RATIONAL CURVES AND ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The systems of complex analytic second order ordinary differential equations whose solutions close up to become rational curves are characterized by the vanishing of an explicit differential invariant, and form an infinite dimensional family of integrable systems.

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1. Introduction

1.1. **The problem.** For which complex analytic ordinary differential equations are all solutions rational curves (i.e. topological spheres)? Use the phrase *integral curve* to mean solution of a system of ordinary differential equations. We might wonder if the differential equations may be analytically continued along any integral curve, perhaps in some new system of coordinates, so that that integral curve closes up to become a rational curve.

Definition 1. A system of second order complex analytic ordinary differential equations is *straight* if the associated path geometry is locally isomorphic to a path geometry whose integral curves are all rational curves.

See definition 3 on page 9 for the definition of a *path geometry*. We shall first consider some examples to motivate the definition. Our problem: to characterize straightness. I shall shortly give an explicit local condition (which is easy to check), characterizing straightness for second order systems of ordinary differential equations. Henceforth all manifolds and Lie groups are complex, and all maps, vector bundles, functions, sections of vector bundles and connections are holomorphic.

Example 1. The fundamental example which guides our work is the differential equation

$$\frac{d^2y}{dx^2} = 0,$$

whose solutions are straight lines. This equation is invariant under translations in both x and y, so that we can quotient to define the equation on a complex torus. However, it is straight because it is locally isomorphic to the equation of projective lines in projective space.

This problem of characterizing straight equations is similar to Painlevé's problem on systems with fixed singular points, but the answer is quite different.

1.2. **The solution.**

Definition 2. For a system of $n \geq 1$ second order ordinary differential equations

$$\frac{d^2y^I}{dx^2} = f^I\left(x, y, \frac{dy}{dx}\right).$$

in complex variables $x, y^1, \dots, y^n$, and for any function $g(x, y, \dot{y})$, define dg/dx to mean

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y^I}\dot{y}^I + \frac{\partial g}{\partial \dot{y}^I} f^I(x, y, \dot{y}).$$

Define the *Fels torsion* [17] of the system to be:

$$\Phi^I_J = \phi^I_J - \frac{1}{n}\phi^K_K \delta^I_J$$

where

$$\phi^I_J = \frac{1}{2}\frac{d}{dx}\frac{\partial f^I}{\partial \dot{y}^J} - \frac{\partial f^I}{\partial y^J} - \frac{1}{4}\frac{\partial f^I}{\partial \dot{y}^K}\frac{\partial f^K}{\partial \dot{y}^J},$$

with $f^I = f^I(x, y, \dot{y})$. For a single second order ordinary differential equation (i.e. $n = 1$), say

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right),$$

clearly the Fels torsion vanishes by definition, and we define the *Tresse torsion* (see [38, 9, 1]:

$$\frac{d^2}{dx^2}\frac{\partial^2 f}{\partial \dot{y}^2} - 4\frac{d}{dx}\frac{\partial^2 f}{\partial y \partial \dot{y}} + \frac{\partial f}{\partial \dot{y}}\left(4\frac{\partial^2 f}{\partial y \partial \dot{y}} - \frac{d}{dx}\frac{\partial^2 f}{\partial \dot{y}^2}\right) - 3\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial \dot{y}^2} + 6\frac{\partial^2 f}{\partial y^2}.$$

The Fels torsion depends only on second derivatives of the functions f^I, while the Tresse torsion depends on derivatives of fourth order.

Theorem 1. *A second order ordinary differential equation [system of equations] is torsion-free (i.e. the Tresse [Fels] torsion vanishes) just when it is straight.*

1.3. **Examples.**

Example 2. Lets return to our fundamental example,

$$\frac{d^2y}{dx^2} = 0,$$

whose integral curves are straight lines. Lines analytically continue to become projective lines in projective space. The differential equation looks the same throughout projective space, although one has to make projective linear changes of variable to see what happens out at the hyperplane at infinity. There is no global choice of variable x over which the integral curves can be graphed, because there are no nonconstant functions on a projective line. Thus the variables x, y are *not* distinguished, and we must allow ourselves to use different variables in different regions.

Example 3. The straight linear second order systems with constant coefficients are precisely those of the form

$$\frac{d^2y}{dx^2} = A\frac{dy}{dx} + \left(a - \frac{1}{4}A^2\right)y$$

where A is any constant complex $n \times n$ matrix, and a any complex scalar.

Example 4. None of Painlevé's equations are straight.

Example 5. For any complex constant c, the equations

$$\frac{d^2y^1}{dx^2} = \left(\frac{dy^1}{dx}\right)^2$$

$$\frac{d^2y^2}{dx^2} = \left(\frac{dy^1}{dx}\right)^2 + c\frac{dy^1}{dx}\frac{dy^2}{dx} - \frac{1}{2}c\left(1 - \frac{1}{2}c\right)\left(\frac{dy^2}{dx}\right)^2$$

are straight.

Example 6. Second order equations with one dimensional symmetry group can be brought by coordinate transformation to the form

$$\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right);$$

(for proof see Lie [29]). The conditions on $f(y, \dot{y})$ under which this equation is straight form the fourth order equation

$$\begin{aligned} 0 =& \dot{y}^2 \frac{\partial^4 f}{\partial y^2 \partial \dot{y}^2} + 2\dot{y} f \frac{\partial^4 f}{\partial y \partial \dot{y}^3} + f^2 \frac{\partial^4 f}{\partial \dot{y}^4} + \dot{y} \frac{\partial^3 f}{\partial \dot{y}^3} \frac{\partial f}{\partial y} - 3 \frac{\partial^3 f}{\partial y \partial \dot{y}^2} - 4\dot{y} \frac{\partial^3 f}{\partial y^2 \partial \dot{y}} + 4 \frac{\partial f}{\partial \dot{y}} \frac{\partial^2 f}{\partial y \partial \dot{y}} \\ & - \dot{y} \frac{\partial f}{\partial \dot{y}} \frac{\partial^3 f}{\partial y \partial \dot{y}^2} - 3 \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial y \partial \dot{y}} + 6 \frac{\partial f}{\partial y^2}, \end{aligned}$$

so that the generic straight equation with symmetry group depends on 4 functions of 2 variables. In particular, the generic equation with one dimensional symmetry is not straight, and vice versa; straightness is independent of symmetry group. Straightness is also independent of linearizability (see Merker [32]).

Example 7. All linear second order equations $\frac{d^2y}{dx^2} = a(x)\frac{dy}{dx} + b(x)y$ are straight. However, coupled systems of linear second order equations are generally not straight. Intuitively, the problem is analoguous to the problem of coupled harmonic oscillators being generically nonperiodic, while a single oscillator is periodic, because the coupled oscillators have different periods. Indeed if we consider a single oscillator

$$\frac{d^2y}{dx^2} = \omega^2 y,$$

with ω constant, the solutions are

$$y = a_+ e^{\omega x} + a_- e^{-\omega x},$$

so that if we introduce the variable $X = e^{\omega x}$, then

$$y = a_+ X + \frac{a_-}{X},$$

or

$$a_+ X^2 - Xy + a_- = 0,$$

quadratic equations, so the (generic) solutions are smooth rational curves. Consider a coupled system, say

$$\begin{aligned} \frac{d^2y_1}{dx^2} &= \omega_1^2 y_1 \\ \frac{d^2y_2}{dx^2} &= \omega_2^2 y_2. \end{aligned}$$

If $\omega_1 \neq \omega_2$, then there is no choice of parameter X. For a general coupled system, if all of the frequencies ω_j are rational multiples of a common frequency ω, then we can introduce a parameter $X = e^{\omega x}$, and obtain algebraic equations for the solutions. But the degrees are not low enough to keep the curves rational, unless the frequencies ω_j are all equal.

1.4. **Integrability.** Generic straight systems of second order differential equations are integrable by geometric construction, as we shall see (also see Grossmann [22]). So by calculating torsion, we have an explicit test for integrability; for example, every straight ordinary differential equation (i.e. $n = 1$)

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

for which f is a sum of linear and quadratic terms in $x, y, \dot{y}$ is integrable by use of hypergeometric functions and quadratures, while this is certainly not true for the generic second degree second order ordinary differential equation. Indeed, Cartan

[9] can apparently integrate any straight ordinary differential equation by differentiation and at most two quadratures. (But see subsection 1.5 for some concerns about Cartan's claim.) Moreover, straight systems remain straight under symmetry reduction obviously, so they form a fascinating class of systems of ordinary differential equations.

The ability to integrate straight equations is particularly exciting when we realize that all of the second order ordinary differential equations of classical mathematical physics are straight; see table 2 on the following page. Therefore all ordinary differential equations of mathematical physics before the 20^{th} century, except Painlevé equations, are apparently solved by Cartan's method. Special function theory is just one special case of the theory of straight equations. There are some well known modern equations of mathematical physics which are not integrable without adjoining new transcendental functions, even if we allow inversions of integrals of elementary functions, and thus are not obtained by algebra and quadratures (see table 3 on page 7); therefore they are not integrable by Cartan's method, or by symmetry reduction, and thus they cannot be straight; and of course, they are not torsion-free. Indeed from the table, we see that torsion is found in all of them, except in certain special cases. These cases turn out to all be well known degeneracies in which the general solution can be expressed in hypergeometric or elliptic functions.

Having tested many equations (as the reader can see), my conclusion is that if Cartan is correct, then his methods integrate in quadratures every equation known to me for which such integration is known to be possible. It seems natural to conjecture that Cartan's method is the final answer to the problem of integration in quadratures for a single second order equation.

1.5. **A note of sour skepticism.** If a second order ordinary differential equation has a Lie group of symmetries of positive dimension, it would appear to invalidate Cartan's approach (which we shall see in section 15.3 on page 31) as Cartan describes it, since the invariants do not provide enough conservation laws. Cartan does not point out this case, but integrability still follows as long as the Lie group has dimension 2 or greater (see Lie [29]). Even if the symmetry group is not solvable, the equation is integrable. For example, consider

$$\frac{d^2y}{dx^2} = 0,$$

whose symmetry Lie algebra turns out to be $\mathfrak{sl}(3, \mathbb{C})$, a simple Lie group. However, one needs to know the symmetry group action explicitly to carry out this integration. The question of the integrability of a straight scalar second order ordinary differential equation in the presence of a one dimensional Lie group of symmetries is apparently not settled, in contrast to Cartan's claim.

The equations of mathematical physics described above, as a consequence of the theorems we shall prove below, are all locally equivalent under point transformations to the standard equation $d^2y/dx^2 = 0$, and therefore have simple Lie group of point symmmetries, so that Lie's method of reduction does not apply. The symmetry groups are not explicit, and finding them explicitly appears to be as difficult as solving the equations directly. Moreover, Cartan's approach as he outlines it also does not apply, since it depends on local invariants under point transformations. It may be that Cartan has up his sleeve some deeper methods that apply in these

Common name	**Equation**
Airy	$\frac{d^2y}{dx^2} = xy$
Anger	$\frac{d^2y}{dx^2} + \frac{\frac{dy}{dx}}{x} + \left(1 - \frac{a^2}{x^2}\right) y = \frac{x-a}{\pi x^2} \sin \pi a$
Bessel	$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 + a)y = 0$
Bessel (modified)	$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + a)y = 0$
Bessel (spherical)	$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + (x^2 + a)y = 0$
Bessel (modified spherical)	$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - (x^2 + a)y = 0$
confluent hypergeometric	$x \frac{d^2y}{dx^2} + (c - x) \frac{dy}{dx} - ay = 0$
Coulomb wave	$\frac{d^2y}{dx^2} + \left(1 - \frac{a}{x} - \frac{b}{x^2}\right) y = 0$
Eckart	$\frac{d^2y}{dx^2} + \left(\frac{a\, e^{dx}}{1+e^{dx}} + \frac{b\, e^{dx}}{(1+e^{dx})^2} + c\right) y = 0$
ellipsoidal	$\frac{d^2y}{dx^2} = \left(a + b \sin(x)^2 + c \sin(x)^4\right) y$
elliptic	$x(1 - x^2) \frac{d^2y}{dx^2} + \left(1 - x^2\right) \frac{dy}{dx} - 2x^2 \frac{dy}{dx} - xy$
error function	$\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 2ay$
Euler	$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = 0$
Gauß hypergeometric	$x(x-1) \frac{d^2y}{dx^2} + ((a + b + 1)\, x - c) \frac{dy}{dx} + aby = 0$
Halm	$\left(1 + x^2\right)^2 + \frac{d^2y}{dx^2} + a \frac{dy}{dx} = 0$
Hermite	$\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 2ay$
Lienard	$\frac{d^2y}{dx^2} + (ay + b) \frac{dy}{dx} + \left(\frac{1}{9} a^2 y^3 + \frac{1}{3} aby^2 + cy + d\right) = 0$
Liouville	$\frac{d^2y}{dx^2} + g(y) \left(\frac{dy}{dx}\right)^2 + f(x) \frac{dy}{dx} = 0$
Mathieu	$\frac{d^2y}{dx^2} + (a - 2b \cos 2x)\, y = 0$
Titchmarsh	$\frac{d^2y}{dx^2} + (b - x^a)\, y = 0$

Table 2. Some straight equations from mathematical physics; a, b, c, d any constants, f, g any functions. See Polyanin & Zaitsev [34], Zwillinger [39]

"degenerate" cases, but he gives no indication. Nonetheless, it is amazing that the basic ordinary differential equations of mathematical physics (before Painlevé) are straight, that straightness is a rare property, and that no one has previously noticed this.

