}$. Taking exterior derivative,

$$du_1 - u_1\gamma + u_1^2\omega^1 = u_2 \left(\omega^2 - u\omega^1\right).$$

The orbits of the symmetry group in B are curves, which are permuted by the action of the structure group G, and any curves thereby permuted project to the same curve on M. Let X be the vector field on M generating the group action, and also write X for the associated vector field on B. By the Cartan formula,

$$0 = \mathcal{L}_X d\gamma = d\left(X \lrcorner\, d\gamma\right)$$

so that $X \lrcorner\, d\gamma$ is a closed 1-form, say dH for H a locally defined smooth function on the surface M, and

$$\mathcal{L}_X H = X \lrcorner\, dH = X \lrcorner\, X \lrcorner\, d\gamma = 0$$

so H is invariant under the flow of X. The differential is $dH = X \lrcorner\, d\gamma \neq 0$, since $d\gamma \neq 0$. So up on B, $dH = H_1\left(\omega^2 - u\omega^1\right)$ for nonvanishing function H_1 on B. Every scalar invariant on M must be a function of H, since H is a scalar invariant with nonvanishing differential. By similar calculation,

$$dH_1 + H_1\left(u_1\omega^1 - \gamma\right) = h\left(\omega^2 - u\omega^1\right).$$

So by moving up the fibers (i.e. in the γ direction), we can arrange $H_1 = \pm 1$. Changing sign of X if needed, we consider the subbundle B_1 of B on which $H_1 = 1$. There we have

$$\begin{aligned} dH &= \omega^2 - u\omega^1 \\ \gamma &= u_1\omega^1 - h\, dH. \end{aligned}$$

Note that we assumed dH was actually the differential of a function H, but we could really repeat the above steps without the hypothesis that such a function H exists; we only needed that the 1-form $X \lrcorner\, d\gamma$ was closed.

Fix a particular curve of the foliation $\omega^1 = 0$ and let t be the function defined (locally) by asking at each point how much time it takes for the given curve to pass through that point under the flow of X. Then clearly $\omega^1 = g\, dt$ for some function g on our bundle B, and $X \lrcorner\, dt = 1$. Calculate that in H, t coordinates,

$$\begin{aligned} X &= \frac{\partial}{\partial t} \\ h &= \frac{g'}{g} \\ g &= g(H) \\ u &= u(H) \end{aligned}$$

and with a bit more calculation, use X invariance of $\omega^1, \omega^2, \gamma$, and the structure equations to find

$$\begin{aligned} u &= -\int \frac{1}{g} \left(\int g\kappa \, dH \right) dH \\ \omega^1 &= g \, dt \\ \omega^2 &= dH + ug \, dt \\ \gamma &= u' g \, dt - \frac{g'}{g} dH, \end{aligned}$$

where the curvature is $d\gamma = \kappa(H)\omega^1 \wedge \omega^2$. The symmetry group is translation in t. Conversely, if we draw any functions $t = g(H), t = \kappa(H)$, these structure equations produce a web geometry with symmetry. The coordinate functions t, H are invariants defined up to adding constants. The function $g(H)$ is a differential invariant, up to translation in the H variable.

Exercise 5.2 What are the 3 foliations by curves?

The moral of the story: using the differential forms coming from the method of equivalence, we can sometimes construct (nearly) canonical coordinates, in which a wide variety of questions can be answered by direct calculation, a process called *integrating the structure equations*.

5.3 Example: Finsler surfaces

Let $\Sigma^3 \subset TM$ be a three manifold in the tangent bundle of a surface M, cutting across the fibers transversely (hence on curves in each tangent space). We call Σ a Finsler metric if these curves are strictly convex, closed, and symmetric about the origin of each tangent space. (Such Σ is the bundle of unit vectors of a smooth family of Banach space structures on the tangent spaces of M.) More generally, we call Σ a Finsler structure if it is locally a Finsler metric, i.e. we do not require the curves to be symmetric or to close up. This distinction will reappear soon. Let $\pi : TM \to M$ be the canonical bundle map. To each point $v \in \Sigma$ we can associate the line $\pi'(v)^{-1} v \subset T_v \Sigma$. Thus we have a choice of line in each tangent space of Σ.

Suppose that Σ is merely a 3 manifold on which we have a choice of line in each tangent space, never passing through 0. Consider the choices of coframes

$$\eta : T_v\Sigma \to \mathbb{R}^3$$

for which $\eta^1 = 1$ on the line, and the translate of the line passing through 0 is defined by

$$\eta^1 = \eta^2 = 0$$

Any two such coframes can be brought into correspondence by a linear map in the group

$$G = \left\{ \begin{pmatrix} 1 & a^1_2 & 0 \\ 0 & a^2_2 & 0 \\ a^3_1 & a^3_2 & a^3_3 \end{pmatrix} \middle| \, a^2_2 a^3_3 \neq 0 \right\}.$$

Our distinguished coframes form a G-structure $B \to \Sigma$. After getting rid of the inessential torsion, as in the previous example, we find that the structure equations are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} 0 & \gamma^1_2 & 0 \\ 0 & \gamma^2_2 & 0 \\ \gamma^3_1 & \gamma^3_2 & \gamma^3_3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} T^1_{31}\omega^3 \wedge \omega^1 \\ T^2_{31}\omega^3 \wedge \omega^1 \\ 0 \end{pmatrix}$$

(The reader will notice occasional factors of 2 or 1/2 being reabsorbed into the definitions of some functions.) We can write the essential torsion as the vector

$$[T] = \begin{pmatrix} T^1_{31} \\ T^2_{31} \end{pmatrix} \in \mathbb{R}^2 \cong H^{0,2}(\mathfrak{g})$$

The representation on torsion is

$$g = \begin{pmatrix} 1 & a^1_2 & 0 \\ 0 & a^2_2 & 0 \\ a^3_1 & a^3_2 & a^3_3 \end{pmatrix} \mapsto \frac{1}{a^3_3} \begin{pmatrix} 1 & a^1_2 \\ 0 & a^2_2 \end{pmatrix}$$

How do we figure this out? The straightforward, algebraic route uses the definitions given in section 3.2 on page 23. Another approach, which is easier in this case: we differentiate the structure equations. For example

$$d\omega^1 = -\gamma^1_2 \wedge \omega^2 + T^1_{31}\omega^3 \wedge \omega^1$$

implies (by taking d)

$$\begin{aligned} 0 &= \gamma^1_2 \wedge d\omega^2 + \dots \pmod{\omega^2} \\ &= \gamma^1_2 \wedge T^2_{31}\omega^3 \wedge \omega^1 + dT^1_{31} \wedge \omega^3 \wedge \omega^1 - T^1_{31}\gamma^3_3 \wedge \omega^3 \wedge \omega^1 \pmod{\omega^2} \end{aligned}$$

Therefore on vectors on which $\omega^1 = \omega^2 = \omega^3 = 0$ (i.e. vertical vectors)

$$dT^1_{31} = T^1_{31}\gamma^3_3 - T^2_{31}\gamma^1_2$$

Working out a similar expression for dT^2_{31} by differentiating the other structure equation, we find that on vertical vectors

$$d\begin{pmatrix} T^1_{31} \\ T^2_{31} \end{pmatrix} = \begin{pmatrix} \gamma^3_3 & -\gamma^1_2 \\ 0 & \gamma^3_3 - \gamma^2_2 \end{pmatrix} \begin{pmatrix} T^1_{31} \\ T^2_{31} \end{pmatrix} \tag{5.1}$$

or in other words

$$dT = -\rho(\gamma)T \mod \omega$$

where ρ is the representation of G on $H^{0,2}(\mathfrak{g})$. We see that the intrinsic torsion is right equivariant, because of the minus sign.

Intrepretations of the 3 types of torsion are indicated in table 5.1. How do I know this? I will show that Finsler structures must have $T^2_{31} \neq 0$, and leave the other cases to the reader.

Orbit	*Interpretation*
$T^1_{31} = T^2_{31} = 0$	Σ is a nonzero vector field
$T^2_{31} = 0, T^1_{31} \neq 0$	Σ is an affine line field
$T^2_{31} \neq 0$	Σ is a Finsler structure

Table 5.1. Interpretations of torsion

We take local coordinates on our surface M, say x, y. We can write local coordinates x, y, ξ, η on TM, defined by identifying a vector

$$v = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

with the point with coordinates

$$(x, y, \xi, \eta)\,.$$

To write down Σ, we need to sweep out a curve in each tangent space $T_m M$. For instance, we could use coordinates x, y, θ on Σ, where θ indicates the angle of a vector, so that the inclusion

$$\Sigma \subset TM$$

is written

$$\xi = \rho(x, y, \theta)\cos\theta$$
$$\eta = \rho(x, y, \theta)\sin\theta$$

The function $\rho(x, y, \theta)$ is a positive function; it describes the radius of the vector belonging to Σ which sits at point x, y and points in direction θ. At least locally on Σ such a thing will be unique. The line field on Σ as described above has as line at (x, y, θ)

$$\begin{aligned} dx &= \rho \cos\theta \\ dy &= \rho \sin\theta \end{aligned}$$

We can see that the coframing

$$\begin{aligned} \eta^1 &= \frac{1}{\rho} (\cos\theta \, dx + \sin\theta \, dy) \\ \eta^2 &= \frac{1}{\rho} (-\sin\theta \, dx + \cos\theta \, dy) \\ \eta^3 &= d\theta \end{aligned}$$

is a section of the bundle of coframes B, i.e. write $u \in B$ as

$$u = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} : T_x M \to \mathbb{R}^3.$$

But

$$d\eta^2 = \eta^1 \wedge \eta^3 \pmod{\eta^2}$$

Because this holds for one choice of section of the bundle B, and when we pull back the ω^i 1-forms on B by this section, we get the "reproducing property",

$$\eta^* \omega^i = \eta^i$$

where

$$\eta = \left(\eta^1, \eta^2, \eta^3\right) : M \to B$$

i.e. essentially the same equations hold, we must have $T^2_{31} \neq 0$, because $T^2_{31} = -1$ along the graph of this section η of B.

In the case of a Finsler structure, we can reduce to the bundle B_1 determined by

$$\begin{pmatrix} T^1_{31} \\ T^2_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The group G_1 is

$$G_1 = \left\{ \begin{pmatrix} 1 & a^1_2 & 0 \\ 0 & a^2_2 & 0 \\ a^3_1 & a^3_2 & a^2_2 \end{pmatrix} \middle| \, a^2_2 \neq 0 \right\}.$$

On B_1 equation 5.1 reduces to the equation on vertical vectors:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} -\gamma^3_3 & \gamma^1_2 \\ 0 & \gamma^2_2 - \gamma^3_3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

So $\gamma^1_2 = 0$ and $\gamma^2_2 = \gamma^3_3$, on vertical vectors, which fits in nicely with the group G_1, as it should, forcing γ to be valued in the Lie algebra of G_1, i.e. a G_1 pseudoconnection. By vertical here I mean tangent to the fibers of $B_1 \to \Sigma$. So this means that on arbitrary (not necessarily vertical) vectors,

$$\begin{aligned}\gamma^1_2 &= s_i \omega^i \\ \gamma^3_3 &= \gamma^2_2 + t_i \omega^i\end{aligned}$$

for some functions $s_i, t_i : B_1 \to \mathbb{R}$. Since γ^1_2 only ever appears wedged with ω^2, we can replace it with a different choice of γ^1_2 without loss of generality so that $s_2 = 0$. Similarly we can force $t_3 = 0$. We can rewrite our structure equations now with these results, on B_1:

$$\begin{aligned} d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} &= -\begin{pmatrix}0 & \gamma^1_2 & 0\\ 0 & \gamma^2_2 & 0\\ \gamma^3_1 & \gamma^3_2 & \gamma^3_3\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} + \begin{pmatrix}0\\ \omega^1\wedge\omega^3\\ 0\end{pmatrix} \\ &= -\begin{pmatrix}0 & 0 & 0\\ 0 & \gamma^2_2 & 0\\ \gamma^3_1 & \gamma^3_2 & \gamma^2_2\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} + \begin{pmatrix}s_1\omega^1\wedge\omega^2 + s_3\omega^3\wedge\omega^2\\ \omega^1\wedge\omega^3\\ 0\end{pmatrix}\end{aligned}$$

(We are a little dishonest here: the $t_i\omega^i \wedge \omega^2$ terms have been absorbed in to γ^3_i 1–forms, between the second to last and the last lines above.)

Again we can use any coframing η^1, η^2, η^3 of the sort we have written above in local coordinates, and check that this is a section of B_1, and that it has $s_3 \neq 0$, i.e.

$$d\eta^1 = \eta^3 \wedge \eta^2 \pmod{\eta^1}$$

so on the graph of η in B_1, we have $s_3 = 1$. From differentiating the structure equation for $d\omega^2$, and working modulo ω^1, we find

$$ds_3 = 2s_3\gamma^2_2 \pmod{\omega^1, \omega^2, \omega^3}$$

In other words, on the fibers of B_1, we find s_3 scaling as we travel in directions in which $\gamma^2_2 \neq 0$. We can therefore restrict to the subbundle $B_2 \subset B_1$ on which $s_3 = 1$. We find that on B_2,

$$d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} = -\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \gamma^3_1 & \gamma^3_2 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} + \begin{pmatrix}\omega^3\wedge\omega^2\\ \omega^1\wedge\omega^3 + u_1\omega^1\wedge\omega^2 + u_3\omega^3\wedge\omega^2\\ 0\end{pmatrix}$$

Differentiating these equations, we find that on vertical vectors

$$du_1 = -\gamma^3_2$$

so that we can arrange that $u_1 = 0$, say on a subbundle B_3. We find the structure equations on that bundle are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1^3 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} s_1 \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^2 \\ \omega^1 \wedge \omega^3 + u \omega^3 \wedge \omega^2 \\ v \omega^3 \wedge \omega^2 \end{pmatrix}$$

for some function v. (Again the poor reader has to keep pace with a little absorbing of torsion into the γ_j^i and a bit of renaming.) We find that on vertical vectors

$$ds_1 = \gamma_1^3$$

so we can find a subbundle B_4 on which $s_1 = 0$. On this subbundle, γ_1^3 is forced to vanish on the fibers, so we have completely eliminated the structure group down to the identity group. We write our structure equations again:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} \omega^3 \wedge \omega^2 \\ \omega^1 \wedge \omega^3 + u \omega^3 \wedge \omega^2 \\ v \omega^3 \wedge \omega^2 + w_2 \omega^2 \wedge \omega^1 + w_3 \omega^3 \wedge \omega^1 \end{pmatrix}$$

If we differentiate the first of these equations, we find that $w_3 = 0$. To preserve Cartan's notation, we rename

$$\begin{aligned} u &\text{ to } I \\ v &\text{ to } J \\ w_2 &\text{ to } K \end{aligned}$$

and have Cartan's structure equations of Finsler surfaces:

$$\begin{aligned} d\omega^1 &= \omega^3 \wedge \omega^2 \\ d\omega^2 &= \left(\omega^1 - I \omega^2\right) \wedge \omega^3 \\ d\omega^3 &= \omega^2 \wedge \left(K \omega^1 - J \omega^3\right) \end{aligned}$$

The reader can prove the following assertions as exercises. You don't need coordinates to do any of them; just use the structure equations.

Exercise 5.3 To every immersed curve on a Finsler surface can be associated its lift to Σ, a curve on which $\omega^2 = 0$ and $\omega^1 \neq 0$.

Exercise 5.4 The length of the curve is $\int \omega^1$ integrated over its lift.

Exercise 5.5 Taking the first variation, we find the geodesic equation

$$\omega^2 = \omega^3 = 0.$$

Exercise 5.6 The structure equation

$$d\omega^1 = \omega^3 \wedge \omega^2$$

ensures that closed geodesics belonging to a smooth connected family must all have the same length.

Exercise 5.7 The invariants I, J both vanish precisely when the Finsler metric is Riemannian.

Exercise 5.8 The "unit circles" in each tangent space of our surface are the curves in Σ on which

$$\omega^1 = \omega^2 = 0.$$

Exercise 5.9 The integral

$$\int \omega^3$$

over these circles gives them a notion of length, and that length is constant when $J = 0$, since

$$d\omega^3 = \omega^2 \wedge (K\omega^1 - J\omega^3)$$

Exercise 5.10 The invariant K is called the Gauss–Finsler curvature—it measures the infinitesimal focusing of geodesics; to see this consider a family of geodesics on a Finsler surface emanating from a point, and standard Sturm–Liouville theory.

Exercise 5.11 If K is constant, then from the structure equations we can calculate that

$$d \begin{pmatrix} I \\ J \end{pmatrix} = \begin{pmatrix} J \\ -KI \end{pmatrix} \omega^1 \quad \mod \omega^2, \omega^3$$

This implies that on the geodesics we have conservation of the quantity

$$KI^2 + J^2$$

and that along geodesics

$$\begin{aligned} \dot{I} &= J \\ \dot{J} &= -KI \end{aligned}$$

Exercise 5.12 So for example, if the Gauss–Finsler curvature is constant equal to -1, then

$$\begin{aligned} \dot{I} &= J \\ \dot{J} &= I \end{aligned}$$

so

$$I = c_1 e^t + c_2 e^{-t}$$

for some constants c_1, c_2, which implies that I is unbounded along complete geodesics, unless it vanishes everywhere.

Exercise 5.13 If I is bounded and $K = -1$ is constant and the Finsler surface is a complete metric space, then I vanishes everywhere. Similarly, J vanishes everywhere, so the Finsler surface is a Riemannian surface of constant negative curvature, locally isometric to the hyperbolic plane, a result of Akbar-Zadeh [3].

Exercise 5.14 Compact Finsler surfaces of constant negative Gauss–Finsler curvature $K \equiv -1$ are quotients of the hyperbolic plane. Remark: If $K > 0$ is a positive constant, and the Finsler surface is complete, then it is diffeomorphic to a sphere, but surprisingly there is an infinite dimensional family of such Finsler surfaces. This result is difficult to prove, see Bryant [17]. A related remark: There are also many Finsler surfaces with $K = 0$, as we will see in section 12.2 on page 252. If $K = 0$ and I is constant, then some cover of the surface is an open subset of a homogeneous Finsler structure (see section 8 on page 169).

Exercise 5.15 On Finsler surfaces with constant Gauss-Finsler curvature $K = 1$, if we set $\Omega^1 = \omega^2, \Omega^2 = \omega^3$, and $\Gamma = -\omega^1 + I\omega^2 + J\omega^3$, then $\Omega^1, \Omega^2, \Gamma$ satisfy the structure equations of a Riemannian metric. In particular, if the quotient of Σ by geodesic flow is a smooth surface, show that it has a Riemannian metric. Find the curvature of that metric.

Exercise 5.16 Finsler structures with constant $I \neq 0$ are never Finsler surfaces, since as we move up the fiber of $\Sigma \to M$ i.e. on a vector field X with $\omega^1(X) = \omega^2(X) = 0$ and $\omega^3(X) = 1$, we find

$$\mathfrak{L}_X \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -I \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

so the flow on TM is not periodic (since the trace of the matrix is $-I \neq 0$), and hence the fibers of $\Sigma \to M$ do not close up to form circles in the tangent spaces of M. Therefore, any complete Finsler surface of vanishing Gauss–Finsler curvature has nonconstant I or is flat.

Exercise 5.17 Surfaces with $J = 0$ have two constants of motion on geodesics, I and KI^2, so the geodesics can be calculated explicitly, unless

$$dI \wedge dK = 0.$$

Thus on Finsler surfaces with $J = 0$, geodesic flow is not ergodic.

Exercise 5.18 On surfaces with $J = 0$, an analogue of the Gauss–Bonnet theorem holds, since

$$d\omega^3 = K\omega^2 \wedge \omega^1.$$

5.4 The recursive nature of reduction

Looking back at the Finsler surface example, we see that we applied reduction repeatedly. In practice, it is clear what is happening, but the theory is confusing. We can take a sequence of nested closed subgroups

$$\cdots \subset G_{j+1} \subset G_j \subset \cdots \subset G_1 \subset G$$

and write their Lie algebras as

$$\cdots \subset \mathfrak{g}_{j+1} \subset \mathfrak{g}_j \subset \cdots \subset \mathfrak{g}_1 \subset \mathfrak{g}$$

and for $j > k$ write

$$T \mapsto (T)^j_k$$

for the quotient map

$$H^{0,2}(\mathfrak{g}_j) \to H^{0,2}(\mathfrak{g}_k).$$

Also, write

$$T \mapsto [T]_k$$

for the quotient map

$$V \otimes \Lambda^2(V^*) \to H^{0,2}(\mathfrak{g}_k).$$

Describe a sequence of spaces Y_j and sections by

$$Y_1 = H^{0,2}(\mathfrak{g})$$

and take $S_1 \subset V \otimes \Lambda^2(V^*)$ any G_1 section for $V \otimes \Lambda^2(V^*) \to Y_1$, and looking at the quotient map

$$()^2_1 : H^{0,2}(\mathfrak{g}_2) \to H^{0,2}(\mathfrak{g}_1)$$

apply the section to it:

$$Y_2 = H^{0,2}(\mathfrak{g}_2)_{S_1}$$

and then proceed recursively, so that $S_j \subset V \otimes \Lambda^2(V^*)$ is a G_j section for $V \otimes \Lambda^2(V^*) \to Y_j$ and

$$Y_{j+1} = H^{0,2}(\mathfrak{g}_{j+1})_{S_j}.$$

It is clear that in order that this process be defined, each group G_j must occur as the stabilizer in G_{j-1} of an element of Y_j. Therefore, it must be the intersection of G_{j-1} with an algebraic subgroup of $\mathrm{GL}(V)$ (i.e. defined by polynomial equations). At each step, we are making our groups G_j smaller by adding polynomial equations on them. Therefore we can guarantee that this process must terminate, by the Hilbert basis theorem, since eventually $G_j = G_{j+1}$ (i.e. we have added enough polynomials to generate the entire ideal). Keep in mind that reduction may fail at any stage, unless we can guarantee that the result of each reduction has constant type.

5.5 Reduction using a larger group

Suppose that the torsion representation $H^{0,2}(\mathfrak{g})$ of $G \subset \mathrm{GL}(V)$ is the restriction of a representation of a larger group H,

$$G \subset H \subset \mathrm{GL}(V).$$

For example, if $G \subset H$ is a normal subgroup, then H has a canonical representation on $H^{0,2}(\mathfrak{g})$, and on all of the Spencer cohomology of $\mathfrak{g}$, since adjoint H action is a representation on $\mathfrak{g}$.

Suppose that the torsion of a G-structure $B \subset FM$ is of constant H type, i.e.

$$[T] : B \to H^{0,2}(\mathfrak{g})$$

hits only the part of $H^{0,2}(\mathfrak{g})$ where the stabilizer in H is conjugate to some subgroup $H_1 \subset H$. Take a section $S \subset V \otimes \Lambda^2(V^*)$ for H_1, i.e. S is transverse to the fibers of

$$[\,] : V \otimes \Lambda^2(V^*) \to H^{0,2}(\mathfrak{g})$$

and locally its image in $H^{0,2}(\mathfrak{g})$ is transverse to the H orbits. We might not find any point of B where $[T]$ lies in the image of S in $H^{0,2}(\mathfrak{g})$, i.e. the pushout B_S might be empty. But every point of B can be multiplied by an element of H so that it lies in a point of S. This element of H is determined up to the action of H_1, and a sheet of S. Therefore we have a kind of reduction

$$B_S = \{(x, u, H_1 \cdot h, s) \mid (x, u) \in B,\ H_1 \cdot h \in H_1 \backslash H,\ s \in S,\ r_h [T](x, u) = s\}$$

and the obvious maps $B_S \to H_1 \backslash H$ and $B_S \to S$.

*Example 33 (*CR *geometry).* This is precisely the sort of reduction that Élie Cartan [22, 23] employed in his work on *CR* geometry (see section 7.4 on page 111 for more on the topic): he considered a 3-manifold sitting inside a complex surface. Each tangent space of the 3-manifold contains a unique complex line. Therefore the 3-manifold has a G-structure, where G is the group of real linear transformations of a 3-dimensional real vector space which leave invariant a real 2-plane, and act as complex linear transformations on that 2-plane. Call this a *CR geometry* (where C stands for Cauchy, and R for Riemann). We can let H be the group of linear transformations fixing the 2-plane, ignoring the complex structure on it. Then the H-structure is just a 2-plane field. If the H-structure has constant type, then it is either a contact structure (see section 7.9 on page 147) or a foliation by surfaces. In the first case, the G-structure is called *Levi-pseudoconvex* and in the second it is called *Levi-flat.* All Levi-flat *CR* geometries are locally isomorphic.

6

General features of G-structures

6.1 Coordinate expressions

6.1.1 Example: time dependent force on a line

Following Cartan [24] p. 129, consider a force field depending on time t, and the position x and velocity dx/dt of a particle on a line. The force gives a differential equation

$$\frac{d^2x}{dt^2} = F\left(t, x, \frac{dx}{dt}\right).$$

We want to consider equivalence of such equations, up to changing the x variable arbitrarily, and preserving the time variable up to translation. Consider the pair of 1-forms $dx - x'\,dt, dx' - F\,dt$ on the manifold $M = \mathbb{R}^3_{t,x,x'}$; they will determine the differential equation. On each tangent space of M, we want to preserve the differential dt, and also preserve dx up to multiples, so that we can make changes in the x variable independent of t. We also need to recall the differential equation, so we need the preserve $dx - x'\,dt, dx' - F\,dt$ up to linear combinations. But there is a unique multiple $a\,dx$ of dx for which $a\,dx - dt$ belongs to these linear combinations, $a = 1/x'$, so we can arrange on the set where $x' \neq 0$ to preserve both dt and dx/x', and to preserve the span of $dx - x'\,dt, dx' - F\,dt$. So we have differential forms

$$\begin{aligned}
\omega^1 &= dt \\
\omega^2 &= \frac{dx}{x'} \\
\omega^3 &= u\,(dx' - F\,dt) + v\left(\frac{dx}{x'} - dt\right)
\end{aligned}$$

defined up to choice of u and v, where u and v are arbitrary, with $u \neq 0$. This class of coframings on M is a G-structure, where G is the group of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v & v & u \end{pmatrix}.$$

The bundle B has explicit coordinates t, x, x', u, v, for which the above equations for ω^i give the soldering forms.

6.1.2 Torsion in coordinates

Suppose that $B \subset FM$ is a G-structure on a manifold M of dimension n. Let us work locally, taking M to be an open subset of a vector space V. On $FM = M \times \mathrm{GL}(V)$ we have coordinates x, h where h is an invertible $n \times n$ matrix. This associates to each point (x, h) the frame

$$\begin{aligned} T_x M &\to \mathbb{R}^n \\ a^i \frac{\partial}{\partial x^i} &\mapsto \left(h^1_j a^j, \dots, h^n_j a^j\right) \end{aligned}$$

Trivialize $B = M \times G$ and parameterize B as the image of the map

$$(x, u) \mapsto (x, u f(x)).$$

The form ω becomes

$$\omega = u f(x)\, dx$$

and if we write $w = u^{-1}$ and $G(x) = f(x)^{-1}$ then

$$d\omega^i = du^i_j w^j_k \wedge \omega^k + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k$$

where

$$T^i_{jk} = u^i_l \left(\frac{\partial f^l_m}{\partial x^p} - \frac{\partial f^l_p}{\partial x^m} \right) G^p_q w^q_j G^m_r w^r_k.$$

This provides an explicit expression for the torsion. In particular, at the points where $u = f = I$

$$T^i_{jk} = \left(\frac{\partial f^i_k}{\partial x^j} - \frac{\partial f^i_j}{\partial x^k} \right).$$

We can always pick the coordinates so that a given point of B is represented by $u = f(0) = I$, so this is a typical perspective on torsion: failure of commutation of first derivatives. The tangent space to B at this point is

$$T_{(0,I)} B = \left\{ \left(v, \frac{\partial f}{\partial x} v + A \right) \;\middle|\; v \in V, A \in \mathfrak{g} \right\}.$$

For the flat G-structure, the tangent space is

$$T_{(0,I)}B_{\text{flat}} = \{\,(v, A) \mid v \in V, A \in \mathfrak{g}\} = V \oplus \mathfrak{g}.$$

So our G-structure $B \subset FM$ is tangent to the flat G-structure exactly when $\frac{\partial f}{\partial x}$ belongs to $\mathfrak{g} \otimes V^*$, which is the case exactly when $[T]\,(0, I) = 0$, by definition of intrinsic torsion.

In the example of Finsler surfaces, we used this sort of presentation of a G-structure in local coordinates to examine its torsion, and justify our constant type hypotheses. Clearly one could employ local coordinate expressions like these throughout the computations of the equivalence method. The advantage is that one can explicitly calculate torsion, so all of the differential invariants uncovered by the equivalence method receive explicit calculable expressions. For example, in studying the equivalence of Riemannian metrics, one obtains explicit, but very long, coordinate expressions for the curvature tensor. In harder examples, like exceptional holonomy metrics [14], writing out the curvature tensor (or similar torsion invariants) is practically impossible.

Exercise 6.1 What is the torsion of a force field in one dimension (see subsection 6.1.1 on page 57), in local coordinates?

6.2 Fattening structure groups

6.2.1 Fattening principal bundles

If $G \subset H \subset \mathrm{GL}\,(V)$ are subgroups, and $B \subset FM$ is a G-structure, then we can fatten it up to a H-structure by taking

$$B(H) = (B \times H)\,/G$$

the quotient under the right G action

$$g \in G, (u, h) \in B \times H \mapsto \left(g^{-1}u, g^{-1}h\right) \in B \times H$$

and using the map

$$(u, h) \in B \times H \mapsto h^{-1}u \in FM$$

to map $B(H) \to FM$. We have an embedding

$$u \in B \mapsto (u, 1) \in B \times H \mapsto B(H).$$

This fattened up $B(H)$ is defined for any principal bundle, not only for B a G-structure; the definition of the map $B(H) \to FM$ is equally useful for G-structures which are not embedded.

Example 34.

$$B(\mathrm{GL}\,(V)) = FM.$$

Example 35. If $B_0 \subset B_1$ is a G-structure living inside a H-structure B_1, then

$$B_0(H) = B_1.$$

Lemma 3. *The map $B \times H \to M$ given by using the map $B \to M$ on the B factor, and forgetting the H factor, descends to a map $B(H) \to M$, which is a fiber bundle. The right action of H given by*

$$r_h\left(u_0, h_0\right) = \left(u_0, h_0 h\right).$$

commutes with the right G action above, so that it descends to $B(H)$. This turns $B(H) \to M$ into a principal right H bundle, and the map $B(H) \to FM$ makes $B(H)$ into an H-structure, which is embedded just when the original G-structure $B \to FM$ is.

Proof. Locally the results follow from carrying out the construction on M a vector space, and thereby on any of its open subsets; the global results are elementary.

Example 36 (e-structures). The most important example is the simplest: an e-structure $B = M \subset FM$ (where e means the group with one element, $e = \{1\}$). If we have a manifold M, an e-structure is a choice of coframe, at each point (up to e action, i.e. up to no action at all, an actual coframe), say ω. No danger of confusion in calling it ω, by the reproducing property For any morphism of Lie groups $G \subset \mathrm{GL}\left(V\right)$, we have a G-structure $B(G)$.

Note that we can carry out the fattening up construction even when the G-structure is not embedded; we only use the morphisms of groups $G \to H \to \mathrm{GL}\left(V\right)$.

6.2.2 The soldering form

The soldering form survives fattening up: we have started with a map $B \to FM$ and produced a map $B(H) \to FM$, and the soldering form is pulled back from FM. Alternately, we can define a 1-form ω_* on $B \times H$ by $\omega_{*(u,h)} = h^{-1}\omega$. Then calculate that under the right G action

$$r_g^* \omega_* = \omega_*$$

while under the right H action

$$r_h^* \omega_* = h^{-1} \omega_*,$$

and from these equations, and the vanishing of ω_* on the fibers of $B \times H \to B(H)$, we see that ω_* is pulled back from $B(H)$, where it is the soldering form. Calculate

$$d\omega_* = -\left(h^{-1}\, dh + \mathrm{Ad}_h^{-1} \gamma\right) \wedge \omega_* + \frac{1}{2}\rho(h)(T)\omega_* \wedge \omega_*,$$

where ρ is the representation of H on $\Lambda^2(V^*) \otimes V$, and $h^{-1}\,dh$ is the left invariant Maurer–Cartan 1-form on H, defined at a point $h \in H$ by

$$h^{-1}\,dh = \left(L_h^{-1}\right)'$$

where L_h means left multiplication by h.

6.2.3 The pseudoconnection

Pseudoconnections do not quite survive fattening up. We can take a pseudoconnection 1-form γ at a single point of B and push it forward to a pseudoconnection at the corresponding point of $B(H)$ by

$$\gamma_*(v) = \gamma(v)$$

if v is tangent to $B \subset B(H)$, and for $A \in \mathfrak{g}$ ask that

$$\gamma_*\left(\vec{A}\right) = A.$$

These two equations determine γ_* uniquely:

$$\gamma \in \Lambda^1(T_u B) \otimes \mathfrak{g} \mapsto \gamma_* \in \Lambda^1(T_u B(H)) \otimes \mathfrak{h}.$$

Notice that this only defines γ_* at points in the image of $B \to B(H)$, so it doesn't even define a pseudoconnection on all of $B(H)$.

6.2.4 Connections

Suppose that γ is a connection defined on all of B. Another approach to γ_*, more constructive, is to define on $B \times H$ the form

$$\gamma_* = h^{-1}\,dh + \mathrm{Ad}_h^{-1}\,\gamma \in \Omega^1(B \times H) \otimes \mathfrak{h}$$

where γ is the connection from B. Under the right H action

$$r_h^H(u_0, h_0) = (u_0, h_0 h)$$

we find that

$$r_h^{H*}\gamma_* = \mathrm{Ad}_h^{-1}\,\gamma_*.$$

Under the right G action

$$r_g^G(u_0, h_0) = \left(g^{-1}u_0, g^{-1}h_0\right)$$

we find

$$r_g^{G*}\gamma_* = \gamma_*,$$

and γ_* vanishes on the fibers of $B \times H \to B(H)$, so γ_* is defined on $B(H)$.

The problem for pseudoconnections γ is that this formula attempts to extend γ off of B to be H equivariant. If γ is not G invariant, then it can not be extended to be H invariant, and then there is no obvious way to extend it.

6.2.5 Torsion

An inclusion of Lie algebras gives maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}^{(1)} & \longrightarrow & \mathfrak{g} \otimes V^* & \xrightarrow{\delta} & V \otimes \Lambda^2 (V^*) & \xrightarrow{[]} & H^{0,2}(\mathfrak{g}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{h}^{(1)} & \longrightarrow & \mathfrak{h} \otimes V^* & \xrightarrow{\delta} & V \otimes \Lambda^2 (V^*) & \xrightarrow{[]} & H^{0,2}(\mathfrak{h}) & \longrightarrow & 0 \end{array}$$

providing a surjection on the $(0, 2)$ cohomologies

$$H^{0,2}(\mathfrak{g}) \to H^{0,2}(\mathfrak{h})$$

(smaller Lie algebra implies larger torsion, i.e. not as much room to absorb torsion) so that the torsion of any G-structure B determines the torsion of the H-structure $B(H)$.

Lemma 4. *The torsion T_* of $B(H)$ at a point $(u, 1) \in B$ is just the image of the torsion T of B under the map $H^{0,2}(\mathfrak{g}) \to H^{0,2}(\mathfrak{h})$.*

6.2.6 Reduction

If $S_H \to S_G \to V \otimes \Lambda^2 (V^*)$ are two immersions, and S_G is a section, and $S_H \to V \otimes \Lambda^2 (V^*)$ a section, then obviously reduction of constant type G-structures commutes with fattening up.

6.2.7 Prolongation

We will define prolongations and show that prolongations commute with fattening up in subsection 7.1.4 on page 95.

6.2.8 Generalizations

These constructions can be useful in more general circumstances. Imagine that $G \subset H$ but that H is not contained in $\mathrm{GL}(V)$. If we have W an H representation, and $V \subset W$ equivariant under G, then we can interpret ω_* as being W-valued, and the same equations above apply.

Example 37. Let $G = \mathrm{GL}(n, \mathbb{R}), V = \mathbb{R}^n, H = \mathrm{GL}(n, \mathbb{C}), W = \mathbb{C}^n$. Then a G-structure must just be $B = FM$ the frame bundle. Fattening up gives $B(H) = FM(\mathrm{GL}(n, \mathbb{C})) = F_{\mathbb{C}}M$ the complexified frame bundle, whose elements are choices of complex linear isomorphism $u : T_x M \otimes \mathbb{C} \to \mathbb{C}^n$. If we let $P_{\mathbb{C}}$ be the group of complex $n \times n$ matrices preserving a complex line, then $F_{\mathbb{C}}M/P_{\mathbb{C}} = \mathbb{P}_{\mathbb{C}}M$ is the projectivized complexified tangent bundle. Inside it is the real projectivized tangent bundle. The point of view here is that the structure equations on $F_{\mathbb{C}}M$ look like those on FM,

$$d\omega = -\gamma \wedge \omega$$

except that ω is now valued in $\mathbb{C}^n$, and γ in $n \times n$ complex matrices. Anything $P_{\mathbb{C}}$-equivariant will descend to $\mathbb{P}_{\mathbb{C}} M$. We could similarly complexify any G-structure: define $B_{\mathbb{C}}$ by complexifying Lie groups: $H = G_{\mathbb{C}}$. This is not going to complexify the underlying manifold of course. See LeBrun & Mason [57] for an application of this idea.

6.3 Flattening out along a curve

Since torsion is (roughly) a vector-valued 2-form, we expect to be unable to see it if we merely travel along a curve.

Definition 14. *Suppose that $B \subset FM$ is a G-structure, with $FM \to M$ the usual V-valued frame bundle, V a fixed vector space, and suppose that $\phi : C \to M$ is an immersed curve. The pullback:*

$$\begin{array}{ccc} \phi^* B & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & M \end{array}$$

is the set of triples (x, m, u) so that $m = \phi(x)$ and $u \in B$. At each point of the pullback, there is a linear map

$$u\phi' : T_x C \to V.$$

This determines a map

$$T\phi : \phi^* B \to \mathbb{P}(V)$$

by taking (x, m, u) to the line $u\phi'(T_x C) \subset V$, and identifying $\mathbb{P}(V)$ with the space of lines through 0 in V. Quotienting by the diagonal G action on $\phi^* V \times \mathbb{P}(V)$, we can identify this map $T\phi$ with a section of a bundle of projective spaces over C. This is just keeping track of the tangent line. If our curve is oriented, we could keep track of its oriented tangent line, so map to the sphere $V \backslash 0 / \mathbb{R}^+$ instead of $\mathbb{P}(V)$.

We will say that an immersed curve $\phi : C \to M$ is of constant type *if $T\phi : \phi^* C \to \mathbb{P}(V)$ is always in the same G-orbit in $\mathbb{P}(V)$, and this G-orbit will be said to be the type of the curve.*

Example 38. In the flat plane, with its e-structure (e the group with 1 element), constant type means precisely always pointing in the same direction, so a straight line.

Example 39. Every curve in a Riemannian manifold is of constant type. In a pseudo-Riemannian manifold, there are three types: positive definite, negative definite, and null.

Example 40. In a contact manifold (see section 7.9 on page 147), there are 2 types: lines tangent to the contact planes, and those transverse.

Lemma 5. *Let $B \subset FM$ be a G-structure. If a curve has constant type, then every point of the curve lies in a region $U \subset M$ of the manifold in which there is a flat structure $B_0 \subset FU$ which has the same fibers as B at all points of the curve in that region.*

Proof. In local coordinates, write $\omega = uf(x)\,dx$. By linear change of coordinates, we can arrange that $f(0) = 1$ at the origin, and linear change of coordinates in $\mathbb{R}^n$ to arrange that the type of our curve is the type of $\frac{\partial}{\partial x^1}$. This ensures that our curve is tangent to the x^1-axis, and we can then change coordinates so that C is the x^1-axis. The curve has constant type, which means that if $\ell = \operatorname{span}\left\langle \frac{\partial}{\partial x^1} \right\rangle$ is the direction of the curve, then $f(x)\ell \in G\ell \subset \mathbb{P}^{n-1}$ for every $x \in C$. The space $G\ell$ is clearly homogeneous, $G\ell = G/G_0$ where G_0 is the stabilizer of ℓ in G. Therefore we can choose a lift of the map

$$x \in C \mapsto f(x)\ell \in G/G_0$$

up to G, say $g(x)$, which then satisfies $g(x)\ell = f(x)\ell$. But the map f is determined only up to multiplication on the left by elements of G, so we can arrange $f(x)\ell = \ell$ all along the curve C. Therefore $f(x)$ must have first column $f_1^i(x) = \delta_1^i$.

Our problem is to provide a map $\phi(x)$ so that

$$\phi'(x) = f(x)$$

along C. If we can do this, then $\omega = uf(x)\,dx = u\,d\phi$ along C. But this is easy: take

$$\phi^i(x) = \sum_{j>1} x^j f_j^i\left(x^1\right) + \int_0^{x^1} f_1^i(s)\,ds.$$

We can see that this satisfies $\phi'(x) = f(x)$ along C, and also that

$$\phi\left(x^1, 0, \dots, 0\right) = \left(\phi^1, 0, \dots, 0\right)$$

preserves the curve C. After this diffeomorphism ϕ (which is a diffeomorphism near the origin), we have arranged that $\omega = u\,dx$ all along C.

Example 41 (Riemannian geometry). In a Riemannian manifold, every immersed curve has constant type, so we can construct a flat Riemannian metric near such a curve, at least locally, so that the flat metric and the given metric agree on all tangent vectors to the ambient manifold at each point of the curve. We can not do this globally if the curve is closed, unless it has trivial holonomy.

Doesn't development make this result obvious? If we develop, we get a map between curves, and match up frames above the curves. But then suppose that one curve is a geodesic, i.e. the tangent line is always identified with a fixed line in V by the frames. If the other curve is not a geodesic, then the frames cannot be matched up by any diffeomorphism which matches up the curves. Besides, development requires a choice of pseudoconnection.

Theorem 2. *Let $B \subset FM$ be a G-structure equipped with a connection γ. Note that this connection on B imposes a connection γ_* on all of FM, as in section 6.2 on page 59. If a curve has constant type, then every point of the curve lies in a region $U \subset M$ of the manifold in which there is a flat structure $B_0 \subset FU$ which has the same fibers as B at all points of the curve in that region, so that the flat connection γ_{0*} agrees with γ_* at all points of the frame bundle above the curve C.*

Proof. As in the previous result, we can take local coordinates so that the curve C is $x^2 = \cdots = x^n = 0$, and so that the bundle B is given by $U \times G$, mapped into the frame bundle by $(x, u) \in B \mapsto (x, uf(x)) \in FM$, with $f(x) = 1$ at all points of C. The connection γ can be written

$$\gamma = -du\, u^{-1} + \mathrm{Ad}_u\,(\Gamma(x)\, dx)$$

with $\Gamma : U \to \mathfrak{g}$. Gauge transformation: $(x, u) \mapsto (x, ug(x))$ on B affects γ by

$$\Gamma(x) \mapsto \Gamma(x) - dg\, g^{-1}.$$

Solving the required ordinary differential equation

$$\Gamma^i_{j1}(x) = \frac{\partial g^i_k}{\partial x^1} \left(g^{-1}\right)^k_j$$

on C brings us to $\Gamma^i_{j1} = 0$. Following this by the gauge transformation

$$g^i_j(x) = \delta^i_j + \sum_{J>1} x^J \Gamma^i_{jJ}$$

brings us to $\Gamma = 0$ along C. Since $f(x)$ is really only defined up to such transformations, we can still suppose that $f(x) = 1$ along C.

6.4 Example: principal bundles

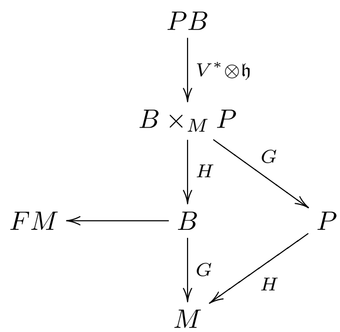
Suppose that B an G-structure on a manifold M, and that $P \to M$ is a right principal H bundle. Write the Lie algebra of G as $\mathfrak{g}$ and that of H as $\mathfrak{h}$. For any $A \in \mathfrak{h}$, write $\vec{A}$ for the vector field on P

$$\vec{A}(p) = \left.\frac{d}{dt} p e^{tA}\right|_{t=0}.$$

Let PB be the set of ordered tuples (x,u,p,w) so that $(x,u) \in B$, $p \in P_x$, $w : T_p P \to \mathfrak{h}$, and

$$\vec{A} \lrcorner w = A$$

for all $A \in \mathfrak{h}$.



Each arrow is labelled with a group, so that the arrow represents a principal bundle with that structure group. Since our commutative diagram of bundles contains only one map from any manifold to any other, we will label the maps using the notation $[X \to Y]$ for the map from X to Y.

The bundle $PB \to P$ is a right principal $G \times (V^* \otimes \mathfrak{h})$ bundle. We have actions on PB: an action of G

$$r_g(x,u,p,w) = \left(x, g^{-1}u, p, w\right)$$

and of H

$$r_h(x,u,p,w) = \left(x, u, pg, Ad_g^{-1} w \left(r_g^{-1}\right)'\right)$$

and of $V^* \otimes \mathfrak{h}$:

$$r_S(x,u,p,w) = \left(x, u, p, w - Su \left[P \to M\right]' (p)\right).$$

We define 1-forms on PB by

$$\begin{aligned} \omega =& u \left[PB \to M\right]' \\ \eta =& w \left[PB \to P\right]' \end{aligned}$$

These are the soldering forms of our structure. This ω is just the soldering form on B (or on FM) pulled back to PB. It is therefore invariant under the H action on PB, and the $V^* \otimes \mathfrak{h}$ action. In fact, it still satisfies the same structure equations, pulled back:

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega.$$

We easily uncover

$$
\begin{aligned}
r_g^* \omega &= g^{-1} \omega \\
r_h^* \omega &= \omega \\
r_S^* \omega &= \omega \\
r_g^* \eta &= \eta \\
r_h^* \eta &= \mathrm{Ad}_h^{-1} \eta \\
r_S^* \eta &= \eta - S\omega
\end{aligned}
$$

for

$$g \in G, h \in H, S \in V^* \otimes \mathfrak{h}.$$

We can differentiate these equations to find the Lie derivatives of the soldering forms under the flows of vertical vector fields. We find

$$
\begin{aligned}
\mathcal{L}_{\vec{A}} \omega =& - A\omega \\
\mathcal{L}_{\vec{B}} \omega =& 0 \\
\mathcal{L}_{\vec{S}} \omega =& 0 \\
\mathcal{L}_{\vec{A}} \eta =& 0 \\
\mathcal{L}_{\vec{B}} \eta =& - ad_h \eta \\
\mathcal{L}_{\vec{S}} \eta =& - S\omega
\end{aligned}
$$

for

$$A \in \mathfrak{g}, B \in \mathfrak{h}, S \in V^* \otimes \mathfrak{h}.$$

Then the left hooks:

$$
\begin{aligned}
\vec{A} \lrcorner \omega =& 0 \\
\vec{B} \lrcorner \omega =& 0 \\
\vec{S} \lrcorner \omega =& 0 \\
\vec{A} \lrcorner \eta =& A \\
\vec{B} \lrcorner \eta =& 0 \\
\vec{S} \lrcorner \eta =& 0
\end{aligned}
$$

This determines the structure equations:

$$
\begin{aligned}
d\omega + \gamma \wedge \omega &= \frac{1}{2} T\omega \wedge \omega \\
d\eta + \eta \wedge \eta &= \theta \wedge \eta
\end{aligned}
$$

where

$$\theta \in \Omega^1 (PB) \otimes V^* \otimes \mathfrak{h}.$$

Conversely, as we will see in section 6.7 on page 86, any system of 1-forms on a manifold X which satisfies these structure equations, with the η and ω

components being independent, must be locally the structure equations of a principal right H bundle over a manifold with G-structure: in other words, the manifold X is *locally* equipped with such a structure. It might not be a global bundle. Worse, we can only read off the connected component of the identity in each of the groups G and H.

The associated vector bundles are given by the representations of the group H. If W is such a representation, then

$$W_P = (P \times W)/G,$$

using the diagonal H action

$$(p, v)h = (ph, h^{-1}v),$$

is a vector bundle. A section of W_P corresponds to a map

$$f : P \to W$$

satisfying

$$f(ph) = h^{-1} f(p)$$

for all $h \in H$. Taking derivatives:

$$\vec{B} \lrcorner df(p) = -Bf(p)$$

by H invariance, for all $B \in \mathfrak{h}$. Therefore, if we pull f back to PB:

$$df = -\eta f + \omega \nabla f$$

where

$$\nabla f : PB \to V^* \otimes W$$

is the *covariant derivative*. We can take higher derivatives by differentiating further, but they are only defined on the prolongations of PB.[1]. For example,

$$d\nabla f + \nabla f \eta = f\theta + \nabla f \gamma + \nabla^2 f \omega$$

on $PB^{(1)}$.

A *linear first order operator* taking sections of W_P to those of U_P is a map

$$L : PB \to (W^* + W^* \otimes V) \otimes U$$

which is G and H and $V^* \otimes \mathfrak{h}$ equivariant.

---

[1] Prolongation will be defined in section 7.1 on page 93

6.5 The Sussmann orbit theorem

We will need to understand a little about Lie algebra actions.

Definition 15. *Write* $e^{tX}(m) \in M$ *for the flow of a vector field* X *through a point* m *after time* t*. Let* $\mathfrak{F}$ *be a family of smooth vector fields on a manifold* M*. The* orbit *of* $\mathfrak{F}$ *through a point* $m \in M$ *is the set of all points* $e^{t_1 X_1} e^{t_2 X_2} \dots e^{t_k X_k}(m)$ *for any vector fields* $X_j \in \mathfrak{F}$ *and numbers* t_j *(positive or negative) for which this is defined.*

Example 42. The vector field $\frac{\partial}{\partial \theta}$ on the Euclidean plane (in polar coordinates) has orbits the circles around the origin, and the origin itself.

Example 43. The set of smooth vector fields supported in a disk has as orbits the open disk (a 2-dimensional orbit) and the individual points outside or on the boundary of the disk (zero dimensional orbits).

Example 44. On Euclidean space, the set of vector fields supported inside a ball, together with the radial vector field coming from the center of the ball, forms a set of vector fields with a single orbit.

Theorem 3 (Sussmann [84]). *The orbit of any point under any family of smooth vector fields is an immersed submanifold (in a canonical topology to be defined in the proof). If two orbits intersect, then they are equal. Let* $\bar{\mathfrak{F}}$ *be the largest family of smooth vector fields which have the same orbits as the given family* $\mathfrak{F}$*. Then* $\bar{\mathfrak{F}}$ *is a Lie algebra of vector fields, and a module over the algebra of smooth functions.*

Remark 7. Obviously, one could localize these results, replacing globally defined vector fields with subsheaves of the sheaf of locally defined smooth vector fields.

Proof. We can replace $\mathfrak{F}$ by $\bar{\mathfrak{F}}$ without loss of generality, so we can assume that $\mathfrak{F} = \bar{\mathfrak{F}}$. Therefore, if $X, Y \in \mathfrak{F}$, we can suppose that $e^X_* Y \in \mathfrak{F}$ since the flow of $e^X_* Y$ is $e^{te^X_* Y} = e^X e^{tY}$, which must preserve orbits. We refer to this process as *pushing around* vector fields.

Fix attention on a specific orbit. For each point $m_0 \in M$, take as many vector fields as possible $X_1, \dots, X_k$, out of $\mathfrak{F}$, which are linearly independent at m. Refer to the number k of vector fields as the *orbit dimension*. Pushing around convinces us that the orbit dimension is a constant throughout the orbit. Refer to the map

$$(t_1, \dots, t_k) \in \text{open } \subset \mathbb{R}^k \mapsto e^{t_1 X_1} \dots e^{t_k X_k} m_0 \in M$$

(defined in any open set on which it is an embedding) as a *distinguished chart* and its image as a *distinguished set*. The tangent space to each point $e^{t_1 X_1} \dots e^{t_k X_k} m_0$ of a distinguished set is spanned by the linearly independent vector fields

$$X_1, e_*^{t_1 X_1} X_2, \dots, e^{t_1 X_1} \dots e_*^{t_{k-1} X_{k-1}} X_k,$$

which belong to $\mathfrak{F}$, since they are just pushed around copies of the X_j. Let Ω be a distinguished set. Suppose that $Y \in \mathfrak{F}$ is a vector field, which is not tangent to Ω. Then at some point of Ω, Y is not a multiple of those pushed around vector fields, so the orbit dimension must exceed k.

Therefore all vector fields in $\mathfrak{F}$ are tangent to all distinguished sets. So any point inside any distinguished set stays inside that set under the flow of any vector field in $\mathfrak{F}$, at least for a short time. So such a point must also stay inside the distinguished set under compositions of flows of the vector fields, at least for short time. Therefore a point belonging to two distinguished sets must remain in both of them under the flows that draw out either of them, at least for short times, and therefore belongs to another distinguished set lying inside both of them. Therefore the intersection of distinguished sets is distinguished.

We define an open set of an orbit to be any union of distinguished sets; so the orbit is locally homeomorphic to Euclidean space. Every open subset of M intersects every distinguished set in a distinguished set, so intersects every open set of the orbit in an open set of the orbit. Thus the inclusion mapping of the orbit into M is continuous. Since M is metrizable, the orbit is also metrizable, so a submanifold of M. The distinguished charts give the orbit a smooth structure. They are smoothly mapped into M, ensuring that the inclusion is a smooth map.

Example 45. Let $\alpha = dy - z\,dx$ in $\mathbb{R}^3$. The vector fields on which $\alpha = 0$ have one orbit: all of $\mathbb{R}^3$, since they include $\partial_z, \partial_x + z\partial_y$, and therefore include the bracket:

$$[\partial_z, \partial_x + z\partial_y] = \partial_y.$$

Exercise 6.2 Rolling a ball in the plane, you can acheive any desired rotation of the ball, by a sequence of moves, with each move rolling straight along the x direction, or straight along the y direction.

Definition 16. *Take a map $\phi : M_0 \to M_1$, and vector fields X_j on M_j, $j = 0, 1$. Write $\phi_* X_0 = X_1$ to mean that for all $m_0 \in M_0$, $\phi'(m_0) X_0(m_0) = X_1(\phi(m_0))$. For families of vector fields, write $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$ to mean that*

1. *for any $X_0 \in \mathfrak{F}_0$ there is an $X_1 \in \mathfrak{F}_1$ so that $\phi_* X_0 = X_1$ and*
2. *for any $X_1 \in \mathfrak{F}_1$ there is a vector field $X_0 \in \mathfrak{F}_0$ so that $\phi_* X_0 = X_1$.*

Example 46. The vector field ∂_x on $\mathbb{R}$ has $\mathbb{R}$ as orbit. Consider the inclusion $(0, 1) \subset \mathbb{R}$ of some open interval. The orbit of ∂_x on $(0, 1)$ is $(0, 1)$. The orbits are mapped to each other by the inclusion, but not surjectively.

Example 47. If $M_0 = \mathbb{R}^2_{x,y}$ and $M_1 = \mathbb{R}^1_x$, and $\phi(x, y) = x$, and $\mathfrak{F}_0 = \{\partial_x, \partial_y\}$ and $\mathfrak{F}_1 = \{\partial_x, 0\}$, then clearly $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$.

Theorem 4. *If $\mathfrak{F}_j$ are sets of vector fields on manifolds M_j, for $j = 0, 1$, and $\phi : M_0 \to M_1$ satisfies $\phi_* \mathfrak{F}_0 = \mathfrak{F}_1$, then ϕ takes $\mathfrak{F}_0$-orbits into $\mathfrak{F}_1$-orbits. On each orbit, ϕ has constant rank. If the vector fields in both families are complete, then ϕ is a fiber bundle mapping on each orbit.*

Proof. By restricting to an orbit in M_0, we may assume that there is only one orbit. The map ϕ is invariant under the flows of the vector fields, so must have constant rank.

Henceforth, suppose that the vector fields are complete. Given a path $e^{t_1 X_1} \dots e^{t_k X_k} m_0$ down in M_1, we can always lift it to one in M_0, so ϕ is onto.

It might not be true that $\phi_* \bar{\mathfrak{F}}_0 = \bar{\mathfrak{F}}_1$, but nonetheless we can still push around vector fields, because the pushing upstairs in M_0 corresponds to pushing downstairs in M_1. So without loss of generality, both $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are closed under "pushing around".

As in the above proof, for each point $m_1 \in M_1$, we can construct a distinguished chart

$$(t_1, \dots, t_k) \mapsto e^{t_1 X_1} \dots e^{t_k X_k} m_1.$$

These X_k are vector fields on M_1. Write Y_k for some vector fields on M_0 which satisfy $\phi_* Y_k = X_k$. Clearly ϕ is a surjective submersion. Let $U_1 \subset M_1$ be the associated distinguished set; on U_1these t_j are now coordinates. Let $U_0 = \phi^{-1} U_1 \subset M_0$. Let Z be the fiber of $\phi : M_0 \to M_1$ above the origin of the distinguished chart. Map

$$u_0 \in U_0 \mapsto (u_1, z) \, U_1 \times Y$$

by $u_1 = \phi(u_0)$ and

$$z = e^{-t_k Y_k} \dots e^{-t_1 Y_1} u_0.$$

Clearly this gives M_0 the local structure of a product. The transition maps have a similar form, composing various flows, so $M_0 \to M_1$ is a fiber bundle.

Example 48. Take any 2-plane field on $SO(3)$ transverse to the leaves of the Hopf fibration $SO(3) \to S^2$, and lift each vector field X_1 from S^2 to a vector field X_0 on $SO(3)$, by asking that X_0 be tangent to the 2-plane field. A two dimensional orbit would have to be diffeomorphic to S^2, since S^2 is simply connected. The Hopf fibration admits no section, so therefore all orbits must be three dimensional, hence open and disjoint, and cover $SO(3)$, which is connected. Hence every 2-plane field transverse to the Hopf fibration has all of $SO(3)$ as orbit, even though the 2-plane field may be holonomic on an open set. The same idea works for any nontrivial circle bundle over a compact manifold in any dimension: either the circle bundle becomes trivial on a covering space, or there is only one orbit of any plane field transverse to the fibers.

Definition 17. *A* right Lie algebra action *of a Lie algebra $\mathfrak{g}$ on a manifold M means a choice, for each Lie algebra element $A \in \mathfrak{g}$, of a vector field $\vec{A}$ on M so that*

$$\left[\vec{A}, \vec{B}\right] = \overrightarrow{[A, B]}.$$

Similarly, a left Lie algebra action *has*

$$\left[\overleftarrow{A}, \overleftarrow{B}\right] = -\overleftarrow{[A, B]}.$$

If G is a Lie group acting on a manifold, we define in the usual way the action of its Lie algebra. By the orbit *of a point under a Lie algebra action, we mean the points reachable from that point by riding the flows of the vector fields $\vec{A}_j$, one after the other, taking in total a finite number of rides (i.e. the orbit thinking of the Lie algebra as a family of vector fields). The number of these rides might depend on which point you are at, and where you want to go. The space of orbits is written $M/\mathfrak{g}$.*

Example 49. Write L_x for left multiplication $L_x y = xy$ and R_x for right multiplication $R_x y = yx$. A Lie group G has Lie algebra $\mathfrak{g}$ acting on it on the right via

$$A \in \mathfrak{g} \mapsto \vec{A}(g) = L'_g(1)A$$

and on the left via

$$A \in \mathfrak{g} \mapsto \overleftarrow{A}(g) = R'_g(1)A.$$

These have flows as vector fields:

$$e^{\vec{A}} g = g e^{A}$$

and

$$e^{\overleftarrow{A}} g = e^{A} g$$

(hence the arrows). The quotient in either case is $\mathfrak{g}\backslash G = G/\mathfrak{g} = G/G_0$, the space of path components of G.

Exercise 6.3 Prove that these are the correct flows.

Corollary 5. *Suppose that $\mathfrak{g}$ is a Lie algebra acting on a manifold M. Let $\mathfrak{g}(m) = \left\{\vec{A}(m) | A \in \mathfrak{g}\right\}$ The orbits $\mathfrak{g}m$ for $m \in M$ satisfy*

$$T_m \mathfrak{g} m = \mathfrak{g}(m).$$

Proof. Inclusion of these spaces into the tangent spaces is clear, since we can clearly move in these directions. It suffices to restrict attention to a single orbit. We can assume that no element of the Lie algebra acts trivially, by quotienting out those elements. The vector fields $\vec{A}$ vary in the adjoint representation under flows of one another, just as they would if we were on a Lie group, since the bracket relations are the same. Therefore the spaces $\mathfrak{g}(m)$ are invariant under the flows, and so these $\mathfrak{g}(m)$ have constant rank, and satisfy the conditions of the Frobenius theorem. The Frobenius theorem cuts up our manifold into invariant submanifolds, i.e. into orbits, unless $\mathfrak{g}(m) = T_m M$.

Definition 18. *An action of a Lie algebra* $\mathfrak{g}$ *on a manifold* M *is called* locally transitive *if for each point* $m \in M$

$$T_m M = \left\{ \vec{A}(m) | A \in \mathfrak{g} \right\}.$$

An action is called complete *if all of the vector fields* $\vec{A}$ *are complete, i.e. their flows are defined for all time.*

Corollary 6 (Palais [71]). *An action of a finite dimensional Lie algebra comes from the action of a Lie group just when it is complete, which occurs just when it is generated by a finite set of complete vector fields.*

Proof. Let $\mathfrak{g}$ be a Lie algebra with complete action on a manifold M. Take G to be any Lie group with Lie algebra $\mathfrak{g}$. (See any text on Lie groups, for example [69], p. 284, for proof that there is one.) On $G \times M$, we get $\mathfrak{g}$ to act via $\vec{A} = \vec{A}_G \oplus \vec{A}_M$. Calculating the bracket, we see that this is a Lie algebra action. Mapping $G \times M \to G$ by forgetting M, we apply the previous result to see that the orbits inside $G \times M$ are smooth manifolds.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the complete vector fields belonging to $\mathfrak{g}$. Clearly $\mathfrak{g}_0$ is invariant under rescaling. It is also closed, since a limit of complete vector fields will be complete. Under "pushing around", complete vector fields remain complete. Therefore $\mathfrak{g}_0$ is invariant under the adjoint representation of G, and consequently, under brackets with any elements of $\mathfrak{g}$. Since $\mathfrak{g}_0$ generates $\mathfrak{g}$, and is closed under brackets, it must span $\mathfrak{g}$. We want to show that $\mathfrak{g}_0$ is closed under addition, so that $\mathfrak{g}_0 = \mathfrak{g}$.

Take a basis of $\mathfrak{g}$, say $X_1, \dots, X_s \in \mathfrak{g}_0$. Then inside G we can write

$$e^{A_j X_j} = e^{a_1 X_1} \dots e^{a_s X_s},$$

for some functions a_j depending smoothly on A_j, defined for A_j small enough. Indeed we can determine the function $a(A)$ by the Campbell–Baker–Haussdorf formula explicitly; in particular $a(A) = A + \mathcal{O}\left(A^2\right)$. The flow on M of $e^{A_j X_j}$ is given by the same expression, because we have the same Campbell–Baker–Haussdorff formula. Therefore there is a neighborhood of 0 in $\mathfrak{g}$ so that each element $A_j X_j$ in that neighborhood has flow defined on M for at least one unit of time at every point. Composing these flows ensures completeness. Therefore $\mathfrak{g}_0 = \mathfrak{g}$ and all of the vector fields in the Lie algebra are complete.

The orbits inside $G \times M$ are therefore covering spaces of G. Replacing G by a covering Lie group if needed, we can arrange that some orbit is diffeomorphic to G. But G acts on $G \times M$, by acting on G on the left and leaving M alone, permuting orbits. Therefore all orbits are sent diffeomorphically to G by $G \times M \to G$. Define the group action of G by: we define $m_0 g = m_1$ when the orbit of $(1, m_0)$ contains (g, m_1). Clearly this is well defined and smooth, because the map from each orbit to G is a local diffeomorphism. It is easy to check that it is a group action.

Theorem 5 (Palais [71]). *Let H be a group of transformations of a manifold, and $\mathfrak{h}$ the set of vector fields X whose flows e^{tX} belong to the group, for all $t \in \mathbb{R}$. If $\mathfrak{h}$ generates a finite dimensional Lie algebra, then H is a finite dimensional Lie group and $\mathfrak{h}$ is the Lie algebra of H.*

Proof. There must be a Lie group H^0 generated by $\bar{\mathfrak{h}}$, by corollary 6 on the preceding page. This group must be a subgroup of H, since it is generated by the flows of the $\mathfrak{h}$ vector fields. The adjoint action of H on vector fields preserves completeness, so preserves $\mathfrak{h}$, so H^0 is a normal subgroup. Put a topology on H by declaring that open sets are the unions of sets of the form hU, where $h \in H$ and U open $\subset H^0$. On H^0, this agrees with the given topology, since the open sets of the form hU which strike H^0 have to have $h \in H^0$, so are open in H^0. Check that this topology is Hausdorff, because H^0 is. The quotient H/H^0 is discrete, so metrizable, and therefore H is metrizable.

Lets show that H has a countable basis of open sets. First, take any countable basis of open sets for M. Then for each continuous map $\phi : M \to M$, write down the infinite sequence of pairs (U, V) of basis elements, so that $\phi(U) \subset V$. To any finite sequence of pairs (U, V) of basis elements, we associate the set of maps ϕ whose sequence contains this finite sequence. This is a basis for the topology of uniform convergence. Replacing M by an appropriate jet bundle, we can find a countable basis for the topology of convergence uniformly with any number of derivatives, and putting these bases together we get a countable basis for the topology of smooth maps. Therefore H has a countable basis of open sets. The smooth structure on H^0 translates in H to give a smooth structure.

6.6 Tangential approximation and the exponential map

Let us imagine a G-structure $\pi : B \to M$ and pick a point $m \in M$. Write B_m for the fiber of B above m, let $M_{(m)} = T_m M$ be the tangent space, let $B_{(m)} = M_{(m)} \times B_m$. Then $\pi_{(m)} : (v, u) \in B_{(m)} \to v \in M_{(m)}$ is the obvious bundle map. The obvious right G action on $B_{(m)}$ is $r_g(w, u) = \left(w, g^{-1} u\right)$. Map $B_{(m)} \to FM_{(m)}$ by taking each (w, u) to $u : T_v M_{(m)} \cong T_m M \to V$. Clearly $B_{(m)}$ is equivalent (not canonically) to the standard flat G-structure over $M_{(m)}$. On $B_{(m)}$, we have the standard flat connection. Write the soldering and connection forms on $B_{(m)}$ as

$$
\begin{aligned}
\omega_{(m)} &= u\, dw \\
\gamma_{(m)} &= -du\, u^{-1}.
\end{aligned}
$$

The map $B_{(m)} \to M_{(m)}$ has distinguished fiber: over $0 \in M_{(m)} = T_m M$. Given a pseudoconnection γ on B, and any curve $\Gamma(t)$ in $B_{(m)}$ starting at a point of this fiber, there is a unique curve $C(t)$ in B so that

$$\Gamma(0) = (0, C(0))$$
$$\Gamma'(t) \lrcorner \left(\omega_{(m)}, \gamma_{(m)}\right) = C'(t) \lrcorner (\omega, \gamma)\,,$$

at least for short times. For any point $(w, u) \in B_{(m)}$ we start with the curve $\Gamma(t) = (tw, u)$ and we define the exponential map exp : open $\subset B_{(m)} \to B$ by

$$\exp(w, u) = C(1).$$

Clearly

$$\exp(0, u) = (0, u)$$

and the map exp extends to a smooth map $\exp : TM \times_B B \to B$.

Exercise 6.4 The exponential map is smooth on a neighborhood $U_{(m)}$ of $\pi_{(m)}^{-1}(0) \subset B_{(m)}$. It may help to define, for $v \in V$, the vector field $\vec{v}$ on B by $\vec{v} \lrcorner (\omega, \gamma) = (v, 0)\,.$

Proposition 13. *Suppose that*

$$\begin{array}{ccc} B_0 & \xrightarrow{\Phi} & B_1 \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{\phi} & M_1 \end{array}$$

is an equivalence of G-structures, preserving a pseudoconnection. Then the equivalence is completely determined by taking the exponential maps to identify with the flat G-structures in tangential approximations, and identifying the flat tangential approximations with an equivalence.

Corollary 7. *If M_0 is connected, then any equivalence $\phi : M_0 \to M_1$ preserving pseudoconnections is uniquely determined by its derivative $\phi'(m)$ at a single point.*

Proof. We can see immediately that the above expression determines ϕ near m, and we can take m to be any point of M_0. The exponential map will be defined near any point of B, and any point of B', in some open sets. These open sets will cover, and so we can take any compact subset of B and make a finite subcover. So if we had two such maps $\phi, \psi : M_0 \to M_1$ agreeing to first order at a point m, but disagreeing at some other point n, we could take any path from m to n, start at corresponding points of B and B', and take a path through B from our given point u over m to some point v over n covering the path down on M, and pick finitely many points on that path, so that we cover that path with finitely many open sets on which $\exp^{-1}$ is defined. Once we have determined $\phi = \psi$ on one of them, it follows on the next overlapping one, and so on.

Corollary 8. *On an n-dimensional manifold, every G-structure with pseudoconnection has at most a $n+d$-parameter group of pseudoconnection preserving symmetries, where $d = \dim G$.*

Definition 19. *A pseudoconnection for a G-structure $B \to M$ is called* complete *if the exponential map at each point m is defined everywhere on $B_{(m)}$.*

6.6.1 Examples of incomplete connections

Compact manifolds with pseudoconnections, or even with connections, even flat connections, may fail to be complete. A famous mathematician once remarked that, as a graduate student, he had proven a number of elegant results about compact 1-dimensional Riemannian manifolds. He boasted about them to his thesis adviser, who deflated him by explaining that the circle is the only compact 1-dimensional Riemannian manifold. Lets study the circle in depth.

Exercise 6.5 Prove that all connections on the frame bundle of a line or circle are locally identified by diffeomorphisms.

Exercise 6.6 On $M = \mathbb{R}$, take $\Gamma(x)$ any smooth function and let γ be the connection on FM given by

$$\gamma = -du\, u^{-1} + \Gamma(x)\, dx.$$

Define

$$K(x) = \int^x e^{\int^y \Gamma(z)\, dz} dy.$$

Check that on geodesics of γ,

$$\frac{d^2}{dt^2} K(x(t)) = 0.$$

Therefore on geodesics

$$K(x(t)) = at + b$$

for some constants a and b. Solve for the geodesics:

$$x(t) = K^{-1}\left(at + K\left(x\left(0\right)\right)\right).$$

If $\Gamma(x)$ has period 2π, then the connection γ is defined on the frame bundle of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For instance, taking $\Gamma = 0$ gives the usual connection: geodesic flow is linear. Taking $\Gamma = \Gamma_0$ constant gives geodesics

$$x(t) = \begin{cases} at + x_0, & \text{if } \Gamma_0 = 0 \\ \frac{1}{\Gamma_0} \log\left(e^{\Gamma_0 x_0} + at\right), & \text{otherwise.} \end{cases}$$

Note that the constant a is arbitrary. The geodesic is defined for times

$$\begin{cases} -\frac{e^{\Gamma_0 x_0}}{a} < t < \infty & \text{if } a > 0 \\ -\infty < t < \frac{e^{\Gamma_0 x_0}}{|a|} & \text{if } a < 0. \end{cases}$$

Exercise 6.7 Take $\Gamma(x)$ any smooth real-valued function of period 2π, expand in a Fourier series:

$$\Gamma(x) = \sum_k \Gamma_k e^{\sqrt{-1}kx}$$

and show that the time T_N take to travel the N-th trip around the circle satisfies

$$T_N = T_1 e^{2\pi\Gamma_0(N-1)}.$$

What happens if we change direction? Add up to see that the connection γ is complete just when $\Gamma_0 = 0$.

