THIRD ORDER ODES SYSTEMS AND ITS CHARACTERISTIC CONNECTIONS

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ABSTRACT. We compute the characteristic Cartan connection associated with a system of third order ODEs. Our connection is different from Tanaka normal one, but still is uniquely associated to the system of third order ODEs. This allows us to find all fundamental invariants of a system of third order ODEs and, in particular, determine when a system of third order ODEs is trivializable. As application differential invariants of equations on circles in $\mathbb{R}^n$ are computed.

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1. INTRODUCTION

1.1. **Differential equation as a structure on a filtered manifold.** The main purpose of this article is to study geometry of systems of ordinary differential equations of third order. The geometry of ordinary differential equations or, more generally, of differential equation of finite type is based on the general theory of geometric structures on filtered manifolds. First it was developed by Tanaka in [8], [9]. Recall that *a filtered manifold* is a smooth manifold M equipped with a filtration of the tangent bundle TM compatible with the Lie bracket of vector fields. At any point $x \in M$ the associated graded vector space $\operatorname{gr} T_xM$ can be endowed with a Lie algebra structure. This nilpotent Lie algebra $\mathfrak{m}$ called *a symbol of a filtered manifold* (at the point x). In the following we consider only the so-called filtered manifolds of *constant type*, assuming that the graded nilpotent Lie algebras $\operatorname{gr} T_xM$ are isomorphic to each other for all points $x \in M$.

By *a symbol of a geometric structure on M* we understand a graded Lie algebra $\mathfrak{g}$ with the negative part $\mathfrak{g}_- = \sum_{i<0} \mathfrak{g}_i$ equal to the symbol $\mathfrak{m}$ of the filtered manifold M of constant type. The algebra Lie $\mathfrak{g}$ here is the subalgebra of a so-called universal Tanaka prolongation $\mathfrak{g}(\mathfrak{m})$. Roughly speaking, this mean that for any element $X \in \mathfrak{g}_i, i \geq 0$ the equality $[X, \mathfrak{g}_-] = 0$ implies $X = 0$.

An arbitrary equation $\mathcal{E}$ can be viewed as surface in jet space. The canonical restriction of the contact distribution on jet space define the structure of filtered manifold on $\mathcal{E}$.

1.2. **The problem of equivalence.** One of the main problems in the theory of differential equations is the problem of equivalence. Two differential equations are called equivalent if one can be transformed to another by a certain change of variables. We consider equations up to point transformations, i.e. we allow arbitrary changes of both dependent and independent variables.

First classical approach to the equivalence problem of ODEs was developed by Sophus Lie. In [5] he obtains partial results about second order ODE. The complete answer was given later by Tresse [10]. Invariants of the third order ODE were computed by Chern in his paper [1]. A modern approach to the equivalence problem of ODEs can be found in the papers [2] and [4] where were constructed characteristic

Cartan connection for the one equation of arbitrary order and for the system of ODE of the second order.

The general approach to the equivalence problem for the holonomic differential equations can be found in [3]. The key fact there is the existence of a full functor from the category of holonomic differential equations to the category of Cartan connections. This reduce the equivalence problem for differential equations to the equivalence problem for the corresponding Cartan connections.

1.3. **Normalization of Cartan connections.** Let P be the principal H-bundle. Let ω be a Cartan connection of type (G, G_-), where G is a Lie group with a semisimple graded Lie algebra $\mathfrak{g}$ and G_- is a subgroup of G with the Lie algebra $\mathfrak{g}_-$. In the paper [9] Tanaka build a set of normal Cartan connections on the principal bundle P as follows. He used the scalar product defined with the help of the Killing form to construct adjoint Lie algebra codifferential ∂^*. Then a Cartan connection is normal iff the structure function $C : P \to \mathrm{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g})$ belongs to the kernel of the operator ∂^* and the structure function has not negative components. As usual define a Laplacian $\Delta = \partial^* \partial + \partial \partial^*$. The structure function C decomposes as $C = H(C) + \Delta(C)$. The component $H(C)$ is called the harmonic part of the structure function. The key fact about it is that $H(C)$ is the fundamental system of invariants (see definition 5 for details). In the case of the geometry of holonomic differential equations the Lie algebra $\mathfrak{g}$ is not necessary semisimple. However in [3] is shown that we still can found the scalar product on $\mathfrak{g}$ such that the normal Tanaka conditions define the unique Cartan connection associated to a holonomic differential equation.

In the present paper we associate with every system of ODEs of third order a characteristic Cartan connection which differ from a normal Tanaka Cartan connection. The reason to do so is a relation between conformal geometry and geometry of the system of the third order ODEs. Conformal manifold is determined by the family of conformal circles, which was shown by Yano[11]. Each conformal circle is determined by the point on it, the direction and the curvature, i.e. by the point in the third jet space. The system of appropriate differential equations of the third order give us the bridge between the conformal geometry and the geometry of the differential equation. It is appeared that a characteristic Cartan connection, which is build in the paper, is in close relations with the normal conformal Cartan connection. The relation of the conformal geometry and the geometry of third order ODEs is the topic of the our next paper.

The paper is organized as follows. In the second section we naturally associate the system of the third order ODEs with the pair of distributions. This pair of distributions give rise to the filtered manifold associated with the system of the third order ODEs. We write down the symbol of the system of ODEs of the third order, the notion of adopted coframe and adopted Cartan connection. In the third section of the paper discuss the problem of equivalence. When we working in the case of semisimple Lie algebras and normal Cartan connections, the harmonic part of the curvature give us the fundamental system of differential invariants. We show that in general case fundamental differential invariants are contained in the $\ker \partial$ part of the curvature, where ∂ is the Lie algebra cohomology differential. In the fourth section we build the characteristic Cartan connection uniquely associated to the the system of ODE of the third order. This allow use to obtain the results about equivalence of such equations and to describe the structure of the fundamental invariants of the system of third order ODEs. In particular, this gives the explicit answer to the question, when a given system is trivializable, i.e. equivalent to the system $y_i''' = 0$.

2. Geometry of the systems of third order ODEs

Consider an arbitrary system of m ordinary differential equations of third order:

$$y_i'''(x) = f_i(y_j''(x), y_k'(x), y_l(x), x), i, j, k, l = 1, \dots, m. \tag{1}$$

We associate a filtered manifold with this system in the following way. Let $J^3(\mathbb{R}^{m+1}, 1)$ be the third jet space of unparametrized curves. Then the equations (1) can be considered as a submanifold $\mathcal{E}$ in $J^3(\mathbb{R}^{m+1}, 1)$. We introduce the following coordinate system on the surface $\mathcal{E}$:

$$(x, y_1, \dots, y_m, p_1 = y_1', \dots, p_m = y_m', q_1 = y_1'', \dots, q_m = y_m'').$$

There is a natural one-dimensional distribution E whose integral curves are the lifts of solutions of equations (1). Let π_1^2 be the canonical projection from the surface $\mathcal{E}$ to the first jet space $J^1(\mathbb{R}^{m+1}, 1)$. We denote a kernel of a differential $d\,\pi_1^2$ as V. In coordinates distributions E, V have the form:

$$E = \left\langle \frac{\partial}{\partial x} + p_i \frac{\partial}{\partial y_i} + q_i \frac{\partial}{\partial p_i} + f^i \frac{\partial}{\partial q_i} \right\rangle,$$
$$V = \left\langle \frac{\partial}{\partial q_i} \right\rangle,$$

where $i, j = 1, \dots, m$.