1.6. **Optimism returns, with topology in tow.** The theory of second order equations of mathematical physics appears from this point of view to be nearly topological, in the sense that all of the above equations are locally point equivalent to $d^2y/dx^2 = 0$, i.e. to the contact 3-manifold x, y, p with contact planes $dy = p\,dx$ and two Legendre foliations (a) $dy = p\,dx, dp = 0$ and (b) $dx = dy = 0$. The global study of such "flat" double Legendre folations is thus at the core of physics, while being locally completely elementary. This flavour of contact topology (i.e. with flat double Legendre folation) is entirely mysterious.

Common name	Equation	When torsion is found
Emden–Fowler	$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + x^2y^a$	$a \neq 0, 1$
modified Emden–Fowler	$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + y^a$	$a \neq 0, 1$
Lagerstrom	$x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + bxy\frac{dy}{dx} = 0$	$b \neq 0$
Painlevé I	$\frac{d^2y}{dx^2} = 6y^2 + x$	
Painlevé II	$\frac{d^2y}{dx^2} = 2y^3 + xy + a$	
Painlevé III	$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}^2}{y} - \frac{\frac{dy}{dx}}{x} + \frac{ay^2+b}{x} + cy^3 + \frac{d}{y}$	$(a,b,c,d) \neq (0,0,0,0)$
Painlevé IV	$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}^2}{2y} + \frac{3y^2}{2} + 4y^3x + 2\left(x^2-a\right)y + \frac{b}{y}$	
Painlevé V	$\frac{d^2y}{dx^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\frac{dy}{dx}^2 - \frac{\frac{dy}{dx}}{x} + \frac{(y-1)^2\left(ay+\frac{b}{y}\right)}{x^2} + \frac{cy}{x} + \frac{dy(y+1)}{y-1}$	$(a,b,c,d) \neq (0,0,0,0)$
Painlevé VI	$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\frac{dy}{dx}^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} + \frac{y(y-1)(y-x)\Gamma}{x^2(x-1)^2}$, $\Gamma = a + \frac{bx}{y^2} + \frac{c(x-1)}{(y-1)^2} + \frac{dx(x-1)}{(y-x)^2}$	$(a,b,c,d) \neq \left(0,0,0,\frac{1}{2}\right)$
van der Pol	$\frac{d^2y}{dx^2} = a\left(1-y^2\right)\frac{dy}{dx} - y$	$a \neq 0$

TABLE 3. Some equations of mathematical physics which are not straight; a, b, c, d any constants

1.7. **Why we study second order systems, not first order ones.** All first order systems of ordinary differential equations

$$\frac{dy^I}{dx} = f^I(x, y)$$

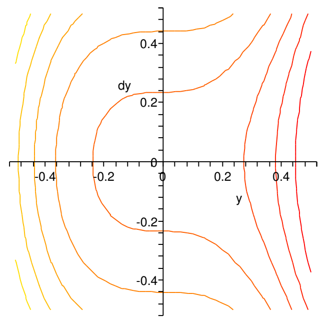

FIGURE 1. The family of cubic curves in the plane

are straight, since the Frobenius theorem tells us that we can change coordinates to arrange that $f^I = 0$. Moreover, higher order systems can be rewritten as first order systems, so it might appear that they are always straight. But this is not the case, since changes of coordinates acting on second order equations are not quite so powerful.

1.8. **An equation which is not straight.** For example consider the equation

$$\frac{d^2y}{dx^2} = 6y^2,$$

which is not torsion-free. Check that the function $\dot{y}^2 - 4y^3$ is constant along integral curves. Therefore the integral curves are precisely the curves

$$\dot{y}^2 = 4y^3 + A$$

for any constant A. These curves are elliptic curves (hence not rational), filling out the phase space, except for the curve with $A = 0$, which is a cuspidal cubic curve, hence rational; see figure 1. Integrating, we find

$$\int \frac{dy}{\sqrt{4y^3 + A}} = x + B,$$

an elliptic integral on each elliptic curve; the constant B just translates the elliptic curve along the x variable, and is defined up to periods. Going backwards, $y(x)$ is an elliptic function on each elliptic curve; in fact it is the Weierstraß $\wp$-function: $y(x) = \wp(x - c)$ with modular parameters $g_2 = 0$ and $g_3 = A$. Globally we can analytically continue all of the integral curves to the 3-manifold $\mathbb{C} \times \mathbb{P}^2$, with x a coordinate function on $\mathbb{C}$, and $(y, \dot{y})$ an affine chart on $\mathbb{P}^2$ and each integral curve is an elliptic curve, except for the 1-parameter family of curves with $A = 0$ and B arbitrary. The picture extends to the line at infinity on $\mathbb{P}^2$, because the elliptic curves are smooth there too. We have to avoid the surface of points $(x, y, \dot{y})$ for $(y, \dot{y})$ in the cuspidal cubic, where the curves don't behave nicely. The ordinary differential equation is not straight, since the elliptic curves are not rational. It doesn't matter how we treat points at infinity, or the cuspidal cubic, because even after cutting out those points, or any other hypersurface, what remains of each integral curve is still an elliptic curve minus some points, and therefore birational to an elliptic curve, and not to a rational curve. If we could find a birational isomorphism to a differential equation with a rational integral curve, then we would be able to map the rational curve to the corresponding elliptic curve, without branching, which is impossible.

Rationality of integral curves of a system of second order ordinary differential equations is birationally invariant, i.e. we can cut out hypersurfaces from the phase space and glue in other ones. The surprise is that for some second order ordinary differential equations, but *not for most*, it is possible to find a local isomorphism to an equation with rational solutions. This is a consequence of our theorems for which we know of no simple explanation. The Frobenius theorem straightens out the curves by a local coordinate transformation of $(x, y, \dot{y})$ space to some other 3-manifold, but this is not local in (x, y) space.

We apologize to the reader that so much of this paper depends on results of other papers, which we have tried to summarize. In particular, most of the results on scalar second order ordinary differential equations have been previously presented in the literature, but are presented again here to make the exposition clearer and more self contained. This paper contains only a small collection of new results, but the required information is scattered throughout the literature, and a proper discussion of these ideas seems to demand a review of part of that literature. The approach we take here is very similar to Hitchin [23] and Dunajski & Todd [15]. Merker [32] has a different approach, which characterizes very explicitly the systems of ordinary differential equations equivalent to $d^2y/dx^2 = 0$. The papers [20, 33] concern questions and constructions closely related to this paper.

2. Path geometry

We will want a global definition of second ordinary differential equations. A *path geometry* on a manifold usually means a differential system locally given by a system of second order ordinary differential equations, so that through any point, in each direction, there is a unique immersed curve (called an *integral curve*) solving those ordinary differential equations, passing through that point, tangent to that direction. In local coordinates $x, y^1, \dots, y^n$ on the manifold, the integral curves are the solutions of an equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

Hitchin [23] shows that complex surfaces containing rational curves provide a source of important path geometries. He demonstrates that straight path geometries play a role in the Penrose twistor programme, being the most elementary example.

I shall give a slightly more general definition of path geometry:

Definition 3. A *path geometry* on a complex manifold M is a pair of foliations whose leaves are respectively called the *integral curves* and *stalks*, so that near any point of M, there is a coordinate chart with coordinate functions $x, y_1, \dots y_n, \dot{y}_1, \dots, \dot{y}_n$ and functions $f^I(x, y, \dot{y})$ in which the integral curves intersect the coordinate chart precisely in the solutions of

$$dy^I = \dot{y}^I\, dx,\ d\dot{y}^I = f^I\left(x, y, \dot{y}\right)\, dx, \qquad I = 1, \dots, n,$$

and the stalks intersect the coordinate chart precisely in the solutions of

$$dy^I = dx = 0, \qquad I = 1, \dots, n.$$

The functions f^I can change from one coordinate chart to another.

We shall refer to the path geometry consisting of lines in projective space (for which M is the space of pointed lines in projective space) as the *model*.

Locally, there is a smooth quotient space of stalks, with coordinates x, y, but I shall not require this space to be smooth globally. Following Cartan's terminology, call the space of stalks the *space of points*, and our original manifold M the *space of elements*. Even if the space of points is smooth, the path geometry may appear on it as a multivalued ordinary differential equation, in local coordinates, and there might be no paths in certain directions (i.e. $f(x, y, \dot{y})$ might not be defined for certain values of $\dot{y}$). I shall mollify this multivaluedness only slightly by assuming that the space of elements is connected. I shall not require existence or uniqueness of an integral curve in each direction at a given point x, y in the space of points; however the discreteness of the set of such curves follows from the assumptions we have made above.

In the process of travelling along an integral curve, one might carry out changes of variables. So when we think of analytically continuing the differential equation along an integral curve, we have in mind that we might repeatedly change variables as we do so. In this sense the x variable appearing in the local coordinate presentation is not distinguished, so the integral curves might not globally be graphs of functions of an x variable. The two foliations are the only meaningful global data.

One motivation for this paper is that (as we shall see) both stalks and integral curves are canonically locally identified with projective spaces, modulo projective transformations. This might remind us of Riemannian geometry, where geodesics are canonically equipped with arclength parameterization, defined up to choice of a constant; the Riemannian manifold is complete just when the parameterization is a local diffeomorphism from the real line. The geometry of more general second order equations is more complicated, and more slippery, so we have local projective parameterizations defined only up to projective transformation. Therefore it is natural to ask when both the stalks and the integral curves are projective spaces. We shall say integral curves or stalks are *rational* if they are globally parameterized by projective spaces. For integral curves, this is the natural analogue of completeness. We shall prove:

Theorem 2. *A stalk [integral curve] of a path geometry is rational just when it is compact with finite fundamental group. Moreover this occurs just when the canonical local identifications with projective spaces extend globally to a diffeomorphism.*

Therefore rationality (of the leaves of either foliation) is a topological condition, but with strong global consequences. We shall prove:

Theorem 3. *The only path geometry on any connected complex manifold whose integral curves and stalks are all rational is the path geometry on projective space whose integral curves are projective lines.*

Summing up, we have a topological criterion for isomorphism with the model. Note that we do not assume that our complex manifold is compact or Kähler. We will also prove:

Theorem 4. *A path geometry is locally isomorphic to the path geometry of projective space just when it is locally isomorphic to some path geometry with rational* *integral curves, and also locally isomorphic to some path geometry with rational stalks.*

We will also locally characterize path geometries with rational integral curves and those with rational stalks:

Theorem 5. *A path geometry is straight just when it is torsion-free.*

Theorem 6 (LeBrun [23])**.** *If all of the stalks are rational, then (1) there is a smooth space of points which bears a projective connection, (2) the space of elements is invariantly mapped by local biholomorphism to the projectived tangent bundle of the space of points, and (3) each integral curve is locally identified with the family of tangent lines to a unique geodesic of the projective connection. Conversely, every projective connection on any manifold gives rise to a path geometry on its projectivized tangent bundle, with rational stalks (the projectived tangent spaces). A path geometry is locally isomorphic to a path geometry with rational stalks, and therefore is locally a projective connection, just when it satisfies*

$$\frac{\partial^4 f^i}{\partial \dot{y}^I \dot{y}^J \dot{y}^K \dot{y}^L} = 0,$$

for any four indices $I, J, K, L = 1, \dots, n$*, i.e. has the form*

$$\frac{d^2 y}{dx^2} = \sum_{|\alpha| \le 3} f_\alpha\left(x, y\right) \left(\frac{dy}{dt}\right)^\alpha,$$

with α *a multi-index* $\alpha = \left(\alpha_1, \dots, \alpha_n\right)$.