Exercise 6.8 Calculate the symmetry group and symmetry Lie algebra for $\Gamma(x) = \Gamma_0$ a constant. You should find that the symmetry Lie algebra consists in the vector fields

$$\left(a + be^{-\Gamma_0 x}\right) \partial_x.$$

Show that the locally defined symmetries act transitively on the frame bundle (which makes it more surprising that the connection is not complete), but that the globally defined symmetries are the translations, and do not act transitively on the frame bundle.

Exercise 6.9 Apply the formula

$$\Gamma^i_{jk} = \frac{1}{2}\left(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right) g^{kl}$$

for the Levi-Civita connection to a 1-dimensional manifold to obtain $g_{11} = e^{2\int \Gamma\, dx}$. Show that a connection on the frame bundle of the circle is complete just when it is the Levi-Civita connection of a metric on the circle, and the metric can be recovered from the connection up to a conformal factor. Show that up to equivalence, there is a unique complete connection on the frame bundle of the circle.

6.6.2 Connections and exponential map

Lemma 6. *A pseudoconnection γ is a connection precisely when*

$$r_g \exp = \exp r_g.$$

Proof. For $v \in V$ and $A \in \mathfrak{g}$, let $\vec{v}$ and $\vec{A}$ be the vector fields on B, defined by

$$\vec{v} \lrcorner \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix},$$
$$\vec{A} \lrcorner \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ A \end{pmatrix}.$$

Calculate

$$\begin{aligned} r'_g \cdot \left(\vec{v} + \vec{A}\right) \lrcorner (\omega, \gamma) &= \left(\vec{v} + \vec{A}\right) \lrcorner \left(r_g^* \omega, r_g^* \gamma\right) \\ &= \left(\vec{v} + \vec{A}\right) \lrcorner \left(g^{-1}\omega, \operatorname{Ad}_g^{-1} \gamma + \text{semibasic}\right) \\ &= \left(\overrightarrow{g^{-1}v} + \overrightarrow{\operatorname{Ad}_g^{-1} A}\right) \lrcorner (\omega, \gamma) \end{aligned}$$

for all v, A just when $r^*\gamma$ is a connection. So

$$r_{g*}\left(\vec{v} + \vec{A}\right) = \overrightarrow{g^{-1}v} + \overrightarrow{\operatorname{Ad}_g^{-1} A}$$

precisely for γ a connection. Henceforth suppose that γ is a connection. Thus the flow lines of $\vec{v} + \vec{A}$ are taken to those of $\overrightarrow{g^{-1}v} + \overrightarrow{\operatorname{Ad}_g^{-1} A}$ by G action on B. Therefore they have the same images down on M. They relate to the exponential map by

$$\exp(w, u) = e^{\overrightarrow{u(w)}}(u)$$

for $w \in T_m M$ and $u \in B_m$ (where e^X means time 1 flow of the vector field X). Under G action

$$\begin{aligned} r_g \exp(w, u) &= e^{r_g^* \overrightarrow{u(w)}} (r_g u) \\ &= e^{\overrightarrow{g^{-1} u(w)}} (r_g u) \\ &= \exp\left(\left(r_g u\right)^{-1} g^{-1} u(w), r_g u\right) \\ &= \exp\left(\left(g^{-1} u\right)^{-1} g^{-1} u(w), g^{-1} u\right) \\ &= \exp\left(u^{-1} g g^{-1} u(w), g^{-1} u\right) \\ &= \exp\left(w, g^{-1} u\right). \end{aligned}$$

6.6.3 Local coordinates

In local coordinates, identify M with an open subset of V, and write

$$\begin{aligned} \omega &= u\, dx \\ \gamma &= -du\, u^{-1} + \operatorname{Ad}_u \left(\Gamma(x, u)\, dx\right) \\ \omega_{(m)} &= \bar{u}\, d\bar{x} \\ \gamma_{(m)} &= -d\bar{u}\, \bar{u}^{-1} \end{aligned}$$

the curves in $B_{(m)}$ that give rise to the exponential are just straight lines $\varGamma(t) = (t\bar{x}, \bar{u})$, while in B their images satisfy

$$\check{\varGamma}'(t) \lrcorner (\omega, \gamma) = \varGamma'(t) \lrcorner \left(\omega_{(m)}, \gamma_{(m)}\right) = (\bar{u}\,(\bar{x}))\,.$$

Writing $\check{\varGamma}(t) = (x(t), u(t))$, we find

$$\begin{aligned}
&\check{\varGamma}(t) \lrcorner \omega = ux'(t) = \bar{u}\,(\bar{x}) \\
&\check{\varGamma}(t) \lrcorner \gamma = -u'(t)u(t)^{-1} + u(t)\,(\varGamma(x,u)x'(t))\,u(t)^{-1} = 0.
\end{aligned}$$

Solving, we find

$$\begin{aligned}
x'(t) &= u(t)^{-1}\bar{u}\,(\bar{x}) \\
u'(t) &= u(t)\,(\varGamma(x,u)x'(t))
\end{aligned}$$

a coupled system of ordinary differential equations. Taking second derivatives,

$$x''(t) = -\varGamma(x,u)x'x',$$

so that $\varGamma(x,u)$ is independent of u precisely when γ is a connection, and this happens precisely when $x(t)$ satisfies a second order ordinary differential equation on M.

6.6.3.1 Tangential approximation more generally

This idea of tangential approximation generalizes to allow approximation of any G-structure $B \to M$ by an arbitrary homogeneous G-structure $B' \to M'$ with pseudoconnection, as long as the homogeneous model has symmetry group acting transitively on fibers of the G-structure bundle B'. We are asking for a homogeneous bundle, but also existence of an invariant pseudoconnection. Those are strong assumptions. As we will see, for Riemannian geometry, these are the space forms, while for Kähler geometry, they are the complex space forms.

It is not clear if in any sense B "converges" to $B_{(m)}$ under some kind of rescaling.

Example 50. A Lie group H bears a canonical e-structure (where e means the group $e = \{1\}$) given as follows: at each point $h \in H$ we identify $T_h H \to \mathfrak{h} = T_1 H$ by left translation by h^{-1}:

$$\left(L_h^{-1}\right)'(h) : T_h H \to \mathfrak{h}.$$

The map exp is just the usual exponential map familiar from the theory of Lie groups.

6.6.4 The exponential map downstairs

Imagine a manifold M with connection γ for a G-structure B on M. The exponential map on the total space on B determines an exponential map downstairs as follows: pick any vector $v \in T_m M$, and any $u \in B_m$. We have seen that the curve $t \mapsto \exp(tw, u)$ is right invariant in the choice of u:

$$r_g \exp(tw, u) = \exp\left(tw, r_g u\right).$$

This shows that the image down on M is independent of u:

$$\pi\left(\exp(tw, u)\right) = \pi\left(\exp\left(tw, r_g u\right)\right),$$

determining a curve $\exp(tw) \in M$, the usual exponential map.

6.6.5 Complete connections

Definition 20. *Suppose that $B \to M$ is a G-structure with pseudoconnection γ. For any $v \in V$ and $A \in \mathfrak{g}$, define the vector fields $\vec{v}$ and $\vec{A}$ by*

$$\vec{v} \lrcorner \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$
$$\vec{A} \lrcorner \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

We will say that the pseudoconnection γ is complete *if these vector fields are complete. Note that the $\vec{A}$ generate the action of the structure group, so they are always complete, and it is only the $\vec{v}$ that we might have to think about. Another way to present the information: define the vector field E on $B \times V$ by*

$$E(u, v) = (\vec{v}, 0).$$

The flow of this vector field is called the geodesic flow. *Completeness of B is just completeness of the vector field E.*

Proposition 14. *A local diffeomorphism $\phi : M_0 \to M_1$ between manifolds, which matches up G-structures and complete pseudoconnections on those manifolds is a covering map.*

Proof. Suppose that the G-structures are $B_0 \to M_0$ and $B_1 \to M_1$. If $\phi\left(u_0\right) = u_1$, then the exponential maps match up at these points (this requires completeness). Taking a small open set Ω in the flat tangential approximation, we can make it small enough so that the exponential map will take it diffeomorphically into each of the bundles. So $B_0 \to B_1$ is a covering map, and therefore $M_0 \to M_1$ is a covering map.

6.6.6 Detecting homogeneity

Lemma 7. *Let $B \to M$ be a G-structure with pseudoconnection γ with only constants appearing in the structure equations. Then the vector fields $\vec{v}$ and $\vec{A}$ span a Lie algebra. The Lie algebra $\mathfrak{g}$ of G is a subalgebra, spanned by the $\vec{A}$ vector fields. If we write*

$$d \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = -C \begin{pmatrix} \omega \\ \gamma \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \gamma \end{pmatrix}$$

then C is the matrix of structure constants of the Lie algebra $\mathfrak{h}$.

Proof. Let $\Omega = (\omega, \gamma)$. Apply the Cartan equation repeatedly; for example:

$$(\vec{v}_0, \vec{v}_1) \lrcorner d\Omega = \mathfrak{L}_{\vec{v}_0} (\vec{v}_1 \lrcorner \Omega) - \mathfrak{L}_{\vec{v}_1} (\vec{v}_0 \lrcorner \Omega) - [\vec{v}_0, \vec{v}_1] \lrcorner \Omega.$$

Simplifying both sides gives

$$[\vec{v}_0, \vec{v}_1] \lrcorner \Omega$$

a constant, so $[\vec{v}_0, \vec{v}_1]$ must be a constant multiple of $\vec{v}$ and $\vec{A}$ vector fields.

Example 51. The structure equations of the usual round metric on the 2-sphere are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, d\gamma = \omega^1 \wedge \omega^2.$$

The Lie algebra is $\mathfrak{so}(3)$.

Proposition 15. *Let $B \to M$ be a G-structure on a path connected manifold with complete pseudoconnection γ, and only constants in the structure equations. Let $\mathfrak{h}$ be the Lie algebra of infinitesimal symmetries. Then $B = \Gamma \backslash H \to M = \Gamma \backslash H/G$ for some discrete subgroup $\Gamma \subset H$ of a Lie group H containing G, with Lie algebra $\mathfrak{h}$, $\Gamma \cap G = 1$, H/G path connected and Γ acting on H/G as the deck transformations of the normal covering map $H/G \to M$. Moreover (ω, γ) pulls back to H under $H \to \Gamma \backslash H$ to be the left invariant Maurer–Cartan form on H. In particular, if M is simply connected, then $\Gamma = 1, B = H$, and $M = H/G$.*

Proof. Consider the Lie algebra $\mathfrak{h}$ spanned by the vector fields of the form $\vec{v}$ and $\vec{A}$. By Palais' theorem 6 on page 73, there is a connected Lie group H_0 which induces this action, which we take to be a right action. As vector spaces, the Lie algebra of H_0 is $\mathfrak{h} = V \oplus \mathfrak{g}$. We can assume that H_0 acts faithfully, by quotienting out any elements that act trivially. The vector fields $\vec{A}$ come from the action of the structure group G. Thus the identity component of G is contained in H_0. Let H be the group of diffeomorphisms of B which are generated by H_0 and G. As a set,

$$H = \bigsqcup_{g \in G/G \cap H_0} H_0 g,$$

which is a countable disjoint union of manifolds, and is a group because H_0 is connected, and

$$g e^{tA} = e^{t \operatorname{Ad}_g A} g$$

for $g \in G$ and $A \in \mathfrak{h}$. Clearly the group operations are smooth, so H is a Lie group with Lie algebra $\mathfrak{h}$, acting on the right on B. Pick a point $u \in B$ and map $h \in H \mapsto uh \in B$. The left invariant vector fields on H are taken to the vector fields generating the Lie algebra action, so to constant multiples of the $\vec{v}$ and $\vec{A}$ vector fields. Therefore (ω, γ) pulls back to the Maurer–Cartan form, just by precisely the identification $\mathfrak{h} = V \oplus \mathfrak{g}$. The map $H \to B$ is a local diffeomorphism, since it matches up these coframings. Clearly H acts locally transitively on B. Since $M = B/G$ is a connected manifold, it is path connected, so B/H is also path connected. Therefore H acts transitively on B, and letting Γ be the stabilizer of a point $u \in B$, we have $B = \Gamma \backslash H$ and $M = \Gamma \backslash H/G$. Since G is the structure group of $B \to M$, no element of G can have a fixed point on B, so $G \cap \Gamma = 1$. Therefore $H/G \to M = \Gamma \backslash H/G$ is a covering map with Γ as the group of deck transformations.

By definition of H, $\pi_0(G)$ acts transitively on $\pi_0(H)$, so $\pi_0(H/G) = 1$. Since $\pi_0(M) = 1$, the covering map $H/G \to M = \Gamma \backslash H/G$ is a normal covering.

Corollary 9. *Any two complete G-structures $B_j \to M_j$ with pseudoconnections γ_j, $j = 0, 1$, with the same structure groups and only constants appearing in the structure equations, and all the constants matching up, have covering maps*

$$\begin{array}{ccccc} B_0 & \longleftarrow & B & \longrightarrow & B_1 \\ \downarrow & & \downarrow & & \downarrow \\ M_0 & \longleftarrow & M & \longrightarrow & M_1 \end{array}$$

a common covering G-structure. In particular, if the M_j are simply connected, then there is an equivalence preserving the pseudoconnections.

Lemma 8. *Let $B \to M$ be a G-structure with connection γ with only constants appearing in the structure equations. Let $\mathfrak{h}$ be the Lie algebra spanned by the $\vec{v}$ and $\vec{A}$ vector fields, $v \in V$ and $A \in \mathfrak{g}$. Then M is covered by open sets so that on each open set, the Lie algebra of infinitesimal connection-preserving symmetriesof the G-structure is isomorphic to $\mathfrak{h}$.*

Proof. Let $\mathfrak{h} = V \oplus \mathfrak{g}$ be that Lie algebra. Clearly $\mathfrak{g}$ is a subalgebra. Let H_0 be any Lie group with Lie algebra $\mathfrak{h}$. Let Ω_0 be the left invariant Maurer–Cartan form on H_0. Inside $B \times H_0$, the equation $\Omega = \Omega_0$ is holonomic, so by the Frobenius theorem, $B \times H_0$ is foliated by submanifolds on which $\Omega = \Omega_0$. Clearly each one is locally the graph of a diffeomorphism from an open subset of B to an open subset of H_0, identifying Ω and Ω_0. Moreover, such diffeomorphisms are unique up to left H_0 action.

The left invariant vector fields on H_0 are precisely those which satisfy $\vec{A} \lrcorner \Omega_0 = A$, while the right invariant ones are precisely those which satisfy

$$\mathcal{L}_{\overleftarrow{A}} \Omega_0 = 0.$$

These right invariant vector fields locally map via the local diffeomorphisms $B \to H_0$ to an isomorphic Lie algebra of vector fields satisfying the same equation, but only on various open sets in B. To see that they do map, note that the local diffeomorphisms are determined up to left action of H_0, and under left action, right invariant vector fields are taken to other right invariant vector fields. Back on H_0, right invariant vector fields commute with the $\vec{A}$ left invariant vector fields for $A \in \mathfrak{h}$. Moreover, they form a basis for each tangent space. Therefore on some open subsets covering B, there are vector fields defined, satisfying $\mathcal{L}\Omega = 0$, and commuting with the $\vec{v}$ and $\vec{A}$ vector fields, for any $v \in V$ and $A \in \mathfrak{g}$, and forming a basis at each point of B. Henceforth, we can dispense with H_0.

The central problem is that, while these vector fields on B commute with the $\vec{v}$ and $\vec{A}$, nonetheless they might not extend to all of B, because of monodromy problems. Lets take one such vector field, say X on B, defined in some open set $U \subset B$. Map $\pi|_U : U \to M$. Define $\bar{X} : U \to \pi^* TM$ by $\bar{X}(u) = \pi'(u) X(u)$. Because $\left[\vec{A}, X\right] = 0$ for $\vec{A} \in \mathfrak{g}$, we have

$$X\left(e^{t\vec{A}} u\right) = \left(e^{t\vec{A}}\right)'(u) X(u).$$

Calculate that for $A \in \mathfrak{g}$:

$$\begin{aligned}
\bar{X}\left(e^{t\vec{A}} u\right) &= \pi'\left(e^{t\vec{A}} u\right)\left(e^{t\vec{A}}\right)'(u) X(u) \\
&= \left(\pi e^{t\vec{A}}\right)'(u) X(u) \\
&= \pi'(u) X(u) \\
&= \bar{X}(u).
\end{aligned}$$

Therefore $\bar{X}$ is invariant under $\mathfrak{g}$ action, where that is defined. Assume that the open set $U \subset B$ on which X is defined is connected and has connected fibers over M. Then $\bar{X}$ is constant on those fibers, since they are the orbits of the $\mathfrak{g}$ action. Therefore $\bar{X}$ is now a vector field defined on a connected open subset of M, and clearly $\pi_* X = \bar{X}$ so $e^{t\bar{X}} \pi = \pi e^{tX}$.

Given any vector field Z on M, define a vector field Z^F on FM by

$$Z^F(u) = \left.\frac{d}{dt}\right|_{t=0} F\left(e^{tZ}\right)(u).$$

Lets check that $X = \bar{X}^F$. To check this, note that where defined,

$$e^{tX*}\omega = \omega.$$

Therefore

$$\begin{aligned}
u\pi'(u) &= \omega_u \\
&= \left(e^{tX*}\omega\right)_u \\
&= \omega_{e^{tX}u}\left(e^{tX}\right)'(u) \\
&= e^{tX}u\pi'\left(e^{tX}u\right)\left(e^{tX}\right)'(u) \\
&= e^{tX}u\left(\pi e^{tX}\right)'(u) \\
&= e^{tX}u\left(e^{t\bar{X}}\pi\right)'(u) \\
&= e^{tX}u\left(e^{t\bar{X}}\right)'(\pi(u))\pi'(u) \\
&= \left(F\left(e^{-t\bar{X}}\right)e^{tX}u\right)\pi'(u)
\end{aligned}$$

Therefore

$$u = F\left(e^{-t\bar{X}}\right)e^{tX}u,$$

so that

$$F\left(e^{t\bar{X}}\right) = e^{tX},$$

and thus $\bar{X}^F = X$. This $\bar{X}$ is defined in an open subset of M, so $\bar{X}^F$ is defined in the preimage of that open set in B, and we can extend each vector field X from its original open set to a G-invariant open set, setting $X = \bar{X}^F$.

Exercise 6.10 Why do we need γ to be a connection? Show that if G is connected, it is enough to assume that γ is a pseudoconnection.

Proposition 16. *Let $B \to M$ be a G-structure with connection γ with only constants appearing in the structure equations. Suppose that every component of M is compact with finite fundamental group. Then γ is complete.*

Remark 8. Consequently we can apply proposition 15 on page 81 to each component of M.

Proof. Let $\mathfrak{h}$ be the Lie algebra spanned by the $\vec{v}$ and $\vec{A}$ vector fields, $v \in V$ and $A \in \mathfrak{g}$. Without loss of generality, we can assume M is connected. We cover M by open sets, so that on each open set we have a Lie algebra of infinitesimal symmetries, and the Lie algebra is isomorphic across any of those open sets. When we travel along a path in M, we can smoothly continue any of our infinitesimal symmetries along that path, uniquely. However, if our path is a loop, we may encounter monodromy, giving a morphism of groups

$$\pi_1(M) \to \operatorname{Aut}(\mathfrak{h}).$$

So the infinitesimal symmetries are defined on a finite covering space of M. Replace M by this covering space. Because M is compact, these infinitesimal symmetries are complete vector fields. For each one of them, say Z, we have Z^F defined on B, and the various Z^F give a basis of each tangent space of B. In particular, they generate a symmetry group that acts locally transitively on B.

Given any one of the vector fields $\vec{v}$ on B, we can imagine picking up its flow lines and moving them around with the symmetry group. Because the group is locally transitive, I can take any two points on a single flow line, and move one to another by a symmetry. This will just slide the flow line along itself. But then the time for which the flow is defined through each of those points must be the same, and therefore must be infinite, so the flow is complete.

We can generalize slightly:

Definition 21. *A group Γ* defies *a group G if every morphism $\Gamma \to G$ has finite image; otherwise it* yields *to G.*

Example 52. If Γ is finite, it defies every group. If G is finite, every group defies it. The groups $\mathbb{Z}, \mathbb{Z}^2, \dots$ yield to one another, and defy sums and free products of finite cyclic groups.

Example 53. $\Gamma = \left\langle x, y | x^2 = y^2 = 1 \right\rangle$ defies $G = SL\left(2, \mathbb{R}\right)$, since the only involutions in $SL\left(2, \mathbb{R}\right)$ are ± 1. It yields to $SO\left(3\right)$, which has lots of involutions.

Corollary 10. *Let $B \to M$ be a G-structure with connection γ with only constants appearing in the structure equations. Let $\mathfrak{h}$ be the Lie algebra spanned by the $\vec{v}$ and $\vec{A}$ vector fields, $v \in V$ and $A \in \mathfrak{g}$. Suppose that every component of M is compact with fundamental group defying* Aut $(\mathfrak{h})$. *Then γ is complete.*

Proof. The proof is identical to the proof of corollary 16 on the preceding page.

Example 54 (Web geometry again). Recall the structure equations of web geometry from section 4.1 on page 27:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

and $d\gamma = \kappa \omega^1 \wedge \omega^2$. Proposition 10 on page 43 showed that unless $\kappa = 0$, the web geometry cannot be homogeneous. On the other hand, if $\kappa = 0$, then the web geometry is flat and locally homogeneous, since there are only constants appearing in the structure equations. If the web geometry lives on a compact surface, then the surface is a torus, and the web geometry is the quotient of the standard flat web geometry from the plane by a discrete group of symmetries.

6.7 Structure equations locally define G-structures

Suppose that X is a manifold equipped with 1-forms $\omega \in \Lambda^1(X) \otimes V$ and $\gamma \in \Lambda^1(X) \otimes \mathfrak{g}$ and a 0-form $T \in \Lambda^0(X) \otimes V \otimes \Lambda^2(V^*)$ so that

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega$$

so that the components of ω^i and γ^i_j are all independent 1-forms and form a basis for each cotangent space of X.

Definition 22. *Call this X a* local G-structure with pseudoconnection. *A* local G-structure *is a choice of local G-structure with pseudoconnection, but with γ only defined up to adding multiples of ω to γ.*

Example 55. The purpose of this section is to show that all local G-structures come about locally from G-structures. This is often used by writing down a choice of e-structure, i.e. a coframing on a manifold. For instance, consider a function $u(x, t)$ satisfying the sine-Gordon equation

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u.$$

Following Chern & Tenenblat [35], let

$$\omega^1 = \sin u \, dt$$
$$\omega^2 = dx + \cos u \, dt$$
$$\gamma = -\frac{\partial u}{\partial x} \, dx.$$

The 1-forms ω^1, ω^2 already provide a coframing, and (as we will see in this section) we can fatten this to an $SO(2)$-structure. But since in fact the 1-forms ω^1, ω^2 formally satisfy the equations of an $SO(2)$-structure, with γ plugged in instead of a pseudoconnection 1-form, we can see that the fattened up $SO(2)$-structure has no torsion, and its connection 1-form pulls back to γ under the fattening up map including this e-structure into the $SO(2)$-structure. We can even calculate the curvature: $d\gamma = -\omega^1 \wedge \omega^2$ tells us that the Gauss curvature of the surface is -1.

Definition 23. *If M is a manifold, V a vector field on M and $x_0 \in M$, write $e^V x_0$ for the point x_1 (should one exist) so that there is an absolutely continuous path $x(t)$ with $x(0) = x_0, x(1) = x_1$ and*

$$\frac{dx}{dt} = V(x(t)).$$

Lemma 9. *If X is a local G-structure, define for each $A \in \mathfrak{g}$ a vector field $\vec{A}$ on X by*

$$\vec{A} \lrcorner \omega = 0, \vec{A} \lrcorner \gamma = A.$$

This is a right Lie algebra action.

Proof. Calculate that

$$\mathfrak{L}_{\vec{A}}\omega = \vec{A} \lrcorner d\omega + d\left(\vec{A} \lrcorner \omega\right) \\ = -A\omega$$

Taking exterior derivative of the structure equations on X, and plugging in two of these $\vec{A}$ vectors, one finds

$$\vec{A} \wedge \vec{B} \lrcorner (d\gamma + \gamma \wedge \gamma) = 0.$$

Calculate

$$\vec{A} \wedge \vec{B} \lrcorner d\gamma = \mathfrak{L}_{\vec{A}}\left(\vec{B} \lrcorner \gamma\right) - \mathfrak{L}_{\vec{B}}\left(\vec{A} \lrcorner \gamma\right) - \left[\vec{A}, \vec{B}\right] \lrcorner \gamma$$

so that

$$\left[\vec{A}, \vec{B}\right] \lrcorner (\omega, \gamma) = \overrightarrow{[A, B]} \lrcorner (\omega, \gamma).$$

This shows that the map $A \mapsto \vec{A}$ is a right Lie algebra action.

Definition 24. *A local G-structure X is said to* fiber *if there is a fiber bundle map $X \to M$, so that the path components of the fibers are unions of leaves of the foliation $\omega = 0$.*

Lemma 10. *Let X be a local G-structure. The orbits of the Lie algebra action are precisely the leaves of the foliation $\omega = 0$.*

Proof. See corollary 5 on page 72.

The quotient $X/\mathfrak{g}$ is paracompact (since X is), but not necessarily Hausdorff.

Lemma 11. *If X fibers, then $X/\mathfrak{g}$ is a manifold, and $X \to X/\mathfrak{g}$ is a fiber bundle.*

Proof. Locally, for an open subset $U \subset M$, its inverse image in X is diffeomorphic to $U \times F$, with F the fiber, and the Lie algebra vector fields are tangent to F. The orbits are the path components of F, so $U \times F/\mathfrak{g} = U \times \pi_0(F)$.

Globally, we have to patch these things together, which we do with maps patching the U open sets together to make M, similar maps putting X together, and clearly the maps putting $X/\mathfrak{g}$ together make it a covering of M.

Proposition 17. *Given a local G-structure X which fibers, $\rho : X \to M$, there is a unique map $F\rho : X \to FM$ which satisfies*

$$F\rho^*\omega = \omega \text{ and } \pi F\rho = \rho$$

(where $\pi : FM \to M$ is the frame bundle). Moreover, this map $F\rho$ is an immersion and a bundle map, and there is a unique right Lie algebra action of $\mathfrak{g}$ on X for which $F\rho$ is equivariant.

Proof. For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on X (sometimes we will write it $\vec{A}_X$ to distinguish from other actions) by

$$\begin{aligned} \vec{A} \lrcorner \omega &= 0 \\ \vec{A} \lrcorner \gamma &= A. \end{aligned}$$

Calculate that

$$\mathcal{L}_{\vec{A}} \omega = -A\omega$$

from which we can conclude that for as long as the flow of $\vec{A}$ is defined,

$$e^{t\vec{A}*} \omega = e^{-tA} \omega.$$

This shows that the map $A \mapsto \vec{A}$ is a right Lie algebra action.

Write $\pi : FM \to M$ for the usual V-valued coframe bundle. Define the map $F\rho : X \to FM$ by

$$F\rho(x) = u(x)$$

where

$$m = \rho(x)$$

and

$$u(x)\rho'(x) = \omega_x.$$

This is well-defined, because ω is semibasic for the map $X \to M$. Moreover, $\pi F\rho = \rho$. Recall that the soldering form (also called ω) on FM is defined by

$$\omega_u = u\pi'(u).$$

It now follows that

$$F\rho^* \omega = \omega.$$

Next we need to check that $F\rho$ is equivariant under the Lie algebra action. First,

$$\begin{aligned} e^{t\vec{A}*} \omega_x &= e^{-tA} \omega_x \\ &= \omega_{e^{t\vec{A}}x} \left(e^{t\vec{A}} \right)' (x). \end{aligned}$$

Next,

$$\begin{aligned} u\left(e^{t\vec{A}}x\right) \rho'\left(e^{t\vec{A}}x\right) &= \omega_{e^{t\vec{A}}x} \\ &= e^{-tA} \omega_x \left(e^{-t\vec{A}}\right)' \left(e^{t\vec{A}}x\right) \\ &= e^{-tA} u(x) \rho'(x) \left(e^{-t\vec{A}}\right)' \left(e^{t\vec{A}}x\right) \\ &= e^{-tA} u(x) \rho'\left(e^{t\vec{A}}x\right). \end{aligned}$$

Putting this together,

$$u\left(e^{t\vec{A}}x\right) = e^{-tA}u(x)$$

which is the same as saying

$$r_g F\rho = F\rho r_g,$$

equivariance where defined. The fibers of $X \to M$ are now mapped equivariantly, and therefore the map $F\rho$ is an immersion on fibers. But it is also a bundle map over M, so it must be an immersion.

If we had two maps $\rho_0, \rho_1 : X \to FM$ satisfying

$$\rho_j^*\omega = \omega \text{ and } \pi\rho_j = \rho$$

then take $x \in X$ and let $u_j = \rho_j(x)$. Taking π of both sides gives

$$\begin{aligned} m_j &= \pi\rho_j(x) \\ &= \rho(x) \end{aligned}$$

so $m_0 = m_1$. Using the pullback of ω:

$$\begin{aligned} \omega_x &= \rho_j^*\omega_x \\ &= \omega_{u_j}\rho_j'(x) \\ &= u_j\pi'(u_j)\,\rho_j'(x) \\ &= u_j\,(\pi\rho_j)'\,(x) \\ &= u_j\rho'(x) \end{aligned}$$

and since ρ is a submersion, $u_0 = u_1$.

Proposition 18. *Given a local G-structure X, define a Lie algebra action on $X \times G$ by*

$$A \in \mathfrak{g} \mapsto \vec{A}_X - \overleftarrow{A}_G$$

where $\vec{A}_X$ is the right Lie algebra action on X, and $\overleftarrow{A}_G$ is the left Lie algebra action on G.

Define $X[G] = (X \times G)/\mathfrak{g}$. Write $[x, g]$ for the Lie algebra orbit of a a point $(x, g) \in X \times G$. The group G acts on $X \times G$ via the action

$$(x_0, g_0)\,g = (x_0, g_0 g)$$

and this descends to $X[G]$. This action on $X[G]$ is free. Its orbit through a point $[x_0, g_0]$ is precisely the set of points $[x_1, g_1]$ so that x_0 and x_1 are in the same Lie algebra orbit. If X fibers, then the map

$$F\rho : X \times G \to FM, (x_0, g_0) \mapsto g_0^{-1}F\rho(x_0)$$

descends to $X[G]$ where it is continuous and G equivariant, and the map

$$X \to X[G], x_0 \mapsto (x_0, 1)$$

is continuous and equivariant under the Lie algebra action, and the map

$$\rho : X[G] \to M, [x, g] \mapsto \rho(x)$$

is well-defined, continuous and G invariant.

Proof. To see the freedom of the action, use the map $F\rho : X[G] \to FM$. The rest is elementary.

Notice that we could replace M by $X/\mathfrak{g}$ here. So if X fibers, even if we don't know how, we can canonically map $X[G] \to FX/\mathfrak{g}$.

Lemma 12. *If X fibers, then $X[G] \to X/\mathfrak{g}$ is a principal G-bundle; in particular $X[G]$ is a smooth manifold, and the G-action on it is smooth. Moreover, the map $X[G] \to FX/\mathfrak{g}$ is an embedded G-structure.*

Proof. We have seen that $X/\mathfrak{g} = X[G]/G$, and that the action of G on $X[G]$ is free. We need to show that $X[G]$ is Hausdorff, then show that it is a manifold, and then exhibit local, smooth, trivializing maps for $X[G] \to X/\mathfrak{g}$. The map $X[G] \to X/\mathfrak{g}$ is continuous, and $X/\mathfrak{g}$ is Hausdorff, so if two points $[x_0, g_0], [x_1, g_1] \in X[G]$ do not map to the same point of $X/\mathfrak{g}$, we can easily surround them with disjoint open sets. The same holds for the map $F\rho$. If they map to the same point of $X/\mathfrak{g}$, we have seen that they are in the same G-orbit, say $[x_1, g_1] = [x_0, g_0 g]$. They agree under $F\rho$, so

$$F\rho\,[x_0, g_0] = F\rho\,[x_0, g_0 g]$$

but this gives

$$g_0^{-1} F\rho\,(x_0) = g^{-1} g_0^{-1} F\,(x_0)$$

so that $g = 1$ and the points are the same in $X[G]$. Therefore $X[G]$ is Hausdorff. Consider the continuous maps

$$\phi_{g_0} : x \in X \to [x, g_0] \in X[G].$$

Taking such a map ϕ_{g_0} and any point x_0, we need to show that this map has an inverse, at least when restricted to a neighborhood of x_0, and that, in that neighborhood the map is a homeomorphism. Being close to $[x_0, g_0]$ in $X[G]$ means being at a point $(x, g) \in X \times G$ where, after flowing around with the local Lie algebra action, we can arrange x close to x_0 and g close to g_0. Every $(x, g) \in X \times G$ with g near g_0 can be written $\left(x, e^A g_0\right)$ for a unique $A \in \mathfrak{g}$ close to 0. But then under the Lie algebra action of $B \mapsto \vec{B}_X - \overleftarrow{B}_G$, we can flow to the point $\left(e^{\vec{A}} x, g_0\right)$, as long as A belongs to yet another small neighborhood of the origin, which can be chosen to depend only on x_0. This provides a local inverse for our map $x \in X \mapsto (x, g_0) \in X[G]$ in a small open subset of $X[G]$, which depends on the sufficiently small neighborhood of x_0 we started working in. It is easy to check that the inverse is continuous.

We employ these ϕ_{g_0} maps to provide a smooth structure: a function is smooth if it pulls back to a smooth function under all of these maps. Smoothness of transitions is easy. Smoothness of the map $X \times G \to X$ is also easy, and this implies smoothness of the G-action.

To find a local section, take any local section of $X \to X/\mathfrak{g}$ (which is locally a fiber bundle, i.e. a disjoint union of fiber bundles), say $\sigma : \text{open} \subset X/\mathfrak{g} \to X$. Given a point $[x_0, g_0]$ with $\sigma(x_0)$ defined, let $\sigma[G](x) = [x, g_0]$. Clearly this is smooth.

Lemma 13. *The map $x \in X \mapsto [x, 1] \in X[G]$ is a local diffeomorphism. If X fibers, then this map is a local equivalence, in the sense that it is a local diffeomorphism, and identifies the 1-forms ω.*

Theorem 6. *Every local G-structure with pseudoconnection admits near any point a local diffeomorphism to a G-structure, preserving structure equations.*

Proof. Locally, the foliation by $\omega = 0$ is a fiber bundle.

Proposition 19. *A local G-structure [with pseudoconnection] X is a G-structure [with pseudoconnection] precisely when there is a G action on X extending the Lie algebra action, so that*

$$r_g^* \omega = g^{-1} \omega$$

for all $g \in G$, and so that X is a principal G bundle.

Proof. In this case, it is easy to see that the map $F\rho$ is an embedding.

7

Prolonging G-structures into involution

Prolongation means differentation. The purpose of prolongation is to find "enough" differential invariants to tell geometric objects apart. A good criterion for "enough" will emerge in the next chapter, when we discuss symmetries. For now, we can see why we need to prolong: there is no intrinsic torsion on a Riemannian manifold, so our process has not yet told us anything of substance about Riemannian manifolds (except that they are all the same, to first order). Clearly we need to differentiate in order to feel the curvature. Cartan's approach is to build a new G-structure on top of the G-structure we have already built, i.e. if B is a structure on a manifold M, build a structure on the manifold B, and so on to form a tower of structures. Each point of the k-th structure represents a Taylor expansion of the original geometry up to k-th order.

7.1 Prolongation

We do not have a unique choice of pseudoconnection except when $\mathfrak{g}^{(1)} = 0$. Let us instead parameterize the choices. Suppose that we have succeeded in carrying out reduction on a G-structure $\pi : B \to M$, having taken a structure with constant type, and having reduced the structure group so that the resulting G-structure has constant torsion on each fiber of $B \to M$, say torsion $T : M \to V \otimes \Lambda^2 \left(V^*\right)$. To be more precise, we should write $T : M \to S \subset V \otimes \Lambda^2 \left(V^*\right)$, where S is our section.

Recall the exact sequence

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 \left(V^*\right) \xrightarrow{[]} H^{0,2}\left(\mathfrak{g}\right) \longrightarrow 0.$$

Consider $\mathfrak{g}^{(1)}$ as a group under addition, and consider its action on $V \oplus \mathfrak{g}$ given for $Q \in \mathfrak{g}^{(1)}$ by

$$(v, A) \in V \oplus \mathfrak{g} \mapsto (v, Q \cdot v + A)$$

where $Q \cdot v$ is the obvious contraction: in a basis e_i of V:

$$(Q \cdot v)^i = Q^i_{jk} v^k.$$

Fix a choice Γ of pseudoconnection on B. Then any other pseudoconnection must look like $\gamma = \Gamma + Q\omega$ where $Q \in \mathfrak{g}^{(1)}$. We form a bundle $B^{(1)} \to B$ whose fibers are the possible choices of pseudoconnection at each point of B. Obviously $B^{(1)} \to B$ is a trivial bundle, because every section is just $\Gamma + Q\omega$. Nonetheless, it is not canonically trivial, because we do not have a canonical choice of pseudoconnection Γ to start with. Let $\pi^{(1)} : B^{(1)} \to B$ be the obvious map. On $B^{(1)}$, define the 1-forms ω and γ by

$$\begin{aligned} \omega &= \pi^{(1)*}\omega \\ \gamma_{\Gamma+Q\omega} &= \pi^{(1)*}\Gamma + Q\pi^{(1)*}\omega. \end{aligned}$$

Note that now γ is *not* a pseudoconnection on B, but rather a 1-form canonically defined on $B^{(1)}$. Down on B we had the structure equations

$$d\omega = -\Gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega.$$

Now we work on $B^{(1)}$, and find exactly the same equation, with everything just pulled back to $B^{(1)}$. Once we pull back to $B^{(1)}$, we find the (peculiar!) equation $\gamma = \Gamma + Q\omega$, and since $Q \in \mathfrak{g}^{(1)}$, we know $(Q\omega) \wedge \omega = 0$, so

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega.$$

With some effort, we have managed to keep the equations the same as on B, but made γ mean something different.

Map $\mathfrak{g}^{(1)} \to \mathrm{GL}(V \oplus \mathfrak{g})$ by

$$Q \mapsto Q', Q'(v, A) = (v, A + Qv).$$

Proposition 20. *Map*

$$U = \Gamma + Q\omega \in B^{(1)} \mapsto u\pi'(u) \oplus U \in FB$$

Then $B^{(1)} \to B$ is a $\mathfrak{g}^{(1)}$-structure, called the prolongation *of $B \to M$. Two G-structures of the same constant type are equivalent precisely if their prolongations are.*

7.1.1 Partial reduction

To define the prolongation, we used constancy of torsion on fibers of B, i.e. we employed a section S to carry out reduction. Otherwise, there would be too much freedom in the choice of pseudoconnection form Γ so that the bundle would not be a $\mathfrak{g}^{(1)}$ principal bundle. One can nonetheless make use of such partial reductions, where the torsion T might not be constant on the fiber of $B \to M$, but only have constant $f(T)$, where f is some G-equivariant map into some G-space. There is a natural generalization of this process to circumstances when only partial reductions have been carried out, but we will not belabour our exposition with it.

7.1.2 Higher prolongations

Subsequent prolongations are defined inductively, after carrying out reduction at each step (assuming that the type is constant at each step). Prolongations are an analogue of coordinate systems adapted to the geometry of the G-structure, but defined only up to a certain order in their Taylor expansions. We will discuss the higher order prolongations in section 7.5 on page 115.

7.1.3 Higher order structures

G-structures are principal subbundles of the frame bundle FM. We could impose geometry on a manifold by taking principal subbundles of the bundle $FM^{(1)} \to FM$ instead of $FM \to M$, called *second order structures*. An example is the notion of projective structure (see section 7.11 on page 154).

7.1.4 Fattening up and prolonging

The map constructed in section 6.2 on page 59 taking pseudoconnection 1-forms on a G-structure B to those on its "fattening up" $B\,(H)$ for any group H with $G \subset H \subset \mathrm{GL}\,(V)$ does not always map

$$B^{(1)} \to B(H)^{(1)}$$

since the condition for a pseudoconnection 1-form to belong to the first prolongation is that its torsion must belong to a specified section.

Exercise 7.1 If sections S_H and S_G are used to carry out reduction, and we can smoothly map $S_H \to S_G$, so that the diagram

$$\begin{array}{ccccc} S_H & \longrightarrow & S_G & \longrightarrow & V \otimes \Lambda^2\,(V^*) \\ & & & & \downarrow \\ & & & & H^{0,2}\,(\mathfrak{g}) \\ & & & & \\ & & & & H^{0,2}\,(\mathfrak{h}) \end{array}$$

commutes, using the obvious mapping $H^{0,2}\,(\mathfrak{g}) \to H^{0,2}\,(\mathfrak{h})$ on torsion, then fattening commutes with prolonging, i.e. there is a natural map $\gamma \in B^{(1)} \mapsto \gamma_* \in B(H)^{(1)}$ so that the diagram

$$\begin{array}{ccc} B^{(1)} & \longrightarrow & B(H)^{(1)} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B(H) \end{array}$$

commutes.

7.1.5 Structure equations of the prolongation

Let us formulate structure equations on the prolongation. Note that γ is well defined on $B^{(1)}$, while on B we found that there were choices of pseudoconnections γ. But on $B^{(1)}$, γ is not a pseudoconnection; instead $\omega + \gamma$ is the soldering form of $B^{(1)} \to B$. Searching for a pseudoconnection on $B^{(1)}$ is the next step. We find, by straightforward unwinding of definitions, that the right action of $Q \in \mathfrak{g}^{(1)}$ given by $r_Q U = U - Q\omega$ (where $U = \Gamma + Q_0\omega$ is a point of $B^{(1)}$) satisfies

$$\begin{aligned} r_Q^* \omega &= \omega \\ r_Q^* \gamma &= \gamma - Q\omega. \end{aligned}$$

There is a natural action of G on $B^{(1)}$, commuting with the bundle map $B^{(1)} \to B$, which is (for $g \in G$):

$$r_g U = \mathrm{Ad}_g^{-1} \left(U \left(r_g^{-1} \right)' \right)$$

Form the semidirect product $G \rtimes \mathfrak{g}^{(1)}$ with multiplication

$$\left(g_1, Q_1\right)\left(g_2, Q_2\right) = \left(g_1 g_2, Q_1 + g_1 Q_2\right)$$

where gQ means the element of $\mathrm{Sym}^2\left(V^*\right) \otimes V$ defined by

$$gQ(u, v) = g\left(Q\left(g^{-1}u, g^{-1}v\right)\right).$$

We can write a point of $B^{(1)}$ as (u, U) where $u \in B$ and $U = \Gamma + Q\omega$ is the value of some choice of pseudoconnection at u. The two group actions fuse together to an action of the semidirect product:

$$r_{(g,Q)}\left(u, U\right) = \left(g^{-1}u, \mathrm{Ad}_g^{-1}\left(U - Qu\pi'\right)\left(r_g^{-1}\right)'\right).$$

This action makes $B^{(1)} \to M$ into a principal right $G \rtimes \mathfrak{g}^{(1)}$-bundle. It is *not* a $G \rtimes \mathfrak{g}^{(1)}$-structure. Under the G action

$$\begin{aligned} r_g^* \omega &= g^{-1}\omega \\ r_g^* \gamma &= \mathrm{Ad}_g^{-1} \gamma. \end{aligned}$$

Therefore under the $G \rtimes \mathfrak{g}^{(1)}$ action

$$r^*_{(g,Q)} \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} g^{-1}\omega \\ \mathrm{Ad}_g^{-1}(\gamma - Q\omega) \end{pmatrix}.$$

Proposition 21. *Since $T : M \to S$ is defined on M, its differential can be written*

$$dT^i_{jk} = \nabla_m T^i_{jk} \omega^m.$$

The structure equations of the prolongation are

$$\begin{aligned} d\omega^i + \gamma^i_j \omega^j &= \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k \\ d\gamma^i_j + \gamma^i_k \wedge \gamma^j_k + \xi^i_{jk} \wedge \omega^k &= \frac{1}{2} T^i_{jkl} \omega^k \wedge \omega^l + T^{im}_{jkl} \gamma^k_m \wedge \omega^l \\ &\quad + \frac{1}{6} \left(\nabla_l T^i_{jm} + \nabla_j T^i_{ml} + \nabla_m T^i_{lj} \right. \\ &\quad \left. + T^i_{kj} T^k_{lm} + T^i_{km} T^k_{jl} + T^i_{kl} T^k_{mj} \right) \omega^l \wedge \omega^m \end{aligned}$$

where

$$\xi \in \Omega^1 \left(B^{(1)} \right) \otimes \mathfrak{g}^{(1)}$$

is some 1-form satisfying $\vec{Q} \lrcorner \xi = Q$ (a pseudoconnection for $B^{(1)} \to B$) and

$$\begin{aligned} T^i_{jkl} &= T^i_{kjl} \\ T^{im}_{jkl} &= T^{im}_{kjl}. \end{aligned}$$

Remark 9. It is easy to arrive at these structure equations: formally, we just take the exterior derivatives of the structure equations for ω that we already had on B. These structure equations will not help us compute examples, since in each example it will be easier just to write down the structure equations on B, and then arrive at the same equations for $d\omega$ on $B^{(1)}$ as above, and then just differentiate them to recover equations for $d\gamma$.

Clearly $\vec{Q}$ means the vector field on $B^{(1)}$ giving the infinitesimal action of $Q \in \mathfrak{g}^{(1)}$ and $\vec{A}$ is the vector field on $B^{(1)}$ giving the infinitesimal action of $A \in \mathfrak{g}$. This ξ is defined up to replacing with $\xi + f\omega$ for any $f : B^{(1)} \to \mathfrak{g}^{(2)}$. It is a pseudoconnection 1-form for the $\mathfrak{g}^{(1)}$-structure $B^{(1)} \to B$.

Remark 10. $\vec{A} \lrcorner \xi = 0$ for all A just when $\gamma \oplus \xi$ forms a pseudoconnection for $B^{(1)} \to M$. This is not the case for CR 3-manifolds; see section 7.4 on page 111.

Definition 25. *Henceforth it is useful to rename the torsion on B the* 1-torsion*, and the torsion on $B^{(1)}$ the* 2-torsion*, etc. and use the all encompassing name* torsion *for any of these. Say that a G-structure is* 1-flat *if its 1-torsion vanishes, and* k-flat *if its 1-torsion, 2-torsion, etc. up to and including its k-torsion all vanish.*

7.1.6 Torsion and Spencer cohomology

If 1-torsion vanishes everywhere, then the prolongation has 2-torsion in $H^{1,2}(\mathfrak{g})$. Similarly, if the 2-torsion from the prolongation vanishes everywhere, then the next prolongation has 3-torsion in $H^{2,2}(\mathfrak{g})$. Practically speaking, one rarely encounters torsion vanishing higher than this stage without vanishing altogether at all orders, indeed without flatness.

Example 56 (The octave projective plane). $G = \text{Spin}(9,1)$ has an irreducible representation on $\mathbb{R}^{16}$, and $\dim H^{0,2}(\mathfrak{so}(9,1)) = 1200$. The 1-torsion of a $\text{Spin}(9,1)$-structure lives in that 1200 dimension space. If the 1-torsion vanishes everywhere, then we should get 2-torsion in $H^{1,2}(\mathfrak{so}(9,1))$, which has 5400 dimensions. But in fact the second Bianchi identity forces flatness of every 1-torsion-free $\text{Spin}(9,1)$-structure, up to local equivalence; every 1-torsion-free $\text{Spin}(9,1)$-structure is locally isomorphic to the octave projective plane.

7.1.7 The two actions of the structure group on the frame bundle FB

Note: do not confuse the action of G on $B^{(1)}$ with the action on the frame bundle of B which is induced from the action on B. Take r_g acting on B, and prolong it to frames to produce

$$Fr_g : FB \to FB.$$

We have an embedding

$$\begin{aligned} B^{(1)} &\to FB \\ (x,u,w) &\mapsto u \circ \pi' \oplus w \end{aligned}$$

but the prolongation of r_g to frames on B does not necessarily leave $B^{(1)}$ invariant, and it is not the action of G on $B^{(1)}$. Indeed,

$$Fr_g(x,u,w) = \left(x, g^{-1}u, w \circ \left(r_g^{-1}\right)'\right)$$

7.1.8 Recursion

Now that we have obtained the prolongation, and found out how its torsion behaves under the action of each factor group in the semidirect product $G \rtimes \mathfrak{g}^{(1)}$, the next step (assuming constant type) is to attempt reduction with respect to $G \rtimes \mathfrak{g}^{(1)}$. Repeatedly, we carry out reduction as far as possible, and then prolong. The process will halt if the next prolongation is identical to the previous one, i.e. if $\mathfrak{g}^{(k)} = 0$.

Remark 11. When quotienting out by the action of a semidirect product of groups $G_0 \rtimes G_1$, one can always quotient out first by the normal subgroup G_1, and then by the quotient group G_0. In our case, that means that we can always try to normalize the torsion first by $\mathfrak{g}^{(1)}$ action, and then by G action.

7.2 Example: the general linear group

Consider a manifold with no structure at all. If the manifold has dimension n, then this can be thought of as a $\mathrm{GL}(n, \mathbb{R})$ structure. We will find the structure equations of the prolongations. This isn't going to be very useful, since there is no geometric structure on the manifold; the equivalence method focuses on local geometric invariants and there aren't any. But we get an idea of what it looks like to carry out prolongation to all orders.

First, for notation, fix a vector space V and consider the spaces

$$\mathrm{Sym}^p(V^*) \otimes V$$

of V-valued polynomials on V of degree p, where $p \geq 0$. Keep in mind that

$$\mathfrak{gl}(V)^{(p)} = \mathrm{Sym}^{p+1}(V^*) \otimes V,$$

for $p = -1, 0, \dots$. Think of these as vector fields on V with polynomial coefficients of degree $p+1$. For any vector $v \in V$, define the differential operator $D_v : \mathrm{Sym}^p(V^*) \otimes V \to \mathrm{Sym}^{p-1}(V^*) \otimes V$ by differentiating the polynomial part. Following Guillemin [42], define a bracket by

$$[s_1 \otimes v_1, s_2 \otimes v_2] = (s_2 D_{v_2} s_1) \otimes v_1 - (s_1 D_{v_1} s_2) \otimes v_2.$$

Careful: this is the negative of the usual Lie bracket on vector fields, applied to polynomial vector fields; in fact, we are not quite following Guillemin, but have instead reversed his sign convention. This makes the formal sum

$$\mathfrak{gl}(V)^{(\bullet)} = \sum_{p=-1}^{\infty} \mathfrak{gl}(V)^{(p)}$$

into a Lie algebra, the "negative" of Lie algebra of polynomial vector fields.

Lemma 14. *Let $\omega^{(-1)} = \omega, \omega^{(0)} = \gamma$. Prolongations give rise to 1-forms $\omega^{(k)}$ and bundles $FM^{(k)}$ so that on $FM^{(p)}$ the 1-forms*

$$\omega^{(-1)}, \omega^{(1)}, \dots, \omega^{(p-1)}$$

are defined, with

$$\omega^{(k)} \in \Omega^1\left(FM^{(p)}\right) \otimes \mathrm{Sym}^{k+1}(V^*) \otimes V = \mathfrak{gl}(V)^{(k)}.$$

The structure group of $FM^{(p)}$ is the group $\mathrm{GL}(V)^{(p \geq 0)}$ of $(p+1)$-jets of diffeomorphisms $V \to V$ fixing the origin. The 1-forms $\omega^{(-1)}, \dots, \omega^{(p-1)}$ vary in the representation of $\mathrm{GL}(V)^{(p \geq 0)}$ given by treating $\omega^{(-1)} + \dots + \omega^{(p-1)}$ as a polynomial vector field of degree at most p. The structure equations are

$$d\omega^{(\bullet)} = -\frac{1}{2}\left[\omega^{(\bullet)}, \omega^{(\bullet)}\right],$$

which is to say

$$d\omega^{(p)} = -\frac{1}{2} \sum_{k=-1}^{p+1} \left[\omega^{(p-k)}, \omega^{(k)}\right].$$

Proof. For $p = -1, 0$, it is easy to check (paying very careful attention to minus signs). The Jacobi identity on a Lie algebra is just the equation $d^2 = 0$ on the Maurer–Cartan form, so the consistency of the above rules follows from the consistency of the Jacobi identity on the ("negative" of the) Lie algebra of polynomial vector fields.

Given $FM^{(p)}$, we let $FM^{(p+1)} \to FM^{(p)}$ be the prolongation, so on $FM^{(p+1)}$, we have the same structure equations up to

$$d\omega^{(p-1)} = -\frac{1}{2} \sum_{k=-1}^{p} \left[\omega^{(p-1-k)}, \omega^{(k)}\right].$$

Under the structure group, $\omega^{(-1)}, \dots, \omega^{(p-1)}$ transform by hypothesis in the required representation.

Question 5. Finish this.

If there was a Lie group of polynomial diffeomorphisms $V \to V$, we would like to say that this would be its Lie algebra; morally, it is the Lie algebra of the diffeomorphism group, when we concentrate our attention only on order-by-order formal geometry.

For the reader curious about the sign convention, it comes about as follows: we would like to pretend that there is an infinite prolongation $FM^{(\infty)}$, and that the diffeomorphism group acts simply transitively on $FM^{(\infty)}$. (We don't claim that this is true, but useful to pretend.) Then the Lie algebra of the diffeomorphism group should consist, formally, in the vector fields on M. We would like to pretend that picking any point of $FM^{(\infty)}$, we can identify $FM^{(\infty)}$ with the diffeomorphism group, and thereby identity $\omega^{(\bullet)}$ with the left-invariant Maurer–Cartan 1-form. This follows the pattern in our examples so far, that homogeneous G-structures have symmetry group identified with the last nontrivial prolongation, with Maurer–Cartan 1-form agreeing with the 1-form $\omega^{(\bullet)}$. But $\omega^{(\bullet)}$ is certainly invariant under symmetries, and symmetries act on the left, so it would have to be the left invariant Maurer–Cartan form $\lambda = g^{-1}\, dg$. The bracket on left invariant vector fields is determined from the differential of the left invariant Maurer-Cartan form, by the equation

$$\left[\vec{A}, \vec{B}\right] = \overrightarrow{[A, B]}$$

where

$$d\lambda = -\frac{1}{2}\left[\lambda, \lambda\right],$$

and is the negative of the bracket on right invariant vector fields. But right invariant vector fields generate the left action, so in our case it is the right invariant vector fields that should be the vector fields on M with their usual Lie bracket. Consequently, the negative of the usual Lie bracket on vector fields should be the bracket on left invariant vector fields on the diffeomorphism group, and should give the structure equations for the Maurer–Cartan form.

To express this story in terms of real-valued 1-forms, instead of Lie algebra valued 1-forms, lets write each polynomial vector field in coordinates on $V = \mathbb{R}^n$ as

$$X(x) = \sum_\alpha \left. \frac{\partial^\alpha X^i}{\partial x^\alpha} \right|_{x=0} \frac{x^\alpha}{\alpha!} \frac{\partial}{\partial x^i}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ is a multindex, and

$$\begin{aligned} e_j &= (0, \dots, \underbrace{1}_{j}, \dots, 0) \\ \alpha! &= \alpha_1! \dots \alpha_n!, \\ |\alpha| &= \alpha_1 + \dots \alpha_n, \\ x^\alpha &= \left(x^1\right)^{\alpha_1} \dots \left(x^n\right)^{\alpha_n}, \\ \frac{\partial^\alpha}{\partial x^\alpha} &= \frac{\partial^{\alpha_1}}{\partial \left(x^1\right)^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial \left(x^n\right)^{\alpha_n}}. \end{aligned}$$

The usual Lie bracket on (formal) vector fields is

$$\begin{aligned} [X, Y] &= \sum_\gamma \left. \frac{\partial^\gamma}{\partial x^\gamma} [X, Y]^i \right|_{x=0} \frac{x^\gamma}{\gamma!} \frac{\partial}{\partial x^i} \\ &= \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} \left(\frac{\partial^{\alpha+e_j} Y^i}{\partial x^{\alpha+e_j}} \frac{\partial^\beta X^j}{\partial x^\beta} - \frac{\partial^{\alpha+e_j} X^i}{\partial x^{\alpha+e_j}} \frac{\partial^\beta Y^j}{\partial x^\beta} \right) \frac{x^\gamma}{\gamma!} \frac{\partial}{\partial x^i}. \end{aligned}$$

So the structure equations of a $\mathrm{GL}(n, \mathbb{R})$-structure can be written

$$d\omega_\gamma^i = - \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} \omega_{\alpha+e_j}^i \wedge \omega_\beta^j$$

(keeping in mind the necessary sign change).

Question 6. Here is where things get dodgy in this section.

We have a natural pairing between them:

$$\alpha \in \mathrm{Sym}^p(V^*) \otimes V,\ \beta \in \mathrm{Sym}^q(V^*) \otimes V \mapsto \alpha \cdot \beta \in \mathrm{Sym}^{p+q-1}(V^*) \otimes V$$

defined by taking the V factor from β and plugging it into a V^* slot from α, and then averaging over all permutations.

Our prolongations will give rise to forms $\omega^{(-1)} = \omega, \omega^{(0)} = \gamma$, etc., so that on $FM^{(j)}$ the 1-forms

$$\omega^{(-1)}, \omega^{(1)}, \dots, \omega^{(j-1)}$$

are defined, with

$$\omega^{(k)} \in \Omega^1\left(FM^{(j)}\right) \otimes \mathrm{Sym}^{k+1}\left(V^*\right) \otimes V = \mathfrak{gl}\left(V\right)^{(k)}.$$

The structure equations are

$$d\omega^{(k)} + \sum_{p=0}^{k+1} \omega^{(p)} \wedge \omega^{(k-p)} = 0.$$

Note that we never obtain a coframing, since there are never enough 1-forms on any bundle $FM^{(j)}$ defined to form a basis. We can absorb all torsion on each bundle.

In index notation, the 1-form $\omega^{(p)}$ has $p+1$ lower indices, say $\omega^{(p)} = \left(\omega^i_\alpha\right)$, with $\alpha = (\alpha_1, \dots, \alpha_{p+1})$. Write

$$|\alpha| = \alpha_1 + \dots + \alpha_{p+1}$$
$$\alpha! = \alpha_1! \dots \alpha_{p+1}!.e_j \qquad = (0, \dots, \underset{j}{1}, \dots, 0)$$

Proposition 22.

$$d\omega^i_\alpha = - \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \omega^i_{\beta+e_j} \wedge \omega^j_\gamma$$

Proof.

Question 7. This stuff needs a careful rewrite, using proper multi-index notation.

In index notation, the form $\omega^{(p)}$ has $p+1$ lower indices, say

$$\omega^{(p)} = \left(\omega^i_{j_1 \dots j_{p+1}}\right)$$

which is symmetric in all lower indices. Given a multiindex

$$J = (j_1, \dots, j_{p+1})$$

we define

$$|J| = p+1.$$

We define the symbol

$$\sum_{J_1 J_2 | J} f\left(J_1, J_2\right)$$

to mean the sum of the expression

$$\frac{1}{|J_1|!\,|J_2|!} f\left(J_1, J_2\right)$$

over all choices of partitions

$$J_1 = \left(j_{s_1}, \dots, j_{s_q}\right)$$
$$J_2 = \left(j_{t_1}, \dots, j_{t_{p-q}}\right)$$

where

$$s_1, \dots, s_q, t_1, \dots, t_{p+1-q}$$

is any permutation of

$$1, \dots, p+1.$$

We will adapt this notation to partitions of J into any number of pieces $J_1, J_2, \dots, J_N$ in the obvious manner.

Lemma 15. *If $f(J_1, J_2)$ is symmetric in all entries of J_1 and also in all entries of J_2 then*

$$\sum_{J_1 J_2 | Jk} f(J_1, J_2) = \sum_{J_1 J_2 | J} f((J_1, k), J_2) + f(J_1, (J_2, k))$$

(a Leibnitz rule for "differentiating in k").

Proof. Partitioning J, k instead of J, we either have to put k with J_1 or with J_2. If we put it with J_1, then it can occur anywhere in J_1, but by symmetry of F it suffices to always put k at the end of J_1, and the factorials take care of ensuring that we count correctly.

Lemma 16. *If $f(J_1, J_2, J_3)$ is symmetric in all entries of J_1, and also in all entries of J_2, and also those of J_3, then*

$$\sum_{J_1 J_2 | J} \sum_{J_{11} J_{12} | J_1} f(J_{11}, J_{12}, J_2) = \sum_{J_1 J_2 J_3 | J} f(J_1, J_2, J_3)$$

Proof. Again it is just a matter of keeping track of the factorials:

$$\frac{1}{|J_1|! \, |J_2|!} \frac{1}{|J_{11}|! \, |J_{12}|!} |J_1|! = \frac{1}{|J_{11}|! \, |J_{12}|! \, |J_2|!}$$

and seeing that there are $|J_1|!$ possible ways to divide up J_1.

The structure equations are

$$d\omega^i_J = -\sum_{J_1 J_2 | J} \omega^i_{J_1 k} \wedge \omega^k_{J_2}.$$

To see that this is correct, first we calculate by hand that

$$0 = d\omega^i + \omega^i_j \wedge \omega^j$$
$$0 = d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_{jk} \wedge \omega^k$$
$$0 = d\omega^i_{jk} + \omega^i_l \wedge \omega^l_{jk} + \omega^i_{lk} \wedge \omega^l_j + \omega^i_{jl} \wedge \omega^l_k + \omega^i_{jkl} \wedge \omega^l$$

These agree with the claimed result.

Then we suppose by induction that we have

$$d\omega^i_J = -\sum_{J_1 J_2 | J} \omega^i_{J_1 k} \wedge \omega^k_{J_2}$$

for any set of indices J with at most p indices in it. Our induction hypothesis tells us all of the exterior derivatives of all of the expressions in this equation, except the last:

$$\omega^i_{Jk} \wedge \omega^k$$

(i.e. with $J_2 = \emptyset$), because this term involves a 1-form

$$\omega^i_{Jk}$$

with more than p indices. Therefore we rewrite our induction hypothesis equation as

$$0 = d\omega^i_J + \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \omega^k_{J_2} + \omega^i_{Jk} \wedge \omega^k$$

where the sum is again over partitions, but this time we don't allow J_2 to be empty.

Taking exterior derivatives, we find

$$0 = \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \left(d\omega^i_{J_1 k} \wedge \omega^k_{J_2} - \omega^i_{J_1 k} \wedge d\omega^k_{J_2} \right) + \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k.$$

We use the induction hypothesis to compute the exterior derivatives in the first term:

$$\begin{aligned} 0 = & \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \left(-\sum_{J_{11} J_{12} | J_1 k} \omega^i_{J_{11} l} \wedge \omega^l_{J_{12}} \right) \wedge \omega^k_{J_2} \\ & - \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \left(-\sum_{J_{21} J_{22} | J_2} \omega^k_{J_{21} l} \wedge \omega^l_{J_{22}} \right) \\ & + \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k \end{aligned}$$

Using our lemma, we find

$$\begin{aligned} 0 = & -\sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \sum_{J_{11} J_{12} | J_1} \omega^i_{J_{11} kl} \wedge \omega^l_{J_{12}} \wedge \omega^k_{J_2} \\ & - \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \sum_{J_{11} J_{12} | J_2} \omega^i_{J_{11} l} \wedge \omega^l_{J_{12} k} \wedge \omega^k_{J_2} \\ & + \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \sum_{J_{21} J_{22} | J_2} \omega^k_{J_{21} l} \wedge \omega^l_{J_{22}} + \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k. \end{aligned}$$

We are dividing J into J_1 and J_2 and then dividing J_1 or J_2. We can instead divide J into three parts. In the first term, we needed $J_2 \neq \emptyset$, so if we change notation to

$$\begin{aligned} J_{11} &\mapsto J_1 \\ J_{12} &\mapsto J_2 \\ J_2 &\mapsto J_3 \end{aligned}$$

then we need $J_3 \neq \emptyset$. Similarly for the second term, we had $J_2 \neq \emptyset$, so if we change notation to

$$\begin{aligned} J_1 &\mapsto J_1 \\ J_{21} &\mapsto J_2 \\ J_{22} &\mapsto J_3 \end{aligned}$$

then we need to have either $J_2 \neq \emptyset$ or $J_3 \neq \emptyset$. This gives

$$\begin{aligned} 0 = &- \sum_{J_1 J_2 J_3 | J}^{J_3 \neq \emptyset} \omega^i_{J_1 kl} \wedge \omega^l_{J_2} \wedge \omega^k_{J_3} \\ &- \sum_{J_1 J_2 J_3 | J}^{J_3 \neq \emptyset} \omega^i_{J_1 l} \wedge \omega^l_{J_2 k} \wedge \omega^k_{J_3} \\ &+ \sum_{J_1 J_2 J_3 | J}^{J_2 \neq \emptyset \text{ or } J_3 \neq \emptyset} \omega^i_{J_1 k} \wedge \omega^k_{J_2 l} \wedge \omega^l_{J_3} \\ &+ \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k . \end{aligned}$$

The $J_3 \neq \emptyset$ terms in the third sum kill those in the second, so we get

$$\begin{aligned} 0 = &- \sum_{J_1 J_2 J_3 | J}^{J_3 \neq \emptyset} \omega^i_{J_1 kl} \wedge \omega^l_{J_2} \wedge \omega^k_{J_3} \\ &+ \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \omega^k_{J_2 l} \wedge \omega^l \\ &+ \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k . \end{aligned}$$

Any term in the first sum, say

$$\omega^i_{L_1 kl} \wedge \omega^l_{L_2} \wedge \omega^k_{L_3}$$

with $L_2 \neq \emptyset$ must appear in the sum once with $J_2 = L_2$ and $J_3 = L_3$ and then once again with $J_2 = L_3$ and $J_3 = L_2$. By symmetry of these ω^i_J in all lower indices, the two appearances of these terms have opposite signs, so they contribute nothing. Therefore

$$\begin{aligned} 0 = &- \sum_{J_1 J_3 | J}^{J_3 \neq \emptyset} \omega^i_{J_1 k l} \wedge \omega^l \wedge \omega^k_{J_3} \\ &+ \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \omega^k_{J_2 l} \wedge \omega^l \\ &+ \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k. \end{aligned}$$

We can rewrite this as

$$0 = \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k l} \wedge \omega^k_{J_2} \wedge \omega^l + \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 k} \wedge \omega^k_{J_2 l} \wedge \omega^l + \left(d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \right) \wedge \omega^k.$$

or as

$$\begin{aligned} 0 = \Bigg(& d\omega^i_{Jk} + \omega^i_{Jl} \wedge \omega^l_k \\ &+ \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 l k} \wedge \omega^l_{J_2} \\ &+ \sum_{J_1 J_2 | J}^{J_2 \neq \emptyset} \omega^i_{J_1 l} \wedge \omega^l_{J_2 k} \Bigg) \wedge \omega^k. \end{aligned}$$

Therefore there exist 1-forms ω^i_{Jkl} symmetric in all lower indices, so that

$$d\omega^i_{Jk} + \sum_{J_1, J_2 | J, k} \omega^i_{J_1, l} \wedge \omega^l_{J_2} = 0.$$

7.3 Example: pseudo-Riemannian geometry

Let Q be a nondegenerate quadratic form of signature p, q on a vector space V, and take G the group of orientation preserving linear transformations of V fixing Q:

$$G = SO\left(p, q\right)$$

and $\mathfrak{g} = \mathfrak{so}\left(p, q\right)$. We have the explicit isomorphism

$$A \in \mathfrak{g} \mapsto A' \in \Lambda^2\left(V^*\right)$$

defined by

$$A'(v, w) = Q\left(Av, w\right).$$

Similarly, $V = V^*$ using Q. Therefore

$$\mathfrak{g} \otimes V^* = V \otimes \Lambda^2\left(V^*\right)$$

as G representations. This ensures that $\mathfrak{g}^{(1)} = 0$ and that $H^{0,2}(\mathfrak{g}) = 0$. Therefore, $\mathfrak{g}^{(k)} = 0$ for all $k > 0$. In particular, there are no prolongations of G-structures: we can stop right away. On the first bundle $B \to M$ of the $SO(p,q)$-structure, we already have a unique torsion-free pseudoconnection, called the *Levi-Civita connection*. Since $\mathfrak{g}^{(k)} = 0$ for $k > 0$, we find

$$C^{a,b}(\mathfrak{g}) = 0$$

for $a > 1$, and consequently, $H^{a,b}(\mathfrak{g}) = 0$ for $a > 1$. We only have to find $H^{a,b}(\mathfrak{g})$ for $a = 0, 1$ and $b = 0, \dots, n = \dim V$.

The cohomology $H^{1,2}(\mathfrak{g})$ is the collection of

$$\rho \in C^{1,2}(\mathfrak{g}) = \mathfrak{g} \otimes \Lambda^2(V^*)$$

so that

$$Q(\rho(x,y)z,w) + Q(\rho(z,x)y,w) + Q(\rho(y,z)x,w) = 0$$

(the first Bianchi identity). Since the prolongation $\mathfrak{g}^{(1)}$ vanishes, we have $B^{(1)} = B$, for B any $SO(p,q)$-structure, and we can write out the prolonged structure equations as equations on B, i.e. our structure equations now look like

$$d\omega = -\gamma \wedge \omega$$
$$d\gamma = -\gamma \wedge \gamma + \frac{1}{2} R\omega \wedge \omega$$

since the prolongation connection 1-form is $\xi = 0$ (because ξ is valued in $\mathfrak{g}^{(1)} = 0$) and the torsion is $T = 0$ (because it is valued in $H^{0,2}(\mathfrak{g}) = 0$). We have written the torsion of the prolongation as

$$R : B \to H^{1,2}(\mathfrak{g})$$

to conform with Riemannian geometry: it is the curvature of the connection γ. To see that γ is a connection, rather than a pseudoconnection:

Lemma 17. *For any Lie group G for which $\mathfrak{g}^{(1)} = 0$, every torsion-free G-structure has a unique torsion-free connection.*

Proof. The choice of torsion-free pseudoconnection γ is unique, since $\mathfrak{g}^{(1)} = 0$. We calculate from $r_g^* \omega = g^{-1} \omega$ that

$$d\omega = -\left(g r_g^* \gamma g^{-1}\right) \wedge \omega.$$

This shows that

$$g r_g^* \gamma g^{-1} = \gamma,$$

by uniqueness of γ, so that γ is a connection.