Define a distribution C as the direct sum of the distributions E and V. Then C and its subsequent brackets define a filtration of a tangent bundle $T\mathcal{E}$:

$$C = C^{-1} \subset C^{-2} \subset C^{-3} = T\mathcal{E},$$

where $C^{-i-1} = C^{-i} + [C^{-i}, C^{-1}]$.

It is easy to see that the symbol of the filtrated manifold $\mathcal{E}$ is a nilpotent Lie algebra $\mathfrak{m}$ isomorphic to the Lie algebra of vector fields generated by

$$\mathfrak{m}_{-1} = \left\langle \frac{\partial}{\partial x} + p_i \frac{\partial}{\partial y_i} + q_i \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_i} \right\rangle. \tag{2}$$

The splitting $E \oplus V$ of the distribution C determines a G_0-structure of type $\mathfrak{m}$, where G_0 is the subgroup of $\mathrm{Aut}_0(\mathfrak{m})$ The action of the group G_0 on $\mathfrak{m}$ is completely determined by its action on $\mathfrak{m}_{-1}$. The latter has the following form in the basis (2):

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, a \in \mathbb{R}^*, B \in GL_m(\mathbb{R})$$

The symbol $\mathfrak{g}$ is the universal prolongation of the pair $(\mathfrak{m}, \mathfrak{g}_0)$. It has the following form:

$$\mathfrak{g} = (\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})) \leftthreetimes (V_2 \otimes W).$$

In other words, $\mathfrak{g}$ is equal to the semidirect product of the Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})$ and an abelian ideal V. The ideal V has the form $V_2 \otimes W$, where V_2 is an irreducible $\mathfrak{sl}_2(\mathbb{R})$-module of dimension 3 and $W = \mathbb{R}^m$ is the standard representation of $\mathfrak{gl}_m(\mathbb{R})$.

Let us fix a basis of the Lie algebra $\mathfrak{sl}_2$ and $\mathfrak{sl}_2$-module V_2. Let x, y, h be the standard basis of an algebra $\mathfrak{sl}_2(\mathbb{R})$ with relations:

$$[x, y] = h, [h, x] = 2x, [h, y] = -2y.$$

In the matrix form this basis is the following:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let v_0, v_1, v_2 be a basis of the module V_2 such that $x_2 = v_1, xv_1 = v_0, xv_0 = 0$.

Define the grading of the Lie algebra $\mathfrak{g}$ as follows:

$$\begin{aligned} \mathfrak{g}_1 &= \langle y \rangle, \\ \mathfrak{g}_0 &= \langle h, \mathfrak{gl}_m \rangle, \\ \mathfrak{g}_{-1} &= \langle x \rangle + \langle v_2 \otimes W \rangle, \\ \mathfrak{g}_{-2} &= \langle v_1 \otimes W \rangle, \\ \mathfrak{g}_{-3} &= \langle v_0 \otimes W \rangle. \end{aligned}$$

To build a natural Cartan geometry associated to the equation (1) we will use the fact [7] that under some additional conditions (which are satisfied for geometric structures arising from holonomic differential equations, see [3]) there exists a full functor from the category of G_0-structures of type $\mathfrak{m}$ to the category of Cartan connections of type (G, H), where G and H are the Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively which are determined from G_0 in natural manner.

The group G is a semisimple product:

$$G = (SL_2(\mathbb{R}) \times GL_m(\mathbb{R})) \leftthreetimes (V_2 \otimes W) .$$

Let H be the following subgroup of G:

$$H = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \times A, a \in \mathbb{R}^*, b \in \mathbb{R}, A \in GL_m(\mathbb{R}).$$

Note that the corresponding subalgebra $\mathfrak{h}$ is exactly the nonnegative part of the Lie algebra $\mathfrak{g}$: $\mathfrak{h} = \sum_{i \geq 0} \mathfrak{g}_i$.

Definition 1. We say that a coframe $\{\omega^i_{-3}, \omega^i_{-2}, \omega^i_{-1}, \omega_x\}$ on $\mathcal{E}$ *is adapted to equation* (1) if:

- the annihilator of forms $\omega^i_{-3}, \omega^i_{-2}, \omega_x$ is V;
- the annihilator of forms $\omega^i_{-3}, \omega^i_{-2}, \omega^i_{-1}$ is E;
- the annihilator of forms ω^i_{-3} is C^{-2}.

Let $\overline{\pi}\colon P \to \mathcal{E}$ be a principle H-bundle and let $\overline{\omega}$ be and arbitrary Cartan connection of type G/H on P. Connection $\widetilde{\omega}$ can be written as:

$$\overline{\omega} = \overline{\omega}^i_{-3} v_0 \otimes e_i + \overline{\omega}^i_{-2} v_1 \otimes e_i + \overline{\omega}^i_{-1} v_2 \otimes e_i + \overline{\omega}_x x + \overline{\omega}_h h + \overline{\omega}^i_j e^j_i + \overline{\omega}_y y.$$

Definition 2. We say that a Cartan connection $\overline{\omega}$ on a principal H-bundle is adapted to equations (1), if for any local section s of π the set $\{s^*\overline{\omega}_x, s^*\overline{\omega}^i_{-1}, s^*\overline{\omega}^i_{-2}, s^*\overline{\omega}^i_{-3}\}$ is an adapted co-frame on $\mathcal{E}$.

We have described the set of Cartan connection adapted to the system of third order ODEs. However, we can chose the representative in different ways. The next two sections is devoted to the building of such canonical connection which we call characteristic.