This paper uses methods from my paper [31]; the main theorem in that paper exhausts the study of complete parabolic geometries, but there are many more geometries to which the methods apply. Path geometries turn out to impose parabolic geometries, but these will not be assumed complete. Indeed completeness would be equivalent to assuming that all integral curves and all stalks of the parabolic geometry are rational.

Hitchin [23], Bryant, Griffiths & Hsu [3], Fels [17], and Grossman [22], use a more restrictive definition of path geometry, requiring that there be a smooth space of points and a smooth space of integral curves; we do not require either of these, but the reader can easily see that those authors did not employ these hypotheses in their calculations, only in their conclusions.

3. Elementary remarks on linearization

Recall the concept of linearization of a system of ordinary differential equations: given a system

$$\frac{d^2 y^I}{dx^2} = f^I\left(x, y, \frac{dy}{dx}\right),$$

we linearize about a point (x, y) by first taking the solution $y = y(x)$ through that point, and then changing coordinates so that the solution becomes just $y(x) = 0$, and the point $(0, 0)$, and then we expand f^I into a Taylor expansion and keep the lowest order terms. It is thus elementary to see that

Theorem 7. *A system of second order ordinary differential equations is torsion-free just when its linearization about any point is torsion-free, which occurs just when its linearization about any point has the form*

$$\frac{d^2y}{dx^2} = A\frac{dy}{dx} + \left(a - \frac{1}{4}A^2\right) y.$$

4. A first glance at surface path geometries

A path geometry will be called a *surface* path geometry to denote that there is one y variable (and there is always only one x variable), i.e. that the space of points is a (not necessarily Hausdorff) surface. Therefore the space of elements M is a 3-fold. We will see that path geometries are quite different in this special (lowest possible) dimension.

Theorem 8. *If the stalks [integral curves] of a surface path geometry are compact, and there is a submersion from a nonempty open set $\pi : U \text{ open } \subset M^3 \to S^2$ and whose fibers are stalks [integral curves] respectively, and the space of elements M is connected, then all of the stalks [integral curves] are rational.*

Proof. Take each point $m \in U$, construct the integral curve $C_m \subset M$ through m, and map $m \in U \mapsto \pi'(m)T_m C_m \in \mathbb{P}TS$. In local coordinates $x, y, \dot{y}$, evidently this is a local biholomorphism, mapping stalks to fibers $\mathbb{P}T_sS$. By compactness of stalks, this map is onto. Stalks are connected by definition, so the map is a covering map on each stalk, and therefore is a biholomorphism, because the fibers $\mathbb{P}T_sS = \mathbb{P}^1$ are simply connected. By analytic continuation, all stalks are rational. □

4.1. The structure equations.

I will draw freely from Bryant, Griffiths & Hsu [3]. They prove that given any surface path geometry, M bears a canonical choice of principal bundle $E \to M$ (which they call $B_{G_3} \to M$), a principal right $G^{\text{pt,line}}$-bundle, where $G^{\text{pt,line}} \subset G = \mathbb{P}\,\text{GL}\,(3, \mathbb{C})$ is the subgroup fixing a projective line in the projective plane and a point on that line, i.e. the group of matrices of the form

$$g = \begin{bmatrix} g^0_0 & g^0_1 & g^0_2 \\ 0 & g^1_1 & g^1_2 \\ 0 & 0 & g^2_2 \end{bmatrix},$$

with Lie algebra $\mathfrak{g}^{\text{pt,line}}$. (The square brackets indicate that the matrix is defined up to rescaling, being an element of $\mathbb{P}\,\text{GL}\,(3, \mathbb{C})$). Moreover, they define a canonical 1-form ω (which they write as ϕ) on E valued in $\mathfrak{g} = \mathfrak{sl}\,(3, \mathbb{C})$ (the Lie algebra of $G = \mathbb{P}\,\text{GL}\,(3, \mathbb{C})$), so that

(1) $\omega_e : T_eE \to \mathfrak{g}$ is a linear isomorphism
(2) $\omega \pmod{\mathfrak{g}^{\text{pt,line}}}$ is semibasic for $E \to M$, and

$$d\omega = -\frac{1}{2}\left[\omega, \omega\right] + \nabla\omega$$

where (writing $\omega = \left(\omega^i_j\right)$)

$$\nabla\omega = \begin{pmatrix} 0 & K_1\omega^1_0 \wedge \omega^2_0 & \omega^2_0 \wedge \left(L_1\omega^1_0 + L_2\omega^2_1\right) \\ 0 & 0 & K_2\omega^2_0 \wedge \omega^2_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $r_g : E \to E$ for the right action of an element $g \in G^{\text{pt,line}}$ on E.

(3)
$$r_g^* \omega = \operatorname{Ad}_g^{-1} \omega,$$
(4) Given any local section σ of $E \to M$, the integral curves are precisely the solutions of the exterior differential system $\sigma^* \omega_0^2 = \sigma^* \omega_1^2 = 0$.

Bryant, Griffiths & Hsu don't actually state the equation (3), but it follows immediately as a simple calculation from the transformation properties of the various components of ω as given in their article. They also don't state (4), but it is clear from their remarks on the top of p. A.2.

5. Review of Cartan connections

I won't repeat the entire story of Cartan connections, most of which is contained in my paper [31], but just repeat the definition and prove a few results which were only sketched in that paper.

Definition 4. A *Cartan pseudogeometry* on a manifold M, modelled on a homogeneous space G/G_0, is a principal right G_0-bundle $E \to M$, (with right G_0 action written $r_g : E \to E$ for $g \in G_0$), with a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$, called the *Cartan pseudoconnection* (where $\mathfrak{g}, \mathfrak{g}_0$ are the Lie algebras of G, G_0), so that ω identifies each tangent space of E with $\mathfrak{g}$. For each $A \in \mathfrak{g}$, let $\vec{A}$ be the vector field on E satisfying $\vec{A} \lrcorner \omega = A$. A Cartan pseudogeometry is called a *Cartan geometry* (and its Cartan pseudoconnection called a *Cartan connection*) if
$$\vec{A} = \left. \frac{d}{dt} r_{e^{tA}} \right|_{t=0}$$
for all $A \in \mathfrak{g}_0$, and $r_g^* \omega = \operatorname{Ad}_g^{-1} \omega$ for all $g \in G_0$.

Lemma 1. *The 1-form ω of Bryant, Griffiths & Hsu is a Cartan connection on M, modelled on $G/G^{pt,line} = \mathbb{P}T\mathbb{P}^2 = \mathbb{F}(1,2)$, the flag variety of pointed lines in projective space.*

Proof. We have only to check that $\vec{A}$ is the infinitesimal generator of the right action, for $A \in \mathfrak{g}^{\text{pt,line}}$. This follows immediately from the simple calculation that
$$\mathcal{L}_{\vec{A}} \omega = -[A, \omega].$$
□

Remark 1. Before going over the theory of Cartan geometries, I would like to meet the concerns of Cartan geometry experts. Cartan (and followers, see Sharpe [36]) worked with Cartan geometries directly in terms of the differential form we have called ω. Cartan geometries modelled on rational homogeneous varieties are known as *parabolic geometries*. Recently, there have been great advances made the theory of invariantly defined vector bundles on manifolds with parabolic geometries. See Čap [5] for an excellent overview. The first step in this recent theory is to move away from Cartan's differential forms on principal bundles, and instead look at associated vector bundles. Associated vector bundles are called *tractor bundles*. One uses the 1-form ω to determine differential operators between tractor bundles. This makes it easier to describe many invariantly defined linear differential operators on tractor bundles, and sequences of such operators. Our concern in this article is with submanifolds inside a manifold M with a Cartan geometry. These submanifolds, in our case, will be the integral curves and stalks. We are not concerned principally with vector bundles on M, but with vector bundles on the integral curves and stalks. Cartan's point of view has the great strength that one can write various exterior differential systems on the bundle E, in terms of the differential form ω. Recall that exterior differential systems are systems of partial differential equations, described in terms of differential forms. One easily sees when the integral manifolds of these exterior differential systems project to submanifolds of M satisfying invariantly defined differential equations, often nonlinear ones. One cannot so easily write down exterior differential systems in terms of vector bundles and connections. The Cartan geometries that we will induce on integral curves and stalks will not be parabolic geometries. Even though the Cartan geometry that we will examine on the space of elements is parabolic, we can not employ the powerful tools of the parabolic theory (Lie algebra cohomology, BGG-sequences, tractor bundles, etc.), because the Cartan geometries we wil study on integral curves and stalks are *not* parabolic. I could have employed parabolic geometries on the stalks and integral curves, as I did in [31], but they would be flat and modelled on the projective line, so the parabolic theory would be trivial. Summing up: Cartan's approach makes it easy to think about differential equations for submanifolds. This paper is about submanifolds (integral curves and stalks) defined by differential equations. Therefore we have followed Cartan.

We will employ a host of results on vector bundles and Cartan geometries, all of which have the same proof (well known and widely known), so we give the proof in just one case:

Lemma 2. *Consider a Cartan geometry $\pi : E \to M$. The tangent bundle is*

$$TM = E \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0) .$$

Proof. At each point $e \in E$, the 1-form $\omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism, taking $\ker \pi'(e) \to \mathfrak{g}_0$. Therefore $\omega_e : T_e E / \ker \pi'(e) \to \mathfrak{g}/\mathfrak{g}_0$ is a linear isomorphism. Also $\pi'(e) : T_e E / \ker \pi'(e) \to T_{\pi(e)} M$ is an isomorphism. Given a function $f : E \to \mathfrak{g}/\mathfrak{g}_0$, define v_f a section of the vector bundle $TE / \ker \pi'$, by the first isomorphism, and a section $\bar{v}_f$ of $\pi^* TM$ by the second. Calculate that $\bar{v}_f$ is G_0-invariant just when f is G_0-equivariant, i.e. just when

$$r_g^* f = \operatorname{Ad}_g^{-1} f.$$

This makes an isomorphism of sheaves between the sections of the tangent bundle TM and the G_0-equivariant functions $E \to \mathfrak{g}/\mathfrak{g}_0$, i.e. the sections of $E \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)$, so that they must be identical vector bundles. □

Definition 5. For $G_0 \subset G$ a closed subgroup, let $\omega \in \Omega^1(G)$ be the left invariant Maurer–Cartan 1-form. Then ω is a Cartan connection on the principal right G_0-bundle $G \to G/G_0$, and the induced Cartan geometry on G/G_0 is called the model Cartan geometry. A Cartan geometry modelled on G/G_0 is called *flat* if it is locally isomorphic to the model Cartan geometry.

Definition 6. The expression $\nabla\omega = d\omega + \frac{1}{2}[\omega, \omega]$ is called the *curvature* of the Cartan geometry; equations on the curvature will be called *structure equations.*

Theorem 9 (Sharpe [36])**.** *A Cartan geometry is flat just when its curvature vanishes.*

Proposition 1. *Pick a flat Cartan geometry $E \to M$ on a compact, connected and simply connected manifold M, modelled on G/G_0 with G connected and G/G_0 connected and simply connected. Then the Cartan geometry is isomorphic to the model.*

Proof. By theorem 3 of McKay [31], some covering space of M maps locally diffeomorphically to G/G_0, and the Cartan geometry on that covering space is pulled back. Because M is simply connected, that covering space is M itself. Because M is compact, the local diffeomorphism is a covering map. Because G/G_0 is connected and simply connected, the map is a diffeomorphism. □

Definition 7. If $G_0 \subset G$ is a closed subgroup of a Lie group, and $\Gamma \subset G$ is a discrete subgroup, acting freely and discontinuously on G/G_0, then we can let $E = G, M = \Gamma \backslash G/G_0, \omega = g^{-1}, dg$, determining a flat Cartan geometry called a *locally Klein geometry.*

Say that a group G *defies* a group H if every morphism $G \to H$ has finite image. We will not repeat the proof of:

Theorem 10 (McKay [31])**.** *A flat Cartan geometry, modelled on G/G_0, defined on a compact connected base manifold M with fundamental group defying G, is a locally Klein geometry.*

Definition 8. If V is a vector space, a V-valued *coframing* on a manifold E is 1-form $\omega \in \Omega^1(E) \otimes V$, so that at each point $e \in E$, $\omega_e : T_e E \to V$ is a linear isomorphism. An isomorphism of coframings is a diffeomorphism matching up the 1-forms. If $G_0 \subset G$ is a closed Lie subgroup of a Lie group, with Lie algebras $\mathfrak{g}_0 \subset \mathfrak{g}$, and ω is a $\mathfrak{g}$-valued coframing, let $\bar{\omega} = \omega \mod \mathfrak{g}_0 \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{g}_0)$. A *local Cartan geometry* modelled on G/G_0 is a $\mathfrak{g}$-valued coframing ω and a function $K : E \to \Lambda^2(\mathfrak{g}/\mathfrak{g}_0) \otimes \mathfrak{g}$, for which

$$d\omega + \frac{1}{2}[\omega, \omega] = K \bar{\omega} \wedge \bar{\omega}.$$

Definition 9. If $E \to M$ bears a Cartan geometry with Cartan connection ω, then ω and the curvature K of ω are together called the *associated local Cartan geometry.* We say that a local Cartan geometry is isomorphic to a Cartan geometry if it is isomorphic to the associated local Cartan geometry.