By applying some well known representation theory of special orthogonal groups, we find that

$$H^{1,2}(\mathfrak{g}) = \mathbb{R} \oplus \operatorname{Sym}_0^2(V) \oplus \operatorname{Weyl}(p,q)$$

where $\operatorname{Sym}_0^2(V)$ means the traceless quadratic forms, which form an irreducible $SO(p,q)$ representation, and $\operatorname{Weyl}(p,q)$ is an $SO(p,q)$ representation known as the Weyl curvature representation. The Weyl curvature representation is irreducible except in case $\dim V = 4$, where it can split depending on p and q. Using the form Q we can write

$$\begin{aligned}
\operatorname{Ricci}(x,y) &= \operatorname{tr} R(x,\cdot)y \in \operatorname{Sym}^2(V^*) \\
s &= \operatorname{tr}_Q \operatorname{Ricci} \in \mathbb{R} \\
R_0 &= \operatorname{Ricci} - \frac{s}{n} Q \in \operatorname{Sym}_0^2(V^*) \\
W(x,y)z &= R(x,y)z \\
&\quad - \frac{1}{n-2}\left(R_0(,x)^* Q(y,z) - R_0(,y)^* Q(x,z) + xR_0(y,z) - yR_0(x,z)\right) \\
&\quad - \frac{s}{n(n-1)}\left(xQ(y,z) - yQ(x,z)\right) \in \operatorname{Weyl}(p,q)
\end{aligned}$$

The dimensions of the various representations are detailed in table 7.1, with $\kappa(p,q)$ the space of values of the Riemann curvature. The structure equations

$n = p + q$	$\dim \operatorname{Sym}_0^2(\mathbb{R}^n)$	$\dim \operatorname{Weyl}(p,q)$	$H^{1,2}(\mathfrak{so}(p,q))$
1	0	0	0
2	0	0	1
3	5	0	6
4	9	10	20
5	14	35	50
6	20	84	105
7	27	168	196
8	35	300	336
9	44	495	540
10	54	770	825
$n > 2$	$\frac{n(n+1)}{2} - 1$	$\frac{(n-3)n(n+1)(n+2)}{12}$	$\frac{(n-1)n^2(n+1)}{12}$

Table 7.1. Dimensions of representations in the decomposition of the Riemann curvature tensor.

can be rewritten via the 1-form

$$\Omega = \begin{pmatrix} \gamma & \omega \\ 0 & 0 \end{pmatrix}$$

(which is valued in $\mathfrak{so}(p,q) \rtimes \mathbb{R}^{p+q}$) as

$$d\Omega + \Omega \wedge \Omega = \begin{pmatrix} \frac{1}{2} R\omega \wedge \omega & 0 \\ 0 & 0 \end{pmatrix}$$

so that when $R = 0$ we see that these are the equations satisfied by the Maurer–Cartan 1-form on the group of rigid motions of pseudo-Euclidean space $M = V$ with quadratic form Q. This exhibits the notion that all $SO(p,q)$-structures are "deformations" of the flat one.

Let us consider which Riemannian geometries have unusually large symmetry groups. If the symmetry group acts transitively on the bundle of orthonormal frames then the invariants R_0 and W must be constant on that bundle (recall that they are functions on B, and tensors on M). But they vary according to an irreducible representation of $SO(p,q)$. Either they vanish or the constant values of R_0 and W span invariant one dimensional subspaces; therefore they vanish. (This argument also holds in dimension 4, even though the Weyl can split, because it has no trivial components). Moreover, the scalar curvature must also be constant. Therefore the structure equations reduce to

$$\begin{aligned} d\omega &= -\gamma \wedge \omega \\ d\gamma &= -\gamma \wedge \gamma + \frac{1}{2} R\omega \wedge \omega \end{aligned}$$

with R constant satisfying $W = 0$, i.e.

$$R(x,y)z = \frac{s}{n(n-1)} \left(xQ(y,z) - yQ(x,z) \right).$$

We can therefore write the structure equations as

$$\begin{aligned} d\omega &= -\gamma \wedge \omega \\ d\gamma &= -\gamma \wedge \gamma + \frac{s}{n(n-1)} \omega \wedge \omega^*. \end{aligned}$$

Packing these into a matrix

$$\Omega = \begin{pmatrix} \gamma & \omega \\ -\kappa\omega^* & 0 \end{pmatrix}$$

(not quite the same choice of Ω we made just above), picking

$$\kappa = \frac{s}{n(n-1)},$$

we find that our structure equations are expressed by

$$d\Omega + \Omega \wedge \Omega = 0.$$

These are the structure equations satisfied by the Maurer–Cartan 1-form of a Lie group, the group of symmetries of this G-structure. If $\kappa = 0$, we have seen that these are the symmetries of pseudo-Euclidean space. If $\kappa > 0$, then the Lie algebra of this group is $\mathfrak{so}\,(p+1,q)$, and up to coverings, each component of the manifold is

$$M = SO\,(p+1,q)\,/SO\,(p,q)\,.$$

We can realize this manifold explicitly: let $V' = \mathbb{R}_x \oplus V$ with quadratic form

$$dx^2 + Q.$$

Then the group $SO\,(p+1,q)$ is the group of symmetries of this quadratic form, and the subgroup $SO\,(p,q)$ is the subgroup preserving the unit vector $(1,0) \in V'$. This vector has length 1, and the group $SO\,(p+1,q)$ acts transitively on such vectors, with isotropy group $SO\,(p,q)$, so M is the "sphere" of vectors of length 1. Consequently, M is an affine hyperquadric.

On the other hand, if $\kappa < 0$ then the Lie algebra is $\mathfrak{so}\,(p,q+1)$, and up to coverings each component of the manifold is

$$M = SO\,(p,q+1)\,/SO\,(p,q)\,.$$

Here, we let $V' = V \oplus \mathbb{R}_y$ with quadratic form

$$Q - dy^2.$$

Then M is the manifold of vectors with length -1, and again an affine hyperquadric.

There are many more homogeneous examples; for instance we can take any Lie group and pick a nondegenerate quadratic form in one of its tangent spaces, and carry the form around the group by left translation. But only the examples given above have symmetry group acting transitively on the manifold of orthonormal coframes.

Lets classify the homogeneous pseudo-Riemannian manifolds of dimensions 1, 2 and 3. Every 1 dimensional manifold is locally isometric to the line, so homogeneous precisely when it is a line or circle. Homogeneous surfaces must have constant Gauß curvature, so must be space forms. A homogeneous 3 dimensional pseudo-Riemannian manifold is one of three types: a three dimensional space form (isometry group of dimension 6), a product of a line or circle with a surface of constant curvature (isometry group of dimension 4), or a 3 dimensional Lie group with left invariant metric (isometry group of dimension 3). The classification of 3 dimensional Lie groups is due to Bianchi, and is well known.

Let us consider how to generalize the classification of space forms above to infinite dimensions on a Hilbert manifold, with Q a positive definite continuous quadratic form. Then we can no longer use the splitting into irreducible representations, because we have no reason to imagine that R is trace class (i.e. has a trace). But we can introduce the *sectional curvature*

$$K(P) = Q(R(x, y)y, x)$$

where $x, y \in V$ are an orthonormal basis for a fixed plane $P \subset V$. Thus the sectional curvature is a function on the bundle B. We can then recover the Riemann curvature tensor from K by a well known trick from Riemannian geometry (see any textbook on the subject), and show that if R is invariant under the orthogonal group $O(Q)$ then R is in fact of the form

$$R(x, y)z = \kappa \left(xQ(y, z) - yQ(x, z)\right)$$

for a constant κ. Hence the same classification of space forms, even in infinite dimensions. The method of equivalence is not often used in infinite dimensions.

7.4 Example: CR 3-manifolds

Picture a smooth real hypersurface M in a complex surface, in other words a submanifold of 3 real dimensions inside an ambient complex manifold of 2 complex (hence 4 real) dimensions. Every tangent space of M is a real 3-plane in a complex 2-plane, and so by easy linear algebra contains a unique complex line. This imposes a CR-structure on M, which means (in this setting) a choice of 2-plane field on M with a complex structure on each 2-plane. This can be expressed as a G-structure, where G is the group of real linear transformations of a 3-dimensional real vector space preserving a 2-plane and a complex structure on that 2-plane, so G consists of the matrices of the form

$$\begin{pmatrix} a^1_1 & -a^2_1 & a^1_3 \\ a^2_1 & a^1_1 & a^2_3 \\ 0 & 0 & a^3_3 \end{pmatrix}$$

with real entries. Lets try a complex notation: write these matrices as complex matrices

$$\begin{pmatrix} a^1_1 & a^1_2 \\ 0 & a^2_2 \end{pmatrix}$$

with a^2_2 being a real number. In this notation, letting $B \to FM$ be G-structure, the structure equations on B are clearly

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} t^1_{1\bar{1}}\omega^{1\bar{1}} + t^1_{12}\omega^{12} + t^1_{\bar{1}2}\omega^{\bar{1}2} \\ \sqrt{-1}t^2_{1\bar{1}}\omega^{1\bar{1}} + t^2_{12}\omega^{12} + t^2_{\bar{1}2}\omega^{\bar{1}2} \end{pmatrix}$$

(writing $\omega^{AB} = \omega^A \wedge \omega^B$), so that ω^1 is complex-valued, while ω^2 is real-valued, and $t^2_{1\bar{1}} \in \mathbb{R}$ and $t^2_{12} = \left(t^2_{\bar{1}2}\right)^*$. But we can absorb torsion into the γ^A_B to get to

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{-1}t\omega^{1\bar{1}} \end{pmatrix}$$

with t real-valued. Calculate that

$$dt = t\left(\gamma_1^1 + \gamma_{\bar{1}}^{\bar{1}} - \gamma_2^2\right) \quad \pmod{\omega^1, \omega^{\bar{1}}, \omega^2}$$

so that the torsion $t \in H^{0,2}(\mathfrak{g})$ transforms in a representation which is not trivial. Returning to the definition of the torsion representation, we can calculate that under right action by $g \in G$,

$$r_g^* t = \frac{\left|g_1^1\right|^2}{g_2^2} t,$$

which we could guess from the equation above for dt. Therefore if t does not vanish anywhere, we can reduce the structure group. We can say that M is *Levi-flat* if $t = 0$ everywhere, and *Levi-pseudoconvex* if $t \neq 0$ everywhere. The geometric meaning of Levi pseudoconvexity is not very clear.

7.4.1 Levi-flat hypersurfaces

The structure equations of a Levi-flat hypersurface are

$$d\begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix} = -\begin{pmatrix}\gamma_1^1 & \gamma_2^1 \\ 0 & \gamma_2^2\end{pmatrix} \wedge \begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix}$$

with no torsion occuring, looking identical for any Levi-flat hypersurfaces. We might guess then that all Levi-flat hypersurfaces are locally equivalent. We will reconsider this question later, in section 8 on page 169. From the structure equations, we can see that $\omega^2 = 0$ is a holonomic equation (i.e. satisfies the conditions of the Frobenius theorem) on the bundle B, and involves only semibasic 1-forms, so that any Levi-flat CR-structure endows the underlying 3-manifold with a foliation by surfaces (the images in M of the leaves of the foliation $\omega^2 = 0$ in B). These surfaces are tangent to the 2-planes of our 2-plane field in the tangent spaces of M, so inherit a $\mathrm{GL}(1, \mathbb{C})$-structure, since we have a complex structure in each of those 2-planes. Recall that we started off this section by thinking about real hypersurfaces in complex surfaces. Levi-flat real hypersurfaces M are then foliated by real surfaces, which we can see are holomorphic curves, since their tangent planes are complex lines in the tangent spaces of M. Conversely, picking any family of complex curves in a complex surface, chosen so that they foliate a 3-manifold in the complex surface, we get a Levi-flat hypersurface.

7.4.2 Levi-pseudoconvex hypersurfaces

Suppose that $t \neq 0$ at every point. Then $t = 1$ is a subbundle $B_1 \subset B$, principal for G_1 the group of matrices of the form

$$\begin{pmatrix} g_1^1 & g_2^1 \\ 0 & \left|g_1^1\right|^2 \end{pmatrix}$$

with complex entries, and $g_1^1 \neq 0$. The structure equations are now:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ 0 & \gamma_2^2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{-1}\omega^{1\bar{1}} \end{pmatrix}$$

On this subbundle, $\gamma_2^2 - \gamma_1^1 - \gamma_{\bar{1}}^{\bar{1}}$ must be semibasic, say

$$\gamma_2^2 = \gamma_1^1 + \gamma_{\bar{1}}^{\bar{1}} + a_1 \omega^1 + a_{\bar{1}} \omega^{\bar{1}} + a_2 \omega^2 + a_{\bar{2}} \omega^{\bar{2}}$$

with $a_{\bar{j}}$ the conjugate of a_j, since γ_2^2 is real valued. We can absorb these a_j (and hence $a_{\bar{j}}$) by redefining γ_1^1 and γ_2^1. The structure equations:

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ 0 & \gamma_1^1 + \gamma_{\bar{1}}^{\bar{1}} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{-1}\omega^{1\bar{1}} \end{pmatrix}.$$

There are no longer any functions appearing here, so we might guess that all Levi pseudoconvex hypersurfaces are equivalent, since nothing has popped up to tell them apart. It will turn out that we simply haven't differentiated enough times to see the local invariants that can tell them apart. (In subsection 8.2 on page 172, we will explain a technique of Cartan to tell us whether or not we really "need" to differentiate to detect local invariants at higher order, and in this example Cartan's technique will tell us that we do.)

7.4.3 The prolongation

The pseudoconnection 1-forms γ_j^i are determined up to

$$\begin{aligned} \gamma_1^1 &\mapsto \gamma_1^1 + a_1 \omega^2 \\ \gamma_2^1 &\mapsto \gamma_2^1 + a_1 \omega^1 + a_2 \omega^2. \end{aligned}$$

Equivalently, we could say that the prolongation $\mathfrak{g}_1^{(1)}$ of the Lie algebra $\mathfrak{g}_1$ consists of the choices of a_1 and a_2. These a_1, a_2 coefficients will parameterize the choices of pseudoconnection, and thus parameterize the structure group $\mathfrak{g}_1^{(1)}$ of the bundle $B_1^{(1)} \to B_1$, the prolongation of the G_1-structure. We can pick a specific choice Γ of pseudoconnection on B_1, and write $\gamma = \Gamma + a\omega$ on $B_1^{(1)} \to B$, so that a^1 and a^2 are thought of as parameterizing the fibers of $B^{(1)} \to B$.

Taking the structure equations for $d\omega$ and performing exterior derivative to both sides, we find

$$
\begin{aligned}
0 = & \left(d \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}} \end{pmatrix} + \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}} \end{pmatrix} \wedge \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}} \end{pmatrix} \right. \\
& \left. + \begin{pmatrix} \sqrt{-1}\gamma^1_2 \wedge \omega^{\bar{1}} + 2\sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} \wedge \omega^1 & 0 \\ 0 & -\sqrt{-1}\gamma^1_2 \wedge \omega^{\bar{1}} + \sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} \wedge \omega^1 \end{pmatrix} \right) \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}
\end{aligned}
$$

from which we can conclude, using Cartan's lemma, that there are 1-forms ξ_1 and ξ_2 so that

$$
\begin{aligned}
0 &= d\gamma^1_1 + \sqrt{-1}\gamma^1_2 \wedge \omega^{\bar{1}} + 2\sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} \wedge \omega^1 + \xi_1 \wedge \omega^2 + R\omega^{\bar{1}} \wedge \omega^1 \\
0 &= d\gamma^1_2 + \gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{1}} + \xi_1 \wedge \omega^1 + \xi_2 \wedge \omega^2
\end{aligned}
$$

for some function R. We can check explicitly that on fibers of $B_1^{(1)} \to B_1$, these forms ξ_1 and ξ_2 satisfy $\xi_j = da_j$. In particular, the ξ_j are pseudoconnection 1-forms for $B_1^{(1)} \to B$.

Taking exterior derivative, we find

$$
dR = 2\sqrt{-1}\left(\xi_{\bar{1}} - \xi_1\right)
$$

modulo ω, γ, i.e. on each fiber of $B_1^{(1)} \to B_1$, R varies according to an affine action of the structure group $\mathfrak{g}_1^{(1)}$. Therefore there is a distinguished subbundle $B_2 \subset B_1^{(1)}$ on which $R = 0$, which is a principal right G_2 bundle, with G_2 the subgroup of $\mathfrak{g}_1^{(1)}$ given by requiring a_1 to be real. On B_2, effectively (i.e. by excusable change of notation) ξ_1 has become real-valued, but there are new torsion terms coming from what was its imaginary part:

$$
\begin{aligned}
0 &= d\gamma^1_1 + \sqrt{-1}\gamma^1_2 \wedge \omega^{\bar{1}} + 2\sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} \wedge \omega^1 + \xi_1 \wedge \omega^2 + \sqrt{-1}\tau \wedge \omega^2 \\
0 &= d\gamma^1_2 + \gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{1}} + \xi_1 \wedge \omega^1 + \xi_2 \wedge \omega^2 + \sqrt{-1}\tau \wedge \omega^1.
\end{aligned}
$$

Absorbing torsion, I can arrange $\tau = T\omega^1 + \bar{T}\omega^{\bar{1}}$. Calculating exterior derivatives determines that

$$
dT = -\frac{3}{2}\xi_2
$$

modulo ω, γ, i.e. on the fibers of $B_2 \to B_1$. Therefore again T varies in an affine action of the structure group, and there is a subbundle $B_3 \subset B_2$ on which $T = 0$, and on that subbundle, ξ_2 becomes semibasic, i.e. new torsion emerges: ξ_2 becomes a multiple of the ω, γ 1-forms. Lets write ξ instead of ξ_1. Taking exterior derivatives, absorbing torsion, we obtain the final structure equations:

$$
\begin{aligned}
d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} &= - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{-1}\omega^{1\bar{1}} \end{pmatrix} \\
d \begin{pmatrix} \gamma^1_1 \\ \gamma^1_2 \end{pmatrix} &= - \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} - \begin{pmatrix} \sqrt{-1}\gamma^1_2 \wedge \omega^{\bar{1}} + 2\sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} \wedge \omega^1 \\ \gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{1}} + s\omega^{\bar{1}} \wedge \omega^2 \end{pmatrix} \\
d\xi &= \left(\gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}}\right) \wedge \xi + \sqrt{-1}\gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{2}} - \left(p\omega^{\bar{1}} + \bar{p}\omega^1\right) \wedge \omega^2
\end{aligned}
$$

for a uniquely determined complex-valued functions s and p. Note that now the exterior derivatives of all of the 1-forms ω, γ, ξ are expressed in terms of those same 1-forms, and some functions. We will see later on that this is precisely the signal to us that the equivalence method can be stopped here.

We can calculate the exterior derivatives of these structure equations to find expressions for ds and dp, which are

$$d \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} \gamma_1^1 + 3\gamma_{\bar{1}}^{\bar{1}} & 0 \\ -\sqrt{-1}\gamma_2^{\bar{1}} & 2\gamma_1^1 + 3\gamma_{\bar{1}}^{\bar{1}} \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix} + \begin{pmatrix} p & \nabla_{\bar{1}} s & \nabla_2 s \\ \nabla_1 p & \nabla_{\bar{1}} p & \nabla_2 p \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^{\bar{1}} \\ \omega^2 \end{pmatrix}.$$

7.4.4 Homogeneous examples

Cartan [22, 23] classified all of the homogeneous cases. Lets just do the simplest one: suppose that the symmetry group acts transitively on the bundle B_3 which we have constructed above. Then the invariants s and p must be constants, since they are invariant. From the equations above for s and p, being constant forces them both to vanish. Plugging in $s = p = 0$ to the structure equations gives

$$\begin{aligned} d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} &= - \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ 0 & \gamma_1^1 + \gamma_{\bar{1}}^{\bar{1}} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{-1}\omega^{1\bar{1}} \end{pmatrix} \\ d \begin{pmatrix} \gamma_1^1 \\ \gamma_2^1 \end{pmatrix} &= - \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} - \begin{pmatrix} \sqrt{-1}\gamma_2^1 \wedge \omega^{\bar{1}} + 2\sqrt{-1}\gamma_2^{\bar{1}} \wedge \omega^1 \\ \gamma_2^1 \wedge \gamma_{\bar{1}}^{\bar{1}} \end{pmatrix} \\ d\xi &= \left(\gamma_1^1 + \gamma_{\bar{1}}^{\bar{1}}\right) \wedge \xi + \sqrt{-1}\gamma_2^1 \wedge \gamma_2^{\bar{1}} \end{aligned}$$

Exercise 7.2 We don't have any room left: there can only be one example, at least locally. Prove this by defining a developing map, analogous to that for a manifold with connection as discussed in section 2.7 on page 22.

To learn the identity of this example, see subsection 8.6 on page 189. See Jacobowitz [48], for more information on CR-manifolds in all dimensions, including an excellent explanation of the equivalence method, and proof that the generic Levi pseudoconvex CR 3-manifold is not the boundary of a domain in $\mathbb{C}^2$.

7.5 Higher prolongations

We call a Lie algebra $\mathfrak{g} \subset \operatorname{End}(V)$ of linear transformations of a finite dimensional vector space V *finite type* if the sequence of prolongations $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \dots$ has only finitely many nonzero terms. (See the appendix on Spencer cohomology for the definition of prolongations.)

If the Lie algebra $\mathfrak{g}$ of our group G has finite type, then eventually we will arrive at a prolongation $B^{(k)}$ for which there is an invariant choice of coframing. Be careful: the choice of which constant type our structure has at each stage will decide which abelian groups occur as the structure groups of each prolongation, since we will have to reduce structure group before each prolongation. But the structure groups will be subgroups of the $\mathfrak{g}^{(k)}$ (i.e. vector subspaces), and consequently finite type of $\mathfrak{g}$ implies that the sequence of prolongations will stop. If the torsion of each prolongation up to $B^{(k-1)}$ vanishes, then the torsion of the k-th prolongation lies in the Spencer cohomology group $H^{k,2}(\mathfrak{g})$.

Remark 12 (How far do we have to go?). By finiteness of Spencer cohomology (see the appendix) there are only finitely many nonzero torsion representations $H^{k,2}(\mathfrak{g})$ (whether G has finite type or not); finitely many obstructions to flatness that emerge from the method of equivalence.

Remark 13 (Not very far). There are no important examples of higher order finite type structures beyond $B^{(1)}$. This is perhaps explained by the absence of irreducible homogeneous spaces of higher than second order.

Exercise 7.3 If $M = H/H_0$ is a homogeneous space, let $N \subset H_0$ be a normal subgroup of H. Show that $M = (H/N)/(H_0/N)$.

Recall that a homogeneous space $M = H/H_0$ can be regarded as having an H_0-structure. We do this by taking a point $m_0 \in M$ with stabilizer H_0, and a point $u_0 \in F_{m_0}M$, and mapping $h \in H \mapsto hu_0 \in FM$. Henceforth, suppose that H_0 contains no normal subgroup of H. If H_0 is discrete, we say that the homogeneous space M has order 0. If not, then we let H_1 be the subgroup of H_0 acting trivially on $\mathfrak{h}/\mathfrak{h}_0$ (which we can identify with the tangent space $T_{m_0}M$). Our map $H \to FM$ descends to a map $H/H_1 \to FM$. We can view H/H_1 as our H_0/H_1-structure, and take its prolongation.

Exercise 7.4 Let $H_2 \subset H_1$ be the elements of H_1 which act trivially on $T_{u_0}FM$. Show how to map $H/H_2 \to (H/H_1)^{(1)}$.

In this way, we proceed inductively, and say that $M = H/H_0$ has order k if H_k is discrete.

Exercise 7.5 Show that if the dimension of H_1 equals that of H_0, then the identity component of H_0 is a normal subgroup of H.

Theorem 7 (Cartan [19], Kobayashi & Nagano [54]). *A homogeneous space $M = H/H_0$ is called* irreducible *if H_0/H_1 acts irreducibly on $\mathfrak{h}/\mathfrak{h}_0$. Every irreducible homogeneous space has order at most 2.*

7.6 Cartan geometries, homogeneity and completeness

7.6.1 Definition of Cartan geometries

It is occasionally useful to generalize the concept of finite type G-structure to:

Definition 26. *A* Cartan geometry *on a principal right G-bundle $B \to M$ is a Lie algebra valued coframing $\Omega \in \Omega^1(B) \otimes \mathfrak{h}$ so that*

1. *the Lie algebra $\mathfrak{g}$ of G is a Lie subalgebra of $\mathfrak{h}$ and*
2. *G occurs as a closed subgroup of some Lie group H with Lie algebra $\mathfrak{h}$ and*
3. *for each $A \in \mathfrak{g}$, the vector field $\vec{A}$ on B generating the G action, i.e.*

$$\vec{A}(u) = \left.\frac{d}{dt}\right|_{t=0} ue^{-tA},$$

   satisfies $\vec{A} \lrcorner \Omega = A$ and
4. *under G action,*

$$r_g^* \Omega = \operatorname{Ad}_g^{-1} \Omega$$

Call M the base space*, B the* bundle*, G the* structure group*, $\mathfrak{h}$ the* principal Lie algebra*, Ω the* Cartan connection*. For each $A \in \mathfrak{h}$, define the vector field $\vec{A}$ on B by $\vec{A} \lrcorner \Omega = A$. Say that the Cartan geometry is* complete *if all of the $\vec{A}$ vector fields are complete. The* curvature *of a Cartan geometry is the function $\kappa : B \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{h}^*)$ defined by*

$$d\Omega = -\frac{1}{2}[\Omega, \Omega] + \kappa \Omega \wedge \Omega.$$

We call a Cartan geometry flat *if its curvature vanishes.*

Sharpe [76, 77] is the standard reference on Cartan connections.

Remark 14. DANGER: a Cartan connection is *not* a connection!

Exercise 7.6 Check that the vector fields $\vec{A}$ form a Lie algebra just when the curvature is constant.

Exercise 7.7 (Surfaces with Riemannian metric) A surface with Riemannian metric has structure equations

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}0 & \gamma\\ -\gamma & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$
$$d\gamma = K\omega^1 \wedge \omega^2.$$

Check that the form

$$\Omega = \begin{pmatrix} 0 & \gamma & \omega^1 \\ -\gamma & 0 & \omega^2 \\ -K_0\omega^1 & -K_0\omega^2 & 0 \end{pmatrix}$$

for K_0 any constant is a Cartan geometry on the orthonormal frame bundle. Calculate the curvature.

Example 57. Prolong any G-structure B enough times, and if if at each stage one can choose an equivariant section for the torsion, and if finally the prolongation stops, because $\mathfrak{g}^{(k)} = 0$, then on the bundle $B^{(k)} \to M$, its soldering and connection forms give a Cartan geometry.

Definition 27. *A* locally Klein geometry *is a choice of Lie group H, closed subgroup G and discrete subgroup Γ so that Γ acts freely and properly on the left on H/G. The associated Cartan geometry is: B is the quotient $B = \Gamma\backslash H$, the principal Lie algebra $\mathfrak{h}$ is the Lie algebra of H, $M = \Gamma\backslash H/G$, and Ω is the Maurer–Cartan 1-form of H. It is called a* Klein geometry *if $\Gamma = 1$. To each Cartan geometry with principal Lie group H and structure group G, the Klein geometry H/G is called its* model.

This definition is due to Sharpe [76], chapter 4. Note that the definition includes a choice of group H, but only the Lie algebra of H plays a substantial role. This leads to annoying subtleties.

7.6.2 Example: Levi pseudoconvex CR 3-manifolds

We follow Robert Bryant [15].

Exercise 7.8 The structure equations from subsection 7.4.2 on page 112 can be organized into 1-form

$$\Omega = \begin{pmatrix} -\frac{1}{3}\left(2\gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}}\right) & -\sqrt{-1}\gamma^{\bar{1}}_{\bar{2}} & -\sqrt{-1}\xi \\ \omega^1 & \frac{1}{3}\left(\gamma^1_1 - \gamma^{\bar{1}}_{\bar{1}}\right) & \sqrt{-1}\gamma^1_2 \\ -\sqrt{-1}\omega^2 & \omega^{\bar{1}} & \frac{1}{3}\left(\gamma^1_1 + 2\gamma^{\bar{1}}_{\bar{1}}\right) \end{pmatrix}$$

(which is a Cartan connection on the bundle $B_3 \to M$) and if we write

$$\nabla\Omega = d\Omega + \frac{1}{2}[\Omega, \Omega]$$

then

$$\nabla\Omega = \begin{pmatrix} 0 & -\sqrt{-1}\bar{s}\omega^1 \wedge \omega^2 & \sqrt{-1}\left(p\omega^{\bar{1}} \wedge +\bar{p}\omega^1\right) \wedge \omega^2 \\ 0 & 0 & \sqrt{-1}s\omega^{\bar{1}} \wedge \omega^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The reader is naturally curious as to how we came up with Ω; this comes from the study of the CR-geometries with largest symmetry group. The general formalism will be explained in subsection 8.5.1 on page 182.

7.6.3 Flat Cartan geometries

Exercise 7.9 (Surfaces with Riemannian metric) Continuing exercise 7.7 on page 117, if the Cartan geometry is a locally Klein geometry, show that $K = K_0$ constant Gauss curvature. Give an example with $K = K_0$ which is not locally Klein. For those which are locally Klein, rescale the metric to get $K_0 = -1, 0$ or 1, and then show that the principal Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{so}(2) \rtimes \mathbb{R}^2$ or $\mathfrak{so}(3)$.

Lemma 18. *A locally Klein geometry is a complete flat Cartan geometry.*

Proof. You need to show that $\Gamma \backslash H \to \Gamma \backslash H/G$ is a principal right G bundle. See Sharpe [76], pp.154–156 for details.

Definition 28. *If $M_0 \to M_1$ is a local diffeomorphism, and Ω_1 is a Cartan connection on a bundle $B_1 \to M_1$, the* pullback *Cartan geometry is the pulled back form on the pullback bundle.*

Lemma 19. *Under a covering map $M_0 \to M_1$, the pullback of a Cartan geometry is complete just when the original is complete.*

Proof. The bundles $B_0 \to B_1$ are covering spaces, and the vector fields $\vec{A}$ are matched up by the covering map.

Definition 29. *A* double coset space *is a space $\Gamma \backslash H/G$, where $G \subset H$ is a closed subgroup, and $\Gamma \subset H$ is a countable subgroup, perhaps not acting freely or properly on H/G.*

Example 58. Take $H = SO(2) \rtimes \mathbb{R}^2$, $G = SO(2)$ and $\Gamma = \pm 1 \subset G$. Then $\Gamma \backslash H/G$ is the plane with points x and $-x$ identified, a cone with a sharp point, a double coset space but not a locally Klein geometry.

Exercise 7.10 Give other examples of double coset spaces which are not locally Klein geometries.

Definition 30. *If $\Gamma \backslash H/G$ is a double coset space, and Γ is endowed with a free and proper discontinuous action on a manifold M, commuting with a local diffeomorphism $M \to H/G$, define the* quotient *$\Gamma \backslash M$ the Cartan geometry on the quotient bundle of the pullback bundle.*

Example 59. Returning to the previous example, take $\tilde{M}$ the plane with origin removed, and M the cone with the sharp point removed.

Theorem 8. *Every flat Cartan geometry is a quotient $M = \Gamma \backslash \tilde{M}$ of a pullback $\tilde{M} \to H/G$.*

Proof. Take $B \to M$ the bundle, Ω the Cartan connection. Let H be any Lie group with Lie algebra $\mathfrak{h}$ containing G as a closed subgroup. Let Ω_0 be the left invariant Maurer–Cartan form of H. Let H act on $B \times H$ on the left, by leaving B alone and acting on H by left translation. Let G act on $B \times H$ by acting diagonally on the right. On $B \times H$, the equation $\Omega = \Omega_0$ is holonomic, so by the Frobenius theorem, $B \times H$ is foliated by its leaves. Both the H and G actions preserve the equation $\Omega = \Omega_0$, so both permute the leaves. Define a G-leaf to be the G-orbit of a maximal leaf. Each leaf must be invariant under the vector fields $\vec{A} \oplus \vec{A}$, since these satisfy $\Omega = \Omega_0$. Therefore each leaf must be invariant under the identity component of G, and each G-leaf $\tilde{B}$ is a union of leaves, parameterized by the topological components of G. Thus a G-leaf is an immersed submanifold, and there is a unique G-leaf through each point of $B \times H$. Any two G-leaves are permuted by H action. Each G-leaf is acted on freely and properly by G, since B and H are. So each G-leaf is a smooth principal right G-bundle $\tilde{B} \to \tilde{M}$ over its quotient by G-action.

Pick a G-leaf $\tilde{B}$. Lets see why $\tilde{B} \to B$ is onto. Suppose that $u' \in B$ is in the boundary of the image of $\tilde{B} \to B$. Pick any other G-leaf $\tilde{B}'$, passing through some point $(u', h') \in B \times H$. Then the map $\tilde{B}' \to B$ is a local diffeomorphism near (u', h'). Therefore it must hit some point $u \in B$ in the range of of $\tilde{B} \to B$, say $(u, h) \in \tilde{B}$. But then some H translate of $\tilde{B}'$ must intersect $\tilde{B}$, and therefore equal $\tilde{B}$. Therefore $\tilde{B} \to B$ is onto, and thus $\tilde{M} \to M$ is onto.

Pick a point $u_0 \in B$. Consider the map $\mathfrak{h} \to B$ given by $A \mapsto e^{\vec{A}} u_0$. This map might not be defined globally, but is a coordinate chart in some open neighborhood $W \subset \mathfrak{h}$ of the origin. Perhaps replacing W by a smaller open neighborhood of the origin, we can arrange that $A \in W \mapsto e^A \in H$ is also a coordinate chart. Take $W_B \subset B$ the image of W. Let $W_{\tilde{B}}$ be the preimage of W_B via $\tilde{B} \to B$. Let $u_0 \times F$ be the preimage in $\tilde{B}$ of u_0, $F \subset H$ a discrete subset (not empty). An explicit diffeomorphism:

$$(A, h) \in W \times F \mapsto \left(e^{\vec{A}} u_0, e^A h \right) \in W_{\tilde{B}}.$$

shows that $\tilde{B} \to B$ is a covering map.

Let $\tilde{u}_0 = (u_0, 1) \in \tilde{B}$. Given a smooth path in B starting at u_0, lift it to a path in $\tilde{B}$, starting at $\tilde{u}_0$, by requiring $\Omega = \Omega_0$. The path, say $u(t)$, gives rise to a Lie equation $u'(t) \lrcorner \Omega = h'(t) \lrcorner \Omega_0$ for a path $h(t) \in H$. We can solve Lie equations locally using short time existence of solutions of ordinary differential equations, and globally by left invariant patching of local solutions. Alternatively, we can just lift paths via the observation that $\tilde{B} \to B$ is a covering map. In particular, we can lift loops $u(t)$ to paths $(u(t), h(t)) \in \tilde{B}$, and then map to $h(1)$. This defines a map taking loops based at u_0 to elements of H. It is easy to see that this map is homotopy invariant, so takes $\pi_1(B) \to H$. Moreover, given two loops $u_1(t)$ and $u_2(t)$, we join them together to form a loop $u(t)$, and find that $u(t)$ gives rise to the path $h(t)$ which is $h_1(t)$ for $0 \le t \le 1$ and $h_1(1) h_2(t-1)$ for $1 \le t \le 2$, by left translation. Therefore $\pi_1(B) \to H$ is a homomorphism. Clearly it vanishes precisely on

$\pi_1\left(\tilde{B}\right)$. So $\tilde{B} \to B$ is a normal covering map. $\Gamma = \pi_1(B)/\pi_1\left(\tilde{B}\right) \subset H$ acts on $\tilde{B}$, as a subgroup of H acting on the left, so commuting with G acting on the right. Therefore the action of Γ descends to an action on $\tilde{M}$. Check that Γ acts freely. To see that Γ acts properly, follow Sharpe [76] lemma 3.12, p. 154. Therefore $\tilde{M} \to M$ is a normal covering map. Because the map $\tilde{B} \to B$ is a G-bundle morphism, $\pi_1(B) \to H$ vanishes on the image of $\pi_1(G) \to \pi_1(B)$, so descends to a map $\pi_1(M) \to H$. Moreover, one can easily check that its kernel is precisely $\pi_1\left(\tilde{M}\right)$, so $\Gamma = \pi_1(M)/\pi_1\left(\tilde{M}\right)$.

Question 8. I think there is a problem here: I think that Γ is not a subgroup of G. In fact, if you look carefully, it looks like in the case of $S^n \to \mathbb{RP}^n$ covering, the group of deck transformations ± 1 is mapped to $1 \in G = \mathbb{PGL}\left(\mathbb{RP}^n\right)$.

Remark 15. The proof also shows that Γ is containing in the identity component of H.

7.6.4 Complete flat Cartan geometries

Lemma 20. *A Cartan geometry with connected base space is complete and flat just when it is a locally Klein geometry, possibly for some Lie group H' with the same Lie algebra $\mathfrak{h}$ as the principal group H, and which also contains G as a closed subgroup, with the same Lie algebra inclusion $\mathfrak{g} \subset \mathfrak{h}$, with H'/G connected.*

Proof. This generalizes proposition 15 on page 81 and has the same proof, and is proven in Sharpe [76].

Definition 31. *The* kernel *N of a Cartan geometry modelled on H/G is the largest subgroup of G which is normal in H. We will say that a Cartan geometry is* effective *if $N = 1$. We will say it is* locally effective *if N is discrete.*

Example 60 (Reducing by the kernel). Given a Klein geometry H/G, construct the Klein geometry $G/N \subset H/N$, which has the same base manifold $(H/N)/(G/N) = H/G$. Similarly for a double coset space, $\Gamma \backslash H/G = (\Gamma/(\Gamma \cap N)) \backslash (H/N)/(G/N)$.

Exercise 7.11 Show that the kernel N of a Klein geometry $G \subset H$ is a closed subgroup of G, and

$$N = \bigcap_{h \in H} hGh^{-1}.$$

Definition 32. *Let $\Gamma \backslash H/G$ (with $\Gamma, G \subset H$) be a locally Klein geometry. The* associated effective *locally Klein geometry is $\Gamma' \backslash H'/G'$ where $\Gamma' = \Gamma/(\Gamma \cap N)$, $H' = H/N$ and $G' = G/N$.*

Definition 33. *If Ω on $B \to M$ is a Cartan geometry, then $\Omega + \mathfrak{n}$, $B/N \to M$ is the* associated effective *Cartan geometry; where $\mathfrak{n}$ is the Lie algebra of the kernel N.*

Lemma 21. *A Cartan geometry is complete just when its associated effective Cartan geometry is complete.*

Proof. The 1-form $\Omega + \mathfrak{n}$ is invariant under N action, and vanishes on the fibers of $B \to B/N$, so is defined on B/N, and one easily checks that it is a Cartan connection. A vector field $\vec{A}$ up on B will map to $\vec{A}$ on B/N, intertwining the flows. Therefore if $\vec{A}$ is complete upstairs, then it is complete downstairs. If complete downstairs, but not upstairs, there must be some point $u \in B$ and time T so that the flow of some $\vec{A}$ through u is defined on B for times up to but not equal to T. (Switching $\vec{A}$ and $-\vec{A}$, we can assume $T > 0$.) Downstairs, the flow is defined at time T, so we can look at the corresponding point $\bar{u}$ downstairs and watch its flow even past time T. The problem is clearly local in B/N, so we can assume that $B \to B/N$ is a trivial bundle, $B = M \times G \to B/N = M \times G/N$. Write the quotient by N as $g \in G \mapsto \bar{g} in G/N$. Upstairs the vector field $\vec{A}$ must be

$$\vec{A}(m, g) = (X(m, \bar{g}), L_{g*} A_0(m)) ,$$

for some $A_0 : M \to \mathfrak{g}$. Downstairs it is

$$\vec{A}(m, \bar{g}) = \left(X(m, \bar{g}), L_{\bar{g}*} \bar{A}_0(m)\right) .$$

We get a flow line upstairs $(m(t), g(t))$, so that $g(t)$ does not converge as $t \to T$, but $(m(t), \bar{g}(t))$ converges. Thus we have a smooth function $A_0\,(m(t))$ defined for t even bigger than T, and the equation upstairs

$$\frac{dg}{dt} = L_{g(t)*} A_0\,(m(t)) .$$

But this ordinary differential equation must have a solution for times even bigger than T, because it is a Lie equation, so we can solve it locally on the group G, and then patch local solutions together by left translation.

Lemma 22. *Every infinitesimal symmetry of a complete Cartan connection is a complete vector field; the Lie algebra of infinitesimal symmetries is the Lie algebra of the Lie group of symmetries.*

Proof. The vector fields $\vec{A}$ commute with infinitesimal symmetries X. If the Cartan connection is complete, then these $\vec{A}$ will permute the integral curves of X, so the amount of time we can flow along X is independent of which point we start at. But we can also take any piece of an integral curve of X, and just slide its initial point to its final point by flowing along finitely many of the $\vec{A}$ vector fields, each for a finite time, by Sussmann's orbit theorem

3. Therefore however long we can flow along X, we can always flow twice as long as that. This ensures completeness of all infinitesimal symmetries, and the rest follows by Palais's result 6 on page 73, combined with the results in corollary 14 on page 177 which will show us that the symmetry group is a finite dimensional Lie group.

Question 9. I should point out somewhere below that the symmetry group of a Cartan connection is a finite dimensional Lie group, and change this last lemma to point to that spot.

For example, the Killing vector fields (infinitesimal symmetries) of a complete Riemannian manifold are complete vector fields.

Lemma 23. *A flat Cartan connection on a compact, connected base manifold with finite fundamental group is a locally Klein geometry.*

Proof. Let $\tilde{M} \to M$ be the covering map of theorem 8 on page 119. Because M has finite fundamental group, so does $\tilde{M}$, and therefore $\tilde{M} \to M$ is a finite covering, so $\tilde{M}$ is compact. By compactness of $\tilde{M}$, $\tilde{M} \to H/G$ is a covering map (to its image–it need not be onto). Therefore $B \leftarrow \tilde{B} \to H$ are covering maps (to their images). So Ω is a complete Cartan connection.

Lemma 24. *Every compact manifold M which is the base of a flat Cartan geometry modelled on H/G either (1) has fundamental group $\pi_1(M)$ yielding to the identity component of H/N (where N is the kernel) or (2) is a locally Klein geometry $M = \Gamma\backslash\tilde{H}/G$, where $\tilde{H}$ is a finite covering group of H containing G and Γ, and $\Gamma \subset H$ is finite. In particular, either $n < 4$ or infinitely many compact manifolds of dimension n bear no flat Cartan connection.*

Proof. We can quotient out the kernel, so assume $N = 1$. By Levi decomposition (see Onishchik & Vinberg [69] p. 284), every Lie group H admits a covering group which splits into a semidirect product of simple factors with a solvable factor. No simple group can live in the solvable factor, so simple subgroups must live in the simple factors. Looking at the lowest degree representations of various finitely presented simple groups (alternating groups, for instance), compared to the lowest degree representations of the simple factors of our Lie group, we can find lots of finitely presented simple groups which are not faithfully represented in any of these simple factors. Moreover, the kernel of a covering map between Lie groups is abelian, so there are lots of finitely presented simple groups, both finite and infinite, which defy H. Their free products also defy H.

Every finitely presented group is the fundamental group of some compact manifold of any dimension 4 or more (Massey [61] p. 143); take M any compact manifold with fundamental group defying H. Suppose that M bears a flat Cartan connection. As in the previous proof, some finite normal covering of M, say $\tilde{M}$, is a compact covering of H/G. Therefore $\pi_1(H/G)$ must share a finite index normal subgroup with $\pi_1(M)$.

If we start with M having finite fundamental group defying H, then we find that $\pi_1(H/G)$ must be finite as well. But if we start with M having infinite fundamental group defying H, then $\pi_1(H/G)$ must be infinite as well.

Remark 16. This approach is not very explicit, but it effortlessly demonstrates the existence of counterexamples.

Exercise 7.12 Give an example of an incomplete flat Cartan connection on a compact base manifold, and one on a simply connected noncompact base manifold.

7.6.5 The tangent bundle in a Cartan geometry

Lemma 25. *Given a Cartan connection Ω on $B \to M$, let $V = \mathfrak{h}/\mathfrak{g}$, and get G to act on V via the representation on $\mathfrak{h}$. Consider the diagonal right G action:*

$$r_g(u,v) = \left(ug, \rho(g)^{-1}v\right).$$

Then $TM = (B \times V)/G$.

Proof. We follow Sharpe [76], p. 163. Write the bundle map as $\pi : B \to M$. Define $\omega \in \Omega^1(B) \otimes V$ by $\omega = \Omega \pmod{\mathfrak{g}}$. Define a map

$$(u,v) \in B \times V \mapsto \pi'(u)\bar{\Omega}^{-1}v \in TM.$$

Check G-invariance, so that the map is defined on the quotient $(B \times V)/G$, which is a vector bundle over M. Nonzero vectors remain nonzero under the map, so it is an injection, and clearly of full rank, so a vector bundle isomorphism.

Exercise 7.13 Show that the curvature form $\nabla\Omega = \kappa\Omega \wedge \Omega$ is semibasic, i.e. vanishes on the fibers of $B \to M$. Define $\kappa\bar{\Omega} \wedge \bar{\Omega}$. Show that the curvature represents a section of the vector bundle $\left(B \times \mathfrak{h} \otimes \Lambda^2(V^*)\right/ G \to M$.

7.7 Example: conformal geometry again

Take $\langle,\rangle$ a nondegenerate quadratic form of signature p,q on a vector space V. Recall from section 4.2 on page 29 that as $CO(p,q)$ representations

$$\mathfrak{co}(\mathfrak{p},\mathfrak{q})^{(1)} = V^*$$

and $\mathfrak{co}(p,q)^{(1)} = 0$.

7.7.1 Surfaces

The prolongations of conformal structures on surfaces are more easily understood by treating such structures as complex structures (at least after choosing an orientation, perhaps on a covering space). One simply complexifies the general linear group example, using only complex linear 1-forms. We see identical structure equations, with all Spencer cohomology vanishing, so no local invariants emerge at any prolongation. Indeed, in example 67 on page 171 Cartan's count will tell us that real analytic conformal structures on surfaces are flat and have infinite dimensional local symmetry pseudogroup, so no local invariants. Moreover, by the Newlander–Nirenberg theorem (see [66],[44],[72]) the same result is true for continuously differentiable conformal structures on surfaces.

7.7.2 Higher dimensions

We will follow Cartan [21].

Proposition 23. *If* $\dim V \geq 3$ *then* $\mathfrak{co}\left(p,q\right)^{(k)} = 0$ *for all* $k \geq 2$*. Moreover,*

$$H^{1,2}\left(\mathfrak{co}\left(p,q\right)\right) = \operatorname{Weyl}\left(p,q\right)$$

is identified with the Weyl curvature under the map

$$H^{1,2}\left(\mathfrak{so}\left(p,q\right)\right) \to H^{1,2}\left(\mathfrak{co}\left(p,q\right)\right)$$

induced by the inclusion

$$\mathfrak{so}\left(p,q\right) \subset \mathfrak{co}\left(p,q\right).$$

Proof. Suppose that $\dim V \geq 3$. Take any $\eta \in \mathfrak{co}\left(p,q\right)^{(2)}$. So $\eta \in \operatorname{Sym}^3\left(V^*\right) \otimes V$ and for any $x \in V$ we have

$$\eta(x,\cdot,\cdot) \in \mathfrak{co}\left(p,q\right)^{(1)}.$$

Therefore (in the notation we defined in our previous discussion of conformal geometry)

$$\eta(x,\cdot,\cdot) = \lambda'_x$$

for some $\lambda_x \in V^*$, i.e.

$$\begin{aligned}\eta(x,y,z) &= \lambda'_x(y,z) \\ &= \lambda_x(y)z + \lambda_x(z)y - \lambda_x^* \left\langle y,z\right\rangle \\ &= \lambda_y(x)z + \lambda_y(x)z - \lambda_y^* \left\langle x,y\right\rangle\end{aligned}$$

(by symmetry in x,y). Now pick x,y,z orthogonal and linearly independent. We find $\lambda_x(z) = 0$. Therefore for any pair of vectors $x,z \in V$, if $\langle x,z\rangle = 0$ then $\lambda_x(z) = 0$. This implies that

$$\lambda_x(z) = \mu \langle x, z \rangle$$

for some fixed $\mu \in \mathbb{R}$ i.e. for arbitrary x, y, z we have

$$\begin{aligned} \eta(x,y,z) &= \lambda_x(y)z + \lambda_x(z)y - \lambda_x^* \langle y, z \rangle \\ &= \mu \langle x, y \rangle z + \mu \langle x, z \rangle y - \mu x \langle y, z \rangle \end{aligned}$$

and this is not symmetric in x, y, z unless $\lambda = 0$. Therefore $\mathfrak{co}\,(p,q)^{(2)} = 0$.

Let $\mathfrak{g} = \mathfrak{co}\,(p,q)$. From the exact sequence

$$0 \longrightarrow 0 = \mathfrak{g}^{(2)} \longrightarrow \mathfrak{g}^{(1)} \otimes V^* \xrightarrow{\delta} \mathfrak{g} \otimes \Lambda^2 (V^*) \xrightarrow{[]} H^{1,2}(\mathfrak{g}) \longrightarrow 0$$

we have

$$\delta : \mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 (V^*)$$

an injection. Indeed we can write elements of $\mathfrak{g}^{(1)} \otimes V^*$ as

$$\lambda'(x,y,z) = \lambda(x,z)y + \lambda(y,z)x - \lambda^*(z) \langle x, y \rangle$$

for some $\lambda \in V^* \otimes V^*$, with the notation $\lambda^*(z)$ defined by

$$\langle \lambda^*(z), w \rangle = \lambda(w, z).$$

Calculating δ we find

$$\begin{aligned} \langle \delta\lambda'(x,y,z), w \rangle = \; & \lambda(x,z) \langle y, w \rangle + \lambda(w,y) \langle x, z \rangle \\ & + (\lambda(y,z) - \lambda(z,y)) \langle x, w \rangle \\ & - \lambda(w,z) \langle x, y \rangle - \lambda(x,y) \langle w, z \rangle . \end{aligned}$$

Pick $w = y$ and x, y, z orthogonal.

$$\langle \delta\lambda'(x,y,z), y \rangle = \lambda(x,z) \langle y, y \rangle$$

so this vanishes only for $\lambda = 0$. Therefore, we see explicitly that

$$\delta : \mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 (V^*)$$

is an injection.

In order to investigate

$$H^{1,2}(\mathfrak{g}) = \frac{\ker \delta : \mathfrak{g} \otimes \Lambda^2 (V^*) \to V \otimes \Lambda^3 (V^*)}{\operatorname{im} \delta : \mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 (V^*)}$$

we need now to study the numerator. For

$$\eta \in \mathfrak{g} \otimes \Lambda^2 (V^*)$$

define

$$T\eta \in V^* \otimes V^*$$

by

$$T\eta\,(x,y) = \operatorname{tr}\eta\,(x,\cdot,y)\,.$$

Calculate

$$T\delta\lambda'\,(x,y) = -\lambda(x,y) + \langle x,y\rangle \operatorname{tr}\lambda^*.$$

Obviously, if $\operatorname{tr}\lambda^* = 0$, this recovers λ. More generally, we can always recover λ from $\delta\lambda'$ using a complicated manipulation involving T. Explicitly, if we write

$$\Phi\lambda = -T\delta\lambda'$$

then

$$\lambda = \left(\frac{1}{n-1}\Phi^2 + \frac{n-2}{n-1}\Phi\right)\lambda.$$

Therefore, we can recover λ from $\delta\lambda$ explicitly. Consequently, for each $\eta \in \mathfrak{g}\otimes\Lambda^2\,(V^*)$ we can replace it by a unique $\eta + \delta\lambda'$ so that $T\eta = 0$. Therefore we can identify

$$H^{1,2}\,(\mathfrak{co}\,(p,q)) \cong \ker T \cap \ker\delta \subset C^{1,2}\,(\mathfrak{co}\,(p,q)) \cap \ker\delta.$$

We can identify this as an $\mathfrak{so}\,(p,q)$ representation with the Weyl curvature $\operatorname{Weyl}(p,q)$ under the map

$$H^{1,2}\,(\mathfrak{so}\,(p,q)) \to H^{1,2}\,(\mathfrak{co}\,(p,q))$$

coming from the inclusion

$$\mathfrak{so}\,(p,q) \subset \mathfrak{co}\,(p,q)\,.$$

To prove this, we simply consider the operators δ and T which impose the first Bianchi identity, and wipe out the other components of $C^{1,2}\,(\mathfrak{co}\,(p,q))$.

Note that the Weyl curvature representation is $\operatorname{Weyl}(p,q) = 0$ if $\dim V = 3$. Moreover, if $\dim V = 4$, it splits into subrepresentations, while in all higher dimensions it is an irreducible $SO\,(p,q)$ and $CO\,(p,q)$ representation.

Corollary 11. *Suppose again that* $\dim V \geq 3$. *By the vanishing of* $\mathfrak{g}^{(k)}$ *for* $k > 1$ *we see that all higher torsion representations*

$$H^{k,2}\,(\mathfrak{co}\,(p,q)) = 0$$

vanish for $k > 2$. *Therefore, we have only to look at* $H^{2,2}\,(\mathfrak{co}\,(p,q))$.

Proposition 24. *If* $\dim V \geq 4$ *then*

$$H^{2,2}\,(\mathfrak{co}\,(p,q)) = 0.$$

Proof. The cohomology group is

$$H^{2,2}(\mathfrak{co}(p,q)) = \frac{\ker \delta : \mathfrak{g}^{(1)} \otimes \Lambda^2(V^*) \to \mathfrak{g} \otimes \Lambda^3(V^*)}{\operatorname{im} \delta : \mathfrak{g}^{(2)} \otimes V^* \to \mathfrak{g}^{(1)} \otimes \Lambda^2(V^*)}.$$

The denominator vanishes, since $\mathfrak{g}^{(2)} = 0$. If we have

$$\eta \in \mathfrak{g}^{(1)} \otimes \Lambda^2(V^*)$$

then

$$\delta\eta(x,y,z,w) = \eta(x,y,z,w) + \eta(x,w,y,z) + \eta(x,z,y,w).$$

Essentially as before,

$$\eta(x,y,z,w) = \lambda(x,z,w)y + \lambda(y,z,w)x - \lambda^*(z,w)\langle x,y\rangle$$

for some $\lambda \in V^* \otimes \Lambda^2(V^*)$. Calculate $\delta\eta$ and take x,y,z,w with x perpendicular to y,z,w and y independent of x,z,w. You find

$$\lambda(x,z,w) = 0$$

so that as long as $\dim V \geq 4$ we can ensure that $\lambda(x,z,w) = 0$ whenever x is perpendicular to z and w. Therefore there is some $\vartheta \in V^*$ so that

$$\lambda(x,y,z) = \langle x,y\rangle \vartheta(z) - \langle x,z\rangle \vartheta(y).$$

Now calculate

$$\langle \delta\eta(x,y,z,w), t\rangle$$

plugging in this ϑ. If $\vartheta \neq 0$ then plug in x,y,z,w,t so that $\vartheta(y) \neq 0$ and $\vartheta(z) = \vartheta(w) = 0$. You find that

$$\langle x,z\rangle \langle t,w\rangle = \langle x,w\rangle \langle t,z\rangle$$

for any x,t vectors. Taking w,z perpendicular, and $x = z, t = w$ we have a contradiction. But the kernel of ϑ is a hyperplane in V, so we can take z,w perpendicular. Therefore $\vartheta = 0$, and $\lambda = 0$ and so $\eta = 0$, and the cohomology vanishes.

7.7.3 Dimension 3

Proposition 25. *If* $\dim V = 3$ *then*

$$H^{2,2}(\mathfrak{co}(p,q)) = \operatorname{Sym}^2_0(V^*) \otimes \mathbb{R}\langle,\rangle^* \otimes \Lambda^3(V^*)$$

where $\langle,\rangle^*$ *is the dual quadratic form on* V^* *to the quadratic form* $\langle,\rangle$ *on* V.

Proof. Carrying out the same calculations as before we find that

$$H^{2,2}\left(\mathfrak{co}\left(p,q\right)\right) = \ker \delta : \mathfrak{g}^{(1)} \otimes \Lambda^2\left(V^*\right) \to \mathfrak{g} \otimes \Lambda^3\left(V^*\right).$$

Any representation of $CO\left(p,q\right)$ gives a representation of $SO\left(p,q\right)$, and if we split that representation into irreducibles over $SO\left(p,q\right)$ then we obtain a splitting of the $CO\left(p,q\right)$ representation, together with a weight for each irreducible indicating how it scales with the diagonal matrices in $CO\left(p,q\right)$. Conversely, we can produce all irreducible representations of $CO\left(p,q\right)$ by taking the $SO\left(p,q\right)$ representations and adjoining all possible weights to them. We will assign V weight one. We will write $\mathbb{R}_k$ for the one dimensional representation of $CO\left(p,q\right)$ of weight k. For instance

$$\Lambda^3\left(V^*\right) = \mathbb{R}_{-3}.$$

We find

$$\begin{aligned} \mathfrak{g}^{(1)} &= V^* \\ \Lambda^2\left(V^*\right) &= \mathbb{R}_{-3} \otimes V \\ \mathfrak{g} &= \mathbb{R}_0 \otimes \mathfrak{so}\left(p,q\right) \end{aligned}$$

with weights

$$\begin{aligned} \left[V\right] &= 1 \\ \left[\mathfrak{so}\left(p,q\right)\right] &= 0 \end{aligned}$$

Tensoring gives a splitting into irreducibles:

$$\delta : \mathrm{Sym}^2_0\left(V\right) \otimes \mathbb{R}_{-1} \oplus \mathbb{R}_{-3} \oplus \mathbb{R}_{-2} \otimes V^* \to \mathbb{R}_{-3} \oplus \mathbb{R}_{-2} \otimes V^*.$$

The representation

$$\mathrm{Sym}^2_0\left(V\right)$$

(symmetric traceless quadratic forms) is an irreducible representation of dimension 5, and so the term containing it must be in the kernel. We have only to show that nothing else is.

Again we write

$$\eta(x,y,z,w) = \eta(x,y,z,w) = \lambda(x,z,w)y + \lambda(y,z,w)x - \lambda^*(z,w)\left\langle x,y\right\rangle$$

with

$$\lambda \in V^* \otimes \Lambda^2\left(V^*\right).$$

If we take $\lambda \in \Lambda^3\left(V^*\right)$ then we find that $\delta\eta \neq 0$ unless $\lambda = 0$. Therefore, this is the factor of $\mathbb{R}_{-3}$, not belonging to the kernel of δ.

We are still missing the $\mathbb{R}_{-2} \otimes V^*$ part. Using the cross product defined by

$$\left\langle x \times y, z\right\rangle = dVol(x,y,z)$$

we can write

$$\lambda(x,y,z) = a\langle x, y\times z\rangle + \zeta_0(x, y\times z) + \zeta_1(x, y\times z)$$

decomposing into irreducibles, where ζ_0 is symmetric and traceless and ζ_1 antisymmetric, and a is a constant. We already know that the a part will not belong to the kernel, so we take $a = 0$. Then we can easily calculate that $\delta\eta = 0$ precisely when $\zeta_1 = 0$.

7.7.4 The structure equations

We will now write out the structure equations of conformal geometry in any dimension greater than two. First, we saw that

$$d\omega = -\gamma \wedge \omega$$

with $\omega \in \Omega^1(B) \otimes V$ our soldering form, and $\gamma \in \Omega^1\left(B^{(1)}\right) \otimes \mathfrak{co}(p,q)$. We can split γ into pieces

$$\gamma = \alpha + \sigma 1_{p+q}$$

where

$$\begin{aligned} \alpha &\in \Omega^1\left(B^{(1)}\right) \otimes \mathfrak{so}(p,q) \\ \sigma &\in \Omega^1\left(B^{(1)}\right) \end{aligned}$$

and 1_{p+q} means the $(p+q)\times(p+q)$ identity matrix. These satisfy

$$d\gamma = -\gamma\wedge\gamma - \varpi' \wedge \omega + \frac{1}{2}T^{(1)}\omega\wedge\omega$$

for some $\varpi' \in \Omega^1\left(B^{(2)}\right) \otimes \mathfrak{co}(p,q)^{(1)}$. But note that $\mathfrak{co}(p,q)^{(2)} = 0$, the structure group of the bundle $B^{(2)} \to B^{(1)}$, so $B^{(2)} = B^{(1)}$, and the prolongation stops there. This gives

$$\begin{aligned} d\alpha &= -\alpha\wedge\alpha + \omega\wedge\varpi + \varpi^*\wedge\omega^* + \frac{1}{2}W\omega\wedge\omega \\ d\sigma &= -\varpi\wedge\omega \end{aligned}$$

for a 1-form

$$\varpi \in \Omega^1\left(B^{(1)}\right) \otimes V^*.$$

By taking exterior derivative of these equations, we see that this 1-form ϖ satisfies

$$d\varpi = -\varpi\wedge(\alpha + \sigma 1_{p+q}) + \frac{1}{2}C\omega\wedge\omega.$$

We have seen that W is the Weyl curvature, which vanishes in dimension 3, where C is the only torsion. On the other hand, when the dimension exceeds three, W is the only torsion, so that C is determined by the covariant derivatives of W.

7.7.5 The group actions

$$r_g^* \begin{pmatrix} \omega \\ \alpha \\ \sigma \\ \varpi \end{pmatrix} = \begin{pmatrix} g^{-1}\omega \\ \operatorname{Ad}_g^{-1} \alpha \\ \sigma \\ \varpi g \end{pmatrix},$$

$$r_\lambda^* \begin{pmatrix} \omega \\ \alpha \\ \sigma \\ \varpi \\ W \\ C \end{pmatrix} = \begin{pmatrix} \omega \\ \alpha + \lambda^* \otimes \omega^* - \omega \otimes \lambda \\ \sigma - \lambda\omega \\ \varpi - \lambda\left(\alpha + \sigma 1_{p+q}\right) \\ W \\ C - \lambda W \end{pmatrix},$$

where

1. $g \in CO(p,q)$ and $\lambda \in V^*$ and
2. the structure group is reductive, and the Weyl curvature lives in an irreducible representation, $H^{1,2}(\mathfrak{co}(p,q))$, so we are using as section the unique equivariant choice of linear section of

   $$\mathfrak{co}(p,q) \otimes \Lambda^2(V^*) \to H^{1,2}(\mathfrak{co}(p,q)).$$

   This is just the same as saying that we are requiring our Weyl curvature to be in $\mathfrak{co}(p,q) \otimes \Lambda^2(V^*)$, and be traceless and satisfy the first Bianchi identity:

   $$W^i_{ikl} = 0, W^i_{jkl} + W^i_{klj} + W^i_{ljk} = 0.$$

Note that W, the Weyl curvature, is a tensor, since it lives in the representation $V \otimes V^* \otimes \Lambda^2(V^*)$, and is invariant under the structure group of the prolongation. On the other hand, C is a tensor just when $W = 0$, because of the manner in which C varies under the structure group of the prolongation. But if $W = 0$, and the dimension is $n > 3$, then that forces $C = 0$, by differentiating the structure equations. So we find that C is a tensor just on 3-manifolds.

7.7.6 Structure equations as Cartan connection

Question 10. I really should think carefully about the idea of graded Lie algebras making their way into the theory. The notation from graded Lie algebras, particularly Ruchti [73] and Agaoka [2], might clarify a lot about how things were organized in the Cartan connection. It might also make it easy to bring it Yamaguchi's results on simple Lie groups as symmetry pseudogroups of exterior differential systems. It might be convenient to write $\mathfrak{h} = \bigoplus \mathfrak{h}_i$ and write $\mathfrak{h}_{ij} = \mathfrak{h}_i \oplus \mathfrak{h}_j$ etc., and $\mathfrak{h}_{\geq i} = \bigoplus_{j \geq i} \mathfrak{h}_j$ etc.

We now present a mysterious reformulation of the structure equations, whose origins will be revealed in subsection 8.5.1 on page 182. For now, we merely state the startling:

Proposition 26. *Let $V' = \mathbb{R} \oplus V \oplus \mathbb{R}$ with coordinates (u, v, w). Let $\langle , \rangle'$ be the quadratic form on V':*

$$\langle , \rangle' = -dw\, du - du\, dw + \langle , \rangle .$$

On the bundle $B^{(1)}$ we have the structure equations

$$d\Omega + \Omega \wedge \Omega = \begin{pmatrix} 0 & \frac{1}{2} C\omega \wedge \omega & 0 \\ 0 & \frac{1}{2} W\omega \wedge \omega & \frac{1}{2} \left(C\omega \wedge \omega \right)^* \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\Omega = \begin{pmatrix} -\sigma & -\varpi & 0 \\ \omega & \alpha & -\varpi^* \\ 0 & \omega^* & \sigma \end{pmatrix} \in \Omega^1 \left(B^{(1)} \right) \otimes \mathfrak{so} \left(p+1, q+1 \right)$$

is a Lie algebra valued 1-form, valued in the Lie algebra of the isometry group of the quadratic form $\langle , \rangle'$.

In subsection 8.5.1 on page 182, after careful study of the flat case, we will see that it is natural to organize our differential forms in this fashion.

7.7.7 The flat case

If the symmetry group acts transitively on the bundle $B^{(1)}$ then the invariants W and C must be constant. But since they belong to representations of the structure group which contain no trivial subrepresentations, they must vanish. Therefore the structure equations are

$$d\Omega + \Omega \wedge \Omega = 0.$$

These are also the structure equations of the group $SO\left(p+1, q+1\right)$, so any conformal structure with vanishing Weyl and Cotten invariants must be locally equivalent to the flat example:

$$SO\left(p+1, q+1\right) / H_0$$

where H_0 is the maximal parabolic subgroup of elements of $SO\left(p+1, q+1\right)$ which preserve the element

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{RP}\left(V'\right)$$

which we identify with the line through the origin and through

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in V'.$$

The group H_0 consists precisely in the matrices of the form

$$\begin{pmatrix} s^{-1} & y^* & z \\ 0 & a & y \\ 0 & 0 & s \end{pmatrix}$$

which belong to $SO\,(p+1,q+1)$. How did we find this group H_0? We simply set $\omega = 0$ in the structure equations above; that gives the structure group of the bundle, and the structure equations above become the structure equations of H_0. (We will be somewhat cavalier about factors of ± 1.)

7.7.8 Null lines

The line preserved by H_0 in V' is a null line. Null lines belong to a single orbit of the action of $O\,(p+1,q+1)$ on $\mathbb{RP}\,(V')$. Therefore the space of null lines in V' is precisely the manifold $O\,(p+1,q+1)\,/H_0$ on which we have created the homogeneous flat conformal structure. Moreover, we have a canonical projective embedding of this manifold of null lines into $\mathbb{RP}\,(V')$, embedded as a hyperquadric (because the equation for a point of V' to belong to a null line is quadratic).

7.7.9 The symmetry group

The orthogonal group $O\,(p+1,q+1)$ acts on the flat conformal structure, and the subgroup $\pm 1 \subset P$ acts trivially. So in fact the group

$$\mathbb{P}O\,(p+1,q+1) = O\,(p+1,q+1)\,/\pm 1$$

acts on the space of null lines, and turns out to be the symmetry group.

7.7.10 Algebraic geometry of hyperquadrics

Every smooth nonempty real hyperquadric comes about in this way, i.e. as the projectivized zero locus of a quadratic form $\langle , \rangle'$ for some quadratic form $\langle , \rangle$. The automorphisms of projective space leaving the hyperquadric invariant form precisely the group of equivalences of the flat conformal structure on the hyperquadric.

7.7.11 The diffeomorphism type of a hyperquadric

Let us now find the hyperquadric as a manifold. Take a splitting $V = V^+ \oplus V^-$ of V into a maximal positive definite subspace for $\langle , \rangle$ (possibly empty) and a maximal negative definite subspace (possibly empty). Let $x_1, \dots, x_p$ and $y_1, \dots, y_q$ be orthonormal coordinates on V^+ and V^- respectively. We can then let

$$x_0 = \frac{u - w}{\sqrt{2}} \text{ and } y_0 = \frac{u + w}{\sqrt{2}}$$

and find that our space V' is split into

$$V' = \left(\mathbb{R}_{x_0} \oplus V^+\right) \oplus \left(\mathbb{R}_{y_0} \oplus V^-\right)$$

maximal positive definite and negative definite spaces, with quadratic form

$$\langle , \rangle' = \sum dx_\mu^2 - \sum dy_\nu^2.$$

The null lines are those on which $|x| = |y|$. We can pick out a vector on each null line by the condition that $|x| = 1$, unique up to ± 1. So the hyperquadric on which our conformal structure lives is

$$\left(S^p \times S^q\right) / \{\pm 1\}$$

where the ± 1 action is the simultaneous antipodal map.

Up to covering spaces, this is the unique conformal structure for which the symmetry group acts transitively on all prolongations. Even in infinite dimensions, the same approach works.

Exercise 7.14 The conformal structure on $\left(S^p \times S^q\right) / \pm 1$ is locally, but not globally, equivalent to the standard flat conformal structure.

7.7.12 Circles

One motivation for looking at the group H_0, which we found by setting $\omega = 0$ in our structure equations, is that the ω terms occur everywhere in the curvature. So both flat and curved manifolds will give rise to the same structure equations once we set $\omega = 0$. Indeed setting all but one of the ω^j to 0 (say, leaving ω^1) kills off all of the curvature terms, since they are always wedge products of ω 1-forms. But the structure equations for $d\omega^j$ then force α_1^j to be semibasic, say $\alpha_1^j = k^j \omega^1$, and that gives new curvature, a kind of geodesic curvature. Assume that it vanishes, say $\alpha_1^j = 0$. This forces ϖ^j semibasic by the structure equations. Then ask for $\varpi^j = 0$ to kill off that kind of curvature. Finally, this gives equations $\omega^j = \alpha_1^j = \varpi^j = 0$ for $j > 1$ which turns out to make all curvature terms disappear, and so, if the conformal geometry is complete, then the integral manifolds (in the sense of exterior differential systems) of the equations $\left(\omega^j = \alpha^j_1 = \varpi^j = 0 \,|\, j > 1\right)$ are acted on by a Lie group whose structure equations are just the result of plugging these equations into the structure equations above, so with Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(p-1, q)$. Indeed this group acts transitively and with discrete stabilizer. The bundle $B^{(1)}$ is foliated by these group orbits.

Exercise 7.15 What curves on $M = (S^p \times S^q) / \{\pm 1\}$ live underneath the integral manifolds of the equations $0 = \omega^j = \alpha^j_1 = \varpi^j$ for $j > 1$?

Exercise 7.16 As a vector space, V is equipped with the Maurer–Cartan 1-form dv defined by $v \lrcorner dv = v$. Define a 1-form $\dot{\omega} \in \Omega^1\left(B^{(1)}\right) \otimes V$ by

$$\dot{\omega} = dv + (\alpha + \sigma 1_{p+q})\, v.$$

Define a vector field E on $B^{(1)} \times V$ by

$$E \lrcorner \begin{pmatrix} \omega \\ \alpha \\ \sigma \\ \varpi \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Call the flow of E the *circle flow* of a conformal geometry. Show that its flow lines are permuted by the action of the structure group of $B^{(1)} \to M$. Show that these flow lines project to M by immersion, and are precisely the images in M of the integral manifolds of $(\omega - v\, dt = \alpha v = \varpi - v^*\, dt = 0)$. These curves on M are called the *circles*. Show that the circles satisfy a third order system of ordinary differential equations.

7.7.13 Global consequences

Lemma 26. *Suppose that M is a conformally flat pseudo-Riemannian manifold with metric of signature (p, q). Then the orthonormal frame bundle B admits a canonical embedding into the bundle $B\,(CO\,(p, q))^{(1)}$ of the conformal structure, for which the Weyl curvature tensor pulls back to the Weyl curvature component of the Riemann curvature tensor.*

Proof. Start with the structure equations of pseudo-Riemannian geometry:

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l. \end{aligned}$$

Fattening up the structure group to $CO\,(p, q)$, we have the inclusion $B \to B\,(CO\,(p, q))$. But on $B\,(CO\,(p, q))$, the 1-forms α and σ are not really defined. However, we can create a map $B \to B(CO\,(p, q))^{(1)}$, picking out a pseudoconnection at each point, by using the Levi-Civita pseudoconnection $\gamma : T_u B \to \mathfrak{so}(p,q)$ and fattening it up to a choice of α, σ determined by

$$\begin{aligned} \alpha : T_{(u,h)} B \times H &\to \mathfrak{so}(p,q) \\ \sigma : T_{(u,h)} B \times H &\to \mathbb{R} \end{aligned}$$

by

$$\alpha + \sigma 1_{p+q} = h^{-1}\, dh + \mathrm{Ad}_h^{-1} \gamma.$$

It is easy to see that this defines 1-forms α and σ on $B(CO(p,q))$. It is easy to check that they are pseudoconnection 1-forms, by splitting each $h \in CO(p,q)$ into $h = sa$ with $s \in \mathbb{R}^{\times}$ and $a \in SO(p,q)$.

