Lie Groups and Algebras I,II
Lecture Notes for MTH 915
03/04

Ulrich Meierfrankenfeld

May 3, 2004

Contents

Chapter 1

Basic Properties of Lie Algebras

1.1 Definition

Let $\mathbb{K}$ be a field. With a $\mathbb{K}$-space we mean a vector space over $\mathbb{K}$. For $\mathbb{K}$-space V, $\mathrm{End}(V)$ denotes the ring of $\mathbb{K}$-linear maps from V to V. For $a, b \in \mathrm{End}(V)$ define $[a, b] := ab - ba$. $[a, b]$ is called the commutator or bracket of a and b. The bracket operation has an amazing property

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in \mathrm{End}(V)$. Indeed,

$$\begin{aligned}
&[a, [b, c]] + [b, [c, a]] + [c, [a, b]] \\
&= a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\
&= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\
&= 0
\end{aligned}$$

Also note that $[,]$ is $\mathbb{K}$-bilinear and that $[a, a] = 0$. These observations motivate the following definitions:

Definition 1.1.1 [**def:algebra**] *Let $\mathbb{K}$ be a field, A a (left) vector space over $\mathbb{K}$ and $\cdot : A \times A \to A$ a K-bilinear map. Then $(A, \cdot)$ is called a* $\mathbb{K}$-algebra. *If $\cdot$ is associative, then A is called an* associative algebra.

Definition 1.1.2 [**def: lie algebra**] *A $\mathbb{K}$-algebra $(A, [,])$ is called a* Lie algebra *over $\mathbb{K}$ provided that*

(i) [**a**] *$[,]$ is symplectic, that is$[a, a] = 0$ for all $a \in A$.*

(ii) [**b**] *$[,]$ fullfills the Jacobi identity*

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in A$.

From now on $\mathbb{K}$ is always a field and L a Lie algebra over $\mathbb{K}$.

The prime example for a Lie algebra is $(\mathrm{End}(V), [,])$. We denote this Lie algebra by $\mathfrak{gl}(V)$. L is called abelian if $[a, b] = 0$ for all $a, b \in L$. Any $\mathbb{K}$-space V becomes an abelian Lie algebra if one defines $[a, b] = 0$ for all $a, b \in V$.

Let $(A, \cdot)$ be any associative $\mathbb{K}$-algebra and define $[a, b] := ab - ba$ for all $a, b \in A$. Just as for End(V) none shows that $(A, [,])$ is a Lie algebra over $\mathbb{K}$. We denote this Lie algebra by $\mathfrak{l}(A)$.

Similar as for groups, rings and modules one defines homomorphisms, subalgebras, generations, ideals, ... For example a subalgebra of an algebra A is a $\mathbb{K}$-subspace I of A such that $i \cdot j \in I$ for all $i, j \in I$. Note that this equivalent to requiring that $(I, \cdot)$ is $\mathbb{K}$-algebra. If I is a $\mathbb{K}$-subspace of A with $i \cdot a \in I$ and $a \cdot i \in I$ for all $a \in A, i \in I$ then I is called an ideal. In this case the quotient A/I is a $\mathbb{K}$-algebra. The kernel $\ker \phi$ of an homomorphism $\phi : A \to B$ of $\mathbb{K}$-algebras is an ideal in A and the First Isomorphism Theorem holds: $A/\ker \phi \cong \phi(A)$ as $\mathbb{K}$-algebras.

In general one needs to distinguish between left and right ideals. This is not necessary for Lie algebras:

Lemma 1.1.3 [**alternating**]

(a) [**a**] $[,]$ *is alternating, that is* $[x, y] = -[y, x]$ *for all* $x, y \in L$.

(b) [**b**] *Let* I *be a* $\mathbb{K}$*-subspace of* L. *Then* I *is an ideal (in* L*) iff* I *is a right ideal and iff* I *is a left ideal.*

Proof: (a) $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$.

(b) follows immediately from (a). □

We remark that if $\mathrm{char}\,\mathbb{K} \neq 2$, then $xy = -yx$ for all x, y in an algebra A implies $xx = 0$. Indeed $xx = -xx$ and so $2xx = 0$. As 2 is invertible we get $xx = 0$.

Let V be a $\mathbb{K}$-space and $\mathcal{W}$ a set of subspaces of V with $0 \in \mathcal{W}$ and $V \in \mathcal{W}$ and $\mathcal{W}$. Put

$$\mathrm{End}(\mathcal{W}) = \{\phi \in \mathrm{End}(V) \mid \phi(W) \leq W \ \forall W \in \mathcal{W}\}.$$

Note that End($\mathcal{W}$) is a subalgebra of End(V). We denote the corresponding Lie algebra by $\mathfrak{gl}(\mathcal{W})$. Suppose that V has a finite basis $(v_1, v_2, \ldots, v_n)$ and $\mathcal{W}$ consist of the $n + 1$ subspace $\mathbb{K}v_1 + \mathbb{K}v_2 + \mathbb{K}v_i$, $0 \leq i \leq n$. The reader should verify that $\mathfrak{gl}(\mathcal{W})$ now consist of all the upper triangular matrices (with respect to the given basis).

Let f be a bilinear form on V, that is a $\mathbb{K}$-bilinear function $f : V \times V \to K$. Define

$$\mathfrak{cl}(f) = \{\alpha \in \mathfrak{gl}(V) \mid f(\alpha v, w) + f(v, \alpha w) = 0 \ \forall v, w \in V$$

We claim that $\mathfrak{cl}(f)$ is a Lie subalgebra of $\mathfrak{gl}_(V)$. Clearly it's a $\mathbb{K}$-subspace. Let $\alpha, \beta \in \mathfrak{cl}(f)$ and $v, w \in V$. Then

$$\begin{aligned} f([\alpha,\beta]v,w) &= f(\alpha\beta v,w) - f(\beta\alpha,w) \\ &= -f(\beta v,\alpha w) + f(\alpha v,\beta w) \\ &= f(v,\beta\alpha w) - f(v,\alpha\beta w) \\ &= -f(v,[\alpha,\beta]w) \end{aligned}$$

So $[\alpha,\beta] \in \mathfrak{cl}(f)$ and $\mathfrak{cl}(f)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

1.2 Structure constants

Let L be a Lie algebra over $\mathbb{K}$ and $\mathcal{B}$ a basis for L. So everry $l \in L$ can be uniquely written as $l = \sum_{b\in\mathcal{B}} k_b b$, where $k_b \in \mathbb{K}$ and all but finitely many of the $k_b's$ are zero. Hence we can define $a_{ij}^k \in \mathbb{K}$, $i, j, k \in \mathcal{B}$, by

$$[i,j] = \sum_{k\in B} a_{ij}^k k.$$

The a_{ij}^k's are called the structure constants of L with respect to $\mathcal{B}$. Since $[,]$ is bilinear the structure constants uniquely determine $[,]$. Since $[,]$ is symplectic, alternating and fulfils the Jacobi identity we have for all $i, j, k, l \in \mathcal{B}$.

$$a_{ii}^k = 0$$

$$a_{ij}^k + a_{ji}^k = 0$$

$$\sum_m a_{ij}^m a_{km}^l + a_{jk}^m a_{im}^l + a_{ki}^m a_{jm}^l = 0.$$

Conversely, given a set $\mathcal{B}$ and $a_{ij}^k \in \mathbb{K}$, $i, j, k \in \mathcal{B}$ which fulfill the above three identities one easily obtains a Lie algebra with basis $\mathcal{B}$ and the a_{ij}^k as structure constants.

As an example consider the case of a 2-dimensional Lie-algebra L with basis x, y. Put $a := [x,y]$. Then $[L,L] = \mathbb{K}a$. If $a = 0$ then L is abelian.

Suppose that L is not abelian and choose $b \in L \setminus \mathbb{K}a$. Then also (a,b) is a basis for L and $[a,b] = ka$ for some $0 \neq k \in \mathbb{K}$. Replacing b by $k^{-1}b$ we may assume $[a,b] = a$. So up to isomorphism there exists at most one 2-dimensional non abelian Lie Algebra. For later use we record:

Lemma 1.2.1 [**2 dim**] *If L is 2-dimensional and non-abelian, then L has a basis (a,b) with $[a,b] = a$.* □

To show existence of such a Lie-algebra we could compute the structure constant and verify the above identies. But its easier to exhibit such Lie-algebra as a subalgebra of $\mathfrak{gl}(\mathbb{K}^2)$. Namely choose

$$a := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad b := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

1.3 Derivations

Definition 1.3.1 [**def:derivation**] *Let A be a $\mathbb{K}$-algebra. Then a* derivation *of A is a map $\delta : A \to A$ such that*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in A$. $\mathfrak{der}(A)$ denotes the set of all derivations of A.

Lemma 1.3.2 [**derivations are lie**] *Let A be a $\mathbb{K}$-algebra. Then $\mathfrak{der}(A)$ is a subalgebra of $\mathfrak{gl}(A)$.*

Obviously $\mathfrak{der}(A)$ is a $\mathbb{K}$-subspace of $\mathfrak{gl}(A)$. Now let $\gamma, \delta \in \mathfrak{der}(A)$ and $a, b \in A$. Then

$$\begin{aligned} [\gamma, \delta](ab) &= \gamma\delta(ab) - \delta\gamma(ab) \\ &= \gamma(a\delta(b)) + \gamma(\delta(a)b) - \delta(a\gamma(b)) - \delta(\gamma(a)b) \\ &= \gamma(a)\delta(b) + a\gamma(\delta(b)) + \gamma(\delta(a))b + \delta(a)\gamma(b) \\ &\qquad -\delta(a)\gamma(b) - a\delta(\gamma(b)) - \delta(\gamma(a))b - \gamma(a)\delta(b) \\ &= a(\gamma(\delta(b)) - d(\gamma(b))) + (\gamma(\delta(a)) - \delta(\gamma(\alpha)))b \\ &= [\gamma, \delta](a)b + a[\gamma, d](b) \end{aligned}$$

Lemma 1.3.3 [**left multiplication**] *Let A be an assocative $\mathbb{K}$-algebra and for $a \in A$ define $\mathrm{l}(a)a : A \to A, b \to ab$, $\mathrm{r}(a) : A \to A, b \to ba$ and $\mathrm{ad}\,(a) = \mathrm{l}(a) - \mathrm{r}(a)$. Let $a, b, c \in A$*

(a) [**a**] $\mathrm{l}(a), \mathrm{r}(a)$ *and* $\mathrm{ad}\,(a)$ *all are $\mathbb{K}$-linear.*

(b) [**b**] $[a, bc] = [a, c] + [a, b]c$. *That is* $\mathrm{ad}\,(a)$ *is a derivation of A*

(c) [**c**] $\mathrm{l} : A \to \mathrm{End}(A), a \to \mathrm{l}(a)$ *is a homomorphism.*

(d) [**d**] $\mathrm{r} : A \to \mathrm{End}(A), a \to \mathrm{r}(a)$ *is an anti-homomorphism.*

Proof: (a) Obvious.
(b) We compute

$$b[a, c] + [a, b]c = bac - bca + abc - bac = a(bc) - (bc)a = [a, bc]$$

Also $\mathrm{ad}\,(a)(b) = ab - ba = [a,b]$ and the preceeding equation says that $\mathrm{ad}\,(a)$ is a derivation of A.

(c) and (d) are readily verifed as A is associative. □

Lemma 1.3.4 [**inner derivations**] *Define* $(\,ad)(a) : L \to L, l \to [a,l]$ *and let* $a \in L$

(a) [**a**] $\mathrm{ad}\, a$ *is a derivation of* L.

(b) [**b**] *Let* $\delta \in \mathfrak{der}(L)$. *Then* $[\delta, \mathrm{ad}\, a] = \mathrm{ad}\,(\delta(a))$.

(c) [**c**] $\mathrm{ad} : L \to \mathfrak{gl}(L)$ *is a homomorphism.*

(d) [**d**] $\mathrm{ad}\,(L)$ *is an ideal in* $\mathfrak{der}(L)$.

Proof: Let $a, b, c \in A$. Then

$$\mathrm{ad}\,(a)([b,c]) = [a,[b,c]] = -[b,[c,a]] - [c,[a,b]] = [b, \mathrm{ad}\, a(c)] + [\mathrm{ad}\,(a)(b), c]$$

and so $\mathrm{ad}\,(a)$ is a derivation.

Let δ be a derivation of A. Then

$$\begin{aligned}
[\delta, \mathrm{ad}\,(a)](b) &= \delta(\mathrm{ad}\,(a)(b)) - \mathrm{ad}\,(a)(\delta(b)) \\
&= \delta([a,b]) - [a, \delta(b)] \\
&= [\delta(a), b] + [a, \delta(b)] - [a, \delta(b)] \\
&= \mathrm{ad}\,(\delta(a))(b)
\end{aligned}$$

Thus (b) holds.

From (b) applied to the derivation $\mathrm{ad}\, b$ in place of δ we have $[\mathrm{ad}\, b, \mathrm{ad}\, a] = \mathrm{ad}\,([b,a])$ so (c) holds.

Finally (d) follows from (b). □

A derivation of the form $\mathrm{ad}\,(a)$ is called an *inner derivation.* All other derivations of a Lie Algebra are called *outer derivations*.

1.4 Modules

In this section A is an associative or Lie algebra over the field $\mathbb{K}$.

Definition 1.4.1 [**def:rep for associative**] *Let* A *be an associative* $\mathbb{K}$*-algebra and* V *a* $\mathbb{K}$*-space.*

(a) [**a**] *A representation for* A *over* V *is a homomorphism* $\Phi : A \to \mathrm{End}(V)$.

(b) [**b**] *An action for A on V is a bilinear map $A \times V \to V, (a, v) \to av$ such that*

$$(ab)v = a(bv)$$

for all $a, b \in A$, $v \in V$.

Definition 1.4.2 [**def:rep for lie**] *Let V be a $\mathbb{K}$-space.*

(a) [**a**] *A* representation *for L is a homomorphism $\Phi : L \to \mathfrak{gl}(V)$.*

(b) [**b**] *An* action *for L on V is a bilinear map $L \times V \to V, (a, v) \to av$ such that*

$$[a, b]v = a(bv) - b(av)$$

for all $a, b \in L$, $v \in V$.

If A is a associative algebra, then by 1.3.3 the left multiplication l is a representaion for A on A. And if L is a Lie algebra then by 1.3.4 ad is a representation of L on L.

Lemma 1.4.3 [**rep=action**] *Let A be an associative or a Lie algebra and V a $\mathbb{K}$-space.*

(a) [**a**] *Let Φ be a representation for A over V. Then $A \times V \to V, (a, v) \to \Phi(a)(v)$ is an action for A on V.*

(b) [**b**] *Suppose $A \times V \to V, (a, v) \to av$ is an action for A on V. Define $\Phi : A \to \mathrm{End}(V)$ by $\Phi(a)(v) = av$ for all $a \in A$, $v \in V$. Then Φ is a representation for A over V.*

Proof: Straightforward. □

If A acts on V we say that V is a *module* for A.

Lemma 1.4.4 [**associative to lie**] *Let V be a module for the associative algebra A. Then with the same action V is also a module for $\mathfrak{l}(A)$. In particular, left multiplication is an action of the Lie-Algebra $\mathfrak{l}(A)$ on A.*

Proof: Let $a, b \in A$ and v in v. Then

$$[a, b]v = (ab - ba)v = a(bv) - b(av).$$

□

Definition 1.4.5 [**def:centralizer**] *Let V be a module for the associative or Lie algebra A.*

(a) [**a**] $C_V(A) = \{v \in V \mid av = 0 \,\forall a \in A\}$.

(b) [**b**] $C_A(V) = \{a \in A \mid av = 0 \,\forall v \in V\}$

(c) [**c**] *If $C_A(V) = 0$ we say that V is a* faithful *A-module.*

If Φ is the representation corresponding to the module V, then $C_A(V) = \ker \Phi$. In particular, $C_A(V)$ is an ideal in A. Note that V is also a module for $A/C_A(V)$ via $(a + C_A(V))v = av$. Even more, V is faithful for $A/C_A(V)$.

Put $Z(A) := \{a \in A \mid ab = 0 \forall b \in A\} = C_A(A)$. Then $Z(A)$ is an ideal in A called to center of A. Note that L is abelian iff $L = Z(L)$.

If X is a subsets of A and Y a subset of the A-module V, then we denote by XY the $\mathbb{K}$-subspace of V generated by $\{xy \mid x \in X, y \in Y\}$. We say that Y is X-invariant if $xy \in X$ for all $x \in X$, $y \in Y$. $\langle Y \rangle$ denotes the additive subgroup of V generate by Y, while $\mathbb{K}Y$ denotes the $\mathbb{K}$-subspace of V generated by Y.

Lemma 1.4.6 [**product of subspaces**] *Let A be a Lie or an associative algebra, V an A-module, X a subset of A and Y an X-invariant subset of V. Then $\mathbb{K}Y$ is X-invariant.*

Proof: Let $x \in X$ and put $Z = \{z \in V \mid xz \in \mathbb{K}Y\}$. Then Z is an $\mathbb{K}$-subspace of V and since $Y \subseteq Z$, $\mathbb{K}Y \subseteq Z$. Thus $x\mathbb{K}Y \subseteq \mathbb{K}Y$ and $\mathbb{K}Y$ is X-invariant. □

Lemma 1.4.7 [**submodules and ideals**] *Let V an L-module and $I \subseteq L$.*

(a) [**a**] *I is an ideal in L if and only if I is L-submodule of L.*

(b) [**b**] *If I is an ideal in L then IV and $C_V(I)$ are L-submodule of V.*

Proof: Clearly I is a submodule iff its a left ideal. As left ideals are the same as ideals, (a) holds.

For (b) let $v \in V$, $i \in I$ and $l \in L$. Then $l(iv) = ([l, i])v + i(lv) \in IV$. In particular, IV is a L-submodule. Moreover, if $v \in C_V(I)$ we get $i(lv) = 0$ and so $lv \in C_V(I)$ and $C_V(I)$ is an L-submodule. □

1.5 The universal enveloping algebra

We assume the reader to be familiar with the definitions of tensor products and symmetric powers, see for example [La].

Definition 1.5.1 [**universal**]

(a) [**a**] *Let V be a $\mathbb{K}$-space. Then a tensor algebra for V is an associative algebra T with 1 together with an $\mathbb{F}$-linear map Φ such that whenever T' is an associative $\mathbb{K}$-algebra with one and $\Phi' : V \to T'$ is $\mathbb{K}$-linear, then there exists an unique $\mathbb{K}$-algebra homomorphis $\Psi : T \to T'$ with $\Phi' = \Psi \circ \Phi$.*

(b) [**b**] *Let V be a $\mathbb{K}$-space. Then a symmetric algebra for V is a commutative and associative algebra T with 1 together with an $\mathbb{F}$-linear map Φ such that whenever T' is a commuative and associative $\mathbb{K}$-algebra with one and $\Phi' : V \to T'$ is $\mathbb{K}$-linear, then there exists an unique $\mathbb{K}$-linear map $\Psi : U \to U'$ with $\Phi' = \Psi \circ \Phi$.*

(c) [**c**] *Let L be a Lie algebra over $\mathbb{K}$. Then an* universal enveloping algebra *for L is an associative $\mathbb{K}$-algebra U with one together with a homomorphism $\Phi : L \to \mathfrak{l}(U)$ such that whenever U' is an associative $\mathbb{K}$-algebra with one and $\Phi' : L \to \mathfrak{l}(U')$ is an homomorphism, then there exists an unique homomorphism of $\mathbb{K}$-algebra $\Psi : U \to U'$ with $\Phi' = \Psi \circ \Phi$.*

Lemma 1.5.2 [**existence of universal**]

(a) [**a**] *Let V be a $\mathbb{K}$-space. Then V has a tensor algebra $\mathfrak{T}(V)$ and $\mathfrak{T}(V)$ is unique up to isomorphism.*

(b) [**b**] *Let V be a $\mathbb{K}$-space. Then V has a symmetric algebra $\mathfrak{S}(V)$ and $\mathfrak{S}(V)$ is unique up to isomorphism.*

(c) [**c**] *Let L be a Lie algebra. Then L has a universal enveloping algebra $\mathfrak{U}(L)$ and $\mathfrak{U}(L)$ is unique up to isomorphism.*

Proof: The uniqueness statements follows easily from the definitions.

(a) Define $\mathfrak{T} = \bigoplus_{i=0}^{\infty} \bigotimes^i V$ and define a multiplication on $\mathfrak{T}$ by

$$(v_1 \otimes \ldots \otimes v_m)(w_1 \otimes \ldots w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots w_n$$

The its is staighforward to check that $\mathfrak{T}$ is an associative algebra with 1. If T' is an associative algebra with 1, and $\Phi' : V \to T'$ is linear. Define $\Psi : \mathfrak{T} \to T'$ by $\Psi(v_1 \otimes \ldots \otimes v_m) = \Phi'(v_1)\Phi'(v_2)\ldots\Phi'(v_m)$.

(b) Let $\mathfrak{S}^n V$ be the n-th symmetric power of V and sefine $\mathfrak{S} := \bigoplus_{i=0}^{\infty} \mathfrak{S}^i V$. Proceed as in (a).