3. Characteristic Cartan connection and fundamental differential invariants

Let as in the section two $\overline{\pi}\colon P \to \mathcal{E}$ be a principle H-bundle and let $\overline{\omega}$ be and arbitrary Cartan connection of type G/H on P:

$$\overline{\omega} = \overline{\omega}^i_{-3} v_0 \otimes e_i + \overline{\omega}^i_{-2} v_1 \otimes e_i + \overline{\omega}^i_{-1} v_2 \otimes e_i + \overline{\omega}_x x + \overline{\omega}_h h + \overline{\omega}^i_j e^j_i + \overline{\omega}_y y.$$

Let $\overline{\Omega} = \mathrm{d}\overline{\omega} + \frac{1}{2}[\overline{\omega}, \overline{\omega}]$ be the curvature of the Cartan connection $\overline{\omega}$:

$$\overline{\Omega} = \overline{\Omega}^i_{-3} v_0 \otimes e_i + \overline{\Omega}^i_{-2} v_1 \otimes e_i + \overline{\Omega}^i_{-1} v_2 \otimes e_i + \overline{\Omega}_x x + \overline{\Omega}_h h + \overline{\Omega}^i_j e^j_i + \overline{\Omega}_y y.$$

Definition 3. The structure function of a Cartan connection ω is a function $C : P \to \mathrm{Hom}(\wedge^2 \mathfrak{g}/\mathfrak{g}_-, \mathfrak{g})$ such that:

$$C(p) = (g_1, g_2) \to \Omega_p(\omega_p^{-1}(g_1), \omega_p^{-1}(g_2)).$$

We can obtain the structure function of a Cartan connection explicitly. Let $\{e_1, \dots, e_{n+k}\}$ be a basis of Lie algebra $\mathfrak{g}$ such that $\{e_{n+1}, ..., e_{n+k}\}$ form a basis of the subalgebra h. In our case $\{e_{n+1}, ..., e_{n+k}\} = \{h, y, e_i^j\}$. An arbitrary element $\varphi \in \mathrm{Hom}(\wedge^2 \mathfrak{g}/\mathfrak{g}_-, \mathfrak{g})$ defined by constants C_{ij}^k, where

$$\varphi(e_i, e_j) = \sum_{k=1}^{n+k} C_{ij}^k e_k, \quad (1 \le i, j \le n).$$

The structure function $C : P \to \mathrm{Hom}(\wedge^2 \mathfrak{g}/\mathfrak{g}_, g)$ defines functions $C_{ij}^k(p)$. If

$$\omega = \sum w_i e_i, \quad \Omega = \sum \Omega^k e_k,$$

then the functions $C_{ij}^k(p)$ can be found from the decomposition of the curvature tensor Ω in terms of forms ω_i :

$$\Omega^k = \sum C_{ij}^k \omega_i \wedge \omega_j.$$

Let Ω^i be one of the 2-forms $\overline{\Omega}^i_{-3}, \overline{\Omega}^i_{-2}, \overline{\Omega}^i_{-1}, \overline{\Omega}_x, \overline{\Omega}_h, \overline{\Omega}^i_j$. We can write it explicitly as:

$$\Omega^i = \sum_{p,q=1}^{3} \Omega^i[\overline{\omega}^j_{-q}, \overline{\omega}^k_{-p}] \overline{\omega}^j_{-q} \wedge \overline{\omega}^k_{-p} + \sum_{p=1}^{3} \Omega^i[\overline{\omega}_x, \overline{\omega}^k_{-p}] \overline{\omega}_x \wedge \overline{\omega}^k_{-p}.$$

Then $\Omega^i[\overline{\omega}^j_{-q}, \overline{\omega}^k_{-p}]$ and $\Omega^i[\overline{\omega}_x, \overline{\omega}^k_{-p}]$ are the coefficients of the structure function of the Cartan connection ω. The grading of Lie algebra $\mathfrak{g}$ induces degree of the coefficients $\Omega^i[\overline{\omega}^j_{-q}, \overline{\omega}^k_{-p}]$ and $\Omega^i[\overline{\omega}_x, \overline{\omega}^k_{-p}]$.

Definition 4. We say that Cartan connection associated with the equation (1) is characteristic if the following conditions on a curvature is satisfied:

- all coefficients of degree ≤ 1 is equal to 0;
- in degree 2 we have $\overline{\Omega}_h[\overline{\omega}_x \wedge \overline{\omega}^i_{-1}] = 0$, $\overline{\Omega}^i_j[\overline{\omega}_x \overline{\wedge} \overline{\omega}^k_{-1}] = 0$, $\overline{\Omega}_x[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}] = 0$, $\overline{\Omega}^i_{-1}[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}] = 0$;
- in degree 3 we have $\overline{\Omega}_y[\overline{\omega}_x \wedge \overline{\omega}^i_{-1}] = 0$, $\overline{\Omega}_h[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}] = 0$, $\overline{\Omega}^i_j[\overline{\omega}_x \wedge \overline{\omega}^k_{-2}] = 0$;
- in degree 4 we have $\overline{\Omega}_y[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}] = 0$.

In other worlds these conditions define the subspace U and Cartan connection is characteristic if and only if it belongs to U.

Theorem 1. *There exists a unique characteristic Cartan connection associated to the equation* (1).

Proof. We will proceed with parametric computations of characteristic Cartan connection in the forth section of the paper. We will fix a section $s : \mathcal{E} \to P$ and prove that locally for every equation there exists a unique Cartan connection ω with structure function pullback $s^*C : \mathcal{E} \to \mathrm{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{g})$ takes values in the space U. We show that the characteristic Cartan connection is uniquely defined globally with this data.

Take a covering U_α of the space E and construct a Cartan connection $\wedge_\alpha$ on each trivial fibre bundle $\pi_\alpha : U_\alpha \times H \to U_\alpha$. Let s_α and s_β be the trivial sections of the fibre bundles π_α and π_β. Let $\widetilde{\omega}_\alpha = s^*_\alpha \omega_\alpha$ and $\widetilde{\omega}_\beta = s^*_\beta \omega_\beta$. Since forms $\omega_a l$ and ω_β is uniquely defined there exists a unique function

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to H,$$

such that

$$\omega_\beta = \mathrm{Ad}(\varphi^{-1})\omega_\alpha + \varphi^*\omega_H,$$

where ω_H is Maurer-Cartan form of the Lie group H. The functions $\varphi_{\alpha\beta}$ define a principle H-bundle with the Cartan connection ω.

To prove that structure function C of the Cartan connection ω takes values in the space U it is sufficient to show that U is $\mathrm{Ad}(H)$-invariant.

Note that action of G_0 preserve zero condition on the structure function of the characteristic connection. We need only check that the space U is $\exp(y)$-invariant or equally $\mathrm{ad}(y)$ invariant. Action of the element y has degree one. Conditions on the curvature of the Theorem 1 is $\mathrm{ad}(y)$-invariant up degree 2, since all components of degree less than 2 is zero. Finally, the condition of degree 3 and 4 is $\mathrm{ad}(y)$-invariant, since the coefficients $\overline{\Omega}_y[\overline{\omega}_x \wedge \overline{\omega}^i_{-1}]$, $\overline{\Omega}_h[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}]$, $\overline{\Omega}^i_j[\overline{\omega}_x \wedge \overline{\omega}^k_{-2}]$ and $\overline{\Omega}_y[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}]$ can be obtained only from $\overline{\Omega}_h[\overline{\omega}_x \wedge \overline{\omega}^i_{-1}]$, $\overline{\Omega}_x[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}]$, $\overline{\Omega}^i_j[\overline{\omega}_x \wedge \overline{\omega}^k_{-1}]$, $\overline{\Omega}_h[\overline{\omega}_x \wedge \overline{\omega}^i_{-2}]$ and $\overline{\Omega}_y[\overline{\omega}_x \wedge \overline{\omega}^i_{-1}]$ which all are zero for characteristic Cartan connection. This ends the proof of a global existence of the form ω. □

Let V be an arbitrary finite-dimensional vector space and f a smooth function $f : P \to V$. Denote by $L_0(f)$ the space of all functions of the form $< f, v^* >$, where $v^* \in V^*$ and by $L(f)$ the algebra generated by elements from $L_0(f)$ and all their covariant derivatives. For example, the algebra $L(C)$, where C is structure function of the Cartan connection ω, consists of local invariants of the connection ω.