This map satisfies

$$\sigma = 0, \alpha = \gamma$$

on B. Taking exterior derivative, and plugging into the structure equations, we find

$$\begin{aligned} \varpi_i &= s_{ij}\omega^j \\ R^i_{jkl} &= W^i_{jlk} + \delta^i_l s_{jk} - \delta^i_k s_{jl} - g^{im} g_{jl} s_{mk} + g^{im} g_{jk} s_{ml}, \end{aligned}$$

where g_{ij} are the components of the quadratic form $\langle , \rangle$ on V. The Ricci tensor is represented by s_{ij}.

Theorem 9. *Every complete, connected, conformally flat pseudo-Riemannian manifold of metric signature (p,q) with dimension $p+q \geq 3$ is a quotient of*

- $S^p \times S^q$ *if* $p \neq 1$ *and* $q \neq 1$
- $S^p \times \mathbb{R}$ *if* $p > 1$ *and* $q = 1$
- $\mathbb{R} \times S^q$ *if* $p = 1$ *and* $q > 1$

by a discrete subgroup of (the appropriate covering group of) $SO(p+1, q+1)$.

Remark 17. N.B.: the relevant notion of completeness is *not* the one from pseudo-Riemannian geometry. Indeed Euclidean space is conformally flat, and complete as a Riemannian manifold, but not conformally complete, because the circle flow on the bundle $B^{(1)}$ will not be complete. We can see this from the conformal embedding $\mathbb{R}^n \to S^n$ into the sphere. Similarly, the flat torus is complete as a Riemannian manifold, but incomplete as a conformal manifold, since flat $\mathbb{R}^n$ is a covering space, and is incomplete. In fact, the torus of dimension at least 3 admits no complete flat conformal structure, of any signature, since it does not have one of the above listed spaces as a covering space.

Proof. Apply lemma 20 on page 121.

Exercise 7.17 Use lemma 24 on page 123 to show that a compact manifold can not admit a flat conformal structure if its fundamental group defies $\mathbb{P}O(p+1,q+1)$.

Exercise 7.18 Classify all homogeneous complete flat conformal structures of dimension at least three.

Exercise 7.19 The proof certainly requires dimension at least 3, since the structure equations behave very differently for dimension 2. What happens in 2-dimensional conformal geometry? See Forster [38]. Why do we expect that analysis will be required?

Exercise 7.20 What happens to this proof if we start with M not simply connected, for example a flat 3-torus? Does it at least construct a conformal map from the 3-sphere to the 3-torus?

Exercise 7.21 Are there any compact, simply connected Lorentzian manifolds? Are there conformally flat ones?

Remark 18. The symmetry group of a positive definite conformal structure alway preserves a Riemannian metric with that conformal geometry, except for the standard conformal structure on the sphere or the standard flat conformal geometry on Euclidean space, as proven by R. Schoen [75]. Nothing is known about this question in pseudo-Riemannian geometry. Fefferman & Graham [37] discovered a beautiful fundamental relation between Riemannian conformal geometry and negatively curved Riemannian Einstein manifolds.

7.7.14 Duality

In flat conformal geometry, we see that ϖ and ω have very similar roles in the structure equations. Indeed we can see that transposing the Ω object flips around the roles of ω and ϖ. Note that putting in $\omega = 0$ gives the structure equations of the stabilizer of the null line

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{RP}\left(V'\right).$$

Similarly, putting in $\varpi = 0$ gives the structure equations of the stabilizer of the hyperplane

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \in \mathbb{RP}\left(\left(V'\right)^*\right).$$

This is a hyperplane of signature (p, q) with a null direction, i.e. a degenerate hyperplane. All such hyperplanes are permuted by the action of $\mathbb{P}O(p+1,q+1)$, so that the same structure equations determine a conformal structure on the space of degenerate hyperplanes. This is not surprising, since each degenerate hyperplane is perpendicular to a unique null line. In the picture of $M = S^p \times S^q / \pm 1$, this duality is just the antipodal map on S^p.

We can obviously generalize this observation: consider any conformal geometry. The Cotton invariant vanishes just when the equation $\varpi = 0$ is holonomic, i.e. satisfies the conditions of the Frobenius theorem. Therefore if we have vanishing Cotton invariant, then the bundle $B^{(1)}$ is foliated by the leaves of $\varpi = 0$, and on each leaf we have the structure equations

$$\begin{aligned} d\omega &= -\left(\alpha + \sigma 1_{p+q}\right) \wedge \omega \\ d\alpha &= -\alpha \wedge \alpha + \frac{1}{2} W \omega \wedge \omega \\ d\sigma &= 0. \end{aligned}$$

Therefore each of these leaves is itself foliated by a family hypersurfaces determined by the equation $\sigma = 0$. On each of these, the structure equations are

$$\begin{aligned} d\omega &= -\alpha \wedge \omega \\ d\alpha &= -\alpha \wedge \alpha + \frac{1}{2} W \omega \wedge \omega, \end{aligned}$$

the structure equations of a Ricci-flat metric in the given conformal class.

Exercise 7.22 If the Ricci curvature vanishes in some metric in the conformal class, does the Cotton invariant vanish?

7.7.15 Holomorphic conformal geometry

A *holomorphic conformal structure* is a holomorphic $CO(n, \mathbb{C})$-structure on a complex manifold. The structure equations are the same as above.

Theorem 10. *A complete holomorphic conformal structure is flat, and the quotient of the hyperquadric* $\left(x_0^2 + \dots + x_{n+1}^2 = 0\right) / \mathbb{C}^\times \subset \mathbb{CP}^{n+1}$ *by a discrete subgroup of* $\mathbb{P}O(n+2, \mathbb{C}) = CO(n+2, \mathbb{C}) / \mathbb{C}^\times$.

Proof. We take $B \to M$ to be the bundle of the conformal structure, and $B^{(1)} \to B$ its prolongation. Without loss of generality, we can replace M with its universal covering space, so assume M is simply connected.

Lets compare with the flat example, M_0 a smooth hyperquadric in projective space $M_0 \subset \mathbb{CP}^{n+1}$. Write $B_0 \to M_0$ for the conformal structure, and $B_0^{(1)} \to B_0$ for its prolongation. Recall that the symmetry group is $H = \mathbb{P}O(n+2, \mathbb{C})$. The symmetry group can be identified with $B_0^{(1)}$, because it acts transitively on $B_0^{(1)}$ with trivial stabilizer at any point. Under this identification, $M_0 = H/H_0$, $B_0 = H/H_1$ and $B_0^{(1)} = H$, where H_0 is the stabilizer of a null line in $\mathbb{C}^{n+2}$, and $H_1 = \mathbb{C}^n$ is the group of "null boosts" along that null line, i.e. these are the connected subgroups with Lie algebras

$$\mathfrak{h} = \left\{ \begin{pmatrix} \sigma & -\varpi & 0 \\ \omega & \alpha & v^* \\ 0 & \omega^* & -\sigma \end{pmatrix} \right\},$$

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} \sigma & -\varpi & 0 \\ 0 & \alpha & \varpi^* \\ 0 & 0 & -\sigma \end{pmatrix} \right\},$$

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} 0 & -\varpi & 0 \\ 0 & 0 & \varpi^* \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

The integral manifolds of the differential system

$$\omega^J = \alpha_1^J = \varpi_J = 0, J > 1$$

are just left translates of the subgroup H_- with Lie algebra given by these equations, i.e. the subgroup $\mathbb{P}SL\left(2, \mathbb{C}\right) \times SO\left(n-1, \mathbb{C}\right)$, which is the subgroup fixing a circle. Let $H_+ \subset H_-$ be the subgroup fixing the circle and a point on the circle.

Now consider a general holomorphic conformal structure. Lets check that the nonnull circles are projective lines. Above each nonnull circle C we have a subbundle B_C of $B^{(1)}$ on which

$$\omega^J = \alpha_1^J = \varpi_J = 0$$

for $J > 1$, and B_C bears the coframing $\omega^1, \sigma, \varpi_1, \alpha_J^I$, with the structure equations of $\mathbb{P}SL\left(2, \mathbb{C}\right) \times SO\left(n-1, \mathbb{C}\right)$:

$$\begin{aligned} d\omega^1 &= -\sigma \wedge \omega^1 \\ d\sigma &= -\varpi_1 \wedge \omega^1 \\ d\varpi_1 &= \sigma \wedge \varpi_1 \\ d\alpha_J^I &= -\alpha_K^I \wedge \alpha_J^K. \end{aligned}$$

This is a complete flat Cartan connection so by lemma 20 on page 121, a covering must be isomorphic to the corresponding Klein geometry, i.e. the Cartan connection $B_{\mathbb{P}^1} \to \mathbb{P}^1$ we would find in the hyperquadric case. Therefore, $\mathbb{P}^1$ is a covering space of C. By the classification of complex curves (see Forster [38]), the map $\mathbb{P}^1 \to C$ must be a biholomorphism, so the geodesic is rational. Looking back at the hyperquadric, $B_{\mathbb{P}^1}$ is invariantly identified with H_-. We identify B_C with H_- by equivalence.

To understand how the circle $C = \mathbb{CP}^1$ sits inside the manifold M, we need to construct its normal bundle. Using the same method as lemma 25 on page 124,

$$\begin{aligned} TM|_C &= H_+ \backslash (H_- \times \mathbb{C}^n) \\ NC &= H_+ \backslash \left(H_- \times \left(0 \oplus \mathbb{C}^{n-1}\right)\right) \\ TC &= H_+ \backslash \left(H_- \times \left(\mathbb{C}^1 \oplus 0\right)\right) \end{aligned}$$

using diagonal H_+ action. The tangent and normal bundles of a nonnull geodesic are therefore independent of the geodesic itself (up to isomorphism), and are isomorphic to the tangent and normal bundles of the geodesics of the hyperquadric.

We recall the Birkhoff–Grothendieck theorem [40],[45]: every vector bundle on the complex projective line is a sum of line bundles, and each line bundle is $\mathcal{O}(d)$ for some integer d, up to holomorphic bundle isomorphism. The positive line bundles have holomorphic sections, the negative ones have only the zero section, and $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2) = \mathcal{O}(d_1 + d_2)$. The dual of $\mathcal{O}(d)$ is $\mathcal{O}(-d)$. In particular, the normal bundle of $C = \mathbb{CP}^1 \subset M_0$ is $NC = \bigoplus^{n-1} \mathcal{O}(1)$ (look at an affine chart), while the tangent bundle is $TC = \mathcal{O}(2)$ (look at changing affine chart on $\mathbb{CP}^1$).

Now map $H_+ \to B_C \to B^{(1)}$, and pull back the Weyl curvature W^i_{jkl}. The Weyl curvature tensor has one upper and three lower indices, each of which can be taken to be the "tangent" index 1, or the "normal" indices bigger than 1. Therefore the Weyl curvature restricted to the geodesic C is a section of a vector bundle on C, which is a sum of line bundles, each a tensor product of either $\mathcal{O}(1)$ or $\mathcal{O}(2)$ with three line bundles, each either $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$. All of the resulting line bundles are negative, so there are no nonzero holomorphic sections. All of the Weyl curvature components are forced to vanish, at every point above every geodesic, and therefore everywhere.

Once the Weyl curvature is out of the way, the Cotton invariant C is a tensor, and the same argument applies to it, so that it vanishes. Therefore the conformal structure is flat. Proposition 15 on page 81 shows that the manifold M is locally equivalent to a covering space of the hyperquadric.

By the Lefschetz hyperplane theorem (see [62], theorem 7.4, p. 41), the fundamental group of a hyperquadric is isomorphic to the fundamental group of its intersection with any hyperplane, which is a hyperquadric in one lower dimension. A hyperquadric in the projective plane is a rational curve (projection to a line). Therefore hyperquadrics in all dimensions are simply connected.

7.8 Tensors and other objects defined on the base, and their covariant derivatives

Once we calculate the representation $H^{0,2}(\mathfrak{g})$ (and split it into indecomposable pieces as a G representation), we want to interpret the resulting components of the torsion.

7.8.1 Tensors on the base as equivariant functions on the bundle

Lemma 27. *A G equivariant map $f : B \to V$ on a G-structure $\pi : B \to M$ (i.e. $f(r_g u) = g^{-1} f(u)$ determines a vector field X on M by*

$$u(X(x)) = f(u)$$

for $u \in B$ a linear isomorphism $u : T_x M \to V$. Every vector field comes about this way, from a unique function f given by the same equation. In the same way, a G equivariant map $f : B \to V^*$ determines a 1-form α by

$$f \circ \omega = \pi^* \alpha.$$

Here, $f \circ \omega$ means the 1-form which eats a tangent vector $v \in TB$ and spits out $f(\omega(v))$. In the same manner, the G equivariant maps $f : B \to W$ to any $\mathrm{GL}(V)$ *representation W are precisely in correspondence with tensors of type W on M.*

Proof. Given $f : B \to V$ and a point $u \in B$ which is a linear map $u : T_x M \to V$ we can let $X(u) = u^{-1} f(u)$. Then under the right G action

$$\begin{aligned} X(r_g u) &= \left(g^{-1} u\right)^{-1} f\left(g^{-1} u\right) \\ &= u^{-1} g g^{-1} f(u) \\ &= u^{-1} f(u). \end{aligned}$$

The rest is similar.

Lemma 28. *We will always write the equivariant function $B \to V$ associated to a vector field X as $X^\bullet : B \to V$. It is represented in a basis of V by components $X^i : B \to \mathbb{R}$ so that if we set*

$$\nabla X^i = dX^i + \gamma^i_j X^j$$

then

$$\nabla X^i = \nabla_j X^i \omega^j$$

for some functions $\nabla_j X^i$. Conversely, if the structure group G is connected then any functions X^i which satisfy these equations are equivariant.

Proof. The equivariance condition is

$$X^\bullet(r_g u) = g^{-1} X^\bullet(u)$$

so that

$$X^\bullet(r_{e^{tA}} u) = e^{-tA} X^\bullet(u)$$

for $A \in \mathfrak{g}$. Differentiating

$$\mathcal{L}_{\vec{A}} X^\bullet = -A X^\bullet$$

so that $dX^\bullet + \gamma X^\bullet$ is semibasic. The converse for a connected group follows immediately.

7.8.2 Covariant derivatives

If we wish to avoid components, we can write $\nabla X^{\bullet}$ as

$$\nabla X^{\bullet} = \nabla_{\bullet} X^{\bullet} \omega.$$

For a 1-form, we must have

$$\nabla f_i = df_i - \gamma_i^j f_j = \nabla_j f_i \omega^j$$

for functions $\nabla_j f_i$. For a 2-form, represented by $f_{ij} = -f_{ji}$ components,

$$\nabla f_{ij} = df_{ij} - \gamma_i^k f_{kj} - \gamma_j^k f_{ik} = \nabla_k f_{ij} \omega^k.$$

For a general tensor represented by functions

$$f^{i_1 \dots i_m}_{j_1 \dots j_n}$$

we find

$$\begin{aligned} \nabla f^{i_1 \dots i_m}_{j_1 \dots j_n} &= df^{i_1 \dots i_m}_{j_1 \dots j_n} + \sum f^{i_1 \dots i_{p-1} k i_{p+1} \dots i_m}_{j_1 \dots j_n} \gamma_k^{i_p} - \sum f^{i_1 \dots i_m}_{j_1 \dots j_{q-1} k j_{q+1} \dots j_n} \gamma_{j_q}^k \\ &= \nabla_k f^{i_1 \dots i_m}_{j_1 \dots j_n} \omega^k. \end{aligned}$$

We will call the operator ∇ the covariant derivative. There are three subtleties here:

1. If γ is a chosen connection, then this covariant derivative is just the usual notion, corresponding to the usual Ehresmann ∇ operator (see any textbook on differential geometry).
2. In general, γ might only be a pseudoconnection. Then we have, for any tensor from M, a well defined collection of functions on the bundle B, and we can take ∇ as many times as we like. But only if γ is a connection can we ensure that these ∇_I functions are in fact the component functions of a tensor from M. This is clear if we consider the representations of the structure group under which component functions must vary, and remember that γ is a connection just when it is equivariant under the structure group. It will not always be possible to pick a connection, and so the results of the computation, while sometimes useful, are tricky to interpret geometrically.
3. Even more generally, we might not even have picked any γ pseudoconnection at all. Instead, we might work purely formally on the prolongation $B^{(1)}$, the bundle of all choices of pseudoconnection, so that we can imagine that we are carrying out the universal computation of covariant derivatives, for all pseudoconnections at once.

7.8.2.1 Example: invariant differential operators on complex projective spaces

Lets find the linear differential operators on functions on complex projective space $\mathbb{CP}^n$ which are invariant under isometries. (The problem of classifying invariant differential operators, for tensors in general, on manifolds with arbitrary G-structures, is very natural since it can give rise to operators to which we might apply the Böchner technique, if we are very lucky.) Isometries constitute the symmetry group of the usual $U(n)$-structure (Kähler metric) of the complex projective space. This group is $\mathbb{P}\mathrm{SU}(n+1) = U(n+1)/U(1)$. We can identify the group with the total space B of the $U(n)$-structure, since it acts simply transitively on B. The structure equations are

$$\begin{aligned} d\omega_\mu &= -\gamma_{\mu\bar{\nu}} \wedge \omega_{\bar{\nu}} \\ d\gamma_{\mu\bar{\nu}} &= -\gamma_{\mu\bar{\sigma}} \wedge \gamma_{\sigma\bar{\nu}} + \omega_\mu \wedge \omega_{\bar{\nu}}. \end{aligned}$$

Take a real-valued function $u : \mathbb{CP}^n \to \mathbb{R}$, and pull it back to the group. It now satisfies

$$\begin{aligned} \nabla u &= du \\ &= \nabla_\mu u \, \omega_{\bar{\mu}} + \nabla_{\bar{\mu}} u \, \omega_\mu. \end{aligned}$$

Take the exterior derivative to see that

$$\begin{aligned} \nabla \nabla_\mu u &= d\nabla_\mu u - \nabla_\nu \gamma_{\bar{\nu}\mu} \\ &= \nabla_{\mu\nu} u \, \omega_{\bar{\nu}} + \nabla_{\mu\bar{\nu}} u \, \omega_\nu \end{aligned}$$

and the complex conjugate of this equation. Take another exterior derivative to find

$$\begin{aligned} \nabla \nabla_{\mu\nu} u &= d\nabla_{\mu\nu} u - \nabla_{\sigma\nu} u \, \gamma_{\bar{\sigma}\mu} - \nabla_{\mu\sigma} u \, \gamma_{\bar{\sigma}\nu} \\ &= \left(D_{\mu\nu\bar{\sigma}} u + \frac{1}{2} \delta_{\sigma\mu} \nabla_\nu u \right) \omega_\sigma + D_{\mu\nu\sigma} u \, \omega_{\bar{\sigma}} \end{aligned}$$

and

$$\begin{aligned} \nabla \nabla_{\mu\bar{\nu}} u &= d\nabla_{\mu\bar{\nu}} u - \nabla_{\sigma\bar{\nu}} u \, \gamma_{\bar{\sigma}\mu} - \nabla_{\mu\bar{\sigma}} u \, \gamma_{\sigma\bar{\nu}} \\ &= D_{\mu\nu\bar{\sigma}} u \, \omega_\sigma + \left(D_{\mu\bar{\nu}\sigma} u - \frac{1}{2} \delta_{\nu\mu} \nabla_\sigma u \right) \omega_{\bar{\sigma}} \end{aligned}$$

with these $D_{\dots}$ symmetric in all lower indices. These D things are not the covariant derivatives; indeed the equations show that

$$\nabla_{\mu\bar{\nu}\sigma} u = D_{\mu\bar{\nu}\sigma} u - \frac{1}{2} \delta_{\nu\mu} \nabla_\sigma u.$$

The quantities $\nabla_\mu u, \nabla_{\bar{\mu}} u, \nabla_{\mu\nu} u$ etc. vary in certain representations of the structure group $U(n)$. Constructing invariant linear differential operators just means identifying $U(n)$-invariant linear operators on these representations. Similarly, invariant linear differential equations are just $U(n)$-invariant subspaces of these representations. For example, the equations

$$\begin{aligned} \nabla_\mu u &= 0 \text{ Cauchy–Riemann equations} \\ \nabla_{\mu\bar{\mu}} u &= 0 \text{ Laplace equation} \\ D_{\mu\bar{\mu}\nu} u + \nabla_\nu u &= 0 \text{ etc.} \end{aligned}$$

7.8.2.2 Example: invariant differential equations in conformal geometry

A *k-density* on a manifold M is a function f on the frame bundle FM which satisfies

$$r_g^* f = |\det g|^{-k} f.$$

Exercise 7.23 A compactly supported 1-density can be integrated over a manifold, even if the manifold is not orientable.

The k-densities are the sections of the vector bundle $(FM \times \mathbb{R}) / \mathrm{GL}(V)$, where the action of $\mathrm{GL}(V)$ is

$$r_g(u, F) = \left(r_g u, |\det g|^{-k} F\right).$$

A positive definite conformal structure, structure group $CO(n)$, has structure equations

$$\begin{aligned} d\omega &= -(\alpha + \sigma 1_{p+q}) \wedge \omega \\ d\alpha &= -\alpha \wedge \alpha + \omega \wedge \varpi + \varpi^* \wedge \omega^* + \frac{1}{2} W \omega \wedge \omega \\ d\sigma &= -\varpi \wedge \omega \\ d\varpi &= -\varpi \wedge (\alpha + \sigma 1_{p+q}) + \frac{1}{2} C \omega \wedge \omega \end{aligned}$$

(computed in section 7.7 on page 124). Let $B^{(1)} \to B \to M$ be the bundles of a conformal structure. Differentiating the equation for a k-density gives

$$df + k\sigma = \nabla_i f \omega^i,$$

for some functions $\nabla_i f : B \to \mathbb{R}$. Differentiating this equation:

$$d\nabla_i f = -f\varpi_i + \nabla_j f \left(\alpha_i^j + (1-k)\sigma\delta_i^j\right) + \nabla_{ij} f \omega^j.$$

Check the group equivariance for $\lambda \in V^*$:

$$r_\lambda^* \nabla_{ij} f = \nabla_{ij} f - k\lambda_i \nabla_j f - (\nabla_k f - kf\lambda_k)\left(\lambda_{lm} g^{km} g_{ij} - \delta_j^k \lambda_i - \lambda_j \delta_i^k\right).$$

Exercise 7.24 The expression $\nabla_{ii} f = 0$ is an invariant differential equation just when the conformal manifold has dimension 2 and the density f has weight 0, i.e. is a function.

Exercise 7.25 Classify conformally invariant fourth order linear operators on 4-manifolds.

7.8.3 Intrinsic torsion on the base

The intrinsic torsion always defines an object on the base: a section of the torsion bundle. It is often helpful to try to identify this bundle, relating it to other more well-recognized bundles.

Example 61 (Flag 3-manifolds). A flag geometry determines and is determined by a family of subbundles

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = TM$$

so that W_j has rank j. The torsion of a 3-dimensional flag geometry is a section of the line bundle $(TM/W_2) \otimes \Lambda^2 \left(W_2^*\right)$.

Example 62 (Almost symplectic geometry). If $G = \mathrm{Sp}\left(2n, \mathbb{R}\right)$ acts in its irreducible $\mathbb{R}^{2n}$ representation, a G-structure on a manifold M has torsion a section of $\Lambda^3 \left(T^* M\right)$.

If we are facing a connected structure group G then we can figure out what objects are defined on the base from the torsion by calculating the structure equations

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega$$

and then splitting the torsion into indecomposable pieces, say T_α, each of which will then satisfy an equation like

$$\nabla T_\alpha = dT_\alpha + \rho(\gamma) T_\alpha = \nabla_k T_\alpha \omega^k.$$

We will refer to this as the covariant derivative of T_α. The expression $\rho(\gamma)$ is the Lie algebra representation of $\mathfrak{g}$ on T_α.

Example 63. Consider a pseudo-Riemannian manifold, i.e. an $SO\left(p,q\right)$-structure $\pi : B \to M$. Lets write $\frac{\partial}{\partial \omega^j}$ for the vector satisfying

$$\frac{\partial}{\partial \omega^j} \lrcorner\, \omega^i = \delta^i_j$$
$$\frac{\partial}{\partial \omega^j} \lrcorner\, \gamma^i_j = 0.$$

In pseudo-Riemannian geometry, the curvature $R^i_{jkl} \frac{\partial}{\partial \omega^i} \otimes \omega^j \wedge \omega^k \wedge \omega^l$ is defined on the base manifold M. So are the various pieces of it: scalar, Ricci, and Weyl. To see this, we can use the general fact that the curvature of a Riemannian manifold is the torsion of the prolongation of its orthonormal frame bundle, and torsion is always equivariant under the structure group.

7.8.4 Lie derivatives and covariant derivatives

Fix a G-structure $\pi : B \to M$ and fix a pseudoconnection γ on it. For a vector field X on M, we define a vector field $\hat{X}$ on B by $\omega\left(\hat{X}\right) = u(X)$ and $\gamma\left(\hat{X}\right) = 0$. This is only well-defined on B if there is a choice made of pseudoconnection γ. Similarly, $\hat{X}$ is well-defined on $B^{(1)}$, if we have a choice of pseudoconnection ξ for $B^{(1)} \to B$, and we ask that $\hat{X} \lrcorner \xi = 0$. In the same way, $\hat{X}$ is defined at each prolongation, given a choice of pseudoconnection for that prolongation. We can therefore imagine $\hat{X}$ as being formally defined on some kind of infinite prolongation.

The infinite prolongation can be described as a filtered graded differential algebra, the direct limit of the algebras of differential forms on the various prolongations, mapping to each other by pullback. Roughly speaking, the spectrum of this algebra is the infinite prolongation, but of course the filtered graded differential algebra structure is really what is relevant. This $\hat{X}$ is a derivation of that algebra. This only makes sense for smooth G-structures, so it is probably preferable to think of the infinite prolongation only formally, and actually describe $\hat{X}$ on some finite prolongation, equipped with a choice of pseudoconnection.

Suppose that the structure equations are

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j - \xi^i_{jk} \wedge \omega^k + \frac{1}{2} T^{(1)i}_{jkl} \omega^k \wedge \omega^l . \end{aligned}$$

(Warning: structure equations don't have to look quite like this. For example, $d\gamma$ could involve $\gamma \wedge \omega$ terms.)

We write X^i for the functions on B associated to a vector field X on M and compute for two vector fields X and Y on M:

$$[X, Y]^i = X^j \nabla_j Y^i - Y^j \nabla_j X^i - T^i_{jk} X^j Y^k .$$

We can also check that $\pi_* \hat{X} = X$, so that

$$\widehat{[X, Y]} - \left[\hat{X}, \hat{Y}\right]$$

is tangent to the fibers of $\pi : B \to M$. Finally, we compute using the Cartan formula,

$$\left[\hat{X}, \hat{Y}\right] \lrcorner \gamma^i_j = T^{(1)i}_{jkl} X^k Y^l .$$

We can then calculate $\mathcal{L}_{\hat{X}} \omega^i = T^i_{jk} X^j \omega^k + \nabla_j X^i \omega^j$. We could write this as $\mathcal{L}_{\hat{X}} \omega = T X^\bullet \omega + \nabla_\bullet X^\bullet$. From here we can easily calculate that, for example, the Lie derivative of a 2-tensor h on M pulls back to

$$\pi^* \mathcal{L}_X h = \left(\mathcal{L}_X h\right)_{ij} \omega^i \otimes \omega^j$$

where

$$(\mathcal{L}_X h)_{ij} = X^k \nabla_k h_{ij} \omega^i \otimes \omega^j + h_{lj} \left(T^l_{ki} X^k + \nabla_i X^l \right) + h_{il} \left(T^i_{kj} X^k + \nabla_j X^l \right).$$

In this way, we can carry out calculus on manifolds without ever taking local coordinates.

Example 64 (Riemannian geometry). If our G-structure $\pi : B \to M$ is an $O(n)$-structure, i.e. a Riemannian manifold, then the metric g pulls back to the bundle B to give $\pi^* g = \omega^i \otimes \omega^i$. From the above:

$$(\pi^* \mathcal{L}_X g)_{ij} = \nabla_i X^j + \nabla_j X^i.$$

Beware: when we reduce a G-structure, say $B \to M$ to a G_1-structure for $G_1 \subset G$ a subgroup, say to $B_1 \subset B$, the operation ∇ on B_1 is related in a complicated manner to ∇ on B. Many calculations in differential geometry can be expressed as relating these two operators.

7.9 Example: contact and polycontact geometry

The 1-jet of a function $y = f(x)$ at a point $x = x_0$ means the data consisting of the point x_0, the value $f(x_0)$ and the first derivative $f'(x_0)$. Write $J^1(\mathbb{R}^n, \mathbb{R}^t)$ for the space of 1-jets of functions from open subsets of $\mathbb{R}^n$ to $\mathbb{R}^t$. This manifold $M = J^1(\mathbb{R}^n, \mathbb{R}^t)$ bears a special structure. To see it, first we can look at coordinates x^μ on $\mathbb{R}^n$, y^i on $\mathbb{R}^t$, and impose coordinates on M by asking that for a given function $f(x)$ and point x_0, the corresponding point in the jet space have coordinates $\left(x^\mu, y^i, p^i_\mu\right)$ where x^μ are the coordinates of the point x_0, y^i are the coordinates of the point $f(x_0)$ and

$$p^i_\mu = \left. \frac{\partial f^i}{\partial x^\mu} \right|_{x=x_0}.$$

Then a submanifold of M on which the dx^μ are independent is locally the graph of the 1-jets of a function $f(x)$ just when $dy^i - p^i_\mu \, dx^\mu = 0$ on M. We can store this information by saying that M bears a field of planes of corank t in its tangent planes: the field cut out by the equations $dy^i - p^i_\mu \, dx^\mu = 0$ on tangent vectors. The ideal generated by the 1-forms $dy^i - p^i_\mu \, dx^\mu$ and their exterior derivatives is called the *polycontact ideal*.

We have an obvious choice of G-structure on M: on the vector space $V = T_{(0,0,0)} M = \mathbb{R}^n \oplus \mathbb{R}^t \oplus \mathbb{R}^{n \times t}$ (with obvious (x, y, p) coordinates, since $V = M$ in this case) let G_0 be the group of linear transformations of V preserving the plane $dy = 0$. Then M, the 1-jet manifold, bears a G_0-structure.

Let us now consider any G-structure for this group G, i.e. any field of planes of codimension t on any manifold M. We will ask when this field is locally equivalent to the field on $J^1(\mathbb{R}^n, \mathbb{R}^p)$. The structure equations are then

$$d\begin{pmatrix}\omega^i\\ \omega^\mu\\ \omega^i_\mu\end{pmatrix} = -\begin{pmatrix}\gamma^i_j & 0 & 0\\ \gamma^\mu_j & \gamma^\mu_\nu & \gamma^{\mu\nu}_j\\ \gamma^i_{\mu j} & \gamma^i_{\mu\nu} & \gamma^{i\nu}_{\mu j}\end{pmatrix} \wedge \begin{pmatrix}\omega^j\\ \omega^\nu\\ \omega^j_\nu\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\tau^i\\ 0\\ 0\end{pmatrix}$$

where

$$\tau^i = T^i_{\mu\nu}\omega^\mu \wedge \omega^\nu + T^{i\nu}_{\mu j}\omega^\mu \wedge \omega^j_\nu + T^{i\nu\sigma}_{jk}\omega^j_\nu \wedge \omega^k_\sigma.$$

WARNING: we are using a notation in which the Greek and Roman letters are used to distinguish between the two different objects ω^i and ω^μ. We assume that they range over some disjoint set of labels.

It is easy to see that this torsion determines a section of $\Omega^2(M) \otimes TM/F$ where F is the plane field. This is the first invariant we run into. To have a plane field F locally isomorphic to the plane field on the manifold of 1-jets above we must have T identical to the torsion from the manifold of 1-jets. Let us assume henceforth that this is the case.

On the manifold of 1-jets, a section of the G-structure is given by the coframing

$$\begin{pmatrix}\vartheta^i\\ \vartheta^\mu\\ \vartheta^i_\mu\end{pmatrix} = \begin{pmatrix}dy^i - p^i_\mu\, dx^\mu\\ dx^\mu\\ dp^i_\mu\end{pmatrix}$$

with our ideal being generated by the ϑ^i, and further we have the equations

$$d\vartheta^i = -\vartheta^i_\mu \wedge \vartheta^\mu \pmod{\vartheta^j}.$$

(This tells us what the torsion T is. For example, $T \neq 0$.) There are other coframings satisfying the same equations, i.e. the same normalization of T, given by

$$\begin{pmatrix}\omega^i\\ \omega^\mu\\ \omega^i_\mu\end{pmatrix} = \begin{pmatrix}g^i_j & 0 & 0\\ g^\mu_j & g^\mu_\nu & g^{\mu\nu}_j\\ g^i_{\mu j} & g^i_{\mu\nu} & g^{i\nu}_{\mu j}\end{pmatrix}\begin{pmatrix}\vartheta^j\\ \vartheta^\nu\\ \vartheta^j_\nu\end{pmatrix}$$

where the coefficients, to preserve the same torsion, must satisfy the conditions

$$\begin{aligned} g^i_{\mu\nu}g^\mu_\sigma &= g^i_{\mu\sigma}g^\mu_\nu\\ g^i_j\delta^\nu_\mu &= g^{i\nu}_{\sigma j}g^\sigma_\mu - g^i_{\sigma\mu}g^{\sigma\nu}_j\\ g^{i\nu}_{\mu j}g^{\mu\sigma}_k &= g^{i\sigma}_{\mu k}g^{\mu\nu}_j. \end{aligned}$$

These three equations emerge from asking that

$$d\omega^i = -\omega^i_\mu \wedge \omega^\mu \pmod{\omega^j}.$$

Let G be the group of matrices of this form. We will define a *polycontact structure* to be a G-structure satisfying the torsion equations

$$d\omega^i = -\omega^i_\mu \wedge \omega^\mu \pmod{\omega^j}$$

for any section $\omega^i, \omega^\mu, \omega^i_\mu$ of the G-structure. If $t = 1$ (e.g. intuitively if there is only one y variable) then we call such a G-structure a *contact structure.*

We have structure equations

$$d \begin{pmatrix} \omega^i \\ \omega^\mu \\ \omega^i_\mu \end{pmatrix} = - \begin{pmatrix} \gamma^i_j & \omega^i_\mu & 0 \\ \gamma^\mu_j & \gamma^\mu_\nu & \gamma^{\mu\nu}_j \\ \gamma^i_{\mu j} & \gamma^i_{\mu\nu} & \gamma^{i\nu}_{\mu j} \end{pmatrix} \wedge \begin{pmatrix} \omega^i \\ \omega^\mu \\ \omega^i_\mu \end{pmatrix}.$$

The 1-forms

$$\gamma^i_j, \gamma^\mu_\nu, \gamma^{\mu\nu}_j, \gamma^i_{\mu j}, \gamma^i_{\mu\nu}, \gamma^{i\nu}_{\mu j}$$

constitute the pseudoconnection 1-form. The pseudoconnection is valued in the Lie algebra, so

$$\begin{aligned} \gamma^i_{\mu\nu} &= \gamma^i_{\nu\mu} \\ \delta^i_j \gamma^{\nu\sigma}_k &= \delta^i_k \gamma^{\sigma\nu}_j \\ \gamma^{i\nu}_{\mu j} &= \delta^\nu_\mu \gamma^i_j - \delta^i_j \gamma^\nu_\mu. \end{aligned}$$

The second of these equations tells us that if the number of dimensions indexed by Roman letters (i.e. the number t) is more than one, then

$$\gamma^{\mu\nu}_j = 0$$

while if there is exactly one Roman indexed dimension ($t = 1$, i.e. a contact structure) then

$$\gamma^{\mu\nu}_j = \gamma^{\nu\mu}_j.$$

The geometric significance is enormous: if $t > 1$ (not a contact structure), then the requirements $\gamma^{\mu\nu} = 0$ say precisely that the equations $\omega^i = \omega^\mu = 0$ cut out a foliation. Locally, this foliation is a fiber bundle.

Exercise 7.26 If it is a fiber bundle globally, then each symmetry of the polycontact structure acts on the base of the fiber bundle. In particular, the symmetries of the polycontact structure on the 1-jet bundle must act on the base, as diffeomorphisms of the 0-jet bundle.

Exercise 7.27 More generally, suppose that we start with a manifold M with a codimension t field of planes $F \subset TM$, which is a polycontact structure. Now suppose that M turns out to be a global fiber bundle, say $p : M \to \bar{M}$, following the ideas above, so that the fibers are cut out by equations $\omega^i = \omega^\mu = 0$. Then we can map M to the Grassmann bundle $\operatorname{Gr}\left(n, \bar{M}\right)$ of k-planes in the tangent spaces of $\bar{M}$, by taking $m \in M$ to the n-plane

$$\Pi(m) = p'(m) \cdot \left(\omega^i = 0\right).$$

Prove that this map $\Pi : M \to \operatorname{Gr}\left(n, \bar{M}\right)$ is a local equivalence of G-structures, and a global equivalence just when the fibers of $M \to \bar{M}$ are compact with the map $M \to \operatorname{Gr}\left(n, \bar{M}\right)$ an isomorphism on fundamental groups of fibers.

Exercise 7.28 Contact structures are more flexible than polycontact structures. Look at an odd dimensional sphere S^{2n+1} sitting as the unit sphere inside $\mathbb{C}^{n+1}$, and take the contact structure which is the field of maximal complex subspaces in the real tangent spaces of the sphere. Show that the unitary group $U(n+1)$ acts as symmetries of that contact structure, preserving no foliation. In particular, this contact manifold is *not* a fiber bundle. Show that this is a contact structure.

So for contact structures, we have the structure equations

$$d \begin{pmatrix} \omega \\ \omega^\mu \\ \omega_\mu \end{pmatrix} = - \begin{pmatrix} \gamma & \omega_\mu & 0 \\ \gamma^\mu & \gamma^\mu_\nu & \gamma^{\mu\nu} \\ \gamma_\mu & \gamma_{\mu\nu} & \delta^\nu_\mu \gamma - \gamma^\nu_\mu \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\nu \\ \omega_\nu \end{pmatrix}$$

with $\gamma^{\mu\nu}$ and $\gamma_{\mu\nu}$ symmetric in their indices. There are no invariants emerging here. Moreover, calculating Spencer cohomology convinces us that no invariants will emerge at any order, since the Spencer cohomology groups all vanish. Indeed it is well known that all contact structures are locally equivalent.

Remark 19. One can show that all contact structures are locally isomorphic, by constructing the symplectification and using the Moser homotopy proof to match up symplectic structures, followed by a little more work to match up the contact structures; see [7].

On our contact manifold, our plane field is a family of hyperplanes. Any local section of the G-structure will give rise to a local choice of 1-form ω vanishing on these hyperplanes. We will say that a hypersurface $H \subset M$ is *noncharacteristic* if it has none of these hyperplanes as tangent spaces, i.e. $\omega \neq 0$ on H. On this H, any local section of our G-structure will give 1-forms

$$\omega, \omega^\mu, \omega_\mu$$

and among these there must be one linear relation. This relation can not be $\omega = 0$, so it must be of the form

$$A\omega + A_\mu \omega^\mu + A^\mu \omega_\mu = 0$$

with one of the A_μ or the A^μ not zero. The structure group has Lie algebra containing

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma^\mu_\nu & \gamma^{\mu\nu} \\ 0 & \gamma_{\mu\nu} & -\gamma^\nu_\mu \end{pmatrix},$$

the symplectic Lie algebra, which acts transitively on covectors, so we can arrange by change of local section that the relation is $A\omega + \omega_n = 0$. The structure Lie algebra also contains

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_\mu & 0 & 0 \end{pmatrix}$$

which enables us to arrange a section for which the relation is $\omega_n = 0$. Now consider the subbundle of coframings on H which satisfy this condition.

Differentiating the structure equations reveals that on this bundle of coframes

$$\begin{aligned} \gamma_n &= 0 \\ \gamma_n &= 0 \\ \gamma_{n\nu} &= 0 \\ \delta^\nu_n \gamma - \gamma^\nu_n &= 0 \end{aligned}$$

modulo semibasic 1-forms. We find that we can absorb the semibasic 1-forms in these expressions. The resulting structure equations (using the new convention that Greek letters will run from 1 to $n-1$) are

$$d \begin{pmatrix} \omega \\ \omega^\mu \\ \omega^n \\ \omega_\mu \end{pmatrix} = - \begin{pmatrix} \gamma & \omega_\nu & 0 & 0 \\ \gamma^\mu & \gamma^\mu_\nu & 0 & \gamma^{\mu\nu} \\ \gamma^n & \gamma^n_\nu & \gamma & \gamma^{n\nu} \\ \gamma_\mu & \gamma_{\mu\nu} & 0 & \delta^\nu_\mu \gamma - \gamma^\nu_\mu \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\nu \\ \omega^n \\ \omega_\nu \end{pmatrix}.$$

The foliation

$$\omega = \omega^\mu = \omega_\nu$$

by curves is invariantly defined on H, since

$$0 = d\omega = d\omega^\mu = d\omega_\nu \pmod{\omega, \omega^\sigma, \omega_\tau}.$$

These curves are called the *bicharacteristic curves* of H.

Lemma 29. *If the bicharacteristic foliation of a noncharacteristic hypersurface in a contact manifold is a submersion with connected fibers to a manifold Z, then the plane field $\omega = 0$ quotients to a hyperplane field on Z.*

Proof. We calculate that

$$d\omega = 0 \pmod{\omega, \omega^\mu, \omega_\mu}.$$

If X is a vector field tangent to the fibers of $H \to Z$, we find

$$\mathcal{L}_X \omega = -\left(X \lrcorner \gamma\right) \omega$$

(here ω means the 1-form obtained from taking a local section of the bundle over H, and pulling back ω via that local section). So therefore the plane field $\omega = 0$ is carried invariantly along X, so projects to a smooth plane field on Z.

Example 65. One can imagine the fibers of a map $H \to Z$ not being connected, but still being bicharacteristics, so that the induced geometry on Z would be a web geometry.

Theorem 11. *If the bicharacteristic foliation of a hypersurface H in a contact manifold is the foliation by stalks of a submersion $H \to Z$ then Z is also a contact manifold.*

Proof. In fact, for any section of our bundle over H, we find that $\omega, \omega^{\mu}, \omega_{\mu}$ are semibasic for the bundle map $H \to Z$. But they are only defined up to the contact structure group in the next lowest dimension.

This is not the easiest approach to contact reduction, but it is a simple example of the equivalence method point of view.

Question 11. Every holomorphic contact manifold admits a reduction of structure group to a maximal compact subgroup of the contact structure group. Is there a natural choice of reduction, perhaps using the calculus of variations, especially when the underlying complex manifold is Kähler?

7.10 Induced structures on tensor bundles

Suppose that B is a G-structure on a manifold M. Let $\pi : B \to M$ be the bundle map. Take $\rho : G \to \mathrm{GL}(W)$ any G-representation. We can construct a vector bundle $W' = (B \times W)/G$ where the quotient is taken via the right action

$$g \in G, (u, w) \in B \times W \mapsto \left(g^{-1}u, \rho(g)^{-1}w\right).$$

The map $B \times W \to W'$ is a principal G-bundle, but not a G-structure. The vertical directions of the bundle are spanned by the vector fields

$$\vec{A}_{(u,w)} = \left(\vec{A}_u, -Aw\right).$$

Let γ be any pseudoconnection for B, ω the soldering form. Pull them back to $B \times W$. On $B \times W$ define $\dot{\omega} = dw + \rho(\gamma)w$. We are writing dw for the identity map on W, thought of as a 1-form valued in W. Since we know the vertical directions of the bundle map $B \times W \to W'$, we can see that ω and $\dot{\omega}$ are semibasic 1-forms for the bundle map:

$$\omega \in \Omega^1(B \times W) \otimes V, \dot{\omega} \in \Omega^1(B \times W) \otimes W.$$

The definition of $\dot{\omega}$ depends on the choice of pseudoconnection γ. To eliminate the need to make such a choice, we can consider the map

$$B^{(1)} \times W \to FW'$$

defined by $\gamma \mapsto \omega \oplus \dot{\omega}$ using the above expressions. This is a $G \rtimes \mathfrak{g}^{(1)}$ structure on W'. Its structure equations are completely determined from the structure equations on $B^{(1)}$, since the soldering forms are just ω and $\dot{\omega}$, and the pseudoconnection 1-form γ and the pseudoconnection ξ of the prolongation $B^{(1)} \to B$ determine a pseudoconnection on $B^{(1)} \times W$. The relevant representation of $G \rtimes \mathfrak{g}^{(1)}$ is $V \oplus W$, where $(g, Q) \in G \rtimes \mathfrak{g}^{(1)}$ acts on $(v, w) \in V \oplus W$ via

$$(g, 0)(v, w) = (gv, \rho(g)w)$$
$$(1, Q)(v, w) = (v, \rho(Q \cdot v)\, w)\,.$$

7.10.1 Example: the almost complex structure on the tangent bundle of a Riemannian manifold

If $G = SO(n)$, Riemannian geometry, then the structure equations on the bundle B (the bundle of orthonormal coframes) are

$$d\omega^i = -\gamma^i_j \wedge \omega^j$$
$$d\gamma^i_j = -\gamma^i_k \wedge \gamma^k_j + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l.$$

(We could lower all of the indices, if we like.) Consider the tangent bundle $TM = V'$. We have $B^{(1)} = B$ so our prescription above is simply that on $B \times V$, we have 1-forms $\omega, \gamma, \dot{\omega}_V = dv + \gamma v$, so that in components

$$\dot{\omega}^i = dv^i + \gamma^i_j v^j$$

and we can calculate that

$$d\dot{\omega}^i = -\gamma^i_j \wedge \dot{\omega}^j + \frac{1}{2} R^i_{jkl} v^j \omega^k \wedge \omega^l.$$

Notice that $\mathfrak{g}^{(1)} = 0$, so that the representation is $G = SO(n)$ acting on $V \oplus V = \mathbb{R}^n \oplus \mathbb{R}^n$ by $g(v, w) = (gv, gw)$, the sum of the two representations. In particular, we see that the almost complex structure $J(v, w) = (-w, v)$ is defined on this representation, determining an almost complex structure on the tangent bundle.

Exercise 7.29 Calculate from the structure equations above (and use the Newlander–Nirenberg theorem, see [66],[44],[72]) to show that the almost complex structure on TM is a complex structure just when the Riemannian manifold M is flat.

7.10.2 Example: the exponential map

If a manifold M has a G-structure B with a fixed choice of pseudoconnection γ, we map $B \times V \to TM$ and obtain a G-structure on TM. The soldering

1-forms are ω and $\dot{\omega} = dv + \gamma v$, for $v \in V$. If the structure equations of M are written as

$$\begin{aligned} d\omega &= -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega \\ d\gamma &= -\gamma \wedge \gamma + \xi \wedge \omega \end{aligned}$$

where ξ is a multiple of ω and γ, then we find the structure equations of TM to be

$$\begin{aligned} d\omega &= -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega \\ d\dot{\omega} &= -\gamma \wedge \dot{\omega} + \xi v \wedge \omega \\ d\gamma &= -\gamma \wedge \gamma + \xi \wedge \omega. \end{aligned}$$

We can define a vector field E on $B \times V$, called the *geodesic field* by

$$E \lrcorner \begin{pmatrix} \omega \\ \dot{\omega} \\ \gamma \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}.$$

at a point $(u, v) \in B \times V$. Its flow is the *geodesic flow*. We can calculate

$$\mathcal{L}_E \begin{pmatrix} \omega \\ \dot{\omega} \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma v + T v \omega \\ (E \lrcorner \xi)\, v\omega - (\xi v)\, v \\ (E \lrcorner \xi)\, \omega - \xi v \end{pmatrix}.$$

Exercise 7.30 Show that the vector field E is G-invariant just when γ is a connection, in which case E descends to a vector field on TM.

Exercise 7.31 How do the structure equations look for Riemannian geometry?

7.11 Example: projective structures

Question 12. Think carefully about the indexing conventions being used in this section. They could probably be made consistent.

Two connections on the tangent bundle of a manifold are called *projectively equivalent* if they have the same unparameterized geodesics. Connections modulo this equivalence are called *projective structures*. See Kobayashi & Nagano [53] for some historical references and an approach to this topic.

7.11.1 The flat example: projective space

First, let us consider projective space $\mathbb{P}^n$ (either real or complex—the computations will be the same). We bring up projective space because it is glued together out of affine charts, and the transition functions are affine transformations, so preserve straight lines, i.e. geodesics. The geodesic-preserving transformations of projective space are precisely the projective linear transformations, forming the group $\mathbb{PGL}(n+1)$ (a well-known result in geometry due to David Hilbert).[1]

We will think of $\mathbb{P}^n$ as the space of tuples

$$\begin{pmatrix} x^0 \\ \vdots \\ x^n \end{pmatrix}$$

of numbers, not all zero, modulo rescaling. Write the corresponding point of $\mathbb{P}^n$ as

$$\begin{bmatrix} x^0 \\ \vdots \\ x^n \end{bmatrix}.$$

$\mathbb{P}^n$ is acted on transitively by the group $\mathbb{PGL}(n+1)$ of projective linear transformations, i.e. linear transformations of the x variables modulo rescaling. We will write $[g]$ for the element of $\mathbb{PGL}(n+1)$ determined by an element $g \in \mathrm{GL}(n+1)$. The stabilizer of the point

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the group H_{pt} consisting of $[g]$ where g is a matrix of the form

$$[g] = \begin{bmatrix} g_0^0 & g_j^0 \\ 0 & g_j^i \end{bmatrix}$$

where $i, j = 1, \dots, n$. The Lie algebra of $\mathbb{PGL}(n+1)$ is just $\mathfrak{sl}(n+1)$, so consists of the matrices of the form

$$\begin{pmatrix} A_0^0 & A_j^0 \\ A_0^i & A_j^i \end{pmatrix}$$

[1] Note that over the complex numbers, this result holds only with the assumption of continuity, and preservation of orientation of the complex lines. Over the real numbers, not even continuity need be assumed.

with $A_0^0 + A_i^i = 0$. We define the Maurer–Cartan 1-form $\omega \in \Omega^1\left(\mathbb{PGL}\left(n+1\right)\right) \otimes \mathfrak{sl}\left(n+1\right)$ by $\omega = g^{-1}\,dg$. This form satisfies $d\omega = -\omega \wedge \omega$. Splitting into components, we calculate

$$
\begin{aligned}
d\omega_0^i &= -\left(\omega_j^i + \delta_j^i \omega_k^k\right) \wedge \omega_0^j \\
d\omega_j^i &= -\omega_k^i \wedge \omega_j^k + \omega_j^0 \wedge \omega_0^i \\
d\omega_i^0 &= \left(\omega_i^j + \delta_i^j \omega_k^k\right) \wedge \omega_j^0
\end{aligned}
$$

If we let $\omega^i = \omega_0^i, \gamma_j^i = \omega_j^i + \delta_j^i \omega_k^k$, and $\omega_i = \omega_i^0$ then we find

$$
\begin{aligned}
d\omega^i &= -\gamma_j^i \wedge \omega^j \\
d\gamma_j^i &= -\gamma_k^i \wedge \gamma_j^k + \left(\omega_j \delta_k^i + \omega_k \delta_j^i\right) \wedge \omega^k \\
d\omega_i &= \gamma_i^j \wedge \omega_j .
\end{aligned}
$$

The group H_{pt} is a semidirect product: each element factors into two elements of the form

$$
\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.
$$

It will be helpful later to see how each of these factors acts on our differential forms. This is not difficult, since the form $\omega = g^{-1}\,dg$ satisfies

$$
r_h^* \omega = \operatorname{Ad}_h^{-1} \omega .
$$

We leave to the reader to calculate that if we write g for the matrix

$$
\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}
$$

and λ for the matrix

$$
\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}
$$

then

$$
\begin{aligned}
r_g^* \omega^i &= \left(g^{-1}\right)_j^i \omega^j \\
r_g^* \gamma_j^i &= \left(g^{-1}\right)_k^i \gamma_l^k g_j^l \\
r_g^* \omega_i &= \omega_j g_i^j \\
r_\lambda^* \omega^i &= \omega^i \\
r_\lambda^* \gamma_j^i &= \gamma_j^i + \left(\lambda_j \delta_k^i + \lambda_k \delta_j^i\right) \omega^k \\
r_\lambda^* \omega_i &= \omega_i - \lambda_j \gamma_i^j - \lambda_i \lambda_j \omega^j .
\end{aligned}
$$

We can reconsider the projective geometry above in terms of bundles. The group $\mathbb{PGL}(n+1)$ acts transitively on $\mathbb{P}^n$, and also on the frame bundle $F\mathbb{P}^n$. The stabilizer of a point of $\mathbb{P}^n$ is H_{pt}; the stabilizer of a frame at a point is H_{frm} consisting of matrices of the form

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

We leave this as an exercise. (It is easy to show that $\mathbb{P}^n$ has tangent spaces $T_P\mathbb{P}^n = P^* \otimes \left(\mathbb{R}^{n+1}/P\right)$, and then this result is not difficult.)

So we can identify

$$\begin{array}{ccc} F\mathbb{P}^n & \longleftrightarrow & \mathbb{PGL}(n+1)/H_{\text{frm}} \\ \downarrow & & \downarrow \\ \mathbb{P}^n & \longleftrightarrow & \mathbb{PGL}(n+1)/H_{\text{pt}}. \end{array}$$

We have another bundle over $\mathbb{P}^n$, $\mathbb{PGL}(n+1)$ itself, which we can put on the top at the right side. We will build a corresponding bundle on the left side.

Consider the geodesics of projective space. These are the projective lines. If we think of projective space as the space of lines through 0 in a vector space, its geodesics correspond to 2-planes in that vector space. Thus the space of geodesics is $\operatorname{Gr}(2, n+1) = \mathbb{PGL}(n+1)/H_{\text{geod}}$ where H_{geod} consists of matrices of the form

$$[g] = \begin{bmatrix} g^0_0 & g^0_1 & g^0_J \\ g^1_0 & g^1_1 & g^1_J \\ 0 & 0 & g^I_J \end{bmatrix}$$

where $I, J = 2, \dots, n$. Above the space of geodesics is the space of pointed geodesics, which is the space of choices of a 2-plane in our vector space with a line in that 2-plane, so it is $\mathbb{PGL}(n+1)/H_{\text{pt+geod}}$ where $H_{\text{pt+geod}} \subset H_{\text{pt}}$ consists of matrices of the form

$$[g] = \begin{bmatrix} g^0_0 & g^0_1 & g^0_J \\ 0 & g^1_1 & g^1_J \\ 0 & 0 & g^I_J \end{bmatrix}.$$

7.11.2 When do connections have the same geodesics

Now we turn to the real problem at hand: declaring two connections on the tangent bundle of a manifold to be equivalent if the have the same unparameterized geodesics, what are the invariants of such an equivalence class?

Proposition 27. *Given any connection γ on the frame bundle of a manifold, the connections $\tilde{\gamma}$ with the same geodesics are precisely those of the form*

$$\tilde{\gamma}^i_j = \gamma^i_j + \left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right) \omega^k + a^i_{jk} \omega^k$$

where $\lambda = \lambda_k \omega^k$ is any 1-form on M, pulled back to FM, and $a^i_{jk} \omega^j \wedge \omega^k$ is any section of $\Lambda^2 \left(T^* M\right) \otimes TM$, pulled back to FM.

Proof. Consider any manifold M^{n+1} with an equivalence class of connections γ for the tangent bundle, where equivalent means having the same unparameterized geodesics. Let us take our soldering 1-form ω on FM and write its components $\omega^0, \dots, \omega^n$. We will use Greek indices for $0, \dots, n$ and Roman for $1, \dots, n$. (This is not in agreement with the conventions used in the statement of the proposition we are proving, but it is convenient for the moment.) Take one such geodesic, say $\Gamma \subset M$ (which we take to be embedded, by taking a sufficiently small piece of it). Define the principal right bundle $\Gamma_1 \subset FM$ to be the set of all frames $(x, u) \in FM$ so that $x \in \Gamma$ and $u : T_x M \to \mathbb{R}^{n+1}$ takes $T_x \Gamma$ to $\mathbb{R} \oplus 0 \subset \mathbb{R}^{n+1}$. The structure group of this bundle is the group of matrices of the form

$$\begin{pmatrix} a^0_0 & a^0_i \\ 0 & a^i_j \end{pmatrix}.$$

The equation of a geodesic in FM is $\omega^j = \gamma = 0$. The tangent space to Γ_1 satisfies the equations $\omega^j = 0$, and therefore $d\omega^j = -\gamma^j_0 \wedge \omega^0 = 0$. So $\omega^j = \gamma^j_0 = 0$ on Γ_1, and by dimension count, this is the tangent space to Γ_1. If we have another connection $\tilde{\gamma}$ with the same geodesics, we would have the same ideals:

$$\left(\omega^j, \gamma^k_0\right) = \left(\omega^j, \tilde{\gamma}^k_0\right)$$

since Γ_1 is independent of the choice of connection, depending only on the curve Γ. Since both γ and $\tilde{\gamma}$ are connections, they must agree on vertical vectors, and so their difference must be semibasic:

$$\tilde{\gamma}^\mu_\nu = \gamma^\mu_\nu + a^\mu_{\nu\sigma} \omega^\sigma.$$

But then the equality of the two ideals forces

$$a^j_{00} = 0.$$

Since the indices can be permuted freely, this implies that

$$a^\mu_{\nu\sigma} = 0$$

whenever $\nu = \sigma \neq \mu$. We check that the expression D defined by

$$D \lrcorner \, \omega^\mu = a^\mu_{\nu\sigma} \omega^\nu \otimes \omega^\sigma$$

defines a tensor D, a section of $T^* M \otimes T^* M \otimes TM$. (This requires that our γ and $\tilde{\gamma}$ be connections, not pseudoconnections, so that they transform in the adjoint representation.) We can split this into a symmetric and an antisymmetric part: $D = A + S$, sections of $\Lambda^2 (T^* M) \otimes TM$ and $\mathrm{Sym}^2 (T^* M) \otimes TM$. If $v \in T_x M$ is not zero, pick a frame (x, u) above x at which $u(v) = e_\mu$ is a fixed direction, and find $D(v, v) = S(v, v)$ must be a multiple of v from the above reasoning. Define a 1-form by $\lambda(v)v = \frac{1}{2} S(v, v)$. It is easy to show that this λ is well defined. Moreover, we now have

$$\tilde{\gamma}^\mu_\nu = \gamma^\mu_\nu + \left(\lambda_\nu \delta^\mu_\tau + \lambda_\tau \delta^\mu_\nu\right) \omega^\tau + a^\mu_{\nu\sigma} \omega^\sigma$$

where $\lambda = \lambda_\tau \omega^\tau$ when we pull back to FM.

Corollary 12. *Given any connection, there is a torsion-free connection with the same geodesics (and with the same parameterization for those geodesics).*

Proof. Set

$$a^i_{jk} = \frac{1}{2} T^i_{jk},$$

in other words just subtract off the torsion. Since the antisymmetric A part of the difference D in connection is arbitrary, this is permissable without changing the unparameterized geodesics.

Exercise 7.32 Show that the geodesics end up with the same parameterization.

Example 66. Which conformal changes of metric preserve geodesics? On the frame bundle of a Riemannian manifold, the structure equations are

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ \gamma^i_j + \gamma^j_i &= 0. \end{aligned}$$

A metric conformal to the given one has soldering 1-forms

$$\tilde{\omega}^i = e^f \omega^i$$

for some function f on the manifold. Write the exterior derivative of f as

$$df = f_i \omega^i.$$

Calculate that the connection 1-forms are

$$\tilde{\gamma}^i_j = \gamma^i_j + f_i \omega^j - f_j \omega^i.$$

To preserve unparameterized geodesics, this must have the form

$$\tilde{\gamma}^i_j = \gamma^i_j + \lambda_j \omega^i + \delta^i_j \lambda_k \omega^k$$

for some functions λ_k. If $i \neq j$ then

$$f_i \omega^j - f_j \omega^i = \lambda_j \omega^i$$

forcing $f_j = 0$. Therefore $df = 0$, and we have the result:

Proposition 28. *Every conformal change of Riemannian metric preserving unparameterized geodesics is a rescaling by a constant.*

The conformal transformations of a fixed metric modulo constant rescaling embed into the space of projective structures.

7.11.3 Structure equations of projective structures

Suppose that our manifold M^n has a projective structure $[\nabla]$, i.e. a choice of connection ∇ on the tangent bundle, up to equivalence, or equivalently a foliation of the projectived tangent bundle by the geodesics of some unknown connection. We will use the V valued frame bundle, as usual, and when convenient we identify $V = \mathbb{R}^n$ by taking any basis of V. Let $B \to FM$ be the bundle of triples (m, u, Γ) so that $u \in FM$, $u : T_m M \to V$ a linear isomorphism, and Γ is any connection 1-form $\Gamma : T_u FM \to \mathfrak{gl}(V)$, i.e. $\vec{A} \lrcorner \Gamma = A$, so that this Γ can occur as the value at u, $\Gamma = \tilde{\gamma}_u$, of a torsion-free connection $\tilde{\gamma}$ with the given geodesics. By the proposition above, B is a principal V^* bundle under the right action

$$\lambda \in V^*, (m, u, \Gamma) \in B \mapsto r_\lambda(m, u, \Gamma) = (m, u, \Gamma + \langle \lambda, \omega \rangle 1_V + \lambda \otimes \omega) .$$

Define $\pi : (m, u, \Gamma) \in B \mapsto u \in FM$ and define a 1-form γ on B by

$$w \lrcorner \gamma_{(m,u,\Gamma)} = (\pi'(m, u, \Gamma)w) \lrcorner \Gamma.$$

Just as for projective space:

$$\begin{aligned} r_\lambda^* \omega^i &= \omega^i \\ r_\lambda^* \gamma^i_j &= \gamma^i_j + \left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right) \omega^k . \end{aligned}$$

We also have an action of $\mathrm{GL}(V)$ on B :

$$r_g (m, u, \Gamma) = \left(m, g^{-1}u, Ad_g^{-1} \Gamma \left(r_g^{-1}\right)' \left(g^{-1}u\right)\right)$$

for which

$$\begin{aligned} r_g^* \omega &= g^{-1} \omega \\ r_g^* \gamma &= Ad_g^{-1} \gamma \end{aligned}$$

again just like projective space. As we have done before, we can differentiate these equations, for example

$$\mathcal{L}_{\vec{A}} \omega = -A\omega$$

etc., to find equations that govern the exterior derivatives, using the formula

$$\mathcal{L}_X \xi = d\left(X \lrcorner \xi\right) + X \lrcorner d\xi$$

for any vector field X and differential form ξ. Applying these equations, and the fact that $\mathrm{GL}(V)$ equivariant local sections of $B \to FM$ are torsion-free connections, we get

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k + \nabla \gamma^i_j \end{aligned}$$

for some 1-forms ω_j satisfying

$$\omega_j\left(\vec{\lambda}\right) = \lambda_j \text{ and } \omega_j\left(\vec{A}\right) = 0,$$

for $\lambda \in V^*$ and $A \in \mathfrak{gl}(V)$ (i.e. pseudoconnection 1-forms for $B \to FM$), and some forms

$$\nabla \gamma^i_j = \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l.$$

By taking exterior derivative of these equations, we find

$$\nabla \gamma^i_j \wedge \omega^j = 0.$$

We still have some freedom in picking ω_i. Indeed, one can see from these equations that $\tilde{\omega}_i = \omega_i + a_{ij}\omega^j$ can be used in place of ω_i with the effect that $\nabla \gamma^i_j$ changes to

$$\nabla \gamma^i_j - a_{jl}\omega^l \wedge \omega^i - \delta^i_j a_{kl} \omega^l \wedge \omega^k.$$

Using the antisymmetric part of a_{ij} we can arrange

$$\nabla \gamma^i_i = 0.$$

This implies, together with $\nabla \gamma^i_j \wedge \omega^j = 0$, that K^i_{jil} is symmetric in j and l. Now using the symmetric part of a_{ij}, we can arrange

$$K^i_{jil} = 0$$

as long as $n > 1$.

Proposition 29. *Suppose that M^n is a manifold with projective structure $[\nabla]$. Let $B \to FM$ be the bundle whose $\mathrm{GL}(n)$-equivariant sections are the torsion-free connections on M with geodesics given by the projective structure. There are 1-forms $\omega^i, \gamma^i_j, \omega_i$ and functions K^i_{jkl} and K_{ijk} on B satisfying*

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k + \nabla \gamma^i_j \\ d\omega_i &= \gamma^k_i \wedge \omega_k + \nabla \omega_i \\ \nabla \gamma^i_j &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l \\ \nabla \omega_i &= \frac{1}{2} K_{ijk} \omega^j \wedge \omega^k \end{aligned}$$

with

$$
\begin{aligned}
0 &= K^i_{jkl} + K^i_{jlk} \\
0 &= K^i_{ikl} \\
0 &= K^i_{jil} \\
0 &= K^i_{jkl} + K^i_{klj} + K^i_{ljk} \\
0 &= K_{jkl} + K_{jlk} \\
0 &= K_{jkl} + K_{klj} + K_{ljk}.
\end{aligned}
$$

Moreover the ω^i are the soldering 1-forms on FM and the γ^i_j are the soldering 1-forms on B, and the ω_i 1-forms and the K functions are uniquely determined by the above equations. Under the $\mathrm{GL}(n)$ *action on $B \to FM \to M$ we have*

$$
\begin{aligned}
r_g^* \omega^* &= g^{-1} \omega^* \\
r_g^* \gamma &= Ad_g^{-1} \gamma \\
r_g^* \omega_* &= \omega_* g.
\end{aligned}
$$

Under the $\mathbb{R}^{n*}$ action on $B \to FM$ we have

$$
\begin{aligned}
r_\lambda^* \omega^* &= \omega^* \\
r_\lambda^* \gamma &= \gamma + \langle \lambda, \omega \rangle \cdot 1 + \lambda \otimes \omega
\end{aligned}
$$

Proof. Taking exterior derivatives of the equations we have already established gives these results immediately.

Question 13. Are these a complete set of equations on the invariants—in other words does the Cartan–Kähler theorem tell us that we can pick the values of invariants arbitrarily at one point, subject to these equations, and construct a projective structure with those values for invariants at a point?

Exercise 7.33

$$
\begin{aligned}
r_g^* \nabla \gamma &= g^{-1} \nabla \gamma g \\
r_g^* \nabla \omega &= \nabla \omega g
\end{aligned}
$$

Lemma 30.

$$
\begin{aligned}
r_\lambda^* \omega_i &= \omega_i - \lambda_j \gamma^j_i - \lambda_i \lambda_j \omega^j \\
r_\lambda^* \nabla \gamma^i_j &= \nabla \gamma^i_j \\
r_\lambda^* \nabla \omega_i &= \nabla \omega_i - \lambda_j \nabla \gamma^j_i.
\end{aligned}
$$

Proof. These equations come from differentiating the equation for $r_\lambda^* \gamma$ (and it helps to have the equations from projective space, in order to guess the right answer for $r_\lambda^* \omega_i$).

We have a tower of principal bundles $B \to FM \to M$. Let us put these together into a single principal bundle $B \to M$. The actions of $\lambda \in \mathbb{R}^{n*}$ and $g \in \mathrm{GL}(n)$ can be put together into a single action of H_{pt} by letting

$$[h] = \begin{bmatrix} h^0_0 & h^0_i \\ 0 & h^i_j \end{bmatrix}$$

act on $(x, u, U) \in B$ by

$$r_{[h]}(x, u, U) = r_\lambda r_g (x, u, U)$$

where

$$\begin{aligned} \lambda_i &= h^0_i / h^0_0 \\ g^i_j &= h^i_j / h^0_0. \end{aligned}$$

In terms of $[h]$ the action is

$$r_{[h]}(x, u, U) = \left(x, g^{-1} u, Ad_g^{-1} U \left(r_g^{-1} \right)' \left(x, g^{-1} u \right) - \langle \lambda, \omega \rangle \, 1 - \lambda \otimes \omega \right).$$

This makes $B \to M$ into a right principal H_{pt} bundle.

Lemma 31. *A projective structure $B \to FM$ admits a canonical $\mathbb{R}^{n*}$-equivariant embedding $B \to FM^{(1)}$, making it an $\mathbb{R}^{n*}$-structure on FM, a second order structure.*

Proof. Each triple $(x, u, U) \in B$ already belongs to $FM^{(1)}$ by definition of projective structure and prolongation. Moreover, we have imposed the same actions on B and on $FM^{(1)}$.

Proposition 30. *Suppose that B is a manifold equipped with a coframing by 1-forms $\omega^i, \gamma^i_j, \omega_j$ and equipped with functions K^i_{jkl} and K_{ijk} satisfying the structure equations*

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j \right) \wedge \omega^k + \nabla \gamma^i_j \\ d\omega_i &= \gamma^k_i \wedge \omega_k + \nabla \omega_i \\ \nabla \gamma^i_j &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l \\ \nabla \omega_i &= \frac{1}{2} K_{ijk} \omega^j \wedge \omega^k \end{aligned}$$

with

$$
\begin{aligned}
0 &= K^i_{jkl} + K^i_{jlk} \\
0 &= K^i_{ikl} \\
0 &= K^i_{jil} \\
0 &= K^i_{jkl} + K^i_{klj} + K^i_{ljk} \\
0 &= K_{jkl} + K_{jlk} \\
0 &= K_{jkl} + K_{klj} + K_{ljk}.
\end{aligned}
$$

Suppose that the foliation $\omega^i = 0$ is a fiber bundle $B \to M$ with connected fibers and that the foliation $\omega^i = \gamma^i_j = 0$ is a fiber bundle $B \to F$ over a manifold F, with connected fibers. There is a local diffeomorphism $F \to FM$, and a unique projective structure $\tilde{B} \to FM$ with a map $B \to \tilde{B}$ which is a local diffeomorphism, matching up the given differential forms and functions on B with those on $\tilde{B}$.

Proof. Apply theorem 6 on page 91, twice.

Question 14. What are the objects defined on the base, and the characteristic classes?

Proposition 31. *The curvature $\nabla\gamma^i_j$ of a projective structure vanishes everywhere precisely when it is flat, which occurs just when it is locally equivalent to the projective structure on projective space.*

Proof. Equivalent projective structures must have agreement in their curvatures. But on projective space, we have exhibited the projective structure explicitly (it is $\mathbb{PGL}(n+1) \to \mathbb{P}^n$), and see directly that the curvatures vanish. So the vanishing of curvatures is a necessary condition for isomorphism with projective space. Taking exterior derivatives of the structure equations, one finds that if the K^i_{jkl} all vanish, then so do the K_{ijk}.

7.11.4 Completeness

Definition 34. *Suppose that $B \to M$ is a projective structure. For each $v \in \mathbb{R}^n$ define the vector field $\vec{v}$ on B by*

$$
\vec{v} \lrcorner \begin{pmatrix} \omega^i \\ \gamma^i_j \\ \omega_i \end{pmatrix} = \begin{pmatrix} v^i \\ 0 \\ 0 \end{pmatrix}.
$$

A projective structure $B \to M$ is called complete *if all of the vector fields $\vec{v}$ are complete.*

Exercise 7.34 Define the exponential map of a projective structure. (Start by figuring out what bundle it should be defined on.) Show that it is a local diffeomorphism.