Theorem 11. *Every local Cartan geometry is locally isomorphic to a Cartan geometry.*

Remark 2. This theorem is a well-known folk theorem, but we know of no source for a proof, so we give a proof to make our exposition more self-contained.

Proof. Consider the foliation of E by the submanifolds $\bar{\omega} = 0$. Since our result is local, we can assume that this foliation is a fiber bundle $E \to M$, and also that this fiber bundle is trivial. Consider the vectors fields $\vec{A}_E$ on E defined by the equation $\vec{A}_E \lrcorner \omega = A$, for any $A \in \mathfrak{g}_0$. Clearly these provide a Lie algebra action, whose orbits are the fibers of $E \to M$. Taking any local section of $E \to M$, say $\sigma : M \to E$, the map

$$(m, A) \in M \times \mathfrak{g}_0 \mapsto e^A m \in E$$

is defined near $A = 0$, and a local diffeomorphism there. Therefore we can find an open set of the form $U_M \times U_{\mathfrak{g}_0}$ with $U_M \subset M$ and $U_{\mathfrak{g}_0} \subset \mathfrak{g}_0$ open sets, on which the map is defined and is a diffeomorphism to its image. Because our results are local, we can assume that $M = U_M$, and the map is a global diffeomorphism, with image all of E. Moreover, we can assume that the exponential map identifies $U_{\mathfrak{g}_0}$ with an open subset $U_{G_0} \subset G_0$. We therefore have $E = M \times U_{G_0} \subset M \times G_0$. On $M \times G_0$, define a 1-form Ω by

$$\Omega_{(m,g_0)} = \mathrm{Ad}_{g_0}^{-1} \omega_{(m,1)} r'_{g_0^{-1}} (m, g_0) .$$

Check that $\Omega = \omega$ on $M \times 1$ and that $\mathcal{L}_{\vec{A}}\Omega = -[A, \Omega]$, for $A \in \mathfrak{g}_0$, so that by uniqueness of solutions of ordinary differential equations, $\Omega = \omega$ on E. □

6. Inducing a Cartan connection on integral curves

Let $C \to M$ be any immersed integral curve. Consider the pullback subbundle $E|_C$. Since $0 = \omega_0^2 = \omega_1^2$ along C on every local section of $E \to M$, and $0 = \omega_0^2 = \omega_1^2$ on the fibers, we find that $0 = \omega_0^2 = \omega_1^2$ on all of $E|_C$. Moreover, $E|_C \to C$ is a principal right $G^{\text{pt,line}}$-bundle.

Lemma 3. *On $E|_C \to C$, ω is a flat Cartan connection.*

Proof. The structure equations are identical to those of $E \to M$, except that $\nabla\omega = 0$ because $\omega_0^2 = \omega_1^2 = 0$. □

7. Classification of Cartan connections on rational curves

Definition 10. A *projective representation* is a morphism of complex Lie groups $\alpha : G \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{C})$. A projective representation is *transitive* if G acts transitively on $\mathbb{P}^n$. Given a transitive projective representation, set $G_0 = \ker\alpha$, $E = G$, and $\omega = g^{-1}\,dg$ the left invariant Maurer–Cartan 1-form on G. Call this the Cartan geometry associated to the transitive projective representation.

Theorem 12. *Every flat Cartan geometry on $\mathbb{P}^n$, with connected model G/G_0, is isomorphic to its model, hence isomorphic to the Cartan geometry associated to a transitive projective representation.*

Proof. Obviously, the construction beginning with a transitive projective representation determines a Cartan geometry. Conversely, any flat Cartan geometry on $\mathbb{P}^n$ is obtained by taking a local biholomorphism to the model $\mathbb{P}^n \to G/G_0$, so a covering map (since $\mathbb{P}^n$ is compact). The deck transformations must be biholomorphisms of $\mathbb{P}^n$, so projective linear transformations. However, every projective linear transformation has a fixed point, so they can't act as deck transformations unless the covering group is trivial. Therefore the covering map is a biholomorphism. □

Corollary 1. *Every Cartan geometry on a rational curve is associated to a transitive surjective projective representation.*

Proof. Any Cartan geometry on $\mathbb{P}^1$ is flat, since the curvature is a semibasic 2-form. No complex Lie subgroup of $\mathbb{P}\,\mathrm{GL}\,(2, \mathbb{C})$ acts transitively on $\mathbb{P}^1$. Therefore the projective representation $G \to \mathbb{P}\,\mathrm{GL}\,(2, \mathbb{C})$ is surjective. □

8. A cornucopia of vector bundles

If M^3 bears a surface path geometry, then the stalks are curves transverse to the integral curves. Let $\Theta \subset TM$ be the field of 2-planes spanned by the tangent lines to integral curves and tangent lines to stalks. In local coordinates, $x, y, \dot{y}$, we see that $\Theta = (dy = \dot{y}\, dx)$, so a contact structure.

Proposition 2. *Let C be an immersed integral curve in a complex 3-fold M with path geometry. Let $E \to M$ be the Cartan geometry associated to the path geometry. Let $\Theta \subset M$ be the canonical contact structure. Let S be the space of points. (If S is not a smooth surface, then equations below involving S are meaningless, but the right hand sides still define vector bundles.) Let $\nu_S C$ the normal bundle of the immersion $C \to M \to S$ (for which a similar proviso applies). Let $G = \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{C})$, G^{pt} the subgroup preserving the point $[e_0]$, G^{line} the subgroup preserving the line through $[e_0]$ and $[e_1]$, $G^{pt,line}$ the subgroup preserving the point and the line, and write their Lie algebras as $\mathfrak{g}, \mathfrak{g}^{pt}$, etc. Then*

$$\begin{aligned} TM|_C &= E|_C \times_{G^{pt,line}} \left(\mathfrak{g}/\mathfrak{g}^{pt,line}\right) \\ \Theta|_C &= E|_C \times_{G^{pt,line}} \left(\left(\mathfrak{g}^{line} + \mathfrak{g}^{pt}\right)/\mathfrak{g}^{pt,line}\right) \\ \nu_M C &= E|_C \times_{G^{pt,line}} \left(\mathfrak{g}/\mathfrak{g}^{line}\right) \\ TS|_C &= E|_C \times_{G^{pt,line}} \left(\mathfrak{g}/\mathfrak{g}^{pt}\right) \\ TC &= E|_C \times_{G^{pt,line}} \left(\mathfrak{g}^{line}/\mathfrak{g}^{pt,line}\right) \\ \nu_S C &= E|_C \times_{G^{pt,line}} \left(\mathfrak{g}/\left(\mathfrak{g}^{pt} + \mathfrak{g}^{line}\right)\right) \end{aligned}$$

Proof. As in the proof of lemma 2. □

Intuitively, the equations above allow us to pretend to work with the space of points S, even if it isn't Hausdorff, by instead working with various vector bundles.

Recall that $\mathbb{P}^1$ bears line bundles $\mathcal{O}(p)$, defined as follows: think of $\mathbb{P}^1$ as the space of lines through 0 in $\mathbb{C}^2$, and let $\mathcal{O}(-1)$ be the bundle whose fiber above a line L is just L; then let $\mathcal{O}(p) = \mathcal{O}(-1)^{\oplus -p}$. So a local section of $\mathcal{O}(p)$ is a choice of map $z \in \text{open } \subset \mathbb{C}^2 \backslash 0 \to f(z)z$ for which $f(\lambda z) = \lambda^p f(z)$. Moreover, the line bundles $\mathcal{O}(p)$ have global nonzero sections just when $p > 0$. Another way to present these line bundles: Let

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \backslash 0.$$

Write B for the group of matrices of the form:

$$g_0 = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

(those which preserve the complex line through e_0). Consider the principal right B bundle $\mathrm{SL}\,(2, \mathbb{C}) \to \mathbb{P}^1$ given by the map $g \in \mathrm{SL}\,(2, \mathbb{C}) \mapsto g e_0 \in \mathbb{P}^1$. Given an open subset $U \subset \mathbb{P}^1$, let $\mathrm{SL}\,(2, \mathbb{C})_U \to U$ be the pullback bundle.

Lemma 4. *Sections of $\mathcal{O}(p)_U \to U$ correspond to maps $F : \mathrm{SL}\,(2, \mathbb{C})_U \to \mathbb{C}$ for which*

$$F(g g_0) = a^p F(g),$$

for all $g_0 \in B$.

Proof. Pick a local section f of $\mathcal{O}(p)$, i.e. a choice of map $f : \hat{U} \subset \mathbb{C}^2 \backslash 0 \to \mathbb{C}$, where $\hat{U}$ is the preimage of U under $\mathbb{C}^2 \backslash 0 \to \mathbb{P}^1$, and with $f(az) = a^p z$. Define $F(g) = f(ge_0)$. Conversely, given F, define $f(z) = F(g_z)$ where

$$g_z = \begin{pmatrix} z_1 & 0 \\ z_2 & z_1^{-1} \end{pmatrix},$$

defined for all $z_1 \neq 0$ for which g_z lies in the domain of F. □

In giving this proof, we are merely trying to avoid abstract Borel–Weil–Bott theory, and give concrete expressions for these line bundles.

Corollary 2. *On any rational integral curve,*

$$\begin{aligned} TM|_C &= \mathcal{O}(2) \oplus \mathcal{O}^{\oplus 2} \\ \Theta|_C &= \mathcal{O}(2) \oplus \mathcal{O}(-1) \\ TM/\Theta|_C &= \mathcal{O}(1) \\ \nu_M C &= \mathcal{O}^{\oplus 2} \\ TS|_C &= \mathcal{O}(2) \oplus \mathcal{O}(1) \\ TC &= \mathcal{O}(2) \\ \nu_S C &= \mathcal{O}(1) \end{aligned}$$

Proof. Calculate these on the model. (Note that the normal bundle of $C = \mathbb{P}^1 \subset M = \mathbb{P}T\mathbb{P}^2$ has sections coming from the tangent bundle of the dual space $\mathbb{P}^{2*}$; from which it is easy to see that this normal bundle is trivial.) By the classification of Cartan geometries on rational curves, the Cartan geometry on every rational integral curve is isomorphic to the one found on the integral curves of the model, making these vector bundles identical. □

9. Kodaira deformation theory

We give a brief review of Kodaira's theory [26, 27, 28].

Definition 11. Let Y and M be complex manifolds and let $\pi_M : M \times Y \to M$ and $\pi_Y : M \times Y \to Y$ be the obvious maps. A *family* of closed complex submanifolds of the complex manifold M parameterized by Y is a complex submanifold $F \subset M \times Y$ such that the $\pi_Y|_F : F \to Y$ is a proper submersion. Let $X_y = F \cap M \times \{y\}$.

Definition 12. A *morphism* of families $F_j \subset M \times Y_j$ (in the same manifold M), $j = 0, 1$, is a map $\phi : Y_0 \to Y_1$ so that $(m, y_0) \to (m, \phi(y_0))$ takes F_0 to F_1.

Definition 13. We say that a submanifold $X \subset M$ belongs to a family $\{X_y\}_{y \in Y}$ if $X = X_y$ for some $y \in Y$.

Definition 14. A family $\{X_y\}_{y \in Y}$ is *locally complete* if, should one of the submanifolds X_y belong to another family of complex submanifolds $\{X_z\}_{z \in Z}$, say $X_{y_0} = X_{z_0} \subset M$, then there is morphism of families $U \to Y$ defined on an open neighborhood $U \subset Z$ of z_0, taking $z_0 \mapsto y_0$.

Definition 15. A closed complex submanifold $X \subset M$ with normal bundle ν_X is *free* if $H^1(X, \nu_X) = 0$.