(c) Let I be the ideal in $\mathfrak{T}(L)$ generated by all the $a \otimes b - b \otimes a - [a, b]$, $a, b \in L$. Then T/I is a universal enveloping algebra. □

Lemma 1.5.3 [**basis for tensor**]

(a) [**a**] *Let I be set and for $i \in I$ let V_i be a $\mathbb{K}$-space with basis $\mathcal{B}_i$. Put $\mathcal{B} = \otimes_{i \in I} \mathcal{B}_i = (\otimes_{i \in I} b_i \mid b_i \mid b_i \in \mathcal{B}_i \forall i \in I)$. Then $\mathcal{B}$ is a basis for $\bigotimes_{i \in I} V_i$.*

(b) [**b**] *Let V be a $\mathbb{K}$-space with ordered basis $\mathcal{B}$. Let $n \in \mathbb{N}$. Then $(b_1 b_2 \ldots b_n \mid b_1 \leq b_2 \leq \ldots \leq b_n, b_i \in \mathcal{B})$ is a basis for $\mathfrak{S}^n V$.*

Proof: Wellknown. See for example [La]. □

Let A be any associative $\mathbb{K}$-algebra. Note that the definition of an universal enveloping algebra implies that the map

$$\mathrm{Hom}(\mathfrak{U}(L), A) \to \mathrm{Hom}(L, \mathfrak{l}(A)), \quad \alpha \to \alpha \circ \Phi$$

is a bijection. For the case that $A = \mathrm{End}(V)$ for a $\mathbb{K}$-space V we conclude:

Lemma 1.5.4 [**modules for universal**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping alebra. Let V an L-module. Then there exists a unique action of $\mathfrak{U}$ on V with $\phi(l)v = lv$ for all $l \in L$. The resulting map between the set of L-modules and the set of $\mathfrak{U}$-modules is a bijection.*

□

Lemma 1.5.5 [**d spans u**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping algebra for L. Also let $\mathcal{B}$ be an ordered basis for L. Put $\mathfrak{U}_m = \sum_{i=0}^{m} \phi(L)^i$. Then*

$$\mathfrak{U}_m = \mathbb{K}\langle \phi(b_1) \ldots \phi(b_i) \mid 0 \leq i \leq m, b_j \in \mathcal{B}, b_1 \leq b_2 \leq \ldots \leq b_i \rangle.$$

Proof: By induction on m. Since we interpret the empty product as 1, the statement is true for $m = 0$. Suppose its is true for $m - 1$. Let $b_1, b_2 \in b_m \in \mathcal{B}$. Also let $0 \leq i < m$ and put $a = b_1 b_2 \ldots b_{i-1}$ and $c = b_{i+2} \ldots b_m$. Then

$$b_1 b_2 \ldots b_m = a b_i b_{i+1} c = a b_{i+1} b_i c + a[b_i, b_{i+1}]$$

Thus

$$b_1 b_2 \ldots b_m + \mathfrak{U}_{m-1} = b_1 \ldots b_{i-1} b_{i+1} b_i b_{i+2} \ldots b_n + \mathfrak{U}_{m-1}$$

and so for all $\pi \in \mathrm{Sym}(m)$,

$$b_1 b_2 \ldots b_m + \mathfrak{U}_{m-1} = b_{\pi(1)} \ldots b_{\pi(m)} + \mathfrak{U}_{m-1}$$

Choosing π such that $b_{\pi(1)} \leq b_{\pi(2)} \leq \ldots \leq b_{\pi(m)}$ and we see that the lemma also holds for m. □

Lemma 1.5.6 [**action of l on s**] *Let $\mathcal{B}$ be an ordered basis for the Lie algebra L. Identify L with is image in $S := \mathfrak{S}(L)$. Let $b \in \mathcal{B}$ and $s \in \mathcal{B}^n$. Define $b \leq s$ if either $n = 0$ or $s = \prod_{i=1}^{n} b_i, b_i \in \mathcal{B}$ with $b \leq b_i$ for all $1 \leq i \leq n$. Then there exists a unique action $\cdot$ of L on S such that $b \cdot s = bs$ for all $b \in \mathcal{B}$, $n \in \mathbb{N}$ and $s \in \mathcal{B}^n$ with $b < s$.*

Proof: Put $S_m = \sum_{i=0}^{m} L^m \leq S$. To show the uniqueness of $\cdot$ we show by induction on m that the restriction of $\cdot$ to $L \times S_m$ is unique and that

1° [**1**] *$w(b,s) := b \cdot s - bs \in S_m$ for all $b \in L$ and $s \in S_m$.*

Note that (1°) implies that $b \cdot s = bs + w(b,s) \in S_{m+1}$

If $m = 0$, the $S_0 = \mathbb{K}$ and $b \cdot s = bs = sb$ for all $s \in S_0$. Suppose now that $m \geq 1$. Let $s = dt \in \mathcal{B}$ with $d \in \mathcal{B}$, $t \in \mathcal{B}^{m-1}$ and $d \leq t$. We need to compute $b \cdot s$ uniquely and show that $b \cdot s - bs \in S_m$. Note that $dt = d \cdot t$.

If $b \leq d$, then $b \leq s$. So

2° [**2**] *$b \cdot s = bs$, whenever $b \leq s$*

Also $b \cdot s - bs = 0 \in S_m$.

If $b > d$, then since $\cdot$ is an action

3° [**3**]

$$b \cdot s = b \cdot (d \cdot t) = d \cdot (b \cdot t) + [b,d] \cdot t$$

By induction on m, $b \cdot t$ and $[b,d] \cdot t$ are uniquely determined. Moreover, $b \cdot t = bt + w(b,t)$ with $w(b,t) \in S_{m-1}$ and $[b,d] \cdot t \in S_m$. Since $c \leq bt$ we have $d \cdot (bt) = dbt = bs$. Also by induction $d \cdot w$ is uniquely determined and contained in S_m. Thus the formula

4° [**4**] *$b \cdot s = dbt + d \cdot w(b,t) + [b,d] \cdot t$, whenever $b \not\leq s$*

uniquely determines $b \cdot s$. Moreover $w(b,s) = d \cdot w(b,t) + [b,d] \cdot t \in S_m$.

Thus $\cdot$ is unique and (1°) holds.

To prove existence we define $b \cdot s$ for $b \in \mathcal{B}$ and $s \in \mathcal{B}^m$ by induction on m via (2°) and (4°). Once $b \cdots s$ is defined for all $s \in \mathcal{B}^m$, define $l \cdot s$ for all $l \in L$ and $s \in S_m$ by linear extension. Note also that (1°) will hold inductively. So all terms are on the right side of (4°) are defined at the time its used to define the left side.

We need to verify that $\cdot$ is an action.

Let $a, b \in L$ and $s \in S$. We say that $\{a,b\}$ acts on v if $a \cdot (b \cdot s) - b \cdot (a \cdot s) = [a,b] \cdot s$. Note that set of $s \in S$ on which $\{a,b\}$ acts is a $\mathbb{K}$-subspae of V.

Suppose inductively that we have shown

5° [**5**] *For all $a, b \in \mathcal{L}$ and all $s \in S_{m-1}$, $\{a,b\}$ acts on s.*

Let $a, b \in \mathcal{B}$ and $s \in \mathrm{B}^m$. We need to show that $\{a,b\}$ acts on s. This is obviously the case then $a = b$. So suppose $a \neq b$

Suppose that $a \leq s$ or $b \leq s$. Without loss $a > b$. Then $b \leq s$. Using the definition of $a \cdot u$ for $u = b \cdot s$ (compare (3°)) we get

6° [**6**] *If $a \leq s$ or $b \leq s$ then $\{a,b\}$ acts on all $s \in \mathcal{B}^m$.*

Suppose next that $a > s$ and $b > s$. Let $s = dt = d \cot t$ be as above. Then

Since $d \leq bs$ (6°) gives that $\{a, d\}$ acts on bs. By induction $\{a, d\}$ also acts on $w(b, s) \in S_{m-1}$ and so $\{a, d\}$ acts on $b \cdot s = bs + w(b, s)$. This allows us to compute (using our inductive assumption (5°) various times):

$$\begin{aligned} a \cdot (b \cdot (d \cdot t)) &= a \cdot (d \cdot (b \cdot t) + [b, d] \cdot t) \\ &= d \cdot (a \cdot (b \cdot t)) + [a, d] \cdot (b \cdot t) + [b, d] \cdot (a \cdot t) + [a, [b, d]] \cdot t \end{aligned}$$

Since the situation is symmetric in a and b the above equation also holds with the roles of a and b interchanged. Subtracting these two equations we obtain:

$$\begin{aligned} a \cdot (b \cdot dt) - b \cdot (a \cdot dt) &= d \cdot (a \cdot (b \cdot t) - b \cdot (a \cdot t)) + [a, [b, d]] \cdot t - [b, [a, d]] \cdot t \\ &= d \cdot ([a, b] \cdot t) + [a, [b, d]] \cdot t + [b, [d, a]] \cdot t \\ &= [a, b] \cdot (d \cdot t) + ([d, [a, b]] + [a, [b, d]] + [b, [d, a]]) \cdot t \\ &= [a, b] \cdot dt \end{aligned}$$

Thus $\{a, b\}$ acts on $s = dt$ and so by induction L acts on S. □

Theorem 1.5.7 (Poincare-Birkhof-Witt) [**pbw**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping algebra of L. Let $\mathcal{B}$ be the ordered basis of L and view $\mathfrak{S}(L)$ as an L- (and so as an $\mathfrak{U}(L)$-) module via 1.5.6.*

(a) [**a**] *The map $\Psi : \mathfrak{U}(L) \to \mathfrak{S}(L), u \to u \cdot 1$ is a isomorphism of $\mathbb{K}$-spaces.*

(b) [**b**]

$$\mathcal{D} := (\phi(b_1)\phi(b_2)\ldots\phi(b_n) \mid n \in \mathbb{N}, b_1 \leq b_2 \leq \ldots \leq b_n \in \mathcal{B})$$

is a basis for $\mathfrak{U}$.

(c) [**c**] *ϕ is one to one.*

Proof: Let $b_1, b_2, \ldots b_n$ be a nondecreasing sequence in $\mathcal{B}$. The definition of the action of L on $\mathfrak{S}(L)$ implies that $\phi(b_1)\phi(b_2)\ldots\phi(b_n) \cdot 1 = b_1 b_2 \ldots b_n$. Hence $\Psi(\mathcal{D})$ is a basis for $\mathfrak{S}(L)$. Thus Ψ is onto and $\mathcal{D}$ is linearly independent in $\mathfrak{U}$. By 1.5.5 $\mathbb{K}\mathcal{D} = \mathfrak{U}$ and so $\mathcal{D}$ is a basis for $\mathfrak{U}$. Hence Ψ sends a basis of $\mathfrak{U}$ to a basis of $\mathfrak{S}(L)$ and so is an isomorphism. Also $\phi(\mathcal{B})$ is linearly independent and so ϕ is one to one. □

From now on $\mathfrak{U}$ denotes a universal envelpong algebra for L. In view of the Poincare-Witt-Birkhoff Theorem we may and do identify L with its image in $\mathfrak{U}$. In particular for $n \in \mathbb{N}$ we obtain the $\mathbb{K}$-subspace L^n of $\mathfrak{U}$ Also according to 1.5.4 we view every L-module V as an $\mathfrak{U}$-module. Indeed if $a_1, a_2, \ldots, a_n \in L$ and $v \in V$, then $a_1 a_2 \ldots a_n \in U$ just acts

$$(a_1a_2\dots a_n)v = a_1(a_2(\dots(a_nv)\dots)).$$

In particular the adjoint action of L on L extends to an action of $\mathfrak{U}$ on L. We denote this action by $U \times L \to L, u \to u * l$. For example $a, b, l \in L$ we have $a * l = [a, l]$ and $(ab) * l = [a, [b, l]]$. With this notations we have

$$L^n * L = \underbrace{[L, [L, \dots [L}_{n\text{-times}}, L]] \dots]].$$

Lemma 1.5.8 [**[l,l,n]**] $L^n * L \leq L^{n+1}$.

Proof: The proof is by induction on n.. The statement is clearly true for $n = 0$. Suppose now that $L^{n-1} * L \leq L^n$. Let $l \in L$ and $a \in L^{n-1} * L$. Then $a \in L^n$ and so $l * a = la - al \in L^{n+1}$. Thus $L^n * L = L * (L^{n-1} * L) \leq L^{n+1}$ and the lemma is proved. □

1.6 Nilpotent Action

Let R be a ring and $X \subseteq R$. We say that X is nilpotent if $X^n = 0$ for some $n \in \mathbb{N}$. Note that for $R = \text{End}(V)$ we have $X^n = 0$ iff $X^nV = 0$.

Now let A be an associative or Lie algebra and V a module for A. Then we say that $X \subseteq A$ acts nilpotenly on V if the image of X in $\text{End}(V)$ is nilpotent. Note that that X acts *nilpotently* on V if and only if $X^nV = 0$ for some $n \in \mathbb{N}$.

We say that L is nilpotent if L acts nilpotently on L, that is if $L^n * L = 0$ for some n. Note that for associative algebra A a subalgebra B is nilpotent if an only if the action of B on A by left multiplication is nilpotent. Indeed if $B^n = 0$, then $B^nA = 0$ and if $B^nA = 0$ then $B^{n+1} = 0$. The analog of this statement is not true for Lie algebras. For example consider that Lie algebra L with basis x, y such that $[x, y] = x$. Then $\mathbb{K}y$ is an abelian and so a nilpotent subalgebra of L, but y does not act nilpotently on L. An the other hand if I is an ideal in L, then I is nilpotent if and only if I acts nilpotently on L.

We remark that if X acts nilpotently on V then all elements in X act nilpotently on V. The main goal of this section is to show that for finite dimensional Lie-algebras, the converse holds. That is if all elements of the finite dimensional Lie-algebra L act nilpotently on V, then also L acts nilpotenly on V.

We say that L acts trivially on V if $LV = 0$.

Lemma 1.6.1 [**nilpotent and chains**] *Let A be an associative or Lie algebra. Let A be an L-module. Then the following are equivalent:*

(a) [**a**] *A acts nilpotently on V.*

(b) [**b**] *There exists a finite chain of A submodules $0 = V_n \leq V_{n-1} \leq \dots V_0 = V$ such that A acts trivially on each V_i/V_{i+1}.*

(c) [**c**] *There exists a finite chain of A submodules $0 = V_n \leq V_{n-1} \leq \ldots V_0 = V$ such that A acts nilpotently on each V_i/V_{i+1}.*

Proof: (a)$\Longrightarrow$ (b): Just put $V_i = A^i V$.

(b)$\Longrightarrow$ (c): This holds since trivial action is nilpotent.

(c)$\Longrightarrow$ (a): For $0 \leq i < n$ choose m_i with $A^{m_i}(V_i/V_{i+1}) = 0$. Then $A^{m_i}V_i \leq V_{i+1}$. Put $m = \sum_{i=0}^{n-1} m_i$. Then $A^m V = 0$. □

Lemma 1.6.2 [**nilpotent implies nilpotent**] *Suppose L acts nilpotenly on the L-module V. Then $L/C_L(V)$ is nilpotent.*

Proof: Let $L^n V = 0$ for some $n \geq 1$. Then by 1.5.8 $(L^{n-1} * L)V = 0$. Thus $L^{n-1} * L \leq C_L(V)$ and $L/C_L(V)$ is nilpotent. □

Lemma 1.6.3 [**nilpotent + nilpotent**] *Let A be an associative or Lie algebra. Let V be an A-module, D, E subalgebras of A with $[E, D] \leq D$. If E and D acts nilpotently on V, then $E + D$ acts nilpotently on V.*

Proof: In the case that A is associative, we replace A by $\mathfrak{l}(A)$. So A is now a Lie algebra. Since $[E, D] \leq D$, D is an ideal in $E + D$. By 1.4.7(b) DV is an $E + D$-submodule. By induction, $D^n V$ is a $E + D$-submodule. D acts trivially and so $E + D$ acts nilpotenly on $D^n V/D^{n+1}V$ for all n. Thus the lemma follows from 1.6.1. □

Lemma 1.6.4 [**associative and nilpotent**] *Let A be an associative $\mathbb{K}$-algebra.*

(a) [**a**] *Let $D, E \leq A$ be nilpotent with $[E, D] \leq D$. Then $D + E$ is nilpotent.*

(b) [**b**] *Let $D \leq A$ be nilpotent. Then D acts nilpotently on on $\mathfrak{l}(A)$.*

Proof: (a) By 1.6.3 $D + E$ acts nilpotently on A and so is nilpotent.

(b) Since $D^n = 0$ we have $\mathrm{l}(D)^n = 0$ and $\mathrm{r}(D)^n = 0$. Also since A is associative $\mathrm{l}(D)^n$ and $\mathrm{r}(D)^n$ commute. Thus (a) implies that $\mathrm{l}(D) + \mathrm{r}(D)$ is nilpotent. Since $\mathrm{ad}\,(a) = \mathrm{l}(a) - \mathrm{r}(a)$ we have $\mathrm{ad}\,(D) \leq \mathrm{l}(D) + \mathrm{r}(D)$ and so $\mathrm{ad}\,(D)$ is nilpotent in $\mathrm{End}(A)$. and D so acts n nilpotently $\mathfrak{l}(A)$. □

Corollary 1.6.5 [**nil on V and in L**] *Suppose that L acts faithfully on V and that $X \subseteq L$ acts nilpotently on V. Then X acts nilpotently on L.*

Proof: Let $\Phi : L \to \mathfrak{gl}(V)$ be the corresponding representation. Then by the definition of nilpotent action, $\Phi(X)$ is nilpotent in $\mathrm{End}(V)$. From 1.6.4 the adjoint action of $\Phi(X)$ on $\mathfrak{gl}(V)$ is nilpotent. Thus $\Phi(X)$ acts nilpotently on $\Phi(L)$ and as Φ is one to one, X acts nilpotently on L. $\square$

Lemma 1.6.6 [**normalizer of nilpotent**] *Suppose L acts nilpotenly on V $W \subset V$ with $0 \in W \neq V$. Then there exists $v \in V \setminus W$ with $Lv \leq W$.*

Proof: Since L is nilpotent on V and $0 \in W$, we can choose $n \in \mathbb{N}$ minimal with $L^n V \subseteq W$. Since $M \neq L$, $n \neq 0$. By minmality of n, $L^{n-1}V \not\leq W$. Pick $v \in L^{n-1}V \setminus W$. Then $Lv \leq L(L^{n-1}V) = L^n V \leq W$. $\square$

For a subalgebra $A \leq L$ put $N_L(A) = \{l \in L \mid [l, A] \leq A\}$. Note that $N_L(A)$ is subalgebra of L and that A is an ideal in $N_L(A)$.

Corollary 1.6.7 [**normalizer of nilpotent II**] *Suppose tthat M is a subalgebra of L acting nilpotenly on L. If $M \neq L$, then $M \lneq N_L(M)$.*

By 1.6.6 (applied with (M, L, M) in the roles of (L, V, W)) there exists $d \in L \setminus M$ with $[M, d] \leq M$. Then $d \in N_L(M)$. $\square$

Definition 1.6.8 [**def:subideal**] *Let A be a $\mathbb{K}$-algebra and $I \subseteq A$. We write $I \trianglelefteq A$ if I is an ideal in A. We say that I is a* subideal *in A and write $I \trianglelefteq \trianglelefteq A$ if there exists chain $I = I_0 \trianglelefteq I_1 \trianglelefteq \ldots \trianglelefteq I_n \trianglelefteq I_n = A$.*

Lemma 1.6.9 [**subideals in nilpotent**] *Suppose L is nilpotent. Then every subalgebra in L is an subideal in L.*

Proof: Let n be minimal with $L^n * L = 0$ and $A \leq L$. Let $Z = L^{n-1} * L$. Then $L * Z = 0$, that is $Z \leq Z(L)$. Thus $[A, Z + A] = [A, A] \leq A$ and $A \trianglelefteq Z + A$. Put $\overline{L} = L/Z$. Since $L^{n-1} * L \leq Z$, $\overline{L}^{n-1} * \overline{L} = 0$. By induction on n we may assume $Z + A/Z \trianglelefteq \trianglelefteq L/Z$. Thus $Z + A \trianglelefteq \trianglelefteq L$ and so $Z \trianglelefteq \trianglelefteq L$ $\square$

Theorem 1.6.10 [**elementwise nilpotent**] *Let L be a finite dimensional Lie algebra and V a L-module. If all elements of L act nilpotently on V, then L acts nilpotently on V.*

Proof: We may assume without loss that L is faithful on V. The proof is by induction on $\dim V$. Let M be a maximal subalgebra of L. By induction M acts nilpotently on V. So by 1.6.5 M acts nilpotenly on L. 1.6.6 implies that there exists $d \in N_L(M) \setminus M$. Note that $\mathbb{K}d$ is a subalgebra and $M + \mathbb{K}d$ are subalgebras of L. By maximality of M, $L = M + \mathbb{K}d \leq N_L(M)$. As d is nilpotent on V, $\mathbb{K}d$ is nilpotent on V as well. Thus 1.6.3 implies that L is nilpotent on V $\square$.