Theorem 12. *A complete, flat projective structure is the quotient of the standard projective structure on projective space, or (if we are working over the real numbers, rather than the complex numbers) to the standard projective structure on the sphere (induced as the double cover of projective space), by a discontinuous group action. A flat projective structure on a compact manifold with finite fundamental group is a quotient of the sphere by a finite subgroup of* $SL(n+1,\mathbb{R})$. *The symmetry group of a projective structure on an* n*-dimensional manifold has dimension no more than* n^2+2n; *this dimension of symmetry group is acheived only by the standard projective space or the sphere.*

Proof. A symmetry group of dimension n^2+2n occurs on real projective space: $\mathbb{PGL}(n+1)$, and on the sphere: $SL(n+1)$. On the other hand, the symmetry group of a projective structure must act faithfully on the bundle $B \to M$ of the projective structure (see theorem 15 on page 176), and each of the orbits of the symmetry group must be an embedding of the symmetry group into B. If the group has the same dimension as B, then the orbits on B must all be open, and so the curvature functions K^i_{jkl} and K_{ijk} must be locally constant. But they occur in irreducible representations of the structure group, and must transform under the structure group in those representations. Therefore they must all vanish. The rest follows from the results of section 7.6.

Remark 20. Note that the appropriate notion of completeness is not the completeness in some Riemannian metric, but rather that the geodesic flow is complete on the bundle B. For example, the standard flat Riemannian metric on Euclidean space is complete as a Riemannian metric, but the induced projective structure (which is the standard flat projective structure) is not complete. To see this, note that it sits as an affine chart inside projective space, and the flows that would need to be complete generate the entire projective linear group. Now consider the flat torus; the flat metric induces a projective structure. Lift it up to the covering space, the standard flat structure, which is not complete. If the torus' projective structure was complete, then the projective structure upstairs would be complete too. Therefore one can have incomplete flat projective structures on both simply connected and on compact manifolds, but not on compact, simply connected manifolds.

Remark 21. By lemma 24 on page 123, if the fundamental group of a compact n dimensional manifold defies $\mathbb{PGL}(n+1,\mathbb{R})$ (a purely algebraic property of the fundamental group) then the manifold has no flat projective structure.

7.11.5 Geodesics

Given any immersed curve $\phi : C \to M$ in a manifold M with projective structure $B \to FM \to M$, consider the pullback bundle $\phi^* B \to C$ and inside it the subbundle B_C consisting of choices of (m,u,Γ) for which $u(T_m C) = \mathbb{R} \oplus 0 \subset \mathbb{R}^n$. Then on B_C we have the equations $\omega^J = 0$ for $J > 1$ and $\omega^1 \neq 0$. Taking exterior derivative, we find $\gamma^J_1 = k^J \omega^1$ for some functions $k^J : \Gamma_0 \to \mathbb{R}$.

Exercise 7.35 $k = \sum_{J>1} k^J \frac{\partial}{\partial \omega^J} \left(\omega^1\right)^2$ is defined on the curve C, forming a section of $\phi^* TM/\phi(TC) \otimes \left(T^* C\right)^{\otimes 2}$. Call this the *geodesic curvature* of C.

Definition 35. *An immersed curve in a manifold with projective structure is called a* geodesic *if its geodesic curvature vanishes.*

A choice of connection γ^i_j on a manifold M picks out a coframing at each point of $FM : \omega^i, \gamma^i_j$, and thus a section of $FM^{(1)} \to FM$.

Lemma 32. *Suppose that M is a manifold bearing both a torsion-free connection γ and a projective structure $B \to FM$. Then the geodesics of the connection are the same as those of the projective structure (as unparameterize curves on M) precisely when, under pulling back under the section of $FM \to FM^{(1)}$, the image is contained in B. Equivalently, a torsion-free connection with a given projective structure is just a reduction of structure group, from H_{pt} to $\mathrm{GL}(n, \mathbb{R})$, killing the action of the λ variables.*

Proof. This is just the definition of B.

Lets follow the structure equations for such a connection. The ω^i and γ^i_j match up on FM and B by the reproducing property. The ω_i must pull back to be multiples of the ω^i and γ^i_j, and the equation $\vec{A} \lrcorner \omega_i = 0$ forces ω_i to be a multiple of the ω^j, say

$$\omega_i = a_{ij} \omega^j$$

giving

$$d\gamma^i_j = -\gamma^i_k \wedge \gamma^k_j + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l$$

with

$$R^i_{jkl} = K^i_{jkl} + 2a_{jk}\delta^i_l + 2a_{lk}\delta^i_j.$$

Question 15. What happens when there is torsion in the connection?

7.11.6 Holomorphic projective structures

Remark 22. A complex curve (possibly singular) is called *rational* when it is the image under a holomorphic map of the projective line $\mathbb{P}^1$.

Proposition 32. *A holomorphic projective structure on a complex manifold is complete just when the geodesics are immersed rational curves.*

Proof. Consider the equations of geodesics $\omega^I = \gamma^I_1 = 0$ on the bundle B of a projective structure. These equations are holonomic (i.e. satisfy the conditions of the Frobenius theorem), so their leaves foliate B. We have seen that the leaves are just the B_C bundles over geodesics C. On B_C the 1-forms

$$\omega^1, \gamma^1_1, \gamma^1_J, \gamma^I_J, \omega_1, \omega_J$$

form a coframing, with the structure equations

$$
\begin{aligned}
d\omega^1 &= -\gamma^1_1 \wedge \omega^1 \\
d\gamma^1_1 &= 2\omega_1 \wedge \omega^1 \\
d\gamma^1_J &= -\gamma^1_1 \wedge \gamma^1_J - \gamma^1_K \wedge \gamma^K_J + \omega_J \wedge \omega^1 \\
d\gamma^I_J &= -\gamma^I_K \wedge \gamma^K_J + \delta^I_J \omega_1 \wedge \omega^1 \\
d\omega_1 &= \gamma^1_1 \wedge \omega_1 \\
d\omega_J &= \gamma^1_J \wedge \omega_1 - \gamma^K_J \wedge \omega_K
\end{aligned}
$$

which are the Maurer–Cartan structure equations of H_{geod}. Note that $B_C \to C$ is an $H_{\text{pt+geod}}$ bundle, and that $H_{\text{geod}}/H_{\text{pt+geod}} = \mathbb{P}^1$ is simply connected. By lemma 20 on page 121, if the projective structure is complete, then $B_C \to C$ is a complete flat Cartan connection, so C is covered by $\mathbb{P}^1$. By the classification of complex curves (see Forster [38]), the map $\mathbb{P}^1 \to C$ must be a biholomorphism, so the geodesic is rational.

Conversely, if C is a rational curve, lemma 23 on page 123 says that the Cartan connection $B_C \to C$ is complete, and therefore the vector field dual to ω^1 is complete on B_C. Since the manifolds B_C for the various geodesics C foliate B, the vector field dual to ω^1 is complete on B.

We need now only show completeness of all of the vector fields dual to the $\omega^i, \gamma^i_j, \omega_j$ coframing on B. We have completeness of those dual to $\omega^1, \gamma^1_1, \gamma^1_J, \gamma^I_J, \omega_1, \omega_J$. The vector fields dual to γ^*_*, ω_* generate the action of the structure group, so they must be complete. We need only check on the vector fields dual to the ω^I for $I > 1$. By just changing the ordering of the indices in the proofs above (using a 2 index instead of a 1 index, etc.), we get completeness of all of the dual vector fields.

Definition 36. *Say that a geodesic C of a projective structure is* complete *just when the Cartan connection on B_C (given in the proof of the last proposition) is complete.*

Corollary 13. *On a complex manifold with holomorphic projective structure, a geodesic is complete just when it is rational with normal bundle $\bigoplus^{n-1} \mathcal{O}(1)$.*

Proof. Split $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$, and get the structure group $H_{\text{pt+geod}}$ to act on $B_C \times \mathbb{C}^n$ in the obvious manner on B_C, and on $\mathbb{C}^n$ by taking

$$
[g] = \begin{bmatrix} g^0_0 & g^0_1 & g^0_J \\ 0 & g^1_1 & g^1_J \\ 0 & 0 & g^I_J \end{bmatrix},
$$

rescaling to get $g^0_0 = 1$, and then having $[g]$ act on vectors in $\mathbb{C}^n$ as the matrix

$$\begin{pmatrix} g^1_1 & g^1_J \\ 0 & g^I_J \end{pmatrix}.$$

Using the methods of lemma 25 on page 124, we see that

$$\begin{aligned} TC &= \left(B_C \times \mathbb{C}^1\right)/H_{\text{pt+geod}}, \\ C^*TM &= \left(B_C \times \mathbb{C}^n\right)/H_{\text{pt+geod}}, \\ NC &= C^*TM/TC = \left(B_C \times \mathbb{C}^{n-1}\right)/H_{\text{pt+geod}}. \end{aligned}$$

Because B_C is identified with $B_{\mathbb{P}^1}$, as above, these bundles are all isomorphic to those obtained for the flat case $M = \mathbb{P}^n$.

Theorem 13. *The only complete holomorphic projective structures on any complex manifolds are the quotients of the standard projective structure on* $\mathbb{P}^n$ *by discrete groups of projective transformations.*

Proof. To be complete, all geodesics must be rational, with tangent bundles $\mathcal{O}(2)$ and normal bundles $\bigoplus^n \mathcal{O}(1)$. Therefore the tensor represented by K^i_{jkl} lives in a vector bundle which is a sum of line bundles of the form $\mathcal{O}(d_1 - d_2 - d_3 - d_4)$, where each d_1, d_2, d_3, d_4 is either 1 or 2. But that forces the line bundles to be negative, so there are no nonzero holomorphic sections, and the projective structure is flat and complete. By lemma 20 on page 121 it is locally Klein.

Remark 23. Compact complex surfaces which admit a projective structure have been classified by Gunning [43] and Kobayashi & Ochiai [55].

Question 16. Why is $\mathbb{P}SL(n+1, \mathbb{R})$ a maximal Lie group of transformations of $\mathbb{P}^n$? This must be connected with the projective structure bundle $B \to F\mathbb{P}^n \to \mathbb{P}^n$. Any bigger group would have to have orbit inside $F\mathbb{P}^{n(1)}$ of larger dimension. So larger $\mathfrak{g}_1$ than $\mathbb{R}^{n*}$. The structure equations should then force $\mathfrak{g}_2 \neq 0$. Then some induction argument should apply.

8

Symmetries of G-structures

Remark 24. Every example so far has either had a finite dimensional Lie group of symmetries (e.g. Riemann surfaces have finite dimensional symmetry groups), or some kind of "infinite dimensional" symmetry group. We won't try to formulate a theory of infinite dimensional groups in this book; see Milnor [63] and Kamran & Robart [49]. However, Kim & Zaitsev [50] provide examples of G-structures so wildly behaved that their symmetry group would appear to admit no such structure, being just a union of infinitely many finite but arbitrarily large subgroups of a single infinite dimensional Lie group.

8.1 Symmetries as integral manifolds of an exterior differential system

Symmetries are autoequivalences, so in some sense a special case of the method of equivalence. We have seen that a map between manifolds matching their G-structures

$$\begin{array}{ccc} B & \xrightarrow{F\phi} & B' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\phi} & M' \end{array}$$

is equivalent to a map between the bundles matching up the soldering form. Consider a G-structure $B \to M$. We have

$$d\omega = -\gamma \wedge \omega + T\omega \wedge \omega$$

for some pseudoconnection γ. Symmetries of this G-structure are represented by maps

$$F\phi : B \to B$$

so that

$$F\phi^* \omega = \omega$$

Consider two copies of B, say B_1 and B_2, and on them we put soldering forms ω_1 and ω_2, pseudoconnections γ_1 and γ_2, and torsion T_1, T_2. The graph of a symmetry of the G-structure is a submanifold of $B_1 \times B_2$, on which $\omega_1 = \omega_2$ and so $d\omega_1 = d\omega_2$. The 1-form components of ω_1 and γ_1 are independent 1-forms on the graph. In this way, we have translated the problem of finding symmetries to one of solving an exterior differential system (a system of equations in differential forms), with an independence condition.

We restrict to the submanifold on which

$$[T_1] = [T_2]$$

(hoping that it is actually a submanifold; if the instrinsic torsions are of different types, then of course, this locus is empty.) We may attempt to study the resulting equations using the Cartan–Kähler theorem.[1]

The solutions of this differential system will usually be local symmetries, between open subsets of M.

Definition 37. *A* pseudogroup *on a topological space M is a set Γ of homeomorphisms between open sets of M, say $\phi : U \to V$, so that*

1. *if $\phi : U \to V$ belongs to Γ, and $U' \subset U$ is an open subset, then the restriction $\phi|_{U'} : U' \to \phi(U')$ belongs to Γ*
2. *the composition of two elements of Γ (when defined) belongs to Γ*
3. *inverses of elements of Γ belong to Γ*
4. *domains U of elements ϕ of Γ cover M*
5. *if $\phi : U \to V$ is a homeomorphism, and U_α is an open cover of U, and $\phi|_{U_\alpha}$ belongs to Γ for all α, then ϕ belongs to α.*

A Lie pseudogroup *is a pseudogroup whose elements are precisely the solutions of a system of partial differential equations and inequalities invariant under the pseudogroup.*

Clearly the symmetries of a G-structure form a Lie pseudogroup.

The space of integral elements of these equations at each point is exactly identified with the first prolongation $\mathfrak{g}^{(1)}$, since we need to write $d\omega_0 - d\omega_1 = \pi \wedge \omega_0$ where π can be anything of the form

$$\pi = p\omega$$

for $p \in \mathfrak{g}^{(1)}$.

Theorem 14. *If the structure equations on $B^{(k)}$ of an analytic G-structure $B \to M$ are in involution, with Cartan characters $s_1 \geq s_2 \geq \cdots \geq s_k > s_{k+1} = 0$ and the G-structure is $(k+1)$-flat, then it is flat, and its general symmetry depends on s_k functions of k variables; in particular, the local symmetry pseudogroup is infinite dimensional.*

[1] Ivey and Landsberg [47] provide a nice explanation of the Cartan–Kähler theorem, which is proven in complete detail and maximal generality by Bryant et. al. [12]. The Cartan–Kähler theorem is only valid in the real analytic category.

To say that the structure equations are in involution just means that the Lie algebra $\mathfrak{g}^{(k)}$ is involutive in the sense of the appendix to these notes.

Example 67 (Complex manifolds). For a complex manifold of complex dimension n, the last nonzero character is $s_n = 2n$, and the local symmetries are just the local invertible holomorphic maps, which are determined by their restrictions to any total real submanifold, hence $2n$ real functions of n real variables. Any real analytic functions can be locally extended off of a real analytic totally real submanifold to holomorphic functions. Nonanalytic smooth functions can not be extended. Any two complex manifolds of the same dimension are locally isomorphic. In particular, any two conformal structures on surfaces are locally isomorphic.

Example 68 (Flag geometry). Consider a manifold M equipped with a flag of smooth subbundles of its tangent bundle:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = TM$$

with V_k a vector bundle of rank k. This is a G-structure, with G the stabilizer of a flag. The reader might find it easier to follow along with this example by assuming $n = 3$. The structure equations are

$$d\omega^i = -\gamma^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k,$$

and we can absorb torsion to arrange that $T^i_{jk} = 0$ for $j \geq i$ or $k \geq i$, and $T^i_{jk} + T^i_{kj} = 0$. In the case of a torsion-free flag geometry, the structure equations are just

$$d\omega^i = -\gamma^i_j \wedge \omega^j.$$

Lets check for involution. The Cartan characters are $s_1 = n, s_2 = n - 1, \ldots, s_n = 1$, counting independent 1-forms in the columns of the matrix $\gamma = \left(\gamma^i_j\right)$. Thus

$$s_1 + 2s_2 + \cdots + ns_n = (1)n + 2(n-1) + 3(n-2) + \cdots + n(1).$$

The prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra is given by objects ξ^i_{jk} for which $\xi^i_{jk} = \xi^i_{kj}$ and $\xi^i_{jk} = 0$ for $i > j$ or $i > k$, so the number of independent components in such a ξ is

$$\begin{aligned} \dim \mathfrak{g}^{(1)} &= \sum_{i=1}^{n} \dim \operatorname{Sym}^2 \left(\mathbb{R}^{n-i+1}\right) \\ &= \sum_{i=1}^{n} \frac{(n-i+1)(n-i+2)}{2} \\ &= \sum_{j=1}^{n} \frac{j(j+1)}{2}. \end{aligned}$$

We leave the reader to juggle the combinatorics to show that

$$s_1 + 2s_2 + \cdots = \dim \mathfrak{g}^{(1)},$$

i.e. involution. Therefore

Proposition 33. *Any two torsion-free real analytic n-dimensional flag geometries are locally equivalent.*

8.2 Are we there yet? How to stop prolonging

Involutivity (in the real analytic category) implies transitivity of the pseudogroup of local symmetries on B, so that no further invariantly defined reduction of B. Moreover, involutivity implies that no higher order torsion can emerge (since the Spencer cohomology of involutive tableaux vanish) and implies involutivity of all prolongations, so that the symmetry pseudogroup acts transitively on all prolongations, and therefore there is no reason to prolong. Even in the smooth category, prolonging would not reveal any new local invariants, since formally to all orders the G-structures appear equivalent, and there is no differential invariant that can tell them apart. That does not imply that they are equivalent, even locally.

Example 69 (Flat web geometries). Lets find the possible symmetry groups of a flat web geometry, continuing our discussion from subsection 4.1.2 on page 28. The standard flat web geometry has symmetry group $\mathbb{R}^2 \rtimes \mathbb{R}^\times$ because the standard flat connection is invariantly determined on it (as in theorem 1 on page 19). Note that we can't exchange the foliations.

We can quotient the plane by a lattice of translations, bringing the standard flat web geometry down onto a torus. Translations preserve the flat web geometry on the torus. Similarly, we could quotient onto a cylinder, and still have translations. We could also cut out a point of the plane, and still have dilations around that point acting as symmetries, so we could quotient by a discrete group of dilations to produce a torus with a one parameter symmetry group.

Consider a general flat web geometry. The symmetry group must embed in the G-structure bundle B (by fixing a point of B, and letting each symmetry move it around, to map symmetries to points of B; see theorem 15), so that the structure equations of B pull back under the embedding to become the structure equations (i.e. Maurer–Cartan equations) of the symmetry group. So the symmetry Lie algebra of any flat web geometry sits inside the symmetry Lie algebra of the standard flat web geometry, i.e. the Lie algebra of the group $\mathbb{R}^2 \rtimes \mathbb{R}^\times$ of translations and rotations. We get an easy classification of the possible symmetry Lie algebras.

Exercise 8.1 Classify the homogeneous flat web geometries on surfaces.

Example 70 (Levi-flat CR *3-manifolds).* Let us apply these techniques to Levi-flat *CR* 3-manifolds, a topic discussed previously in section 7.4 on page 111. The structure equations of a Levi-flat 3-manifold are

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\gamma^1_1 & \gamma^1_2\\ 0 & \gamma^2_2\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

The Cartan characters are determined just from the γ pseudoconnection matrix: keeping in mind that the complex entries have each a real and imaginary part, that gives $s_1 = 3$ (second column) and $s_2 = 2$ (first column). The prolongation of the Lie algebra is given by

$$\begin{pmatrix}\gamma^1_1 & \gamma^1_2\\ 0 & \gamma^2_2\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = 0$$

which gives

$$\begin{aligned}\gamma^1_1 &= a^1_{11}\omega^1 + a^1_{12}\omega^2\\ \gamma^1_2 &= a^1_{21}\omega^1 + a^1_{22}\omega^2\\ \gamma^2_2 &= a^2_{22}\omega^2\end{aligned}$$

with $a^1_{21} = a^1_{12}$ complex coefficients, and a^2_{22} real. In particular, the Lie algebra has prolongation $\mathfrak{g}^{(1)} = \mathbb{C}^3 \oplus \mathbb{R}$ with 7 real dimensions. Cartan's test: $s_1 + 2s_2 = 3 + 2\cdot 2 = 7$, so in involution with general solution depending on $s_2 = 2$ functions of 2 variables. Thus the Cartan–Kähler theorem predicts that there is a symmetry group acting transitively on the Levi-flat hypersurface, as symmetries of the CR-structure, and that any two Levi-flat CR-structures are equivalent with general equivalence depending on 2 functions of 2 variables. Since the Cartan–Kähler theorem is only valid in the analytic category, and is local, we can only be sure of:

Proposition 34. *Any real analytic Levi-flat 3-manifold is flat.*

In fact, Malgrange [60] proved that all Levi-flat 3-manifolds are flat.

Remark 25. Symmetry group versus symmetry pseudogroup Note that the symmetry group might not really be as large as the symmetry pseudogroup. The global symmetry group of a complex curve is always finite dimensional, while its pseudogroup of local symmetries is infinite dimensional, since local symmetries are just holomorphic local changes of variable. Applying this to the leaves of a Levi-flat hypersurface, we see that the global symmetry group must depend on functions of one variable at most; indeed the largest symmetry group is that of a family of rational curves, with symmetries depending on one function of 6 variables, rather than the two functions of 2 variables predicted by Cartan–Kähler.

Exercise 8.2 Use the Cartan–Kähler theorem to show that the symmetry pseudogroup of a real analytic contact structure is transitive. Note that theorem 14 on page 170 does not apply, since the torsion does not vanish.

Example 71 (Levi pseudoconvex CR 3-manifolds). Let us see an example in which we *have* to prolong. Indeed higher order invariants appear. Recall the structure equations from subsection 7.4.2 on page 112:

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\gamma^1_1 & \gamma^1_2\\ 0 & \gamma^1_1+\gamma^{\bar 1}_{\bar 1}\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} + \begin{pmatrix}0\\ \sqrt{-1}\omega^{1\bar 1}\end{pmatrix}.$$

Following the discussion in the appendix on prolongations of Lie algebras, we calculate the prolongation of the Lie algebra of the structure group as follows:

$$\begin{pmatrix}\gamma^1_1 & \gamma^1_2\\ 0 & \gamma^1_1+\gamma^{\bar 1}_{\bar 1}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = 0$$

gives, by Cartan's lemma:

$$\begin{aligned}\gamma^1_1 &= a_1\omega^2\\ \gamma^1_2 &= a_1\omega^1 + a_2\omega^2.\end{aligned}$$

These are 2 complex numbers, so the Cartan integer is 4. The Lie algebra has $s_1 = 3, s_2 = 1$ (keeping track of real and imaginary parts), so $s_1+2s_2 = 5 \neq 4$. Therefore the structure equations are not in involution, and we must prolong. We saw in subsection 7.4.2 on page 112 that indeed new differential invariants appear on the prolongation.

8.3 Finite type

Exercise 8.3 (Finsler surfaces) If the symmetry group of a Finsler structure acts transitively on the bundle $B \to \Sigma$ we constructed previously, then the invariants I, J, K must be constants on that bundle, since they are invariant under symmetries. Suppose that the functions in Cartan's structure equations for Finsler surfaces from page 52 are constant. Show that $J = IK = 0$, so $J = 0$ and either $I = 0$ or $K = 0$. In case $I = J = 0$ (which occurs exactly when the Finsler geometry is Riemannian), find structure equations

$$\begin{aligned}d\omega^1 &= \omega^3\wedge\omega^2\\ d\omega^2 &= \omega^1\wedge\omega^3\\ d\omega^3 &= K\omega^2\wedge\omega^1.\end{aligned}$$

Show that these are the structure equations (i.e. Maurer–Cartan equations) $d\Omega = -\Omega\wedge\Omega$ for the symmetry groups

$$\begin{aligned} SL(2,\mathbb{R}) &\quad \text{if } K<0 \\ SO(2) \rtimes \mathbb{R}^2 &\quad \text{if } K=0 \\ SO(3) &\quad \text{if } K>0 \end{aligned}$$

giving the hyperbolic plane, Euclidean plane and sphere, where we take

$$\Omega = \begin{pmatrix} 0 & \omega^3 & \omega^1 \\ -\omega^3 & 0 & \omega^2 \\ K\omega^1 & K\omega^2 & 0 \end{pmatrix}.$$

If we suppose that $I \neq 0$, then we have $J = K = 0$, and a one parameter family of exotic homogeneous Finsler structures, which, as noted in section 5.3 on page 47, do not come about from Finsler surfaces.

In general, if we have a G-structure $B \to M$ and if we find an invariantly determined pseudoconnection, then the symmetry group of the G structure is just the symmetry group of the soldering form and pseudoconnection, which is of dimension at most $\dim B$. It is of dimension equal to $\dim B$ precisely when the universal cover of B is a Lie group, and the coframing is just the coframing of Maurer-Cartan forms. Let us see why.

Even if the symmetry group is trivial, as long as the symmetry Lie algebra is not trivial, we can look at the flows of Lie algebra elements, and see that the torsion must be invariant under those flows. So if the infinitesimal symmetries point in all directions, at a generic point, then the torsion functions on B must be constants. So we find structure equations expressing $d\omega$ and $d\gamma$ in terms of constant multiples of ω and γ.

Exercise 8.4 Suppose that $d\omega$ and $d\gamma$ are constant multiples of $\omega \wedge \omega, \gamma \wedge \omega$ and $\gamma \wedge \gamma$. Show that the dual vector fields to the components of ω and γ form a Lie algebra. Show that B is locally identified with some Lie group, uniquely up to left action of the Lie group, so that (ω, γ) is identified with the left invariant Maurer–Cartan form. Show that the infinitesimal symmetries are identified with the right invariant vector fields. Hence the Lie algebra of infinitesimal symmetries has dimension equal to $\dim B$.

Similarly, by prolonging we see that if we have any group G of finite type, say $\mathfrak{g}^{(k)} = 0$, then the symmetry group of any G-structure must have dimension no larger than

$$\dim V + \dim \mathfrak{g} + \dim \mathfrak{g}^{(1)} + \cdots + \dim \mathfrak{g}^{(k-1)}.$$

Moreover, the Lie algebra of the symmetry group will be filtered, with a contribution from each of these prolongations. The part of the symmetry Lie algebra which sits inside

$$\mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \dots$$

is the Lie algebra of the isotropy group of a point, while that which sits inside

$$\mathfrak{g}^{(p)} \oplus \mathfrak{g}^{(p+1)} \dots$$

is the symmetry group of a choice of point and frame adapted up to order p, i.e. a point of $B^{(p-1)}$.

Example 72 (Riemannian manifolds). The symmetry group of a Riemannian manifold is of dimension no larger than

$$\dim V + \dim \mathfrak{g} = n + \frac{n(n-1)}{2}$$

because $\mathfrak{g}^{(1)} = 0$. The sphere, hyperbolic space, and Euclidean space meet this dimension bound, with nonisomorphic symmetry groups, each having Lie algebra composed of:

$$V \times \mathfrak{g} = \mathbb{R}^n \times \mathfrak{so}(n).$$

Keep in mind that this is not a direct or even semidirect sum of Lie algebras. For instance, the symmetry group of the n-sphere is $O(n+1)$, which is simple for $n \neq 1, 3$, and the subgroup fixing a point is $O(n) \times \pm 1$. This gives the filtration

$$\mathfrak{so}(n+1) = \mathbb{R}^n \oplus \mathfrak{so}(n)$$

but (e.g. since $SO(n+1)$ is simple for $n > 3$) this is not a semidirect sum of Lie algebras, since $\mathbb{R}^n$ is not a Lie subalgebra. The generic flat torus V/Λ has itself as symmetry group, so the Lie algebra of the symmetry group is V. The generic Riemannian manifold has trivial symmetry group. Since the Weyl representation is irreducible (except in dimension 4), as is the traceless Ricci representation, any pseudo-Riemannian manifold with symmetry group acting transitively on the oriented orthonormal frame bundle will have to have vanishing traceless Ricci curvature (i.e. must be an Einstein manifold) and vanishing Weyl curvature (i.e. be conformally flat). Differentiating the structure equations, we find that it must have constant scalar curvature, so with a little work one can show that it is a space form (see section 7.3 on page 106.) In particular, if it is Riemannian, then it is locally equivalent (i.e. locally isometric) to a sphere, hyperbolic space, or Euclidean space.

8.3.1 Embedding the symmetry group

Theorem 15 (Kobayashi [51]). *The symmetry group H of an e-structure on a connected manifold M embeds into M via the map $h \in H \mapsto hm_0 \in M$, for any choice of point $m_0 \in M$.*

Proof. Let ω be the soldering form. An e-structure is just a coframing, which is just ω itself. The dual vector fields to the elements of this coframing must commute with all symmetries. Their flows will head in all directions out of each point. If a symmetry fixes a point, then it must fix the points obtained by following any of these flows, so must fix all points nearby. Therefore the set of fixed points of any symmetry is both open and closed, so a symmetry fixing a point must be the identity. If an infinitesimal symmetry vanished at a point, then its flow would be a symmetry fixing that point, so fixing all points. Therefore the symmetry Lie algebra is injectively mapped to the tangent space at each point, and so the map $H \to M$ is an injective immersion.

To show that this map is an embedding, following Abraham & Marsden [1] page 264, we need to show that if $m_j \to m$ and $h_j m_j \to m'$ are convergent sequences in M, then h_j is convergent in H. For each $v \in V$ define the vector field $\vec{v}$ by $\vec{v} \lrcorner \omega = v$. Then for a fixed point m, we define $\exp_m(v) = e^{\vec{v}} m$; this is defined for v near 0 in V. Check that this is diffeomorphism, since $\exp'_m(0)^* \omega = dv$. Then we define h by

$$h\left(\exp_m\left(v\right)\right) = \exp_{m'}\left(v\right).$$

This defines h near m, and clearly h is a limit of symmetries, so a symmetry. We now replace m by any point where h is defined, and use the same formula at the new point. This agrees with h at points where h was already defined, because it does so for the h_j which approach h at those points. Extend the definition of h over all of M, and it is a symmetry.

Corollary 14. *Suppose that $B \to M$ is a G-structure, which has prolongations $B^{(k)}$ defined for all $k > 0$. The symmetry group of $B \to M$ has dimension at most* $\dim B^{(\infty)}$. *If that dimension is finite, then the symmetry group is a Lie group, and an embedded submanifold of $B^{(\infty)}$*

Proof. The manifold $B^{(\infty)}$ is really just the last $B^{(k)}$ for which $\mathfrak{g}^{(k)} \neq 0$, and carries an e-structure. Apply the previous theorem and theorem 5 on page 74.

Example 73. The symmetry group of a compact Riemannian manifold is a compact Lie group.

Corollary 15. *The symmetry group of a Cartan geometry $B \to M$ modelled on a Klein geometry H/G embeds into B. If its dimension equals that of B, then the curvature is constant, and therefore lives in a trivial G-representation inside* $\mathfrak{h} \otimes \Lambda^2 \left(\mathfrak{h}/\mathfrak{g}^*\right)$

Corollary 16. *A flat Cartan geometry on a bundle $B \to M$ modelled on a Klein geometry H/G has symmetry group of dimension at most that of H. If the base manifold M is connected, then the following are equivalent:*

1. *the dimension of the symmetry group reaches that of H*
2. *the Cartan geometry is complete and homogeneous*
3. *$B \to M$ is a locally Klein geometry, i.e. after possibly replacing H by a covering Lie group $\tilde{H} \to H$ with the same Lie algebra, also containing G as a closed subgroup, we have $B = \Gamma \backslash \tilde{H} \to M = \Gamma \backslash \tilde{H}/G$ where $\Gamma \subset \tilde{H}$ acts trivially on $\mathfrak{h}$ in the adjoint representation.*

Proof. If the dimension of the symmetry group is equal to $\dim H$, then the symmetry group acts locally transitively, preserving the integral curves of the vector fields $\vec{A}$ defined in subsection 7.6.1 on page 117. Sliding those integral curves along themselves we see that their flows are defined for all time, hence complete.

If the Cartan connection is complete and flat, then by lemma 20 on page 121, it has the form $B = \Gamma\backslash\tilde{H} \to M = \Gamma\backslash\tilde{H}/G$, but possibly for a different group H with the same Lie algebra, and containing the same subgroup G. The map $\tilde{H} \to B = \Gamma\backslash\tilde{H}$ pulls back the Cartan connection to the Maurer–Cartan form, so matches up the Lie algebra of infinitesimal symmetries of the Cartan connection with the right invariant vector fields. These generate the left action on $\tilde{H}$. They must commute with the left action of Γ, so Γ must act trivially in the adjoint representation on $\mathfrak{h}$.

Example 74 (Symmetries of connections). A connection on the frame bundle FM of a connected n-dimensional manifold M is a Cartan geometry modelled on H/G, with $H = \mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$ and $G = \mathrm{GL}(n, \mathbb{R})$. Therefore the symmetry group of the connection has dimension at most $\dim H = n^2 + n$. It can only reach this bound if the curvature and torsion vanish, since they live in representations which have no trivial subrepresentation. If the symmetry group reaches $n^2 + n$ dimensions, the manifold must have the form $M = \Gamma\backslash\tilde{H}/G$, for $\Gamma \subset \tilde{H}$ acting trivially in the adjoint representation, and $\tilde{H}/G \to H/G$ must be a covering map of connected manifolds. Since $H/G = \mathbb{R}^n$ is simply connected, $\tilde{H}/G = H/G = \mathbb{R}^n$.

Exercise 8.5 Calculate that no element of H acts trivially in the adjoint representation. Show that the same must be true for any covering group $\tilde{H} \to H$.

Therefore $M = H/G = \mathbb{R}^n$, the standard flat connection has the largest symmetry group of any pseudoconnection.

Exercise 8.6 Apply the same reasoning to conformal geometry.

Remark 26 (Symmetries and path components). What if the manifold has many components? Think of a countable (or finite) set of spheres, all of the same radius, or maybe different radii. If they have different radii, then the symmetry group is a direct product of the symmetries of each sphere. If they are of the same radius, you also have to allow permutations of spheres, so you get a discrete permutation group appearing in a semidirect product with a Lie group. The same idea works with arbitrary G-structures.

Example 75 (Compact subgroups of Lie groups).

Proposition 35. *Let G be a simple Lie group of dimension p. Every compact subgroup $K \subset G$ has dimension q satisfying either $q = p$ (i.e. K is a union of path components of G) or*

$$q \leq p + \frac{1}{2} - \frac{1}{2}\sqrt{8p+1}.$$

Proof. The manifold G/K has a G-invariant Riemannian metric by a standard argument (find an K-invariant positive definite quadratic form on the tangent space $\mathfrak{g}/\mathfrak{k}$, using center of mass and convexity of the space of positive definite forms, then extend by G-invariance). So G must act on G/K by isometries of that metric. Let $N \subset G$ be the elements acting trivially. Then $N \subset G$ is normal. By simplicity of G, N is discrete or $N = G$. (Recall that a simple Lie group may fail to be simple as a group, but its normal subgroups are discrete and abelian). If $N = G$, so that G acts trivially on G/K, then $G = K$ because $gK = K$ for any $g \in G$. Therefore we may assume N discrete, so that G is immersed into the group of isometries of G/K. The isometry group is embedded into the orthonormal frame bundle, so has dimension at most $n + n(n-1)/2$ where $n = \dim G/K = p - q$. Therefore

$$p \leq n + \frac{n(n-1)}{2},$$

which, plugging in $n = p - q$, easily gives

$$q \leq p + \frac{1}{2} - \frac{\sqrt{8p+1}}{2}.$$

See table 8.1 for examples. In particular, $\mathrm{SU}(3) \subset G_2$ is a maximal subgroup,

G	$p = \dim G$	$\left\lfloor p + \frac{1}{2} - \frac{\sqrt{8p+1}}{2} \right\rfloor$
$\mathrm{SU}(2)$	3	1
$\mathrm{SU}(3)$	8	4
$\mathrm{SU}(4)$	15	10
$\mathrm{SU}(5)$	24	17
$\mathrm{SU}(6)$	35	27
$\mathrm{SU}(7)$	48	38
$\mathrm{SU}(8)$	63	52
$\mathrm{SU}(9)$	80	67
$\mathrm{SU}(10)$	99	85
$SO(n)$ $(n \neq 4)$	$\frac{n(n-1)}{2}$	$\frac{(n-2)(n-1)}{2}$
E_6	78	66
E_7	133	117
E_8	248	226
F_4	52	42
G_2	14	9

Table 8.1. Bounds on dimensions of compact subgroups of some simple Lie groups

as is $O(n-1) \subset SO(n)$. These bounds on subgroup dimensions are not optimal, but they are easy to come by: no roots or weights were calculated. The reader might ponder whether $U(n) \subset \mathrm{SU}(n+1)$ (obvious embedding) is maximal.

8.4 Homogeneous G-structures

More generally, the symmetry group always embeds into the last bundle $B^{(k)}$, as long as the G-structure has finite type. Let us examine this more carefully.

8.4.1 Replacing the bundle by an orbit

Suppose that H is a Lie group acting transitively on a manifold M, preserving a G-structure $B \subset FM$. Pick a point $m_0 \in M$, and let H_0 be the stabilizer of m_0, so $M = H/H_0$. Let H_1 be the subgroup fixing m_0 and every tangent vector at m_0, so $H_1 \subset H_0$ is a closed normal subgroup. Pick a point $u_0 \in B$, and map $h \in H \mapsto hu_0 \in B$. Clearly H_1 fixes u_0, so this map descends to a map $H/H_1 \to B$. We use the fact that $u_0 : T_m M \to V$ is a linear isomorphism as follows: give H_0/H_1 a representation on V by

$$h \in H_0/H_1, v \in V \mapsto u_0 h'(m_0)\, u_0^{-1} v.$$

We have the composition $H/H_1 \to B \to FM$ and the identification $M = H/H_0$ to make $H/H_1 \to H/H_0$ an H_0/H_1-structure. Alternatively, we can just use the adjoint representation to get H_0/H_1 to act on $T_{H_1}(H/H_1) = \mathfrak{h}/\mathfrak{h}_1$. We identify these two representations by mapping

$$A \in \mathfrak{h}/\mathfrak{h}_1 \mapsto u_0\left(\vec{A}\right) \in V$$

where $A \in \mathfrak{h}$ generates the vector field $\vec{A}$ on M. We can identify an element $hH_1 \in H/H_1$ with a linear map $T_{hH_0}H/H_0 \to \mathfrak{h}/\mathfrak{h}_1$ by

$$w \in T_{hH_0}H/H_0 \mapsto L_{h*}w \in \mathfrak{h}/\mathfrak{h}_1$$

using left translation, we get $H/H_1 \to FH/H_0$ a H_0/H_1-structure. Clearly $H/H_1 \to B$ is a smooth injective map, immersing H/H_1 into the orbit of u_0 under H.

Example 76 (Homogeneity and connections).

Proposition 36. *Suppose that the symmetry group H of a G-structure $B \to M$ acts transitively on M, with stabilizer $H_0 \subset H$ of a point $m_0 \in M$, and H_1 the subgroup of H_0 consisting of elements acting trivially on $T_{m_0}M$ (as above). Suppose that $\mathfrak{h}_j$ is the Lie algebra of H_j for $j = 0, 1$. Let $W = \mathfrak{h}_0/\mathfrak{h}_1$. Suppose that for this H_0-representation, (1) the prolongation is trivial: $\mathfrak{h}_0^{(1)} = 0$, and (2) the subrepresentation $\mathfrak{h}_0 \otimes W^* \subset W \otimes \Lambda^2(W^*)$ has an H_0-invariant complement. Then there is an H-invariant connection on B.*

Proof. We start by replacing the G-structure B by the H_0-structure $H/H_0 \to FM$ as above. Then we build the canonical connection on H (see example 28 on page 25). We then push this connection forward to B (see subsection 6.2.4 on page 61).

Exercise 8.7 The unitary group acts transitively on complex projective space, preserving the complex structure. Show that there must be a connection on complex projective space, invariant under unitary transformations, for which the complex structure is parallel.

8.4.2 Torsion

We can normalize torsion as usual, but we also have the action of the original structure group G, which permutes the orbits of H on B. We can use the action of G to normalize torsion even further, and to replace the subgroup $H_0/H_1 \to G$ by any conjugate subgroup. We will see this in action shortly.

8.4.3 Prolonging

Rather than prolonging the original G-structure, we can prolong the H_0/H_1-structure H/H_1. We obtain a new bundle $(H/H_1)^{(1)} \to H/H_1$, and we can pick any point b of this bundle, and look at its orbit under H, H/H_2, and continue by induction. Of course, the structure groups of prolongations are always vector spaces, so connected, and therefore either the dimension goes up at each step, or else the structure group becomes trivial, and the process stops, with a prolongation bundle H/H_∞ which is equipped with an H invariant pseudoconnection. By theorem 15 on page 176, each H-orbit in H/H_∞ is a copy of H. Therefore $H_\infty = 1$, i.e. the bundle is the symmetry group itself, acting on itself by left translation.

After all of the reduction and prolongation, structure equations look like

$$d\eta = A\eta \wedge \eta$$

and these A must be constants, since they are invariant under the transitive action of H. (Obviously, constant type hypotheses are justified at each reduction.) But then, these A must be the structure constants of the Lie algebra of the symmetry group, and the η, being left invariant, must be multiples of the Maurer–Cartan 1-forms.

Exercise 8.8 Show that the moving frame of vector fields dual to the component 1-forms of η will give infinitesimal symmetries of the G-structure. Show that these are all of the infinitesimal symmetries (use the Cartan formula $\mathcal{L}_X \alpha = X \lrcorner d\alpha + dX \lrcorner \alpha$ for differential forms).

Exercise 8.9 Classify all G-structures $B \to M$ which have an abelian group of symmetries acting transitively on M. (It is not known which G-structures have abelian symmetry groups.)

Question 17. Cartan's classification of the irreducible second order homogeneous spaces seems important here. I should recall what it is.

8.5 Example: conformal geometry

8.5.1 Guessing the symmetry group

Recall our discussion of conformal geometry from section 7.7 on page 124. The reader is naturally curious as to how we guess that we can put the 1-forms ω, α, σ and ϖ together into a single matrix, which we called Ω, and magically find that conformally flat manifolds have infinitesimal symmetries forming the Lie algebra $\mathfrak{so}\,(p+1, q+1)$. Let us examine the method here. The underlying idea is that we want to take all of the standard results and techniques from the theory of Lie algebras, which are usually expressed in terms of the Lie bracket, and rewrite them in the notation of the left invariant Maurer–Cartan 1-form and its exterior derivative.

8.5.2 The symmetry Lie algebra

Start with M a manifold with flat conformal structure. The structure equations:

$$\begin{aligned} d\omega &= -\left(\alpha + \sigma 1_{p+q}\right) \wedge \omega \\ d\alpha &= -\alpha \wedge \alpha + \omega \wedge \varpi + \varpi^* \wedge \omega^* \\ d\sigma &= -\varpi \wedge \omega \\ d\varpi &= -\varpi \wedge \left(\alpha + \sigma 1_{p+q}\right). \end{aligned}$$

Because the structure equations only contain constants, as we have seen above, these must be the structure equations of a Lie algebra $\mathfrak{h}$, i.e. the ω, α, σ and ϖ will together, in some combination, compose the Maurer–Cartan 1-form of a Lie algebra.

8.5.3 Detecting semisimplicity

Exercise 8.10 Suppose that a Lie group has Maurer–Cartan 1-forms ϑ^i and structure equations

$$d\vartheta^i = -c^i_{jk}\vartheta^j \wedge \vartheta^k.$$

Show that the dual vector fields X_i to the 1-forms ϑ^i satisfy

$$[X_j, X_k] = c^i_{jk} X_i.$$

Show that the *Killing form*

$$B(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$$

is given by

$$B(X_i, X_j) = c^l_{jk} c^k_{il}.$$

We can now determine whether the symmetry group of a homogeneous G-structure is semisimple by direct calculation: the Lie algebra is semisimple just when the Killing form is definite as a quadratic form on the Lie algebra. This is just a calculation on the matrix

$$(B_{ij}) = \left(c^l_{jk} c^k_{il}\right).$$

Exercise 8.11 By hand, calculate the Killing form matrix for constant curvature Riemannian geometry on surfaces. Show that $\det(B_{ij})$ vanishes just when the curvature is zero, so that the symmetry group is semisimple just when the curvature is not zero. Show that the Killing form is negative definite just when the curvature is positive.

Recall that the Killing form of a semisimple Lie group is negative definite just when the identity component of the Lie group is compact (see Fulton & Harris [39] p. 434). Determining compactness of a nonsemisimple Lie group can be much more complicated. More generally, the Killing form of a semisimple Lie algebra is negative definite on the Lie algebra of the maximal compact subgroup of the adjoint form, and positive definite on the complement.

Exercise 8.12 Using a computer, calculate the Killing form matrix for the standard flat positive definite conformal geometry in low dimensions. Write $6[-6] + 4[6]$ to mean eigenvalues -6 of multiplicity 6 and 6 of multiplicity 4; compare with table 8.2 on the following page. The Killing form is negative definite on the Lie algebra of the maximal compact subgroup of the adjoint form, and positive definite on its complement, so we can read off the dimension of the maximal compact subgroup: $n(n+1)/2$.

8.5.4 Finding abelian subgroups

To find the maximal abelian subgroup, follow the process which we will explain in detail in subsection 12.2.1 on page 254: look for choices of equations on differential forms which will render all of the right hand sides of all of the structure equations to 0. For example, if we set $\omega = \varpi = 0$, our structure equations simplify to $d\alpha = -\alpha \wedge \alpha, d\sigma = 0$. We need to add some more equations on the α to kill off the rest of the equations. Recall that α is valued in $\mathfrak{so}(p, q)$. Let $\mathfrak{t}_0 \subset \mathfrak{so}(p, q)$ be a maximal abelian Lie subalgebra, and split $\mathfrak{so}(p, q)$ into a sum of linear subspaces:

n	$\dim \mathfrak{h}$	*Spectrum of* B
3	10	$6\,[-6] + 4\,[6]$
4	15	$10\,[-8] + 5\,[8]$
5	21	$15\,[-10] + 6\,[10]$
6	28	$21\,[-12] + 7\,[12]$
7	36	$26\,[-14] + 8\,[14]$
8	45	$36\,[-16] + 9\,[16]$
9	55	$45\,[-18] + 10\,[18]$
10	66	$55\,[-20] + 11\,[20]$
n	$\frac{(n+1)(n+2)}{2}$	$\frac{n(n+1)}{2}\,[-2n] + (n+1)\,[2n]$

Table 8.2. The spectrum of the Killing form of the symmetry group of flat conformal geometry

$$\mathfrak{so}\,(p,q) = \mathfrak{t}_0 \oplus \mathfrak{t}_0^{\perp}.$$

We write α as $\alpha = \alpha_0 + \alpha_1$ in terms of this splitting. Consider the equations $\mathfrak{I} = (0 = \omega = \varpi = \alpha_1)$.

Exercise 8.13 Let $\mathfrak{t}$ be the set of elements of $\mathfrak{h}$ on which these equations $\mathfrak{I}$ are satisfied. From the structure equations, why is $\mathfrak{t}$ an abelian subalgebra?

Clearly $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{R}$.

8.5.5 Checking that an abelian subgroup is maximal abelian

Next we have to check that $\mathfrak{t}$ is a maximal abelian subalgebra. Define $Z\,(\mathfrak{t})$ to be the set of elements z of $\mathfrak{h}$ so that $d\zeta(z,h) = 0$ for all $h \in \mathfrak{t}$, where $\zeta = 0$ is any one of our equations from $\mathfrak{I}$.

Exercise 8.14 Show that $Z\,(\mathfrak{t})$ is the centralizer of $\mathfrak{t}$, i.e. the set of all elements of $\mathfrak{h}$ which commute with every element of $\mathfrak{t}$.

For example,

$$d\omega = -\left(\alpha + \sigma 1_{p+q}\right) \wedge \omega$$

so that plugging in the element of $\mathfrak{t}$ dual to σ, we get the equations $0 = \omega$ satisfied by the elements of $Z\,(\mathfrak{t})$. Similarly, using the same element on the equation

$$d\varpi = -\varpi \wedge \left(\alpha + \sigma 1_{p+q}\right) \wedge \varpi$$

we find the equation $0 = \varpi$ on $Z\,(\mathfrak{t})$.

Exercise 8.15 On $Z(\mathfrak{t})$, we have now established that $\omega = \varpi = 0$. But therefore, on $Z(\mathfrak{t})$,

$$d\alpha = -\alpha \wedge \alpha.$$

Use this to show that the map

$$v \in Z(\mathfrak{t}) \mapsto v \lrcorner \alpha \in \mathfrak{so}(p, q)$$

is a Lie algebra morphism.

This morphism maps $\mathfrak{t}$ onto $\mathfrak{t}_0$, by definition of $\mathfrak{t}$. Therefore it maps

$$\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathfrak{t}_0 \\ \downarrow & & \downarrow \\ Z(\mathfrak{t}) & \longrightarrow & Z(\mathfrak{t}_0). \end{array}$$

The subalgebra $\mathfrak{t}_0 \subset \mathfrak{so}(p, q)$ is already maximal abelian, so $Z(\mathfrak{t}_0) = \mathfrak{t}_0$, and therefore $Z(\mathfrak{t})$ must map under α to $\mathfrak{t}_0$, i.e. satisfy $\alpha_1 = 0$, and therefore must equal $\mathfrak{t}$. Therefore $\mathfrak{t} \subset \mathfrak{h}$ is maximal abelian.

8.5.6 How to see all of this from the structure equations

Summing up, if we have a finite type G-structure, with only constants appearing in the structure equations of all prolongations, and if we write the 1-forms in the infinitely prolonged structure equations as η^i, ζ^α (so that the structure equations say that $d\eta^i, d\zeta^\alpha$ are constant coefficient multiples of wedge products built from η^j, ζ^β), then the equations $(\zeta = 0)$ cut out a subgroup just when all $d\zeta$ vanish modulo ζ. Our subgroup is abelian if we also have $d\eta = 0$ modulo ζ. Denote by $\frac{\partial}{\partial \eta^i}$ the vectors dual to the η^i, and write

$$Z^\perp(\zeta) = \operatorname*{span}_{i,\alpha} \left(\frac{\partial}{\partial \eta^i} \lrcorner d\zeta^\alpha \right).$$

Our subgroup is maximal abelian if furthermore

$$Z^\perp(\zeta) = \operatorname*{span}_{\alpha} (\zeta^\alpha).$$

All of this can be checked just by looking at the structure equations, without even calculating anything.

8.5.7 Roots

We have identified that $\mathfrak{t}$ is a maximal abelian subgroup. Next we calculate roots.

Definition 38. *Recall the definition of a* root vector*: if* $\mathfrak{t} \subset \mathfrak{h}$ *is a maximal abelian subgroup of a semisimple Lie algebra, and* $\lambda \in \mathfrak{t}^*$*, we say that* $w \in \mathfrak{h}$ *is a root vector with weight* λ *if*

$$[h, w] = \lambda(h) w$$

for all $h \in \mathfrak{t}$*.*

Exercise 8.16 Suppose that the Lie group H has left invariant Maurer–Cartan 1-form ϑ. Show that w is a root vector with weight λ just when

$$0 = w \mathbin{\lrcorner} \left(h \mathbin{\lrcorner} d\vartheta + \lambda(h)\vartheta\right),$$

for all $h \in \mathfrak{t}$. (If λ is the weight of some root vector, then λ is called a *root*.)

Example 77 (Surfaces of nonzero constant curvature). The structure equations of a Riemannian geometry on a surface are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \gamma \end{pmatrix} = - \begin{pmatrix} -\gamma \wedge \omega^2 \\ \gamma \wedge \omega^1 \\ \kappa \omega^1 \wedge \omega^2 \end{pmatrix}.$$

We suppose that κ is a constant. A maximal abelian subgroup is cut out by $\omega^1 = \omega^2 = 0$. The equations of a root with weight λ are therefore

$$0 = w \mathbin{\lrcorner} \left(\frac{\partial}{\partial \gamma} \mathbin{\lrcorner} d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \gamma \end{pmatrix} + \lambda \begin{pmatrix} \omega^1 \\ \omega^2 \\ \gamma \end{pmatrix} \right)$$

where

$$\frac{\partial}{\partial \omega^1}, \frac{\partial}{\partial \omega^2}, \frac{\partial}{\partial \gamma}$$

is the dual basis to $\omega^1, \omega^2, \gamma$, and $\lambda \in \mathbb{R}$. In this simple case, $\mathfrak{t}$ is just the span of $\frac{\partial}{\partial \gamma}$. We compute out that the roots are solutions of

$$0 = w \mathbin{\lrcorner} \begin{pmatrix} -\omega^2 + \lambda \omega^1 \\ \omega^1 + \lambda \omega^2 \\ \lambda \gamma \end{pmatrix}.$$

We can easily see that w must be zero unless we allow *complex roots*:

Root	*Root vector*
$-\sqrt{-1}$	$\frac{\partial}{\partial \omega^1} - \sqrt{-1} \frac{\partial}{\partial \omega^2}$
0	$\frac{\partial}{\partial \gamma}$
$\sqrt{-1}$	$\frac{\partial}{\partial \omega^1} + \sqrt{-1} \frac{\partial}{\partial \omega^2}$

Lets find the roots of the symmetry group of the flat conformal geometry. The maximal abelian subgroup $\mathfrak{t}$ splits as $\mathfrak{t}_0 \oplus \mathbb{R}$, and in the same way, we can represent each weight in $\mathfrak{t}^*$ as a sum of two pieces $\lambda \oplus \Lambda$ with $\lambda \in \mathfrak{t}_0^*$ and $\Lambda \in \mathbb{R}$. The equations for root vectors are

$$0 = w \lrcorner \left(h \lrcorner d \begin{pmatrix} \omega \\ \alpha \\ \sigma \\ \varpi \end{pmatrix} + \lambda(h) \begin{pmatrix} \omega \\ \alpha \\ \sigma \\ \varpi \end{pmatrix} \right)$$

for $h \in \mathfrak{t}_0$ and a similar equation for the $\mathbb{R}$ part of $\mathfrak{t}$. Calculating these out, and putting them together gives:

$$0 = w \lrcorner \begin{pmatrix} -h\omega + \lambda(h)\omega \\ -[h, \alpha] + \lambda(h)\alpha \\ \lambda(h)\sigma \\ -h\varpi + \lambda(h)\varpi \end{pmatrix}$$

and

$$0 = w \lrcorner \begin{pmatrix} (\Lambda + 1)\omega \\ \Lambda\alpha \\ \Lambda\sigma \\ (\Lambda - 1)\varpi \end{pmatrix}.$$

If $w \lrcorner \alpha \neq 0$ then this forces $\alpha(w)$ to be a root vector of $\mathfrak{so}(p, q)$ with weight λ, and forces $\Lambda = 0$, and then $\omega = \varpi = \sigma = 0$ on w. This provides at $\Lambda = 0$ a copy of the root lattice of $\mathfrak{so}(p, q)$.

If we consider $w \lrcorner \alpha = 0$, and still ask for $\Lambda = 0$, there is one more possibility: we are forced to have $\omega = \alpha = \varpi = 0$ on such a root vector, but could still have $\sigma \neq 0$. Hence one additional dimension of root vectors at the weight $\lambda = \Lambda = 0$.

Recall that a vector $v \in V$ in a representation of a Lie algebra $\mathfrak{h}$ is called a *weight vector* of weight $\lambda \in \mathfrak{t}^*$ if $hv = \lambda(h)v$ for all $h \in \mathfrak{t}$. A *weight* is a choice of λ which is the weight of some nonzero weight vector. *Complex weight vectors* and *complex weights* are defined similarly. As above, a weight for the adjoint representation is called a *root*.

In our case, if $\Lambda \neq 0$, this forces $w \lrcorner \alpha = 0$ and $w \lrcorner \sigma = 0$, but allows $\omega \neq 0$ if $\Lambda = -1$, and $\varpi \neq 0$ if $\Lambda = 1$. In each case, λ is forced to be a (possibly complex) weight of the representation $\mathbb{R}^{p+q}$.

We now have the following picture of the roots of the symmetry group: there are three layers, $\Lambda = -1, 0, 1$, and the $\Lambda = 0$ layer is just the roots of $\mathfrak{so}(p, q)$, but with multiplicity increased by one at the origin. The $\Lambda = \pm 1$ layers are just the weights of the $\mathbb{R}^{p+q}$ representation of $\mathfrak{so}(p, q)$; see figures 8.1 and 8.2.

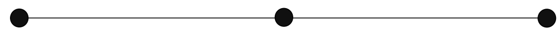


Fig. 8.1. The roots of $SO(3,\mathbb{C})$

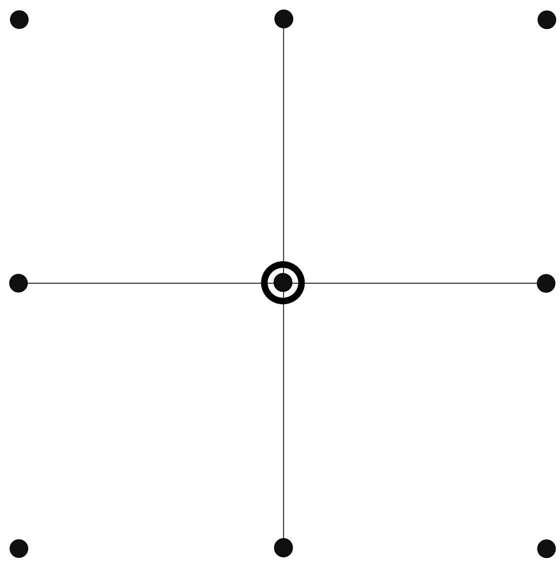


Fig. 8.2. The roots of $SO(5,\mathbb{C})$

Exercise 8.17 Draw the roots of the symmetry Lie algebra of the flat conformal structure on $S^p \times S^q$ for small values of p and q, from the drawings of the roots of $\mathfrak{so}(p,q)$ given by Fulton and Harris [39] lecture 18.

From these root diagrams, the reader should be able to identify that the symmetry Lie algebra of a flat conformal structure on a three-manifold must be a real form of $\mathfrak{so}(5,\mathbb{C})$, for example. We found that the maximal compact subgroup had dimension 6, and therefore the symmetry group of a flat conformal structure on a three-manifold must be $\mathfrak{so}(4,1)$. Similarly, for a flat conformal four-manifold, we must have symmetry Lie algebra $\mathfrak{so}(5,1)$.

In the usual picture of the weight lattice (Fulton & Harris [39], p. 283), the weights of the $\mathbb{R}^{p+q}$ representation of $\mathfrak{so}(p,q)$ are at the standard basis vectors of the weight lattice and their negatives. If $p+q$ is odd, then the roots of $\mathfrak{so}(p,q)$ are at $0, \pm e_i, \pm e_i \pm e_j$, all with unit multiplicity, except that 0 has multiplicity $(p+q-1)/2$. So putting these together, say letting the Λ direction have standard basis vector e_0, we can see that the resulting root vectors are $0, \pm e_0, \pm e_i, \pm e_0 \pm e_i, \pm e_i \pm e_j$, all with unit multiplicity, except that 0 has multiplicity $(p+q-1)/2+1$, and so these are the roots of $\mathfrak{so}(p+q+2,\mathbb{C})$. We leave the case of $p+q$ even to the reader.

8.6 Example: homogeneous Levi pseudoconvex *CR* 3-manifolds

Torsion-free Levi pseudoconvex CR 3-manifolds provide a simpler example. Recall our structure equations from subsection 7.4.4 on page 115:

$$d\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} = -\begin{pmatrix}\gamma_1^1 & \gamma_2^1\\ 0 & \gamma_1^1+\gamma_{\bar 1}^{\bar 1}\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}+\begin{pmatrix}0\\ \sqrt{-1}\omega^{1\bar 1}\end{pmatrix}$$

$$d\begin{pmatrix}\gamma_1^1\\ \gamma_2^1\end{pmatrix} = -\begin{pmatrix}0 & \xi\\ \xi & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}-\begin{pmatrix}\sqrt{-1}\gamma_2^1\wedge\omega^{\bar 1}+2\sqrt{-1}\gamma_2^{\bar 1}\wedge\omega^1\\ \gamma_2^1\wedge\gamma_{\bar 1}^{\bar 1}\end{pmatrix}$$

$$d\xi = \left(\gamma_1^1+\gamma_{\bar 1}^{\bar 1}\right)\wedge\xi+\sqrt{-1}\gamma_2^1\wedge\gamma_{\bar 2}^{\bar 1}$$

8.6.1 Semisimplicity

Exercise 8.18 Split $\omega^1, \gamma_1^1, \gamma_2^1$ into real and imaginary parts. Read off from the structure equations the structure constants c_{jk}^i, and then calculate the Killing form $B_{ij} = c_{jk}^l c_{il}^k$ in the basis

$$\operatorname{Re}\omega^1, \operatorname{Im}\omega^1, \omega^2, \operatorname{Re}\gamma_1^1, \operatorname{Im}\gamma_1^1, \operatorname{Re}\gamma_2^1, \operatorname{Im}\gamma_2^1, \xi.$$

You should find

$$B = \begin{pmatrix}0&0&0&0&0&0&2&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&-6\\ 0&0&0&12&0&0&0&0\\ 0&0&0&0&4&0&0&0\\ 0&2&0&0&0&0&0&0\\ 2&0&0&0&0&0&0&0\\ 0&0&-6&0&0&0&0&0\end{pmatrix}.$$

The spectrum of B is

$$[-6]+2[-2]+2[2]+[4]+[6]+[12].$$

Of course, the values that occur in the spectrum depend on the basis chosen for the structure equations, but the signs do not. Zero is not in the spectrum, so the symmetry group is semisimple. The symmetry group is 8 dimensional, since there are 8 real-valued 1-forms in our basis. There is a 3 dimensional space of negative eigenvectors, so the maximal compact subgroup of the symmetry group has 3 dimensions.

8.6.2 Finding an abelian subgroup

Lets follow subsection 8.5.6. To find an abelian subgroup, try setting some 1-forms to 0, and keep going until all of the exterior derivatives of all of the remaining 1-forms vanish. In our case, we could try $\omega^1 = 0$, but then $d\omega^1$ wouldn't vanish. So lets try $\omega^1 = \omega^2 = 0$. Then $d\gamma^1_2 = \gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{1}}$. The easiest way to kill this off is to ask for $\omega^1 = \omega^2 = \gamma^1_2 = 0$. This still leaves $d\xi = \left(\gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}}\right) \wedge \xi$. The easiest way to kill this off is to make $\xi = 0$ too, so our equations are

$$\omega^1 = \omega^2 = \gamma^1_2 = \xi = 0,$$

an abelian subgroup. We call our abelian Lie subalgebra $\mathfrak{t}$, so

$$\mathfrak{t}^\perp = \operatorname{span}\left\{\omega^1, \omega^2, \gamma^1_2, \xi\right\}.$$

8.6.3 Checking maximality of our abelian subalgebra

To see that $\mathfrak{t}$ is maximal abelian, we take each 1-form in this list of equations, for example $\omega^1 = 0$, we take the differential of the 1-form on the original group:

$$d\omega^1 = -\gamma^1_1 \wedge \omega^1 - \gamma^1_2 \wedge \omega^2,$$

and, since γ^1_1 is not in $\mathfrak{t}^\perp$, we find that ω^1 is in $Z^\perp(\mathfrak{t})$. Similarly, checking through

$$d\xi = \left(\gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}}\right) \wedge \xi + \sqrt{-1}\gamma^1_2 \wedge \gamma^{\bar{1}}_{\bar{2}}$$

since $\gamma^1_1 + \gamma^{\bar{1}}_{\bar{1}}$ is not in $\mathfrak{t}^\perp$, then ξ must be in $Z^\perp(\mathfrak{t})$. Checking all of the relevant 1-forms, we find that $Z^\perp(\mathfrak{t}) = \mathfrak{t}^\perp$, so the subalgebra is maximal abelian.

8.6.4 Roots

Since $\mathfrak{t}$ is has two real dimensions, parameterized by the real and imaginary parts of γ^1_1, lets let X and Y be the vector fields dual to the real and imaginary parts of γ^1_1. We have to plug $h = X$ and $h = Y$ into the equation

$$0 = w \lrcorner \left(h \lrcorner d \begin{pmatrix} \omega^2 \\ \omega^1 \\ \omega^{\bar{1}} \\ \gamma^1_1 \\ \gamma^{\bar{1}}_{\bar{1}} \\ \gamma^1_2 \\ \gamma^{\bar{1}}_{\bar{2}} \\ \xi \end{pmatrix} + \lambda(h) \begin{pmatrix} \omega^2 \\ \omega^1 \\ \omega^{\bar{1}} \\ \gamma^1_1 \\ \gamma^{\bar{1}}_{\bar{1}} \\ \gamma^1_2 \\ \gamma^{\bar{1}}_{\bar{2}} \\ \xi \end{pmatrix} \right).$$

This turns out to be quite easy, even though it might look daunting. Note that complex-valued 1-forms have to appear along with their conjugates.

Exercise 8.19 Find the roots (in dX, dY coordinates) and root vectors; check table 8.3.

Root	*Root vector*
$(2,0)$	$\frac{\partial}{\partial \omega^2}$
$(1,1)$	$\frac{\partial}{\partial \omega^1}$
$(1,-1)$	$\frac{\partial}{\partial \omega^{\bar{1}}}$
$(0,0)$	$\frac{\partial}{\partial \gamma_1^1}, \frac{\partial}{\partial \gamma_{\bar{1}}^{\bar{1}}}$
$(-1,1)$	$\frac{\partial}{\partial \gamma_2^1}$
$(-1,-1)$	$\frac{\partial}{\partial \gamma_2^{\bar{1}}}$
$(-2,0)$	$\frac{\partial}{\partial \xi}$

Table 8.3. Roots and root vectors

Question 18. Consider the point of view of Ivey & Landsberg, using the same root diagram even for the inhomogeneous case, to see how the invariants can be organized by weight.

Exercise 8.20 Decorate figure 8.3 with the root vectors labelled on each root; use this to find $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ subgroups, checking the structure equations to be sure of the constants.

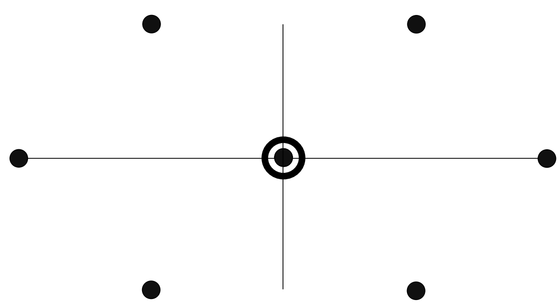


Fig. 8.3. The roots of $SL(3,\mathbb{C})$

The symmetry group is a real form of $\mathrm{SU}(3,\mathbb{C})$ because it has the same roots as $\mathrm{SU}(3,\mathbb{C})$ (up to choice of basis of $\mathfrak{t}$). It has 3 dimensional maximal compact subgroup, and therefore must have Lie algebra $\mathfrak{su}(2,1)$. Therefore, the symmetry group is either $\mathrm{SU}(2,1)$ or $\mathbb{P}\mathrm{SU}(2,1) = \mathrm{SU}(2,1) / \pm 1$. At this stage, we might already guess how to construct the Levi-pseudoconvex hypersurface. The group $\mathrm{SU}(2,1)$ acts on $\mathbb{C}^3$ preserving a pseudo-Hermitian form of signature $(2,1)$. Therefore $\mathbb{P}\mathrm{SU}(2,1)$ acts on $\mathbb{CP}^2$, preserving the real hypersurface on which the pseudo-Hermitian form vanishes. Moreover, $\mathbb{P}\mathrm{SU}(2,1)$ acts freely on that hypersurface, so must be a subgroup of the symmetry group. The symmetry group of any Levi-pseudoconvex CR 3-manifold is at most 8 dimensional, and reaches this bound just when its invariants p and s vanish, i.e. just when it has the required structure equations. But $\dim \mathbb{P}\mathrm{SU}(2,1) = 8$, and therefore $\mathbb{P}\mathrm{SU}(2,1)$ is the identity component of the symmetry group.

Proposition 37. *Every Levi-pseudoconvex CR 3-manifold with $p = s = 0$ is locally isomorphic to the real hypersurface in $\mathbb{CP}^2$ given by the null lines of a pseudo-Hermitian form of signature $(2,1)$ on $\mathbb{C}^3$.*

Exercise 8.21 Consider the pseudo-Hermitian form

$$|z_0|^2 + |z_1|^2 - |z_2|^2 .$$

Show that the set of null points in $\mathbb{CP}^2$ for this form is the 3-sphere $z_2 = 1$, $|z_0|^2 + |z_1|^2 = 1$.

Exercise 8.22 Check that $\mathbb{P}\mathrm{SU}(2,1)$ is a simple group. (We know that it is a simple Lie group, i.e. has simple Lie algebra, so every normal subgroup is a discrete subgroup of the center. You need to check that its center is trivial.)

Corollary 17. *Every Levi-pseudoconvex CR 3-manifold with symmetry group of dimension 8 is $S^3 \subset \mathbb{CP}^2$, the null lines of a pseudo-Hermitian form.*

Corollary 18. *Every Levi-pseudoconvex CR 3-manifold with $p = s = 0$ is locally isomorphic to $S^3 \subset \mathbb{CP}^2$.*

Corollary 19. *Every Levi-pseudoconvex CR 3-manifold with $p = s = 0$ which is compact and has fundamental group defying $\mathbb{P}\mathrm{SU}(2,1)$ is isomorphic to S^3/Γ where $\Gamma \subset \mathbb{P}\mathrm{SU}(2,1)$ is a finite group.*

Proof. It must be a quotient S^3/Γ by lemma 24 on page 123, with $\Gamma \subset \mathbb{P}\mathrm{SU}(2,1)$ a finite group, since S^3 is compact and simply connected.

Exercise 8.23 Classify all homogeneous Levi-pseudoconvex CR 3-manifolds.

8.7 Infinite type

Lie algebras not of finite type are quite rare. In the case of infinite type Lie algebras, the notion of symmetry group can be replaced by that of the symmetry pseudogroup, i.e. treated as a system of partial differential equations for local symmetries, as above.

8.7.1 Elliptic G-structures

It is elementary to see that ellipticity of the differential equations of section 8.1 for symmetries is equivalent to having no linear maps of rank one in the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ (see [12], chapter V): in such a case we call every G-structure *elliptic*. Any elliptic G-structure on a compact manifold, has as symmetry group a finite dimensional Lie group, by standard elliptic theory; see Kobayashi [51]. For example, the global symmetries of a compact complex manifold form a finite dimensional Lie group, because the complex linear maps of a vector space to itself contain no linear maps of rank one. We measure rank thinking of the complex linear maps on $\mathbb{C}^n$ as real linear maps of $\mathbb{R}^{2n}$.

Remark 27. Even if the G-structure is not of finite type, as long as the global symmetry group is a finite dimensional Lie group, its Lie algebra will still have the sum decomposition:

$$\mathfrak{h} \subset V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \dots .$$

8.8 Example: homogeneous foliated surfaces

Question 19. This section is awful.

Consider how we could find examples of homogeneous foliations of surfaces by curves.

8.8.1 Structure equations

Let $G \subset \mathrm{GL}(2, \mathbb{R})$ be the group of linear transformations preserving the line $dx^1 = 0$, i.e. invertible matrices of the form

$$\begin{pmatrix} g^1_1 & 0 \\ g^2_1 & g^2_2 \end{pmatrix} .$$

The structure equations of a foliated surface, i.e. the G-structure $B \subset FM$, will look like

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & 0 \\ \gamma^2_1 & \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} .$$

On the bundle $B^{(1)}$, the γ^i_j are defined.

