ELEMENTARY REMARKS ON THE GEOMETRY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This is a lecture for mathematics graduate students which I gave at Duke University in 1998. Linear differential equations have vector spaces of solutions: a flat geometry. Sometimes the space of solutions of a nonlinear equation has a curved geometry.

1. First order ODE

The easiest ode of all is

$$\dot{x}(t) = 0$$

The solutions are just

$$x(t) = \text{constant}$$

so the solutions form a line. I don't have much to say about this line.

2. Second order ODE

Consider the next easiest:

$$\ddot{x}(t) = 0$$

The solutions are affine linear functions

$$x(t) = \xi t + \tau$$

We get to choose the constants ξ, τ any way we like, so the space of solutions is a plane. By linearity, we can add solutions, so this plane is a vector space. But what about the space of solutions of a more general ordinary differential equation? What geometry lives there?

Consider a more difficult equation like

$$\ddot{x}(t) = -\frac{\dot{x}}{2t}$$

The general solution is given by the equation

$$\xi(\tau + x)^2 = t$$

where ξ, τ are arbitrary constants. The ξ, τ plane is the space of solutions, because a solution $x(t)$ is defined by the values of ξ, τ appearing in the general solution. Look at this equation again, but let ξ and τ be variables, and x, t be constants. Then if we differentiate the equation twice, we find that

$$\ddot{\xi}(\tau) = \frac{3\dot{\xi}^2}{2\xi}$$

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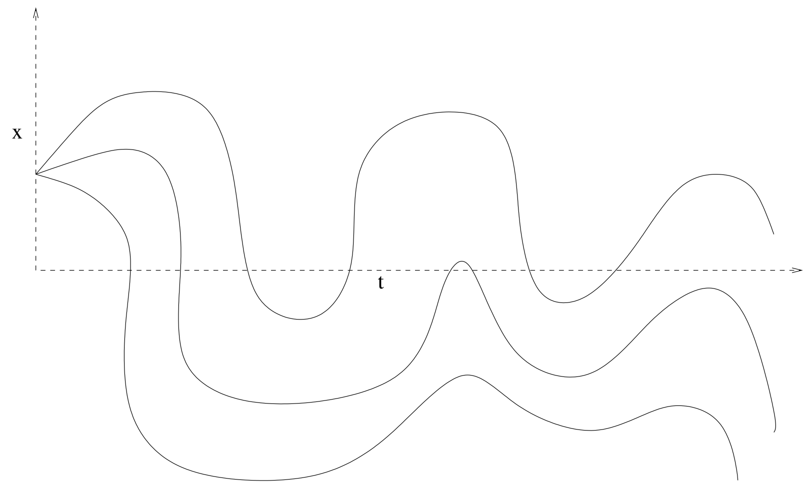
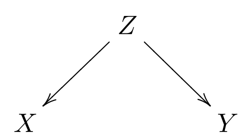


Figure 1. Solutions (of some second order ODE) with different initial $\dot{x}$

on the ξ, τ plane. Classically these were refered to as dual ordinary differential equations.

$$\ddot{x}(t) = -\frac{\dot{x}}{2t} \longleftrightarrow \ddot{\xi}(\tau) = \frac{3\dot{\xi}^2}{2\xi}$$

In a picture, what we are doing looks like a pair of maps



where X is the x, t plane, Y is the ξ, τ plane (the space of solutions), and Z is the space of "pointed solutions". A pointed solution is a choice of a solution of the x, t ordinary differential equation, together with a point on that solution, as in figure 2. We map $Z \to X$ by taking the point, and $Z \to Y$ by taking the solution. The idea behind dual equations is that if X is the x, t plane with a second order ODE on it, then this picture is symmetric in X and Y: the dual equation is defined by switching X and Y.

We can draw the stalks of the map $Z \to X$ as in figure 1: the solutions passing through a point of X. We can also draw the stalks of the map $Z \to Y$ as the points lying on a given solution, as in figure 2, imagining that the solution is fixed, but the point is allowed to vary.