Definition 5. We say that functions f_i are the fundamental system of invariant for the structure with Cartan connection ω if $L(f_i) = L(C)$.

The key to calculation of the fundamental system of differential invariants is to determine which parts of the curvature are expressed throw another. In [3] it is shown that fundamental invariants of holonomic differential equation lie in non-negative harmonic part of the curvature of the normal Cartan connection. In general we have approximately the same situation: there is one to one correspondence between fundamental differential invariants of the characteristic Cartan connection and $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ part of the curvature. The $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ here is the non-negative part of second Lie algebra cohomology group.

Proposition 1. *Let ω be a Cartan connection of type (G, H) on a principal H-bundle P, where (G, H) is an arbitrary pair of Lie group and its subgroup.Assume that the Lie algebra $\mathfrak{g}$ is a graded Lie algebra of the Lie group G with the nonnegative part $\mathfrak{h}$. Assume that ω is characteristic, that mean that the structure function C of the connection ω lies in some subspace W of the $\mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})$ and the subspace W is complementary to the image of the Lie algebra cohomology operator ∂. Then the restriction of the $\ker\partial$ to the space W is a system of fundamental differential invariants.*

Proof. Algebra of differential invariants is generated by the structure function coefficients. We will use Bianchi identity to show that coefficients of the characteristic Cartan connection curvature are obtained from the image of the operator ∂. Since the space S is complementary to the $\ker\partial$ there exists isomorphism between S and the image of ∂.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H. Let e_i be the basis of the Lie algebra G, X_i be the corresponding fundamental vector fields on P and ω^i be the dual coframe. We can write the Cartan connection ω in the form:

$$\omega = \omega^i e_i.$$

Assume that Lie algebra $\mathfrak{g}$ has the structure constants A_{ij}^k that means that:

$$[e_i, e_j] = A_{ij}^k e_k$$

Then the following equality is fulfilled:

$$\mathrm{d}\,\omega^k = -A_{ij}^k \omega^i \wedge \omega^j$$

Write the curvature of the Cartan connection ω in coordinates:

$$\Omega = \sum_{i<j} C_{ij}^k \omega^i \wedge \omega^j e_k. \tag{3}$$

Now apply the Bianchi identity $\mathrm{d}\,\Omega = [\Omega, \omega]$ to the (3):

$$\sum_{i<j} \left(\frac{\partial C_{ij}^k}{\partial X_l} \omega_l \wedge \omega_i \wedge \omega_j e_k + C_{ij}^k \,\mathrm{d}\,\omega_i \wedge \omega_j e_k + C_{ij}^k \omega_i \wedge \mathrm{d}\,\omega_j e_k \right) = \sum_{i<j} C_{i,j}^k [e_k, e_l] \omega_i \wedge \omega_j \wedge \omega_l.$$

Express the covariant derivative of the structure function:

$$\begin{aligned} (4) \quad & \sum_{i<j} \frac{\partial C_{ij}^p}{\partial X_l} \omega_l \wedge \omega_i \wedge \omega_j e_p = \\ & \sum_{i<j} \left(-C_{kl}^p \,\mathrm{d}\,\omega_k \wedge \omega_l - C_{kl}^p \omega_k \wedge \mathrm{d}\,\omega_l + C_{i,j}^k A_{kl}^p \omega_i \wedge \omega_j \wedge \omega_l \right) e_p = \\ & - \sum_{k \neq l} C_{kl}^p A_{ij}^k \omega_i \wedge \omega_j \wedge \omega_l e_p + \sum_{i<j} C_{i,j}^k A_{kl}^p \omega_i \wedge \omega_j \wedge \omega_l e_p. \end{aligned}$$

If we take $\mathrm{Hom}(\wedge^3 \mathfrak{h}, \mathfrak{g})$ part of (4) (i.e. assume that $\omega_l \in \mathfrak{h}^*$) we get that the right side of the (4) is exactly the Lie cohomology differential.

The right side of (4) does not change the degree of the coefficients, but the right side increase the degree. So we get that coefficients which maps to the $\mathrm{im}\,\partial$ can be expressed through the covariant derivative of the coefficients of the lower degree. This proves the proposition. □

Theorem 2. *The the following invariants form the basis of an algebra of differential invariants:*

$$\begin{aligned} (W_2)_j^i &= \mathrm{tr}_0 \left(\frac{\partial f^i}{\partial p^j} - \frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial f^i}{\partial q^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial q^j} \right), \\ (I_2)_{j,k}^i &= \mathrm{tr}_0 \left(\frac{\partial^2 f^i}{\partial q^j \partial q^k} \right), \\ (W_3)_j^i &= \frac{\partial f^i}{\partial y^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial p^j} - \frac{1}{2} \frac{d}{dx} \frac{\partial f^i}{\partial p^j} + \frac{1}{6} \frac{d^2}{dx^2} \frac{\partial f^i}{\partial q^j} - \frac{2}{27} (\frac{\partial f^i}{\partial q^k})^3 - \frac{1}{18} \frac{\partial f^i}{\partial q^k} \frac{d}{dx} \frac{\partial f^k}{\partial q^j} \\ &\quad - \frac{5}{18} \frac{d}{dx} \left(\frac{\partial f^i}{\partial q^k} \right) \frac{\partial f^k}{\partial q^j}, \\ (I_4)_{j,k} &= -\frac{\partial H_k^{-1}}{\partial p_j} + \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} H^x - \frac{\partial}{\partial q_k} \frac{\mathrm{d}}{\mathrm{dx}} H_j^{-1} - \frac{\partial}{\partial q^k} (H_l^{-1} \frac{\partial f^l}{\partial q^j}) + 2 H_j^{-1} H_k^{-1}, \end{aligned}$$

where $H_j^{-1} = \frac{1}{6(m+1)} \left(\frac{\partial^2 f^i}{\partial q^i \partial q^j} \right)$ *and* $H^x = -\frac{1}{4m} \left(\frac{\partial f^i}{\partial p^i} - \frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial f^i}{\partial q^i} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial q^i} \right)$.