The *first step* of a symmetry group H is the order k for which H acts transitively on $B^{(k)}$ and not on $B^{(k+1)}$. The *first step dimension* of a symmetry group H is the dimension of the orbits of H on $B^{(k+1)}$. Consequently, we will divide our study into cases, according to the first step and first step dimension of the symmetry group. If k is the first step, then $\dim H \geq \dim B^{(k)}$, so if the manifold M has dimension n (which in our case is $n = 2$), and the dimension of the structure groups are $d_0 = \dim \mathfrak{g}, d_1 = \mathfrak{g}^{(1)}, \dots$ then

$$\dim H \geq n + \sum_{j=0}^{k} d_j.$$

For foliated surfaces, we let the reader check that

$$\dim \mathfrak{g}^{(k)} = k + 3$$

(easy to check by just carrying out the prolongation on the structure equations a few times, until the pattern becomes clear), and so

$$\dim H \geq \frac{k^2 + 7k + 10}{2}.$$

See table 8.4. At the first step, the first step dimension must be at least $\frac{k^2+7k+10}{2}$, and can not be as large as $\frac{(k+1)^2+7(k+1)+10}{2}$.

First step k	*Dimension lower bound* $\frac{k^2+7k+10}{2}$
-1	2
0	5
1	9
2	14
3	20
4	27
5	35
6	44
7	54
8	65
9	77
10	90

Table 8.4. Limits on dimensions of symmetry groups of foliated surfaces

8.8.2 Induction

Question 20. This subsection is very unclear, because the induction step was described quite a bit earlier.

We start with a G-structure $B \to M$ with symmetry group H acting transitively. After the first step, H is now acting transitively on a G_0-structure

$B_0 \to M$, with different structure group. We proceed inductively to the second step, when once again the H-orbits are smaller than the ambient prolongation of the G_0-structure. This is our algorithm. At each stage, we can use the structure group to normalize torsion. The result is a tower of bundles, each representing a G-structure over the previous one.

At each step, the bundle we construct is larger in dimension, and still no larger in dimension than the symmetry group H itself. Therefore at some stage the sequence will stabilize in dimension, the *last step*, and produce a sequence of covering spaces, each a homogeneous H-space. Once the bundles stabilize in dimension, each of these coverings will determine a coframing on the previous one, up to the action of the covering group, a discrete group. This discrete group structure is H-invariant, hence homogeneous, so all of its torsion coefficients are constant.

Exercise 8.24 If Γ is a discrete group, and $B \to M$ is a Γ-structure, and M is a connected manifold, then the symmetries of $B \to M$ which fix a point of M form a discrete group.

Therefore, H itself must be a covering space of the first bundle in the tower which has stabilized dimension.

Exercise 8.25 Give an example of a manifold which admits infinitely many inequivalent group actions under which it is homogeneous.

Lemma 33. *The symmetry group H of a homogeneous G-structure is diffeomorphic to the first bundle in the tower which has stabilized dimension.*

Proof. The various groups $\mathfrak{g}^{(k)}$ are abelian groups, so their only discrete algebraically closed subgroups are trivial.

8.8.3 Compact stabilizer

Lets return to our search for foliated surfaces. If the stabilizer in H of a point of M is compact, then the action preserves a Riemannian metric, which must be of constant curvature, so the group acting must be a subgroup of the isometry group of a constant curvature surface, and a cover of that group must act on a simply connected surface of constant curvature, so must be a subgroup of $O(3), SL(2,\mathbb{R}) \times \{\pm 1\}$ or $\mathbb{R}^3 \rtimes O(3)$. Moreover the maximal compact subgroup of G is finite, $\pm 1 \times \pm 1$, so the group H will be a two dimensional subgroup of one of these groups. Leaving these cases as an exercise, assume noncompact stabilizer of any point of M.

8.8.4 Groups of first step -1 and first step dimension 2

Any two dimensional group H acting transitively on the surface M must have discrete stabilizer at any point, so after taking a cover (replacing M by H, say), we can arrange finite stabilizer, so again we reduce to the previous case. Therefore, we will assume that our symmetry group H has more dimensions than the manifold M that it acts on.

8.8.5 Example: the standard flat foliated surface

Consider the standard flat G-structure. The surface M is the plane, foliated by parallel lines. The symmetry group is obviously infinite dimensional: take functions $X = X(x), Y = Y(x, y)$ with independent differentials at a point, i.e. $X'(x) \neq 0$ and $\frac{\partial Y}{\partial y} \neq 0$, and near that point you get a local symmetry. It is easy to arrange X and Y to make infinite dimensional families of global symmetries. But it is not clear how to find finite dimensional subgroups of this infinite dimensional group. If we take X and Y to be affine linear functions, that will give a five dimensional group of symmetries, acting transitively, with noncompact stabilizer of any point.

Question 21. Lets start with 5 dimensional groups, instead of 3. It would be nice to see why 5 is the maximum dimension, as well. Point out that the answer is the same over the complex numbers. I think 5 is not the maximal dimension.

Question 22. How do we stop this process? Presumably we show that if H has first step that is very large, then it has infinite dimension, perhaps by computing on the infinite prolongation. Another approach might use a Lie algebra decomposition theorem. Perhaps splitting into a semidirect product of semisimple and solvable. The semisimple has a maximal compact subgroup preserving a Riemannian metric, so having dimension strictly controlled. The KAK decomposition of the semisimple part reduces the problem to finding a dimension bound for an abelian symmetry group A and for a solvable group. Of course abelian groups are solvable, so we only need to understand solvable groups and their homogeneous spaces. Cartan classified the irreducible first and second order homogeneous spaces, and showed that there are no higher order irreducible homogeneous spaces. But if the group is not acting irreducibly in the tangent spaces of the homogeneous space (hence, irreducible), then all bets are off apparently. One can easily prove Cartan's result using the KAK decomposition of a simple Lie group.

8.8.6 Three dimensional groups

Let's start with a three dimensional group H acting transitively as symmetries of a foliated surface. Then H must act on the G-structure B as well. Since the H action commutes with the G action on B (as is always the case with symmetries), the H orbits are permuted by the G action. In particular, the H orbits in B are all of the same dimension. The H orbits in B project to the H orbits in M; but we assumed that H acts transitively on M. Therefore the

H orbits in B are all of dimension at least two. If they are two dimensional, then each H orbit is a G'-structure on M, where G' is finite, and we reduce to the previous cases. Therefore, we can assume that the H orbits are three dimensional, so that H is immersed into B. The 1-forms ω^1 and ω^2 must be independent on the H orbit, since they are semibasic and the H orbit surjects to the surface M. We will also need to have one of the γ^i_j 1-forms independent of ω^1 and ω^2 on the H orbit. Let us suppose that γ^1_1 is one such 1-form. (The reader can work out the other cases.)

So on any orbit of H in B, we must have equations

$$\gamma^i_2 = a^i_j \omega^j + b^i \gamma^1_1$$

and these a^i_j and b^i must be constant on each orbit. However, the γ^i_j are not really determined: they are determined only up to the action of the prolongation, i.e. up to replacing them by

$$\begin{aligned}
\gamma^1_1 &\mapsto \gamma^1_1 - p^1_{11}\omega^1 - p^1_{12}\omega^2 \\
\gamma^1_2 &\mapsto \gamma^1_2 - p^1_{21}\omega^1 - p^1_{22}\omega^2 \\
\gamma^2_2 &\mapsto \gamma^2_2 - p^2_{22}\omega^2
\end{aligned}$$

with $p^1_{12} = p^1_{21}$. Moreover, we can move the orbit inside $B^{(1)}$ by action of $G \rtimes \mathfrak{g}^{(1)}$, and under the G action

$$r_g^* \gamma = \operatorname{Ad}_g^{-1} \gamma,$$

so we calculate for

$$g = \begin{pmatrix} g^1_1 & g^1_2 \\ 0 & g^2_2 \end{pmatrix}$$

that

$$r_g^* \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^2_2 \end{pmatrix} = \begin{pmatrix} \gamma^1_1 & \frac{g^2_2 \gamma^1_2 + g^1_2\left(\gamma^1_1 - \gamma^2_2\right)}{g^2_2} \\ 0 & \gamma^2_2 \end{pmatrix}.$$

From the G-action, we can force γ^1_2 to have no γ^1_1 term, i.e. $b^1 = 0$, unless $b^2 = 1$.

Question 23. Should we always normalize via the G-action first, and then the $\mathfrak{g}^{(1)}$-action? Is G the normal subgroup of $G \rtimes \mathfrak{g}^{(1)}$?

This enables us to force $b^1 = 0$ or $b^2 = 1$ and then (in either case) to force $a^1_1 = a^1_2 = a^2_1 = a^2_2 = 0$, so

$$\begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^2_2 \end{pmatrix} = \begin{pmatrix} 1 & b^1 \\ 0 & b^2 \end{pmatrix} \gamma^1_1$$

with

$$\begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ b^2 \neq 1 \end{pmatrix}.$$

We split into cases:

8.8.6.1 Case I

Question 24. I think I only pursued this case, so I might as well not give it a subsection, and just say that I will only pursue this one case.

Assume

$$\begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The structure equations look like

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma_1^1 & \gamma_1^1 \\ 0 & \gamma_1^1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

Take exterior derivative and apply the Cartan lemma to find

$$d\gamma_1^1 = K\omega^1 \wedge \omega^2$$

and differentiate this to find

$$dK - 2K\gamma_1^1 = \nabla_1 K\omega^1 + \nabla_2 K\omega^2.$$

But the structure is homogeneous, so K must be a constant, which forces $K = 0$, so structure equations

$$d\gamma_1^1 = 0.$$

If we simply replace ω^1, ω^2 by $\varpi^1 = \omega^1 - \omega^2, \varpi^2 = \omega^2$, we see the structure equations of the flat web geometry. So this geometry is locally the standard flat foliation of a surface by curves, but viewed at a funny angle. We leave the many other cases for the reader to pursue.

Question 25. At this point, how do we plug in Cartan's classification of the second order homogeneous spaces and their maximal symmetry groups?

8.9 Upper bounds on order of homogeneous G-structures

The process we have described to catalogue the homogeneous G-structures, possibly with some torsion condition, does not terminate, because the symmetry group could have a large dimension, which is unknown *a priori*; another way to say this is that we have no upper bound on the orders of the possible homogeneous spaces. We will work out the simplest example where we can find a bound:

8.9.1 One dimensional manifolds with any kind of G-structure

Theorem 16. *A homogeneous G-structure on any one dimensional manifold has order at most 2.*

Proof. The structure equations of the frame bundle $F\mathbb{R}$ and its prolongations are

$$d\omega^{(p)} = -\sum_{k=-1}^{p} \frac{(p+1)!}{(p-k)!(k+1)!}\omega^{(p-k)} \wedge \omega^{(k)}.$$

Lets write out the first few:

$$\begin{aligned}
d\omega^{(-1)} &= -\omega^{(0)} \wedge \omega^{(-1)} \\
d\omega^{(0)} &= -\omega^{(1)} \wedge \omega^{(-1)} \\
d\omega^{(1)} &= -\omega^{(2)} \wedge \omega^{(-1)} - \omega^{(1)} \wedge \omega^{(0)} && (8.1) \\
d\omega^{(2)} &= -\omega^{(3)} \wedge \omega^{(-1)} - 2\omega^{(2)} \wedge \omega^{(0)} \\
d\omega^{(3)} &= -\omega^{(4)} \wedge \omega^{(-1)} - 3\omega^{(3)} \wedge \omega^{(0)} - 2\omega^{(2)} \wedge \omega^{(1)} \\
d\omega^{(4)} &= -\omega^{(5)} \wedge \omega^{(-1)} - 4\omega^{(4)} \wedge \omega^{(0)} - 5\omega^{(3)} \wedge \omega^{(1)} && (8.2) \\
d\omega^{(5)} &= -\omega^{(6)} \wedge \omega^{(-1)} - 5\omega^{(5)} \wedge \omega^{(0)} - 9\omega^{(4)} \wedge \omega^{(1)} - 5\omega^{(3)} \wedge \omega^{(2)} \\
&\vdots
\end{aligned}$$

If we impose a structure, of any order, this will put constraints on these 1-forms, and produce relations among them. For a homogeneous structure, the relations will have constant coefficients, and after finitely many steps, the relations will force all subsequent $\omega^{(p)}$ 1-forms to be multiples of earlier ones.

Suppose for example that the structure has order 4, so that the 1-forms $\omega^{(-1)}, \dots, \omega^{(3)}$ are linearly independent on the appropriate prolongation, but that (even on the next prolongation) $\omega^{(4)}$ is not linearly independent, i.e. is a multiple of these. Then the structure equation (8.2) above, forces $\omega^{(5)}$ to be also a multiple, and inductively all of the $\omega^{(p)}$ 1-forms are multiples of $\omega^{(-1)}, \dots, \omega^{(3)}$.

But the important observation is that not only will $\omega^{(4)}$ be a multiple of $\omega^{(-1)}, \dots, \omega^{(3)}$, but so will $d\omega^{(4)}$, because $\omega^{(-1)}, \dots, \omega^{(3)}$ will form a coframing. Moreover, $\omega^{(4)}$ must be a constant coefficient multiple of $\omega^{(-1)}, \dots, \omega^{(3)}$, by homogeneity. This will ensure that the right hand side of the equation above for $d\omega^{(4)}$ involves only constant coefficient multiples of 2-forms already encountered in earlier lines in the same set of equations. In particular, no $\omega^{(3)} \wedge \omega^{(1)}$ term can appear in $d\omega^{(4)}$, because such a term has never appeared before. This is a contradiction, since such a term does appear on the right hand side of (8.2). So there can't be a fourth-order structure.

A similar contradiction appears in the equation for $d\omega^{(3)}$, where we see the $\omega^{(2)} \wedge \omega^{(1)}$ term. So there can't be a third-order structure. Since $\omega^{(1)} \wedge \omega^{(1)} = 0$, there is no term to prevent second-order structures.

In general, $d\omega^{(p)}$ has a term of the form

$$\frac{(p+1)!}{2!(p-2)!}\omega^{(p-1)} \wedge \omega^{(1)}$$

which prevents any higher order structures.

This procedure is very difficult to imitate in general, although the theory is clear. Structure equations of more general G-structures are complicated only by lots of indices interacting via the representation of the Lie algebra of the structure group. Ultimately, one should be able to find bounds on orders of many types of G-structures using only the representation theory of G, but this approach has yet to be developed.

Exercise 8.26 (Lie, Brouwer [11]) Classify transitive actions of connected Lie groups on the line and circle. You should get

1. the projective transformations of the projective line
2. the affine transformations of the affine line
3. the translations of the affine line

up to taking covering spaces.

Exercise 8.27 Classify actions of connected Lie groups on simply connected Riemann surfaces.

Lie classified local Lie group actions on surfaces and wrote down without proof the classification on three dimensional manifolds (see Olver [68], pp. 472–473); he and Engel tried to extend the classification to 4 and 5 dimensional manifolds as well (see Bryant [16], p.69).

Exercise 8.28 Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup of the group Δ of diagonal invertible matrices. Prove that the symmetry group of a G-structure has dimension at most $3n$, with equality occuring only for $G = \Delta$ and just for Δ-structures locally equivalent to the obvious Δ-structure on $\mathbb{RP}^1 \times \cdots \times \mathbb{RP}^1$ (n copies), whose symmetry group is the group of projective transformations on each $\mathbb{RP}^1$ together with the obvious permutations.

8.9.2 Homogeneous foliated surfaces

Consider the foliation of the plane by vertical lines. This is invariant under the transformations

$$(x, y) \mapsto (X(x), Y(x, y)),$$

where X can be any diffeomorphism of the line, and Y any function satisfying $\frac{\partial Y}{\partial y} \neq 0$ at every point. Among this infinite dimensional group of transformations, we can find many finite dimensional subgroups, for example allowing affine transformations of x, say $X(x) = mx + b$, and allowing

$Y(x,y) = ny + p(x)$, where $p(x)$ can be any polynomial of degree less than, say, some fixed degree k. These form a group of dimension $3 + k$. Clearly this group acts faithfully, so there is no bound on the dimension (or order) of Lie groups acting as symmetries of foliated surfaces. Note that the action on the space of leaves of the foliation will always have order at most 2, dimension at most 3. For another example, allow $X = x + b, Y = ny + f(x)$ where $f(x)$ is a Fourier series with integer frequencies in some interval, say from $-N$ to N; in this fashion, we can construct examples on cylinders. See Olver [68] p. 61 for more on foliated surfaces.

Exercise 8.29 Generalize this example to show that the standard flat G-structure for G any one of

$$\mathrm{GL}(n,\mathbb{R}), SL(n,\mathbb{R}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{GL}(n,\mathbb{C}), SL(n,\mathbb{C})$$

(with $n > 1$) has arbitrarily large finite dimensional Lie groups of symmetries; similarly for the standard examples of contact structures (hint: think of 1-jets of curves in the plane), and holomorphic contact structures. Intuitively, this teaches us to expect that if the symmetry pseudogroup can be infinite dimensional, then it has a good chance of having arbitrarily large subgroups.

8.10 Pair structures

Question 26. This section is a huge mess.

Question 27. There is a serious disagreement about what the term “higher order structure” should mean. Certainly it has to do with principal subbundles of frame bundles $FM^{(k+1)}$. But it is a principal subbundle of $FM^{(k+1)} \to M$ or of $FM^{(k+1)} \to FM^{(k)}$? I need to think this out carefully. Following Cartan’s work on CR 3-manifolds, the right concept is clearly the latter. In fact, the pseudoconnection Cartan builds in that case is not a pseudoconnection for the former interpretation. The error in Coleman and Korté is in thinking along the lines of the first interpretation. This is very restrictive.

The concept of Cartan geometry fits naturally into the study of finite type G-structures, since often finite type G-structures give rise to Cartan geometries. For the study of infinite type structures, we will use the notion of graded Lie algebra.

Definition 39. *Let G^0 be a Lie group with Lie algebra $\mathfrak{g}^0$. Let*

$$\mathfrak{g}^\bullet = \bigoplus_{k=-1}^{\infty} \mathfrak{g}^k$$

be a graded Lie algebra (with $\mathfrak{g}^0$ the Lie algebra of G^0). Being graded means that the bracket $[x,y]$ takes $x \in \mathfrak{g}^p$ and $y \in \mathfrak{g}^q$ to $[x,y] \in \mathfrak{g}^{p+q}$. Therefore *the bracket makes each $\mathfrak{g}^k$ into a $\mathfrak{g}^0$-representation; we call $\left(G^0, \mathfrak{g}^\bullet\right)$ a* pair *if all of the $\mathfrak{g}^k$ are equipped with representations of G^0 which extend their $\mathfrak{g}^0$-representations. A* morphism *of pairs $\left(G^0, \mathfrak{g}^\bullet\right) \to \left(H^0, \mathfrak{h}^\bullet\right)$ is a morphism of Lie groups $G^0 \to H^0$ and a G^0-equivariant map of graded Lie algebras $\mathfrak{g}^\bullet \to \mathfrak{h}^\bullet$. A morphism is called a* monomorphism *if all of these maps are monomorphisms, etc.*

We can write V for $V = \mathfrak{g}^{-1}$.

Exercise 8.30 The Jacobi identity for the Lie algebra $\mathfrak{g}^\bullet$ ensures that the expression $[[x, v_0], v_1]$ for $x \in \mathfrak{g}^1$ and $v_j \in V$ is symmetric in v_0, v_1, giving a G^0-morphism

$$\mathfrak{g}^1 \to \left(\mathfrak{g}^0\right)^{(1)}.$$

This generalizes in the obvious manner to maps

$$\mathfrak{g}^p \to \left(\mathfrak{g}^q\right)^{(p-q)}.$$

These maps determine a canonical morphism

$$\left(G^0, \mathfrak{g}^\bullet\right) \to \left(\mathrm{GL}\left(V\right), \mathfrak{gl}\left(V\right)^{(\bullet)}\right),$$

where $V = \mathfrak{g}^{-1}$.

An *embedded* pair is one for which this canonical morphism is a monomorphism. A pair is said to be of *involutive* if the maps

$$\mathfrak{g}^p \to \bigcap_{q<p} \left(\mathfrak{g}^q\right)^{(p-q)}$$

are isomorphisms, for all sufficiently large p. It is said to be of *finite type* if only finitely many terms $\mathfrak{g}^\bullet$ are nonzero.

For any integer p, let

$$\mathfrak{g}^{\ge p} = \bigoplus_{q \ge p} \mathfrak{g}^q.$$

We can build a group $G^\bullet$ and a group $G^{\ge p}$ which formally should have as Lie algebras $\mathfrak{g}^\bullet$ and $\mathfrak{g}^{\ge p}$: we let $G^\bullet$ be the set of real analytic maps ϕ : open $\subset V \to V$ defined near the origin so that at every point $x \in V$ where ϕ is defined $\phi'(x) \in G$ and that the p-th derivative of $\phi(x)$ belongs to $\mathfrak{g}^p$ for all p. Let $G^{\ge p}$ be the subgroup for which ϕ agrees with the identity map up to p-th order at $x = 0$. Let $G^{p,q} = G^{\ge p}/G^{\ge q+1}$.

Question 28. These groups are interesting and relevant, but I really need the semidirect product group $G^0 \rtimes \mathfrak{g}^1 \rtimes \dots$, which I need to carefully define. It will have to be my structure group, as it was for prolongations.

We can define a $(G^0, \mathfrak{g}^\bullet)$-structure in the obvious way, as a tower of bundles $\dots B_{k+1} \to B_k \to \dots \to B_0 \to B_{-1} = M$, so that each B_k is a principal $\mathfrak{g}^k$ bundle over B_{k-1} for $k > 0$, and $B_0 \to M$ is a G^0-structure, with $B_k \to M$ a principal $G^{0,k}$-bundle, and all of these group actions commuting in the obvious manner, together with embeddings $B_k \subset FB_{k-1}$, equivariant via the map

$$G^{k-1,k-1} \to \mathrm{GL}\left(\mathfrak{g}^{k-1}\right)$$

induced from the adjoint action of $\mathfrak{g}^\bullet$.

Question 29. There might have to be a little more equivariance of various obvious group actions.

It is useful to consider such a tower which only extends up to a certain height. If we have specified bundles $B_{-1}, B_0, \dots, B_p$ satisfying these rules, but haven't picked out a $B_{p+1}, B_{p+2}, \dots,$ we will call our tower of bundles a $(G^0, \mathfrak{g}^\bullet)$-structure of *height* $p+1$. If we don't want to specify the pair, we can just call such a thing a *pair structure.*

Even for an infinite order G^0-structure, subsequent prolongations will determine a $(G^0, \mathfrak{g}^\bullet)$-structure (whose height is determined by how many prolongations are defined) for a suitable graded Lie algebra. So pair structures form a reasonable category to work in. Given any Lie group G^0 with representation V, we can construct the canonical pair $(G^0, \mathfrak{g}^\bullet)$ of this representation by letting $\mathfrak{g}^0$ be the Lie algebra of G^0 and taking $\mathfrak{g}^p = \left(\mathfrak{g}^0\right)^{(p)}$. For example, $\mathrm{GL}(n, \mathbb{R})$ has a canonical pair. Generalizing this, if I have a pair $(G^0, \mathfrak{g}^\bullet)$, and fix a subspace $\mathfrak{h}^p \subset \mathfrak{g}^p$ inside some of the graded terms, for some values of $p > 0$, then I can define the *generated pair* to be the largest pair $(G^0, \mathfrak{k}^\bullet)$ which has $\mathfrak{k}^p \subset \mathfrak{h}^p$. (Keep in mind that successive prolongations will impose constraints). For example, if I start with the canonical pair of $\mathrm{GL}(n, \mathbb{R})$, and then pick the subspace $\mathbb{R}^{n*} \subset \mathfrak{gl}(n, \mathbb{R})^{(1)}$, I obtain the canonical pair of projective structures. A pair structure for this pair of height 2 is precisely a projective structure.

Question 30. Or is it a projective connection?

A morphism between pairs $(G^0, \mathfrak{g}^\bullet) \to (H^0, \mathfrak{h}^\bullet)$ is called an *isomorphism to order* p if $\mathfrak{g}^q \to \mathfrak{h}^q$ is an isomorphism for $q < p$.

Theorem 17 (Coleman & Korté [36]). *The canonical morphism*

$$\left(G^0, \mathfrak{g}^\bullet\right) \to \left(\mathrm{GL}(V), \mathfrak{gl}(V)^{(\bullet)}\right)$$

of a pair is isomorphic to order 1 *just when the pair is isomorphic to order* 2 *to the canonical pair of*

1. $\mathrm{GL}(n, \mathbb{R})$*,*
2. *projective structures,*

3. connections on the tangent bundle, or
4. connections on the determinant line bundle of the tangent bundle.

The canonical morphism is isomorphic to order 2 *(order 3 if* $n = 1$*) just when it agrees to all orders.*

There are lots of higher order structures which are not pair structures. This fact is perhaps slightly obscured in the paper of Coleman & Korté.

Proof. We won't follow the proof of Coleman and Korté because, although it involves explicit local coordinate calculations, we feel it is more helpful to expose the representation theory and hide the coordinates, to make the approach easier to generalize.

From section 7.2 on page 99, we can easily see that the successive prolongations of $\mathfrak{gl}(n,\mathbb{R})$ are $\mathfrak{gl}(n,\mathbb{R})^{(p)} = \mathbb{R}^n \otimes \operatorname{Sym}^{p+1}(\mathbb{R}^{n*})$, and their sum is equipped with the Lie bracket

$$[a,b]^i_J = \sum_{J_1 J_2 | J} a^i_{J_1 k} b^k_{J_2} - \sum_{J_1 J_2 | J} b^i_{J_1 k} a^k_{J_2}.$$

The representation $\mathfrak{gl}(n,\mathbb{R})^{(1)}$ of $\mathrm{GL}(n,\mathbb{R})$ splits into irreducible representations

$$\begin{aligned}\mathfrak{gl}(n,\mathbb{R})^{(1)} &= \mathbb{R}^n \otimes \operatorname{Sym}^2(\mathbb{R}^{n*}) \\ &= \mathbb{R}^{n*} \oplus \Gamma_{1,0,\dots,0,2}\end{aligned}$$

following the notation of Fulton & Harris [39] lecture 15; Fulton & Harris make the result very explicit for $\mathfrak{sl}(4,\mathbb{R})$, and the same calculations work in general, without using the Weyl character formula, or any heavy machinery. The representation $\Gamma_{1,0,\dots,0,2}$ is the collection of tensors $Q \in \mathbb{R}^n \otimes \operatorname{Sym}^2(\mathbb{R}^{n*})$ for which $Q^i_{ji} = 0$, and the second order structure consists in a choice of connection on the determinant bundle of the tangent bundle. The representation $\mathbb{R}^{n*}$ is the one we encountered in discussing projective structures in section 7.11 on page 154. This gives our first result.

Corollary 20. *There are no homogeneous spaces* $M = G/H$ *for which a finite dimensional Lie group* G *can act transitively on* M, FM *and* $FM^{(1)}$*, except if* $M = \mathbb{P}^1$ *in which case the group action must factor through* $G \to \mathbb{P}SL\left(\mathbb{P}^1\right)$*. Even* $M = \mathbb{P}^1$ *does not have a group acting on it transitively on* $FM^{(2)}$*.*

Proof. Question 31. Finish this.

A structure properly of order k, for some $k > 2$ is an A-subbundle of $FM^{(k-1)} \to FM^{(k-2)}$ for $A \subset \mathfrak{gl}(n,\mathbb{R})^{(k-1)}$ a $\mathrm{GL}(n,\mathbb{R})$-submodule. In fact, A must be an $\mathfrak{h}$-submodule, where

$$\mathfrak{h} = \mathfrak{gl}(n,\mathbb{R}) \rtimes \mathfrak{gl}(n,\mathbb{R})^{(1)} \rtimes \cdots \rtimes \mathfrak{gl}(n,\mathbb{R})^{(k-2)}.$$

We will then need to make $\mathfrak{h}' = \mathfrak{h} \oplus A$ into a Lie algebra using the induced bracket. The bracket is just the one which we constructed above and in section 7.2 on page 99 (i.e. think of the structure equations as the Maurer–Cartan 1-forms, at least formally, of the diffeomorphism group, and that gives the Lie algebra).

Lets assume that A is a structure properly of order 3. Up to irrelevant constants, the bracket on elements of $\mathfrak{gl}(n, \mathbb{R})^{(1)}$ is

$$\left[\frac{\partial}{\partial x^i} dx^p \cdot dx^q, \frac{\partial}{\partial x^j} dx^r \cdot dx^s\right] = \frac{\partial}{\partial x^i} \left(\delta^p_j dx^q + \delta^q_j dx^p\right) \cdot dx^r \cdot dx^s - \frac{\partial}{\partial x^j} \left(\delta^r_i dx^s + \delta^s_i dx^r\right) \cdot dx^p \cdot dx^q.$$

This gives

$$\begin{aligned} &\frac{\partial}{\partial x^1} dx^2 dx^3 dx^4, i = j = p = 1, q = 2, r = 3, s = 4 \\ 2&\frac{\partial}{\partial x^1} dx^1 dx^2 dx^3, i = j = p = q = 1, r = 2, s = 3 \\ &\frac{\partial}{\partial x^1} \left(dx^1\right)^2 dx^2, i = j = p = q = r = 1, s = 2. \end{aligned}$$

Question 32. In dimension 1, this has to be zero. But I have already shown the required result in dimension 1. So I really need to compare those proofs.

These must all belong to A, since $\mathfrak{h}'$ is closed under Lie bracket. By $\mathrm{GL}(n, \mathbb{R})$ action, we can change the indices on the variables as we like. We don't find terms of the form

$$\frac{\partial}{\partial x^1} \left(dx^1\right)^3.$$

However, we can use linear transformations to get

$$\frac{\partial}{\partial x^1} \left(dx^1\right)^2 \left(dx^1 + dx^2\right) \in A,$$

and then find that

$$\frac{\partial}{\partial x^1} \left(dx^1\right)^2 \left(dx^1 + dx^2\right) - \frac{\partial}{\partial x^1} \left(dx^1\right)^2 dx^2 = \frac{\partial}{\partial x^1} \left(dx^1\right)^3 \in A.$$

This ensures that all monomials lie in A, so $A = \mathfrak{gl}(n, \mathbb{R})^{(2)}$ is not properly a 3rd order structure.

For higher order structures beyond third order, the argument is very similar, involving more indices.

Question 33. I will need to understand the relation of the symmetry group to the graded Lie algebra.

Question 34. The stated theorem of Coleman and Korté is wrong. Obviously we can pick lots of higher order structures, just taking sections of $FM^{(p)}/A$ for $A \subset \mathfrak{gl}(n, \mathbb{R})^{(p)}$ any vector subspace. But that is not quite what they are getting at. We need a new notion to make sense out of their result.

Exercise 8.31 Suppose that you have a higher order structure on a one dimensional manifold M, say at order 4, a submanifold $B \subset FM^{(3)}$. Prove that either $B = FM^{(3)}$ or else the torsion is nonconstant on the fibers of $B \to M$. Therefore we can try to reduce the structure group, and even if we fail to reduce the structure group, at least we can say that there are invariant defined submanifolds of B cut out by level sets of the torsion.

Exercise 8.32 Generalize this last exercise to structures of all orders 4 or more on one dimensional manifolds.

Exercise 8.33 Generalize the last two exercises to structures of all orders 3 or more in dimensions 2 or more.

The problem is that to really convince me that there is no meaning to Coleman & Korté's statements as they made them, I would need to find an example of a properly higher order structure with no invariant choice of induced reduction on any lower order bundle. But we can always try to reduce, at least locally, by normalizing torsion, unless all of the torsion is constant on all of the fibers. Keep in mind how complicated this is: in any region where the torsion has differential of constant rank, we can specify a submanifold in the Spencer cohomology transverse to the torsion, and cut out the submanifold on which torsion lives there. But that won't be a reduction of structure group. To be sure that there is no way to reduce structure group by normalizing torsion, one would have to think carefully about reduction. But lets just for now say that failure to cut out a submanifold of any one of our bundles would require torsion to be constant on fibers; this is not quite the same as failure to obtain an invariant reduction of structure group. Therefore the problem for Coleman and Korté is to convince me that the torsion of a higher order structure cannot be constant unless it is the start of a pair structure.

Question 35. All of this Coleman and Korté stuff is getting close to a paper.

8.11 The symmetry Lie algebra

8.11.1 Equations for infinitesimal symmetries

Suppose that B is a G-structure on a manifold M, and that B is equipped with a choice of pseudoconnection. Lets try to find explicit equations for the symmetries of B as a G-structure, and be careful *not* to require that the symmetries preserve the pseudoconnection.

An infinitesimal symmetry is a vector field X on M whose prolongation to frames preserves $B \subset FM$. The prolongation to frames of a vector field X is the vector field X on FM, defined by the equation

$$X = \left.\frac{d}{dt}\right|_{t=0} Fe^{tX}$$

Recall prolongation to frames of a diffeomorphism (or local diffeomorphism) was defined in section 2.1 on page 5.

With $V = \mathbb{R}^n$ as usual, for each vector field X on M we can define a function $f_X : FM \to V$ by $f_X(u) = u(X(x))$ for $u \in F_m M, m \in M$. This function satisfies

$$r_g^* f_X = g^{-1} f_X,$$

for any $g \in \mathrm{GL}(V)$. Conversely, each function satisfying this equation comes about from a vector field X on M. Differentiating this equation gives

$$df_X + \gamma f_X = \nabla_i f_X \omega^i$$

for some function $\nabla_\bullet f_X : FM \to V \otimes V^*$.

Exercise 8.34 At points where $X = 0$, show that $f_X = 0$ and that $\nabla_\bullet f_X$ descends to an endomorphism on $T_x M$ at such points, well known in dynamical systems: the linearization.

One can check easily that on FM

$$X \lrcorner \omega = f_X.$$

Frequently it is useful to treat $V = \mathbb{R}^n$ as having a fixed basis, so that $\omega = \left(\omega^i\right)$ in components. Then we can write $f_X = \left(X^i\right)$, so that

$$dX^i + \gamma^i_j X^j = \nabla_j X^i \omega^j.$$

We want to see if a vector field X is a symmetry of the G-structure B. We can restrict f_X to B as a G-equivariant function $f_X : B \to V$. Our vector field determines a vector field X on FM and we need to know when X is tangent to a given G-structure $B \subset FM$. Recall from section 6.2 on page 59 that our pseudoconnection γ on B induces a 1-form γ_* on the tangent spaces of FM at points of B, with γ_* valued in $\mathfrak{gl}(V)$.

Lemma 34. *The tangent directions to B are precisely the directions in which γ_* is valued in the Lie algebra $\mathfrak{g}$.*

Proof. Consider any fattening up of structure group $G \subset H$. Returning to the notations of section 6.2 on page 59, we see the diagram

$$\begin{array}{ccc} B\times G & \longrightarrow & B\times H \\ \downarrow & & \downarrow \\ B=B(G) & \longrightarrow & B(H) \end{array}$$

and so the tangent directions to B inside $B(H)$ are just the quotient by the vertical of the tangent directions to $B \times G$ inside $B \times H$. The 1-form γ_* on $T_u B(H)$ (defined only for $u \in B$) pulls back to $B \times H$ at the point $(u, 1)$ to become

$$\gamma_* = h^{-1}\, dh + \operatorname{Ad}_h^{-1} \gamma = dh + \gamma,$$

and γ is valued in $\mathfrak{g}$, so that γ_* lives in the tangent directions to $B \times G$ just when $dh \in \mathfrak{g}$, i.e. just when $\gamma_* \in \mathfrak{g}$. To apply this in our context, set $H = \mathrm{GL}\,(V)$.

Lemma 35.

$$Z \lrcorner \gamma_* = \left(\nabla_j Z^i + T^i_{kj} Z^k\right) \in \mathfrak{gl}\,(V)\,.$$

Proof. We work on FM entirely. First we will show that

$$\mathcal{L}_Z \omega = 0.$$

Write $\pi : FM \to M$ for the obvious bundle map. From the definition of Z and of ω, we can calculate that

$$\begin{aligned} e^{tZ*}\omega_u &= \omega_{e^{tZ}u} \left(e^{-tZ}\right)' \\ &= e^{tZ} u \pi' \left(e^{-tZ}\right)'. \end{aligned}$$

By definition of Z,

$$\begin{aligned} e^{tZ} u &= F\big(e^{tZ}\big) u \\ &= u \left(e^{-tZ}\right)' \end{aligned}$$

From the definition of Z, we see that $\pi e^{tZ} = e^{tZ}$. Therefore returning to the previous calculation,

$$\begin{aligned} e^{tZ*}\omega_u &= u \left(e^{-tZ}\right)' \pi' \left(e^{-tZ}\right)' \\ &= u \left(e^{-tZ} \pi e^{tZ}\right)' \\ &= u\pi' \\ &= \omega_u. \end{aligned}$$

Therefore

$$\mathcal{L}_Z \omega = 0.$$

(Not surprising: Z is an infinitesimal equivalence of the $\mathrm{GL}\,(V)$-structure FM on M.)

Fix a point $u \in B$ and take $\Gamma = \left(\Gamma^i_j\right)$ any pseudoconnection for FM which agrees with γ_* at u. Using the Cartan equation

$$\mathcal{L}_Z \omega = Z \lrcorner\, d\omega + d\left(Z \lrcorner\, \omega\right)$$

we find

$$\begin{aligned} dZ^i &= -Z \lrcorner \left(-\Gamma^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k\right) \\ &= -\Gamma^i_j Z^j Z \lrcorner\, \Gamma^i_j + T^i_{kj} Z^k \omega^j. \end{aligned}$$

It is vital to note that the torsion of Γ at $u \in FM$ is the same as that of γ at $u \in B$. Inside the manifold B, we have

$$dZ^i = -\gamma^i_j Z^j + \nabla_j Z^i \omega^j.$$

A similar relation,

$$dZ^i = -\Gamma^i_j Z^j + \nabla^\Gamma_j Z^i \omega^j$$

occurs in FM. We can unwind the definition of γ_* to see easily that at the point u,

$$\nabla^\Gamma = \nabla.$$

Therefore

$$Z \lrcorner\, \Gamma^i_j = \nabla_j Z^i + T^i_{kj} Z^k$$

Putting things together:

Proposition 38. *A vector field Z on a manifold M with G-structure $B \to M$ is an infinitesimal symmetry of B (not necessarily preserving any pseudoconnection) just when the expression*

$$\nabla_j Z^i + T^i_{kj} Z^j$$

belongs to the Lie algebra of G, for some (and hence any) choice of pseudoconnection.

Another way to say it:

Proposition 39. *Suppose that Z is a vector field on a manifold M with a G-structure $B \to M$. Pick any pseudoconnection γ for this G-structure. Then define the vector field $\hat{Z}$ by*

$$\begin{aligned} \hat{Z} \lrcorner\, \omega &= Z^\bullet \\ \hat{Z} \lrcorner\, \gamma &= 0. \end{aligned}$$

The vector field Z is an infinitesimal symmetry of the G-structure $B \to M$ (not necessarily preserving the pseudoconnection) just when ω evolves along the flow of $\hat{Z}$ via transformations from the group G:

$$e^{t\hat{Z}*} \omega_u = g(t, u)\omega,$$

with $g \in G$.

Proof. See section 7.8.4 on page 146.

Example 78 (Flat structures). On a flat e-structure, $\mathfrak{g} = 0$ so $\gamma = 0$, structure equations are $d\omega = 0$ and a symmetry is just $dX^\bullet = 0$, i.e. $X^\bullet$ is a constant function.

Example 79 (Spheres). On $M = S^n$ the sphere with its usual constant curvature Riemannian metric (or indeed on any Riemannian manifold with unit sectional curvature), the structure equations are

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_j \wedge \gamma^k_j + \omega^i \wedge \omega^j . \end{aligned}$$

There is no 1-torsion. We will write $X^\bullet$ in components as X^i. Write $\nabla_\bullet X^\bullet$ as $\nabla_j X^i$ so that

$$dX^i = -\gamma^i_j X^j + \nabla_j X^i \omega^j .$$

Taking exterior derivative of this equation gives

$$d\nabla_j X^i - \nabla_k X^i \gamma^k_j + \nabla_j X^k \gamma^i_k - X^j \omega^i = \nabla_{jk} X^i \omega^k$$

with $\nabla_{jk} X^i = \nabla_{kj} X^i$, by Cartan's lemma. But $\nabla_j X^i$ must be valued in the Lie algebra of the structure group to provide an infinitesimal symmetry, and our manifold S^n bears an $SO\left(n\right)$-structure. Thus the Lie algebra is $\mathfrak{so}\left(n\right)$, the skew-symmetric matrices, so $\nabla_j X^i + \nabla_i X^j = 0$, and $\nabla_{jk} X^k$ is symmetric in lower indices, and skew symmetric in any pair of upper and lower index, and therefore vanishes. Thus we can specify X^i and $\nabla_j X^i$ satisfying the coupled system of ordinary differential equations

$$\begin{aligned} dX^i + \gamma^i_j X^j &= \nabla_j X^i \omega^j \\ d\nabla_j X^i - \gamma^j_k \nabla_i X^k + \gamma^i_k \nabla_j X^k &= X^j \omega^i - X^i \omega^j . \end{aligned}$$

At most we get to choose $X^i \in \mathbb{R}^n$ and $\nabla_j X^i \in \mathfrak{so}\left(n\right)$ to determine a local solution of these equations, so the Lie algebra of the symmetry group has at most $n + n(n-1)/2$ dimensions. Of course, the symmetry Lie algebra is in fact $\mathfrak{so}\left(n+1\right)$, which has precisely these dimensions.

Example 80 (Almost complex structures). A $\mathrm{GL}\left(n, \mathbb{C}\right)$-structure $B \to M$ on a manifold M is also called an *almost complex structure*. We can write the structure equations in terms of complex-valued 1-forms as

$$d\omega^i = -\gamma^i_j \wedge \omega^j + \frac{1}{2} T^i_{\bar{j}\bar{k}} \omega^{\bar{j}} \wedge \omega^{\bar{k}}$$

where $\omega^{\bar{\imath}}$ means the complex-valued 1-form $\overline{\omega^i}$, i.e. when you plug a vector into it, it spits out the complex conjugate of what ω^i spits out. The intrinsic torsion T lives in the representation $V \otimes_{\mathbb{C}} \Lambda^{0,2}\left(V^*\right)$, which is a subrepresentation of

$V \otimes_{\mathbb{R}} \Lambda^2 (V^*)$, so it determines a tensor, called the *Nijenhuis tensor*. A vector field on an almost complex manifold is given by functions X^i satisfying

$$dX^i + \gamma^i_j X^j = \nabla X^i_j \omega^j + \nabla X^i_{\bar{j}} \omega^{\bar{j}}.$$

The vector field is a symmetry of the almost complex structure just when $\nabla X^i_{\bar{j}} = 0$, forcing $\nabla_\bullet X^\bullet$ to live in the Lie algebra. Determine an almost complex structure on B by asking that γ and ω be complex linear 1-forms.

Exercise 8.35 With torsion absorbed as above, show that this almost complex structure on B is well defined.

The equations $\nabla_{\bar{j}} X^i = 0$ are precisely requiring X^i to be a holomorphic function, since they require just that the differential dX^i be a multiple of ω^i and γ^i_j, which span the complex linear 1-forms on B. Thus a vector field X on M preserves the almost complex structure downstairs just when $X^\bullet : B \to V$ is a holomorphic $\mathrm{GL}(n, \mathbb{C})$-equivariant function.

Example 81 (Symmetries of contact geometry). In section 7.9 on page 147, we found that the structure equations of contact geometry are

$$d \begin{pmatrix} \omega \\ \omega^\mu \\ \omega_\mu \end{pmatrix} - \begin{pmatrix} \gamma & \omega_\mu & 0 \\ \gamma^\mu & \gamma^\mu_\nu & \gamma^{\mu\nu} \\ \gamma_\mu & \gamma_{\mu\nu} & \delta^\nu_\mu \gamma - \gamma^\nu_\mu \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\nu \\ \omega_\nu \end{pmatrix}.$$

A vector field on a contact manifold is represented by functions X^0, X^μ, X_μ satisfying

$$d \begin{pmatrix} X^0 \\ X^\mu \\ X_\mu \end{pmatrix} - \begin{pmatrix} \gamma & 0 & 0 \\ \gamma^\mu & \gamma^\mu_\nu & \gamma^{\mu\nu} \\ \gamma_\mu & \gamma_{\mu\nu} & \delta^\nu_\mu \gamma - \gamma^\nu_\mu \end{pmatrix} \begin{pmatrix} X^0 \\ X^\nu \\ X_\nu \end{pmatrix} + \begin{pmatrix} \nabla_0 X^0 & \nabla_\nu X^0 & \nabla^\nu X^0 \\ \nabla_0 X^\mu & \nabla_\nu X^\mu & \nabla^\nu X^\mu \\ \nabla_0 X_\mu & \nabla_\nu X_\mu & \nabla^\nu X_\mu \end{pmatrix}.$$

The only torsion is the ω_μ term in $d\omega$, so to be a symmetry, we need to have the $\nabla X + TX$ in the Lie algebra, i.e.

$$\begin{pmatrix} \nabla_0 X^0 & \nabla_\nu X^0 + X_\nu & \nabla^\nu X^0 - X^\nu \\ \nabla_0 X^\mu & \nabla_\nu X^\mu & \nabla^\nu X^\mu \\ \nabla_0 X_\mu & \nabla_\nu X_\mu & \nabla^\nu X_\mu \end{pmatrix}$$

in the Lie algebra, so

$$\begin{aligned} X_\nu &= -\nabla_\nu X^0 \\ X^\nu &= \nabla^\nu X^0 \\ \nabla_\nu X_\mu &= \nabla_\mu X_\nu \\ \nabla^\nu X_\mu &= \delta^\nu_\mu \nabla_0 X^0 - \nabla_\mu X^\nu. \end{aligned}$$

Clearly X^0 determines the rest of the symmetry, so symmetries depend on at most one function.

Exercise 8.36 The equation

$$\nabla X^0 = dX^0 + \gamma X^0$$

ensures that X^0 represents a section of the line bundle dual to the bundle of 1-forms vanishing on the contact planes.

8.11.1.1 Lie's method of symmetries

The approach of Sophus Lie to calculating symmetries of geometric structures (particularly systems of partial differential equations) is explained in detail by Olver [67], and in [4, 46, 5]. The general idea is that one can calculate a system of linear differential equations whose solutions are the infinitesimal symmetries of a given system of differential equations. Applied to G-structures, the method reveals just what we have found: the differential equation asking $\nabla_\bullet Z^\bullet + TZ^\bullet$ to belong to the Lie algebra of the structure group. Lie's method is somewhat more general, but practically this covers most of the interesting examples.

8.11.2 The Lie bracket on infinitesimal symmetries

Now that we can write infinitesimal symmetries as functions on the total space of our G-structure, we need to compute the Lie bracket on those symmetries. We have found that symmetries can be described by functions X^i on our bundle, so that

$$dX^i + \gamma^i_j X^j = \nabla_j X^i \omega^j$$

where $\nabla_j X^i$ belongs to $\mathfrak{g}$, the Lie algebra of the structure group.

Proposition 40.

$$[X, Y]^i = -T^i_{jk} X^j Y^k + X^j \nabla_j Y^i - Y^j \nabla_j X^i.$$

Proof. Suppose that we have two such symmetries, given by functions X^i and Y^i satisfying these sort of equations, associated to vector fields X and Y on the base manifold M. We have equations

$$\begin{aligned} X^\bullet &= X \lrcorner\, \omega \\ Y^\bullet &= Y \lrcorner\, \omega \\ X &= \pi_* X \\ Y &= \pi_* Y \\ \mathcal{L}_X \omega &= \mathcal{L}_Y \omega = 0 \end{aligned}$$

which give

$$[X, Y] = \pi_* [X, Y]$$

and

$$\mathfrak{L}_{[X,Y]}\omega = 0$$

so that

$$[X, Y] = [X, Y] .$$

Calculate

$$d\omega (X, Y) = \mathfrak{L}_X (Y \lrcorner \omega) - \mathfrak{L}_Y (X \lrcorner \omega) - \omega ([X, Y])$$

and use

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega$$

to find, putting it all together:

$$[X, Y]^i = -T^i_{jk} X^j Y^k + X^j \nabla_j Y^i - Y^j \nabla_j X^i .$$

Example 82 (The sphere continued). We found $\nabla_{ij} = 0$ on the sphere, so that the bracket has

$$[X, Y]^i = X^j \nabla_j Y^i - Y^j \nabla_j X^i .$$

Exercise 8.37 Calculate that on a unit sectional curvature Riemannian manifold, the Lie bracket of infinitesimal symmetries is determined by the equation

$$\nabla_k [X, Y]^i = \nabla_k X^j \nabla_j Y^i - \nabla_k Y^j \nabla_j X^i + X^j \left(Y^j \delta^i_k - Y^i \delta^j_k \right) - Y^j \left(X^j \delta^i_k - X^i \delta^j_k \right) .$$

In particular, the symmetry Lie algebra must be a subalgebra of the symmetry Lie algebra of S^n, i.e. a subalgebra of $\mathfrak{so}(n + 1)$. It is interesting that the Lie algebra of symmetries of any unit sectional curvature Riemannian manifold lives inside $\mathfrak{so}(n + 1)$, while the manifold itself need not live inside the sphere. For example, slice out a codimension two (or more) great subsphere (or a finite number of them), and take the universal cover, to obtain an onion, which can have a large symmetry group, and is not a submanifold of the original sphere. Still, its symmetry Lie algebra lives in $\mathfrak{so}(n + 1)$.

8.11.2.1 Remarks on symmetries

For numerous techniques for relating infinitesimal symmetries and global geometry on complex and especially Kähler manifolds, see Kobayashi [51]; in particular the Carrell–Liebermann theorem and the Grothendieck–Riemann–Roch theorem, see Griffiths & Harris [40]; and the notion of hyperbolicity, see Kobayashi [52]. The search is underway for analogues of these results for other types of geometry, especially elliptic G-structures.

8.12 Example: 2 plane fields on 4 manifolds

Question 36. This section should be moved much earlier, since I didn't really use very much theory here.

A 2-plane field on a 4-manifold imposes a G_1-structure where G_1 is the group of linear transformations of $\mathbb{R}^4$ preserving a 2-plane, i.e. the group of invertible matrices of the form

$$\begin{pmatrix} a^1_1 & a^1_2 & 0 & 0 \\ a^2_1 & a^2_2 & 0 & 0 \\ a^3_1 & a^3_2 & a^3_3 & a^3_4 \\ a^4_1 & a^4_2 & a^4_3 & a^4_4 \end{pmatrix}.$$

Thus after a little absorption of torsion, the structure equations on the G_1-structure are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = - \begin{pmatrix} \omega^1_1 & \omega^1_2 & 0 & 0 \\ \omega^2_1 & \omega^2_2 & 0 & 0 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 & \omega^3_4 \\ \omega^4_1 & \omega^4_2 & \omega^4_3 & \omega^4_4 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} t^1 \omega^3 \wedge \omega^4 \\ t^2 \omega^3 \wedge \omega^4 \\ 0 \\ 0 \end{pmatrix}.$$

(The reader will notice that we are using notation similar to Cartan, calling the pseudoconnection ω^i_j just as we called the soldering form ω^i.) Differentiating reveals

$$d \begin{pmatrix} t^1 \\ t^2 \end{pmatrix} = - \begin{pmatrix} \omega^1_1 - \omega^3_3 - \omega^4_4 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 - \omega^3_3 - \omega^4_4 \end{pmatrix} \begin{pmatrix} t^1 \\ t^2 \end{pmatrix}$$

modulo semibasic terms (i.e. modulo the ω^j). So we either have $t^1 = t^2 = 0$ or else we can arrange $t^1 = 0, t^2 = 1$.

It is an immediate consequence of the Frobenius theorem that $t^1 = t^2 = 0$ everywhere precisely when the 2-plane field is the field of tangent planes to a foliation by surfaces. Henceforth we will suppose that t^1 and t^2 do not both vanish anywhere.

To figure out what to do with t^1 and t^2, lets look at an example. Consider the example of the 2-jet bundle of real valued functions of a real variable. This bundle has coordinates x, y, p, q with 2-plane field described by

$$\begin{aligned} \eta^1 &= dy - p\,dx \\ \eta^2 &= dp - q\,dx \\ \eta^3 &= dx \\ \eta^4 &= dq. \end{aligned}$$

Exercise 8.38 These η^j are adapted coframes, i.e. a section of a G_1-structure. (To see this, just note that they satisfy the required structure equations.) Moreover, for this section $t^1 = 0, t^2 = 1$.

This guides our choices of how to normalize t^1 and t^2.

Let B_2 be the subbundle of our G_1 structure on which $t^1 = 0$ and $t^2 = 1$. This B_2 is a G_2-structure, and we let the reader identify G_2. The structure equations are now

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = - \begin{pmatrix} \omega_1^1 & 0 & 0 & 0 \\ \omega_1^2 & \omega_3^3 + \omega_4^4 & 0 & 0 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \\ \omega_1^4 & \omega_2^4 & \omega_3^4 & \omega_4^4 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \omega^2 \wedge \left(a_3^1 \omega^3 + a_4^1 \omega^4 \right) \\ \omega^2 \wedge \left(a_3^2 \omega^3 + a_4^2 \omega^4 \right) + \omega^3 \wedge \omega^4 \\ 0 \\ 0 \end{pmatrix}$$

In our example of the 2-jet bundle,

$$a_3^1 = -1, a_4^1 = a_3^2 = a_4^2 = 0.$$

In fact, we can absorb the a_3^2 and a_4^2 functions into ω_4^4.

The differentials of the remaining a functions are

$$d \begin{pmatrix} a_3^1 \\ a_4^1 \end{pmatrix} = \begin{pmatrix} -\omega_1^1 + 2\omega_3^3 + \omega_4^4 & \omega_3^4 \\ \omega_4^3 & -\omega_1^1 + \omega_3^3 + 2\omega_4^4 \end{pmatrix} \begin{pmatrix} a_3^1 \\ a_4^1 \end{pmatrix}$$

which is a Lie algebra representation onto $\mathfrak{gl}(2, \mathbb{R})$, so that either we can arrange $a_3^1 = -1, a_4^1 = 0$ or else we must have $a_3^1 = a_4^1 = 0$. Again we will employ a constant type hypothesis, and break into these two cases. In the case of $a_3^1 = a_4^1 = 0$, the foliation $\omega^1 = 0$ descends to the 4-manifold, so that it is foliated by contact 3-manifolds. Conversely, any foliation of a 4-manifold by 3-manifolds carrying contact structures provides an example. So now we can suppose that $a_3^1 = -1$ and $a_4^1 = 0$. This forces $\omega_1^1 - 2\omega_3^3 - \omega_4^4$ and ω_4^3 to be semibasic. Absorbing torsion, we find

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = - \begin{pmatrix} 2\omega_3^3 + \omega_4^4 & 0 & 0 & 0 \\ \omega_1^2 & \omega_3^3 + \omega_4^4 & 0 & 0 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & 0 \\ \omega_1^4 & \omega_2^4 & \omega_3^4 & \omega_4^4 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \omega^3 \wedge \omega^2 \\ \omega^3 \wedge \omega^4 \\ c\omega^3 \wedge \omega^4 \\ 0 \end{pmatrix}.$$

Taking two exterior derivatives of ω^1 forces $c = 0$. We now have the structure equations

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = - \begin{pmatrix} 2\omega_3^3 + \omega_4^4 & 0 & 0 & 0 \\ \omega_1^2 & \omega_3^3 + \omega_4^4 & 0 & 0 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & 0 \\ \omega_1^4 & \omega_2^4 & \omega_3^4 & \omega_4^4 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + \begin{pmatrix} \omega^3 \wedge \omega^2 \\ \omega^3 \wedge \omega^4 \\ 0 \\ 0 \end{pmatrix}$$

which are the same as those of the example of 2-jets of real functions of a real variable (or 2-jets of sections of a line bundle over a curve). It is clear that the symmetry group of this example is infinite dimensional: local diffeomorphisms of one variable act on the 2-jet space. We can check that the structure equations are in involution using the Cartan–Kähler theorem, so that

Proposition 41. *Every analytic 2-plane field of constant type on a 4-manifold is either (1) a foliation by surfaces or (2) a foliation by contact 3-manifolds or (3) an* Engel*2-plane field (i.e. locally equivalent to the canonical 2-plane field on the 2-jet bundle of a line bundle on a curve).*

We can see that the equations $\omega^1 = \omega^2 = \omega^3 = 0$ cut out a foliation of an Engel 4-manifold into curves, called *characteristic curves.*

Exercise 8.39 Suppose that the foliation by characteristic curves is a fiber bundle $M^4 \to Q^3$ with connected fibers. Consider the family of 3-planes on M described by $\omega^1 = 0$. (The reader will need to figure out why this equation $\omega^1 = 0$ is defined on M, while ω^1 is not.)

Exercise 8.40 Show that these 3-planes descend to a contact structure on Q.

Exercise 8.41 Show that the quotient manifold Q bears a Legendre foliation, i.e. a foliation by curves tangent to the contact plane field.

Exercise 8.42 Reconstruct the Engel 4-manifold from its quotient contact manifold.

Exercise 8.43 Every hypersurface in an Engel 4-manifold transverse to the characteristic curves is a contact manifold.

Exercise 8.44 Every Engel 4-manifold has a finite cover with trivial tangent bundle. For example, there is no Engel structure on $S^4, \mathbb{CP}^2$, or $S^2 \times S^2$.

See papers of Montgomery & Zhitomirskii for more on this subject.

Question 37. We can try to use the Cartan–Kähler theorem to prove the existence of 2-plane fields on 4-manifolds which change type, and do so in some nondegenerate manner. This then makes it possible to arrange such 2-plane fields on lots of 4-manifolds, and with a bit of thought, we might be able to show that they exist on any 4-manifold which has a 2-plane field. Cartan–Kähler is not usually used this way; in fact we might need a more subtle understanding of 2-plane fields than it can provide to get interesting global results.

Question 38. A Kodaira–Merkulov type theorem for Engel 2-plane fields on complex manifolds would be nice, identifying the obstructions and tangent space to moduli space of deformations in terms of cohomology of sheaves. We could look at deforming with fixed complex structure, giving a foliation of the moduli space of Engel structures with varying complex structure. Or fix the bicharacteristics. See papers of Zhitomirskii and of Richard Montgomery.

Question 39. What are the possible homogeneous examples? Use the idea that the symmetry group embeds somewhere, and try the lowest order possibilities.

8.13 Symmetry reduction

8.13.1 Constant type

Definition 40. *Choose a G-structure B on a manifold M. Take $\Sigma \subset M$ a submanifold. The* type *of Σ at a point $m \in \Sigma$ is the subspace $u(T_m\Sigma) \subset V$ for some $u \in B_m$, where this subspace is thereby determined up to G action. We will say that a submanifold has constant type if its type is the same at all points. We will say that a group action has constant type if all of its orbits have the same type at all points.*

8.13.2 The reduced structure

Theorem 18. *Suppose that H is a Lie group of symmetries of a G_0-structure B_0 on a manifold M_0, of constant type. Let $M_1 = H\backslash M_0$. Suppose that M_1 is a smooth manifold, and that $M_0 \to M_1$ is a smooth submersion. Then we can define the pushforward B_1 of B_0 as follows: fix a point $u_0 \in B_0$, let $W_0 = u_0(T_{m_0}Hm_0)$,*

1. *let $V_1 = V_0/W_0$,*
2. *let $G_1 = G/N$ where $G \subset G_0$ is the subgroup leaving $W_0 \subset V_0$ invariant, and $N \subset G_0$ is the subgroup fixing each point of V_0/W_0,*
3. *map $G_0 \to \mathrm{GL}(V_1)$, in the obvious manner,*
4. *let $\phi : M_0 \to M_1$ be the quotient map, and for all $u \in B_0$ which map the tangent space to orbit of H to W_0, let $\bar{u}$ be defined by*
$$\bar{u}(\phi'(m_0)v_0) = u(v_0) \mod W_0,$$
5. *let B_1 be the set of all such $\bar{u}$.*

Then $B_1 \to M_1$ is a G_1-structure.

Proof. Since $G_0 \subset \mathrm{GL}(V_0)$ is a Lie subgroup, it is easy to see that $G_1 \subset \mathrm{GL}(V_1)$ is also a Lie subgroup. Let $B \subset B_0$ be the G-subbundle consisting of those $u \in B_0$ taking the tangent space to the orbit to W_0. By the constancy of type, proposition 8 on page 39 tells us that $B \subset B_0$ is a G_0-structure on M_0. Clearly $B/N \to M_0$ is a principal right G bundle. We define $B_1 = H\backslash B/N$. We have to show that $B_1 \to M_1$ is a principal right G-bundle. Clearly G-action on B/N commutes with H-action, so $G_1 = G/N$-action is defined on B_1, and the quotient is $B_1/G_1 = M_1$. We can map $B \to FM_1$ by mapping $u \mapsto \bar{u}$, and this map is both N and H invariant, and G equivariant, and therefore is defined on B_1.

To put the structure of a manifold onto B_1, we will first work locally. We can assume that M_0 and M_1 are open subsets of vector spaces, that $B_0 = M_0 \times G_0$ and $B_1 = M_1 \times G_1$, and moreover, if we are willing to sacrifice the group action of H for a mere Lie algebra action, then we can suppose that $M_0 = M_1 \times Y$ for some Y an open subset of a vector space. So each point

$m_0 \in M_0$ can be written $m_0 = (m_1, y)$. With these simplifications, we can write the map $B_0 \to FM_0$ as

$$(m_1, y, g_0) \in B_0 = M_1 \times Y \times G_0 \mapsto (m_1, y, g_0 f(m_1, y)) \in FM_0$$

for some smooth function $f : M_1 \times Y \to \mathrm{GL}(V_0)$. We still have the choice of trivialization of B_0 to make, which allows us to change f by a gauge transformation. We may assume that M_1 and Y are simply connected. We wish to arrange that $f(m_1, y) \cdot 0 \times T_y Y = W_0$ at all $(m_1, y) \in M_0$, by using a gauge transformation.

The map f must always take $T_y Y$ to the G_0-orbit of W_0 in the Grassmannian of subspaces of V_0 of dimension equal to that of W_0. This orbit must be $G_0 W_0 = G_0/G$, a homogeneous space, and $G_0 \to G_0/G$ is therefore a fiber bundle. So any map into G_0/G lifts to a map into G_0. Inverting with this choice of element of G_0 retracts our original map to G_0/G into a constant map. Therefore by gauge transformation, we can arrange that $f(m_1, y) \cdot 0 \times T_y Y = W_0$. With this arrangement, $B = M_1 \times Y \times G$, and $B_1 = M_1 \times G$ is a smooth manifold, and is smoothly mapped into FM_1 by $(m_1, g) \mapsto g f(m_1, y) \mod W_0 \in FM_1$, for any choice of $y = y(m_1)$.

We now have to glue together these local pictures, which is just another choice of gauge transformation.

Example 83 (Complex Hopf fibration). $U(n+1)$ acts on $\mathbb{C}^{n+1}$, leaving the unit sphere S^{2n+1} invariant. Let $M_0 = S^{2n+1}$, $G_0 = U(n)$ acting on $V_0 = \mathbb{R} \oplus \mathbb{C}^n$ (acting trivially on $\mathbb{R}$, and in the obvious representation on $\mathbb{C}^n$). Let $H = U(1)$ be the subgroup of $U(n+1)$ consisting of the matrices of the form $e^{\sqrt{-1}\theta} 1_{n+1}$, for $\theta \in \mathbb{R}$. Then clearly $M_1 = H \backslash M_0 = U(1) \backslash S^{2n+1} = \mathbb{CP}^n$. We take $W_0 = \mathbb{R} \oplus 0 \subset V_0$, and then $G = G_0 = U(n)$, and $N = 1$ so $G_1 = G/N = U(n)$, we find a $U(n)$-structure on $\mathbb{CP}^n$. Next we will calculate its structure equations.

8.13.3 Torsion in symmetry reduction

Continuing with the same notation, pick any splitting $V_0 = W_0 \oplus V_1$, not necessarily G_0 invariant. We will relate the torsion of $B \subset FM_0$ (*not* B_0) to that of $B_1 \subset FM_1$. Split up the soldering form ω on B into

$$\omega = \begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix}$$

with $\omega^0 \in \Omega^1(B) \otimes W_0, \omega^1 \in \Omega^1(B) \otimes V_1$. Check that ω^1 is the pullback via the map $B \to B_1$ of the soldering 1-form on B_1. Therefore, it must satisfy the structure equations of B_1,

$$d\omega^1 = -\gamma_1^1 \wedge \omega^1 + \frac{1}{2} T_{11}^1 \omega^1 \wedge \omega^1$$

where γ_1^1 can be any pseudoconnection on B_1. The structure equations on B must be

$$d \begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} = - \begin{pmatrix} \gamma_0^0 & \gamma_1^0 \\ 0 & \gamma_1^1 \end{pmatrix} \wedge \begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} T_{00}^0 \omega^0 \wedge \omega^0 + T_{01}^0 \omega^0 \wedge \omega^1 + \frac{1}{2} T_{11}^0 \omega^1 \wedge \omega^1 \\ \frac{1}{2} T_{11}^1 \omega^1 \wedge \omega^1 \end{pmatrix}.$$

These torsion coefficients are not really well-defined, because they depend on the choice of pseudoconnection, but the essential torsion lives in a G-representation $H^{1,2}(\mathfrak{g})$ which maps to the torsion of B_0, living in $H^{1,2}(\mathfrak{g}_1)$.

The torsion T_{11}^1 can be calculated directly upstairs. Moreover, the torsion of B is clearly H invariant, so is defined on $H \backslash B \to B_0$, a principal G bundle over M_1. We can define a torsion bundle

$$\left((H \backslash B) \times H^{1,2}(\mathfrak{g}) \right) / G \to M_0,$$

which is a vector bundle, and the torsion of B lives in this bundle, which quotients down to the torsion bundle of B_1.

Example 84 (Complex Hopf fibration). Lets continue with our earlier example. First we will need the structure equations of the G-structure B on S^{2n+1}. This is just the bundle

$$\begin{array}{ccc} U(n) & \longrightarrow & U(n+1) \\ & & \downarrow \\ & & S^{2n+1}. \end{array}$$

The Maurer–Cartan 1-form of $U(n+1)$ is (with the index convention that Greek indices run $0, \dots, n$, and Latin indices run $1, \dots, n$)

$$d \begin{pmatrix} \omega_{0\bar{0}} & \omega_{0\bar{b}} \\ \omega_{a\bar{0}} & \omega_{a\bar{b}} \end{pmatrix} = - \begin{pmatrix} \omega_{0\bar{0}} & \omega_{0\bar{c}} \\ \omega_{a\bar{0}} & \omega_{a\bar{c}} \end{pmatrix} \wedge \begin{pmatrix} \omega_{0\bar{0}} & \omega_{0\bar{b}} \\ \omega_{c\bar{0}} & \omega_{c\bar{b}} \end{pmatrix}.$$

The 1-forms $\omega_{a\bar{b}}$ are valued in $\mathfrak{u}(n)$, giving the pseudoconnection, while the 1-forms $\omega_{0\bar{0}}, \omega_{a\bar{0}}$ are semibasic for the bundle map $U(n+1) \to S^{2n+1}$, so give the soldering form. The structure equations of the G-structure on the sphere are therefore

$$d \begin{pmatrix} \omega_{0\bar{0}} \\ \omega_{a\bar{0}} \end{pmatrix} = - \begin{pmatrix} 0 & 0 \\ 0 & (\omega_{a\bar{b}} - \delta_{a\bar{b}} \omega_{0\bar{0}}) \end{pmatrix} \wedge \begin{pmatrix} \omega_{0\bar{0}} \\ \omega_{b\bar{0}} \end{pmatrix} - \underbrace{\begin{pmatrix} \omega_{0\bar{b}} \wedge \omega_{b\bar{0}} \\ 0 \end{pmatrix}}_{\text{torsion}}.$$

Clearly there is torsion; in the notation above, we have $T_{00}^0 \neq 0$. Nevertheless, the torsion on the quotient manifold is T_{11}^1, which in our case is 0. Therefore $\mathbb{CP}^n$ has a torsion-free $U(n)$-structure (with $U(n)$ acting in the $\mathbb{C}^n$ irreducible representation), while S^{2n+1} has torsion in its $U(n)$-structure (with $U(n)$ acting in the $\mathbb{R} \oplus \mathbb{C}^n$ representation).

Question 40. Can we package the entire G-structure B into some kind of equivariant information on a principal bundle over M_1? Here is a guess: we take

$$X = \{(m_0, h_0) \in M_0 \times H | \, h_0 m_0 = m_0\}.$$

Then $X \to M_1$ defined by $(m_0, h_0) \mapsto \phi(m_0)$ should be a principal H-bundle. We can also map $X \to M_0$ by $(m_0, h_0) \mapsto m_0$. We can get H to act on the left by

$$h \in H, (m_0, h_0) \mapsto \left(hm_0, hh_0h^{-1}\right).$$

This should make $X \to M_1$ into a principal right H-bundle. Next, I want to encode the map $B_0 \to FM_0$ into data on X. On the bundle $X \times_{M_1} B_1$, I want to build all possible choices of the manifold M_1 and the G-structure B. For each point on $X \times_{M_1} B_1$, I can look at all of the possible objects $U : T_x X \to V_0$ with

1. $\ker U = \ker [X \to M_0]'$,
2. $u_1 = U [X \to M_1]'$.

We could then try to set $u_0 = U [X \to M_0]'$. The idea is that I can try to take a principal H bundle $X \to M_1$, and some family of these U objects on $X \times_{M_1} B_1$, and see when some such family determines a G-structure on a quotient of X, by a foliation whose leaves are just the kernels of the various U guys, forming the orbits of various closed subgroups of H. Tricky.

Question 41. It might be possible to use G-structure-like notions to handle singularities of analytic varieties, as long as the stabilizer of the tangent cone is of constant type, in some sense. For example, if X is a projective variety embedded in projective space, and Y is a smooth subvariety contained in the singular locus of X, we could say that Y has constant type if the stabilizer of the tangent cone to X and tangent space of Y acts on the tangent space of Y at each point as a group always in the same conjugacy class of the general linear group. Then Y would bear a G-structure.

Remark 28 (Averaging). If a Riemannian manifold M_0 is mapped to any manifold M_1 by a fiber bundle mapping with compact stalks, then we can average out the metric to produce a Riemannian metric on the base.

We can generalize this easily: if M_0 is equipped with a G_0-structure, and a fiber bundle mapping $M_0 \to M_1$, and the fibers are compact with constant type W_0, we can reduce to a G-structure, where $G \subset G_0$ is the stabilizer of W_0. Let $N \subset G$ be the normal subgroup acting trivially on $V_1 = V_0/W_0$, and let $G_1 = G/N$. If G preserves a volume form on W_0, and the homogeneous space $\mathrm{GL}(V_1)/G_1$ is equipped with an affine structure, then we can average out the G_0-structure to produce a G_1-structure.

9

Constructing G-structures with controlled torsion

Lets try to use the Cartan–Kähler theorem to construct G_0-reductions of a G-structure, with torsion satisfying some equations. Suppose that we have selected a G_0-invariant submanifold $X_0 \subset V \otimes \Lambda^2 (V^*)$, and we want to build a G_0-reduction $B_0 \to M$ of a given G-structure $B \to M$, and we want the torsion T_0 of the reduction to lie in X_0. To be more precise, we want to ensure that there is a pseudoconnection on B_0 with torsion T_0 belonging to X_0. We will assume that B has constant type, and that it has been reduced as far as possible. For every example of interest, the torsion T of B will be a constant. In this chapter, we will never refer to the intrinsic torsion of B, since the torsion of B is already reduced, so the expression $[T]$ means

$$T \in V \otimes \Lambda^2 (V^*) \mapsto [T] \in H^{0,2} (\mathfrak{g}_0) .$$

Let Z be the set of pairs (x, p) where $x \in M$, $p \in \mathfrak{g} \otimes V^*$ and $T(x) + \delta p \in X_0$. Map $(x, p) \in Z \mapsto x \in M$. (In most examples, T will be a constant, so Z will be a product $Z = M \times Z_0$.) Let $\Pi : B^{(1)} \to M$ be the obvious bundle map.