Corollary 1.6.11 (Engel) [**engel**] *Let L be a finite dimensional Lie algebra all of whose elements act nilpotently on L. Then L is nilpotent.*

Proof: Apply 1.6.10 to the adjoint module. □

1.7 Finite Dimensional Modules

Definition 1.7.1 [**series**] *Let A be Lie or an associate $\mathbb{K}$-algebra and V an A-module.*

(a) [**a**] *V is called* simple *if V has no proper A-submodules. (that is O and V are the only A-submodules. V is* semisimple *if its the direct sum of simple modules and its* homogeneous *if its the direct sum of isomorphic simple modules.*

(b) [**b**] *A* series *for A on V is a chain $\mathcal{S}$ of A-submodules of V such that*

(a) [**a**] *$0 \in \mathcal{S}$ and $V \in \mathcal{S}$.*

(b) [**b**] *$\mathcal{S}$ is closed under intersections and unions, that is for every nonempty $\mathcal{D} \subset \mathcal{S}$, $\bigcap \mathcal{D} \in \mathcal{S}$ and $\bigcup \mathcal{D} \in \mathcal{S}$.*

(Here a chain is a set of sets which is totally ordered with respect inclusion)

(c) [**c**] *Let $\mathcal{S}$ be an A-series. A* jump *of $\mathcal{S}$ is pair $(D, E))$ such that $D, E \in \mathcal{S}$, $D < E$ and $C \in \mathcal{S}$ with $D \leq C \leq E$ implies $C = D$ or $S = E$. In this case E/D is called a* factor *of $\mathcal{S}$.*

(d) [**d**] *A* composition series *for A on S is a series all of whose factors are simple A-modules.*

(e) [**e**] *Let $\mathcal{S}$ and $\mathcal{T}$ be A-series on V. We say that $\mathcal{S}$ and $\mathcal{T}$ have isomorphic factors if there exists a bijection Φ between the sets of factors of $\mathcal{S}$ and $\mathcal{T}$ such that for each factor F of $\mathcal{S}$, F and ΦF are isomorphic A-modules. Such a Φ is called an isomorphism of the sets of factor.*

(f) [**f**] *Let V and W be A-modules and $\phi \in \mathrm{Hom}(V, W)$. Then ϕ is called A-invariant if $\phi(av) = a\phi(v)$ for all $a \in A$, $v \in V$. $\mathrm{Hom}_A(V, W)$ denotes the set of such ϕ.*

(g) [**g**] *If X and Y are A-submodules o V with $X \leq Y$, then Y/X is called an A-section of V.*

Lemma 1.7.2 [**lifting series**] *Let V be an L-module and W an L-submodule of V. Let $\mathcal{S}$ be a L-series on W and $\overline{\mathcal{T}}$ and L-series on V/W. Let $\mathcal{T}$ be the inverse image of $\overline{\mathcal{T}}$ in V (so $\overline{\mathcal{T}} = \{T/W \mid T \in \mathcal{T}\}$). Then $\mathcal{S} \cup \mathcal{T}$ is a series for L on V. The factors of $\mathcal{S} \cup \mathcal{T}$ are the factors of $\mathcal{S}$ and $\overline{\mathcal{T}}$. In particular, $\mathcal{S} \cup \mathcal{T}$ is an L-composition seres if and only if both $\mathcal{S}$ and $\overline{\mathcal{T}}$ are L-composition series.*

Proof: This follows readily from the definition. We leave the details as an exercise. □

Lemma 1.7.3 (Jordan Hölder) [**jordan hoelder**] *Let A be a Lie or an associative $\mathbb{K}$-algebra. Suppose that there exists a finite composition series for A on V. Then any two composition series for A on V have isomorphic factors.*

Proof: Let $\mathcal{S}$ be a finite composition series for A on V and $\mathcal{T}$ any compostion series. For a jump (B, C) of $\mathcal{T}$ choose $D \in \mathcal{S}$ maximal with $C \not\leq B + D$. Let E be minimal in $\mathcal{S}$ with $D < E$. Then E/D is a factor of $\mathcal{S}$, C/B is a factor of $\mathcal{S}$ and we will show that map $B/C \to E/D$ is an isomorphism of the sets of factor.

By maximality of D we have

$$C \leq B + E.$$

Thus $C = C \cap (B + E) = B + (C \cap E)$ and so

$$C/B \cong C \cap E/C \cap E \cap B = C \cap E/B \cap E.$$

Since $C \not\leq B + D$, $C \cap E \not\leq D$ and since E/D is simple, $E = D + (C \cap E)$. Thus

$$E/D \cong C \cap E/C \cap D.$$

If $B \cap E \not\leq D$, then $E = (B \cap E) + D \leq B + D$ and so $C =\leq B + E \leq B + D$, contrary to our choice of D. Thus $B\mathcal{E} \leq D$ and so $B \cap E = B \cap D$. Suppose that $C \cap D \not\leq B$. Then $C = (C \cap D) + B \leq B + D$, again a contradiction. Thus $C \cap D = B \cap D = B \cap E$ and so

$$C/B \cong C \cap E/B \cap D = C \cap E/B \cap E \cong E/D.$$

It remains to show that our map between the factor sets is a bijection. Let (B', C') be a jump other than (B, C) and say $C' \leq B$. Then $C' \cap E \leq B \cap E = B \cap D \leq D$ and so (B', C') is not mapped to E/D. So our map is one to one.

Since $\mathcal{S}$ is finite we conclude that, $\mathcal{T}$ has finitely many jumps and so also $\mathcal{T}$ is finite and $|\mathcal{T}| \leq |\mathcal{S}|$. But now the situation is symmetric in $\mathcal{T}$ and $\mathcal{S}$. Thus $|\mathcal{S}| \leq |\mathcal{T}|$, $|\mathcal{S}| = |\mathcal{T}|$ and our map is a bijection. □

Lemma 1.7.4 [**submodules for ideals**] *Let L be a Lie algebra L, V an L-module, I an ideal in L, W an I-submodule in V and $l \in L$. Let X be an I submodule of V containing $[I, l]W$*

(a) [**a**] *The map $W \to V/X, w \to lw + X$ is I-invariant.*

(b) [**b**] *The map $W \to V/W, w \to lw + W$ is I-invariant.*

(c) [**c**] *If $[I, l] = 0$ then the map $W \to V, w \to lw$ is I-invariant.*

Proof: (a) Let ϕ be the map in question. Let $i \in I$ and $w \in W$. Then $ilw = liw + [i,l]w \in liw + X$ and so $i\phi(w) = \phi(iw)$.

(b) Since W is an I-submodule and I is an ideal, $[I,l]W \leq W$ and so we can apply (a) with $X = W$.

(c) Apply (b) with $X = 0$.

Definition 1.7.5 [**def:nil v**] *Let V be a finite dimensional L-module.*

(a) [**a**] $\mathrm{Comp}_V(L)$ *is the set of factors of some L-compositions series on V. (Note by the Jordan Hölder Theorem,* $\mathrm{Comp}_V(L)$ *is essentially independent from the choice of the composition series)*

(b) [**b**] $\mathrm{Nil}_L(V) = \bigcap\{C_L(W) \mid W \in \mathrm{Comp}_V(L)\}$

Lemma 1.7.6 [**nil V**] *Let V be a finite dimensional L-module. Then* $\mathrm{Nil}_L(V)$ *is the unique maximal ideal of L acting nilpotently on V.*

Proof: $\mathrm{Nil}_L(V)$ is the intersection of ideals and so an ideal in L. By 1.6.1(b), $\mathrm{Nil}_L(V)$ is nilpotent on V. Now let I be an ideal of L acting nilpotently on V. Also let$W$ a composition factor for L on V. Then $0 \neq C_W(I)$ is an L-submodule of W and so $C_W(I) = W$, $I \leq C_L(W)$ and $I \leq \mathrm{Nil}_L(V)$. □

Corollary 1.7.7 [**Nil L**] *Let L be finite dimensional. Then L has a unique maximal nilpotent ideal* $\mathrm{Nil}(L)$.

Proof: An ideal in L is nilpotent if and only if its acts nilpotently on L. So the lemma follows from 1.7.6 applied to the adjoint module. □

We remark that there may not exist a unique largest nilpotenly acting subideal in L. For example consider $L = \mathfrak{sl}(\mathbb{K}^2)$ and let $V = \mathbb{K}^2$. Let

$$x = E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $[h,x] = 2x$, $[y,h] = 2y$ and $[x,y] = h$.

If $\mathrm{char}\,\mathbb{K} = 2$ we conclude that $\mathbb{K}x + \mathbb{K}h$ is an ideal in $\mathfrak{sl}(\mathbb{K}^2)$ and $\mathbb{K}x$ is an ideal in in $\mathbb{K}x + \mathbb{K}h$. Thus $\mathbb{K}x$ is a subideal acting nilpotently on $\mathbb{K}^2$. The same holds for $\mathbb{K}y$. But $\mathfrak{sl}(\mathbb{K}^2)$ is the subalgebra generated by x and y. Since $\mathfrak{sl}(\mathbb{K}^2)V = V$, $\mathfrak{sl}(\mathbb{K}^2)$ does not act nilpotently on V and so $\mathbb{K}x$ and $\mathbb{K}Y$ are not contained in common nilpotently acting subideal of L.

Definition 1.7.8 [**def:vd**] *Let V be finite dimesional L-module.*

(a) [**a**] $\mathrm{Sim}(L)$ *is the set of all isomorphism classes of finite dimensional simple L-modules.*

(b) [**b**] $\mathrm{Sim}_V = \mathrm{Sim}_V(L)$ *is the set the isomorphism classes of the* L*-composition factors of* V*.*

(c) [**c**] *Let* $\mathcal{D} \subseteq \mathrm{Sim}(L)$*. A* $\mathcal{D}$*-module is an* L*-module* W *with* $\mathrm{Sim}_W \subseteq \mathcal{D}$ *(If* W *is simple this means that the isomorphism class of* W *is in* $\mathcal{D}$*.*

(d) [**d**] $V_{\mathcal{D}}$ *is the sum of all the simple* $\mathcal{D}$*-submodules in* V

(e) [**e**] $V_{\mathcal{D}}(0) = 0$ *and inductively define the submodule* $V_{\mathcal{D}}(n+1)$ *of* L *in* V *by*

$$V_{\mathcal{D}}(n+1)/V_{\mathcal{D}}(n) = (V/V_{\mathcal{D}}(n))_{\mathcal{D}}.$$

(f) [**f**] $V^c_{\mathcal{D}} = \bigcup_{i=0}^{\infty} V_{\mathcal{D}}(i)$*.*

(g) [**g**] *Let* $A \leq L$ *and* $\mathcal{A} \subseteq \mathrm{Sim}(A)$*. Then* $\mathcal{A} \mid^L$ *is the set of isomorphism classes of the finite dimensional simple* L*-modules which are* $\mathcal{A}$*-modules.*

To digest the preceeding definitions we consider an example. Let L be the subalgebra of $\mathfrak{gl}(\mathbb{K}^3)$ consisting of all 3×3 matrices of the form

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $V = \mathbb{K}^3$ viewed as an L-module via left multiplication. Let (e_1, e_2, e_3) be the standard basis for $\mathbb{K}^3$. Let $V_i = \sum_{j=0}^{3} \mathbb{K}e_j$. Then

$$0 = V_0 < V_1 < V_2 < V_3 = V$$

is a composition series for L on V. Put $I_k = V_k/V_{k-1}$. Then I_k is a simple 1-dimensional L-module. Note that $LI_1 = 0$ and $LI_3 = 0$ while $LI_2 \neq 0$. So $I_1 \cong I_3$ but $I_1 \not\cong I_2$ as L-module. For an L-module W let $[W]$ be the isomorphism class of W(that is the class of L-modules isomorphic to W. Then $\mathrm{Sim}_V = \{[I_1], [I_2]\}$. For $k = 1, 2$ let $\mathcal{D}_k = \{[I_k]\}$. Also put $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \mathrm{Sim}_V$. Observe that any L-submodule of V is one of the V_i.

By definition $V_{\mathcal{D}_1}$ is the sum of all the simple L-submodule of V isomorphic to I_1. V_1 is the only simple L-submodule of V and $V_1 \cong I_1$ so $V_{\mathcal{D}_1} = V_1$. To compute $\overline{V}_{\mathcal{D}_1}$, put $\overline{V} = V/V_1$. The only simple submodule of $\overline{V}$ is $I_2 = V_2/V_1$. Since $I_1 \not\cong I_2$ we get $\overline{V}_{\mathcal{D}_1} = 0$. Thus $V_{\mathcal{D}_1}(2) = V_1$. It follows that $V_{\mathcal{D}_1}(j) = V_1$ for all $j \geq 1$ and so also $V^c_{\mathcal{D}_1} = V_1$.

No submodule of V is isomorphic to I_2 and hence $V_{\mathcal{D}_2} = 0$. Thus $V^c_{\mathcal{D}_2} = V_{\mathcal{D}_2}(j) = 0$ for all $j \geq 0$.

V_1 is the only submodule of V isomorphic to I_1 or I_2 and so $V_{\mathcal{D}} = V_1$. V_2/V_1 is the only submodule of V/V_1 isomorphic to I_1 or I_2. So $(V/V_1)_{\mathcal{D}} = V_2/V_1$ and $V_{\mathcal{D}}(2) = V_2$. V/V_2 is isomorphic to I_1 and so $V = V_{\mathcal{D}}(3) = V^c_{\mathcal{D}}$.

Definition 1.7.9 [**def:linear indep**] *Let* V *be* $\mathbb{K}$*-space and* $\mathcal{V}$ *a set of* $\mathbb{K}$*-subspaces of* V*. We say that* $\mathcal{V}$ *is linearly independent if* $\sum \mathcal{V} = \bigoplus \mathcal{V}$*.*

Lemma 1.7.10 [**basic semisimple**] *Let V be an L-modules and $\mathcal{V}$ a set of simple L-submodules in V. Suppose that $V = \sum \mathcal{V}$.*

(a) [**a**] *Let W be an L-submodule of V, Then there exists $\mathcal{W} \subseteq \mathcal{V}$ such that $V = W \oplus \bigoplus \mathcal{W}$.*

(b) [**b**] *Let $X \leq Y$ be L-submodules. Then there exists $\mathcal{W} \subseteq \mathcal{V}$ with $Y/X \cong \bigoplus \mathcal{W}$ as L-modules.*

(c) [**c**] *Every L-section of V is semisimple.*

(d) [**d**] *Every composition factor of V is isomorphic to some member of $\mathcal{V}$.*

Proof: (a) Let $\mathcal{C}$ be the set of linearly independet subsets $\mathcal{W}$ of $\mathcal{V}$ with $W \cap \sum \mathcal{W} = 0$. Order $\mathcal{C}$ by inclusion. If $\mathcal{D}$ is a chain in $\mathcal{C}$, then it is easy to verify that $\bigcap \mathcal{D} \in \mathcal{C}$. So every chain in $\mathcal{C}$ has an upper bound. By Zorn's Lemma, $\mathcal{C}$ has a maximal elements $\mathcal{W}$. Suppose that $V \neq W + \sum \mathcal{W}$. Then there exists $U \in \mathcal{V}$ with $U \not\leq W + \sum \mathcal{W}$. Since U is simple, $U \cap (W + \sum \mathcal{W}) = 0$. But then $\mathcal{W} \cup \{U\} \in \mathcal{C}$, contradicting the maximality of $\mathcal{W}$. Thus $V = W + \sum \mathcal{W}$ and the definition of $\mathcal{C}$ implies that $V = W \oplus \bigoplus \mathcal{W}$.

(b) By (c) there exists an L-submodule Z of V with $V = Y \oplus X$. Put $\overline{V} = V/Z$. Then $Y \cong \overline{V}$. Let $W \in \mathcal{V}$ with $W \not\leq Z$. Then $W \cap Z = 0$ and $\overline{W} = W + Z/Z \cong W$. Let $\overline{\mathcal{V}} = \{\overline{W} \mid W \in \mathcal{V}, W \not\leq Z$. Then $\overline{V} = \sum \overline{\mathcal{V}}$. By (a) applied to $\overline{X} \leq \overline{V}$ there exists $\overline{\mathcal{W}} \subseteq \overline{\mathcal{V}}$ with $\overline{V} = \overline{X} \oplus \overline{\mathcal{W}}$. Hence $Y/X \cong \overline{Y}/\overline{X} = \bigoplus \overline{\mathcal{W}}$ and so (b) holds.

(c) and (d) follow directly from (b). □

Lemma 1.7.11 [**basic vd**] *Let V be a finite dimensional L-module and $\mathcal{D} \subseteq \mathrm{Sim}(L)$.*

(a) [**z**] *Let $A \leq L$ and $\mathcal{A} \subseteq \mathrm{Sim}(A)$. Then V is an $\mathcal{A}$-module if and only if V is an $\mathcal{A} |^L$-module.*

(b) [**a**] *Let $A \leq B \leq C \leq L$ and $\mathcal{A} \subseteq \mathrm{Sim}(A)$. Then $\mathcal{A} |^B|^C = \mathcal{A} |^C$.*

(c) [**b**] *Let W be L-submodule of V. Then V is an $\mathcal{D}$-module if and only if W and V/W are $\mathcal{D}$-modules.*

(d) [**y**] *$V_{\mathcal{D}}$ is the unique maximal semisimple $\mathcal{D}$-submodule in V.*

(e) [**c**] *$V^c_{\mathcal{D}}$ is the unique maximal $\mathcal{D}$-submodule of V.*

(f) [**x**] *Let $\mathcal{E} \subseteq \mathrm{Sim}(L)$. Then $V^c_{\mathcal{D}} \cap V^c_{\mathcal{E}} = V^c_{\mathcal{D}\cap\mathcal{E}}$ and $V^c_{\mathcal{D}} + V^c_{\mathcal{E}} \leq V^c_{\mathcal{D}\cup\mathcal{E}}$.*

(g) [**d**] *Let $I \trianglelefteq\trianglelefteq L$ and $\mathcal{I} \subseteq \mathrm{Sim}(I)$. Put $\mathcal{L} = \mathcal{I} |^L$. Then $V^c_{\mathcal{I}}$ is an L submodule and $V^c_{\mathcal{A}} = V^c_{\mathcal{L}}$.*

(h) [**e**] *Suppose $I \leq Z(L)$, $\mathcal{I} \subseteq \mathrm{Sim}(I)$ and $i \in \mathbb{N}$. Then $V_{\mathcal{I}}(i)$ is an L-submodule of V.*

Proof:

(a) Let $\mathcal{S}$ a composition series for L on V and choose a composition series $\mathcal{R}$ for A on V with $\mathcal{S} \subseteq \mathcal{R}$. Then a factor A/B of $\mathcal{S}$ is a $\mathcal{A} \mid^L$-module iff all the factors C/D of $\mathcal{R}$ with $B \leq C < D \leq A$ are $\mathcal{A}$-modules. Thus V is an $A \mid^L$-module iff each factor T of $\mathcal{S}$ is an $\mathcal{A} \mid^L$-module iff each factor of $\mathcal{R}$ is a $\mathcal{A}$-module iff V is an $\mathcal{A}$-module.

(b) Let X be finite dimensional C-module. Then by (b), the following are equivalent.

X is an $\mathcal{A} \mid^B\mid^C$-module, X is an $\mathcal{A} \mid^B$-module, X is an $\mathcal{A}$-module, X is an $\mathcal{A} \mid^C$-module.

(c) follows from 1.7.2

(d) By 1.7.10(c), $V_{\mathcal{D}}$ is semisimple and by 1.7.10(d), $V_{\mathcal{D}}$ is a $\mathcal{D}$-module. Conversely every semisimple $\mathcal{D}$-module is a sum of simple $\mathcal{D}$-modules and so contained in $V_{\mathcal{D}}$.

(e) Any composition factor of $V_{\mathcal{D}}^c$ is isomorphic to a compostion factor of some $V_{\mathcal{D}}(n+1)/V_{\mathcal{D}}(n)$ and so (by (d)) is $\mathcal{D}$-module. So $V_{\mathcal{D}}^c$ is a $\mathcal{D}$-module. Conversely let W be a $\mathcal{D}$-submodule and $0 = W_0 < W_1 < \ldots < W_n = W$ an L-composition series on W.

We show by induction on i that $W_i \leq V_{\mathcal{D}}(i)$. For $i = 0$ this is obvious. So suppose $W_i \leq V_{\mathcal{D}}(i)$. Since W_i is a maximal submodule of W_{i+1} we either have $W_{i+1} \cap V_{\mathcal{D}}(i) = W_i$ or W_{i+1}. In the latter case, $W_{i+1} \leq V_{\mathcal{D}}(i+1))$. In the former put $\overline{V} = V/V_{\mathcal{D}}(i)$ and note that $\overline{W_{i+1}} \cong W_{i+1}/W_i$ is a simple $\mathcal{D}$-module. Hence $\overline{W_{i+1}} \leq \overline{V}_{\mathcal{D}}$. Hence the defintion of $V_{\mathcal{D}}(i+1)$ implies $W_{i+1} \leq V_{\mathcal{D}}(i+1)$.

In particular, $W_n \leq V_{\mathcal{D}}(n) \leq V_{\mathcal{D}}^c$ and (e) is proved.

(f) This follow easily from (e). We leave the details to the reader.

(g) Suppose first that I is an ideal in L. Let $W = V_{\mathcal{I}}^c$. We claim that W is a L-submodule. Let $l \in L$. Then by 1.7.4(b), $\phi : W \to V/W, w \to lw + W$ is I-invariant. Hence $\phi(W) \cong W/\ker\phi$ and so by (c), $\phi(W)$ is an $\mathcal{I}$-submodule. Now $\phi(W) = lW + W/W$ and so by (c), $lW + W$ is a $\mathcal{I}$. According to (e), W is a maximal $\mathcal{I}$-submodule. Thus $lW + W = W$, $lW \leq W$ and W is an L-submodule. By (a) W is an $\mathcal{L}$-submodule. Thus by (e), $W \leq V_{\mathcal{L}}^c$. Also by (a), $V_{\mathcal{L}}^c$ is an $\mathcal{I}$-submodule and thus by (e), $V_{\mathcal{L}}^c \leq W$.