Each point of Y corresponds to a curve in X: we take the stalk of that point up in Z, and project that stalk down to X, as in figure 4. These are just the solutions of our differential equation. The dual equation describes the curves we get in the same way, after switching X and Y, as in figure 5.

What is the purpose of the dual equation? In geometry, we find curvatures of geometric structures. The duality between X and Y means that the notion of curvature for second order equations must admit some kind of symmetry: it describes the geometry of the equation on X, but also the geometry of the dual

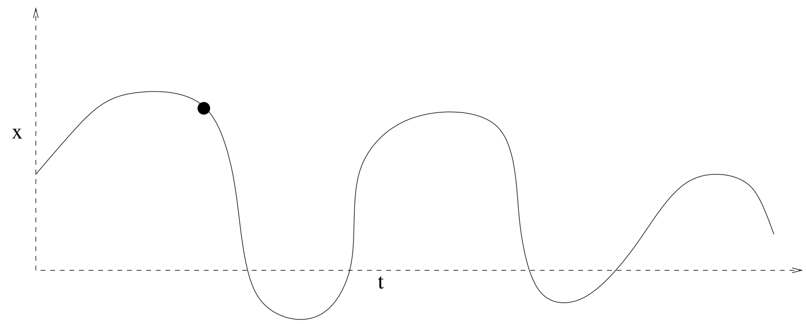


FIGURE 2. A pointed solution

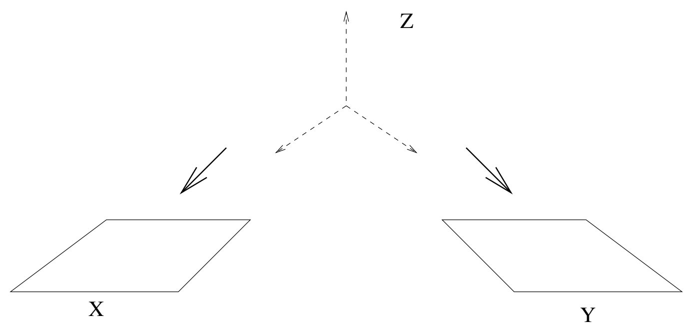


FIGURE 3. Z is three dimensional, X and Y are surfaces

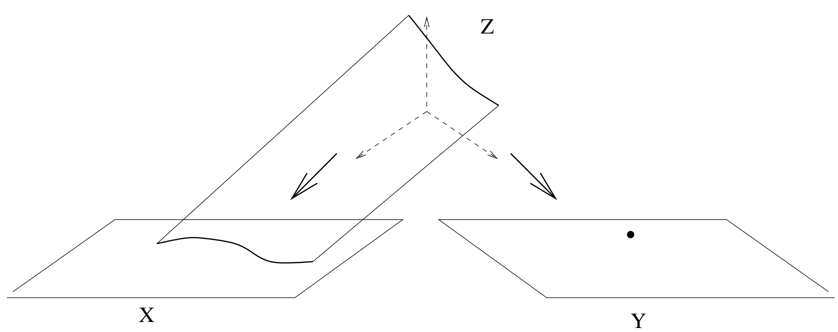


FIGURE 4. A point of Y is a curve on X ...

equation on Y. How do the curvatures help to find the solutions of the differential equations?

In general, we don't know how to solve ordinary differential equations, even second order ODEs for 1 function of 1 variable:

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t))$$

But we can calculate curvatures explicitly, because they are purely defined in terms of calculating derivatives, in this case derivatives of the function f, and putting them together into a geometrically meaningful package.

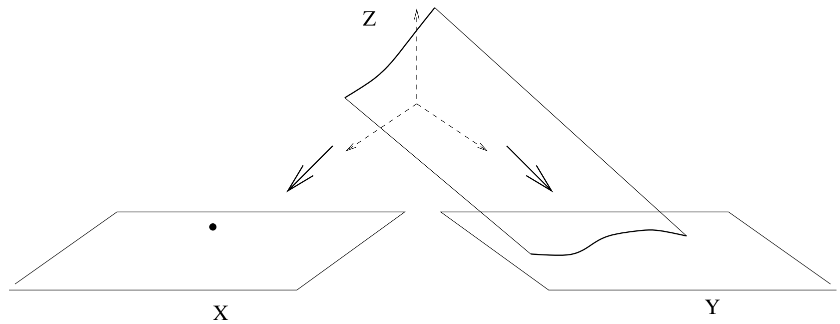


FIGURE 5. ... and point of X is a curve on Y

A. Tresse first found the curvatures to all orders in his prize winning (unpublished) 1893 PhD thesis. The lowest order curvatures of the geometric structure on Z come in pairs κ_1, κ_2. Cartan, in [3] showed that κ_2 vanishes precisely when the differential equation