To handle independence conditions, we will need to adopt a notation. If η is a 1-form valued in a vector space W of dimension N, let $\bigwedge \eta$ be $\eta^1 \wedge \dots \wedge \eta^N$ for some choice of basis of W. If γ is a pseudoconnection on B, write $\gamma^{\mathfrak{g}_0}$ for any projection of γ to $\mathfrak{g}_0$.

Lemma 36. *Suppose that $Z \to M$ is a smooth fiber bundle. On the manifold $\Pi^* Z$, every integral manifold of the exterior differential system $\bar{\gamma} + \bar{p}\omega = 0$, on which the independence condition*

$$\bigwedge \omega \wedge \bigwedge \gamma^{\mathfrak{g}_0} \neq 0$$

is satisfied, locally determines a unique G_0-reduction of $B \to M$ defined over an open subset of M, together with a pseudoconnection with torsion in X_0. Conversely, every such reduction determines a unique integral manifold of that exterior differential system, satisfying that independence condition.

Question 42. In general, I should probably not use H as notation when I fatten up. Instead I should fatten from G_0 to G, and that will make the notation consonant with this section.

Proof. Suppose that $B_0 \subset B$ is a G_0-reduction, defined on some open set $M_0 \subset M$, with a pseudoconnection γ_0 whose torsion belongs to X_0. Without loss of generality, replacing M by M_0, we can assume that $M = M_0$. Consider the set B'_0

Take any pseudoconnection γ for $B \to M$ adapted to the choice of a section S for G. We can fatten up B_0 to $B = B_0\left(G\right)$. Recall the notation from subsection 6.2.3 on page 61 that $\left(\gamma_0\right)_*$ means the 1-form which agrees with γ_0 in directions tangent to B_0, and satisfies $\vec{A} \lrcorner \left(\gamma_0\right)_* = A$ for any $A \in \mathfrak{g}$. For each point $u_0 \in B_0$, let $p \in \mathfrak{g} \otimes V^*$ satisfy $\gamma = \left(\gamma_0\right)_* - p\omega$. This defines $p : B_0 \to \mathfrak{g} \otimes V^*$. However, we can change the choices of pseudoconnections γ_0 and γ, by adding on things like $\gamma_0 + q_0\omega$, $\gamma + q\omega$, where $q_0 \in \mathfrak{g}_0^{(1)}$ and $q \in \mathfrak{g}^{(1)}$. This will change p by $p - \delta q + \delta q_0$, not affecting T_0. Let $B'_0 \to B_0$ be the bundle whose sections are pairs $\left(\gamma_0, p\right)$ of pseudoconnections γ_0 and functions $p : B_0 \to \mathfrak{g} \otimes V^*$ for which the torsion T_0 of γ_0 lies in X_0 and for which $T_0 - \delta p = T$. So $B'_0 \to B_0$ is a fiber bundle, since γ_0 is chosen from a fiber bundle over $X_0 \times B_0$, with fiber $\mathfrak{g}_0^{(1)}$, and p is chosen ???

Question 43. Why is $B'_0 \to B_0$ a smooth bundle? This looks very tricky. If we suppose that there is a choice of γ_0 with required torsion, then at least we can get started, but I think I will need some additional hypothesis to ensure that $B'_0 \to B_0$ is a smooth bundle. But in general, we can just make some local choice of γ_0, as given by hypothesis, and some smooth choice of p, since $\delta p = T_0 - T$ is a nice smooth equation, determining p up to $\mathfrak{g}^{(1)}$ action, so it should be a bundle, and a local section should give us a map $B_0 \to B^{(1)}$.

Map $\phi : B'_0 \to \Pi^* Z$ by $\left(\gamma_0, p\right) \mapsto \left(\gamma, p\right)$ where $\gamma = \left(\gamma_0\right)_* - p\omega$. Then $\phi^*\left(\bar{\gamma} + \bar{p}\omega\right) = \bar{\gamma}_0 = 0$, so $\phi : B'_0 \to \Pi^* Z$ is an integral manifold of $\mathcal{I}$.

Question 44. I am messing up the independence condition. What is independent on B'_0?

Suppose that $\phi : N \subset \Pi^* Z$ is an integral manifold satisfying the independence condition. Pulled back to N, $\gamma + p\omega$ is a 1-form valued in $\mathfrak{g}_0$, and

$$\begin{aligned} d\omega &= -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega \\ &= -\left(\gamma + p\omega\right) \wedge \omega + \frac{1}{2}\left(T + \delta p\right) \omega \wedge \omega. \end{aligned}$$

By theorem 6 on page 91, after perhaps restricting to an open subset of N, there is a unique G_0-structure $B_0 \to M_0$ on some manifold M_0, with these structure equations. The fibers of $B_0 \to M_0$ satisfy $\omega = 0$, so locally map to fibers of $N \to M$. By independence of $\bigwedge \omega$, the map $M_0 \to M$ is a local diffeomorphism. Replacing M_0 by an open subset of itself, we can arrange that M_0 is an open subset of M, and G_0-equivariance ensures us that B_0 is a principal right G_0 subbundle of B.

In most examples, the torsion T of the G-structure $B \to M$ is a constant, so lets assume this. Moreover, calculations are made simpler by choosing a map $P : X_0 \to \mathfrak{g} \otimes V^*$ so that

$$T + \delta P(T_0) = T_0.$$

Call such a map a *torsion gauge*.

Question 45. I think it helps if the torsion gauge is transverse to $\mathfrak{g}^{(1)}$. It is useful in carrying out Cartan's count, I think.

Corollary 21. *Assume that T is constant, and that P is a torsion gauge. On the manifold $B^{(1)}$, every integral manifold of the exterior differential system $\bar{\gamma} + \bar{P}\omega = 0$, on which the independence condition*

$$\bigwedge \omega \wedge \bigwedge \gamma^{\mathfrak{g}_0} \neq 0$$

is satisfied, locally determines a unique G_0-reduction of $B \to M$ defined over an open subset of M, together with a pseudoconnection with torsion in X_0. Conversely, every such reduction determines a unique integral manifold of that exterior differential system, satisfying that independence condition.

Proof. *Question 46.* Finish this.

Question 47. From here on, we treat X_0 as a submanifold of $H^{0,2}(\mathfrak{g})$. Make this clear.

Proposition 42. *Suppose that $B \to M$ is a G-structure with constant torsion T, and that $G_0 \subset G$ is a Lie subgroup, with Lie algebras $\mathfrak{g}_0 \subset \mathfrak{g}$. Write quotients as $A \in \mathfrak{g} \mapsto \bar{A} \in \mathfrak{g}/\mathfrak{g}_0$. Pick a submanifold $X_0 \subset H^{0,2}(g_0)$. Let $P : X_0 \to \mathfrak{g} \otimes V^*$ be a smooth map so that*

$$[T + \delta P([T_0])] = [T_0]$$

for any $[T_0] \in X_0$. Given a G_0-reduction $\iota : B_0 \to B$ with torsion in X_0, let $B_0^T \to B_0$ be the bundle whose sections are pseudoconnections γ_0 on B_0 with torsion $T + \delta P$. Then $B_0^T \to B_0$ is a principal right $\mathfrak{g}_0$-bundle. Map $\phi : B_0^T \to B^{(1)} \times X_0$ by taking $\gamma_0 \mapsto (\gamma, [T_0])$ where $\gamma = \gamma_0 + P\omega$ and T_0 is the intrinsic torsion of B_0. For any G_0-reduction $\iota : B_0 \to B$ with torsion in X_0, ϕ embeds into $B^{(1)} \times X_0$ and satisfies $\bar{\gamma} + \bar{P}\omega = 0$. Moreover, every integral manifold of $\bar{\gamma} + \bar{P}\omega = 0$ transverse to the fibers of $B^{(1)} \to M$ locally fattens up to B_0^T for a unique G_0-reduction B_0 with torsion in X_0.

Proof. Given a G_0-reduction $B_0 \subset B$ with torsion $[T_0]$, let B_0' be the set of 1-forms γ_0 at points of B_0 satisfying $\vec{A} \lrcorner \gamma_0 = A$ for all $A \in \mathfrak{g}_0$, i.e. the bundle whose sections are pseudoconnections. For each such γ_0, we can compute its intrinsic torsion T_0, so

$$d\omega = -\gamma_0 \wedge \omega + \frac{1}{2} T_0 \omega \wedge \omega.$$

Given a pseudoconnection γ for $B \to G$, which is a section of $B^{(1)}$, it satisfies

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T \omega \wedge \omega.$$

So $(\gamma_0)_* - \gamma = q\omega$ for some $q \in \mathfrak{g}^{(1)}$.

Question 48. Finish this.

Question 49. The relevant transversality conditions are difficult to state. I need to be clearer about them.

Question 50. Note that we haven't taken advantage of the action of $(\mathfrak{g}_0 \otimes V^*) \oplus \mathfrak{g}^{(1)}$ on $\Pi^* Z$, corresponding to changing our choices of γ_0 and γ.

Question 51. What is the prolongation of this system?

Question 52. Eventually, I need to join this subsection with the discussion earlier which I called "Counting generality".

Question 53. This tower of bundles is a basic object in the study of reductions of structure group; all of the calculations can be carried out order by order on the tower directly, without having to resort to working with actual reductions.

This will be useful to constructing exterior differential systems for reductions of structure group with required torsion properties, and also for variations of substructures. But it won't become useful until I figure out what sort of differential forms and torsions are defined on each bundle, and how to set up bundles for the torsions to live in, and how to write a differential system to constrain torsion of a subbundle.

Question 54. There seem to be 2 very different problems we might want to solve concerning the construction of reductions. The first is to construct a reduction for which the intrinsic torsion lands in a given invariant submanifold of the Spencer cohomology. The second is to construct a reduction whose torsion is already reduced via a section. Both can be thought of as constructing a reduction together with all possible pseudoconnections with extrinsic torsion in a given subset of $V \otimes \Lambda^2 (V^*)$. So perhaps this is the right problem to study. If we study this general problem, we should get an exterior differential system for it, and then it should be possible to simplify the story when the torsion T of the original, unreduced structure is a constant, and then again when we have a map $P : X_0 \to \mathfrak{g} \otimes V^*$ so that $\delta P (T_0) = T - T_0$.

Question 55. A nice example might be to find out which projective structures arise from Riemannian manifolds. Or even pseudo-Riemannian. The amazing thing is that this question can be solved by exterior differential systems. Keep in mind that the problem of determining whether an affine connection is the Levi-Civita connection of a Riemannian or pseudo-Riemannian metric involves holonomy groups.

Question 56. A fairly simple example, to impose a Riemannian metric on a manifold. There might be a nice way to see that that underlying geometry is described by taking affine combinations, which might be useful to generalize.

9.1 Example: which projective structures arise from Riemannian metrics?

Take $B \subset FM^{(1)}$ a projective structure (see section 7.11 on page 154); recall that a projective structure on a manifold M is a choice of connection on the tangent bundle of M up to reparameterization of geodesics. We want to determine if there is a Riemannian metric on M whose geodesics are the geodesics of this projective structure; we will say that such a metric induces that projective structure. The sections of $B \to FM$ are precisely the pseudoconnections on M with the required geodesics. If we had a Riemannian metric g whose geodesics are those of the projective structure, then the Levi-Civita connection γ_g on the orthonormal frame bundle B_g would extend to a connection $\left(\gamma_g\right)_*$ on FM which would belong to B. So we can map $u \in B_g \mapsto \left(\gamma_g\right)_* \in B$, embedding B_g into B. Recall the structure equations of B (see proposition 29 on page 161):

$$
\begin{aligned}
d\omega^i &= -\gamma^i_j \wedge \omega^j \\
d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k + \nabla \gamma^i_j \\
d\omega_i &= \gamma^k_i \wedge \omega_k + \nabla \omega_i \\
\nabla \gamma^i_j &= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l \\
\nabla \omega_i &= \frac{1}{2} K_{ijk} \omega^j \wedge \omega^k
\end{aligned}
$$

with

$$
\begin{aligned}
0 &= K^i_{jkl} + K^i_{jlk} \\
0 &= K^i_{ikl} \\
0 &= K^i_{jil} \\
0 &= K^i_{jkl} + K^i_{klj} + K^i_{ljk} \\
0 &= K_{jkl} + K_{jlk} \\
0 &= K_{jkl} + K_{klj} + K_{ljk}.
\end{aligned}
$$

On the submanifold B_g, γ must equal the Levi-Civita connection, by the reproducing property. The structure equations of Riemannian geometry are

$$
\begin{aligned}
d\omega^i &= -\gamma^i_j \wedge \omega^j \\
d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l,
\end{aligned}
$$

with the usual Bianchi identities for R^i_{jkl}, and with $\gamma^i_j + \gamma^j_i = 0$. On B_g,

$$\frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l = \left(\omega_j \delta^i_k + \omega_k \delta^i_j \right) \wedge \omega^k + \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l.$$

Symmetrizing in i and j gives

$$0 = \left(\omega_j \delta^i_k + \omega_k \delta^i_j + \omega_i \delta^j_k + \omega_k \delta^j_i - \frac{1}{2} \left(K^i_{jkl} + K^j_{ikl} \right) \right) \wedge \omega^l) \wedge \omega^k.$$

By Cartan's lemma,

$$\omega_j \delta^i_k + \omega_k \delta^i_j + \omega_i \delta^j_k + \omega_k \delta^j_i = \frac{1}{2} \left(K^i_{jkl} + K^j_{ikl} \right) \omega^l + p_{ijkl} \omega^l$$

for some p_{ijkl} with various symmetries. Summing over $i = k$,

$$(n+3)\omega_j = \frac{1}{2} \left(K^i_{jil} + K^j_{iil} \right) \omega^l + p_{ijil} \omega^l.$$

So we can write

$$\omega_i = t_{ij} \omega^j$$

for some coefficients t_{ij}. Plugging back in gives

$$K^i_{jkl} + t_{jk} \delta^i_l + t_{lk} \delta^i_j - t_{jl} \delta^i_k - t_{kl} \delta^i_j = R^i_{jkl}.$$

The first Bianchi identity

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$$

is satisfied by the left hand side. Using the identity $0 = R^i_{jkl} + R^j_{ikl}$,

$$0 = K^i_{jkl} + t_{jk} \delta^i_l + t_{lk} \delta^i_j - t_{jl} \delta^i_k - t_{kl} \delta^i_j + K^j_{ikl} + t_{ik} \delta^j_l + t_{lk} \delta^j_i - t_{il} \delta^j_k - t_{kl} \delta^j_i.$$

Again plug in $k = i$ and sum:

$$\begin{aligned} 0 &= K^i_{jil} + t_{jl} + t_{lj} - n t_{jl} - t_{jl} + K^j_{iil} + t_{ii} \delta^j_l + t_{lj} - t_{jl} - t_{jl} \\ &= K^i_{jil} + K^j_{jil} + t_{ii} \delta^j_l - (n+2) t_{jl} + 2 t_{lj}. \end{aligned}$$

Let $t = t_{ii}$:

$$\begin{aligned} (n+2) t_{jl} - 2 t_{lj} \qquad &= t \delta^j_l + K^j_{iil} \\ -2 t_{jl} + (n+2) t_{jl} &= t \delta^l_j + K^l_{iij} \end{aligned}$$

so that

$$t_{jl} = \frac{1}{(n+2)^2 - 4} \left((n+4) \delta_{jl} t + (n+2) K^j_{iil} + 2 K^l_{iij} \right).$$

We have determined all of the ω_i, up to choice of a function t. If we let $K_{ij} = K^i_{kkj}$, then plugging t_{jl} back in gives:

$$\begin{aligned} 0 = K^i_{jkl} + K^j_{ikl} + \frac{1}{(n+2)^2 - 4} \Big(& 2n\delta_{ij} \left(K_{lk} - K_{kl}\right) \\ & + \delta_{il} \left((n+2)K_{jk} + 2K_{kj}\right) \\ & + \delta_{jl} \left((n+2)K_{ik} + 2K_{ki}\right) \\ & - \delta_{ik} \left((n+2)K_{jl} + 2K_{lj}\right) \\ & - \delta_{jk} \left((n+2)K_{il} + 2K_{li}\right) \Big) . \end{aligned}$$

These identities defined a subset $B' \subset B$, and for every Riemannian metric g inducing the given projective structure, $B_g \subset B'$. So we will always need to restrict to B' first, and check that $B' \to B \to M$ has a nonempty stalk over each point of M.

Question 57. If I start with a homogeneous projective structure, and ask whether it is induced by a Riemannian metric, I will have K a constant, so if these identities are not satisfied, I will know right away. That would be nice to see. The problem is that the only example for which the symmetry group acts transitively on the projective structure bundle is the flat case. So the orbit of the symmetry group in any nonflat case might pass through a region of the bundle in which the identities are not satisfied, even though there is some locus away from there where they are satisfied. So I need to see how to read off the existence of solutions of the identities even at other points, i.e. the orbit of the locus satisfying the identities under the $\mathrm{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^{n*}$ action. This looks like it might be difficult. The $\mathbb{R}^{n*}$ action on K^i_{jkl} is trivial, so we need to ask what the $\mathrm{GL}(n, \mathbb{R})$ orbit of the identities is.

Part III

Exotic Delicacies

10

Characteristic classes

Question 58. I might want to move all of this stuff after the definition of variations of G-structure, so that I can vary bundles.

Question 59. I should be able to connect this theory to the theory of the Atiyah class for connections on complex manifolds, and Gunning's work on Atiyah classes for complex manifolds with projective connection.

10.1 Cartan geometries and G-structures

Lemma 37. *A Cartan geometry arises as the infinite prolongation of a finite type G-structure just when it is locally effective.*

Proof. Let $\pi : B \to M$ be the bundle. As above, define $V = \mathfrak{h}/\mathfrak{g}$, and let $\omega = \Omega \pmod{\mathfrak{g}}$. Check that ω is semibasic for $B \to M$, so treat ω as a section of $\pi^* T^* M \otimes V$. Map $B \to FM$ by $u \mapsto \omega_u$. This is right G-equivariant, so $B \to FM \to M$ is a G-structure, not necessarily embedded. Let $N \subset G$ be the normal subgroup acting trivially on V, and let $G_0 = G/N$. The image of $B \to FM$ is an embedded G_0-structure, call it B_0. Moreover, the quotienting map $\pi_1 : B \to B_0$ is a principal right N-bundle.

Let $\mathfrak{n}$ be the Lie algebra of N. We have Lie algebra morphisms

$$0 \to \mathfrak{g}_0 \to \mathfrak{h}/\mathfrak{n} \to V \to 0.$$

Pick any splitting as vector spaces:

$$\mathfrak{h}/\mathfrak{n} = V \oplus \mathfrak{g}_0.$$

We can then split

$$\Omega + \mathfrak{n} = \omega \oplus \gamma.$$

Check that γ is semibasic for $B \to B_0$, and satisfies $\vec{A} \lrcorner \gamma = A$ for $A \in \mathfrak{g}_0$. So at each point $u \in B$, γ descends to a 1-form on the corresponding point of B_0 (valued in $\mathfrak{g}_0$). Map $u \in B \mapsto \gamma_u \in T^*_{\pi_1(u)} B_0 \otimes \mathfrak{g}_0$.

Calculate modulo $\mathfrak{g}$ that

$$-\frac{1}{2}[\Omega, \Omega] + \mathfrak{g} = -\gamma \wedge \omega - \frac{1}{2}[\omega, \omega].$$

Since $\nabla \Omega$ is semibasic, so is the projection to $\mathfrak{g}$, so we find structure equations

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega$$

with torsion $T\omega \wedge \omega = \pi_{\mathfrak{g}} \nabla \Omega + [\omega, \omega]$.

Under N action, $r_n^* \Omega = \mathrm{Ad}_n^{-1} \Omega$, so $r_n^* \omega = \omega$, and therefore $r_n^* d\omega = d\omega$. Similarly, $r_n^* \nabla \Omega = \mathrm{Ad}_n^{-1} \nabla \Omega$, and therefore T is N invariant, and so

$$(r_n^* \gamma - \gamma) \wedge \omega = 0.$$

Therefore γ varies only by elements of $\mathfrak{g}^{(1)}$, ensuring that $u \mapsto \gamma_u$ takes $B \to B_0^{(1)}$. Let B_1 be the image of this map.

In order to carry out induction, it helps to have better notation, writing $\omega(0)$ instead of ω, and $\omega^{(1)}$ instead of γ, and following the model of section 7.2 on page 99. At each stage, we have the same structure equations as the model, except for torsion terms entering (which are getting more complicated at each prolongation). Therefore we must eventually reach the structure equations of H/N, with lots of torsion added to them.

There is a subtlety here to keep in mind. The G_0-structure might even have infinite order. For example, let H be the biholomorphism group of $\mathbb{CP}^1$, and G the subgroup fixing a point. Then $G_0 = \mathrm{GL}(1, \mathbb{C})$ and the G_0-structure is just the complex structure. However, the next bundle $B_1 \to B_0$ might be a subbundle of $B_0^{(1)}$. Eventually, just as in the model, we must reach a substructure of some $B_k^{(1)}$ which has finite type, since H has finite dimension.

10.2 Characteristic classes of *G*-structures

If $K \subset G$ is a maximal compact subgroup, then every G-structure $B \to FM$ admits a reduction to a K-structure $B_0 \to B$. In general, we may not be able to write down such a reduction, but nonetheless this can help in finding topological obstructions to the existence of a G-structure. A K-structure will induce a metric, and a morphism

$$\mathrm{Sym}^* \left(\mathfrak{so}(n)^*\right)^{SO(n)} \to \mathrm{Sym}^* (\mathfrak{k})^K$$

on characteristic classes.

Example 85 (Web geometry). G is the set of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

with $a \neq 0$. The equation $a = \pm 1$ cuts out a maximal compact subgroup $K \subset G$. Therefore every G-structure (i.e. web geometry) admits reduction to a K structure (i.e. coframing up to sign). The only topological characteristic class of a surface is its Euler characteristic, spanning $\mathfrak{so}(2)^*$. But $\mathfrak{k} = 0$, so the Euler characteristic of any compact surface with web geometry must vanish.

This approach tells us nothing about characteristic classes of conformal structures.

10.3 Characteristic classes of Cartan geometries

For an introduction to characteristic classes, see Chern [30], Spivak [82] and Milnor & Stasheff [64].

Lemma 38. *A Cartan geometry Ω on $B \to M$ determines a connection on $B(H) \to M$ (recall $B(H) = (B \times H)/G$, where G acts diagonally). Conversely, a connection on $B(H)$ comes about from a Cartan geometry just when the kernel of the connection does not meet the tangent space of $B \subset B(H)$.*

Proof. Sharpe [76], p. 365.

Recall the *curvature form* of a Cartan geometry:

$$\nabla \Omega = d\Omega + \frac{1}{2}[\Omega, \Omega].$$

Lets try to imitate Chern–Weil theory of characteristic classes for Cartan geometries.

Example 86. Riemannian geometry on an oriented surface gives a Cartan geometry modelled on H/G where $H = SO(2) \rtimes \mathbb{R}^2$ and $G = SO(2)$. Let M be the surface, and $B \to M$ the orthonormal frame bundle. The structure equations are

$$d\Omega = -\frac{1}{2}[\Omega, \Omega] + \kappa \omega^1 \wedge \omega^2$$

where

$$\Omega = \begin{pmatrix} 0 & -\gamma & \omega^1 \\ \gamma & 0 & \omega^2 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\kappa = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with K the Gauss curvature. The invariant $\frac{K}{2\pi}\omega^1 \wedge \omega^2$ descends to a closed form on M. In this example, the characteristic forms from $B(H)$ are trivial, since the invariant polynomials are generated by the Pontryagin polynomials, given for $A \in \mathfrak{so}(n)$ by

$$\det(I + \lambda\kappa) = \sum_{j=0}^{n} (2\pi\lambda)^{2j} p_j(A).$$

But

$$\det(I + \lambda\kappa) = \lambda\left(\lambda^2 + \left(K\omega^1 \wedge \omega^2\right)^2\right) = \lambda^3$$

giving only $p_0 = 1$, trivial information about Pontryagin classes.

Remark 29. Generalizing this last example, if G is reductive, any Cartan geometry modelled on a homogeneous space H/G will have a Cartan connection Ω valued in $\mathfrak{h}$, which splits into $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^\perp$, so we project Ω to $\mathfrak{g}$ and obtain a connection, which has the usual characteristic forms.

An elementary calculation:

Lemma 39 (The Bianchi identity).

$$d\nabla\Omega = [\nabla\Omega, \Omega].$$

Corollary 22. *Let* $\operatorname{Sym}^*(\mathfrak{h})^H$ *be the algebra of* H*-invariant polynomials. The Chern–Weil morphism*

$$p \in \operatorname{Sym}^k(\mathfrak{h})^H \mapsto p(\nabla\Omega) \in \Omega^{2k}(M)$$

takes invariant polynomials to closed differential forms.

The proof is as usual. So we can always apply the Chern–Weil morphism to $B(H)$, while if G is reductive, we can apply it to B itself.

Example 87 (Conformal geometry). Recall that conformal geometry on a manifold M gives a Cartan geometry $B^{(1)} \to M$ modelled on $H/G = \mathbb{P}O(p+1, q+1)/CO(p,q) \rtimes \mathbb{R}^{(p+q)*}$. The Cartan connection looks like

$$\Omega = \begin{pmatrix} -\sigma & -\varpi & 0 \\ \omega & \alpha & -\varpi^* \\ 0 & \omega^* & \sigma \end{pmatrix}$$

and

$$\nabla\Omega = \begin{pmatrix} 0 & \frac{1}{2}C\omega \wedge \omega & 0 \\ 0 & \frac{1}{2}W\omega \wedge \omega & \frac{1}{2}(C\omega \wedge \omega)^* \\ 0 & 0 & 0 \end{pmatrix}.$$

Calculating the H-invariant polynomials on $\mathfrak{h} = \mathfrak{so}(p+1, q+1)$ we find the Pontryagin forms

$$\det(I + \lambda\nabla\Omega) = (2\pi\lambda)^{2j} p_j(\nabla\Omega)$$

of $B(H)$ look as if they were Pontryagin forms of a pseudo-Riemannian metric on M, but computed in terms of $(1/2)W\omega \wedge \omega$.

Question 60. How do I make use of this to show that the prescence of a flat conformal structure on a compact manifold forces the Pontryagin classes of its tangent bundle to vanish? To show that, I really need to show that the inclusion $B \subset B(H)$ of one bundle in another gives a relation between rational characteristic classes which is just the obvious relation on the invariant polynomials in the Lie algebras. That should not be difficult, just by fattening up a connection.

Question 61. Morse theory is a method for reducing a structure group, everywhere but on some singular set which is made nice by making things generic.

Question 62. This stuff about characteristic classes has to be placed later in the book, perhaps as the final chapter. Certainly it should be after the material about conformal and projective geometry, since I use the conformal geometry as the next example.

Question 63. The theory here badly needs to be developed. I think that the bundle $B \to M$ of a Cartan geometry generally has no expression in terms of $\nabla \Omega$ for its characteristic classes. But in reductive cases, it probably does. There is also the possibility of constructing characteristic classes directly out of invariant theory applied to $\nabla \Omega$, but that should be computing just the characteristic classes of $(B \times H)/G$, a principal H-bundle containing B. So I believe that the more refined approach would be to find the characteristic class of B, and relate them to characteristic classes of FM via the map $B \to B/N \to FM$ which I constructed above. I should be able to take any choice of connection on B/N, or perhaps more generally any choice of section of $B \to B/N$, and use it as a connection, and pullback everything, in particular pulling back the characteristic forms.

Ultimately, the only topology I have is that of the G-bundle $B \to M$. This is contractible to a K-bundle, say $B_0 \subset B$, so I really only have the topology of B_0. The only characteristic classes I should be able to calculate are those of B_0. But I can't always calculate even those, using only the curvature of the Cartan connection. I can certainly calculate the rational characteristic classes of $B(H)$, the fattening up, because the Cartan connection determines a connection on $B(H)$. But it is sometimes possible to calculate more data than that.

11

Twistor theory

11.1 Quotienting G-structures

Let B be a G-structure on a manifold M. If $G_0 \subset G$ is a closed subgroup, we can construct the bundle $B/G_0 \to M$, which is a bundle of homogeneous spaces. For example, if M is a Riemannian manifold, so $G = O(n)$, and we take $G_0 = SO(n)$, then B/G_0 is the bundle of orientations. Instead if keep the same bundle B and Riemannian manifold M, and we take $G_0 \subset G$ the group fixing a given nonzero vector $v_0 \in V$, then B/G_0 is the unit tangent bundle, the bundle of unit vectors in the tangent spaces of M (or, canonically diffeomorphic, the bundle of vectors of some fixed nonzero length).

Naturally we might expect to read off a G_0-structure on B/G_0 from the G-structure on $M = B/G$. We might imagine just rearranging the structure equations.

Example 88 (Foliated surfaces). A very simple toy example, of no practical interest: let M be a surface foliated by curves. The reader can see that the foliation is described by a G-structure, where G is the group of linear transformations of $V = \mathbb{R}^2$ leaving $\mathbb{R}^1 \oplus 0$ invariant. As matrices, these linear transformations have the form

$$\begin{pmatrix} g^1_1 & g^1_2 \\ 0 & g^2_2 \end{pmatrix}$$

so that the structure equations are

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ 0 & \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

Taking exterior derivative gives the equations of the prolongation:

$$d \begin{pmatrix} \gamma^1_1 \\ \gamma^1_2 \\ \gamma^2_2 \end{pmatrix} = - \begin{pmatrix} \xi^1_{11} & \xi^1_{12} \\ \xi^1_{21} & \xi^1_{22} \\ 0 & \xi^2_{22} \end{pmatrix} \wedge \begin{pmatrix} \gamma^1_1 \\ \gamma^1_2 \\ \gamma^2_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \left(\gamma^2_2 - \gamma^1_1\right) \wedge \gamma^1_2 \\ 0 \end{pmatrix}.$$

Consider the subgroup G_0 consisting of the linear transformations which not only leave $\mathbb{R}^1 \otimes 0$ invariant, but actually fix it pointwise, so matrices of the form

$$\begin{pmatrix} 1 & g^1_2 \\ 0 & g^2_2 \end{pmatrix}.$$

Then B/G_0 is the set of choices of nonzero vector tangent to the leaf of the foliation at a given point of M. The 1-form γ^1_1 is semibasic for the bundle map $B \to B/G_0$; lets write $\dot{\omega} = \gamma^1_1$ to remind us that it is now being demoted to a semibasic 1-form, i.e. a part of the soldering 1-form. (We might be tempted to call it ω^3, and this would be just as reasonable. I have opted in this book to use $\dot{\omega}$ to denote any 1-form which "used to be" part of a pseudoconnection, and got "demoted" to being part of the soldering 1-form.) We can just rearrange the structure equations, writing $\dot{\omega}$ for γ^1_1, to get structure equations

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \dot{\omega} \end{pmatrix} = - \begin{pmatrix} 0 & \gamma^1_2 \\ 0 & \gamma^2_2 \\ \xi^1_{11} & \xi^1_{12} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \dot{\omega} \end{pmatrix} + \underbrace{\begin{pmatrix} \omega^1 \wedge \dot{\omega} \\ 0 \\ 0 \end{pmatrix}}_{\text{torsion}}.$$

Notice that ξ^1_{11} and ξ^1_{12} have now become part of the pseudoconnection on the new bundle $B^{(1)} \to B/G_0$. They "used to be" part of the pseudoconnection of the prolongation $B^{(1)} \to B$. The new structure group is not G_0, but rather $G_0 \rtimes \mathfrak{g}^{(1)}$, the structure group of $B^{(1)} \to B/G_0$.

Example 89 (The unit tangent bundle of the 3-sphere). The structure equations of Riemannian geometry on the 3-sphere (i.e. of the usual $SO(3)$-structure) are

$$\begin{aligned} d\omega_i &= -\gamma_{ij} \wedge \omega_j \\ d\gamma_{ij} &= -\gamma_{ik} \wedge \gamma_{kj} + \omega_i \wedge \omega_j \end{aligned}$$

$(i, j, k, l = 1, 2, 3)$. Take

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3,$$

and let $G_0 = SO(2) \subset SO(3)$ be the stabilizer of v_0. The Lie algebra of G_0 consists of the matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma_{23} \\ 0 & -\gamma_{23} & 0 \end{pmatrix}.$$

Thus B/G_0 is the unit tangent bundle of the 3-sphere, and the 1-forms γ_{12} and γ_{13} are semibasic for $B \to B/G_0$. Keeping in mind that $\mathfrak{g}^{(1)} = 0$, so that $B^{(1)} = B$, we see that $B \to B/G_0$ is an $SO(2)$-structure. Write

$$\begin{pmatrix} \dot\omega_2 \\ \dot\omega_3 \end{pmatrix} = \begin{pmatrix} \gamma_{12} \\ \gamma_{13} \end{pmatrix}.$$

Our structure equations are

$$d\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot\omega_2 \\ \dot\omega_3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{23} & 0 & 0 \\ 0 & -\gamma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_{23} \\ 0 & 0 & 0 & \gamma_{23} & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot\omega_2 \\ \dot\omega_3 \end{pmatrix} + \begin{pmatrix} \omega_2 \wedge \dot\omega_2 + \omega_3 \wedge \dot\omega_3 \\ \dot\omega_2 \wedge \omega_1 \\ \dot\omega_3 \wedge \omega_1 \\ \omega_1 \wedge \omega_3 \\ \omega_1 \wedge \omega_2 \end{pmatrix}.$$

We see from the representation that the structure group preserves a contact structure $\omega_1 = 0$, and preserves a complex structure on each contact plane. The symmetry group of the Riemannian geometry was $SO(4)$, and we can now see that $SO(4)$ is also precisely the symmetry group of the induced $SO(2)$-structure on the unit tangent bundle.

Exercise 11.1 Generalize this last example to arbitrary Riemannian manifolds.

We can see the general picture:

Proposition 43. *Suppose that $G_0 \subset G$ is a closed subgroup of a Lie group. To any G-structure $B \to M$ of constant type (so that the prolongation $B^{(1)} \to B$ is defined), we can associate a $G_0 \rtimes \mathfrak{g}^{(1)}$-structure $B^{(1)} \to B/G_0$ by the following process. First, define the obvious map $\Pi : B^{(1)} \to B/G_0$ given by*

$$B^{(1)} \to B \to B/G_0.$$

Next take $\gamma \in \Omega^1\left(B^{(1)}\right)$ as defined in section 7.1 on page 93. Define the 1-form $\dot\omega \in \Omega^1\left(B^{(1)}\right)$ by $\dot\omega = \gamma \pmod{\mathfrak{g}_0}$. Then define a map $B^{(1)} \to F(B/G_0)$ by the same process described in section 6.7 on page 86: to each $U \in B^{(1)}$ we associated the coframe $\Phi(U) \in F(B/G_0)$ which satisfies

$$\Phi(U)\Pi'(U) = \omega \oplus \dot\omega.$$

Finally, since $B^{(1)} \to M$ is already a $G \rtimes \mathfrak{g}^{(1)}$-bundle, the group $G_0 \rtimes \mathfrak{g}^{(1)}$ acts on $B^{(1)}$ in the obvious manner.

Proof. We have to check that this is a $G_0 \rtimes \mathfrak{g}^{(1)}$-structure, i.e. that the map Φ we have written down is well-defined and smooth, and $G_0 \rtimes \mathfrak{g}^{(1)}$-equivariant. To see that Φ is well-defined, we have only to check that ω and $\dot\omega$ are semibasic for $\Pi : B^{(1)} \to B/G_0$. The fibers of Π are spanned by the vector fields $\vec{A}$ for $A \in \mathfrak{g}_0$ and $\vec{Q}$ for $Q \in \mathfrak{g}^{(1)}$. Clearly ω vanishes on these, and $\vec{A} \lrcorner \gamma = A \in \mathfrak{g}_0$, and $\vec{Q} \lrcorner \gamma = 0$, so $\dot\omega$ vanishes on these too. Therefore Φ is well-defined, and we will let the reader show that it is smooth, perhaps by looking at local coordinates. Equivariance under $G_0 \rtimes \mathfrak{g}^{(1)}$ is immediate from the equations in section 7.1 on page 93 for the $G \rtimes \mathfrak{g}^{(1)}$ action on ω and γ.

11.2 Example: anti-self-dual metrics on 4-manifolds

Consider a Riemannian 4-manifold. Let $B \to M$ be the bundle of orthonormal coframes. Its structure equations are

$$d\omega_i = -\gamma_{ij} \wedge \omega_j$$
$$d\gamma_{ij} = -\gamma_{ik} \wedge \gamma_{kj} + \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l,$$

$(i, j, k, l = 1, \dots, 4)$. The Lie algebra $\mathfrak{so}(4)$ splits as $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. This splitting is invariant under the adjoint $SO(4)$ action, although the two summands get swapped under reflections from $O(4)$. Therefore let us assume that our manifold M is oriented, reducing the structure group from $O(4)$ to $SO(4)$. We will still write the reduced bundle as B.

To make use of the splitting of the Lie algebra, note that there is no complex structure on $\mathbb{R}^4$ invariant under $SO(4)$, but if we could reduce the Lie algebra to $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$, then the $\mathfrak{so}(2)$ action would be a complex structure, and the $\mathfrak{so}(3)$ would commute with it. This is the fundamental idea. So we want to take G_0 to be the group whose Lie algebra is $\mathfrak{g}_0 = \mathfrak{so}(3) \oplus \mathfrak{so}(2)$, and see what sort of almost complex structure emerges on B/G_0, and in particular we will ask when B/G_0 is a complex manifold. Throughout the long computations that follow, we will hold on to this simple question: when is B/G_0 a complex manifold?

So far, this is just a story about Lie algebras, asking whether they preserve complex structures. We want to analyze structure equations. It will be convenient to recast these structure equations in a form that reflects the splitting of the Lie algebra. Recall from example 22 on page 18 the morphism $\text{Spin}(4) = \text{Sp}(1)_+ \times \text{Sp}(1)_- \to SO(4)$ and the quaternionic formulation of the structure equations of the flat Spin(4)-structure. Since structure equations depend only on the Lie algebra, and torsion-free structure equations are the same for any torsion-free structures, we must still have the same structure equations

$$d\omega = -\gamma_+ \wedge \omega - \omega \wedge \gamma_-$$

on our 4-manifold, with γ_+ and γ_- connection 1-forms valued in imaginary quaternions. The structure equations for γ will be quite different, because of the curvature. To relate these structure equations to the structure equations of $SO(4)$-structures, we write left multiplication by a quaternion x as L_x, and right multiplication by x as R_x. This allows us to write

$$d\omega = -L_{\gamma_+} \wedge \omega + R_{\gamma_+} \wedge \omega$$

so that to compare $SO(4)$ and $\text{Spin}(4)$ structure equations:

$$\gamma = L_{\gamma_+} - R_{\gamma_-}.$$

In this way, we see the splitting. Sometimes we identify $\mathfrak{sp}(1)_+ = \mathfrak{so}(3)$ and $\mathfrak{sp}(1)_- = \mathfrak{so}(3)$ with the imaginary quaternions, and other times we don't. For differential forms α and β valued in quaternions, say $\alpha = \alpha^0 + \alpha^1 i + \alpha^2 j + \alpha^3 k$ with each α^μ real valued, define

$$\langle \alpha \wedge \beta \rangle = \alpha^\mu \wedge \beta^\mu .$$

We let the reader check that for any quaternion q of unit length

$$\langle q\alpha \wedge q\beta \rangle = \langle \alpha q \wedge \beta q \rangle = \langle \alpha \wedge \beta \rangle$$

This is pretty clear from the similar identity for 0-forms, and the fact that the calculations don't depend on commuting the components of the 0-forms. A fancier way to think about it:

$$\langle \alpha \wedge \beta \rangle = \int_{\mathrm{Sp}(1)_+} \langle \alpha, q \rangle \wedge \langle \beta, q \rangle \; dq,$$

and the identity is clear from translation invariance of the integral. Define semibasic 2-forms Ω_+ and Ω_- valued in the imaginary quaternions, by

$$\begin{aligned} \Omega_+ &= \frac{1}{2} \left(\langle i\omega \wedge \omega \rangle + \langle j\omega \wedge \omega \rangle + \langle k\omega \wedge \omega \rangle \right) \\ \Omega_- &= \frac{1}{2} \left(\langle \omega i \wedge \omega \rangle + \langle \omega j \wedge \omega \rangle + \langle \omega k \wedge \omega \rangle . \right) . \end{aligned}$$

We invite the reader to check the following complicated statements: the Spin (4)-structure equations are

$$\begin{aligned} d\omega &= -\gamma_+ \wedge \omega - \omega \wedge \gamma_- \\ d\gamma_+ &= -\gamma_+ \wedge \gamma_+ + \left(W^- - \frac{s}{24} 1_4 \right) \Omega_- + {}^t S \Omega_+ \end{aligned}$$

where

$$\begin{aligned} S &: B \to \mathfrak{sp}\,(1)_+ \otimes \mathfrak{sp}\,(1)_- \\ W^+ &: B \to \mathrm{Sym}^2_0 \left(\mathfrak{sp}\,(1)_+ \right) \\ W^- &: B \to \mathrm{Sym}^2_0 \left(\mathfrak{sp}\,(1)_- \right) \\ s &: M \to \mathbb{R} \end{aligned}$$

and 1_4 is the 4×4 identity matrix. We can relate these S, W^+, W^-, s to the curvature tensor of M as follows: s is the scalar curvature, the Ricci curvature (R_{ij}) is

$$\frac{s}{4} + \begin{pmatrix} -s_{11} - s_{22} - s_{33} & s_{23} - s_{32} & -s_{13} + s_{31} & s_{12} - s_{21} \\ s_{23} - s_{32} & -s_{11} + s_{33} + s_{22} & -s_{12} - s_{21} & -s_{13} - s_{31} \\ -s_{13} + s_{31} & -s_{12} - s_{21} & -s_{22} + s_{33} + s_{11} & -s_{23} - s_{32} \\ s_{12} - s_{21} & -s_{13} - s_{31} & -s_{23} - s_{32} & -s_{33} + s_{22} + s_{11} \end{pmatrix}$$

where $S = (s_{pq})$ for $p, q = 1, 2, 3$. The components of the Weyl curvature W are linear functions with constant coefficients of the components of W^+ and W^-, and conversely, (too complicated to be worth specifying).

Identify $\mathfrak{so}(4) = \Lambda^2(\mathbb{R}^4)$. The splitting $\mathfrak{so}(4) = \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$ splits $\Lambda^2(\mathbb{R}^4) = \Lambda^2_+ \oplus \Lambda^2_-$. Consider the subgroup $G_0 = \text{Sp}(1)_+ \times U(1)_- \subset \text{Spin}(4)$ consisting of the pairs (g_+, g_-) of unit quaternions commuting with $(0, i)$. Let us suppose that our manifold M actually has a Spin (4)-structure on it (which is not an embedded type of structure, so be careful!), which we will also shamelessly call B. We have its structure equations already. Consider the manifold $Z = B/G_0$. If we had quotiented out all of $G = \text{Spin}(4)$, we would have $M = B/G$, so not surprisingly:

Exercise 11.2 $Z \to M$ is a right principal G_0-bundle. Using the splitting $\Lambda^2(\mathbb{R}^4) = \Lambda^2_+ \oplus \Lambda^2_-$, we can split the bundle of 2-forms

$$\Lambda^2(TM) = \left(B \times \Lambda^2\left(\mathbb{R}^4\right)\right) / \text{Spin}(4)$$

into a sum of two bundles

$$\Lambda^2(TM) = \Lambda^2_+(TM) \oplus \Lambda^2_-(TM)$$

defined by

$$\Lambda^2_\pm(TM) = \left(B \times \Lambda^2_\pm\right) / \text{Spin}(4).$$

The bundle Z is the bundle of unit spheres inside $\Lambda^2_-(TM)$.

Exercise 11.3 Write $\gamma_\pm = \gamma^1_\pm i + \gamma^2_\pm j + \gamma^3_\pm k$. Define $\dot\omega = \gamma^2_- - \gamma^3_- i$, so that $\gamma_- = \gamma^1_- + j\dot\omega$. The 1-forms ω and $\dot\omega$ are semibasic for the bundle map $B \to Z$.

Exercise 11.4 Under right G-action on B,

$$r^*_{(g_+,g_-)} \begin{pmatrix} \omega \\ \Omega_+ \\ \Omega_- \end{pmatrix} = \begin{pmatrix} \bar g_+ \omega g_- \\ \bar g_+ \Omega_+ g_+ \\ \bar g_- \Omega_- g_- \end{pmatrix}.$$

The easiest way to check the representation on $\Omega_\pm$ might be to use the integral

$$\Omega_+ = \frac{1}{2} \int_{\text{Sp}(1)} \langle q\omega, \omega \rangle \, q \, dq.$$

As for the functions representing curvature,

$$r^*_{(g_+,g_-)} \begin{pmatrix} W^+ \\ W^- \\ S \end{pmatrix} = \begin{pmatrix} \bar g_+ W^+ g_+ \\ \bar g_- W^- g_- \\ \bar g_+ S g_- \end{pmatrix}$$

and s is invariant under the structure group.

Now if we restriction to the $G_0 \rtimes \mathfrak{g}^{(1)}$ action on B, we find

$$r^*_{(g_+,g_-)}\dot{\omega} = \dot{\omega} g_-^2.$$

The crucial observation is that ω and $\dot{\omega}$ are transforming under a representation that preserves the complex structure of right multiplication by i, since i commutes with g_- and so with G_0 action. This gives Z an almost complex structure.

To understand the complex structure in the flat case, note first that $M = \mathbb{H}$, $Z = \left(\mathbb{H} \oplus \operatorname{Sp}(1)_-\right)/U(1)_-$, so the complex structure acts on $T_{(0,i)}Z$ by acting on $(v_0 + v_1 i + v_2 j + v_3 k, w_2 j + w_3 k)$ by right multiplication by i. Check that this is just the usual complex structure on $\mathbb{C}^2 \oplus \mathbb{C}$ given by complex coordinates

$$v_0 + v_1\sqrt{-1}, v_2 - v_3\sqrt{-1}, w_2 - w_3\sqrt{-1}.$$

Returning to the general case, we can make the almost complex structure more explicit by writing our structure equations in just these terms. Write

$$\begin{aligned}\Omega^1 &= \omega_0 + \sqrt{-1}\omega_1 \\ \Omega^2 &= \omega_2 - \sqrt{-1}\omega_3\end{aligned}$$

and calculate

$$d\begin{pmatrix}\Omega^1 \\ \Omega^2\end{pmatrix} = -\begin{pmatrix}\sqrt{-1}\left(\gamma_+^1 + \gamma_-^1\right) & -\gamma_+^2 - \sqrt{-1}\gamma_+^3 \\ \gamma_+^2 - \sqrt{-1}\gamma_+^3 & \sqrt{-1}\left(\gamma_+^1 - \gamma_-^1\right)\end{pmatrix} \wedge \begin{pmatrix}\Omega^1 \\ \Omega^2\end{pmatrix} + \underbrace{\begin{pmatrix}0 & \dot{\omega} \\ -\dot{\omega} & 0\end{pmatrix} \wedge \begin{pmatrix}\bar{\Omega}^1 \\ \bar{\Omega}^2\end{pmatrix}}_{\text{torsion}}$$

$$d\dot{\omega} = 2\sqrt{-1}\gamma_-^1 \wedge \dot{\omega} + a_{12}\Omega^1 \wedge \Omega^2 + a_{p\bar{q}}\Omega^p \wedge \Omega^{\bar{q}} + a_{\bar{1}\bar{2}}\Omega^{\bar{1}} \wedge \Omega^{\bar{2}}$$

where

$$\begin{aligned}
a_{12} &= \frac{1}{2}(w^-_{22} + w^-_{33}) - \frac{s}{24} \\
a_{1\bar{1}} &= \frac{1}{4}\left(s_{13} + 2w^-_{13} + \sqrt{-1}\left(s_{12} + w^-_{12}\right)\right) \\
a_{1\bar{2}} &= \frac{1}{4}\left(s_{22} - s_{33} - \sqrt{-1}\left(s_{23} + s_{32}\right)\right) \\
a_{2\bar{1}} &= \frac{1}{4}\left(-s_{22} - s_{33} + \sqrt{-1}\left(s_{23} - s_{32}\right)\right) \\
a_{2\bar{2}} &= \frac{1}{4}\left(2w^-_{13} - s_{13} + \sqrt{-1}\left(2w^-_{12} - s_{12}\right)\right) \\
a_{\bar{1}\bar{2}} &= \frac{1}{2}\left(w^-_{22} - w^-_{33}\right) + \sqrt{-1}w^-_{23}
\end{aligned}$$

Exercise 11.5 Without computing this complicated rewriting of our structure equations, show that Ω_+ is a $(1,1)$-form in the almost complex structure, and that $\Omega_- = \Omega_-^1 i + \Omega_-^2 j + \Omega_-^3 k$ where Ω_-^1 is a $(1,1)$-form, and Ω_-^2 and Ω_+^3 are mixed $(2,0)+(0,2)$ real-valued forms. Conclude that W^- looks like

$$W^- = \begin{pmatrix} w_{11}^- & w_{12}^- & w_{13}^- \\ w_{21}^- & 0 & 0 \\ w_{31}^- & 0 & 0 \end{pmatrix}$$

just when $Z = B/G_0$ is a complex manifold, by the Newlander–Nirenberg theorem (see [66],[44]). Use the G-action to show that $W^- = 0$ just when Z is a complex manifold.

Exercise 11.6 Why are the fibers of $Z \to M$ rational curves, i.e. complex curves diffeomorphic to spheres?

Question 64. It would be nice to show how the same ideas can be used to study quaternionic–Kähler manifolds, using $\mathrm{Sp}(n) \subset \mathrm{Sp}(n) \times \mathrm{Sp}(1)$; see [74].

12

Symmetry groups with orbits of given dimension

Question 65. This chapter is absurdly titled. I need to split some stuff up here.

The fundamental principle: symmetries commute with the action of the structure group. Therefore if a group H acts on a manifold M as equivalences of a G-structure $B \to M$, then the orbits of H in B (under the prolongation of the action) are permuted by the action of G. On each tangent space of each orbit, we will have some relations among the soldering and pseudoconnection 1-forms. We have to be careful: the coefficients of those relations might change when we change the pseudoconnection 1-form. We must therefore quotient out by this action, to see what information from these coefficients is intrinisically defined. The resulting quotient data will also change when we move around the bundle B using the structure group. We can thus try to normalize that data using the structure group. To make this approach work, we have to consider the open subset of B on which the orbits have maximal dimension.

Question 66. An example would be nice.

12.1 Actions of a chosen symmetry group

Suppose that we wish to find all actions of a particular group H on a particular finite type G-structure. We could try to proceed as follows. Given a group H and a manifold M with coframing $\omega \in \Omega^1(M) \otimes V$, we know that an action of H on M is a map

$$\phi : H \times M \to M$$

satisfying

$$\phi(h_0, \phi(h_1, x_0)) = \phi(h_0 h_1, x_0)$$

and

$$\phi(1, x_0) = x_0.$$

The graph of such a map ϕ is a submanifold $X \subset H \times M_0 \times M_1$ where $M_0 = M_1 = M$ and we will write ω_0 for ω on M_0, etc.

On $H \times M_0 \times M_1$ we have a coframing

$$\lambda \oplus \omega_0 \oplus \omega_1 \in \Omega^1 \left(H \times M_0 \times M_1\right) \otimes \left(\mathfrak{h} \oplus V \oplus V\right)$$

where λ is the left invariant Maurer-Cartan 1-form on H:

$$\lambda_x(v) = L_x v \in \mathfrak{h} = T_1 H.$$

To have X be an action, we need it to be the graph of a function

$$\phi : H \times M_0 \to M_1$$

so λ and ω_0 form a coframing on X.

We will also need each fiber of X over H to be an equivalence, matching up the coframings ω_0 and ω_1. So on these fibers:

$$\omega_1 = \omega_0.$$

Therefore on X:

$$\omega_1 = \omega_0 + a\lambda$$

for some function

$$a : X \to \operatorname{Lin}\left(\mathfrak{h}, V\right).$$

Example 90. Consider the structure equations of Riemannian geometry on the flat Euclidean plane. Take $z = x + \sqrt{-1}y$ complex Cartesian coordinate on $\mathbb{R}^2 = \mathbb{C}$. The orthonormal frame bundle has coordinates (z, θ) where $0 \le \theta < 2\pi$. The number θ determines the coframe

$$e^{-i\theta} dz.$$

Then the soldering 1-form ω is

$$\omega = e^{-i\theta} dz$$

and the connection 1-form γ is

$$\gamma = -d\theta$$

with

$$d \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} -i\gamma \wedge \omega \\ 0 \end{pmatrix}.$$

The action of a rotation by angle α on a coframe (z, θ) is

$$(z, \theta) \mapsto (z, \theta + \alpha).$$

Now if we take M to be the bundle of orthonormal coframes, we can let $M_0 = M_1 = M$ with coframe

$$\omega_j, \gamma_j$$

on M_j. Then take $H = S^1$ the circle group, and let H act on M by rotation. The submanifold X is defined by the equations

$$\begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha \end{pmatrix}.$$

Therefore

$$d \begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} e^{i\alpha} i & 0 \\ 0 & 0 \end{pmatrix} d\alpha \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix} + \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} d\alpha \begin{pmatrix} dz_0 \\ d\theta_0 \end{pmatrix} + \begin{pmatrix} 0 \\ d\alpha \end{pmatrix}.$$

Finally we find

$$\begin{pmatrix} \omega_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \gamma_0 \end{pmatrix} + \begin{pmatrix} ie^{-i\theta_0} z_0 \\ 1 \end{pmatrix} \lambda$$

where

$$\lambda = d\alpha.$$

Note that

$$a = \begin{pmatrix} ie^{-i\theta_0} z_0 \\ 1 \end{pmatrix}$$

is constant on group orbits.

If we let $H = \mathbb{C}$ with translation action

$$\begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} z_0 + t \\ \theta_0 \end{pmatrix}$$

with

$$\lambda = dt$$

then we find

$$a = \begin{pmatrix} e^{-i\theta_0} \\ 0 \end{pmatrix}.$$

Returning to the general problem, we want to produce an exterior differential system on $H \times M_0 \times M_1$ whose integral manifolds X, if they satisfy an appropriate independence condition, will be actions of H on M preserving ω.

The right action of H on itself,

$$R_{h_0} h_1 = h_1 h_0$$

acts on λ by

$$R_h \lambda = \mathrm{Ad}_x^{-1} \lambda.$$

The left action

$$L_{h_0} h_1 = h_0 h_1$$

acts on λ by

$$L_h \lambda = \lambda.$$

If X is to be the graph of a group action, then a point

$$(h, x_0, x_1)$$

belongs to X precisely when

$$x_1 = \phi(h, x_0).$$

Therefore

$$h_1 x_1 = \phi(h_1 h, x_0)$$

and so

$$(h_1 h, x_0, h_1 x_1) \in X.$$

This is the left action on X. It projects to H to be the left action on H, and to $H \times M_0$ to be the action

$$L_{h_1}(h, x_0) = (h_1 h, x_0).$$

Therefore it satisfies

$$L_{h_1}^* (\lambda \oplus \omega_0) = \lambda \oplus \omega_0.$$

To have a group action, clearly we also need ω_1 to be invariant under the left action. The right action is

$$R_{h_1}(h, x_0, x_1) = \left(h h_1, h_1^{-1} x_0, x_1\right).$$

It projects to $H \times M_1$ as

$$R_{h_1}(h, x_1) = (h h_1, x_1)$$

so that

$$R_{h_1}^* (\lambda \oplus \omega_1) = \operatorname{Ad}_{h_1}^{-1} \lambda \oplus \omega_1.$$

Again, to have a group action, ω_0 must also be invariant under the right action. On X

$$\omega_1 = \omega_0 + a\lambda.$$

Therefore the function a must be invariant under the left action, and under right action must satisfy

$$R_h^* a = a \operatorname{Ad}_h.$$

Also, the left and right actions must commute. From the fact that X is the graph of a function $\phi : H \times M_0 \to M_1$ we see that X is canonically diffeomorphic to $H \times M_0$, so that

$$a : H \times M_0 \to \operatorname{Lin}(\mathfrak{h}, V).$$

Then left invariance tells us that

$$a : M_0 \to \operatorname{Lin}(\mathfrak{h}, V) .$$

Right invariance says that the function a must satisfy

$$a(hx_0)\operatorname{Ad}_h = a(x_0) .$$

Lemma 40. *Conversely, suppose that $X \subset H \times M_0 \times M_1$ is a submanifold for which the maps*

$$\begin{array}{ccccc} & & & & H \times M_0 \\ & & & \nearrow & \\ X & \longrightarrow & H \times M_0 \times M_1 & & \\ & & & \searrow & \\ & & & & H \times M_1 \end{array}$$

are both local diffeomorphisms, and that on X,

$$\omega_1 = \omega_0 + a\lambda$$

where

$$a : M_0 \to \operatorname{Lin}(\mathfrak{h}, V)$$

and that the function a satisfies

$$a(x_1) = a(x_0)\, Ad_h^{-1}$$

whenever

$$(h, x_0, x_1) \in X.$$

Suppose moreover that

$$(1, x_0, x_0) \in X$$

for every x_0 in an open subset of M_0.

Then X is the graph of a local group action, at least near any point $(1, x_0, x_0)$.

Proof. By a local group action, we mean of course a map ϕ : open $\subset H \times M_0 \to M_1$ satisfying the definition of a group action, wherever it is defined.

To see this, we define ϕ near some element $(1, x, x)$ by letting X be the graph of ϕ. We need to show that if

$$\phi(h_0, x_0) = x_1$$

and

$$\phi\left(h_1, x_1\right) = x_2$$

then

$$\phi\left(h_1 h_0, x_0\right) = x_2$$

as long as these expressions are defined. They hold immediately if $h_1 = 1$, at least for h_0 near enough to the identity, since $h_1 = 1$ forces $x_2 = x_1$. So we need only show that these equations hold for h_1 near the identity, say $h_1 = e^{tA}$ for some $A \in \mathfrak{h}$.

We need to compare

$$y(t) = \phi\left(e^{tA}, x_1\right)$$

with

$$z(t) = \phi\left(e^{tA} h_0, x_0\right).$$

These describe two curves in M_1, with

$$y(0) = z(0)$$

so they start at the same point. Their velocities are

$$\frac{d}{dt}\left(e^{tA}, x_1, y(t)\right) = \left(\overrightarrow{A}, 0, \frac{dy}{dt}\right)$$

and

$$\frac{d}{dt}\left(e^{tA} h_0, x_0, z(t)\right) = \left(\overleftarrow{A}, 0, \frac{dy}{dt}\right)$$

where $\overrightarrow{A}$ is an infinitesimal generator of the right action, and $\overleftarrow{A}$ is an infinitesimal generator of the left action, with $A \in \mathfrak{h}$. We find

$$\begin{aligned} \omega_1\left(\frac{dy}{dt}\right) &= \omega_0\left(0\right) + a\lambda\left(\overrightarrow{A}\right) \\ &= a\left(x_1\right) A \end{aligned}$$

and

$$\begin{aligned} \omega_1\left(\frac{dz}{dt}\right) &= \omega_0\left(0\right) + a\lambda\left(\overleftarrow{A}\right) \\ &= a \operatorname{Ad}_{e^{tA} h_1}^{-1} A \\ &= a\left(x_0\right) \operatorname{Ad}_{h_1}^{-1} A \end{aligned}$$

Now using the equation

$$a\left(x_1\right) \operatorname{Ad}_{h_1} = a\left(x_0\right)$$

for all

$$(h_1, x_0, x_1) \in X$$

we find that these two curves have the same velocity, so they are equal. Therefore

$$\phi\left(e^{tA}, \phi\left(h_1, x_0\right)\right) = \phi\left(e^{tA} h_1, x_0, x_1\right)$$

for all t near 0.

Proposition 44. *Every group action*

$$\phi : H \times M \to M$$

preserving a coframing $\omega \in \Omega^1(M) \otimes V$ *has as its graph an integral manifold of the exterior differential system*

$$\mathcal{I} = \left(da - a_0\omega_0, da - a_1\omega_1 - a\,\mathrm{Ad}_\lambda, \omega_1 - \omega_0 - a\lambda\right)$$

on the manifold

$$H \times M \times M \times \mathrm{Lin}\left(\mathfrak{h}, V\right)_a \times \mathrm{Lin}\left(V \otimes \mathfrak{h}, V\right)_{a_0} \times \mathrm{Lin}\left(V \otimes \mathfrak{h}, V\right)_{a_1}$$

on which the independence condition, that

$$\lambda \oplus \omega_0$$

is a coframing, is satisfied. Furthermore, X *contains the submanifold*

$$\{(1, x, x) \mid x \in M\}$$

Conversely, if X *is an integral manifold of this exterior differential system, satisfying that independence condition, and* X *contains an open subset of that submanifold, then near that submanifold,* X *is the graph of a local group action of* H *on* M *preserving the coframing* $\omega = \omega_0 = \omega_1$*.*

Proof. Given a group action ϕ of H on M preserving ω, we simply let X be its graph inside $H \times M \times M$, and we find that

$$\omega_1 = \omega_0 + a\lambda$$

and that

$$da = a_0\omega_0 = a_1\omega_1 - a\,\mathrm{Ad}_\lambda$$

for some functions a_0 and a_1, by differentiating the conditions that a be left invariant, and that a satisfy the equation

$$a\left(x_1\right) = a\left(x_0\right) \mathrm{Ad}_h^{-1} \tag{12.1}$$

whenever

$$(h, x_0, x_1) \in X.$$

Conversely, if we have such an integral manifold, then we can see that the function a is (locally) defined on M_0, and so left invariant. It is also satisfies the identity 12.1 on the previous page whenever

$$(h, x_0, x_1) \in X$$

is a point close enough to our submanifold, because differentiating this identity gives

$$\begin{aligned} da\left(\overrightarrow{A}\right) &= -\,a\,[A, B] \\ &= -\,a\,\mathrm{Ad}_{\lambda\left(\overrightarrow{A}\right)}\, B \end{aligned}$$

and this identity is satisfied by hypothesis in our differential ideal. Integrating, we recover the identity 12.1 on the preceding page. This enables us to apply the previous lemma.

We have constructed an exterior differential system whose integral manifolds are local actions of a given Lie group by equivalences. But plugging in $\omega_1 = \omega_0 + a\lambda$ we get $a_0 = a_1$ and

$$a_0\left(a\left(A_1\right), A_2\right) = a\left(\left[A_1, A_2\right]\right)$$

for all $A_1, A_2 \in \mathfrak{h}$. These conditions are generally quite singular. To manage this exterior differential system, one is forced to restrict to parts of $\mathrm{Lin}\left(V \otimes \mathfrak{h}, C\right)$ where various rank conditions are satisfied. This is not a problem as long as the result is going to be homogeneous, but in general there could be serious problems with using this formalism.

Question 67. As an example, consider the symmetry groups of conformal structures on 3-manifolds.

Question 68. How can we adapt this approach to study finite dimensional group actions on G-structures which are not of finite type? Or to study Lie pseudogroup actions?

12.2 Counting generality

Question 69. I would like to bring this discussion of Finsler geometry as far forward in the book as I can, as an indication of the sort of techniques that will later be covered in more detail. Then I need to make a new chapter about the existence of G-structures with conditions on torsion.

Consider the case of Finsler surfaces again. We had the structure equations

$$d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} = \begin{pmatrix}\omega^3 \wedge \omega^2\\ \left(\omega^1 - I\omega^2\right)\wedge \omega^3\\ \omega^2 \wedge \left(K\omega^1 - J\omega^3\right)\end{pmatrix} \tag{12.2}$$

with three functions appearing, I, J, K. If we apply the exterior derivative to both sides of all three of these equations, we find equations for the exterior derivatives of I, J, K:

$$d\begin{pmatrix} I \\ J \\ K \end{pmatrix} = \begin{pmatrix} J & I_2 & I_3 \\ -IK - K_3 & J_2 & J_3 \\ K_1 & K_2 & K_3 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

Therefore the Cartan characters of this tableau are

$$s_1 = 3, \ s_2 = 3, \ s_3 = 1$$

(just count the "free derivatives" in each column, and then reorder the columns to make the numbers descending). Cartan's count predicts that the general real analytic coframing satisfying Cartan's structure equations of Finsler geometry depends on a choice of one function of 3 variables (since $s_3 = 1$ is the last nonzero character, this gives 1 function of 3 variables). This is correct since a Finsler metric is a hypersurface in the tangent bundle, so a 3 dimensional manifold in a 4 manifold, and can be written as the graph of a function over any other Finsler metric.

If we asked for I to be constant, then we would find that this forces $J = K = 0$, and so we have $d(I, J, K) = 0$, and by the Frobenius theorem the equations 12.2 give a foliation of the frame bundle of any surface; the leaves of the foliation are three manifolds corresponding to the local Finsler structures with this constant value for I. These are not Finsler metrics on any surface, as we have seen in section 5.3 on page 47.

If we ask for $J = K = 0$, and don't demand I constant, we find $s_1 = 1, s_2 = 1, s_3 = 0$, so that solutions depend on one function of 2 variables. The explicit construction of these solutions can be carried out as follows: first note that the equations $\omega^1 = \omega^2 = 0$ determine a foliation by curves, which are the circles in the tangent spaces of the Finsler surface. But also, the equations $\omega^2 = \omega^3 = 0$, which are the geodesic equations, form a foliation by curves (the geodesics) and if they are the fibers of a fiber bundle, we can let Λ be the surface parameterizing the geodesics, i.e. the base of the bundle. From the structure equations, we see that ω^3 and $\omega^2 \wedge \omega^3$ are well defined on Λ, although ω^1, ω^2 are not. Also, I is defined on Λ. We see from $d\omega^3 = 0$ that $\omega^3 = d\theta$ locally, for some function θ. With a bit of manipulation (inspired by the flat case $I = 0$, which is the Euclidean plane), we can see that locally there are functions s, t, θ and $f(t, \theta)$ so that

$$\begin{aligned} \omega^1 &= ds + df - t\,d\theta \\ \omega^2 &= dt + s\,d\theta \\ \omega^3 &= d\theta \end{aligned}$$

where s, t are coordinates on Λ. Moreover, given any twice continuously differentiable (not necessarily real analytic) function $f(t, \theta)$, we can plug it in to these structure equations to construct our Finsler structure. This is our one function of 2 variables, correctly guessed (but not rigorously) by the Cartan–Kähler analysis. This sort of manipulation of differential forms to produce coordinate expressions is called *integrating structure equations*.

Supposing that these coordinates are globally defined, it is easy from here to recover the Finsler surface. Its geodesics are the curves with constant θ, t. The functions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} - \int_0^\theta f(t,\phi) \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} d\phi$$

are coordinates on the base surface. The curves given by constant x, y must therefore be closed, in order that they can be the unit circles of a Finsler surface. The reader can easily invesigate the Fourier series of f in θ if we make the further assumption that x, y are periodic in θ with period 2π, discovering that f turns out to be periodic in θ, and with no 2π frequency terms. We have found an infinite dimensional family of complete Finsler metrics on the plane with vanishing Gauss-Finsler curvature and vanishing J invariant.

12.2.1 The Frobenius theorem and abelian systems

This reveals a useful idea that applies to many G-structures: since the structure equations tell you how to differentiate some 1-forms, you can easily look for foliations perpendicular to some of those 1-forms, by the Frobenius theorem. Assuming that the foliation is a fiber bundle, you can try to find objects defined on the base.

A exceptionally pretty situation occurs when we find a collection of 1-forms from our structure equations, say ϑ^i, so that all of the differentials $d\theta^A$ of all of the 1-forms in the structure equations vanish modulo the ϑ^i. Call such a family of 1-forms ϑ^i an *abelian system.* Not only does an abelian system satisfy the Frobenius theorem, giving a foliation, but moreover the remaining nonzero 1-forms θ^A in the structure equations can be integrated to functions on the leaves of the foliation. For instance, a Finsler surface of vanishing Gauss-Finsler curvature has the abelian system $\left(\omega^3 = 0\right)$. Note that if the structure equations have only constants appearing in them, then they define the local structure of a Lie group, and an abelian system determines a abelian subgroup, hence the name *abelian.* A minimal abelian system, i.e. containing as few 1-forms as possible, determines a maximal abelian subgroup.

Another useful idea: it helps to have an explicit example; in this case the flat plane. Moreover, one should not attempt to choose coordinates until the equivalence method has finished.

Question 70. This last comment doesn't quite fit here.

Question 71. How does the general theory of Cartan's count relate to the equivalence method? Note that the finiteness of Spencer cohomology implies that there are finitely many obstructions to flatness emerging from the equivalence method. But the Guillemin–Sternberg examples [41], among others, shows that vanishing of these do not suffice to ensure flatness. I should also explain in the preface that I decided not to pay too much attention to the problem of flatness, i.e. calculating enough invariants to determine if a G-structure is flat.

12.2.2 Exterior differential systems for G-structures

Question 72. This section is not very good. Needs more examples.

Pick a submanifold $X \subset H^{0,2}(\mathfrak{g})$ and ask for G-structures on a manifold M with torsion belonging to X. A G-structure is a subbundle of FM, so we can try to set up differential equations on FM whose solutions will be G-structures with torsion in X. The structure equations of $FM^{(j)}$ have already been presented. On FM we have

$$d\omega = -\gamma \wedge \omega$$

with $\omega \in \Omega^1(FM) \otimes V$ and γ not well defined, but such a γ can be chosen and lives in $\gamma \in \Omega^1(FM) \otimes \mathrm{GL}(V)$. We want to find a subbundle $B \subset FM$ on which our intrinsic torsion will belong to X. But the γ that is chosen above might not fill the bill, since it is chosen independently of the choice of bundle $B \subset FM$. But, whatever γ is chosen, it must be possible to alter it, say to

$$\gamma + p\omega$$

where

$$p : FM \to V \otimes V^* \otimes V^*$$

so that $\gamma + p\omega$ is a pseudoconnection for B:

$$d\omega = -(\gamma + p\omega) \wedge \omega + p\omega \wedge \omega.$$

Since the effect of p is trivial if it belongs to $\mathfrak{gl}(V)^{(1)}$, we can suppose that

$$p : FM \to V \otimes \Lambda^2(V^*).$$

On the submanifold B we need $\gamma + p\omega$ to be valued in $\mathfrak{g}$, and we need the intrinsic torsion $[p]$ to be valued in X. So we must satisfy the equations

$$\begin{aligned} [p] &\in X \\ \gamma + p\omega &= 0 \mod \mathfrak{g} \end{aligned}$$

This gives us an approach to defining an exterior differential system: take $\hat{X} = [\,]^{-1} X \subset V \otimes \Lambda^2(V^*)$, and take the differential ideal on $FM \times \hat{X}$ generated by the equations

$$\Pi(\gamma + p\omega) = 0$$

where $p \in \hat{X}$ and

$$\Pi : \mathfrak{gl}(V) \to \mathfrak{gl}(V)/\mathfrak{g}$$

is the obvious projection. So our ideal is generated by the components of this Π in any basis.

Question 73. If we look at the prolongations, I think we will find relative Spencer cohomology.