So (g) holds if I is an ideal. In the general can choose $I \trianglelefteq I_1 \ldots I_{n-1} \trianglelefteq I$. Put $J = I_{n-1}$ and $\mathcal{J} = \mathcal{A} \mid^J$. By induction on n, $V_{\mathcal{I}}^c = V_{\mathcal{J}}$. By (b), $\mathcal{L} = \mathcal{J} \mid^L$ and so by the ideal case $V_{\mathcal{J}}^c = V_{\mathcal{L}}$. Thus $V_{\mathcal{I}}^c = V_{\mathcal{L}}^c$. As the latter is an L-submodule, so is $V_{\mathcal{I}}^c$ and (g) is proved.

(h) Let $l \in L$. By 1.7.4(c), $lV_{\mathcal{I}}$ is a sum of simple $\mathcal{A}$-modules So $lV_{\mathcal{I}} \leq V_{\mathcal{I}}$. Thus $V_{\mathcal{I}}$ is an L-submodule. The definition of $V_{\mathcal{I}}(n+1)$ and induction on n now shows that (h) holds. □

Proposition 1.7.12 [**clifford**] *Let I a subideal in L and V a finite dimensional simple L. Then any two composition factors for I on V are isomorphic. If in addition $I \leq Z(L)$, then V is an homogenous I-module.*

Proof: Let W be as simple I-submodule in V and $\mathcal{I}$ the isomorphism class of W. Then by 1.7.11 $V_{\mathcal{I}}^c$ is a non-trival L-submodule of V. Since V is simple, $V = V_{\mathcal{I}}^c$. Similarly if $I \leq Z(L)$, $V = V_{\mathcal{I}}$ □

Lemma 1.7.13 (Schur) [**schur**] *Let V be a simple L-module. Then* $\mathrm{End}_L(V)$ *is a skew-field. If* $\mathbb{K}$ *is algebraicly closed and V is finite dimensional, then* $\mathrm{End}_L(V) = K^* = \mathbb{K}\mathrm{id}_V$, *where* $\mathbb{K}^*$ *is the image of* $\mathbb{K}$ *in* $\mathrm{End}_{\mathbb{K}}(V)$.

Proof: Let $0 \neq f \in \mathrm{End}_L(V)$. Then $V \neq C_V(f)$ is L-submodule of V and since V is simple, $C_V(f) = 0$. So f is $1-1$. Similarly $V = fV$ and so f is onto. Simple calculations show that $f^{-1} \in \mathrm{End}_L(V)$ and so $\mathrm{End}_L(V)$ is a skew field. Suppose now that V is finite dimensional and $\mathbb{K}$ is algebraicly closed. Then $\mathrm{End}_L(V)$ is a finite field extension of $\mathbb{K}^*$ and so $|End_L(V) = K^*$. □

Lemma 1.7.14 [**simple for abelian**] *Let L be an abelian Lie algebra and V a simple L-module. Put* $\mathbb{D} = \mathrm{End}_L(V)$. *Then* $\mathbb{D}$ *is a field, V is 1-dimensional over* $\mathbb{D}$ *and* $\mathrm{D} = \mathbb{K}^*(L^*)$, *where* $\mathbb{K}^*$ *and* L^* *are the images of* $\mathbb{K}$ *and L in* $\mathrm{End}(V)$. *If* $\mathbb{K}$ *is algebrilcy closed and V is finite dimensional, then* $\mathbb{K} = \mathbb{D}$ *and V is 1-dimensional over* $\mathbb{K}$.

Proof: Note that L^* is abelian and $L^* \leq Z(\mathbb{D})$. Let $\mathbb{E}$ be the subfield of $Z(\mathbb{D})$ generated by $\mathbb{K}^*$ and L^*. Let $0 \neq v \in V$. Then $\mathbb{E}v$ is an L-submodule and since V is simple we get $V = \mathbb{E}v$. Hence V is 1-dimensional over $\mathbb{E}$. Moreover, if $d \in \mathbb{D}$, then $dv = ev$ for some $e \in \mathbb{E}$. Then $(d-e)v = 0$, $d = e$, $\mathbb{E} = \mathbb{D}$ and the $\mathbb{E} = \mathbb{D}$.

Suppose in addition that $\mathbb{K}$ is alegbraicly closed and V is finite dimensional. Then $\mathbb{D}$ is a finite extension of $\mathbb{K}^*$ and so $\mathbb{D} = \mathbb{K}^*$. □

Lemma 1.7.15 [**independence of d spaces**] *Let V be finite dimensional L-module and Δ a partition of* Sim_V. *Then* $(V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta)$ *is linearly independent, that is*

$$\sum\{V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta\} = \bigoplus\{V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta\}.$$

Proof: Let $\mathcal{D} \in \Delta$ and $W = \sum\{V^c_{\mathcal{A}} \mid \mathcal{D} \neq \mathcal{A} \in \Delta\}$. We need to show that $V^c_{\mathcal{D}} \cap W = 0$.

For this put $\mathcal{E} = \bigcup \Delta \setminus \{\mathcal{D}\}$. Then $V^c_{\mathcal{D}}$ is a $\mathcal{D}$-module, W is an $\mathcal{E}$-module and so $V^c_{\mathcal{D}} \cap W$ is an $\mathcal{D} \cap \mathcal{E}$-module. As Δ was a partition, $\mathcal{D} \cap \mathcal{E} = \emptyset$. Hence $V^c_{\mathcal{D}} \cap W = 0$. □

Definition 1.7.16 [**def:trace**] *Let V be a finite dimensional L-module and $u \in \mathfrak{U}$. Then* $\mathrm{tr}_V(u) = \mathrm{tr}(u^*)$, *where u^* is the image of u in* $\mathrm{End}(V)$. tr_V *denotes the corresponding function* $\mathfrak{U} \to \mathbb{K}, u \to t_V(u)$. tr^L_V *denotes the restriction of* tr_V *to L.*

Lemma 1.7.17 [**trace and series**] *Let V be a finite dimensional L-module.*

(a) [**a**] tr_V *is* $\mathbb{K}$*-linear and* $\mathrm{tr}_V(ab) = \mathrm{tr}_V(ba)$ *for all* $a, b \in \mathfrak{U}$.

(b) [**b**] $\mathrm{tr}_V(l) = 0$ *for all* $l \in [L, L]$

(c) [**c**] *Let $\mathcal{W}$ the set of factors of some L-series on V. Then*

$$\mathrm{tr}_V = \sum_{W \in \mathcal{W}} \mathrm{tr}_W$$

(d) [**d**] *If W is an L-module isomorphic to V, then* $\mathrm{tr}_V = \mathrm{tr}_W$.

This follows from elementray facts about traces of linear maps. □

Chapter 2

The Structure Of Standard Lie Algebras

2.1 Solvable Lie Algebras

Put $L^{(0)} = L$ and inductively, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. We say that L is solvable if $L^{(k)} = 0$ for some $k < \infty$.

Lemma 2.1.1 [**basic solvable**]

(a) [**a**] *Let $I \trianglelefteq L$. Then L is solvable if and only if I and L/I are solvable.*

(b) [**b**] *Let $A, B \leq L$ with $A \leq N_L(B)$. Then $A + B$ is solvable if and only if A and B are solvable.*

(c) [**c**] *Suppose that L is finite dimensional. Then L has a unique maximal solvable ideal $Sol(L)$.*

Proof: (a) If $L^{(k)} = 0$, then $I^{(k)} = 0$ and $(L/I)^{(k)} = 0$. If $I^{(n)} = 0$ and $(L/I)^{(m)} = 0$, then $L^{(m)} \leq I$ and $L^{(m+k)} = L^{(m)(k)} = 0$.

(b) Suppose A and B are solvable. Since $A \leq N_L(B)$, $B \trianglelefteq A + B$. Now B and $A + B/B \cong A/A \cap B$ are solvable and so by (a) $A + B$ is solvable.

(c) Since L is finite dimensional, there exists a maximal solvable ideal B in L. Let A be any solvable ideal in L. Then by (b), $A + B$ is solvable ideal and so by maximality of B, $A \leq B$. $\square$

Lemma 2.1.2 [**nilpotent is solvable**]

(a) [**a**] $L^{(k+1)} \leq L^k * L$.

(b) [**b**] *Any nilpotent Lie algebra is solvable.*

(c) [**c**] *If* $\mathrm{Nil}(L) = 0$*, then* $\mathrm{Sol}(L) = 0$*.*

Proof: (a) Clear by induction on k.

(b) follows from (a).

(c) If $\mathrm{Sol}(L) \neq 0$ the last non-trival term of the derived series of L is an abelian and so nilpotent ideal in L. □

Write $L' = [L, L] = L^{(1)}$. We say that L is perfect if $L = L'$. Let $L^{(\infty)}$ be the sum of the perfect ideals in L. Then $L^{(infty)}$ is perfect and so the unique maximal perfect ideal in L.

If L is finite dimensional there exists $k \in \mathbb{N}$ with $L^{(k)} = L^{(k+1)}$. It follows that $L^{(\infty)} = L^{(k)}$, $L/L^{(\infty)}$ is solvable, $L^{(\infty)}$ is the unique ideal minimal such that $L/L^{(\infty)}$ is solvable and $L^{(\infty)}$ is the unique maximal perfect subalgebra in L.

Definition 2.1.3 [**standard**] *We say* $\mathbb{K}$ *is standard if* $\mathrm{char}\, K = 0$ *and* $\mathbb{K}$ *is algebraicly closed. We say that* L *is standard if* $\mathbb{K}$ *is standard and* L *is finite dimensional. We say that the* L*-module* V *is standard if* L *is standard and* V *is finite dimensional.*

Proposition 2.1.4 [**sol and simple**] *Let* V *be a simple, standard* L*-module.*

(a) [**a**] $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Sol}(L) \cap \ker \mathrm{tr}_V = \mathrm{Sol}(L) \cap C_L(V)$.

(b) [**b**] *The elements of* $\mathrm{Sol}(L)$ *act as scalars on* V*.*

Proof: Replacing L by $L/C_L(V)$ we may assume that V is faithful.

(a) Let $I = \mathrm{Sol}(L) \cap \ker \mathrm{tr}_V$. Obviously $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L'$ and $\mathrm{Sol}(L) \cap C_L(V) \leq I$. By 1.7.17(b), $L' \leq \ker \mathrm{tr}_V$. So we need to show that $I \leq C_L(V) = 0$. If not, let k be the derived length of I and put $J = I^{(k-1)}$. Then J is a non-trivial abelian ideal in L and $\mathrm{tr}_V(J) = 0$. Let $0 \neq j \in J$ and let Z be a simple J-submodule in V. Since $\mathbb{K}$ is algebraicly closed, 1.7.14 implies that Z is 1-dimensional over $\mathbb{K}$. Hence there exists $k \in \mathbb{K}$ with $jz = kz$ for all $z \in Z$. By 1.7.12 all composition factors for J on V are isomorphic and so 1.7.17 implies that $0 = \mathrm{tr}_V(j) = \dim V \cdot k$. Since $\mathrm{char}\, \mathbb{K} = 0$ we get $k = 0$. Thus $J \leq \mathrm{Nil}_L(V) \leq C_L(V)$ and (a) is proved.

(b) By (a), $[\mathrm{Sol}(L), L] \leq C_L(W) = 1$. Thus $\mathrm{Sol}(L) \leq Z(L)$. Hence by 1.7.12 $\mathrm{Sol}(L)$ is homogenous on V. Now $\mathrm{Sol}(L)$ is abelian and by 1.7.14 all the simple $\mathrm{Sol}(L)$ submodules in V are 1-dimensional. Hence (b) holds.

Theorem 2.1.5 (Lie) [**lie**] *Let* V *be a standard* L*-module.*

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V).$$

Proof: Let W be a composition factor for L on V. By 2.1.4 $\mathrm{Sol}(L) \cap L' \leq C_L(W)$ and so $\mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V)$. □

Corollary 2.1.6 [**solvable and flags**] *Suppose that L is solvable and V is a standard L-module. Then*

(a) [**a**] $L' \leq \mathrm{Nil}_L(V)$.

(b) [**b**] *If V is simple, then V is 1-dimensional.*

(c) [**c**] *There exists a series of L-submodules $0 = V_0 \leq V_1 \leq \ldots \leq V_n = V$ with $\dim V_i = i$.*

(d) [**d**] $\mathrm{Nil}_L(V) = \{l \in L \mid l$ *acts nilpotently on* $V\}$.

Proof: By 2.1.6(b) L acts as scalars on any composition factor for L on V. Thus (a)-(c) holds.

(d) Clearly each elements of $\mathrm{Nil}_L(V)$ acts nilpotently on V. Now let $l \in L$ act nilpotently on V. Then l also acts nilpotently any every composition factor W of L on V. (b) implies that l centralizes W and so $l \in \mathrm{Nil}_L(V)$.

Corollary 2.1.7 [**[sol l, l] nilpotent**] *Let L be standard. Then*

(a) [**a**] $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}(L)$.

(b) [**b**] *If L is solvable then L' is nilpotent and there exists a series of ideals $0 = L_0 \leq L_1 \leq \ldots \leq L_n = L$ in L with $\dim L_i = i$.*

Proof: Apply 2.1.5 and 2.1.6 to V being the adjoint module L.

2.2 Tensor products and invariant maps

Let V, W and Z be L-module. Then L acts on $V \otimes W$ by

$$l(v \otimes w) = (lv) \otimes w + v \otimes (lw)$$

and L acts on $\mathrm{Hom}(V, W)$ by

$$(l\phi)(v) = l(\phi(v)) - \phi(l(v)).$$

In particular, if we view $\mathbb{K}$ as a trivial L-module, L acts on $V^* := \mathrm{Hom}(V, \mathbb{K})$ by

$$(l\phi)(v) = -\phi(lv).$$

Let $X \subseteq L$ and $\phi \in \mathrm{Hom}(V, W)$. We say that ϕ is X-invariant if $\phi(lv) = l(\phi(v))$ for all $v \in V$ and $l \in X$. Note that this is the case if and only if $l\phi = 0$ for all $l \in X$. $\mathrm{Hom}_X(V, W)$ denotes all the X-invariant $\mathbb{K}$-linear maps from V to W. So $\mathrm{Hom}_X(V, W)$ is just the centralizer of X in $\mathrm{Hom}(V, W)$. Let $f : V \times W \to Z$ be $\mathbb{K}$-bilinear. Then f gives rise to a unique $\mathbb{K}$-linear map $\widetilde{f} : V \otimes W \to Z$ with $\widetilde{f}(v \otimes w) = f(v, w)$. We say that f is X invariant if $\widetilde{f}$ is X invariant. So f is X-invariant if and only if

$$f(lv, w) + f(v, lw) = l(f(v, w))$$

for all $l \in X, v \in V$ and $w \in W$. In the special case that Z is a trivial L-module we see that f is X-invariant if and only if

$$f(lv, w) = -f(v, lw)$$

for all $l \in X, v \in V$ and $w \in W$.

Note that the sets of all l in L which leave f invariant (that f is l-invariant) is equal to $C_L(\widetilde{f})$ and so forms a subalgebra of L.

Let $f : V \times W \to Z$ be $\mathbb{K}$-bilinear. For $X \subset V$ define

$$X^\perp = \{w \in W \mid f(x, w) = 0 \, \forall x \in X\}.$$

Similarly for $Y \subseteq W$ define

$$^\perp Y := \{v \in V \mid f(v, y) = 0 \, \forall y \in Y.$$

f is called non-degenerate, if $V^\perp = {}^\perp W = 0$.

Consider now the case where $V = W$. We say that f is symmetric if (for all $v, w \in W$) $f(v, w) = f(w, v)$, f is alternating if $f(v, w) = -f(w, v)$ and f is sympletic if $f(v, v) = 0$. Note that if f is symplectic then f is alternating. We say that f is $\perp$-symmetric, provided that $f(v, w) = 0$ if and only if $f(w, v) = 0$. Observe that if f is symmetric or alternating, then f is $\perp$-symmetric.

If f is $\perp$-symmetric then $V^\perp = {}^\perp V$ and we define $\mathrm{rad}(f) = V^\perp$.

Lemma 2.2.1 [**basic bilinear**] *$f : V \times W \to Z$ a L-invariant and $\mathbb{K}$-bilinear. Let X be a L-submodule of V then $X^\perp$ is L-submodule of W.*

Proof: Let $w \in X^\perp$, $l \in L$ and $x \in X$. Then $lx \in X$ and so

$$f(x, lw) = lf(x, w) - f(lx, w) = l0 - 0 = 0.$$

Thus $lw \in X^\perp$ and $X^\perp$ is a submodule of W. □

Lemma 2.2.2 [**multiplications are invariant**]

(a) [**a**] *Let V be an L-module. Then the map $\mathfrak{l}(\mathfrak{U}) \times V \to V, (u, v) \to uv$ is L-invariant.(Here we view $\mathfrak{l}(\mathfrak{U})$ as an L-module via the adjoint representation.)*

(b) [**b**] *$L \times L \to L, (a, b) \to [a, b]$ is L-invariant.*

(c) [**c**] *$L \times \mathfrak{U} \to \mathfrak{U}, (a, u) \to au$ is L-invariant. (Here we view $\mathfrak{U}$ as an L-module via left multiplication.)*

(d) [**d**] *$L \times L \to \mathfrak{l}(\mathfrak{U}), (a, b) \to ab$ is L-invariant. (Here we view $\mathfrak{l}(\mathfrak{U})$ as an L-module via the adjoint representation.)*

Proof: (a) Let $a \in L$, $u \in U$ and $v \in V$. Define $f(u, v :) = uv$. Then

$$f(a * u, v) + f(u, av) = [a, u]v + u(av) = a(uv) = af(u, v).$$

(b) and (c) are special cases of (a).
(d) Let a, b, c in L and define $f(b, c) := bc$. Then using 1.3.3 b

$$f(a * b, c) + f(b, a * c) = [a, b]c + b[a, c] = [a, bc] = a * f(b, c).$$

□

2.3 A first look at weights

Definition 2.3.1 [**def:weights**] *A weight for L is a Lie-algebra homomorphism $\lambda : L \to \mathfrak{l}(K)$. $\Lambda(L) = \mathrm{Hom}_{Lie}(L, \mathfrak{l}(\mathbb{K}))$ is the set of all weights of L.*

Note that a weight for Λ is nothing else as $\mathbb{K}$-linear map $\lambda : L \to \mathbb{K}$ with $L' \leq \ker \lambda$. Thus $\Lambda(L) \cong \Lambda(L/L') = (L/L')^*$. For a weight λ we denote by $\mathbb{K}_\lambda$ the L-module with action $L \to \mathbb{K} \to \mathbb{K}, (l, k) \to \lambda(l)k$.

Lemma 2.3.2 [**weights and simple**] *The map $\lambda \to \mathbb{K}_\lambda$ is a one to one correspondence between weights of L and isomorphism classes of 1-dimensional L-modules.*

Proof: Let V be a 1-dimensional L-module. Then $lv = \mathrm{tr}_V(l)v$ for all $l \in L, v \in V$ and so tr_V is a weight and $V \cong \mathbb{K}_{\mathrm{tr}_V}$. Clearly two 1-dimensional L-modules are isomorphic if and only their trace functions are equal. □

Corollary 2.3.3 [**simple for solvable**] *Let L be standard and solvable. Then the map $\lambda \to K_\lambda$ is one to one correspondence between the weights of L and finite dimensional simple L-modules.*

Proof: 2.1.6(b) and 2.3.2. □

Let λ be a weight for L. Since λ corresponds to an isomorphism class of simple L-modules we obtain from Definition 1.7.8 the notations V_λ, $V_\lambda(i)$ and V_λ^c. V_λ is called the weight space for λ on V.

A weight for λ for L on V is a weight with $V_\lambda \neq 0$. $\Lambda_V = \Lambda_V(L)$ is the set of weights for L on V. V_λ^c is called the generalized weight space for λ on V. We also will write $V_\lambda(\infty)$ for V_λ^c

Lemma 2.3.4 [**weights and eigenspaces**] *Suppose that $L = \mathbb{K}l$ is 1-dimensional, λ a weight of for L, $k = \lambda(1)$ and $n \in \mathbb{N}$.*

(a) [**a**] *V_λ is the eigenspace for l on V corresponding to k,*

(b) [**b**] $V_\lambda(n) = C_V((k-l)^n)$.

(c) [**c**] *V_λ^c is the generalized eigenspace for l on V corresponding to k.*

Proof: (a) By definition V_λ is the sum of all L-submodules isomorphic to $\mathbb{K}_\lambda$. Since $(k-l)\mathbb{K}_\lambda = 0$, $V_\lambda \leq C_V(k-l)$. Clearly $C_V(k-l)$ is the sum of submodules isomorphic to $\mathbb{K}_\lambda$ and so $V_\lambda = C_V(k-l)$.

(b) For $n = 0$ both sides are 0. By induction we may assume $W = V_\lambda(n-1) = C_V((k-l)^{n-1})$. Applying (a) to $\bar{V} = V/W$ we get

$$V_\lambda(n)/W = \bar{V}_\lambda = C_{\bar{V}}(k-l) = C_V((k-l)^n)/W.$$

So (b) holds.