Theorem 13 (Kodaira [26, 27, 28])**.** *If $X \subset M$ is an immersed free closed complex submanifold of a complex manifold, then X belongs to a family of submanifolds $\left\{X_y\right\}_{y \in Y}$, and there is an isomorphism $T_X Y = H^0\left(X, \nu_X\right)$. Moreover, this family is locally complete. If $H^1\left(X, TX\right) = 0$, then every manifold in this family is biholomorphic to every other.*

Corollary 3. *Let X be a closed complex manifold with $H^1\left(X, TX\right) = 0$. Let M be a complex manifold, and Y the set of free closed complex submanifolds of M biholomorphic to X. Then Y is either empty or a complex manifold of dimension equal to the dimension of $H^0\left(X, \nu_X\right)$, and a locally complete family.*

To make use of this, we need to know a little sheaf cohomology:

Lemma 5. *For the sheaf $\mathcal{O}\left(p\right)$ on $\mathbb{P}^1$,*

$$\dim H^0\left(\mathbb{P}^1, \mathcal{O}\left(p\right)\right) = \begin{cases} p+1 & p \ge 0 \\ 0 & p < 0 \end{cases}$$

$$\dim H^1\left(\mathbb{P}^1, \mathcal{O}\left(p\right)\right) = \begin{cases} 0 & p \ge 0 \\ -(p+1) & p < 0 \end{cases}$$

See Griffiths & Harris [21] for proof.

Corollary 4. *If a surface path geometry has a rational integral curve [rational stalk], and all integral curves [stalks] compact, then all of its integral curves [stalks] are rational, and the space of integral curves [points] is a smooth surface.*

10. Identifying line bundles

The main tool in my paper [31] is a method for computing which vector bundle a given differential invariant lies in, when it is restricted to a rational curve. We will try to make the method more transparent.

Suppose that $G_0 \subset G$ are connected Lie groups. Fix a Lie group morphism $\mathrm{SL}\left(2, \mathbb{C}\right) \to G$, whose image does not lie in G_0. Let $B \subset \mathrm{SL}\left(2, \mathbb{C}\right)$ be the Borel subgroup, with Lie algebra $\mathfrak{b} \subset \mathfrak{sl}\left(2, \mathbb{C}\right)$. Let $E \to C$ be a Cartan geometry on a rational curve, with Cartan connection ω, modelled on a homogeneous space G/G_0. Let $\omega_B = \omega \mod \mathfrak{b}$. So ω is $\mathfrak{b}$-valued modulo ω_B. Therefore we can write

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_1 \\ 0 & -\omega^0_0 \end{pmatrix} \pmod{\omega_B}.$$

Consider a function F : connected open $\subset E \to \mathbb{C}$. If there is a number p for which

$$dF = p\omega^0_0 \pmod{\omega_B},$$

then call p the *weight* of F.

Proposition 3. *Suppose that $E \to C$ is a Cartan geometry on a rational curve. A function F : connected open $\subset E \to \mathbb{C}$ has integer weight p just when F is a local section of $\mathcal{O}\left(p\right)$ on C.*

Proof. Since $E \to C$ is isomorphic to the model, we can assume $E = G, C = G/G_0 = \mathbb{P}^1$. The action of $\mathrm{SL}\left(2, \mathbb{C}\right)$ on G/G_0 is not trivial, because the image of

$\mathrm{SL}(2,\mathbb{C}) \to G$ is not contained in G_0. There is only one one-dimensional homogeneous space of $\mathrm{SL}(2,\mathbb{C})$; therefore $\mathrm{SL}(2,\mathbb{C})$ acts on G/G_0 as the usual action on $\mathbb{P}^1$. Pull back F to $\mathrm{SL}(2,\mathbb{C})$. In explicit coordinates on B,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

the equation on dF precisely demands that on $\mathrm{SL}(2,\mathbb{C})$, $F(gg_0) = a_0^{-p} F(g)$. □

Corollary 5. *Functions of negative integer weight vanish.*

11. Dual surface path geometries

It is an old observation that every surface path geometry has a dual surface path geometry, given by interchanging the role of integral curves and stalks; see Bryant, Griffiths and Hsu [3] or Crampin and Saunders [14]. We can see directly from the structure equations of the Cartan geometry that this is just an interchange of indices $\omega^\mu_\nu \leftrightarrow \omega^{2-\nu}_{2-\mu}$.

Example 8. For the model, this duality is the duality between lines in the projective plane and points in the dual plane, i.e. between 2-planes through 0 in $\mathbb{C}^3$ and lines through 0 in $\mathbb{C}^{3*}$, given by $\Pi \mapsto \Pi^\perp$.

Example 9 (Hitchin [23] p. 83). The ordinary differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{4y^3}$$

has solutions

$$y^2 = ax^2 + bx + c$$

for any complex constants a, b, c for which $4ac - b^2 = 1$. Now treat x, y as constants, and think of the equation as determining a family of curves $b = b(a), c = c(a)$. Differentiating twice, we find the relationship:

$$\frac{d^2c}{da^2} = \frac{4Q(a\frac{dc}{da} + c + \sqrt{Q})}{(4ac-1)(2a^2\frac{dc}{da} - 2ac + 2a\sqrt{Q} + 1)},$$

the dual ordinary differential equation, where

$$Q = c^2 - 2ac\frac{dc}{da} + a^2\left(\frac{dc}{da}\right)^2 + \frac{dc}{da}.$$

The integral curves of the dual equation consist precisely in the values of a, b, c which will produce a solution $y = y(x)$ of the original equation which passes through a chosen point of the (x, y) plane.

Even though the equations

$$y^2 = ax^2 + bx + c, 4ac - b^2 = 1$$

are quadratic, so the integral curves and stalks are rational curves in the plane, the original differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{4y^3}$$

is *not* torsion-free, detecting the singularity emerging in the ordinary differential equation at $y = 0$. Nonetheless, the dual equation is torsion-free, and is straight, as Hitchin explains.

Example 10. The ordinary differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{2}y(y-1) + \frac{y-\frac{1}{2}}{y(y-1)}\left(\frac{dy}{dx}\right)^2$$

conserves the quantity

$$\lambda = y - \frac{\left(\frac{dy}{dx}\right)^2}{y(y-1)},$$

from which we conclude that

$$\frac{dy}{dx} = \sqrt{y(y-1)(y-\lambda)},$$

i.e. an elliptic curve in phase space, giving

$$x + \varpi = \int \frac{dy}{\sqrt{y(y-1)(y-\lambda)}},$$

where ϖ is the integral over a period. Our elliptic curve has equation

$$\dot{y}^2 = y(y-1)(y-\lambda).$$

Compute (as in Clemens [13] p. 59) that

$$d\left(\frac{\dot{y}}{(y-\lambda)^2}\right) = -\frac{1}{2}\frac{dy}{\dot{y}} - 2(2\lambda-1)\frac{\partial}{\partial\lambda}\frac{dy}{\dot{y}} - 2\lambda(\lambda-1)\frac{\partial^2}{\partial\lambda^2}\frac{dy}{\dot{y}}.$$

Integrating both sides along the elliptic curve, avoiding $\dot{y} = 0$, we find the Picard–Fuchs equation

$$0 = \lambda(\lambda-1)\frac{d^2\varpi}{d\lambda^2} + (2\lambda-1)\frac{d\varpi}{d\lambda} + \frac{1}{4}\varpi,$$

the dual path geometry, so that (x, y) are the variables on the original space, and (λ, ϖ) are the variables on the dual space. It is remarkable that the original equation has nowhere vanishing Tresse torsion (which is easy to calculate, and fits with our theory, since all of the integral curves are elliptic curves), but the Picard–Fuchs equation is torsion-free, so straight.

Example 11. Similarly, returning to our previous example of

$$\frac{d^2y}{dx^2} = 6y^2,$$

the solutions are given implicitly by

$$\int \frac{dy}{\sqrt{4y^3 + A}} = x + B,$$

and the dual equation describes how to vary B as a function of $A = g_3$, in order to keep this elliptic function passing through a fixed point x_0, y_0, i.e. the relation between the modular parameter g_3 (with $g_2 = 0$) and the period of the elliptic integral. This is equivalent to solving for B as a function of A in the equation

$$y_0 = \wp\left(x_0 + B\right)\Big|_{g_3=A}^{g_2=0}.$$

Again there is a Picard–Fuchs equation, but it is more difficult to find, and we will not try to find it.

12. Rationality of integral curves and stalks for surface path geometries

Theorem 14 (Hitchin [23], Bryant, Griffiths, Hsu [3])**.** *If the integral curves [stalks] of a surface path geometry are rational, then $K_1 = 0$ [$K_2 = 0$].*

Proof. Calculating the exterior derivatives of the structure equation in item 2 on page 12, we find

$$\begin{aligned}\nabla \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} &= d \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} + \begin{pmatrix} 5\omega^0_0 + 3\omega^1_1 & 0 \\ 0 & 4\omega^0_0 + 5\omega^1_1 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \\ &= \begin{pmatrix} \nabla^0_1 K_1 \omega^1_0 + \nabla^0_2 K_1 \omega^2_0 + L_1 \omega^2_1 \\ \nabla^0_1 K_2 \omega^1_0 + L_2 \omega^2_0 + \nabla^1_2 K_2 \omega^2_1 \end{pmatrix}.\end{aligned}$$

If C is a rational integral curve, the bundle $E_C \to C$ has $\omega^2_0 = \omega^2_1 = 0$. Consider the copy of $\mathfrak{sl}(2, \mathbb{C})$ determined by

$$0 = \omega^{\bullet}_2 = \omega^2_{\bullet} = \omega^0_0 + \omega^1_1.$$

Our equations simplify to

$$d \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \omega^0_0 \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

modulo semibasic terms, hence weights $-2, 1$ for K_1, K_2 respectively. Therefore $K_1 = 0$. Duality proves the results for stalks. □

Corollary 6. *If the space of elements of a surface path geometry is connected, and both the generic integral curve and the generic stalk are rational, then the path geometry is locally isomorphic to the model.*

Proof. We see that $K_1 = K_2 = 0$, and differentiate to find that all invariants vanish. The structure equations are identical. Let ω be the Cartan connection of the path geometry on $E \to M$, and ω' the Cartan connection of the model, say $E' \to M'$. Then on $E \times E'$, the holonomic differential system $\omega = \omega'$ has integral manifolds giving the graphs of local isomorphisms. □

Proposition 4 (Cartan [9])**.** *Under any local choice of section of $E \to M$, K_1 is Tresse torsion, up to multiplication by a nowhere vanishing function. In particular, Tresse torsion vanishes just when K_1 does.*

13. Normal projective connections

A normal projective connection is complicated to define precisely.

Definition 16. A *local affine connection* on a manifold is a choice of open set and affine connection defined on that open set. A *covered normal projective connection* is a set of local affine connections whose open sets cover the manifold, so that on overlaps of the open sets, the affine connections have the same geodesics modulo reparameterization. A *normal projective connection* is a covered normal projective connection which is not strictly contained in any other covered normal projective connection.

The tricky issue that makes the definition so complicated is already seen in projective space: projective space has no affine connection. Each affine chart has the obvious flat affine connection. On the overlaps of the affine charts, the geodesics (lines) agree. So projective space has a normal projective connection.

Definition 17. The path geometry of a normal projective connection on a manifold S is the manifold $M = \mathbb{P}TS$, with stalks the fibers of the obvious projection $\mathbb{P}TS \to S$, and integral curves the curves in $\mathbb{P}TS$ each of which is composed of tangent lines to a geodesic.

Theorem 15 (Hitchin [23], Bryant, Griffiths, Hsu [3]). *If the stalks [integral curves] are rational, then the path geometry is the path geometry of a normal projective connection on the smooth space of points [integral curves].*

Proof. Fels [17] p. 239 demonstrates very clearly how the requirement of being a projective connection (i.e. the geodesic equation of an affine connection, in some coordinates) is identified with having an equation of the form

$$\begin{aligned}\frac{d^2y}{dx^2} &= f\left(x, y, \frac{dy}{dx}\right)\\ &= \sum_{k=0}^{3} a_k(x,y)\left(\frac{dy}{dx}\right)^k.\end{aligned}$$

The surprising third order term arises because x is not necessarily the natural parameter along the geodesic. Hitchin [23] gives a very clear demonstration that this form of differential equation is ensured by rationality of the integral curves. Cartan [7] shows that this form of differential equation is equivalent to vanishing of K_1, which turns out in local coordinates to be a multiple of $\frac{\partial^4 f}{\partial \dot{y}^4}$. □

Bordag & Dryuma [2] make use of Cartan's invariants of projective connections to analyse second order ordinary differential equations.