12.3 Variations of G-structure

Example 91 (Constant curvature metrics on the disk). Take metrics

$$g_t = \frac{4}{\left(1 + tr^2\right)^2}\left(dx^2 + dy^2\right)$$

where $r^2 = x^2 + y^2 < -1/t$. With coordinates (x, y, θ) on the orthonormal frame bundle, and adding a coordinate t for the variation, we have soldering forms

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \frac{2}{1 + tr^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}.$$

A short calculation gives

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = -\begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} - v\,dt \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

where

$$\gamma = d\theta - 2s_j\omega^j$$
$$v = -\frac{r^2}{1 + tr^2}$$

with

$$s_1 = ty\cos\theta + tx\sin\theta$$
$$s_2 = ty\sin\theta - tx\cos\theta$$

Each choice of a constant value for t gives an $SO(2)$-structure. We view v as the first variation of that $SO(2)$-structure.

A family B of G-structures on a manifold M is a principal G subbundle $B \subset FM \times X \to M \times X$ where X is a manifold parameterizing the family. Write structure equations on $B \times_X FX$ as

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega - v\eta \wedge \omega$$

where $\eta \in \Omega^1(FX) \otimes W$ is the soldering 1-form on FX and $\gamma \in \Omega^1(B) \otimes \mathfrak{g}$ is a pseudoconnection 1-form, i.e. a $\mathfrak{g}$ valued 1-form satisfying

$$\gamma\left(\vec{A}\right) = 0$$

for $A \in \mathfrak{g}$ acting on B by the usual right action on FM, and

$$T : B \times_X FX \to H^{0,2}(\mathfrak{g})$$

is the torsion, and

$$v : B \times_X FX \to (\mathfrak{gl}(V)/\mathfrak{g}) \otimes W^*$$

is called the *first variation* of B.

Example 92. Notice that in previous example, the variation could be written

$$d\omega = -\begin{pmatrix} v\,dt & -\gamma \\ \gamma & v\,dt \end{pmatrix} \wedge \omega$$

so that it is clearly conformal, since the parameters of the variation fit along with γ into the Lie algebra of $\mathfrak{co}(2)$.

Exercise 12.1 Let $B_0 \to M$ be a G-structure, and Z a vector field on M. Suppose that the flow of Z is defined for all time, i.e. Z is a complete vector field. Consider the variation for which $X = \mathbb{R}$, and $B = B_0 \times X$ is mapped into FM via the map $(u, t) \mapsto e^{tZ}u$. Show that the first variation v is

$$v^i_j = \nabla_j Z^i + T^i_{kj} Z^k \pmod{\mathfrak{g}}$$

in $\mathfrak{gl}(V)/\mathfrak{g}$.

Question 74. It should not be too difficult to differentiate to obtain prolonged structure equations of all orders.

Question 75. How do the torsion hypotheses on the G-structures parameterized in this way impose equations on v?

Question 76. It would be nice to understand the calculus of variations for G-structures. There should be easy ways to use representation theory to look for invariant low order Lagrangians. For example, every G-structure can be reduced to a K-structure, where K is a maximal compact subgroup of G. Lagrangians should be integrated over M, not over B.

Definition 41. *Suppose that $H \subset G$ is a Lie subgroup, with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Define $H^{0,2}(\mathfrak{h}, \mathfrak{g})$ by*

$$0 \to H^{0,2}(\mathfrak{h}, \mathfrak{g}) \to H^{0,2}(\mathfrak{h}) \to H^{0,2}(\mathfrak{g}) .$$

A Lagrangian *for reductions of a G-structure to an H-structure is an H equivariant map*

$$H^{0,2}(\mathfrak{h}, \mathfrak{g}) \to \det V^*.$$

In studying k-flat G-structures, we would naturally consider Lagrangians like

$$H^{k+1,2}(\mathfrak{h}, \mathfrak{g}) \to \det V^*$$

involving higher torsion, or even allow covariant derivatives of torsion.

Example 93. Consider Lagrangians in pseudo-Riemannian geometry. If $G = SO(p, q)$, then

$$H^{k,2}(\mathfrak{g}) = 0$$

for $k \neq 1$, while

$$H^{1,2}(\mathfrak{g}) = \mathbb{R}_{scalar} \oplus \mathrm{Sym}^2_0(V) \oplus \mathrm{Weyl}(p, q)$$

splits into scalar curvature, traceless Ricci curvature and Weyl curvature. The $\det V^*$ representation is trivial, so the unique (up to scalar multiple) linear G equivariant map

$$H^{*,2}(\mathfrak{g}) \to \det V^*$$

is the scalar curvature $R^i_{jij}\, dV$. Thus we are led to integrate the scalar curvature, the *Hilbert functional* (see Besse [9] for more information). This leads to Euler–Lagrange equations giving

$$\mathrm{Ricci} = 0$$

if we apply it to the study of reductions to G-structures of the $GL(V)$-structure FM. But if we start with an $SL(V)$-structure on M (a volume form), and look for its G reductions, then the Euler–Lagrange equations are the Einstein equations

$$R_0 = 0.$$

Example 94. If we consider Lagrangians for $SO(p, q)$-reductions of a given $CO(p, q)$ structure, we employ the same analysis to find that the natural Lagrangian is the scalar curvature, the *Yamabe functional* (see Aubin [6] for more information).

Exercise 12.2 Consider almost complex structures. They have a Lagrangian in six dimensions that has no analogue in other dimensions. (Robert Bryant lectured on this once. I don't know of any other source of information on it.)

Question 77. What are all of the groups G for which there is a unique G invariant linear map

$$H^{*,2}(\mathfrak{g}) \to \det V^*$$

up to choice of a scalar?

Question 78. It might be possible not only to calculate Euler–Lagrange equations for Lagrangians, but also to see how representation theory controls the characteristic cohomology.

Question 79. Consider variations of CR geometry with fixed contact structure. What are the invariant low-order Lagrangians, and can we guarantee existence of critical reductions on compact manifolds? Reminds me of papers of Jerrison and Lee, where they put natural metrics on compact CR manifolds.

12.4 Connections and pseudoconnections

Every G-structure admits a connection, but there might be no natural choice of one, since we would have to pick a G equivariant choice of where to put the torsion of the connection in $V \otimes \Lambda^2(V^*)$, so that it would quotient down to the correct intrinsic torsion in $H^{0,2}(\mathfrak{g})$. Sometimes there is an invariant choice of pseudoconnection (which can only happen when $\mathfrak{g}^{(1)} = 0$). It is convenient to know how to read the structure equations to see when a pseudoconnection is actually a connection.

Proposition 45. *If $B \to M$ is a G-structure, with structure equations*

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega$$

for a connection γ, then the intrinsic torsion is equivariant:

$$r_g^* T = \rho(g)^{-1} T$$

and

$$d\gamma = -\gamma \wedge \gamma + (T^2 + Q)\omega \wedge \omega \tag{12.3}$$

for some function $Q : B \to \delta C^{2,1}(\mathfrak{g})$. Here ρ is the representation of G on $\Lambda^2(V^*) \otimes V$ determined by the representation of G on V. The object T^2 is defined by

$$\left(T^2\right)^i_{jkl} = \frac{2}{3}\left(T^i_{jm} T^m_{lk} + T^i_{km} T^m_{jl} + T^i_{lm} T^m_{kj}\right)$$

The G representation $\delta C^{2,1}(\mathfrak{g})$ is defined in the appendix. This function Q is also equivariant. Conversely, if γ satisfies an equation of the form 12.3, and the group G is connected, then γ is a connection. The curvature of γ is

$$\nabla\gamma = (T^2 + Q)\omega \wedge \omega$$

Proof. The proof is by direct calculation, first of the prolongation, and then, using the definitions of soldering and pseudoconnection forms, and the Cartan formula, and differentiating in g the expression

$$r_g^* \gamma - \mathrm{Ad}_g^{-1} \gamma$$

(which vanishes for all g exactly when γ is a connection), finding a condition on $d\gamma$.

From this result, we can easily see that the unique torsion free pseudoconnection of any $SO(n)$ structure is a connection (the Levi-Civita connection), by direct calculation.

Proposition 46. *Given any pseudoconnection, there is a connection with the same torsion precisely when that torsion is equivariant.*

Proof. Remark: keep in mind that the torsion of a pseudoconnection lives in $V \otimes \Lambda^2 (V^*)$, and quotients down to the intrinsic torsion. Suppose that there is a global section of the G-structure, $M \to B$. The result is proven by taking the given pseudoconnection, asking that a connection agree with it on the section, and extending from there by using the equation

$$r_g^* \gamma = \mathrm{Ad}_g^{-1} \gamma$$

to define γ away from the section. When there is no global section of the G-structure, we use affine combinations of connections from a partition of unity to paste them together.

We see from direct calculation that the torsion-free pseudoconnection on any $O(p,q)$ structure is a connection, since the existence of a torsion free pseudoconnection implies existence of a torsion free connection, but also we have uniqueness of the torsion free pseudoconnection.

The difference between any two connections with the same torsion is expressed by a section of the bundle

$$\begin{array}{ccc} (\mathfrak{g}^{(1)} \times B)/G & \longrightarrow & \mathrm{Sym}^2 (T^*M) \otimes_M TM \\ & \searrow & \downarrow \\ & & M \end{array}$$

As an example, if $H^{0,2}(\mathfrak{g}) = 0$ then there is always a torsion-free pseudoconnection for any G-structure. By the last result, there is a torsion-free connection.

Example 95. In conformal geometry, $G = CO(p,q)$, this gives a torsion-free connection, and says that two such connections differ by a 1-form. The study of conformal geometry with a torsion-free connection is called *Weyl geometry*, see O'Raifeartaigh [70].

12.5 Generalized higher order structures

Current connections make sense, and can be easily defined. But current G-structures can not. One would like an approach in terms of currents that would enable one to study generalized G-structures, and generate current characteristic classes, similar to the work of Harvey and Lawson. Since a G-structure is a subbundle, it is a submanifold, and submanifolds can always be generalized to currents. The major stumbling block is transversality. We don't have a mechanism to ensure that a generalized submanifold looks like a generalized section of a bundle. This is because when we glue a bundle together, we don't use only affine maps. But other maps require multiplication (their Taylor expansion about some point has quadratic terms), which is not defined on currents. Therefore the only cases where we can have generalized G-structures are those where $\mathrm{GL}(n, \mathbb{R})/G$ is an affine space.

However, this is always going to work for higher order structures, since $\mathfrak{gl}(V)^{(1)}/\mathfrak{g}^{(1)}$ is always a vector space. More generally, $\mathfrak{g}^{(1)}/\mathfrak{h}^{(1)}$ is a vector space, so generalized higher order reductions of structure group make sense.

12.6 The torsion operator

Torsion is not always a tensor, but a certain part of it, which we will christen the *torsion operator*, always is.

Example 96 (Almost symplectic geometry). A $\mathrm{Sp}(2n, \mathbb{R})$-structure is the same as a choice of 2-form Ω so that $\Omega^n \neq 0$. The intrinsic torsion can be identified with $d\Omega$ invariantly.

Given an element $T \in V \otimes \Lambda^2(V^*)$ we can construct an operator

$$d_T : \Lambda^*(V^*) \to \Lambda^*(V^*)$$

by contracting the V part, and wedging the $\Lambda^2(V^*)$ part, i.e.

$$d_T\eta(v_1, \dots, v_{k+1}) = \sum_{i<j} (-1)^{i+j} \eta(T(v_i, v_j), v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}).$$

(Compare this to the exterior derivative, thinking of these v_i as translation invariant vector fields on V.) If $T = [,]$ for a Lie algebra, then d_T is the well known differential from Lie algebra cohomology. But more general T can be used. This operator clearly increases degree by one, and satisfies the Leibnitz rule:

$$d_T(\xi \wedge \eta) = (d_T\xi) \wedge \eta + (-1)^{\deg \xi} \xi \wedge (d_T\eta).$$

Consequently $d_T^2 = 0$.

Lemma 41. *Suppose that $G \subset \mathrm{GL}(V)$ is a Lie subgroup. Take $Q \in \mathfrak{g} \otimes V^*$ and let $T = \delta(Q) \in V \otimes \Lambda^2(V^*)$. (This δ is the Spencer cohomology differential.) Then the differential d_T vanishes on the G invariant forms $\Lambda^*(V^*)^G$.*

Proof. Let $\eta \in \Lambda^k (V^*)^G$ be a G invariant k-form. The G invariance

$$\eta (gv_1, \dots, gv_k) = \eta (v_1, \dots, v_k)$$

implies that for $A \in \mathfrak{g}$,

$$\sum_j \eta (v_1, \dots, Av_j, \dots v_k) = 0.$$

Therefore, if $T = \delta(A \otimes \xi)$, for some $A \in \mathfrak{g}$ and $\xi \in V^*$, then

$$\begin{aligned}
& d_T \eta (v_1, \dots, v_{k+1}) \\
& = \sum_{i<j} (-1)^{i+j} \eta (T (v_i, v_j), v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}) \\
& = \sum_{i<j} (-1)^{i+j} \eta (A (v_i) \xi (v_j) - A (v_j) \xi (v_i), v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}) \\
& = \sum_{i<j} (-1)^{i+j} \xi (v_j) \eta (A (v_i) v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}) \\
& - \sum_{i<j} (-1)^{i+j} \xi (v_i) \eta (A (v_j), v_1, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots v_{k+1}) \\
& = 0
\end{aligned}$$

Every $Q \in \mathfrak{g} \otimes V^*$ can be written as a sum of elements of the form $A \otimes \xi$, so that the result follows by linearity.

Exercise 12.3 The map

$$d : T \in V \otimes \Lambda^2 (V^*) \mapsto d_T \in \operatorname{Der} (\Lambda^* (V^*))$$

is an injective morphism of $\operatorname{GL}(V)$ representations.

Corollary 23. *The derivation d_T when applied to G invariant forms depends only on the $H^{0,2} (\mathfrak{g})$ class $[T]$ of T:*

$$d_{[T]} : \Lambda^* (V^*)^G \to \Lambda^* (V^*)$$

for $[T] \in H^{0,2} (\mathfrak{g})$. Note that the differential forms in the image will not in general be G invariant. The map $[T] \to d_{[T]}$ gives a G equivariant map

$$d : H^{0,2} (\mathfrak{g}) \to \operatorname{Lin} \left(\Lambda^* (V^*)^G, \Lambda^* (V^*) \right).$$

Proposition 47. *Suppose that $\eta \in \Lambda^* (V^*)$ is a k-form, and that $B \subset FM$ is a G-structure, with soldering form ω. Define the differential form*

$$\eta \circ \omega \in \Omega^k (B)$$

by

$$\eta \circ \omega\,(v_1, \dots, v_k) = \eta\,(\omega(v_1), \dots, \omega(v_k))\,.$$

Then

$$d\,(\eta \circ \omega) = \left(d_{[T]}\eta\right) \circ \omega$$

where $[T] : B \to H^{0,2}(\mathfrak{g})$ *is the torsion of* B*. If* η *is* G *invariant, then* $\eta \circ \omega$ *is the pull back from* M *of a unique* k*-form* $\eta_* \in \Omega^k(M)$*, so that*

$$d\eta_* = \left(d_{[T]}\eta\right)_*\,.$$

i.e. $d_{[T]}\eta \circ \omega$ *is also defined on* M*. Note that*

$$d_{[T]}\eta : B \to \Lambda^{k+1}(V^*)$$

is not in general G *invariant, but only* G *equivariant.*

Proof. Since $\eta \circ \omega$ is G invariant, we can take any k vectors v_i from a tangent space T_xM and pick some vectors w_i from some $T_{(x,u)}B$ so that w_i projects down to v_i, and look at the number

$$\eta \circ \omega\,(w_1, \dots, w_k)\,.$$

Because ω vanishes on the fibers, this number is independent of the choice of w_i vectors. By G invariance, it is independent of which point of B we project from. This defines

$$\eta_*\,(v_1, \dots, v_k) = \eta \circ \omega\,(w_1, \dots, w_k)\,.$$

Antisymmetry of η_* is obvious. Smoothness is easily shown by choosing a local section of $B \to M$.

The rest follows easily from the structure equations

$$d\omega = -\gamma \wedge \omega + \frac{1}{2} T\omega \wedge \omega.$$

A Lie subgroup $G \subset \mathrm{GL}(V)$ is called *admissable* if it is defined by forms, in the sense that every element of $\mathrm{GL}(V)$ leaving invariant every G invariant form must belong to G. For example, $SO(n)$ is not admissable, while $\mathrm{Sp}(2n, \mathbb{R})$ is. The group G is called *strongly admissable* if it is admissable, and the G representation homomorphism

$$d : H^{0,2}(\mathfrak{g}) \to \mathrm{Lin}\left(\Lambda^*(V^*)^G, \Lambda^*(V^*)\right)$$

is an injection (so that all of the torsion is determined by the torsion operator). The torsion of a strongly admissable group is representable by the tensor $d_{[T]}$. See [13] for an application of admissability. There is no classification of the admissable or strongly admissable subgroups of $\mathrm{GL}(V)$.

Question 80. There should be a general theory of invariant operators for invariantly defined vector bundles (i.e. associated vector bundles), which would enable me to ask when an invariant section satisfies a given invariant operator. For example, the harmonicity of the Kähler form on Kähler manifolds.

Question 81. How does the theory of torsion operators generalize when instead of $B \to H^{0,2}(\mathfrak{g})$ we map to something like $B \to U \to Y$ where $U \subset H^{0,2}(\mathfrak{g})$ is a G invariant submanifold, and Y is a G-space, and the map is equivariant? We need to understand this in dealing with torsion of variable type.

12.7 Order of osculation

Suppose that we have two G-structures, $B_0 \subset FM_0$ and $B_1 \subset FM_1$. and we wish to see if they can be brought to k-th order contact by some diffeomorphism.

Theorem 19. *Matching up G-structures to first order requires exactly matching up their torsions. More precisely, take $B_0 \subset FM_0$ and $B_1 \subset FM_1$ G-structures. The intrinsic torsions of B_0 and B_1 belong to the same G orbit in $H^{0,2}(\mathfrak{g})$ at any point $u_0 \in B_0$ (resp. B_1) above a point $x_0 \in M_0$ (resp. $x_1 \in M_1$) precisely when then there is a diffeomorphism $\phi : U_0 \subset M_0 \to U_1 \subset M_1$ between open subsets of M_0 and M_1 with $\phi(x_0) = x_1$ and which matches up B_0 to B_1 to first order (makes them tangent inside FM_0).*

Proof. Suppose (by using our G action) that at the points (x_0, u_0) and (x_1, u_1) the torsions agree:

$$[T_0] = [T_1] .$$

We can take local coordinates x on M_0 and y on M_1 and coordinates (x, u) on B_0 and (y, v) on B_0 so that the two points at which $[T_0] = [T_1]$ are $(0, I)$ and $(0, I)$. The G-structures are specified by maps

$$(x, u) \in B_0 \mapsto (x, uF(x)) \in FM$$
$$(y, v) \in B_1 \mapsto (x, vG(x)) \in FM$$

We can arrange that $F(0) = G(0) = I$. Then the torsions are

$$T_0 = F'(0) \text{ and } T_1 = G'(0).$$

The intrinsic torsions are

$$[T_0] = \delta T_0 = [F'(0)] = [T_1] = \delta T_0 = [G'(0)] .$$

These agree, so

$$\delta (F'(0) - G'(0)) = 0$$

or in other words

$$F'(0) - G'(0) = a$$

where $a \in \mathrm{Sym}^2(V^*) \otimes V$. In components,

$$\left(\frac{\partial F^i_j}{\partial x^k} - \frac{\partial G^i_j}{\partial y^k} \right) - \left(\frac{\partial F^i_k}{\partial x^j} - \frac{\partial G^i_k}{\partial y^j} \right) \bigg|_{x=y=0} = 0$$

so that

$$a^i_{jk} = \frac{\partial F^i_j}{\partial x^k}\bigg|_{x=0} - \frac{\partial G^i_j}{\partial y^k}\bigg|_{y=0}$$

is symmetric in its lower indices. We define a map

$$y = \phi(x) = x + \frac{1}{2} a(x, x)$$

or in indices

$$y^i = x^i + \frac{1}{2} a^i_{jk} x^j x^k$$

so that

$$\phi'(0) = I + ax.$$

Then

$$F\phi(x, h) = \left(\phi(x), h\phi'(x)^{-1}\right)$$

so that

$$\begin{aligned}
F\phi(x, uF(x)) &= \left(\phi(x), uF(x)\phi'(x)^{-1}\right) \\
&= \left(x + a(x, x), uF(x)(I + ax + \dots)^{-1}\right) \\
&= (x + a(x, x), u\left(F(0) + F'(0)x + \dots\right)(I - ax + \dots)) \\
&= (x + a(x, x), u\left(I + F'(0)x + \dots\right)(I - ax + \dots)) \\
&= (x + a(x, x), u\left(I + (F'(0) - a)x + \dots\right)) \\
&= (x + a(x, x), u\left(I + G'(0)x + \dots\right)) \\
&= (y, u\left(G(y) + \cdots\right))
\end{aligned}$$

where dots indicate terms of quadratic and higher order.

Conversely, since torsion is calculated with precisely one derivative, if the intrinsic torsions do not agree at a point, then it is clear that first order contact is impossible.

Question 82. Roughly, the order of osculation of G-structures is the order to which their torsions match. But this must include also the covariant derivatives. I would like a proof.

Question 83. Guillemin and Sternberg used the Levi counterexample to show that some tricky things are happening with infinite order osculation. The examples of Imre Patyi of torsion-free non-flat almost complex structures on Banach manifolds show similar problems. Torsion-free almost symplectic structures on Banach manifolds are always flat, as proven by Weinstein. He used Moser's homotopy method.

12.8 Jets of reductions

Consider a G-structure $B \subset FM$, where $G \subset \mathrm{GL}(V)$, and pick a closed subgroup $G_0 \subset G$. Sections of the bundle B/G_0 are precisely G_0-reductions of the given G-structure.

Example 97. If $B = FM$ then we can pick $G_0 = SO(p,q)$ and get $B/G_0 \to M$ the bundle whose sections are pseudo-Riemannian metrics of signature p, q. Such a bundle is canonically identified with the open subset of $\mathrm{Sym}^2(TM)$ consisting of the quadratic forms of that signature.

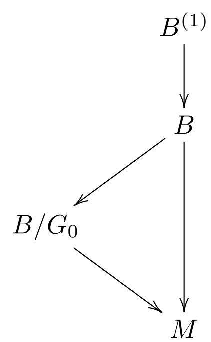
Example 98. Again we take $B = FM$, but let $G_0 = \mathrm{Sp}(n, \mathbb{R})$ and B/G_0 is the bundle of nondegenerate 2-forms in the tangent spaces of M, a subbundle of $\Omega^2(M)$.

Example 99. Take $B = FM$ and G_0 the group of symmetries of a k-plane in V, so that

$$B/G_0 = \mathrm{Gr}(k, TM)$$

is the Grassmann manifold of k-planes in the tangent spaces of M.

We have a diagram



where we map $u \in B$ to $uG_0 \in B/G_0$.

12.8.1 The tower of prolongations

Pick a section S for the 1-torsion, as in section 5.1 on page 37, and use it to prolong $B \to M$. The points of $B^{(1)}$ consist in the triples (x, u, w) so that $u \in B$ and $u : T_x M \to V$ and

$$w : T_u B \to \mathfrak{g}$$

is a pseudoconnection 1-form with torsion belonging to our section. Let us label our maps in our diagram by writing $[X \to Y]$ as the name of the map taking a manifold X to a manifold Y. Then we have

$$[B \to B/G_0]'(u) : T_u B \to T_{uG_0} B/G_0$$

a linear map with kernel given by the tangent space to the G_0 orbit through u in B, which we can think of as $\mathfrak{g}_0$, identifying a vertical vector with an element of the Lie algebra. Composing,

$$[\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_0] \circ w : T_u B \to \mathfrak{g}/\mathfrak{g}_0$$

is a linear map with kernel $\mathfrak{g}_0$. Therefore this map is defined on $T_{uG_0} B/G_0$:

$$\bar{w} = [\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_0] \circ w : T_{uG_0} B/G_0 \to \mathfrak{g}/\mathfrak{g}_0$$

and this associates to each element $w \in B^{(1)}$ a linear map $\bar{w}$ on a tangent space of B/G_0. We can put this together with u to create a $V \oplus \mathfrak{g}/\mathfrak{g}_0$ valued coframe on B/G_0:

$$(x, u, w) \in B^{(1)} \mapsto u \circ [B/G_0 \to M]' (uG_0) \oplus \bar{w} \in F(B/G_0).$$

Two such coframes (x_0, u_0, w_0) and (x_1, u_1, w_1) will map to the same coframe on B/G_0 precisely when

$$x_0 = x_1, u_0 = u_1, \text{ and } w_0 - w_1 = fu\,[B \to M]' (u)$$

where $f \in \mathfrak{g}_0^{(1)}$. Therefore we have bundles as illustrated in figure 12.1 on the next page. On B/G_0, we might have no invariantly defined differential forms. On $B^{(1)}/\mathfrak{g}_0^{(1)}$, we have ω (a 1-form valued in V) and $\bar{\gamma}$ (a 1-form valued in $\mathfrak{g}/\mathfrak{g}_0$).

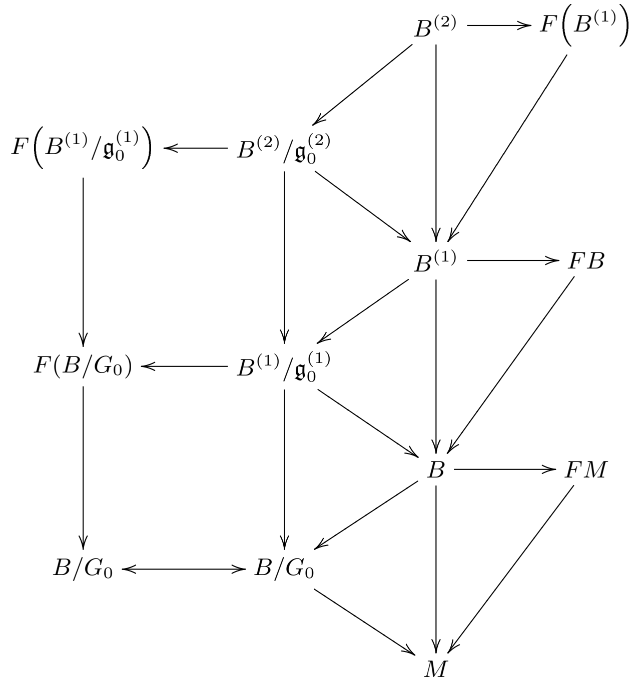


Fig. 12.1. The tower of prolongations

13

Maps

We have objects but no arrows. Suppose that M_0 carries a G_0-structure $B_0 \subset FM_0$, and M_1 a G_1-structure $B_1 \subset FM_1$. Take a smooth map $\phi : M_0 \to M_1$. Any such map is an arrow in our category, but not every arrow is interesting. We can construct the bundle

$$B_\phi = B_0 \times_{M_0} \phi^* B_1 \to M_0.$$

On each tangent space of M_0 we have the linear map $\phi'(x) : T_x M_0 \to T_{\phi(x)} M_1$, and we can write it down in terms of frames from B_0 and B_1, i.e. on B_ϕ we have points looking like

$$(x_0, u_0, u_1) \in B_\phi$$

with

$$x_0 \in M_0,\ u_0 : T_{x_0} M_0 \to V_0,\ u_1 : T_{\phi(x_0)} M_1 \to V_1$$

so that

$$u_1 \circ \phi'(x_0) \circ u_0^{-1} : V_0 \to V_1$$

is a function on B_ϕ

$$\lambda = u_1 \circ \phi' \circ u_0^{-1} : B_\phi \to \operatorname{Lin}(V_0, V_1).$$

Under the action of $G_0 \times G_1$ we have

$$r^*_{(g_0, g_1)} \lambda = g_1^{-1} \lambda g_0.$$

We find the equation

$$\omega_1 = \lambda \omega_0$$

on B_ϕ for the soldering forms ω_0 of B_0 and ω_1 of B_1. We can now apply reduction, as usual, as long as this map has constant type, where the reduction is going to carried out using a section $S \subset \operatorname{Lin}(V_0, V_1)$. (In terms of the reduction theory from chapter 5, this makes

$$\phi = \operatorname{id} : X = \operatorname{Lin}(V_0, V_1) \to Y = \operatorname{Lin}(V_0, V_1)$$

which is a little strange.) In other words, we want $S \subset \operatorname{Lin}(V_0, V_1)$ an immersed submanifold transverse to the $G_0 \times G_1$ orbits it intersects, and with stabilizer $H \subset G_0 \times G_1$. If λ is conjugate to an element of S, at each point of B_ϕ, then we can carry out reduction. This form of reduction is not recursive. The result is not generally an embedded G-structure, but it is a G-structure.

We also have torsion of the bundles B_0 and B_1 available:

$$d\omega_0 = -\gamma_0 \wedge \omega_0 + \frac{1}{2} T_0 \omega_0 \wedge \omega_0$$
$$d\omega_1 = -\gamma_1 \wedge \omega_1 + \frac{1}{2} T_1 \omega_1 \wedge \omega_1$$

Assuming that B_0 and B_1 were already reduced as far as possible, it may still be possible to first reduce B_ϕ by using λ and then reduce further by using the torsion.

Example 100 (Symmetry group orbits). Suppose that $\pi_1 : B_1 \to M_1$ is a principal G_1 bundle, and $\alpha_1 : B_1 \to FM_1$ makes it into a G_1-structure(not necessarily embedded), for some representation V_1 of G_1 (so FM_1 is the bundle of V_1-valued frames). Let H be a Lie group of symmetries of the G_1-structure, i.e. H acts on M_1 and B_1 commuting with π_1 and α_1. Let $u_1 \in B_1$ be any point, and let $m_1 = \pi_1 (u_1)$. Let $G_0 = H_{m_1}$ be the stabilizer of m_1 in H, and $M_0 = H/H_{m_1}$ be the H-orbit of m_1. Let $\phi : M_0 \to M_1$ be the inclusion of the H-orbit. Let $B_0 = H$, and map $B_0 \to B_1$ by $h \mapsto hu_1$. Let V_0 be the image under $u_1 : T_{m_1} M_1 \to V_1$ of the tangent space $T_{m_1} H/H_{m_1}$ to the H orbit of m_1 in M_1. Let G_0 act on V_0 by

$$h \in G_0, v_0 \in V_0 \mapsto u_1 h' (m_1) u_1^{-1} v_0 \in V_0.$$

Map $B_0 \to FM_0$ by

$$h \mapsto \alpha_1 (hu_1)|_{T_{\pi_1(hm_1)} M_0} .$$

Then B_0 is a G_0-structure over M_0, and on B_ϕ, we have $\lambda = 1 : V_0 \to V_1$ the inclusion map.

13.1 Example: mapping real surfaces to complex surfaces

Consider the example of a map from a real surface to a complex surface. The map λ takes $\mathbb{R}^2$ to $\mathbb{C}^2$, and has rank 0,1 or 2. Let us assume constant type.

13.1.1 Rank 0

If the rank is 0 everywhere, then $\lambda = 0$ and the map takes the real surface to a point, so such maps are classified by the points of the complex surface.

13.1.2 Rank 1

If the rank is 1, then the image is a real line, which we can easily arrange by complex linear transformation to be any real line we like, and just as easily arrange

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This reduces the group down from $G_0 \times G_1 = \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{C})$ to the group G_2 of pairs of matrices of the form

$$g_0 = \begin{pmatrix} a_0 & 0 \\ b_0 & c_0 \end{pmatrix}, \quad g_1 = \begin{pmatrix} a_0 & b_1 \\ 0 & c_1 \end{pmatrix}$$

with none of a_0, c_0, c_1 being zero. As equations on differential forms, we can write the structure equations of our surfaces as

$$\begin{aligned} d \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} &= - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 \\ \gamma^2_1 & \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \\ d \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} &= - \begin{pmatrix} \sigma^1_1 & \sigma^1_2 \\ \sigma^2_1 & \sigma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \end{aligned}$$

and we have arranged a subbundle $B_2 \subset B_\phi$ on which

$$\begin{aligned} \eta^1 &= \omega^1 \\ \eta^2 &= 0. \end{aligned}$$

Differentiating gives

$$0 = d\eta^2 = -\sigma^2_1 \wedge \eta^1 = -\sigma^2_1 \wedge \omega^1$$

so that

$$\sigma^2_1 = t\eta^1 = t\omega^1$$

for some function $t : B_2 \to \mathbb{C}$. We can absorb such torsion into σ^2_1, so we obtain the equations

$$\eta^1 - \omega^1 = \eta^2 = \sigma^2_1 = 0.$$

One can easily check, via the Cartan–Kähler theorem, that these equations are involutive. Let us carry this out, without explaining the Cartan–Kähler theorem itself—for a detailed explanation, see Bryant et al. [12] or Ivey & Landsberg [47] or Cartan [25]. The ideal is $\left(\eta^1 - \omega^1, \eta^2, \sigma^2_1\right)$ for which the tableau is

$$d \begin{pmatrix} \eta^1 - \omega^1 \\ \eta^2 \\ \sigma^2_1 \end{pmatrix} = - \begin{pmatrix} \sigma^1_1 - \gamma^1_1 & -\gamma^1_2 \\ 0 & 0 \\ \xi^2_{11} & 0 \end{pmatrix}$$

where the structure equations on the prolongation of B_2 are

$$d\sigma^i_j = -\sigma^i_k \wedge \sigma^k_j + \xi^i_{jk} \wedge \eta^k$$

with all 1-forms being complex linear. The 1-forms in the first column of the tableau matrix are complex, while in the second column they are real, and therefore the characters are $s_1 = 4, s_2 = 1$. We can easily compute the prolongation of these structure equations to find the Cartan integer: $s = 6 = s_1 + 2s_2$, so the equations are in involution, with general solution depending on $s_2 = 1$ arbitrary function of 2 variables. Of course, the general solution is given by taking a real curve in $\mathbb{C}^2$ (3 functions of 1 variable) and then mapping the real surface to it (1 function of 2 variables).

13.1.3 Rank 2

The map λ is now an injection, and so the real surface is immersed into the complex surface. Since we can carry out arbitrary linear transformations from $G_0 = \mathrm{GL}(2, \mathbb{R})$, our map λ is entirely determined up to G_0 action by its image. But then its image is either a complex line (i.e. invariant under $\sqrt{-1}$) or else a totally real 2-plane (i.e. containing no complex line). With some linear algebra, we find that all complex lines are isomorphic under $G_1 = \mathrm{GL}(2, \mathbb{C})$ action, and all totally real 2-planes are also isomorphic. So there are two cases of constant type: either we can arrange

$$\lambda = \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}$$

(the complex line case) or else

$$\lambda = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

(the totally real case). In each case, the structure equations are immediately brought to involution. The tangent spaces of an immersed real surface are complex lines precisely when it is a complex curve. One easily finds that a complex curve depends on 2 real functions of 1 real variable (giving one holomorphic relation between two complex variables only requires describing how the complex value of one depends on the real value of another). However, the Cartan count shows that we have 2 functions of 2 variables worth of solutions, because we require also a map of the given real surface into the complex curve.

On the other hand, a totally real surface is just a generic map, so depends on 4 real functions of 2 real variables.

13.2 Example: Legendre fibrations of contact 3-manifolds

Let M^3 be a contact 3-manifold foliated by Legendre curves, i.e. curves tangent to the contact 2-planes.

13.2.1 Example

For example, take Q any surface, and let $\mathbb{P}T^*Q$ be the bundle of lines in the tangent planes of Q. Then in local coordinates x, y on Q, each line on which $dx \neq 0$ is specified by parameters x, y, p according to an equation $dy - p\,dx = 0$ in $T_{(x,y)}Q$. So x, y, p are local coordinates on $\mathbb{P}T^*Q$. The contact planes in those coordinates have the equation $dy - p\,dx = 0$ (which means something different from what it meant the first time we wrote it). The Legendre curves are $dx = dy = 0$.

13.2.2 Structure equations

Let us return to the general case. Clearly M has a G_0-structure, where G_0 is the group of linear transformations of $\mathbb{R}^3$ preserving a 2-plane and a line in that 2-plane, i.e. matrices of the form

$$\begin{pmatrix} g_1^1 & g_2^1 & g_3^1 \\ 0 & g_2^2 & g_3^2 \\ 0 & 0 & g_3^3 \end{pmatrix}.$$

In other words, M has a flag geometry. Following the discussion of flag geometries in 3-manifolds from section 4.3, we see that the structure equations of the associated G_0-structure $B_0 \to M$ look like

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma_1^1 & \gamma_2^1 & \gamma_3^1 \\ 0 & \gamma_2^2 & \gamma_3^2 \\ 0 & 0 & \gamma_3^3 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t\omega^{12} \end{pmatrix}$$

(with $\omega^{12} = \omega^1 \wedge \omega^2$).

Exercise 13.1 The distinguished 2-plane field is a contact structure just when $t \neq 0$.

Recall that in studying flag geometry, we showed that t varies under the representation of the structure group given by

$$r_g^* t = \frac{g_1^1 g_2^2}{g_3^3} t.$$

Therefore we can arrange that $t = 1$ on a subbundle B, which is a G-structure, where G consists in the matrices of the form

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 \\ 0 & g^2_2 & g^2_3 \\ 0 & 0 & g^1_1 g^2_2 \end{pmatrix}.$$

The structure equations are now

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^1_1 + \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t_1 \omega^{13} + t_2 \omega^{23} + \omega^{12} \end{pmatrix}.$$

We can absorb t_1 by changing the choice of γ^1_1, and similarly absorb t_2 by choice of γ^2_2. Now

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^1_1 + \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega^{12} \end{pmatrix}.$$

At this stage, we carry out the Cartan count: $s_1 = 3$ (independent γ's in the last column), $s_2 = 2$ (independent γ's, not already appearing in the previous step, in the second column), $s_3 = 0$ (nothing more independent of the γ's we have already used from the previous columns). We calculate the prolongation as follows: try setting up some $\delta\gamma^i_j$ so that it belongs to the Lie algebra and so that $\delta\gamma^i_j \wedge \omega^j = 0$ (see appendix A). Writing out these equations

$$\delta \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^1_1 + \gamma^2_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = 0,$$

we find (by repeated use of Cartan's lemma) that

$$\delta \begin{pmatrix} \gamma^1_1 & \gamma^1_2 & \gamma^1_3 \\ 0 & \gamma^2_2 & \gamma^2_3 \\ 0 & 0 & \gamma^1_1 + \gamma^2_2 \end{pmatrix} = \begin{pmatrix} a^1_{12}\omega^2 + a^1_{13}\omega^3 & a^1_{12}\omega^1 + a^1_{22}\omega^2 + a^1_{23}\omega^3 & a^1_{13}\omega^1 + a^1_{23}\omega^2 + a^1_{33}\omega^3 \\ 0 & -a^1_{12}\omega^2 + a^2_{23}\omega^3 & a^2_{23}\omega^2 + a^2_{33}\omega^3 \\ 0 & 0 & \left(a^1_{13} + a^2_{23}\right)\omega^3 \end{pmatrix}.$$

These a^i_{jk} coefficients parameterize the prolongation $\mathfrak{g}^{(1)}$, which therefore has $s = 7$ dimensions. The Cartan test:

$$\begin{aligned} s_1 + 2s_2 + 3s_3 &= 3 + 2 \cdot 2 + 3 \cdot 0 \\ &= 7 \\ &= s. \end{aligned}$$

Therefore the structure equations are in involution, and if the contact structure and foliation are analytic, then they have symmetry pseudogroup acting transitively, with the general symmetry depending on $s_2 = 2$ functions of 2 (from $s_{\underline{2}}$) variables.

13.2.3 Constructing an equivalence with the hyperplane bundle

Suppose that the Legendre foliation of M is a fiber bundle, $q : M \to Q$, over a surface. Given any point $m \in M$, we can attach to it a line Θ_m in $T_q Q$, by sending the contact plane through $q'(m)$. Since the fibers of $q : M \to Q$ are tangent to the contact 2-planes, Θ_m is a line. So $\Theta : M \to \mathbb{P}T^*Q$ is a smooth map, and clearly a fiber bundle map over Q.

Proposition 48. *$\Theta : M \to \mathbb{P}T^*Q$ is an equivalence.*

Proof. Let $B \to M$ be the bundle of our G-structure on M. Let $\tilde{B} \to \mathbb{P}T^*Q$ be the corresponding G-structure on $\mathbb{P}T^*Q$. Suppose that its soldering form is written $\tilde{\omega}$, and $\tilde{\gamma}$ is a choice of pseudoconnection form for it. Let $B_0 \to M$ be the bundle

$$B_0 = B \times_M \Theta^* \tilde{B}.$$

On B_0 we have all of the forms $\omega, \gamma, \tilde{\omega}, \tilde{\gamma}$ defined. Both ω and $\tilde{\omega}$ are semibasic for $B_0 \to M$. Indeed,

$$\tilde{\omega} = \lambda \omega$$

where

$$\lambda(u, \tilde{u}) = \tilde{u} \Theta'(m) u^{-1}.$$

The 1-forms ω^2, ω^3 are semibasic for the map to Q, as are $\tilde{\omega}^2, \tilde{\omega}^3$, and indeed either pair of 1-forms drops to a basis for the tangent space of Q, so they must be invertible multiples of each other.

$$\lambda = \begin{pmatrix} \lambda^1_1 & \lambda^1_2 & \lambda^1_3 \\ 0 & \lambda^2_2 & \lambda^2_3 \\ 0 & \lambda^3_2 & \lambda^3_3 \end{pmatrix},$$

with invertible 2×2 in the lower right corner. Moreover, ω^3 drops to a 1-form on the tangent space of Q which is perpendicular to the line Θ_m, and $\tilde{\omega}^3$ must satisfy the same condition, so they are invertible multiples of each other:

$$\lambda = \begin{pmatrix} \lambda^1_1 & \lambda^1_2 & \lambda^1_3 \\ 0 & \lambda^2_2 & \lambda^2_3 \\ 0 & 0 & \lambda^3_3 \end{pmatrix}$$

with the lower right 2×2 corner invertible. The bundle B_0 has a $G \times G$ action. We can now form a subbundle $B_1 \subset B_0$ on which the lower corner of λ is normalized to

$$\lambda = \begin{pmatrix} \lambda^1_1 & \lambda^1_2 & \lambda^1_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 13.2 What is the structure group $G_1 \subset G \times G$ of the principal bundle B_1?

Calculate

$$d\omega^3 \wedge \omega^3 = \omega^1 \wedge \omega^2 \wedge \omega^3$$

and similarly

$$\begin{aligned} d\tilde{\omega}^3 \wedge \tilde{\omega}^3 &= \tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \tilde{\omega}^3 \\ &= \lambda_1^1 \omega^1 \wedge \omega^2 \wedge \omega^3. \end{aligned}$$

But $\tilde{\omega}^3 = \omega^3$ on B_1, so their exterior derivatives must be equal, so $\lambda_1^1 = 1$ and we find λ invertible.

Exercise 13.3 We can now arrange on a principal G-subbundle $B_2 \subset B_1$ that $\lambda = 1_3$ is the 3×3 identity matrix.

Consequently, $\Theta : M \to \mathbb{P}T^*Q$ is a local equivalence.

Proposition 49. *Every Legendre fibration of a contact 3-manifold is locally equivalent to the bundle of tangent lines of a surface.*

Proposition 50. *Every holomorphic Legendre fibration of a holomorphic contact 3-fold with rational curves as fibers is biholomorphically contactomorphic to the bundle of complex tangent lines of a complex surface.*

Proof. The same calculations work in the holomorphic category, but then the rational curves, i.e. Riemann spheres, are simply connected, so the local diffeomorphism to the fibers of $\mathbb{P}T^*Q$ must be a diffeomorphism.

Part IV

Dessert

13.3 Conclusions and suggestions

The equivalence method provides an embarrasment of riches: a host of differential invariants, computed with (hopefully elementary) representation theory. It is surprising that one can find invariants of geometric structures without integration or solving differential equations, but for G-structures the method accomplishes just that. But the equivalence method does not interpret the invariants. It is not a royal road to geometry. For example, we still don't understand the I or J invariants of Finsler surfaces. Robert Bryant suggests that the method should only be applied to problems where one has a specific geometric question in mind, not just as a means of mindlessly generating differential invariants. As well, it helps to look carefully at how the invariants and the 1-forms move as we travel up the fibers of the G-structure. Perhaps we should always try to relate the results of the equivalence method to local coordinate expressions, so that we can see how to calculate these invariants explicitly. Finding all of the homogeneous examples can be helpful. It is also very useful to look for combinations of soldering and pseudoconnection 1-forms whose differentials vanish modulo those 1-forms, to apply the Frobenius theorem to construct invariant foliations.

There is a certain fiction about the equivalence method: we pretend that we have a manifold in hand, with a G-structure, perhaps explicitly written in local coordinates. We pretend that we want to identify it, i.e. to tell it apart from others. What we really do is to examine all G-structures (on all manifolds) which satisfy some geometric condition, usually a local condition, which we translate into information about the torsion. The problems we might hope to solve include, for instance, finding a class of partial differential equations whose local solutions are somehow identified (preferably quite directly) with the local G-structures satisfying our condition. At best we might hope to exhibit a global construction of all solutions. This is extremely rare; we don't even know what all of the inextendable (or even compact) flat structures are. More often we discover local invariants of G-structures, and try to interpret them geometrically, as with the Riemann curvature tensor.

13.4 Things I don't know about

1. G-structures on supermanifolds, see [59].
2. There is a lot known, particularly from the work of Guillemin and Sternberg, about the flatness of G-structures which are torsion free to all prolongations. The Lewy counterexample arose in this context.
3. Do any Lie group decomposition theorems help to understand the space of all reductions from G to $H \subset G$?
4. Calculus of variations for G-structures, or H-reductions of a given G-structure.

5. Generic behaviour of G-structures, and of those which satisfy some torsion assumption (Thom–Boardman theory).
6. Families of G-structures might occur on families of manifolds. In this regard, I need to revisit the notion of deformation of G-structures. Perhaps articles of Phillip Griffiths?
7. What does the theory of geodesics look like on higher order structures? Consider conformal geometry and CR geometry.

13.5 Pseudogroups

Question 84. I should perhaps write about Olver's and Fel's ideas, and Kamran's, Maurer-Cartan forms, variations, classification of primitives.

Question 85. Consider the pseudogroup of birational maps of a complex algebraic variety. In general, this is not a Lie pseudogroup, since there are no differential equations which force solutions to be birationally defined (as far as I know). Indeed, on complex projective spaces this is not a Lie pseudogroup. But the birational group of complex projective space has a presentation parameterized by finite dimensional manifolds of generators and relations: the linear and quadratic transformations generate, and the relations are finite dimensional. There might be something to this idea.

13.6 Why frame bundles instead of jet bundles

Jet bundles allow us to encode arbitrary systems of partial differential equations as submanifolds. The method of equivalence only allows us to encode geometric information that has a kind of "constant type", i.e. encoding geometry into a bundle of coframes. But jet bundles do not have an obvious choice of natural differential operator, or natural choices of objects to which one may apply such an operator. Bundles of frames have a soldering 1-form, and d is a diffeomorphism invariant differential operator.

Part V

Apéritifs

A

Appendix on Spencer cohomology

The material of this appendix is explained in greater detail by Ivey and Landsberg [47] and by Bryant et. al. [12].

A.1 Definition of tableaux and their prolongations

Suppose that V, W are vector spaces over a field k. Take $A \subset \operatorname{Lin}(V, W)$ any linear subspace of the linear maps from V to W. We call such A a tableau. Intuitively, k is $\mathbb{R}$ or $\mathbb{C}$ and we want to study the problem of finding a smooth map $f : V \to W$ whose derivative $f'(x) \in \operatorname{Lin}(V, W)$ at every point of V lies in the subspace A, $f'(x) \in A$. Relations are forced on the derivatives of all orders by such a condition. These relations are called the prolongations of A.

Definition 42. *The q-th prolongation of $A \subset \operatorname{Lin}(V, W)$, written $A^{(q)}$, is defined by*

$$A^{(0)} = A$$

and

$$A^{(q)} = \left\{ p \in \operatorname{Sym}^{q+1}(V^*) \otimes W \,\middle|\, p(v, \cdot, \dots, \cdot) \in A^{(q-1)}, \text{ any } v \in V \right\}$$

i.e. the q-th prolongation consists of the W valued degree $q + 1$ polynomials p (also thought of as symmetric multilinear maps) so that when we plug a vector $v \in V$ into one of the $q + 1$ slots (i.e. differentiate in the direction v) the polynomial given by the remaining $q - 1$ slots belongs to the prolongation of next lowest order.

Clearly if a smooth map $f : V \to W$ satisfies $f'(x) \in A$ for any x, then its Taylor expansion will consist of multilinear maps from the various prolongations.

Definition 43. *A tableau A is of* finite type *if only finitely many prolongations $A^{(k)}$ are nonzero.*

Consequently, the maps $f : V \to W$ satisfying $f'(x) \in A$ for all x will be polynomials of degree at most $p+1$ where $A^{(p)}$ is the last nonzero prolongation. An infinite type tableau has formal power series solutions with infinitely many nonvanishing terms.

A.2 Calculation

How do we calculate the first prolongation? Take linear coordinates on V and W, and think of A as a family of matrices. For example, if V and W are both 2 dimensional, then A is a family of 2×2 matrices. For example, suppose that A is the set of 2×2 matrices

$$\begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ \gamma_1^2 & \gamma_2^2 \end{pmatrix}$$

(where now these γ_j^i are real numbers), so that $\gamma_1^1 = 2\gamma_2^2$. We can write it as a single 2×2 matrix

$$\begin{pmatrix} 2\gamma_2^2 & \gamma_2^1 \\ \gamma_1^2 & \gamma_2^2 \end{pmatrix},$$

and think of $\gamma_2^1, \gamma_1^2, \gamma_2^2$ just as free variables. Lets write ω^1, ω^2 for the standard basis of 1-forms on V. Elements of the prolongation are maps $p : V \to A$ with the obvious symmetry, $p(v_0)\, v_1 = p(v_1)\, v_0$. If we write

$$p(v_0) = \begin{pmatrix} 2\gamma_2^2(v_0) & \gamma_2^1(v_0) \\ \gamma_1^2(v_0) & \gamma_2^2(v_0) \end{pmatrix}$$

then the γ_j^i are thought of as 1-forms γ_j^i on V. The symmetry condition on $p(v_0)\, v_1$ is just

$$\begin{pmatrix} 2\gamma_2^2(v_0) & \gamma_2^1(v_0) \\ \gamma_1^2(v_0) & \gamma_2^2(v_0) \end{pmatrix} v_1 = \begin{pmatrix} 2\gamma_2^2(v_1) & \gamma_2^1(v_1) \\ \gamma_1^2(v_1) & \gamma_2^2(v_1) \end{pmatrix} v_0$$

which we can write, by thinking of the γ_j^i as 1-forms on V, as

$$\begin{pmatrix} 2\gamma_2^2 & \gamma_2^1 \\ \gamma_1^2 & \gamma_2^2 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By Cartan's lemma, this forces each γ_j^i to be of the form $\gamma_j^i = a_{jk}^i \omega^k$, and plugging that in above gives

$$\begin{aligned} 2a_{22}^2 &= a_{21}^1 \\ a_{12}^2 &= a_{21}^2 \end{aligned}$$

so that

$$\begin{pmatrix} \gamma^1_2 \\ \gamma^2_1 \\ \gamma^2_2 \end{pmatrix} = \begin{pmatrix} a^1_{21}\omega^1 + a^1_{22}\omega^2 \\ a^2_{11}\omega^1 + a^2_{12}\omega^2 \\ a^2_{12}\omega^1 + \frac{1}{2}a^1_{21}\omega^2 \end{pmatrix}$$

giving 4 independent dimensions. The a^i_{jk} appearing here are linear coordinates on $A^{(1)}$, which is 4 dimensional.

A.3 Quotienting out a subspace

Definition 44. *Suppose that $A \subset \operatorname{Lin}(V, W)$ and that we have a subspace $U \subset V$. We define $A^{(q)}/U$ to be the polynomials in $A^{(q)}$ which are constant in the U directions, i.e.*

$$A^{(q)}/U := \left\{ p \in A^{(q)} \;\middle|\; p(v, \cdot, \dots, \cdot) = 0, \text{ any } v \in U \right\}$$

Note that

$$(A/U)^{(q)} := A^{(q)}/U$$

If $F = \{U_i\}$ is a flag in V,

$$0 = U_0 \subset U_1 \subset U_2 \cdots \subset U_n = V$$

we define the characters s_i of the pair (A, F) to be

$$s_k = \dim A/U_{k-1} - \dim A/U_k$$

(how much more of the dimensions of A you see by probing the k directions of U_k instead of just the $k-1$ directions of U_{k-1}). We define the characters of A to be those which occur for the generic flag (which are those for which the character s_1 is maximal, and for which s_2 is maximal subject to the value of s_1, etc.). We define $s = \dim A^{(1)}$.

A.4 Calculating characters

To calculate the characters of a tableau, we write down the tableau in linear coordinates, for example A might be the set of matrices of the form

$$\begin{pmatrix} 2\gamma^2_2 & \gamma^1_2 \\ \gamma^2_1 & \gamma^2_2 \end{pmatrix}.$$

Then we calculate the number s_1 of independent 1-forms in the first column. We calculate the number s_2 of independent 1-forms in the second column, modulo the ones in the first column. We continue in this manner. This determines characters of the flag of subspaces given by U_1 being the x^1 axis in V,

U_2 the x_1, x_2-plane, etc. in these coordinates. Finally, the characters of the tableau are those for which s_1 is maximal, s_2 maximal subject to the value of s_1, etc. For example, in our case we find $s_1 = 2, s_2 = 1$, since there are $s_2 = 2$ independent 1-forms (γ_2^2, γ_1^2) in the first column, and $s_2 = 1$ 1-forms (γ_2^1) in the second independent of those already considered.

Exercise A.1 Why is it that $s_1 = 2, s_2 = 1$ gives the characters of the tableau A, i.e. why is $s_1 = 2$ maximal, and $s_2 = 1$ maximal subject to $s_1 = 2$?

Question 86. Explain borrowing from later columns.

A.5 Elementary theory of characters

Proposition 51. *If* $\dim V = n$*, then*

$$s \leq s_1 + 2s_2 + \cdots + ns_n$$

i.e.

$$\dim A^{(1)} \leq \dim A/U_0 + \cdots + \dim A/U_{n-1}$$

Proof. Taking a pair of subspaces, say $U \subset U^+ \subset V$, it is clear that polynomials constant in U^+ directions are constant in U directions as well. Conversely, polynomials constant in U directions are constant in all of the U^+ directions precisely when they give 0 when differentiated in the U^+/U directions. Taking a basis $x_1, \ldots, x_k$ for U^+/U we have

$$0 \longrightarrow A^{(1)}/U^+ \longrightarrow A^{(1)}/U \longrightarrow \bigoplus^k A/U$$

where the last map is given by differentiating in each of the k directions x_j. This gives the inequality

$$\dim A^{(1)}/U - \dim A^{(1)}/U^+ \leq k \cdot \dim A/U$$

Applying this to a flag gives the result.

A.6 Involutivity

Definition 45. *If for* $A \subset \operatorname{Lin}(V, W)$ *the inequality of the last proposition is an equality we say that* A *is* involutive.

The failure of involutivity, reconsidering the proof of the last proposition, is exactly that differentiating higher order polynomials in our prolongations might not give rise to every polynomial in a lower order prolongation. Otherwise put, we may not be able to integrate a low order polynomial satisfying our equation $f'(x) \in A$ to produce a higher order one. This is related to the difficulty in trying to construct solutions order by order: we are seeing higher order obstructions. Since we measure involutivity using generic flags, this is related to order by order solving differential equations using noncharacteristic initial data.

Proposition 52. *The prolongation of an involutive tableau is involutive.*

A.7 Spencer cohomology

To measure our obstructions, we construct a cohomology.

Definition 46. *Let*

$$C^{p,q}(A) := A^{(p-1)} \otimes \Lambda^q \left(V^*\right)$$

if $p \geq 1$ *and* $q \geq 0$*, and*

$$C^{0,q}(A) := W \otimes \Lambda^q \left(V^*\right)$$

for $q \geq 0$*.*

The elements of $C^{p,q}(A)$ should be thought of as differential forms on V of degree q with coefficients being polynomials of degree p. Thus the differential d is just the usual exterior derivative on differential forms. We will write it as δ to avoid confusion. The resulting cohomology is called *Spencer cohomology*:

$$H^{p,q}\left(A\right) := \frac{\ker\ \delta : C^{p,q}(A) \to C^{p-1,q+1}(A)}{\operatorname{im}\ \delta : C^{p+1,q-1}(A) \to C^{p,q}(A)}$$

By the Poincaré lemma for homogeneous polynomial forms (proven using the usual proof of the usual Poincaré lemma, by integration) we see that the Spencer cohomology of the tableau $A = \operatorname{Lin}\left(V, W\right)$ vanishes:

$$H^{*,*}\left(\operatorname{Lin}\left(V, W\right)\right) = 0.$$

Theorem 20. *Spencer cohomology vanishes precisely for involutive tableaux.*

Theorem 21. *Every tableau has an involutive prolongation.*

Bryant et. al. [12] prove these results.

Question 87. What does relative Spencer cohomology describe? Perhaps something like, if you know how to solve one system of equations, can you solve another one? What are the morphisms, what is excision, dimension, Meyer–Vietoris?

Question 88. How do we calculate Spencer cohomology using a tableau? How do we apply it to equivalence?

B

Appendix on Signs

B.1 Linear Algebra

$$
\begin{aligned}
v_1 \wedge \cdots \wedge v_p &= \sum_{\sigma \in \operatorname{Sym}(p)} \operatorname{sgn} \sigma \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)} \\
\left\langle \partial_1 \wedge \cdots \wedge \partial_n, dx^1 \wedge \cdots \wedge dx^n \right\rangle &= 1 \\
\left\langle v_1 \wedge \cdots \wedge v_n, \xi^1 \wedge \cdots \wedge \xi^n \right\rangle &= \det \left(\left\langle v_i, \xi^j \right\rangle \right) \\
&= \xi^1 \wedge \cdots \wedge \xi^p \left(v_1, \dots, v_p \right) \\
v \lrcorner \, \xi^1 \wedge \cdots \wedge \xi^p &= \sum_{j=1}^{p} (-1)^{j+1} \xi^j(v) \xi^1 \wedge \cdots \wedge \widehat{\xi^j} \wedge \cdots \wedge \xi^p
\end{aligned}
$$

If α and β are exterior forms valued in an algebra A (which could be a Lie algebra, not necessarily associative), of degrees a and b respectively, then

$$
v_1 \wedge \cdots \wedge v_{a+b} \lrcorner \alpha \wedge \beta = \sum_{\sigma \in \operatorname{Sym}(a+b)} \frac{\operatorname{sgn} \sigma}{a!b!} \alpha \left(v_{\sigma(1)}, \dots, v_{\sigma(a)} \right) \beta \left(v_{\sigma(a+1)}, \dots, v_{\sigma(a+b)} \right)
$$

In case of a Lie algebra, we write $\alpha \wedge \beta$ as $[\alpha, \beta]$. In particular, if we have a Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(N, \mathbb{R})$, and α and β both have degree a, then

$$
\rho(\alpha) \wedge \rho(\beta) + (-1)^{ab+1} \rho(\beta) \wedge \rho(\alpha) = \rho \left([\alpha, \beta] \right).
$$

Moreover, the Jacobi identity tells us that

$$
[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + [\beta, [\alpha, \gamma]],
$$

for differential forms of all degrees. Suppose that A is an associative algebra, and for $x \in A$ write L_x for left multiplication by x and R_x for right multiplication by x.

$$\begin{aligned} L_\alpha \wedge \beta &= \alpha \wedge \beta \\ R_\alpha \wedge \beta &= (-1)^{ab}\, \beta \wedge \alpha \\ L_\alpha \wedge L_\beta &= L_{\alpha\wedge\beta} \\ L_\alpha \wedge R_\beta &= (-1)^{ab}\, R_\beta \wedge L_\alpha \\ R_\alpha \wedge R_\beta &= (-1)^{ab}\, R_{\beta\wedge\alpha} \end{aligned}$$

and these wedge products are also associative.

Lemma 42 (Cartan–Poincaré). *If ω^j are linearly independent 1-forms and Ω_j are any 1-forms with*

$$\sum_j \Omega_j \wedge \omega^j = 0$$

then there are real numbers $a_{ij} = a_{ji}$ so that

$$\Omega_i = \sum_j a_{ij}\omega^j \ .$$

More generally, if $\Omega_{i_1\dots i_q}$ are any p-forms, symmetric in the indices i_k, and

$$\sum_j \Omega_{i_1\dots i_{q-1}j} \wedge \omega^j = 0$$

then there are $(p-1)$-forms

$$\alpha_{i_1\dots i_{q+1}}$$

symmetric in the i_k indices so that

$$\Omega_{i_1\dots i_q} = \sum_j \alpha_{i_1\dots i_q j} \wedge \omega^j \ .$$

Lemma 43 ($\bar\partial$ Cartan–Poincaré). *If ω^j are linearly independent $(1,0)$ forms on a complex vector space, and $\Omega_{i_1\dots i_t}$ are any (p,q)-forms, symmetric in the i_k indices, with*

$$\sum_j \Omega_{i_1\dots i_{t-1}j} \wedge \omega^j = 0$$

then there are $(p-1,q)$-forms $\alpha_{i_1\dots i_{t+1}}$ symmetric in the i_k indices so that

$$\Omega_{i_1\dots i_t} = \sum_j \alpha_{i_1\dots i_t j} \wedge \omega^j \ .$$

A similar result holds for ω^j $(0,1)$-forms, by taking complex conjugation.

B.2 Derivatives

$$
\begin{aligned}
\phi_* X(p) &= \phi'\left(\phi^{-1}(p)\right) \cdot X\left(\phi^{-1}(p)\right) \\
[X,Y](p) &= -\left.\frac{d}{dt}\right|_{t=0} \left(e_*^{tX} Y\right)(p) \\
&= \lim_{t \to 0} \frac{Y(p) - \left(e_*^{tX} Y\right)(p)}{t} \\
\mathcal{L}_X Y &= [X,Y] \\
&= \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X \\
\mathcal{L}_X (\xi \otimes \omega) &= (\mathcal{L}_X \xi) \otimes \omega + \xi \otimes (\mathcal{L}_X \omega) \\
\mathcal{L}_X f &= df(X) \\
\mathcal{L}_X \omega &= \lim_{t \to 0} \frac{\left(e^{tX}\right)^* \omega - \omega}{t} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(e^{tX}\right)^* \omega \\
\mathcal{L}_{a^i \partial_i} b^j \partial_j &= \left(a^i \partial_i b^j - b^i \partial_i a^j\right) \partial_j \\
\mathcal{L}_X (\xi \wedge \omega) &= (\mathcal{L}_X \xi) \wedge \omega + \xi \wedge (\mathcal{L}_X \omega) \\
e^{tX} e^{tY} &= e^{t(X+Y) + \frac{t^2}{2}[X,Y] + \dots} \\
d\omega(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i-1} \mathcal{L}_{X_i} \left(\omega\left(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\right)\right) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\right) \\
d(\xi \wedge \omega) &= (d\xi) \wedge \omega + (-1)^p \xi \wedge (d\omega)
\end{aligned}
$$

X, Y, X_i vector fields
ξ, ω differential forms
ϕ a diffeomorphism
a^i, b^i components of vector fields, in coordinates
ξ is a p-form

B.3 Lie groups

B.3.1 Definitions

G is a Lie group, $\mathfrak{g} = T_1 G$ its Lie algebra. Elements $g, h, k \in G$, $A, B, C \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$.

$$\begin{aligned}
\mathrm{ad}_g(h) &= ghg^{-1} \\
\mathrm{Ad}_g &= \mathrm{ad}'_g(1) \\
L_g h &= gh \\
R_g h &= hg \\
\overrightarrow{A}(g) &= L'_g(1)A \\
\overleftarrow{A}(g) &= R'_g(1)A \\
e^{tA} &= e^{t\overrightarrow{A}}(1) \\
[A, B] = \mathrm{ad}_A B &= \left. \frac{d}{dt} \mathrm{Ad}_{e^{tA}} B \right|_{t=0} \\
\lambda(g) &= L'_{g^{-1}} \\
\rho(g) &= R'_{g^{-1}}
\end{aligned}$$

so that

$$\lambda, \rho \in \Omega^1(G) \otimes \mathfrak{g}.$$

B.3.2 Identities

$$
\begin{aligned}
(L_g)_* \overrightarrow{A} &= \overrightarrow{A} \\
(R_g)_* \overleftarrow{A} &= \overleftarrow{A} \\
(L_g)_* \overleftarrow{A} &= \overleftarrow{\mathrm{Ad}_g A} \\
(R_g)_* \overrightarrow{A} &= \overrightarrow{\mathrm{Ad}_g^{-1} A} \\
\overrightarrow{A} \lrcorner \lambda &= A \\
\overrightarrow{A} \lrcorner \rho(g) &= \mathrm{Ad}_g A \\
\overleftarrow{A} \lrcorner \lambda(g) &= \mathrm{Ad}_g^{-1} A \\
\overleftarrow{A} \lrcorner \rho &= A \\
\rho(g) &= \mathrm{Ad}_g \lambda(g) \\
(L_g)^* \lambda &= \lambda \\
(R_g)^* \lambda &= \mathrm{Ad}_g^{-1} \lambda \\
(L_g)^* \rho &= \mathrm{Ad}_g \rho \\
(R_g)^* \rho &= \rho \\
\left[\overrightarrow{A}, \overrightarrow{B}\right] &= \overrightarrow{[A, B]} \\
\left[\overleftarrow{A}, \overleftarrow{B}\right] &= -\overleftarrow{[A, B]} \\
e^{t\overrightarrow{A}}(g) &= g e^{tA} \\
e^{t\overleftarrow{A}}(g) &= e^{tA} g \\
d\lambda &= -\lambda \wedge \lambda \\
&= -\frac{1}{2}[\lambda, \lambda] \\
d\rho &= \rho \wedge \rho \\
&= \frac{1}{2}[\rho, \rho]
\end{aligned}
$$

B.3.3 Matrix groups

$$
\begin{aligned}
\operatorname{Ad}_g A &= gAg^{-1} \\
[A,B] = \operatorname{Ad}_A B &= AB - BA \\
\lambda &= g^{-1}\,dg \\
\rho &= dg\,g^{-1} \\
\overrightarrow{A}(g) &= gA \\
\overleftarrow{A}(g) &= Ag \\
\mathcal{L}_{\overrightarrow{A}}\lambda &= [\lambda, A] \\
\mathcal{L}_{\overleftarrow{A}}\lambda &= 0 \\
\mathcal{L}_{\overrightarrow{A}}\rho &= 0 \\
\mathcal{L}_{\overleftarrow{A}}\rho &= [A, \rho]
\end{aligned}
$$

WARNING: left invariant vector fields generate the *right* action, and vice versa.

Question 89. I should point out the work of Robert Zimmer [**?**] somewhere.

C

Hints

Chapter 2

2.6 The group G is the group of linear transformations of V fixing some subspace $V_0 \subset V$.

2.14 A flat 2-torus has a double covering, and a triple covering; consider the disjoint union of these.

2.21 Try a partition of unity argument; note that an affine combination of connections is always a connection.

2.22 Differentiate:

$$(F\phi)'(x,g) \cdot (\dot{x},\dot{g}) = \left(\phi'(x) \cdot \dot{x}, \dot{g}\phi'(x)^{-1} - g\phi'(x)^{-1}\left[\phi''(x) \cdot x\right]\phi'(x)^{-1}\right).$$

Pull back the standard flat connection

$$\begin{aligned}(\dot{x},\dot{g}) \lrcorner F\phi^*\gamma_{(x,g)} &= \gamma_{F\phi(x,g)} F\phi'(x,g)\,(\dot{x},\dot{g}) \\ &= \gamma_{(\phi(x),g\phi'(x)^{-1})}\left(\phi'(x) \cdot \dot{x}, \dot{g}\phi'(x)^{-1} - g\phi'(x)^{-1}\left[\phi''(x) \cdot x\right]\phi'(x)^{-1}\right) \\ &= -\left(\dot{g}\phi'(x)^{-1} - g\phi'(x)^{-1}\left[\phi''(x) \cdot x\right]\phi'(x)^{-1}\right)\phi'(x)g^{-1} \\ &= -\dot{g}g^{-1} + g\phi'(x)^{-1}\left[\phi''(x)\dot{x}\right]g^{-1}.\end{aligned}$$

Chapter 4

4.3 A G-structure always admits a reduction to a K-structure, where $K \subset G$ is the maximal compact subgroup. How do you get rid of the Klein bottle? For web geometries, G consists of orientation preserving linear transformations.

Chapter 5

5.5 Map a square in a plane with coordinates x, y to the bundle B, so that the $dy = 0$ curves are mapped to $\omega^2 = 0$ geodesics, parameterized by $\omega^2 = dx \pmod{dy}$. Write $\omega^i = a^i \, dx + b^i \, dy$. Calculate $d\omega^i$ and plug in structure equations. Now calculate

$$\frac{\partial}{\partial y} \int \omega^1 .$$

Chapter 6

6.5 Use the Frobenius theorem.

Chapter 7

7.14 Look at structure equations.

7.15 Think about the symmetry group, and the associated subgroup with these structure equations.

7.32 If the parameter is t, then on the lift of the geodesic to FM we have $dt = \omega^1$. Use this as the definition of the parameterization. Show that the lifts which occur are exactly the same.

Chapter 8

8.5 The adjoint representation factors through $\tilde{H} \to H$.

8.23 The first step is to consider the examples for which $s \neq 0$, and show that we can reduce structure group to arrange $s = 1$. See Cartan [23] for the answer.

8.24 If not, then there must be a vector field giving an infinitesimal symmetry, preserving the soldering form.

8.25 A family of nonisomorphic Lie groups.

8.26 We know that $\omega^{(3)}$ is a constant coefficient combination of $\omega^{(0)}, \omega^{(1)}, \omega^{(2)}$. Plug it into the structure equations and absorb torsion.

8.28 Either torsion reduces G, or else the structure equations are n copies of the structure equations encountered in the proof of theorem 16 on page 199.

8.31 The bundle $FM^{(3)} \to FM^{(2)}$ has 1-dimensional fibers. If $B \subset FM^{(3)}$ has codimension more than one, then it cannot be a bundle over $FM^{(2)}$, so is not a purely 3rd order structure. If $B \subset FM^{(3)}$ is a 3rd order structure, its structure group must be a subgroup of the structure group of $FM^{(3)} \to FM^{(2)}$, i.e. $\mathbb{R}$, so either $B = FM^{(3)}$ or B has codimension 1. There must be an equation

$$\omega^{(4)} = a_0 \omega^{(0)} + a_1 \omega^{(1)} + a_2 \omega^{(2)} + a_3 \omega^{(3)}$$

on B. If the coefficients are constant on fibers over M, then they satisfy $da_\mu = a'_\mu \omega^{(0)}$. Differentiating the equation for $\omega^{(4)}$, we find a contradiction coming from the $\omega^{(3)} \wedge \omega^{(2)}$ term, just as we did in the proof of theorem 16 on page 199.

8.32 The $\omega^{(p-1)} \wedge \omega^{(2)}$ term in $d\omega^{(p)}$ always gives rise to trouble.

8.33 The $\omega^{(2)} \wedge \omega^{(2)}$ term is now the crucial one, since it isn't 0 anymore. In fact, our Lie algebra calculations in the proof of theorem 17 on page 203 show that.

8.40 If $X \lrcorner \left(\omega^1, \omega^2, \omega^3\right) = 0$, show that $\mathcal{L}_X \omega^1$ is a multiple of ω^1.

8.44 Every G-structure admits a reduction to a K-structure, where $K \subset G$ is the maximal compact subgroup.

Chapter 11

11.5 Check how they behave under the right multiplication by i.

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List of Notation

List of Authors

Index