Proof. We will use proposition 1. The fundamental differential invariants is in one to one correspondence with the cohomology group $H_+^2(\mathfrak{g}_-, \mathfrak{g})$. For the case of the system of ODEs of the third order the Lie cohomology group $H_+^2(\mathfrak{g}_-, \mathfrak{g})$ was studied in [6]. The main result of this work is that the space $H_+^2(\mathfrak{g}_-, \mathfrak{g})$ has the following decomposition as $\mathfrak{sl}_2$-module:

Degree	Space
-1	$v_6^0 \otimes \wedge^2(W^*) \otimes W$
0	$v_4^0 \otimes S_0^2(W^*) \otimes W$
0	$v_4^0 \otimes \wedge^2(W^*) \otimes W$
1	$v_2^0 \otimes \wedge^2 W^* \otimes W/V_2 \otimes W^*$
2	$x^* \otimes \mathbb{R}y \otimes \mathfrak{sl}(W)$
2	$v_0^0 \otimes S^2(W^*) \otimes W$
3	$x^* \otimes \mathbb{R}y^2 \otimes \mathfrak{gl}(W)$
4	$v_0^0 \otimes S^2(W^*)$
3	v_2^0 if $m = 2$

Here v_k^0 is the lowest vector of corresponding $\mathfrak{sl}_2$-module V_k.

Now we list the result table with the corresponding invariant. We start from degree 2 since all part of curvature of degree less than 2 is zero.

Degree	Space	Part of the curvature	Invariant
2	$x^* \otimes \mathbb{R}y \otimes \mathfrak{sl}(W)$	$\Omega^i_{-1}[\omega_x \wedge \omega^j_{-2}]$	W_2
2	$v_0^0 \otimes S^2(W^*) \otimes W$	$\Omega^i_{-2}[\omega^j_{-1} \wedge \omega^k_{-3}]$	I_2
3	$x^* \otimes \mathbb{R}y^2 \otimes \mathfrak{gl}(W)$	$\Omega^i_{-1}[\omega_x \wedge \omega^j_{-3}]$	W_3
4	$v_0^0 \otimes S^2(W^*)$	$\Omega_y[\omega_x \wedge \omega^j_{-3}]$	I_4
3	v_2^0 if $m = 2$	$\Omega_y[\omega^2_{-1} \wedge \omega^1_{-2}]$	$\equiv 0$

□

Corollary 1. *The system* (1) *is equivalent to the trivial one via point transformations if and only if all invariants I_2, W_2, W_3, I_4 vanish identically.*

Example 1. Differential equations on circles in $\mathbb{R}^n$

As application of the previous results we compute invariants of the system of third order ODES on circles in Euclidean space.

Lemma 1. *Let E be the $(m+1)$-dimensional Euclidean space with the orthonormal basis $\{e_0, \dots, e_n\}$ and the coordinates $\{r_0, r_1, \dots, r_n\}$. Then the equation of circles in E parametrized by the coordinate r_0 is:*

(5)
$$\dddot{r}_i = 3\ddot{r}_i \frac{\sum_{j=1}^m \dot{r}_j \ddot{r}_j}{1 + \sum_{j=1}^m \dot{r}_j^2}, i = 1, \dots, m.$$

This equation is invariant under conformal transformations of E.

Proof. Let the curve $R(t) = (r_0(t), \dots, r_n(t))$ be a circle. Assume now that $r_0(t) = t$. We have

(6)
$$\dddot{R}(t) = a(t)\ddot{R}(t) + b(t)\dot{R}(t), ,$$

since $R(t)$ is 2-dimensional curve. Next, $b(t) = 0$ in our parametrization, since

$$0 = \dddot{r}_0(t) = a(t)\ddot{r}_0(t) + b(t)\dot{r}_0(t) = b(t).$$

To determine $a(t)$ note that

$$(R(t) - C, R(t) - C) = d$$

for some constant d and $C \in E$. Differentiating, we get:

$$(\dot{R}(t), R(t) - C) = 0,$$
$$(\ddot{R}(t), R(t) - C) = -(\dot{R}(t), \dot{R(t)}),$$
$$(\dddot{R}(t), R(t) - C) + 3(\ddot{R}(t), \dot{R}(t)) = 0.$$

Now substitute to (6):

$$(a(t)\ddot{R}(t) + G\dot{R}(t), R(t) -) = -3(\ddot{R}(t), \dot{R}(t)),$$

$$(a(T)\ddot{R}(t), R(t) - C) = -a(t)(\dot{R}(t), \dot{R}(t)) = -3(\ddot{R}(t), \dot{R}(t)).$$

We get that

$$a(t) = 3\frac{(\ddot{R}(t), \dot{R}(t))}{(\dot{R}(t), \dot{R}(t))}.$$

Substituting $a(t)$ into (6) we get our equations. □

Proposition 2. *For differential equation on conformal circles invariants W_2, I_2, W_3 vanish identically. Invariant I_4 has the following form:*

$$(I_4)^i_j = \frac{1}{2}\delta^i_j \frac{1}{1 + \sum_{k=1}^m \dot{r_k}^2} - \frac{1}{2}\frac{\dot{r}_i \dot{r}_j}{\left(1 + \sum_{k=1}^m \dot{r_k}^2\right)^2}$$

Proof. The proof is straightforward applying of the formulas from theorem 2. □

Remark. There are other equations satisfying $W_2 = I_2 = W_3 = 0$. For example, it is an union of a system on circles in R^{n-k} and a system of k trivial equations. It would be interesting to characterize geometrically the class of such equations.

4. Parametric computation of the characteristic Cartan connection

Consider a system of third-order ordinary differential equations of the form

$$(y^i)''' = f^i(x, y^j, (y^k)', (y^l)''),$$

where $i, j = 1, \dots, m$ with $m \geq 2$. It determines a holonomic differential equation $\mathcal{E} \subset J^3(\mathbb{R}^{m+1}, 1)$. Let us use the following coordinate system on the equation $\mathcal{E}$:

$$x, y_1, \dots, y_m, p_1 = y_1', \dots, p_m = y_m', q_1 = y_1'', \dots, q_m = y_m''$$

We choose a coframe θ on the surface $\mathcal{E}$:

$$\theta_x = dx;$$
$$\theta^i_{-1} = dq^i - f^i(x, y, p, q)\, dx, \quad i = 1, \dots, m;$$
$$\theta^i_{-2} = dp^i - q^i\, dx, \quad i = 1, \dots, m;$$
$$\theta^i_{-3} = dy^i - p^i\, dx, \quad i = 1, \dots, m.$$

To connect our computation on the surface $\mathcal{E}$ with the principle bundle P let us use the following uniquely defined section $s\colon \mathcal{E} \to P$ with relations:

$$s^*\overline{\omega}^i_{-3} = \theta^i_{-3},$$
$$s^*\overline{\omega}_h \equiv 0 \mod \langle \theta^i_{-3}, \theta^i_{-2}, \theta^i_{-1} \rangle,$$
$$s^*\overline{\omega}_x \equiv -\theta_x \mod \langle \theta^i_{-3}, \theta^i_{-2}, \theta^i_{-1} \rangle.$$