14. Higher dimensional path geometries

Beyond surface path geometries, the story is more complicated. Following Mark Fels [17], we can define a second order structure on the space of elements M, say $E \to M$. We will not go through the details of the construction of this second order structure, which is quite involved, and is explained in detail in Fels's paper [17]. Lets just say that the second order structure is modelled on the tower of bundles $\mathbb{P}\,\mathrm{GL}(n+2,\mathbb{C}) \to F\mathbb{P}T\mathbb{P}^{n+1} \to \mathbb{P}T\mathbb{P}^{n+1}$. By $F\mathbb{P}T\mathbb{P}^{n+1}$ we mean the bundle of frames on $\mathbb{P}T\mathbb{P}^{n+1}$, i.e. linear isomorphism of tangent spaces of $\mathbb{P}T\mathbb{P}^{n+1}$ to $\mathbb{C}^{2n+1}$. Tanaka has found a Cartan connection associated to any path geometry, using a different normalization of torsion; this will be irrelevant to us, although it would provide a more elegant set of structure equations than Fels's. The interested reader should consult Čap [5] for an elegant explanation and development of Tanaka's construction.

The structure equations of Fels's second order structure are

$$d\omega + \frac{1}{2}[\omega,\omega] = \nabla\omega,$$

where $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ is a 1-form valued in $\mathfrak{g} = \mathfrak{sl}(n+2,\mathbb{C})$, which we can write as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_J \\ \omega^1_0 & \omega^1_1 & \omega^1_J \\ \omega^I_0 & \omega^I_1 & \omega^I_J \end{pmatrix}$$

where $\omega^0_0 + \omega^1_1 + \omega^I_I = 0$, with indices

$$\begin{aligned} \mu, \nu, \sigma, \tau &= 0, \dots, n+1 \\ i, j, k, l &= 1, \dots, n+1 \\ I, J, K, L &= 2, \dots, n+1 \end{aligned}$$

The terms $\nabla\omega$ have the form

$$\nabla\omega^\mu_\nu = \frac{1}{2} t^\mu_{\nu kl} \omega^k_0 \wedge \omega^l_0 + T^\mu_{\nu kl} \omega^k_1 \wedge \omega^l_0 + {}^\sharp t^{\mu K}_{\nu l} \omega^1_K \wedge \omega^l_0 .$$

These torsion terms satisfy a large collection of identities. For example, $0 = \nabla\omega^1_0 = \nabla\omega^I_0$. The torsion terms and many (perhaps all) of their identities are worked out in Fels [17], while Grossman [22] assumes vanishing of the 1-torsion of his second order structure, and therefore his identities do not cover the generality required for our results.

The structure group of the second order structure is the group of projective transformations fixing a pointed line. The second order structure is *not* necessarily a Cartan connection, since $\nabla\omega$ is not necessarily semibasic for the map $E \to M$, i.e. is not necessarily a multiple of the 1-forms $\omega^1_0, \omega^I_0, \omega^I_1$ which span the semibasic 1-forms for that map. However, the mysterious ${}^\sharp t$ terms, which precisely form the obstruction to being a Cartan connection, vanish in every term except $\nabla\omega^0_1, \nabla\omega^0_J$, where they are

$$\begin{aligned} {}^\sharp t^{0K}_{1L} &= \frac{1}{n-1} t^K_{1L1} \\ {}^\sharp t^{0K}_{J1} &= \left(1 + \frac{2}{n-1}\right) t^K_{1J1} . \end{aligned}$$

Lemma 6. *Consider the 1-form ω on Fels's second order structure $E \to M$ for a path geometry on M. The obstructions to ω being a Cartan connection vanish just precisely when the 1-torsion of the second order structure vanishes, i.e. when $\nabla\omega^1_0 = \nabla\omega^I_0 = \nabla\omega^I_1 = 0$.*

Proof. Grossman [22] p. 435 shows that the vanishing of t^I_{1J1} ensures the vanishing of t^I_{1JK}, and that this ensures the vanishing of all ${}^\sharp t$ terms. It is a long, but not difficult, calculation which requires only differentiating the structure equations, made much easier by using my notation. This is precisely the condition for vanishing of the 1-torsion of the second order structure. □

The reader may be curious as to how to obtain the identities. The pattern one observes in differentiating structure equations is straightforward: we have

$$d\omega + \omega \wedge \omega = \nabla\omega$$

so that taking exterior derivative gives the *Bianchi identity*

$$d\nabla\omega = \nabla\omega \wedge \omega - \omega \wedge \nabla\omega .$$

If $\nabla\omega$ satisfies some identity, with constant coefficients, then so must $d\nabla\omega$, from which the above equation gives more identities on $\nabla\omega$. Computing out these equations gives

$$\begin{aligned} 0 = \nabla\omega^1_0 &\implies 0 = \nabla\omega^1_I \wedge \omega^I_0 \\ 0 = \nabla\omega^I_0 &\implies 0 = \nabla\omega^I_1 \wedge \omega^1_0 + \left(\nabla\omega^I_J - \delta^I_J \nabla\omega^0_0\right) \wedge \omega^J_0 \\ 0 = \nabla\omega^0_0 + \nabla\omega^1_1 + \nabla\omega^I_I &\implies 0 = \nabla\omega^0_1 \wedge \omega^1_0 + \nabla\omega^0_J \wedge \omega^J_0. \end{aligned}$$

Another observation which entails many identities is that in the process of the method of equivalence, covariant derivatives ∇ are always semibasic. The 1-forms $\omega^1_0, \omega^I_0, \omega^I_1$ (those under the diagonal of the matrix ω) are semibasic for $E \to M$. But since E is a second-order structure, it is built on a first order structure, say $E \to F \to M$, and the on-diagonal entries $\omega^0_0, \omega^1_1, \omega^I_J$, together with the below diagonal, are semibasic for $E \to F$. Therefore

$$\begin{aligned} \nabla \begin{pmatrix} \omega^1_0 \\ \omega^I_0 \\ \omega^I_1 \end{pmatrix} &= 0 \mod \left(\omega^1_0, \omega^K_0, \omega^K_1\right)^2, \\ \nabla \begin{pmatrix} \omega^0_0 \\ \omega^1_1 \\ \omega^I_J \end{pmatrix} &= 0 \mod \left(\omega^1_0, \omega^K_0, \omega^K_1, \omega^0_0, \omega^1_1, \omega^K_L\right)^2. \end{aligned}$$

Proposition 5. *The second order structure of Fels determines and is determined by a unique path geometry. Moreover, every second order structure modelled on the tower of bundles $\mathbb{P}\,\mathrm{GL}\,(n+2,\mathbb{C}) \to F\mathbb{P}T\mathbb{P}^{n+1} \to \mathbb{P}T\mathbb{P}^{n+1}$ is Fels's second order structure of a path geometry if and only if it satisfies $\nabla\omega^1_0 = \nabla\omega^I_0$.*

Proof. We will sketch the proof, which depends on the details of Fels's argument from Fels [17]. Given the path geometry, we leave it to Fels to construct the second order structure. Given the second order structure, say $E \to M$, pick any local section σ. Following Fels's definition of E (which is complicated), one finds that $\left(0 = \sigma^*\omega^I_0 = \sigma^*\omega^1_0\right)$ is the foliation by integral curves, while the foliation by stalks is $\left(0 = \sigma^*\omega^1_0 = \sigma^*\omega^I_0\right)$. So if there is a path geometry inducing the second order structure, then this is it. Retracing Fels's steps, we can see that the second order structure is now completely determined, since Fels's algorithm for constructing the second order structure depends only on having a given path geometry and forcing the equations $0 = \nabla\omega^1_0 = \nabla\omega^I_0$, which is enough to determine the rest of his equations on torsion. □

15. Rational integral curves

Clearly an integral curve is rational just when it is compact with finite fundamental group, by the classification of complex curves (see Forster [19]).

Theorem 16. *If the generic integral curve of a path geometry on a manifold M^{2n+1} is rational, then the path geometry is* torsion-free *(i.e. its Fels/Tresse torsion vanishes).*

Proof. We prove the result for $n > 1$, i.e. for the Fels torsion, since the result for $n = 1$ is proven above. Let $\iota : C \to M$ be an immersed integral curve, $E \to M$ the bundle constructed by Fels, with 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{sl}\,(n+2,\mathbb{C})$. On the pullback

This paper	Fels	Grossman
$I, J, K, L = 2, \dots, n+1$	$i, j, k, l = 1, \dots, n$ $i = I - 1, \text{etc.}$	$i, j, k, l = 1, \dots, n$ $i = I - 1, \text{etc.}$
$\begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_J \\ \omega^1_0 & \omega^1_1 & \omega^1_J \\ \omega^I_0 & \omega^I_1 & \omega^I_J \end{pmatrix}$	$\begin{pmatrix} \frac{1}{n+2}\left(\alpha + \Omega^i_i\right) & \sigma & -\beta_j \\ \omega & \frac{1}{n+2}\left(\Omega^i_i - (n+1)\,\alpha\right) & -\kappa_j \\ \theta^i & \pi^i & -\Omega^i_j + \frac{\delta^i_j}{n+2}\left(\Omega^k_k + \alpha\right) \end{pmatrix}$	$\begin{pmatrix} \epsilon & \gamma & \pi_j \\ \lambda & \alpha + \epsilon & \zeta_j \\ \theta^i & \eta^i & \beta^i_j + \delta^i_j \epsilon \end{pmatrix}$
t^I_{1J1}	$\tilde{P}^i_j$	T^i_j
t^I_{1JK}	$2\tilde{Q}^i_{jk}$	S^i_{jk}
t^0_{1K1}	$-Y_k$	$*$
t^0_{1KL}	$2X_{kl}$	$*$
T^0_{1KL}	W_{kl}	$*$
T^0_{JKL}	$-\left(\lambda_{jk}\right)_{\theta^l}$	P_{jkl}
t^1_{JKL}	$-\left(\xi^2_{jk}\right)_{\theta^l}$	Z_{jkl}
t^I_{JK1}	$-\tilde{Q}^i_{jk} + T^i_{jk} - \frac{\delta^i_j}{n+2} T^l_{lk}$	$*$
t^I_{JKL}	$-R^i_{jkl} + \frac{\delta^i_j}{n+2} R^m_{mkl}$	$*$
T^I_{JKL}	$-\tilde{S}^i_{jkl}$	B^i_{jkl}

TABLE 4. Dictionary of notation between this paper, Fels [17], and Grossman [22]; $*$ indicates that no notation was provided for this quantity.

bundle $\iota^* E \to C$, $\omega^I_0 = 0$. The structure equations simplify greatly. Indeed ω forms a flat Cartan connection on C, with $\omega \in \Omega^1 (\iota^* E) \otimes \mathfrak{g}^{\text{line}}$, modelled on $G^{\text{line}} / G^{\text{pt,line}}$, the Cartan connection for a line in projective space, which is easy to check. Taking exterior derivative of the structure equations, we find that on E the invariant t^I_{1J1} satisfies

$$\begin{aligned} \nabla t^I_{1J1} &= dt^I_{1J1} + 2 \left(\omega^0_0 - \omega^1_1\right) t^I_{1J1} + \omega^I_K t^K_{1J1} - t^I_{1K1} \omega^K_J \\ &= \nabla^0_K t^I_{1J1} \omega^K_0 + \nabla^1_K t^I_{1J1} \omega^K_1 , \end{aligned}$$

for some functions $\nabla^0_K t^I_{1J1}$ and $\nabla^1_K t^I_{1J1}$. Fixing the subalgebra $\mathfrak{sl}(2, \mathbb{C}) \subset G^{\text{line}}$ given by the structure equations $0 = \omega^0_J = \omega^1_J = \omega^I_J = \omega^0_0 + \omega^1_1$, we find

$$dt^I_{1J1} = -4 \omega^0_0 t^I_{1J1},$$

so that t^I_{1J1} has weight -4. This ensures the vanishing of t^I_{1J1} at every point of E; Grossman [22] p. 435 takes exterior derivatives of the structure equations, to show that vanishing of most of the other invariants follows, leaving only

$$\begin{aligned} \nabla \omega &= d\omega + \frac{1}{2} [\omega, \omega] \\ &= \begin{pmatrix} 0 & 0 & T^0_{JKL} \omega^K_1 \wedge \omega^L_1 \\ 0 & 0 & T^1_{JKL} \omega^K_1 \wedge \omega^L_1 \\ 0 & 0 & T^I_{JKL} \omega^K_1 \wedge \omega^L_1 \end{pmatrix} . \end{aligned}$$