$$\ddot{x}(t) = f(t, x, \dot{x})$$

is a 3rd degree polynomial in $\dot{x}$:

$$\ddot{x}(t) = a(t, x) + b(t, x)\dot{x} + c(t, x)\dot{x}^2 + d(t, x)\dot{x}^3$$

for some choice of coordinates x, t. The curvature κ_1 vanishes precisely when the dual equation has such a form. They both vanish precisely when either one, and hence both, equations are linear (in some coordinates).

Note that the curvatures κ_1 and κ_2 are defined on Z. The amazing result of Cartan is that if the κ_1 curvature vanishes, then κ_2 becomes well defined on the space Y of solutions, i.e. it is constant as we move along a solution. So if $\kappa_1 = 0$, then κ_2 is a conservation law. We can differentiate it, and get more conservation laws, enough to solve the original differential equation

$$\ddot{x}(t) = f(t, x, \dot{x})$$

explicitly. Cartan calculated that the vanishing of κ_1 is exactly the condition

$$\frac{d^2}{dt^2} f_{pp} - 4\frac{d}{dt} f_{px} + f_p \left(4f_{px} - \frac{d}{dt} f_{pp} \right) - 3f_x f_{pp} + 6f_{xx} = 0$$

for the function

$$f(t, x, p)$$

The fact that we can explicitly calculate curvatures (if we have to) means that we can prove results like this by calculation. This is a single 4th order nonlinear pde, about which little is known, but the general solution depends on 4 functions of 2 variables, an absolutely huge family of explicitly integrable ordinary differential equations.

3. THIRD ORDER ODE

What is next? Try

$$\dddot{x}(t) = 0$$

the easiest third order ODE. The solutions are parabolas,

$$x(t) = at^2 + bt + c$$

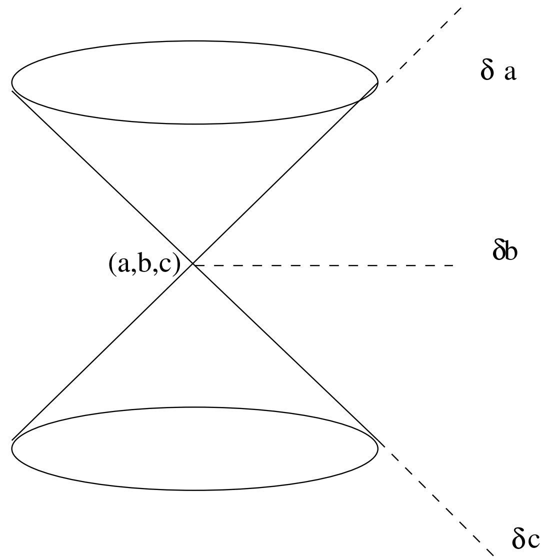


FIGURE 6. The discriminant cone for $\dddot{x} = 0$

Again the solutions are a vector space. We get to choose the constants a, b, c arbitrarily. Picking any one solution, say the zero solution, $x(t) = 0$, we can look at how other solutions

$$X(t) = at^2 + bt + c$$

intersect it. This is described by the discriminant

$$b^2 - 4ac$$

More generally, a solution

$$x_0(t) = at^2 + bt + c$$

intersects another

$$x_1(t) = (a + \delta a)t^2 + (b + \delta b)t + (c + \delta c)$$

with a double point precisely when

$$\delta b^2 - 4 \cdot \delta a \cdot \delta c = 0$$

This is the equation of a cone. We can draw this as in figure 6 At each point (a, b, c) of this 3 dimensional space, we have a cone of vectors $(\delta a, \delta b, \delta c)$ which satisfy the discriminant equation

$$\delta b^2 - 4 \cdot \delta a \cdot \delta c = 0$$

A physicist when faced with a cone like this thinks of relativity: the light cone. But in our case the cones are in a 3 dimensional space, so the space of solutions of

$$\dddot{x}(t) = 0$$

is just Minkowski $2 + 1$ space.