Define a pullback $\omega\colon \mathcal{E}\to\mathfrak{g}$ by the formula $\omega = s^*\overline{\omega}$. Let $\overline{\Omega}$ be a curvature tensor of $\overline{\omega}$, and let $\Omega = s^*\overline{\Omega}$. We see that

$$\begin{aligned}\Omega &= \Omega^i_{-3}v_0\otimes e_i + \Omega^i_{-2}v_1\otimes e_i + \Omega^i_{-1}v_2\otimes e_i + \Omega_x x + \Omega_h h + \Omega^j_i e^i_j + \Omega_y y\\ &= (d\omega^i_{-3} + \omega_x\wedge\omega^i_{-2} + 2\omega_h\wedge\omega^i_{-3} + \omega^i_j\wedge\omega^j_{-3})v_0\otimes e_i\\ &+ (d\omega^i_{-2} + \omega_x\wedge\omega^i_{-1} + \omega^i_j\wedge\omega^j_{-2} + 2\omega_y\wedge\omega^i_{-3})v_1\otimes e_i\\ &+ (d\omega^i_{-1} - 2\omega_h\wedge\omega^i_{-1} + \omega^i_j\wedge\omega^j_{-1} + 2\omega_y\wedge\omega^i_{-2})v_2\otimes e_i\\ &+ (d\omega_x + 2\omega_h\wedge\omega_x)x + (d\omega_h + \omega_x\wedge\omega_y)h\\ &+ (d\omega^i_j + \omega^i_k\wedge\omega^k_j)e^j_i + (d\omega_y - 2\omega_h\wedge\omega_y)y.\end{aligned}$$

An arbitrary Cartan connection adapted to equation (1) has the form:

$$\begin{aligned}\omega^i_{-3} &= \theta^i_{-3},\\ \omega^i_{-2} &= \alpha^i_j\theta^j_{-2} + A^i_j\theta^j_{-3},\\ \omega^i_{-1} &= \beta^i_j\theta^j_{-1} + B^i_j\theta^j_{-2} + C^i_j\theta^j_{-3},\\ \omega_x &= -\theta_x + D_j\theta^j_{-2} + E_j\theta^j_{-3},\\ \omega_h &= F^{-1}_j\theta^j_{-1} + F^{-2}_j\theta^j_{-2} + F^{-3}_j\theta^j_{-3},\\ \omega^i_j &= G^{i,x}_j\theta_x + G^{i,-1}_{jk}\theta^k_{-1} + G^{i,-2}_{jk}\theta^k_{-2} + G^{i,-3}_{jk}\theta^k_{-3},\\ \omega_y &= H^x\theta_x + H^{-1}_j\theta^j_{-1} + H^{-2}_j\theta^j_{-2} + H^{-3}_j\theta^j_{-3}.\end{aligned}$$

In degree 0 of the curvature we have two nonzero components:

$$\begin{aligned}\Omega^i_{-3} \mod \langle\theta_{-2}\wedge\theta_{-2},\theta_{-3}\rangle &= \theta^x\wedge\theta^i_{-2} - \alpha^i_j\theta^x\wedge\theta^j_{-2}\\ \Omega^i_{-2} \mod \langle\theta_{-2},\theta_{-3}\rangle &= \theta^x\wedge\theta^i_{-1} - \beta^i_j\theta^x\wedge\theta^j_{-1}.\end{aligned}$$

Assume these two equalities is zero and get $\alpha^i_j = \delta^i_j$ and $\beta^i_j = \delta^i_j$.

We have three nonzero components in degree 1. The first component is:

$$\begin{aligned}&\Omega^i_{-3} \mod \langle\theta_{-2}\wedge\theta_{-3},\theta_{-3}\wedge\theta_{-3}\rangle =\\ &-\theta_x\wedge A^i_j\theta^j_{-2} + D_j\theta^j_{-2}\wedge\theta^i_{-2} + G^{i,x}_i\theta_x\wedge\theta^j_{-3} + G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-3} + 2F^{-1}_j\theta^j_{-1}\wedge\theta^i_{-3}.\end{aligned}$$

The second component is:

$$\begin{aligned}&\Omega^i_{-2} \mod \langle\theta_{-2}\wedge\theta_{-2},\theta_{-3}\rangle =\\ &\quad A^i_j\theta_x\wedge\theta^j_{-2} + D_j\theta^j_{-2}\wedge\theta^i_{-1} - \theta_x\wedge B^i_j\theta^j_{-2} + G^{i,x}_j\theta_x\wedge\theta^j_{-2} + G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-2}.\end{aligned}$$

The third component is:

$$\begin{aligned}&\Omega^i_{-1} \mod \langle\theta_{-2},\theta_{-3}\rangle =\\ &\frac{\partial f^i}{\partial q^j}\theta_x\wedge\theta^j_{-1} + B^i_j\theta_x\wedge\theta^j_{-1} - 2F^{-1}_j\theta^j_{-1}\wedge\theta^i_{-1} + G^{i,x}_j\theta_x\wedge\theta^j_{-1} + G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-1}.\end{aligned}$$

After applying zero conditions on these parts of the curvature we obtain $A^i_j = G^{i,x}_j = \frac{1}{2}B^i_j = -\frac{1}{3}\frac{\partial f^i}{\partial q^j}$, $\quad D_j = F^{-1}_j = G^{i,-1}_{jk} = 0$;

Proceed now to the second degree.

$$\begin{aligned}&\Omega^i_{-1} \mod \langle\theta_{-2}\wedge\theta_{-2},\theta_{-3}\rangle =\\ &\frac{\partial f^i}{\partial p_j}\theta_x\wedge\theta^j_{-2} + 2\frac{dA^i_j}{dx}\theta_x\wedge\theta^j_{-2} + 2\frac{\partial A^i_j}{\partial q_k}\theta^k_{-1}\wedge\theta^j_{-2} + C^i_j\theta_x\wedge\theta^k_{-2} - 2F^{-2}_j\theta^j_{-2}\wedge\theta^i_{-1}+\\ &\qquad G^{i,-2}_{jk}\theta^k_{-2}\wedge\theta^j_{-1} + 2H^x\theta_x\wedge\theta^i_{-2} + 2H^{-1}_j\theta^j_{-1}\wedge\theta^i_{-2} + G^{i,x}_k\theta_x\wedge B^k_j\theta^j_{-2}.\end{aligned}$$

We have:

$$\Omega^i_{-1}[\theta_x \wedge \theta^j_{-2}] = \frac{\partial f^i}{\partial p_j} + 2\frac{dA^i_j}{dx} + C^i_j + 2H^x + 2A^i_k A^k_j.$$

Assuming the previous tensor is zero, we obtain:

$$C^i_j = -\left(\frac{\partial f^i}{\partial p_j} + 2\frac{dA^i_j}{dx} + 2H^x + 2A^i_k A^k_j\right).$$