(c) follows from (b). □

Lemma 2.3.5 [**weights and invariant maps**] *Let $f : V \times W \to Z$ be L-invariant, $\mathbb{K}$-bilinear map of L-modules. Let λ and μ be weights of L and $i, j \in \mathbb{N} \cup \{\infty\}$. Then*

$$f\left(V_\lambda(i), W_\mu(j)\right) \leq Z_{\lambda+\mu}(i+j-1).$$

Proof: We first consider the case $i = j = 1$. Let $l \in L$, $v \in V_\lambda$ and $w \in W_\mu$. Then

$$\begin{aligned} lf(v,w) = &\quad f(lv,w) + f(v,lw) &= f(\lambda(l)v,w) + f(v,\mu(l)w) \\ = &\quad \lambda(l)f(v,w) + \mu(l)f(v,w) &= (\lambda+\mu)(l)f(v,w). \end{aligned}$$

So the lemma holds in this case.

Also the lemma is obviously true for $i = 0$ or $j = 0$. If the lemma holds for all finite i and j it also holds for $i = \infty$ or $j = \infty$.

So assume $1 \leq i < \infty$ and $1 \leq j < \infty$. By induction on $i + j$ we also may assume that

$$f(V_\lambda(i-1), W_\mu(j)) \leq Z_{\lambda+\mu}(i+j) \text{ and } f(V_\lambda(i), W_\mu(i-1)) \leq Z_{\lambda+\mu}(i+j).$$

Put $\bar{X} = V_\lambda(i)/V_\lambda(i-1)$, $\bar{Y} = W_\mu(j)/W_\mu(j-1)$ and $\bar{Z} = Z/Z_{\lambda+\mu}(i+j)$. Then we obtain a well defined L-invariant map $\bar{f} : \bar{X} \times \bar{Y} \to \bar{Z}$ with $\bar{f}(\bar{v}, \bar{w}) = \overline{f(v,w)}$ for all $v \in V_\lambda(i)$ and $w \in W_\mu(j)$. Note that $\bar{X} = \bar{X}_\lambda$ and $\bar{Y} = \bar{Y}_\mu$. So by the "$i = j = 1$"-case we get that $\bar{f}(\bar{X}, \bar{Y}) \leq \bar{Z}_{\lambda+\mu}$. Taking inverse images in V, W and Z we see that the lemma holds. □

Corollary 2.3.6 [**weight formula**] *Let V be an L modules, $A \leq L$, λ and μ weights for A and $i, j \in \mathbb{N} \cup \{\infty\}$*

(a) [**a**] $L_\lambda(i)V_\mu(j) \leq V_{\lambda+\mu}(i+j-1)$

(b) [**b**] $[L_\lambda(i), L_\mu(j)] \leq L_{\lambda+\mu}(i+j-1)$.

Proof: By 2.2.2 the map $(l, v) \to v$ is L- and so also A-invariant. Hence (a) follows from 2.3.5. (b) is just a special case of (a). □

2.4 Minimal non-solvable Lie algebras

Proposition 2.4.1 [**minimal non solvable**] *Let L be a standard Lie algebra such that all proper subalgebras are solvable but L is not solvable. Then*

(a) [**a**] $L = L'$.

(b) [**b**] $\mathrm{Sol}(L)$ *is the unique maximal ideal in L.*

(c) [**c**] $L/\mathrm{Sol}(L)$ *is simple.*

(d) [**d**] $\mathrm{Sol}(L) = C_L(W)$*, where W is any non-trivial, finite dimensional simple L-module.*

(e) [**e**] $\mathrm{Sol}(L) = \mathrm{Nil}_L(V)$*, where V is any non-trivial, finite dimensional L-module.*

(f) [**f**] $\mathrm{Sol}(L) = \mathrm{Nil}(L)$.

Proof: (a) If $L' \neq L$, then both L' and L/L' are solvable. Thus L is solvable, a contradiction.

(b) Let I be any proper ideal in L. Then I is solvable and so $I \leq Sol(L)$.

(c) By (b), $L/\mathrm{Sol}(L)$ has no proper ideals.

(d). Since W is non-trivial, $C_L(W) \neq L$. Thus $C_L(W) \leq \mathrm{Sol}(L)$. By (a), $\mathrm{Sol}(L) \leq L'$ and so by 2.3.3, $\mathrm{Sol}(L) \leq C_L(W)$.

(e) Note that by (a) $L/C_L(V)$ is perfect. If L acts nilpotently on V, then 1.6.2 implies that $L/C_L(V)$ is nilpotent and perfect, and so trivial. This contradictions shows that $L \neq \mathrm{Nil}_L(V)$. By (d) $\mathrm{Sol}(L) \leq \mathrm{Nil}_L(V)$ and so (b) implies $\mathrm{Sol}(L) = \mathrm{Nil}_L(V)$.

(f) Apply (e) to $V = L$. □

Theorem 2.4.2 [**minimal simple**] *Let L be a non-solvable, standard simple Lie-algebra all of whose proper subalgebra are solvable. Then $L \cong \mathfrak{sl}(\mathbb{K}^2)$.*

Proof: For $X \in L$ let $\widetilde{X} = \mathrm{Nil}_X(L)$. Also let $\mathcal{N}$ be the set of elements in L acting nilpotently on L.

1° [**1**] *Let $X \not\leq L$, then $X' \leq \widetilde{X} = X \cap \mathcal{N}$. and $\widetilde{X}$ is is a nilpotent ideal in X.*

Since $X \neq L$, X is solvable by assumption. Thus 2.1.5 $X' \leq \mathrm{Nil}_X(L)$ and by 2.1.6(d), $\widetilde{X} = X \cap \mathcal{N}$. Since L is non-abelian, $L \neq Z(L)$ and since L is simple, $Z(L) = 1$. Thus L is a faithful L-module and so by 1.6.2 $\widetilde{X}$ is nilpotent.

Let A and B be distinct maximal subalgebras of L and $D = A \cap B$

2° [**2**] $L = A + B$ *and* $\dim L/A = 1 = \dim A/D = \dim B/D = \dim L/B$.

Since A is solvable 2.1.6 applied to $V = L/A$ implies that there exists a 1-dimensional A submodule W/A in L/A. Let $w \in W \setminus A$. Then $[A, W] \leq W$ as W is a A-submodule of L. Also $W = A + \mathbb{K}w$ and so $[w, W] = [w, A] \leq W$. Thus $[W, W] \leq W$, that is W is a subalgebra of L. The maximality of A implies $L = W$. So $\dim L/A = 1$. By symmetry $\dim L/B = 1$. Since $A \neq B$, $L = A + B$. Thus $A/D = A/A \cap B \cong A + B/B = L/B$ and (2°) holds.

3° [**3**] *$\widetilde{D}$ is an ideal in A.*

If $\widetilde{D} = \widetilde{A}$, this is obvious. So suppose that $\widetilde{A} \neq \widetilde{D}$. Since $\widetilde{D}$ acts nilpotenly on $\widetilde{A}$ we get from 1.6.7 that $\widetilde{D} \lneq N_{\widetilde{A}}(\widetilde{D})$. By (1°), $\widetilde{A} \cap D = (A \cap \mathcal{N}) \cap D = \widetilde{D}$ and so $N_{\widetilde{A}}(\widetilde{D}) \nleq D$. By (2°) A/D is 1-dimensioanal and so

$$A = N_{\widetilde{A}}(\widetilde{D}) + D \leq N_A(\widetilde{D}).$$

Thus (3°) holds.

4° [**4**] $A' \cap D = B' \cap D = \widetilde{D} = 0.$

By (3°), $\widetilde{D}$ is an ideal in A and by symmetry also in B. Thus $\widetilde{D}$ is an ideal in $L = A+B$, and as L is simple, $\widetilde{D} = 0$. By (1°) $A' \cap D \leq \widetilde{A} \cap D = \widetilde{D}$. Thus (4°) holds.

5° [**5**] *D is abelian and A' is at most 1-dimensional.*

By (4°) $D' \leq A' \cap D = 0$. Also $A' \cong A'/A' \cap D \cong A' + D/D \leq A/D$ and so by (2°), A' is at most 1-dimensional.

Let $a \in A \setminus D$ with $a \in A'$ if possible. If $A' = 0$ the $[a, A] = 0$ and if $A' \neq 0$, then by (5°), $A' = \mathbb{K}a$. In any case $\mathbb{K}a$ is an ideal in A. Let $\lambda_A = \mathrm{tr}^D_{\mathbb{K}a}$. Similarly, define $b \in B$ and λ_B.

6° [**6**] *$L = \mathbb{K}a \oplus D \oplus \mathbb{K}b$ and $\lambda_A = -\lambda_B$.*

The first statement follows immediately from (2°). In particular $\mathrm{tr}^D_L = \lambda_A + \mathrm{tr}^D_D + \lambda_B$. Since D is abelian, $\mathrm{tr}^D_D = 0$. Since $L = [L, L]$, $\mathrm{tr}^D_L = 0$. Thus (6°) holds.

7° [**7**] *$\ker \lambda_A = \ker \lambda_B = 0$ and D is one-dimensional.*

Note that $[\ker \lambda_A, \mathbb{K}a + D] = 0$ and since $A = \mathbb{K}a + D$ we get $\ker \lambda_A \leq Z(A)$. By (6°), $\ker \lambda_A = \ker \lambda_B$ and so $\ker \lambda_A \leq Z(A) \cap Z(B) \leq Z(L) = 0$. If $D = 0$, $\dim L = 2$ and L is solvable by 1.2.1, a contradiction. Thus $D \neq 0$. Since $\dim D/\ker \lambda_A = \dim \lambda_A(D) \leq \dim \mathbb{K} = 1$ we conclude that D is 1-dimensional.

In particular, we have $\lambda_A \neq 0$ and so λ_A is onto and there exists $d \in D$ with $\lambda_A(d) = 1$.

Also note that $0, \lambda_A$ and λ_B are the weights of D on V and are pairwise distinct. Also $\mathbb{K}a \leq L_{\lambda_A}(D), \mathbb{K}b \leq L_{\lambda_B}(D)$ and $D \leq L_0(D)$. Thus 1.7.15 implies that $L_{\lambda_A}(D) = \mathbb{K}a$,

$L_{\lambda_B}(D) = \mathbb{K}b$ and $L_0(D) = D$. Now 2.3.6 shows that $[a,b] \in D$. Suppose that $[a,b] = 0$. Then A is an ideal in L, a contradiction. Thus $[a,b] = kd$ for some non-zero $k \in \mathbb{K}$. Replacing b by $k^{-1}b$ we may assume $[a,b] = d$. Also from $[d,a] = a$ and $\lambda_B = -\lambda_A$ we have $[d,b] = -b$. Thus

8° [**8**] *a, b, d is a basis for L, $[a,b] = d$, $[d,a] = a$ and $[b,d] = b$.*

From (8°) we see that L is unique up to isomorphism. Since $\mathfrak{sl}(\mathbb{K}^2)$ fullfils the assumptions of the theorem we get $L \cong \mathfrak{sl}(\mathbb{K}^2)$. □

2.5 The simple modules for $\mathfrak{sl}(\mathbb{K}^2)$

In this section $L = \mathfrak{sl}(\mathbb{K}^2)$. Let $x = E_{12}, y = E_{21}$ and $h = E_{11} - E_{22}$. Then (x, y, h) is basis for L with $[h,x] = 2x$, $[y,h] = 2y$ and $[x,y] = h$. We call (x, y, h) the Chevalley basis for L.

Lemma 2.5.1 [**autos for sl2**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h).*

(a) [**a**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = x, \Phi(y) = y$ and $\Phi(h) = -h$. Then Φ is an anti-automorphism of L.*

(b) [**b**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = y, \Phi(y) = x$ and $\Phi(h) = -h$. Then Φ is an automorphism of L.*

(c) [**c**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = y, \Phi(y) = x$ and $\Phi(h) = h$. Then Φ is an anti-automorphism of L.*

Proof: Readily verified from commutator relations of (x, y, h). □

Lemma 2.5.2 [**u for sl2**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h) and let $i \in \mathbb{Z}_+$. Then the following holds in $\mathfrak{U}$.*

(a) [**a**] $hy^i = y^i(h - 2i)$

(b) [**b**] $xy^i = y^i x + iy^{i-1}(h - (i-1))$.

(c) [**c**] $yx^i = xy^i - ix^{i-1}(h + i - 1)$

Proof: Readily verified using the commutator relations and induction on i. □

Corollary 2.5.3 [**u for sl2 in char 0**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h). Suppose* $\operatorname{char}\mathbb{K} = 0$ *and define $x^{(i)} = \frac{1}{i!}x^i$ and $y^{(i)} = \frac{1}{i!}y^i$. Let $i \in \mathbb{Z}_+$.Then*

(a) [**a**] $hy^{(i)} = y^{(i)}(h - 2i)$.

(b) [**b**] $xy^{(i)} = y^{(i)}x + y^{(i-1)}(h - (i-1))$.

This follows immediately from 2.5.2 □

Theorem 2.5.4 [**modules for sl2**] *Suppose $\mathbb{K}$ is standard, $L = \mathfrak{sl}(\mathbb{K}^2)$, V is a an L-module and (x, y, h) is the Chevalley basis for L. Let $k \in \mathbb{K}$ and $0 \neq v \in V$.*

(a) [**a**] *If V is finite dimensional, then there exists $0 \neq v \in V$ and $k \in \mathbb{K}$ with $xv = 0$ and $hv = kv$.*

(b) [**b**] *Suppose that there exist $0 \neq v \in V$ and $k \in \mathbb{K}$ with $xv = 0$ and $hv = kv$. Let $m \in \mathbb{N}$ be minimal with $y^{m+1}v = 0$, if such an m exists and $m = \infty$ otherwise. Also let $W = \mathfrak{U}v$ be the smallest L-submodule of V containing v. Put $v_i = \frac{v^y}{y!}$. Then*

  (a) [**z**] *$(v_i \mid i \in \mathbb{N}, i \leq m)$ is a basis for W.*

  (b) [**a**] $yv_i = (i+1)v_{i+1}$.

  (c) [**b**] $hv_i = (k - 2i)v_i$.

  (d) [**c**] $xv_i = (k - (i-1))v_{i-1}$, *where* $v_{-1} = 0$.

  (e) [**d**] $xyv_i = (i+1)(m-i)v_i$

  (f) [**e**] *If $m < \infty$, then $m = k = \dim W - 1$.*

Proof: (a) Let $A = \mathbb{K}x + \mathbb{K}h$. Then A is solvable and $A' = \mathbb{K}x$. Let V_0 be a simple A-submodule in V. Then by 2.1.6 V_0 is 1-dimensional. Let $0 \neq v_0 \in V_0$. Then $xv_0 = 0$ and $hv_0 = kv_0$ for some $k \in \mathbb{K}$.

(b) $yv_i = yy^{(i)}v_0 = (i+1)y^{(i+1)}v_0 = (i+1)v_{i+1}$ and so (b:b) holds.

From 2.5.3(a) we have

$$hv_i = hy^{(i)}v_0 = y^{(i)}(h - 2i)v_0 = y^{(i)}(k - 2i)v_0 = (k - 2i)v_i$$

and so (b:c) holds. From 2.5.3(b)

$$(3)\quad xv_i = xy^{(i)}v_0 = (y^{(i)}x + y^{(i-1)}(h-(i-1))v_0 = 0 + y^{(i-1)}(k-(i-1))v_0 = (k-(i-1))v_{i-1}$$

and (b:d) holds. (b:e) follows from (b:b) and (b:d).

By (b:c) v_i is an eigenvector with eigenvalue $k - 2i$ for h. Thus the non-zero v_i's are linearly independent. From (b:b),(b:c) and (b:d) the $\mathbb{K}$-space spanned by $v_i's$ invariant under L and so equal to W. Thus (b:a) holds.

Suppse now that $m < \infty$. By (b:d) with $i = m + 1$ we get

$$0 = x0 = xv_{m+1} = (k - m)v_m$$

As $v_m \neq 0$ and $\mathbb{K}$ is a field, $k = m$. Thus (b:f) holds. □

2.6 Non-degenerate Bilinear Forms

In this section we establish some basic facts about non-degenerate bilinear forms that will be of use later on.

Lemma 2.6.1 [**basic non-deg bilinear**] *Let V and W be finite dimensional K-spaces and $f : V \times W \to K$ be non-degenerate and $\mathbb{K}$-bilinear.*

(a) [**a**] *There exists a unique $\mathbb{K}$-isomorphism $t : W^* \to V, \alpha \to t_\alpha$ with $\alpha(w) = f(t_\alpha, w)$ for all $\alpha \in W^*, w \in W$. In particular, $\dim V = \dim W$.*

(b) [**b**] *Let $(w_i \mid i \in I)$ be a basis for W. Then there a unique basis $(v_i \mid i \in I)$ of V with $f(v_i, w_j) = \delta_{ij}$ for all $i, j \in I$.*

(c) [**c**] *Let X be a subspace of V. Then $\dim X + \dim X^\perp = \dim V$. In particular, $X = V$ if and only if $X^\perp = 0$.*

Proof: (a) Note first that if we define $\phi_i \in W^*$ by $\phi_i(w_j) = \delta_{ij}$, then $(\phi_i \mid i \in I)$ is a basis for W^*. In particular, $\dim W = \dim W^*$. Since f is non-degenerate the map $\Psi : V \to W^*$ with $\Psi(v)(w) = f(v, w)$ is one to one. Thus $\dim V =\leq \dim W^* = \dim W,$. By symmetry $\dim W \leq \dim V$. So $\dim V = \dim W$ and Ψ is an isomorphism. Putting $t = \Psi^{-1}$ we see that (a) holds

(b) Just put $w_i = t_{\phi_i}$.

(c) The form $X \times W/X^\perp, (x, w + X^\perp) \to f(x, y)$ is well defined and non-degenerate. Thus by (a) $\dim X = \dim W/X^\perp$ and so (c) holds. □

Definition 2.6.2 [**def:omega**] *A quadratic form on the $\mathbb{K}$-space V is a map $q : V \to \mathbb{K}$ such that $q(kv) = k^2 q(v)$ for all $k \in \mathbb{K}, v \in V$ and such that the function $s : V \times V \to \mathbb{K}, (v, w) \to q(v + w) - q(v) - q(w)$ is $\mathbb{K}$-bilinear. Note that s is symmetric. We call s the bilinear form associated to q. Let $u \in V$ with $q(u) \neq 0$. Define $\check{u} = q(u)^{-1}u$ and*

$$\omega_u : V \to V, v \to v - s(v, \check{u})u = v - \frac{s(v, u)}{q(u)}u.$$

Lemma 2.6.3 [**omega u**] *Let V be a $\mathbb{K}$-space, $q : V \to \mathbb{K}$ a quadratic form with associated bilinear form s, $u \in V$ with $q(u) \neq 0$.*

(a) [**c**] *$s(v, v) = 2q(v)$ for all $v \in V$.*

(b) [**d**] *$q(\check{u}) = q(u)^{-1} \neq 0$, $\check{\check{u}} = u$, $s(u, \check{u}) = 2$ and $\omega_u(u) = -u$.*

(c) [**a**] *Let $0 \neq k \in \mathbb{K}$. Then $\check{ku} = k^{-1}\check{u}$ and $\omega_{ku} = \omega_u$. In particular, $\omega_u = \omega_{\check{u}}$.*

(d) [**e**] *ω_u is an isometry of q.*

(e) [**f**] *Let σ be an isometry of q. Then $\sigma(\check{u}) = \check{\sigma(u)}$ and $\sigma\omega_u\sigma^{-1} = \omega_{\sigma(u)}$.*

Proof:

(a) We have $q(2v) = 4q(v)$ and $q(v+v) = q(v) + s(v,v) + q(v)$.

(b) $q(\check{u}) = q(q(u)^{-1}u) = q(u)^{-2}q(u) = q(u)^{-1}$. So $\check{\check{u}} = (q(u)^{-1})^{-1}q(u)^{-1}u = u$. Also and $s(u,\check{u}) = \frac{s(u,u)}{q(u)}$. So by (a), $s(u,\check{u}) = 2$ and hence $\omega_u(u) = u - 2u = -u$.