At this stage, we may wonder if the weights of these invariants will kill them as well. Once again taking exterior derivatives, as Grossman demonstrates,

$$\begin{aligned} \nabla \begin{pmatrix} T^0_{JKL} \\ T^1_{JKL} \\ T^I_{JKL} \end{pmatrix} =& d \begin{pmatrix} T^0_{JKL} \\ T^1_{JKL} \\ T^I_{JKL} \end{pmatrix} \\ &+ \begin{pmatrix} \left(2\omega^0_0 + \omega^1_1\right) T^0_{JKL} - \left(T^0_{MKL} \omega^M_J + T^0_{JML} \omega^M_K + T^0_{JKM} \omega^M_L\right) + \omega^0_1 T^1_{JKL} + \omega^0_M T^M_{JKL} \\ \left(\omega^0_0 + 2\omega^1_1\right) T^1_{JKL} - \left(T^1_{MKL} \omega^M_J + T^1_{JML} \omega^M_K + T^1_{JKM} \omega^M_L\right) \\ \left(\omega^0_0 + \omega^1_1\right) T^I_{JKL} + \omega^I_M T^M_{JKL} - \left(T^I_{MKL} \omega^M_J + T^I_{JML} \omega^M_K + T^I_{JKM} \omega^M_L\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \nabla^0_M T^0_{JKL} & \nabla^1_M T^0_{JKL} \\ -T^0_{JKL} & \nabla^0_M T^1_{JKL} & \nabla^1_M T^1_{JKL} \\ 0 & \nabla^0_M T^I_{JKL} & \nabla^1_M T^I_{JKL} \end{pmatrix} \begin{pmatrix} \omega^1_0 \\ \omega^M_0 \\ \omega^M_1 \end{pmatrix} . \end{aligned}$$

When I look on the same copy of $\mathfrak{sl}(2, \mathbb{C})$, structure equations turn to

$$d \begin{pmatrix} T^0_{JKL} \\ T^1_{JKL} \\ T^I_{JKL} \end{pmatrix} = \begin{pmatrix} -\omega^0_0 T^0_{JKL} + \omega^0_1 T^1_{JKL} \\ \omega^0_0 T^1_{JKL} - \omega^1_0 T^0_{JKL} \\ 0 \end{pmatrix} .$$

So T^0_{JKL} doesn't have a weight, since it has a ω^0_1 term, while the weights of T^1_{JKL}, T^I_{JKL} are 1 and 0 respectively. Therefore we cannot conclude that these invariants vanish; we will soon consider how they could come about. □

15.1. **Segré geometries.** We have just seen why rationality of integral curves forces vanishing of torsion. We need to see why vanishing of torsion ensures rationality of integral curves (modulo local isomorphism). Grossman [22] proved that torsion-free path geometries are locally isomorphic to path geometries derived from *Segré structures*, so we will need to define and examine Segré structures to see if the derived path geometries have rational integral curves.

In the model case, of lines in projective space, i.e. the system of ordinary differential equations

$$\frac{d^2y^I}{dx^2} = 0,$$

the space of integral curves is the Grassmannian of lines in projective space, i.e. of 2-planes in $\mathbb{C}^{n+1}$. Therefore we should try to understand the local geometry of the Grassmannian clearly, and look for analogies when studying general second order systems. Recall that the Grassmannian is G/G^{line} where $G = \mathbb{P}\,\text{GL}\,(n+1,\mathbb{C})$, and G^{line} the subgroup of G fixing a projective line.

Definition 18. A Cartan geometry $E \to \Lambda$ modelled on the Grassmannian of 2-planes in $\mathbb{C}^{n+2}$ is called a *Segré geometry*. Let $G^{\text{pt}} \subset G$ be the subgroup of transformations preserving a point on the given projective line, and $G^{\text{pt,line}} = G^{\text{pt}} \cap G^{\text{line}}$ the subgroup fixing the point and the line, so that the model Segré geometry is $\text{Gr}\left(2,\mathbb{C}^{n+2}\right) = G/G^{\text{line}}$. The space $E/G^{\text{pt,line}}$ is called the *space of elements* of the Segré geometry. The fibers of $E/G^{\text{pt,line}} \to E/G^{\text{line}} = \Lambda$ are called the *integral curves* of the Segré geometry.

For the Grassmannian, the space of elements is the space of choices of a line in projective space and a point on that line. The integral curves of the Grassmannian are the choices of points lying in a given line in projective space. Keep in mind that the integral curves of a Segré geometry $E \to \Lambda$ are *not* submanifolds of the base manifold Λ, but rather the fibers of the space of elements M as a bundle $M \to \Lambda$.

Lemma 7. *The integral curves of a Segré geometry are rational curves.*

Proof. They are copies of $G^{\text{line}}/G^{\text{pt,line}} = \mathbb{P}^1$. □

Lemma 8. *The space of elements of a Segré geometry bears a canonical Cartan geometry modelled on G/G^{pt}, for which all integral curves are rational curves.*

Proof. Suppose that $E \to \Lambda$ is a Segré geometry, with Cartan connection ω. Let M be the space of elements of that Segré geometry. One easy checks the hypotheses of a Cartan connection to see that ω is a Cartan connection for $E \to M$, modelled on the space of pointed lines in projective space. □

Definition 19. Let $G' \subset G^{\text{line}}$ be the subgroup fixing a point of the Grassmannian and fixing the tangent space to the Grassmannian at that point. A Segré geometry $E \to \Lambda$ is called *torsion-free* if $\nabla\omega = 0 \pmod{\mathfrak{g}'}$.

Torsion-freedom of a Segré geometry $E \to \Lambda$ ensures that the equations $\omega^1_0 = \omega^I_0 = 0$ are holonomic (i.e. satisfy the conditions of the Frobenius), so that the manifold E is foliated by the integral manifolds of this equation. Moreover, the reader can check that each integral manifold maps to an immersed submanifold of Λ, called naturally a *stalk* of the Segré geometry. Indeed the stalks foliate the space of elements.

Theorem 17. *A path geometry is straight just when it is torsion-free, which occurs just when it is locally isomorphic to the path geometry of a unique torsion-free Segré geometry.*

Proof. We have seen that the integral curves of the Cartan connection of the space of elements of a Segré geometry are rational, hence the path geometry is straight and therefore torsion-free. If we have a torsion-free path geometry, then its Cartan connection satisfies the torsion-freedom condition required of a torsion-free Segré geometry. By theorem 11, the Cartan connection is locally isomorphic to the Cartan connection of a torsion-free Segré geometry. Therefore the space of elements of the Segré geometry is identified locally with the path geometry.

If we have a torsion-free Segré geometry, then its structure equations are precisely those of a torsion-free path geometry on the space of pointed lines, with its integral curves as integral curves, and stalks as stalks. □

Proposition 6 (Grossman [22]). *The general torsion-free Segré geometry on a manifold* Λ *of dimension* $2(n-1)$ *depends on* $n(n+1)$ *arbitrary functions of* $n+1$ *variables.*

Grossman's proof unfortunately employs the Cartan–Kähler theorem, which is not constructive. There is no known construction producing the torsion-free Segré geometries, or even any large family of examples of them. It would be very interesting to classify the homogeneous torsion-free Segré geometries, and those of low cohomogeneity.

15.2. **Segré structures.** Just as for normal projective connections, we need to take care in defining Segré structures.

Definition 20. A *local Segré structure* on a manifold Λ of dimension $2n$ is a choice of an open set $\Omega \subset \Lambda$, two vector bundles U, V on that open set of ranks 2 and n respectively, and an isomorphism $U \otimes V = T\Omega$. The *rank* of a tangent vector is its rank as a tensor in $U \otimes V$. Two local Segré structures are *equivalent* if the give the same ranks to all tangent vectors. A *covered Segré structure* is a set of local Segré structures, equivalent on overlaps of open sets, whose open sets cover the manifold. One covered Segré structure is a *refinement* of another if it strictly contains the other. A Segré structure is a covered Segré structure not strictly contained in any other covered Segré structure.

The Grassmannian of 2-planes in $\mathbb{C}^{n+2}$ has the obvious Segré structure, and we can choose global vector bundles U and V:

$$T_P \operatorname{Gr}\left(2, \mathbb{C}^{n+2}\right) \cong P^* \otimes \left(\mathbb{C}^{n+2}/P\right).$$

Definition 21. A Segré geometry has curvature given by

$$\nabla \omega^\mu_\nu = K^{\mu ab}_{\nu IJ}$$

where $\mu, \nu = 0, \dots, n, a, b = 0, 1, I, J = 2, \dots, n$. Following Machida & Sato [30] (who follow Tanaka [37]), we will say that ω is *normal* if $K^{I01}_{0KL} + K^{I10}_{0KL} = 0$.

Lemma 9 (Machida & Sato [30]). *A Segré geometry determines a Segré structure. Conversely, a Segré structure uniquely determines a normal Segré geometry, reversing the construction. The construction of each from the other is local and smooth.*

We will not give the proof, which is long but not conceptually difficult, following Tanaka's interpretation of Cartan's method of equivalence. The group $\operatorname{SL}(n+2, \mathbb{C})$ acts in the obvious representation on $\mathbb{C}^{n+2}$. The group G^{line} is the group of projective transformations leaving invariant the subspace $\mathbb{C}^2 \subset \mathbb{C}^{n+2}$, and write $\mathbb{C}^n$ for $\mathbb{C}^{n+2}/\mathbb{C}^2$. Under the projection, $E \to \Lambda$, say $e \mapsto \lambda$, ω identifies $T_m M$ with $\mathfrak{g}/\mathfrak{g}^{\text{line}} = \mathbb{C}^{2*} \otimes \mathbb{C}^n$. Thereby, ω determines a tensor product decomposition on each tangent space of Λ. One has to be a little careful, since this is not a splitting into vector bundles defined on Λ. Taking any local section of $E \to M$ defined on some open subset of M, say $\sigma : \text{open } \subset M \to E$, we can use this prescription to define a local Segré structure on that open set. This local prescription turns out to determine a Segré structure.

The bundles U and V are not necessarily globally defined, because the expression

$$E \times_{G^{\text{line}}} \mathbb{C}^2$$

doesn't make sense: G^{line} doesn't act on $\mathbb{C}^2$, being only a subgroup of $G = \mathbb{P}\,\mathrm{SL}\,(n+2, \mathbb{C})$. We can define the bundles $E \times_{G^{\text{line}}} \mathbb{P}^1$ and $E \times_{G^{\text{line}}} \mathbb{P}^{n-1}$, which we think of intuitively as $\mathbb{P}U$ and $\mathbb{P}V$.

The distinction between local and global Segré geometries is not always clearly made, nor is the distinction between Segré structures and Segré geometries. The space of elements of a Segré structure is the total space of the bundle $\mathbb{P}U \to \Lambda$, and the fibers are the integral curves.

Proposition 7 (Grossman [22] p. 415)**.** *Given a Segré geometry $E \to \Lambda$, the 1-forms ω_0^I, ω_1^I are semibasic. The symmetric 2-tensors*

$$\Delta^{IJ} = \omega_0^I \omega_1^J + \omega_1^J \omega_0^I - \omega_0^J \omega_1^I - \omega_1^I \omega_0^J$$

descend from each point of E to determine symmetric 2-tensors at the corresponding point of Λ. Their span is independent of the choice of point in E, depending only on the corresponding point of Λ, defining a smooth vector subbundle of $\mathrm{Sym}^2(T\Lambda)$.

Proof. It is an easy calculation that the Δ^{IJ} transform under the action of G^{line} as combinations of one another just when the torsion vanishes, since we know how ω transforms by definition of a Cartan connection; for details see Grossman [22] p. 416. □

Definition 22. Let U^2 and V^n be vector spaces of dimensions 2 and n respectively. The *Segré variety* $\Sigma \subset \mathbb{P}(U \otimes V)$ is the set of elements of rank 1, i.e. pure tensors $u \otimes v$. In coordinates u_0, u_1 on U, and v^I on V, we have coordinates w_0^I, w_1^I on $U \otimes V$ and the Segré variety is cut out by the equations $w_0^I w_1^J = w_0^J w_1^I$.

Definition 23. The group $G(U, V) = (\mathrm{GL}(U) \times \mathrm{GL}(V))/\Delta$ (where Δ is the group of pairs of scalar multiples of the identity of the form (λ, λ^{-1})) clearly acts as linear transformations on $U \otimes V$ leaving the Segré variety invariant. If $\dim U = \dim V$, we can also take any linear isomorphisms $\phi, \psi : U \to V$, and map $g(u \otimes v) = \psi^{-1}(v) \otimes \phi(u)$, giving an action of

$$G'(U, V) = G(U, V) \sqcup (\mathrm{Iso}(U, V) \times \mathrm{Iso}(U, V))/\Delta.$$

Lemma 10. *The group of linear transformations of $U \otimes V$ preserving the Segré variety is $G(U, V)$, unless* $\dim U = \dim V$*, in which case it is $G'(U, V)$.*

The proof is just some linear algebra.