Next curvature component contains all second order invariants:

$$\Omega^i_{-2} \mod \langle \theta_{-2}\wedge\theta_{-3}, \theta_{-3}\wedge\theta_{-3}\rangle =$$

$$\frac{dA^i_j}{dx}\theta_x \wedge \theta^j_{-3} + \frac{\partial A^i_j}{\partial q_k}\theta^k_{-1}\wedge\theta^j_{-3} - \theta_x \wedge C^i_j\theta^j_{-3} + E_j\theta^j_{-3}\wedge\theta^i_{-1} + G^{i,x}_j\theta_x\wedge A^j_k\theta^k_{-3} +$$
$$2H^x\theta_x\wedge\theta^i_{-3} + 2H^{-1}_j\theta^j_{-1}\wedge\theta^i_{-3} + G^{i,-2}_{jk}\theta^k_{-2}\wedge\theta^j_{-2} + G^{i,-1}_{jk}\theta^{-1}_k \wedge A^j_l\theta^{-3}_l.$$

In coefficient $\Omega^i_{-2}[\theta^k_{-1}\wedge\theta^j_{-3}]$ we get invariant I_2.

$$\Omega^i_{-2}[\theta^k_{-1}\wedge\theta^j_{-3}] = \frac{\partial A^i_j}{\partial q_k} - E_j\delta^i_k + 2H^{-1}_k\delta^i_j = \frac{\partial A^i_j}{\partial q_k} + 2H^{-1}_k\delta^i_j + 2F^{-2}_j\delta^i_k.$$

Explicitly, the invariant I_2 is the following:

$$I_2 = \mathrm{tr}_0\left(\frac{\partial^2 f^i}{\partial q_j \partial q_k}\right),$$

were tr_0 is a traceless part of the tensor.

In the coefficient

$$\Omega^i_{-2}[\theta_x\wedge\theta^j_{-3}] = -C^i_j\frac{dA^i_j}{dx} + A^i_k A^k_j + 2H^k\delta^i_j$$

we obtain a so-called generalized Wilczynski invariant. As shown in [2], a part of differential invariants of systems of ODEs comes from its linearisation. As in [2], we call them generalized Wilczynski invariants. In our case we have two Wilczynski invariants of degree 2 and 3. We denote them as W_2 and W_3 respectively. The second degree generalized Wilczynski invariant is the following:

$$W_2 = tr_0\left(\frac{\partial f^i}{\partial p_j} - \frac{d}{dx}\frac{\partial f^i}{\partial q_j} + \frac{1}{3}\frac{\partial f^i}{\partial q_k}\frac{\partial f^k}{\partial q_j}\right)$$

Normalizing the trace of previous tensor to zero we obtain:

$$H^x = -\frac{1}{4m}(\frac{\partial f^i}{\partial p_i} + 3\frac{dA^i_i}{dx} + 3A^i_k A^k_i).$$

It remains to compute only $\mathfrak{sl}_2\times\mathfrak{gl}_m$ part of the curvature in degree 2.

$$\Omega_x \mod \langle\theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle = E_j\theta_x\wedge\theta^j_{-2} + 2F_j\theta_x\wedge\theta^j_{-2}.$$

Assuming that it vanishes identically we get the following condition:

$$E_j = -2F^{-2}_j.$$

We have:

$$\Omega_h \mod \langle\theta_{-2}, \theta_{-3}\rangle = F^{-2}_j\theta_x\wedge\theta^j_{-1} - \theta_x\wedge\theta^j_{-1}H^{-1}_j.$$

The condition $\Omega^i_h[\theta_x\wedge\theta^i_{-1}] = 0$ gives equality $F^{-2}_j = H^{-1}_j$.

Assuming the trace of the tensor $\Omega^i_{-2}[\theta^j_{-1}\wedge\theta^k_{-3}]$ is equal to zero we get:

$$F^{-2}_k = H^{-1}_k = -\frac{1}{2(m+1)}\frac{\partial A^i_i}{\partial q_k}.$$

The last part of degree 2 calculation is:

$$\Omega^i_j \mod \langle \theta_{-2}, \theta_{-3} \rangle = \frac{\partial A^i_j}{\partial q_k} \theta_x \wedge \theta^{-1}_k + G^{i,-2}_{j_k} \theta_x \wedge \theta^{-1}_k.$$

We obtain $G^{i,-2}_{j_k} = \frac{\partial A^i_j}{\partial q_k}$ from condition $\overline{\Omega}^i_j[\overline{\omega}_x \overline{\wedge} \overline{\omega}^k_{-1}] = 0$.

Proceed now to the degree 3. The first part of degree 3 we need to compute is Ω^i_{-1}:

$$\Omega^i_{-1} \mod \langle \theta_{-2} \wedge \theta_{-3}, \theta_{-3} \wedge \theta_{-3} \rangle =$$
$$\frac{\partial f^i}{\partial y_i} \theta_x \wedge \theta^j_{-3} + \frac{\partial B^i_j}{\partial p_k} \theta^k_{-2} \wedge \theta^j_{-2} + \frac{\partial C^i_j}{\partial x} \theta_x \wedge \theta^j_{-3} + \frac{\partial C^i_j}{\partial q_k} \theta^k_{-1} \wedge \theta^j_{-3} - 2F^{-3}_j \theta^j_{-3} \wedge \theta^i_{-1} -$$
$$2F^{-2}_j \theta^j_{-2} \wedge B^i_n \theta^k_{-2} + G^{i,-3}_{j_k} \theta^k_{-3} + G^{i,-2}_{j_k} \theta^k_{-2} \wedge B^i_j \theta^j_{-2} + G^{i_x}_j \theta_x \wedge C^j_x \theta^k_{-3} + 2H^{-2}_j \theta^j_{-2} \wedge \theta^i_{-2}.$$

Wilczynski invariant W_3 appears as the $\Omega^i_{-1}[\theta_x \wedge \theta^j_{-3}]$ coefficient:

$$\frac{\partial f^i}{\partial y_j} + \frac{dC^i_j}{dx} + A^i_k C^k_j + 2H^x A^i_j.$$

Direct computation shows that:

$$\Omega^i_{-1}[\theta_x \wedge \theta^j_{-3}] = \frac{\partial f^i}{\partial y^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial p^j} - \frac{d}{dx} \frac{\partial f^i}{\partial p^j} + \frac{2}{3} \frac{d^2}{dx^2} \frac{\partial f^i}{\partial q^j} - \frac{2}{27} (\frac{\partial f^i}{\partial q^j})^3$$
$$- \frac{4}{9} \frac{\partial f^i}{\partial q^k} \frac{d}{dx} \frac{\partial f^k}{\partial q^j} - \frac{2}{9} \frac{d}{dx} \left(\frac{\partial f^i}{\partial q^k} \right) \frac{\partial f^k}{\partial q^j} - 2\delta^i_j H^x.$$