(c) $\check{ku} = q(ku)^{-1}ku = k^{-2}q(u)^{-1}ku = k^{-1}\check{u}$ and

$$\omega_{ku}(v) = v - s(v, \check{ku})ku = v - s(v, k^{-1}\check{u})ku = v - s(v,\check{u})u = \omega_u(v).$$

(d)

$$\begin{aligned} q(\omega_u(v)) &= q(v - s(v,\check{u})u) &&= q(v) - s(v,\check{u})s(v,u) + s(v,\check{u})^2 q(u) \\ &= q(v) - s(v,\check{u})(s(v,u) - s(v, q(u)^{-1}u)q(v)) &&= q(v). \end{aligned}$$

(e) $\sigma(\check{u}) = q(\sigma(u)^{-1}\sigma(u)) = q(u)^{-1}\sigma(u) = \sigma(q(u)^{-1}u) = \sigma(\check{u})$ and

$$(\sigma\omega_u\sigma^{-1})(v) = \sigma(\sigma^{-1}(v) - s(\sigma^{-1}(v)v, \check{u}) = v - s(v, \sigma\check{(u)})\sigma(u) = \omega_{\sigma u}(v).$$

□

Lemma 2.6.4 [**1/2 f**] *Suppose f is a non-degenerate symmetric form on the $\mathbb{K}$-space V and that $\operatorname{char}\mathbb{K} \neq 2$. Then $q(v) := \frac{1}{2}f(v,v)$ is a quadratic form and f is its associated bilinear form.*

Proof: $q(v+w) - q(v) - q(w) = \frac{1}{2}(f(v+w) - f(v,v) - f(w,w) = f(v,w)$ □

Lemma 2.6.5 [**f circ**] *Let V and W be finite dimensional $\mathbb{K}$-spaces and $f : V \times W \to \mathbb{K}$ a non-degenerate bilinear form. Define $\Phi : V \otimes W \to (V \otimes W)^*$ by $\Phi(v \otimes w)(v' \otimes w') = f(v,w')f(v',w)$ for all $v, v' \in V$ and $w, w' \in W$.*

(a) [**a**] *According to 2.6.1 choose bases $(v_i, i \in I)$ and $(w_i, i \in I)$ for V and W such that $f(v_i, w_j) = \delta_{ij}$. Then $(\Phi(v_i \otimes w_j))_{ij}$ is the dual of the basis $(v_j \otimes w_i)_{ij}$ of $V \otimes W$*

(b) [**b**] *Φ is an isomorphism.*

(c) [**c**] *Let $f^\circ = \Phi^{-1}(\widetilde{f})$. Then $f^\circ = \sum_{i \in I} v_i \otimes w_i$.*

(d) [**e**] *$\widetilde{f}(f^\circ) = \dim V$*

(e) [**d**] *Suppose that V and W are L-modules and f is L-invariant. Then Φ is L-invariant and $Lf^\circ = 0$.*

Proof: We compute

$$(*) \qquad \Phi(v_i \otimes w_j)(v_k \otimes w_l) = f(v_i, w_l) f(v_l, w_j) = \delta_{il}\delta_{jk}.$$

Thus (a) holds. (b) follows directly from (a).

(c) Let $t = \sum_{i \in I} v_i \otimes w_i$. Then by (*)

$$\Phi(t)(v_k \otimes w_l) = \sum_{i \in I} \Phi(v_i \otimes w_i)(v_k \otimes w_l) = \sum_{i \in I} \delta_{il}\delta_{ik} = \delta_{kl} = f(v_k, w_l) = \widetilde{f}(v_k \otimes w_l)$$

Thus $\Phi(t) = \widetilde{f}$ and so $t = \Phi^{-1}(\widetilde{f}) = f^\circ$

(d) From (c) we compute

$$\widetilde{f}(f^\circ) = \sum_{i \in I} \widetilde{f}(v_i \otimes w_i) = \sum_{i \in I} f(v_i, w_i) = \sum_{i \in I} 1 = |I| = \dim V.$$

(e) That Φ is L-invariant is readily verified. Since f is L-invariant, $L\widetilde{f} = 0$. Since Φ is an L-isomorphism, $Lf^\circ = 0$. □

2.7 The Killing Form

For a finite dimensional L-module V we define $f_V : L \times L \to \mathbb{K}, (a, b) \to \operatorname{tr}_V(ab)$. f_V is called the killing form of L with respect to V. In the case of the adjoint module, f_L is just called the Killing form of L.

Lemma 2.7.1 [**basic killing**] *Let V be a finite dimension L-module.*

(a) [**a**] *f_V is a symmetric, L-invariant bilinear form on L.*

(b) [**b**] *If $I \trianglelefteq L$, then $I^\perp \trianglelefteq L$ and $[I, I^\perp] \le \operatorname{rad}(f_V)$.*

(c) [**c**] *Let $\mathcal{W}$ be the set of factors for some L-series on V. Then*

$$f_V = \sum_{W \in \mathcal{W}} f_W.$$

(d) [**d**] *Let I be an ideal in L. Then $f_I \mid_{L \times I} = f_L \mid_{L \times I}$.*

(e) [**e**] $\operatorname{Nil}_L(V) \le \operatorname{rad}(f_V)$.

(f) [**f**] *If L is finite dimensional, then* $\operatorname{Nil}(L) \le \operatorname{rad}(f_L)$.

(a) Clearly f_V is K-bilinear. Let $a, b \in \mathfrak{U}$. Then $\mathrm{tr}_V(ab) = \mathrm{tr}_V(ba)$ so f_V is symmetric. Thus also shows that $\mathrm{tr}_V([a,b]) = \mathrm{tr}_V(ab - ba) = 0$ and so $\mathrm{tr}_V : \mathfrak{U} \to \mathbb{K}$ is L-invariant. By 2.2.2 the map $m_L : L \times L \to \mathfrak{U}, (a,b) \to ab$ is L-invariant. So also $f_V = \mathrm{tr}_V \circ \mathfrak{m}_L$ i L-invariant.

(b) The first statement follows from 2.2.1. For the second, let $i \in I, j \in I^\perp$ and $l \in L$. Then $[j,l] \in I^\perp$ and so since f_V is L-invariant:

$$f_V([i,j],l) = -f_V(i,[j,l]) = 0.$$

Thus $[i,j] \in \mathrm{rad}(f_V)$.

(c) Follows from 1.7.17(c).

(d) By (c), $f_L = f_I + f_{L/I}$. Then I and so also LI acts trivially on L/I. Thus $f_{I/L}\mid_{L\times I}= 0$ and (d) holds.

(e) Let W be composition factors for L on V. Then $\mathrm{Nil}_L(V)W = 0$ and so also $L\mathrm{Nil}_L(V)W = 0$. Thus for all $l \in L$ and $n \in \mathrm{Nil}_L(V)$ we have $f_W(l,n) = \mathrm{tr}_W(ln) = 0$. So by (c) $f_V(l,n) = 0$ and (e) holds.

(f) This is (e) applied to the adjoint module. □

Theorem 2.7.2 (Cartan's Solvabilty Criterion) [**cartan**] *Suppose L posseses a standard, faithful L-module with $f_V = 0$. Then L is solvable.*

Proof: Suppose L is a counter example with $\dim L$ minimal. Then all proper algebras of L are solvable, but L is not. Thus by 2.4.1 and 2.4.2, $\overline{L} := L/\mathrm{Nil}_L(V) \cong \mathfrak{sl}(\mathbb{K}^2)$. Let (x,y,h) be a Chevalley basis for $\overline{L}$ and choose $\widetilde{x}$ and $\widetilde{y}$ in L which are mapped onto x and y. Since $L \neq \mathrm{Nil}_L(V)$ there exists a non-trivial compostion factor W for L on V. For any such W we have $C_L(W) = Nil_L(V)$ and 2.5.4(b:e) implies that $\mathrm{tr}_W(\widetilde{x}\widetilde{y})$ is a positive integer. Hence 1.7.17(c) implies that $f_V(\widetilde{x},\widetilde{y}) = \mathrm{tr}_V(\widetilde{x}\widetilde{y})$ is a positive integer. This is contradiction to $f_V = 0$ and the theorem is proved. □

Proposition 2.7.3 [**rad=sol**] *Let V be standard,faithful L-module. Then*

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V) \leq \mathrm{Sol}(L)$$

In particular, if L is perfect, then $\mathrm{Sol}(L) = \mathrm{Nil}_L(V) = \mathrm{rad}(f_V)$.

Proof: By Lie's Theorem 2.1.5

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V).$$

By 2.7.1(d), $\mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V)$. Finally, Cartan's Solvabilty Criterion 2.7.2 (applied to $\mathrm{rad}(f_V)$ in place of L), we have that $\mathrm{rad}(f_V)$ is solvable and so $\mathrm{rad}(f_V) \leq \mathrm{Sol}(L)$. □

Corollary 2.7.4 [**basic non-degenerate**] *Let V be a standard L-module with f_V non-degenerate. Then*

(a) [**a**] *V is faithful and $Nil_L(V) = 0$.*

(b) [**b**] *$\mathrm{Sol}(L) = Z(L)$ and $\mathrm{Sol}(L) \cap L' = 0$.*

(c) [**c**] *If L is solvable, then L is abelian.*

Proof: By 2.7.1(c), $C_L(V) \leq \mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V) = 0$. So (a) holds. (b) now follows from 2.7.3. (c) follows from the first statement in (b). □

Corollary 2.7.5 [**faithful=non-degenerate**] *Suppose $\mathrm{Sol}(L) = 0$ and V is a standard L-module. Then f_V is non-degenerate if and only if V is faithful.*

Proof: If f_V is non-degenerate, then V is faithful by 2.7.4(a). Suppose now that V is faithful. Then by 2.7.3 $\mathrm{rad}(f_V) \leq \mathrm{Sol}(L) = 0$ and so f_V is non-degenerate. □

Lemma 2.7.6 [**non-degenerate implies semisimple**] *Suppose that L is finite dimensional and f_L is non-degenerate. Then $\mathrm{Sol}(L) = 0$.*

Proof: By 2.7.1(e), $\mathrm{Nil}(L) \leq \mathrm{rad}(f_L) = 0$. So by 2.1.2(c), $\mathrm{Sol}(L) = 0$. □

Corollary 2.7.7 [**semisimple=non-degenerate**] *Suppose L is standard. Then $\mathrm{Sol}(L) = 0$ if and only if f_L is non-degenerate.*

Proof: If f_L is non-degenerate, then by 2.7.6 $\mathrm{Sol}(L) = 0$. If $\mathrm{Sol}(L) = 0$, then also $Z(L) = 0$ and so the adjoint module is faithful. So by 2.7.5, f_L is non-degenerate. □

If f is a symmetric bilinear form on a vector space W, we write $W = W_1 \bigoplus W_2$ if W_i are subspaces of W with $W = W_1 \oplus W_2$ and $f(w_1, w_2) = 0$ for all $w_i \in W_i$. Note that in this case, W is non-degenerate if and only if $f\ |_{W_i}$ is non-degenerate for $i = 1$ and 2.

Proposition 2.7.8 [**decomposing l**] *Let V be a finite dimensional L module and suppose that f_V is non-degenerate. Let I be an ideal in L with $I \cap \mathrm{Sol}(L) = 0$. Then*

(a) [**a**] *$[I, I^{\perp}] = 0$.*

(b) [**b**] *$L = I \bigoplus I^{\perp}$.*

(c) [**c**] *$I^{\perp} = C_L(I)$.*

Proof: By 2.7.1(b), $[I, I^\perp] \leq \mathrm{rad}(f_V) = 0$. Thus (a) holds and

$$I^\perp \leq C_L(I). \tag{1}$$

Since $I \cap C_L(I)$ is an abelian ideal of L and since $\mathrm{Sol}(L) \cap I = 0$, we get

$$I \cap C_L(I) = 0. \tag{2}$$

From (1) and (2)

$$I \cap I^\perp = 0. \tag{3}$$

From 2.6.1(c) we have $\dim I + \dim I^\perp = \dim L$ and so (3) implies that (b) holds. From (b), (1) and (2) we compute

$$C_L(I) = C_L(I) \cap L = C_L(I) \cap (I + I^\perp) = (C_L(I) \cap I) + I^\perp = I^\perp$$

So (c) holds. □

Theorem 2.7.9 [**composition of l**] *Let V be a finite dimensional L-module and suppose that* $\mathrm{Sol}(L) = 0$ *and f_V is non-degenerate. Then there exists perfect, simple ideals $L_1, L_2, \ldots, L_n$ in L such that*

$$L = L_1 \bigoplus L_2 \bigoplus \ldots \bigoplus L_n.$$

Proof: By induction on $\dim L$. If L is simple we can choose $n = 1$ and $L_1 = L$. So suppose that L is not simple and let I be proper ideal in L. Since $\mathrm{Sol}(L) = 0$ the assumptions of 2.7.8 are fulfilled. Hence $L = I \bigoplus I^\perp$ and $[I, I^\perp] = 0$. In particular, $f_V \mid_I$ and $f_V \mid_{I^\perp}$ are non-degenerate. Also any ideal in I or $I^\perp$ is an ideal in L. By induction we can decompose I and $I^\perp$ into an orthogonal sum of ideals. Thus the same is true for L. Since $\mathrm{Sol}(L) = 0$, the L_i are not abelian and so perfect. □

Corollary 2.7.10 [**decomposing standard**] *Let V be a standard L-module with f_V-non-degenerate. Then $L = L' \bigoplus Z(L)$ and L is semisimple*

Proof: By 2.7.4 $L' \cap \mathrm{Sol}(L) = 0$. So by 2.7.8, $L = L' \bigoplus L'^\perp$. In particular, $[L'^\perp, L] \leq L' \cap L'^\perp = 0$. Thus $L'^\perp = Z(L)$, $L = L^\perp \bigoplus Z(L)$. Thus $\mathrm{Sol}(L')$ is an ideal in L and hence $\mathrm{Sol}(L') \leq L' \cap \mathrm{Sol}(L) = 0$. By 2.7.9 L' is semisimple. Clearly also $Z(L)$ is semisimple and so L is semisimple. □

Corollary 2.7.11 [**standard semisimple**] *Suppose L is standard and* $\mathrm{Sol}(L) = 0$. *Then* *(a)* [**a**] *f_L is non-degenerate.*

(b) [**b**] *There exists perfect,simple ideals $L_1, L_2 \dots L_n$ such that*

$$L = L_1 \bigoplus L_2 \bigoplus \dots \bigoplus L_n.$$

(c) [**c**] *$\{L_1, L_2, \dots L_n\}$ is precisely the set of minimal ideals in L.*

(d) [**d**] *Every ideal in L is a sum of some of the L_i's.*

Proof: (a) follows from 2.7.7.

(b) By (a) we can apply 2.7.9 with V the adjoint module. Thus (b) holds.

(c) Let I be a minimal ideal in L. Since $\mathrm{Sol}(L) = 0$, $Z(L) = 0$ and so by (b), $[I, L_i] \neq 0$ for some i. As I is a minimal ideal and L_i is simple, $I = [I, L_i] = L_i$.

(d) Let I be an ideal in L. Let A be the sum of the L_i's with $L_i \leq I$ and B the sum of the remaining L_i's. Then $A \leq I$, $L = A+B$ and $[A, B] = 0$. Thus $I = I \cap (A+B) = A+(I \cap B)$.

Suppose that $I \cap B \neq 0$. Then $I \cap B$ contains a minimal ideal and so by (c), $L_i \leq I \cap B$ for some i. Since $L_i \leq I$, $L_i \leq A$. Since $L_i \leq B$ and $[A, B] = 0$ we conclude that $[L_i, L_i] = 0$, a contradiction since L_i is perfect.

Thus $I \cap B = 0$ and $I = A$. □

We say that L is semisimple if L is the direct sum of simple ideals. Note that this is the case if and only if the adjoint module is a semisimple L-module.

Corollary 2.7.12 [**sol l and semisimple**] *Let L be standard. Then* $\mathrm{Sol}(L) = 0$ *if and only if L is perfect and semisimple.*

Proof: One direction follows from 2.7.11 while the other is obvious. □

2.8 Non-split Extensions of Modules

In this section A is an associative algebra.

Definition 2.8.1 [**def:extension**]

(a) [**a**] *An extensions of A-modules is a pair of A-modules (W, V) with $W \leq V$.*

(b) [**b**] *An extension of A-modules (W, V) is called split if there exists a A-submodule X of V with $V = W \oplus X$.*

(c) [**c**] *Let B and T be A-modules and (W, V) an extension of A-modules. We say that (W, V) is an extension of B by T if $W \cong B$ and $V/W \cong T$ as A-modules.*

Lemma 2.8.2 [**basic split I**] *An extension (W, V) of A-modules. is split if and only if there exists $\phi \in \mathrm{Hom}_A(V, W)$ with $\phi \mid W = \mathrm{id}_W$.*

Suppose first that $V = W \oplus X$ for some A-submodule X of V. Let ϕ be the projection of V onto X. Then ϕ is A-invariant and $\phi_W = \mathrm{id}_W$.

Suppose next that $\phi : V \to W$ is A-invarinant with $\phi\mid_W = \mathrm{id}_W$. Let $X = \ker\phi$. Then X is submodule of V and $X \cap W = 0$. Let $v \in V$. Then $\phi(v) \in W$ and so $\phi(\phi(v)) = \phi(v)$.Hence $\phi(v - \phi(v)) == 0$. That is $v - \phi(v) \in \ker\phi = X$. So $v = \phi(v) + (v - \phi(v)) \in W + X$. Thus $V = W \oplus X$ and (W, V) splits. □

Lemma 2.8.3 [**basic split II**] *Let (W, V) be an extension of A-modules. Let*

$$\Phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W)$$

be the restriction map. Then

(a) [**a**] *Φ is A-invariant and onto.*

(b) [**b**] $\ker\Phi \cong \mathrm{Hom}(V/W, W)$ *and* $\ker\Phi$ *is submodule of* $\mathrm{Hom}(V, W)$

(c) [**c**] $S := \Phi^{-1}(\mathbb{K}\mathrm{id}_W)$ *is an A-submodule of* $\mathrm{Hom}(V, W)$ *and* $S/\ker\Phi \cong \mathbb{K}$

(d) [**e**] (W, V) *is split if and only if* $(\ker\Phi, S)$ *is split.*

Proof:

(a) Clearly Φ is A-invariant. Note thate there exists K-subspace X of V with $V = W \oplus X$. Let $\alpha \in \mathrm{Hom}(W, W)$ and define $\tilde{a}(w + x) = \alpha(w)$. Then $\Phi(\tilde{\alpha}) = \alpha$. So Φ is onto.

(b) Since Φ is A-invarinat, $\ker\Phi$ is an A-submodule.

Let $\alpha \in \ker\Phi$. Define $\beta : V/W \to V, v + W \to \alpha(v)$. Conversely let $\beta \in \mathrm{Hom}(V/W, W)$ define $\alpha : V \to W, \alpha(v) = \beta(v + W)$. Then $\alpha \in \ker\Phi$.

(c) follows from (a).

(d) Suppose first that (W, V) splits. Let ϕ be as in 2.8.2 d. Then $S = \ker\Phi \oplus \mathbb{K}\phi$ and so $(\ker\Phi, S)$ splits.

Next suppose that $(\ker\Phi, S)$ splits and let Y be an A-submodule of S with $\S = \ker\Phi \oplus Y$. Then $\Phi \mid Y : Y \to \mathbb{K}\mathrm{id}_W, \phi \to \phi\mid_W$ is an A-invariant isomorphism. Hence Y is a trivial A-module, all $\phi \in Y$ are A-invariant and there exists $\phi \in Y$ with $\phi\mid_W = \mathrm{id}_W$. Thus by 2.8.2, (W, V) splits. □

Lemma 2.8.4 [**b simple**] *Let T be A-module and suppose there exists a non-split extension of a finite dimensional A-module by T. Then there exists on-split extension of finite dimensional simple A-module by T.*

Proof: Let (W, V) be a non-split extension with $V/W \cong T$ and W finite dimensional. Since W is finite dimensional we can choose a submodule Y of W maximal such that $(W/Y, V/Y)$ is non-split. Since $(V/Y)/(W/Y) \cong V/W \cong T$, $(W/Y, V/Y)$ has the same properties as (W, V). So we may assume that $Y = 0$. Let B be a simple A-submodule of

W. The maximality of Y implies that $(W/B, V/B)$ is split. So $V/B = W/B \oplus X/B$ for some A-submodule X of V with $B \leq X$. Then $W \cap X = B$ and $W + X = V$. Thus

$$T \cong V/W = X + W/W \cong X/X \cap W = X/B.$$

Hence (B, X) is an extension of B by T. Suppose this extension is split. Then $X = B \oplus Y$ for some A-submodule Y of X. Thus $V = X + W = Y + B + W = Y + W$ and $Y \cap W \leq Y \cap (X \cap W) + Y \cap B = 0$. So $V = W \oplus Y$, contrary to the assumptions. Thus (B, X) is non-split and the lemma is proved. □

Corollary 2.8.5 [**splitting reduction**]

(a) [**a**] *a Suppose there exists a finite dimensional A-module which is not semisimple. Then there exists a non-split extension of finite dimensional A-modules.*

(b) [**b**] *Suppose there exists a non-split extension of finite dimensional A-modules. Then there non-split extension of finite dimensional simple A- module by* $\mathbb{K}$.

Proof:

(a) Let V be a finite dimensional A-module of minimal dimension with respect to not being semisimple. Let W be simple A-submodule of V. Suppose $V = W \oplus X$ for A-submodule X of W. Then by minimalty of V, X is semisimple. But then also V is semisimple.

(b) From 2.8.3 there exist a non-split extension of a finite dimensional module by $\mathbb{K}$. (b) now follows from 2.8.4 □

2.9 Casimir Elements and Weyl's Theorem

In this section will show that a standard module for a perfect, semisimple Lie algebra is semisimple.

Proposition 2.9.1 [**casimir**] *Suppose L is finite dimensional and $f : L \times L \to \mathbb{K}$ is a non-degenerate, L-invariant, $\mathbb{K}$-bilinear form. Define $\Psi : L \otimes L \to \mathfrak{U}$ by $\Psi(a \otimes b) = ab$. Let f° be as in 2.6.5 and put $c_f = \Psi(f^\circ)$.*

(a) [**a**] $c_f \in Z(\mathfrak{U}) \cap L^2$.

(b) [**b**] *Let $(v_i, i \in I)$ and $(w_i, \in I)$ be bases of L with $f(v_i, w_j) = \delta_{ij}$. Then*

$$c_f = \sum_{i \in I} v_i w_i$$

Proof: View $\mathfrak{U}$ as L-module via the adjoint representation. By 2.2.2, Ψ is L-invariant, By 2.6.5(e) $Lf^\circ = 0$ and so $[L, c_f] = 0$. Since $\mathfrak{U}$ is generated by L as an associative algebra, $[\mathfrak{U}, c_f] = 0$. Thus $c_f \in Z(U)$. Also $c_f \in \Psi(L \otimes L) = L^2$. So (a) holds. (b) follows immediately from 2.6.5(c). □

The elements c_f from the preceeding proposition is called the Casimir element of f.