This last example is very rich. We don't expect a vector space structure of solutions to make sense for a general third order ODE

$$\dddot{x}(t) = f(t, x, \dot{x}, \ddot{x})$$

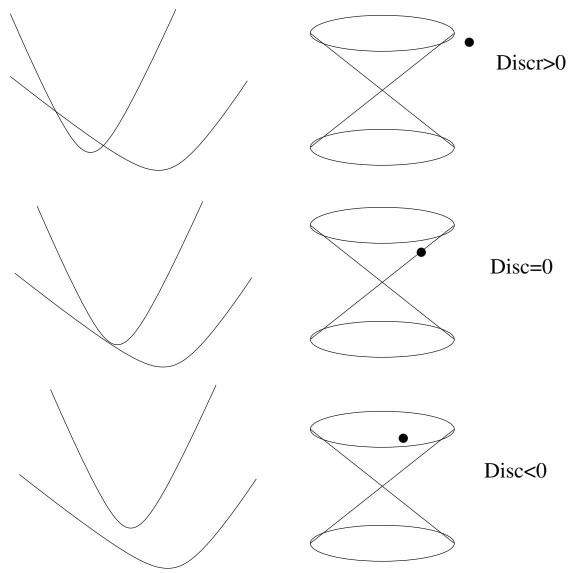


FIGURE 7. The discriminant cone tells how a solution intersects nearby solutions

but what about the discriminant/light cones? Since the equation

$$\dddot{x}(t) = 0$$

is our "model flat geometry", we might hope that even in "curved geometries", i.e. nonlinear ODEs, these cones might survive. The cones measure whether or not a solution with a perturbation of that solution in a double point, and this definition can be used to define the cones in the space of solutions of any third order ODE. But do they form light cones? Light cones in general relativity must be the solutions of a quadratic equation, as in the example of special relativity above. K. Wunschmann and S.S. Chern, [6], figured out when these cones are given by quadratic equations: it depends on a certain type of curvature, which Chern called I. If the equation

$$\dddot{x}(t) = f(t, x, \dot{x}, \ddot{x})$$

has curvature $I = 0$, then the discriminant cones are quadratic. This I is a function of f and its derivatives. Indeed if

$$f = f(t, x, p, q)$$

then

$$I = -f_x - \frac{1}{3} f_p f_q - \frac{2}{27} f_q^3 + \frac{1}{2} \frac{d}{dt} f_p + \frac{1}{3} f_q \frac{d}{dt} f_q - \frac{1}{6} \frac{d^2}{dt^2} f_q$$

The equation $I = 0$ is 3rd order PDE, and its general solution depends on 3 arbitrary functions of 3 variables. Nothing much is known about it.

We begin to see that equations satisfying Chern's $I = 0$ will provide us with some kind of $2+1$ dimensional general relativities. If we calculate the curvatures of these general relativistic space times, we get conservation laws for our differential equations

$$\dddot{x}(t) = f(t, x, \dot{x}, \ddot{x})$$

So in case the curvature I of Chern vanishes, our ODE can be solved explicitly. Not every $2+1$ dimensional general relativity occurs from this construction, and Cartan in [4] determined exactly which general relativistic space times arise in this way.

4. Fourth order ODE

Finally, we come to 4th order equations, which can give rise to very beautiful structures on their spaces of solutions. See R.L. Bryant's amazing paper [1] and N. Hitchin's survey [8] for more of the story, known as twistor theory, which is largely due to Roger Penrose and his collaborators. Hitchin's [7] gives examples of explicit constructions of asymptotically locally Euclidean Einstein metrics using these ideas.

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I have not been able to find a copy of:

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