Denote as W_3 invariant $\Omega^i_{-1}[\theta_x \wedge \theta^j_{-3}] + \frac{1}{2} \frac{d}{dx} W_2$. Invariant W_3 is equivalent fundamental invariant to $\Omega^i_{-1}[\theta_x \wedge \theta^j_{-3}]$. It means that with another fundamental invariant every of them generate all differential invariant of the system of third order ODEs. Explicitly the Wilczynski invariant W_3 is:

$$W_3 = \frac{\partial f^i}{\partial y^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial p^j} - \frac{1}{2} \frac{d}{dx} \frac{\partial f^i}{\partial p^j} + \frac{1}{6} \frac{d^2}{dx^2} \frac{\partial f^i}{\partial q^j} - \frac{2}{27} (\frac{\partial f^i}{\partial q^k})^3 - \frac{1}{18} \frac{\partial f^i}{\partial q^k} \frac{d}{dx} \frac{\partial f^k}{\partial q^j} - \frac{5}{18} \frac{d}{dx} \left(\frac{\partial f^i}{\partial q^k} \right) \frac{\partial f}{\partial q}$$

Note that invariant W_3 has known analogue in the case of one differential equation of third order:

$$\frac{\partial f}{\partial y} + \frac{1}{3} \frac{\partial f}{\partial q} \frac{\partial f}{\partial p} - \frac{1}{2} \frac{d}{dx} \frac{\partial f}{\partial p} + \frac{1}{6} \frac{d^2}{dx^2} \frac{\partial f}{\partial q} - \frac{2}{27} (\frac{\partial f}{\partial q})^3 - \frac{1}{3} \frac{\partial f}{\partial q} \frac{d}{dx} \frac{\partial f^k}{\partial q^j}.$$

The reader can found this invariant for example in the Chern work [1].

Let us compute the third degree normalization conditions.

$$\Omega_h \mod \langle \theta_{-2} \wedge \theta_{-2}, \theta_{-3} \rangle =$$
$$F^{-3}_j \theta_x \wedge \theta^j_{-2} + \frac{dF^{-2}_j}{dx} \theta_x \wedge \theta^j_{-2} + \frac{\partial F^{-2}_j}{\partial q_k} \theta^k_{-1} \wedge \theta^j_{-2} - \theta_x \wedge A^{-2}_j \theta^j_{-2}.$$

Thus:

$$\Omega_h[\theta_x \wedge \theta^j_{-2}] = -H^{-2}_j + F^{-3}_j + \frac{dF^{-2}_j}{dx}.$$

Normalizing this coefficient to 0 we obtain:

$$F^{-3}_j = H^{-2}_j - \frac{dF^{-2}_j}{dx}.$$

Next,

$$\Omega^i_j \mod \langle \theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle =$$

$$\frac{\partial A^i_j}{\partial p_k}\theta^k_{-2}\wedge\theta_x + \frac{\partial G^{i,-2}_{j_k}}{\partial p_l}\theta_x\wedge\theta^k_{-2} + G^{i,-3}_{j_k}\theta_x\wedge\theta^k_{-2} + G^{i,x}_k\theta_x\wedge G^{k-2}_{jl}\theta^l_{-2} + G^{i,-2}_{kl}\theta^{lx}_{-2}\wedge G^{kx}_j\theta_x.$$

We have:

$$\Omega^i_j[\theta_x\wedge\theta^k_{-2}] = -\frac{\partial A^i_j}{\partial p_k} + \frac{\partial G^{i,-2}_{j_k}}{\partial p_l} + G^{i,-3}_{j_k} + G^{i,x}_k G^{k-2}_{jl} - G^{i,-2}_{lk}G^{lx}_j.$$

Assuming this coefficient is equal to 0 we get:

$$G^{i,-3}_{j_k} = \frac{\partial A^i_j}{\partial p_k} - \frac{\partial G^{i,-2}_{j_k}}{\partial p_l} - G^{i,x}_k G^{k-2}_{jl} + G^{i,-2}_{lk}G^{lx}_j.$$

Finally,

$$\Omega_y \mod \langle \theta_{-2}, \theta_{-3}\rangle =$$

$$\frac{\partial H^x}{\partial q_i}\theta^j_{-1}\wedge\theta_x + \frac{dH^{-1}_j}{dx}\theta_x\wedge\theta^j_{-1} + \frac{\partial H^{-1}_j}{\partial q_k}\theta^k_{-1}\wedge\theta^j_{-1} + H^{-1}_j\frac{\partial f^j}{\partial q_k}\theta_x\wedge\theta^k_{-1} + H^{-2}_j\theta_x\wedge\theta^j_{-1}.$$

The coefficient $\Omega_y[\theta^j_{-1}\wedge\theta_x]$ is the following:

$$\frac{\partial H^x}{\partial q_j} - \frac{dH^{-1}_j}{dx} - H^{-1}_k\frac{\partial f^k}{\partial q_j} - H^{-2}_j.$$

Normalizing it to 0 we obtain:

$$H^{-2}_j = \frac{\partial H^x}{\partial q_j} - \frac{dH^{-1}_j}{dx} - H^{-1}_k\frac{\partial f^k}{\partial q_k}.$$

The last coefficient we need in degree 3 is $\Omega_y[\theta^k_{-1}\wedge\theta^j_{-1}]$:

$$\frac{\partial H^{-1}_j}{\partial q_k} - \frac{\partial H^{-1}_k}{\partial q_j} = 0.$$

In the degree 4 we need to compute only one coefficient of curvature:

$$\Omega_y \mod \langle \theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle =$$

$$\frac{\partial H^x}{\partial p_j}\theta^j_{-2}\wedge\theta_x + \frac{\partial H^{-1}_j}{\partial p_k}\theta^k_{-2}\wedge\theta^j_{-1} + \frac{dH^{-2}_j}{dx}\theta_x\wedge\theta^j_{-2} +$$

$$\frac{\partial H^{-2}_j}{\partial q_k}\theta^{-1}_k\wedge\theta^j_{-2} + H^{-1}_j\frac{\partial f^j}{\partial p_k}\theta_x\wedge\theta^k_{-2} + H^{-3}_j\theta_x\wedge\theta^j_{-2} - 2F^{-2}_j\theta^j_{-2}\wedge(H^x\theta_x + H^{-1}_k\theta^k_{-1}).$$

The Cartan connection coefficient $\Omega_y[\theta_x\wedge\theta^j_{-2}]$ has the following form:

$$-\frac{\partial H^x}{\partial p_j} + \frac{dH^{-2}_j}{dx} + H^{-1}_k\frac{\partial f^k}{\partial q_j} - H^{-3}_j.$$

Assuming it is equal to 0 we get:

$$H^{-3}_j = -\frac{\partial H^x}{\partial p_j} + \frac{dH^{-2}_j}{dx} + H^{-1}_k\frac{\partial f^k}{\partial q_j}.$$

Finally, invariant I_4 is the tensor $\Omega_y[\theta^k_{-1}\wedge\theta^j_{-2}]$:

$$-\frac{\partial H^{-1}_k}{\partial p_j} + \frac{\partial H^{-2}_j}{\partial q^k} + 2H^{-1}_j H^{-1}_k.$$

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