Lemma 2.9.2 [**cv**] *Let V be a finite dimensional L-module and suppose that f_V is non-degenerate. Define $c_V = c_{f_V}$.*

(a) [**a**] $\mathrm{tr}_V(c_V) = \dim L$.

(b) [**b**] *Suppose $\mathbb{K}$ is algebraicly closed and V is simple. Then c_V acts as a scalar $k \in \mathbb{K}$ on V. Moreover one of the following holds:*

1. [**a**] $\mathrm{char}\, K \nmid \dim V$ *and* $k = \frac{\dim L}{\dim V}$.
2. [**b**] $\mathrm{char}\, K \mid \dim V$ *and* $\mathrm{char}\, K \mid \dim L$.

Proof: (a) Let Ψ be defined as in 2.9.1. Then by definiton of f_V, $\widetilde{f_V} = \mathrm{tr}_V \circ \Psi$. Thus

$$\mathrm{tr}_V(c_V) = \mathrm{tr}_V(\Psi(f^\circ)) = \widetilde{f}(f^\circ)$$

So (a) follows from 2.6.5(d).

(b) Since $c_V \leq Z(U)$, Schur's Lemma 1.7.13 applied to the image of c_V in $\mathrm{End}_L(V)$ gives that c_V acts as a scalar $k \in \mathbb{K}$. Thus $\mathrm{tr}_V(c_V) = k \dim V$ and so (b) holds. □

Theorem 2.9.3 (Weyl) [**weyl**] *Let L be standard, prefect and semisimple and V a finite dimensional L-module. Then V is semisimple.*

Proof: By 2.8.5 its suffices to show that any finite dimensional L-module extension (W, V) with W simple and $V/W \cong \mathbb{K}$ splits. Since $L/C_L(V)$ is also semisimple we may assume that V is faithful. By 2.7.5 f_V is non-degenerate. So by 2.9.1 $c := c_V \in Z(\mathfrak{U}) \cap L^2$. Since $LV \leq W$, $cV \leq W$. Since W is simple, Schur's lemma 1.7.13 applied to the image of c in $\mathrm{End}_L(V)$ gives that c acts as a scalar k on W. Then

$$\mathrm{tr}_V(c) = \mathrm{tr}_W(c) + \mathrm{tr}_{V/W}(c) = k \dim W + 0.$$

By 2.9.2(a) and since $\mathrm{char}\,\mathbb{K} = 0$, $\mathrm{tr}_V(c) \neq 0$ and so $k \neq 0$. Thus $k^{-1}cV \leq W$ and $k^{-1}c$ acts id_W on W. Thus $k^{-1}c$ induces an L-invarinat $\mathbb{K}$-linear map $\Phi : V \to W$ with $\phi\mid_W = \mathrm{id}_W$. Thus by 2.8.2 , (W, V) splits. □

2.10 Cartan Subalgebras and Cartan Decomposition

Definition 2.10.1 [**def:cartan**] *$H \leq L$ is called selfnormalizing if $H = N_L(H)$. A Cartan subalgebra of L is a nilpotent, selfnormalizing subalgebra of L.*

Lemma 2.10.2 [**existence of cartan**] *Suppose that L is finite dimensional and $|\mathbb{K}| \geq \dim L$. Then L has a Cartan subalgebra.*

Proof: Choose $d \in L$ with $H := L_0^c(\mathbb{K}d)$ minimal. Note that a simple module with weight 0 is a trivial module. So by 1.7.11(h), H is the largest $\mathbb{K}d$-submodule on which d acts nilpotently. In particular, $d \in H$. By 2.3.6 $[L_0^c(\mathbb{K}d), L_0^c(\mathbb{K}d)] \leq L_0^c(\mathbb{K}d)$ so H is a subalgebra. Let $V = L/H$. Then V is an H-module and $C_V(d) = 0$. In particular, the image d^* of d in $\mathrm{End}(V)$ is invertible. Also $N_L(H)/H \leq C_V(d) = 0$ and so $H = N_L(H)$. To complete the proof we may assume that H is not nilpotent and derive a contradiction. Let $D \leq H$ such that $d \in D$ and D is maximal with respect to acting nilpotently on H. Then $D \neq H$ and by 1.6.7 there exists $h \in N_H(D) \setminus D$. Since the number of eigenvalues for h^*d^{*-1} on V is at most $\dim V$ and since $|\mathbb{K}| \geq \dim L > \dim V$, there exists $k \in \mathbb{K}$ such that k is not an eigenvalue of h^*d^{*-1}. Then $h^*d^{-1} - k\mathrm{id}_V$ is invertible and so also $h^* - kd^*$ invertible. Put $l = h - kd$. Then $l \in N_H(D) \setminus D$ and $C_V(l) = 0$. Hence $V_0(^c)(\kappa L) = 0$. As $L_0^c(\mathbb{K}l) + H/H \leq V_0^c(\mathbb{K}L) = 0$ we conclude $L_c^0(\kappa L) \leq H$. The minimality of $H = L_0^c(\mathbb{K}d)$ implies that $H = L_0^c(\mathbb{K}l)$. Thus l acts nilpotently on H. From 1.6.3 we conclude that $D + \mathbb{K}l$ acts nilpotently on H, contradicting the maximal choice of D. □

Lemma 2.10.3 [**cartan decomposition**] *Let V be a standard L-module and N a nilpotent subideal in L. Then*

$$V = \bigoplus_{\lambda \in \Lambda_V(N)} V_\lambda^c$$

Moreover, each of the V_λ^c are L-submodule.

Proof: By 1.7.11(g), the V_λ^c are L-submodules. So it remains to prove the first statement. If V is the direct sum of two proper N-submodules, we may by induction assume that the lemma holds for both summands. But then it also holds for V. So we may and do assume

(*) V is not the direct sum of proper N-submodules.

Let $l \in N$. Since N is nilpotent 1.6.7 implies that $\mathbb{K}l$ is subideal in N. The Jordan Canonical Form of l shows that V is the direct sum of the generalized eigenspaces of l. But the generalized eigenspaces are just the generalized weight spaces of $\mathbb{K}l$. Hence 1.7.11(g) shows that the generalized eigenspaces are N-submodules. So by (*), l has a unique eigenvalue λ_l on V.

Let W be any N-composition factor on V. Then by 2.3.3 $W \cong K_\lambda$ for some weight λ of N. Then $\lambda(l)$ is an eigenvalue for l on V and $\lambda(l) = \lambda_l$. As $l \in N$ was arbitrary, λ is independent of the choice W and so $V = V_\lambda^c$. □

2.11 Perfect semsisimple standard Lie algebras

In this section we will investigate the structure of the perfect semisimple standard Lie algebras. For this we fix the following

Notation 2.11.1 [**not:semisimple**]

(a) [**a**] *L is a perfect, semisimple, standard Lie algebra.*

(b) [**b**] *H is a Cartan subalgebra of L.*

(c) [**c**] *$\Lambda = \Lambda_L(H)$ is the set of weights for H on L.*

(d) [**d**] *$\Phi = \Lambda \setminus \{0\}$. The non-zero weights for H on L are called the roots of H.*

(e) [**e**] *$f = f_L$, and $\perp$ is always meant with respect to f.*

Lemma 2.11.2 [**root decomposition**]

(a) [**a**] $L = \bigoplus_{\alpha\in\Lambda} L^c_\alpha$

(b) [**b**] $H = L^c_0$.

(c) [**c**] $[L^c_\alpha, L^c_\beta] \leq L^c_{\alpha+\beta}$ *for all* $\alpha, \beta \in \Lambda$.

Proof: (a) This follows from 2.10.3 applied with $V = L$,$L = H$ and $N = H$.

(b) By 1.7.11(e), $L_0(H)$ is the largest H-submodule of L, such that all compostion factors for H on $L_0(H)$ are trivial. Since H is nilpotent, all composition factors for H an H are trivial. Thus $H \leq L^c_0$. Suppose $H \neq L_0$ and let A/H be a simple submodule of L_0/H. By definition of L_0, A/H is a trivial module. Thus $[A, H] \leq H$, a contradiction since H is selfnormalizing..

(c) This follows from 2.3.6. □

Lemma 2.11.3 [**simple properties**] *Let $\alpha, \beta \in \Lambda$ and $h \in H$.*

(a) [**a**] *f is non-degenerate.*

(b) [**b**] $\mathrm{tr}_{L^c_\alpha}(h) = \alpha(h) \dim L^c_\alpha$.

(c) [**c**] *If $\alpha(h) \neq 0$, then $[h, L^c_\alpha] = L^c_\alpha$.*

(d) [**d**] *If $\beta \neq -\alpha$, $L^c_\alpha \perp L^c_\beta$.*

(e) [**e**] *$f|_H$ is non-degenerate.*

(f) [**f**] *H is abelian.*

(g) [**g**] $-\alpha \in \Lambda$.

(h) [**h**] $L_\alpha \neq 0$.

Proof: (a) : 2.7.11(a).

(b) Obvious.

(c) By 2.11.2(c), $[h, L^c_\alpha] \leq L^c_\alpha$. Since $\alpha(h) \neq 0$, $C_{L^c_\alpha}(h) = 0$ and so the action of h on L^c_α is invertible.

(d) By 2.3.5 $f(L^c_\alpha, L^c_\beta) \leq (\mathbb{K})^c_{\alpha+\beta}$. If $\alpha + \beta \neq 0$, $(\mathbb{K})^c_{\alpha+\beta} =$ and (d) holds.

(e) By (d), and 2.11.2(a), $L = H \bigoplus H^\perp$. So since f is non-degenerate, (e) holds.

(f) This follows from (e) and 2.7.4(c).

(g) Otherwise (d) would imply $L^c_\alpha \leq L^\perp = 0$.

(h) Follows from the definition of $\Lambda = \Lambda_L(H)$. □

Notation 2.11.4 [**ta for l**] *Since $f|_H$ is nondegenerate 2.6.1 yields a $\mathbb{K}$-isomorphism $t : H^* \to H, \alpha \to t_\alpha$ with $\alpha(h) = f(t_\alpha, h)$ for all $\alpha \in H^*, h \in H$. For $\alpha, \beta \in H^*$ and $h \in H$ define $q(h) = \frac{1}{2} f(h, h)$, $f^*(\alpha, \beta) = f(t_\alpha, t_\beta)$ and $q^*(\alpha) = q(t_\alpha)$. Recall the definition of $\check{h}$ and ω_α in 2.6.2. Note also that $\Lambda \subset \Lambda(H) = H^*$.*

Lemma 2.11.5 [**x,y**] *Let $\alpha \in \Phi$, $x \in L^c_\alpha$ and $y \in L_{-\alpha}$. Put $h = [x, y]$. Then*

(a) [**a**] $h \in H$.

(b) [**b**] *Let $\beta \in \Lambda$. Then there exists $q \in \mathbb{Q}$ with $\beta(h) = q\alpha(h)$.*

(c) [**c**] $h = 0$ *if and only if* $\alpha(h) = 0$.

(d) [**d**] $h = [x, y] = -f(x, y)t_\alpha$.

(e) [**e**] $h = 0$ *if and only if* $x \perp y$.

Proof: (a) follows from 2.11.2.

(b) Put $V := \sum_{n\in\mathbb{Z}} L^c_{\beta+n\alpha}$. By 2.11.2(c), V is invariant under x and y and so under h. We compute:

$$0 = \mathrm{tr}_V(h) = \sum_{n\in\mathbb{Z}} \mathrm{tr}_{L^c_{\beta+n\alpha}} h = \sum_{n\in\mathbb{Z}} (\beta(h) + n\alpha(h)) \dim L^c_{\beta+n\alpha}$$

and so

$$\beta(h) \sum_{n\in\mathbb{Z}} \dim L^c_{\beta+n\alpha} = -\alpha(h) \sum_{n\in\mathbb{Z}} n \dim L^c_{\beta+n\alpha}.$$

So (b) holds.

(c) Suppose $\alpha(h) = 0$, then by (b), $\beta(h) = 0$ for all $\beta \in \Lambda$. Hence h acts nilpotently on L. Since H is abelian we get $\mathbb{K}h \leq \mathrm{Nil}_H(L)$. Since $f|_H$ is non-degenerate, 2.7.4(b) implies $\mathrm{Nil}_H(L) = 0$. So $h = 0$.

(d) Let $a \in H$. Since $y \in L_{-\alpha}$, $[a, y] = -\alpha(a)y$. Since f is L-invariant we obtain:

$$f(h, a) = f([x, y], a)) = f(x, [y, a]) = -f(x, [a, y]) = -f(x, -\alpha(a)y) = \alpha(a)f(x, y)$$

On the otherhand,

$$f(f(x, y)t_\alpha, a) = f(x, y)f(t_\alpha, a) = f(x, y)\alpha(a) = \alpha(a)f(x, y)$$

Since $f\mid_H$ is non-degenerate, this implies $h = f(x, y)t_\alpha$.
(e) follows immediately from (d). □

Lemma 2.11.6 [**dim la**] *Let* $a \in \Phi$.

(a) [**a**] $L_\alpha = L^c_\alpha$ *is 1-dimensional.*

(b) [**c**] *Let* $n \in \mathbb{Z}$. *Then* $n\alpha \in \Phi$ *if and only if* $n = \pm 1$.

(c) [**b**] $[L_\alpha, L_{-\alpha}] = \mathbb{K}t_\alpha$.

(d) [**d**] $f(t_\alpha, t_a) \neq 0$.

Proof: Pick $0 \neq y \in L_{-\alpha}$. By 2.11.3(d), $L = y^\perp + L^c_\alpha$. Hence there exists $x \in L^c_\alpha$ with $x \not\perp y$. Put $h = [x, y]$. By 2.11.5(c) and (e), $h \neq 0$ and $\alpha(h) \neq 0$. Put

$$V := \mathbb{K}y \oplus H \oplus \bigoplus_{n\in\mathbb{Z}_+} L_{n\alpha}.$$

By 2.11.2(c), V is invariant under x. Since $y \in L_{-\alpha}$, $[y, H] \leq \mathbb{K}y$. Also $[y, y] = 0$ and so V is also invariant under y and h. Thus

$$0 = \mathrm{tr}_V(h) = -\alpha(h) + 0 + \sum_{n\in\mathbb{Z}_+} n\alpha(h) \dim L^c_{n\alpha}.$$

Since $\alpha(h) \neq 0$ we can divide by $\alpha(h)$ to obtain:

$$\sum_{n\in\mathbb{Z}_+} n \dim L^c_{n\alpha} = 1$$

Thus $\dim L^c_{n\alpha} = 0$ for $n > 1$ and $\dim L^c_\alpha = 1$. So (a) holds. Also (b) holds for positive n. Applying this result to $-\alpha$ we see that (b) holds. As L_α and $L_{-\alpha}$ are 1-dimensional, $[L_\alpha, L_{-\alpha}]$ is at most 1-dimensional. But $h = [x, y] \neq 0$ and so $[L_\alpha, L_{-\alpha}] = \mathbb{K}h$.

By 2.11.5(d)], $h = f(x, y)t_\alpha$ and hence (c) is proved.

Finally $0 \neq \alpha(h) = \alpha(f(x, y)t_\alpha) = f(x, y)\alpha(t_\alpha) = f(x, y)f(t_\alpha, t_\alpha)$ and so also (d) holds.
□

Lemma 2.11.7 [**xa =slw**] *Let $\alpha \in \Phi$. Define $H_\alpha = \mathbb{K}t_\alpha$, $X_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ and $h_\alpha = \check{t}_\alpha = \frac{2}{f(t_\alpha,t_\alpha)}t_\alpha$. Then X_α is a subalgebra of L, $X_\alpha \cong \mathfrak{sl}(\mathbb{K}^2)$ and $\alpha(h_\alpha) = 2$. More precisely, if $x_\alpha \in L_\alpha$ and $x_{-a} \in L_{-a}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$, then there exists an isomorphism form X_α to $\mathfrak{sl}(\mathbb{K}^2)$ with*

$$x \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h_\alpha \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proof: Note first that by 2.11.6(d), $f(t_\alpha, t_\alpha) \neq 0$, so h_α is well defined. By 2.11.6(c) we can choose $x_{\pm\alpha}$ as in the lemma. Now $\alpha(h_\alpha) = f(t_\alpha, h_\alpha) = f(t_\alpha, \frac{2}{f(t_\alpha,t_\alpha)}t_\alpha) = 2$ and so $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha$ and $[x_{-\alpha}, h_\alpha] = -[h_\alpha, x_{-\alpha}] = -(-\alpha(h_\alpha))y = 2x_{-\alpha}$ and so the lemma holds. □

Notation 2.11.8 [**def: a string**] *Let $\alpha \in \Phi$. We define an equivalence relation $\sim_\alpha$ on Λ by $\beta \sim_\alpha \gamma$ if $\beta - \gamma \in \mathbb{Z}\alpha$. We denote the set of equivalence classes by $\Lambda/\mathbb{Z}\alpha$. The equivalence classes for of $\sim_\alpha$ are called α-strings. If β, γ are in the same α-string we say that $\beta \leq_\alpha \gamma$ if $\gamma - \beta \in \mathbb{N}\alpha$. For $\beta \in \Phi$ let $\beta - r_{\alpha\beta}\alpha$ be the minimal and $\beta + s_{\alpha\beta}\alpha$ be the maximal element (with respect to $\leq_\alpha$) in the α-string through β. For an α string Δ define $L_\Delta = \sum_{\delta\in\Delta} L_\delta$,*

Lemma 2.11.9 [**xa on l**] *Let $\alpha \in \Phi$ and Δ an α-string.*

(a) [**a**] *L_Δ is an X_α-submodule and*

$$L = \bigoplus_{\Delta\in\Lambda/\mathbb{Z}\alpha} L_\Delta.$$

(b) [**b**] *Let $\beta \in \Delta$ and $i \in \mathbb{Z}$ with $i\alpha + \beta \in \Delta$. Then $L_{\beta+i\alpha}$ is the eigenspaces for h_α on L_Δ corresponding to the eigenvector $\beta(h_\alpha) + 2i$.*

(c) [**c**] *Suppose $\alpha \in \Delta$, then*

- *(a)* [**a**] $\Delta = \{-\alpha, 0, \alpha\}$.
- *(b)* [**b**] $L_\Delta = X_\alpha \oplus \ker\alpha$.
- *(c)* [**c**] $\ker\alpha = C_H(X_\alpha) = H \cap H_\alpha^\perp$.

(d) [**d**] *Suppose that $\alpha \notin \Delta$ and let $\beta \in \Delta$.*

- *(a)* [**b**] $\Delta = \{\beta + i\alpha \mid -r_{\alpha\beta} \leq i \leq s_{\alpha\beta}\}$
- *(b)* [**c**] *L_Δ is a simple X_α-module of dimension $|\Delta| = r_{\alpha\beta} + s_{\alpha\beta} + 1$.*
- *(c)* [**d**] $C_{L_\Delta}(L_\alpha) = L_{\beta+s_{\alpha\beta}\alpha}$.
- *(d)* [**e**] $[L_\alpha, L_\beta] = L_{\alpha+\beta}$

Proof: (a) follows immediately from 2.11.2.

(b) Since $\alpha(h_\alpha) = 2$, $L_{\beta+i\alpha}$ is contained in the eigenspace for h_α corresponding to $\beta(h_\alpha) + 2i$. Since the $\beta(h_\alpha) + 2i$, $i \in \mathbb{Z}$ are pairwise distinct we conclude that $L_{\beta+i\alpha}$ is the eigenspace corresponding to $\beta(h_\alpha) + 2i$.

(c) By 2.11.6(b) , $\Delta = \{\alpha, 0, -\alpha\}$. Since $\alpha(h_\alpha) = 2 \neq 0$, $H = \ker\alpha \oplus H_\alpha$ and so $L_\Delta = X_\alpha \oplus \ker\alpha$.

$$C_H(X_\alpha) = C_H(L_\alpha) \cap C_H(L_{-\alpha}) \cap C_H(H) = \ker\alpha \cap \ker -\alpha \cap H = \ker\alpha.$$

By definition of t_α, $f(t_\alpha, h) = \alpha(h)$ and so $\ker\alpha = H \cap t_\alpha^\perp = H \cap H_\alpha^\perp$. Thus all parts of (c) are proved.

(d) For an X_α-module I and $k \in \mathbb{K}$ let $d_\kappa(I)$ be the number of composition factor (in a given composition series) for H_α on I on which h_α acts by multiplication by k. Then

$$d_k(L_\Delta) = \sum_{W \in \mathrm{Comp}_{L_\Delta}(X_\alpha)} d_k(W).$$

Let $W \in \mathrm{Comp}_{L_\Delta}(X_\alpha)$ and let $m_W = \dim W - 1$. Then by 2.5.4

$$d_W(k) = \begin{cases} 1 & \text{if } k \text{ is an integer between } -m_W \text{ and } m_W \text{ with } k \equiv m_W \pmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

In particular, $d_W(0) + d_W(1) = 1$. Thus $d_{L_\Delta}(0) + d_{L_\delta}(1)$ is the number of composition factor for X_α on L_Δ. On the otherhand by (b) $d_{L_\Delta}(k) = 1$ if $k = \beta(h_\alpha) + 2i$ for some $i \in \mathbb{Z}$ such that $\beta + i\alpha$ is a root and $d_{L_\Delta}(k) = 0$ otherwise. Hence $d_{L_\Delta}(0) + d_{L_\delta}(1) \leq 1$. So there exists a unique composition factor for X_α on $\mathbb{L}_\Delta$. Hence L_Δ is simple and (d) follows from 2.5.4 □

Lemma 2.11.10 [**f(ta,hb)**] *Let $\alpha \in \Phi$, Δ an α-string in Λ and $\beta \in \Delta$.*

(a) [**a**] $\omega_\alpha(\beta) \in \Lambda$.