Corollary 7. *A Segré structure on an even dimensional manifold (not of dimension 4) is equivalent to a choice of a smoothly varying family of subvarieties in the projectivized tangent spaces of the manifold, with each variety linearly isomorphic to the Segré variety. Equivalently, a Segré structure is equivalent to a choice of a smoothly varying linear subspace of the symmetric 2-tensors isomorphic at each point to the subspace spanned by the equations cutting out the Segré variety.*

For 4-dimensional manifolds, analoguous to $\mathrm{Gr}\left(2, \mathbb{C}^4\right)$, we can consider a Segré structure to be a choice of a family of Segré varieties in the projectived tangent spaces, together with an analogue of an orientation, picking out which of the two tensor product factors is which.

Grossman took this view of Segré geometries, as families of Segré varieties, which seems quite natural. Nonetheless, it is not clear which point of view will make easier the process of geometrically constructing all of the torsion-free Segré geometries, a task which has yet to be done.

Proposition 8 (Machida & Sato [30])**.** *Every Segré structure determines and is determined by a unique normal Segré geometry, through a local construction. In particular, the concept of torsion-freedom is defined for Segré structures.*

15.3. **Integrability.** Grossman [22] considered in some detail the question of integrability for torsion-free path geometries. We will summarize his results, which generalize Cartan's [9]. Each torsion-free path geometry comes from a torsion-free Segré structure. This Segré structure has an associated normal Segré geometry. Fels [17] shows us how to compute the structure equations of the second order structure $E \to M$, which are precisely the structure equations of the Segré geometry $E \to \Lambda$. Therefore, even though we don't see how to construct explicitly the base manifold Λ of the Segré structure, i.e. the space of solutions, we can compute its curvature, which lives on E. Just by differentiating, we can compute the covariant derivatives of all orders of the curvature. Each of these invariants transforms under the structure group G^{line} of $E \to \Lambda$ in some representation. If we can cobble together a rational invariant (out of these covariant derivatives) which lives in the trivial representation of G^{line}, then it will descend to a function on the unknown manifold Λ, i.e. on the space of integral curves, and therefore it must be a constant on each integral curve. As Cartan and Grossman prove, this process generically succeeds, because there are invariants arising in this manner which, for generic torsion-free Segré structures, have differential nonzero at a generic point. Indeed, in this manner one can find enough conservation laws to reduce the determination of integral curves to quadrature, integrating the original system of ordinary differential equations. Thus we have "integrated by differentiating." This process can fail, but only when too many invariants of the curvature and its covariant derivatives are constant on E. For scalar equations (i.e. surface path geometries), Cartan's methods [8, 3] show that either there is a conserved quantity obtained from the differential equations, or the differential equation has a positive dimensional Lie group of symmetries, so we can hope to reduce the equation using Lie's method. More complicated phenomena are observed in Grossman's thesis [22], where the constancy of one invariant, at least in low dimension, allows one to calculate further higher order invariants which generically still ensure integrability. However, in general it is unknown whether every torsion-free system of equations must either be integrable with differential invariants as conservation laws or have a positive dimensional Lie group of symmetries.

16. Rational stalks

Given a path geometry on a complex manifold M^{2n+1}, let $\Sigma \subset M$ be a stalk. Take the Cartan geometry $E|_\Sigma \to \Sigma$, which is modelled on $G^{\text{pt}}/G^{\text{pt,line}}$. The ω^I_1 are semibasic for this bundle, while $\omega^1_0 = \omega^I_0 = 0$. But at least one ω^1_0 or ω^I_0 term appears in all of the curvature of $E \to M$. Therefore $E|_\Sigma$ is flat.

Theorem 18. *A stalk of a path geometry on a complex manifold M^{2n+1} is rational just when it is compact with fundamental group defying G^{pt}, and this occurs just when its Cartan geometry is isomorphic to the Cartan geometry of the stalks of the model.*

Proof. The Cartan connection is flat, so by theorem 10 on page 15 our compact stalk must be a locally Klein geometry $\Gamma \backslash G^{\text{pt}} / G^{\text{pt,line}}$. But $G^{\text{pt}} / G^{\text{pt,line}} = \mathbb{P}^{n-1}$, so Γ must be a discrete group of projective linear transformations acting as deck transformations on projective space. However, every linear transformation has an eigenspace, so every projective linear transformation has a fixed point. Therefore $\Gamma = \{1\}$. □

Lemma 11. *The normal bundle of a rational stalk (as a submanifold of M) is trivial $\mathcal{O}^{\oplus n}$.*

Proof. First consider the case of the model. Each $\mathbb{P}^n$ fiber of $\mathbb{P}T\mathbb{P}^{n+1}$ lives inside the open set $\mathbb{P}T\mathbb{A}^{n+1} = \mathbb{A}^{n+1} \times \mathbb{P}^n$, so clearly has trivial normal bundle $\nu\mathbb{P}^n = \mathcal{O}^{\oplus n+1}$. Next, in the general case, construct the normal bundle as

$$\nu_M \Sigma = \left(E|_\Sigma \times \left(\mathfrak{g}^{\text{pt}} / \mathfrak{g}^{\text{pt,line}} \right) \right) / G^{\text{pt,line}}.$$

Therefore $\nu_M \Sigma = \mathcal{O}^{\oplus n+1}$. But $E|_\Sigma \to \Sigma$ is isomorphic to the model $G^{\text{pt}} \to \mathbb{P}^n$. □

Theorem 19. *If the space of elements of a path geometry is connected, and all stalks are compact, and one stalk has fundamental group defying G^{pt}, then all stalks are rational, and the space of points is a smooth complex manifold, and the map taking an element to its point is smooth.*

Proof. Follows immediately from Kodaira theory. □

Lemma 12. *If the stalks of a path geometry are rational, then the invariant T^I_{JKL} vanishes.*

Proof. Following Fels [17] p. 235, we compute that

$$\nabla T^I_{JKL} = dT^I_{JKL} + \left(\omega^0_0 + \omega^1_1 \right) T^I_{JKL} + \omega^I_M T^M_{JKL} - T^I_{MKL} \omega^M_J - T^I_{JML} \omega^M_K - T^I_{JKM} \omega^M_L$$

is semibasic for the map $E \to M$. Pick a number N from $2, \dots, n$. Consider the copy of $\mathfrak{sl}(2, \mathbb{C}) \subset G^{\text{pt}}$ given by the equations $\omega^1_1 + \omega^N_N = 0$ together with setting every $\omega^{\bullet}_{\bullet}$ to 0 except for $\omega^1_1, \omega^N_1, \omega^1_N, \omega^N_N$. Calculate that

$$dT^I_{JKL} = T^I_{JKL} \omega^1_1 \left(\delta^I_N - 1 - \delta^N_J - \delta^N_K - \delta^N_L \right).$$

If $I \neq N$, then clearly this is a negative line bundle. Therefore $T^I_{JKL} = 0$ as long as $I \neq N$. But if $I = N$, then switch to a different choice of index N. □

Theorem 20. *All of the stalks of a path geometry are rational just when then the space of points is a smooth complex manifold, and the integral curves of the path geometry project to the geodesics of a unique normal projective connection. In particular, near any point of the point space, the projected integral curves are precisely the geodesics of some affine connection.*

Proof. Kodaira's theorem ensures that the space of stalks is a complex manifold. Following Fels [17] p. 238, the vanishing of T^I_{JKL} is precisely the condition under which the path geometry is locally that of a projective connection on some complex manifold, which is locally identified with Kodaira's moduli space. □

Theorem 21. *A path geometry is locally isomorphic to some path geometry with rational stalks, and also to some path geometry with rational integral curves, just when the path geometry is locally isomorphic to*

$$\frac{d^2y}{dx^2} = 0.$$

Proof. This is a long calculation: once the invariants T^I_{JKL} and ${}^\sharp t$ are forced to vanish, then all of the remaining invariants vanish, and then the Cartan geometry on E is flat. □

Finally, we will prove theorem 3 on page 10: the only path geometry on a connected manifold whose integral curves and stalks are all rational is the model path geometry of lines in projective space.

Proof. The stalks are rational, so the path geometry is a normal projective connection on the space of points S, which is a smooth manifold. The normal projective connection is flat, so a covering space $\tilde{S}$ of S is mapped to projective space, and the normal projective connection pulled back. The integral curves of the path geometry project to the geodesics, so the geodesics are rational curves. Because each geodesic is simply connected, and admits no smooth quotient curve, each geodesic in $\tilde{S}$ maps bijectively to a projective line. Since any two points in projective space lie on a projective line, the map $\tilde{S} \to \mathbb{P}^{n+1}$ is a surjective local diffeomorphism.

Take a point $s \in \tilde{S}$, and suppose it is mapped to a point $p \in \mathbb{P}^{n+1}$. Let B be the blowup of $\mathbb{P}^{n+1}$ at p. So points of B are pairs (ℓ, q) with ℓ a line through p and q a point of that line. Given (ℓ, q), let $\tilde{\ell}$ be the geodesic through s which is mapped to ℓ. Since the map $\tilde{S} \to \mathbb{P}^{n+1}$ is bijective on each geodesic, there is a unique point $\tilde{q}$ on $\tilde{\ell}$ mapping to q. Map $(\ell, q) \in B \to \tilde{q} \in \tilde{S}$. Clearly the map has image consisting precisely in the points which lie on a geodesic through s. Moreover, B is compact, so the image of this map must be as well. Therefore the geodesics through any point of $\tilde{S}$ cover a compact subset of $\tilde{S}$. The composition $B \to \tilde{S} \to \mathbb{P}^{n+1}$ is the blowup map, so a local biholomorphism on a dense open set. Therefore $B \to \tilde{S}$ is holomorphic, and on some open set a local biholomorphism. By Sard's lemma, the map $B \to \tilde{S}$ is onto. Therefore $\tilde{S}$ is compact, and the map $\tilde{S} \to \mathbb{P}^{n+1}$ is a biholomorphism. □

17. Conclusion

How can we apply these ideas in algebraic geometry? If we have a collection of rational curves in a complex manifold S, and the collection is locally described, near one of these curves, as the integral curves of a (possibly multivalued) smooth path geometry on $M = \mathbb{P}TS$, then because the stalks are rational these must be the geodesics of a projective connection. But because the integral curves are rational, this projective connection must be flat, so S is locally identified with $\mathbb{P}^n$, up to local linear projective transformations, and the rational curves identified with projective lines. So families of rational curves must pass through some singularity of S, or be singular, or be laid out in some manner quite different from a path geometry, if $S \neq \mathbb{P}^n$.

Some open problems:

(1) Write software to symbolically integrate "generic" torsion-free systems of ordinary differential equations.

(2) For torsion-free systems for which there are not enough invariants to integrate (using curvature and its covariant derivatives to generate integrals of motion), is there always some other process to integrate the equations, by combination of those integrals of motion together with symmetry reductions?
(3) Find a straightness criterion for higher order equations, for example third order scalar equations [10, 12, 35], fourth order scalar equations [4, 18], and third order systems [16]. Dunajski and Todd [15] have recent solved this problem for n-order scalar equations.
(4) One can adapt the methods of this article to a host of Cartan connections and G-structures; for example for 2-plane fields on a 5-manifold, satisfying a natural nondegeneracy condition (see Cartan [6]), one can ask when their bicharacteristic curves are rational. The crucial idea is to look at a copy of $\mathfrak{sl}(2, \mathbb{C})$ appearing in the structure equations, and see how the torsion (or curvature) varies under it, which we can read off directly from structure equations.
(5) Can something be said about ordinary differential equations whose integral curves are elliptic? The methods employed here seem powerless, since line bundles on elliptic curves have moduli, so we couldn't expect to read them off from the structure equations.
(6) The requirement that a Segré structure be torsion-free is a collection of first order partial differential equations, which has a lot of local solutions (the Cartan–Kähler theorem tells us so). But there is no technique for constructing solutions. We are not interested in flat solutions (i.e. locally isomorphic to the Grassmannian of 2-planes in a vector space), but quite interested to find the nonflat examples with largest possible symmetry groups, which correspond to very special systems of ordinary differential equations.
(7) Cartan's concept of "integrating by differentiating" applies to certain families of ordinary differential equations, which he refered to as *classe C* [9]. Is there actually a relation between straightness and class C? Presumably straightness implies class C, but comments in Bryant [4] p. 35 suggest that there is more to class C than straightness.
(8) We will follow this paper with a paper demonstrating constraints on the characteristic classes of closed Kähler manifolds admitting path geometries. In low dimensions, you might hope to classify the projective 3-folds which admit path geometries, as Jahnke and Radloff [25, 24, **?**] did for normal projective connections and conformal structures.
(9) Perhaps if the path geometry is singular, but all the integral curves are still rational, there is still some local information.

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