(b) [**b**] *Let $\Delta = \{\beta_0, \beta_1, \ldots \beta_k\}$ with $\beta_0 <_\alpha \beta_1 <_\alpha \ldots <_\alpha \beta_k$. Then $\omega_\alpha(\beta_i) = \beta_{k-i}$.*

(c) [**c**] $\beta(h_\alpha) = f^*(\beta, \check{\alpha}) = r_{\alpha\beta} - s_{\alpha\beta} \in \mathbb{Z}$.

Proof: If $\alpha \in \Delta$, this is readily verified. So suppose $\alpha \notin \Delta$. Let $i = \beta(h_\alpha)$. Then i is an eigenvalue for h_α on L_Δ and so by 2.5.4 also $-i$ is an eigenvalue. Since $\alpha(h_\alpha) = 2$ we have $(\beta - i\alpha)(h_\alpha) = -i$ and we conclude from 2.11.9(b) that $\beta - i\alpha \in \Delta$. Also

$$i = \beta(h_\alpha) = f(t_\beta, h_\alpha) = f^*(\beta, \check{\alpha})$$

and so

$$\beta - i\alpha = \beta - f^*(\beta, \check{\alpha})\alpha = \omega_\alpha(\beta)$$

Thus (a) holds.

(b) follows easily from the proof of (a).

(c) From (b), $\omega_\alpha(\beta + s_{\alpha\beta}\alpha) = \beta - r_{\alpha\beta}\alpha$. Hence

$$\beta + s_{\alpha\beta}\alpha - f^*(\beta + s_{\alpha\beta}\alpha, \check{\alpha})\alpha = \beta - r_{\alpha\beta}\alpha$$
$$s_{\alpha\beta} - \beta(h_\alpha) - 2s_{\alpha\beta} = -r_{\alpha\beta}$$
$$r_{\alpha\beta} - s_{\alpha\beta} = \beta(h_\alpha) = f^*(\beta, \check{a}).$$

□

Lemma 2.11.11 [**h=sum ha**] $H = \sum_{\alpha\in\Phi} H_\alpha$.

Proof: Let $h \in H$ with $h \perp H_\alpha = 0$ for all $\alpha \in \Phi$. Then $\alpha(h) = 0$ for all $\alpha \in \Lambda$. So $[h, L_\alpha] = 0$ and $h \in Z(L) = 0$. Thus $h = 0$. Since $f|_H$ is non-degenerate and H is finite dimensional the lemma now follows from 2.6.1(c). □

Lemma 2.11.12 [**q rational**]

(a) [**a**] *$f^*(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$.*

(b) [**b**] *The restriction of $f^*_{\mathbb{Q}}$ of f^* to $\mathbb{Q}\Phi$ is a positive definite symmetric $\mathbb{Q}$-bilinear form on $\mathbb{Q}\Phi$.*

(c) [**c**] *Any $\mathbb{Q}$-basis of $\mathbb{Q}\Phi$ is a $\mathbb{K}$-basis of H^*.*

Proof: Let $h \in H$. Then $f(h,h) = \mathrm{tr}_L(h^2)$. Since h acts trivially on H and acts as $\beta(h)$ on the 1-dimensional space L_β we have

(1) $$f(h,h) = \sum_{\beta\in\Phi} \beta(h)^2.$$

Since $t_\alpha \in H_\alpha = [L_a, L_a]$ we can apply 2.11.5(b) to $h = t_\alpha$. So there exists $q_\beta \in \mathbb{Q}$ such that

(2) $$f(t_\beta, t_\alpha) = \beta(t_\alpha) = q_\beta\alpha(t_\alpha) = q_\beta f(t_\alpha, t_\alpha)$$

Plucking (2) into (1) with $h = t_\alpha$ we obtain

$$f(t_\alpha, t_a) = \sum_{\beta\in\Phi} q_\beta^2 f(t_\alpha, t_\alpha)^2.$$

Since $(ft_\alpha, t_\alpha)) \neq 0$ we can divided by $f(t_\alpha, t_\alpha)$ to conclude $f(t_\alpha, t_\alpha) \in \mathbb{Q}$. From (2) we get $f^*(\beta, \alpha) = f(t_\beta, t_\alpha) \in \mathbb{Q}$. Put $H_{\mathbb{Q}} = \sum_{\alpha\in\Phi} \mathbb{Q}t_\alpha$ and note that $H_{\mathbb{Q}} = t(\mathbb{Q}\Phi)$. Then $f(h, h') \in \mathbb{Q}$ for all $h, h' \in H_{\mathbb{Q}}$. Thus also $\beta(h) = f(t_\beta, h) \in \mathbb{Q}$ for all $b \in \Phi$, $h \in H_{\mathbb{Q}}$. (1) now implies that $f(h,h) \geq 0$. Suppose $f(h,h) = 0$ then $\beta(h) = 0$ for all $\beta \in \Phi$. 2.11.11 shows that $h \in H \cap H^\perp = 0$. So $f\mid_{H_{\mathbb{Q}}}$ is positive definite and so (b) holds.

Let $\mathcal{B}$ be a $\mathbb{Q}$-basis for $H_{\mathbb{Q}}$. By 2.11.11, $\mathbb{K}\mathcal{B} = H$ and so $\mathcal{B}$ contains a $\mathbb{K}$-basis $\mathcal{D}$ for H. Let $h \in H_{\mathbb{Q}}$ with $h \perp \mathbb{Q}\mathcal{D}$. Then $H = \mathbb{K}\mathcal{D} \leq h^\perp$. Hence $h = 0$ and by (b) and 2.6.1(c), $\mathbb{Q}\mathcal{D} = H_{\mathbb{Q}}$. So $\mathcal{B} = \mathcal{D}$ and (c) is proved.

Chapter 3

Rootsystems

3.1 Definition and Rank 2 Rootsystems

Throughout this chapter $\mathbb{F}$ is a subfield of $\mathbb{R}$, E a finite dimensional vector space over $\mathbb{F}$ and $(\cdot,\cdot)$ a positive definite symmetric bilinear form on E. $E^\sharp = E \setminus \{0\}$.

Definition 3.1.1 [**def:root system**] *A subset Φ of $E^\sharp$ is called a root system in E provided that for all $\alpha, \beta \in \Phi$:*

(i) [**a**] $\omega_\beta(\alpha) \in \alpha + \mathbb{Z}\beta$ *(that is $(\alpha, \check{\beta}\,) \in \mathbb{Z}$)*

(ii) [**b**] $\omega_\alpha(\beta) \in \Phi$.

(iii) [**c**] $E = \mathbb{F}\Phi$

(iv) [**d**] $\mathbb{F}\alpha \cap \Phi = \{\alpha, -\alpha\}$.

If only (i) to (iii) hold, then Φ is called a weak root system. If only (i) holds then Φ is called a pre-root system.

Note that by 2.11.10, 2.11.11 and 2.11.12 the non-zero weights of a perfect semisimple standard Lie algebra are a root system in the dual of the Cartan subalgebra. The purpose of this chapter is determine all the roots system up to ismomophism. This will be used in later chapters to complete the classifications of the perfect semisimple standard Lie algebras.

Throughout this chapter Φ denotes a weak root system in E.

Definition 3.1.2 [**def:weyl groups**] *Let $\Delta \subseteq E^\sharp$ and $\alpha, b \in \mathbb{E}^\sharp$.*

(a) [**a**] *$W(\Delta)$ is the subgroups of the isometry group of $(\cdot,\cdot)$ generated by the $\omega_\alpha, \alpha \in \Delta$.*

(b) [**b**] *$W = W(\Phi)$ is called the Weyl group of Φ.*

(c) [**c**] $\langle\Delta\rangle = \bigcup \Delta^{W(\Delta)} = \{w(\delta) \mid w \in W(\Delta), \delta \in \Delta\}$.

(d) [**d**] *$\vartheta_{\alpha\beta}$ is the angle between α and β, that is the real number θ with $0 \leq \theta \leq 180$ and* $\cos\theta = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}}$.

(e) [**e**] $m_{\alpha\beta} = (\alpha,\check{\beta})(\beta,\check{\alpha})$.

Lemma 3.1.3 [**rank 2 root**] *Let $\alpha,\beta \in E^\sharp$. Then*

(a) [**a**] $\cos^2\vartheta_{\alpha\beta} = \frac{1}{4}m_{\alpha\beta}$.

(b) [**b**] $0 \leq m_{\alpha\beta} \leq 4$.

(c) [**c**] *If $\alpha \not\perp \beta$ then* $\frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{(\alpha,\check{\beta})}{(\beta,\check{\alpha})}$.

(d) [**d**] $\mathbb{F}a = \mathbb{F}b$ *iff* $m_{\alpha\beta} = 4$. *In this case* $\alpha = \frac{1}{2}(\alpha,\check{\beta})\beta$.

Proof:

$$\cos^2\vartheta_{\alpha\beta} = \frac{(\alpha,\beta)(\beta,\alpha)}{(\alpha,\alpha)(\beta,\beta)} = \frac{1}{4}(\alpha,\check{\beta})(\beta,\check{\alpha}) = \frac{1}{4}m_{\alpha\beta}$$

and so (a) holds. (b) follows immediately from (a). (c) follows easily from the definition of $\check{\alpha}$. For (d) note that $\mathbb{F}a = \mathbb{F}\beta$ iff $\vartheta_{\alpha\beta} \in \{0^\circ, 180^\circ\}$, that is iff $\cos^2\vartheta_{\alpha\beta} = 1$. By (a) this holds iff $m_{\alpha\beta} = 4$. Suppose now that $\mathbb{F}\alpha = \mathbb{F}b$. Then $(\frac{1}{2}(\alpha,\check{\beta})\beta,\check{\alpha}) = \frac{1}{2}(\alpha,\check{\beta})(\beta,\check{\alpha}) = 2 = (\alpha,\check{\alpha})$ and so $\alpha = \frac{1}{2}(\alpha,\check{\beta})\beta$ □

Lemma 3.1.4 [**sab**] *Let $\{\alpha,\beta\} \in E^\sharp$ with $(\alpha,\check{\beta}) \in \mathbb{Z}$ and $(\beta,\check{\alpha}) \in \mathbb{Z}$ and $(\alpha,\alpha) \geq (\beta,\beta)$. Then one of the following holds:*

$(\alpha,\check{\beta})$	$(\beta,\check{\alpha})$	$\cos\vartheta_{\alpha\beta}$	$\vartheta_{\alpha\beta}$	$\frac{(\alpha,\alpha)}{(\beta,\beta)}$	
0	0	0	90°	?	
1	1	$\frac{1}{2}$	60°	1	
−1	−1	$-\frac{1}{2}$	120°	1	
2	1	$\frac{1}{\sqrt{2}}$	45°	2	
−2	−1	$-\frac{1}{\sqrt{2}}$	135°	2	
3	1	$\frac{\sqrt{3}}{2}$	30°	3	
−3	−1	$-\frac{\sqrt{3}}{2}$	150°	3	
2	2	1	0°	1	$\alpha = \beta$
−2	−2	−1	180°	1	$\alpha = -\beta$
4	1	1	0°	4	$\alpha = 2\beta$
−4	−1	−1	180°	1	$\alpha = -2\beta$

In particular, if $\mathbb{F}\alpha \neq \mathbb{F}\beta$, then $|(\beta,\check{\alpha})| = 1$.

Proof: Note that $|(\alpha,\check{\beta})| \geq |(\beta,\check{\alpha})|$. This follows easily from 3.1.3 □

Definition 3.1.5 [**def:discret**] *For $D \subseteq E$ define*

$$\text{min-d}(D) = \inf\{(e-d, e-d) \mid d \neq e \in D\}$$

and

$$\text{max-d}(D) = \sup\{(e-d, e-d) \mid d \neq e \in D\}.$$

We say that D is discret *if* $\text{min-d}(D) > 0$ *and that D is* bounded *if* $\text{max-d}(D) < \infty$

Lemma 3.1.6 [**discret**] *Let Δ be linear indepdendent subset of E. Then $\mathbb{Z}\Delta$ is discret.*

Proof: Since $\mathbb{Z}Delta$ is closed under subtraction we need to show that $\inf_{t\in\mathbb{Z}\Delta^\sharp}(t,t) > 0$.

Let $d \in \Delta$ and put $\Sigma = \Delta \setminus d$. For $e \in E$ define $f_e \in \mathbb{F}$ and $\tilde{e} \in d^\perp$ by $e = f_e d + \tilde{e}$. Then $\tilde{\Sigma}$ is a linear indepedent. By induction on Δ, $m := \inf_{s\in\mathbb{Z}\Sigma^\sharp}(s,s) > 0$. Let $0 \neq t \in \mathbb{Z}\Delta$. If $\tilde{t} \neq 0$, then $(\tau,\tau) \geq (\tilde{t},\tilde{t}) \geq m$. If $\tilde{t} = 0$, then $t \in \mathbb{F}d \cap \mathbb{Z}\Delta = \mathbb{Z}d$ and so $(t,t) \geq (d,d)$. □

Lemma 3.1.7 [**discret and bounded**] *Let $\Delta \subseteq E$ be discret and bounded. Then $|\Delta|$ is finite and bounded by a function of $\frac{\text{max-d}(D)}{\text{min-d}(D)}$ and* $\dim E$.

Let $l = \text{min-d}(D)$, $u = \text{max-d}(D)$ and $n = \dim \mathbb{E}$. Let $\mathbb{E}_1$ be a 1-dimensional supspace of $\mathbb{E}$ and put $E_2 = E_1^\perp$. Let π_i the the projection of $\mathbb{E}$ onto $\mathbb{E}_i$. Let D_1 be a subset of $\pi_1(D)$ with $(d,e) \geq \frac{l}{4}$ for all $d,e \in D_1$. Since $(d,e) \leq u$ for all $d,e \in \pi_1(D)$ we have

$$(*) \quad |D_1| \leq \frac{4u}{l}$$

In particular, we can choose a maximal such D_1. For $e \in D_1$ let $D(e) = \{d \in D \mid (\pi_1(d), e) < \frac{l}{4}$. The maximality of D_1 implies that

$$(**) \quad D = \bigcup_{e\in D_1} D(e)$$

Now let $e, f \in D(e)$. Then $(\pi_1(e), \pi_1(f)) \leq \frac{l}{2}$ and so $(\pi_2(e), \pi_2(f)) \geq \frac{l}{2}$. In particular, $\pi_2 \mid_{D(e)}$ is $1-1$ and by induction on n, $|\pi_2(D(e))|$ is bounded by a function of $\frac{2u}{l}$ and $n-1$. (*) and (**) now imply the lemma. □

Lemma 3.1.8 [**finite**] *Let Ψ be pre-root system in $\mathbb{E}$. Then $\mathbb{Z}\Psi$ is discret and Ψ is finite.*

Proof: Since E is finite dimensional and $E = \mathbb{F}\Psi$ there exists a finite subset Δ of Ψ with $\mathbb{F}\Delta = \mathbb{F}\Psi$. Let Σ be the basis of $\mathbb{E}$ dual to $\check{\Delta}$. Since $(\check{d}, \psi) \in \mathbb{Z}$ for all $\psi \in \Psi$ we have $\Psi \subseteq \mathbb{Z}\Sigma$. Hence also $\mathbb{Z}\Psi \leq \mathbb{Z}\Sigma$. By 3.1.6, $\mathbb{Z}\Sigma$ is discret and so also Ψ and $\mathbb{Z}\Psi$ are discret.

Put $u = \max_{\delta\in\Delta}(\delta,\delta)$ $\alpha \in \Psi$. Then there exists $\delta \in \Delta$ with $(\alpha,\delta) \neq 0$ and so by 3.1.4

$$(\alpha,\alpha) \leq 4(\delta,\delta) \leq 4u$$

thus Ψ is bounded. So by 3.1.7 Ψ is finite.

Lemma 3.1.9 [**basic ¡¿**] *Let $\Delta \subseteq E^\sharp$.*

(a) [**a**] $W(\langle\Delta\rangle) = W(\Delta)$.

(b) [**b**] $W(\Delta) = W(\check{\Delta})$.

(c) [**c**] $\langle\Delta\rangle\check{} = \langle\check{\Delta}\rangle$.

Proof: Put $\Psi = \langle\Delta\rangle$. Clearly $W(\Delta) \subseteq W(\Psi)$. Let $\alpha \in \Psi$. Then $\alpha = w(\beta)$ for some $w \in W(\Delta)$ and $\beta \in \Delta$. Then by 2.6.3(e) $\omega_\alpha = w\omega_\beta w^{-1} \in W(\Delta)$ and so $W(\Psi) \subseteq W(\Delta)$. Thus (a) holds. (b) follows from $\omega_\alpha = \omega_{\check{\alpha}}$ and (c) follows from $w(\check{\alpha}) = w(\check{\alpha})$. □

Lemma 3.1.10 [**basic root**]

(a) [**a**] *$\check{\Phi}$ is a weak root system in E. If Φ is a root system, so is $\check{\Phi}$.*

(b) [**b**] *Φ is W-invariant, that is $w(\Phi) = \Phi$ for all $w \in W$.*

(c) [**c**] *W acts faithfully on Φ. In particular, Φ is finite.*

Proof: (a) follows immediately from 2.6.1 and the definition of a root system.

(b) Put $T = \{w \in GL(E) \mid w(\Phi) = \Phi)\}$. Let $\alpha \in \Phi$. Note that $\omega_\alpha(\Phi) \subseteq \Phi$ and since $\omega_\alpha^2 = 1$

$$\Phi = \omega_\alpha(\omega_\alpha(\Phi)) \subseteq \omega_\alpha(\Phi) \subseteq \Phi$$

Thus $\omega_\alpha \in T$. As T is a subgroup of $GL(E)$, we have conclude $W \leq T$.

(c) Let $w \in W$ with $w(\alpha) = \alpha$ for all $\alpha \in \Phi$. Since Φ spans E we get $w(e) = 1$ fro all $e \in E$ and so $w = 1$. Hence W acts faithfully on Φ and so the homomorphism from W to $\mathrm{Sym}(\Phi)$ is one to one. By 3.1.8, Φ is finite. Therefore also $\mathrm{Sym}(\Phi)$ and W are finite. □

Definition 3.1.11 [**def:span**] *Let $\Psi \subseteq \Phi$ and R a subring of $\mathbb{F}$ (with $1 \in R$ and so $\mathbb{Z} \leq R$). Then*

(a) [**a**] *Ψ is called a (weak) root subsystem of Φ if Ψ is a (weak) root system in $\mathbb{F}\Psi$.*

(b) [**b**] *Ψ is called R-closed if $\Psi = R\Psi \cap \Phi$.*

(c) [**d**] *$\langle\Psi\rangle_R$ denotes the smallest R-closed subset of Φ containing Ψ. $\langle\Psi\rangle_R$ is called the R-closure of Ψ.*

We often just say “subsystem” for “weak root subsystem”. Note that of Φ is a root subsystem and Ψ a weak roots subsytem then Ψ is already a root subsystem.

Lemma 3.1.12 [**closure**] *Let $\Psi \subseteq \Phi$ and R a subring of $\mathbb{F}$.*

(a) [**a**] *Ψ is a subsystem iff $\omega_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.*

(b) [**b**] *Ψ is a subsystem iff Ψ is invariant under $W(\Psi)$.*

(c) [**c**] *$\langle\Psi\rangle \subseteq \Phi$ and Ψ is the smallest subsystem of Φ containing Ψ.*

(d) [**d**] *Let T be an R-submodule in E. Then $H \cap \Phi$ is an R-closed subsystem of Φ.*

(e) [**e**] *$\langle\Psi\rangle_R$ is a subsystem of Φ and $\langle\Psi\rangle \subseteq \langle\Psi\rangle_R = \Phi \cap R\Psi \subseteq R\Psi$.*

Proof: (a) The forward direction is obvious. For the backward let $\alpha, \beta \in \Psi$. Then $-\alpha = \omega_\alpha(\alpha) \in \Psi$ and all the axiom of a root system are fulfilled.

(b) follows from (a).

(c) Let Σ be a any root subsystem of Φ containing Ψ. Since Σ is invariant under $W(\Sigma)$, $w(\psi) \in \Sigma$ for all $w \in W(\Psi) \leq W(\Sigma)$ and $\psi \in \Psi$. Thus $\langle\Psi\rangle \subseteq \Sigma$. In particular, $\langle\Psi\rangle \subseteq \Phi$.

By definition of $\langle\Psi\rangle$, $\langle\Psi\rangle$ is invariant under $W(\Psi)$. By 3.1.9 $W(\Psi) = W(\langle\Psi\rangle)$ and so $\langle\Psi\rangle$ is invariant under $W(\langle\Psi\rangle)$. Thus by (b), $\langle\Psi\rangle$ is a root subsystem.

(d) Let $\alpha, \beta \in T \cap \Phi$. Then $\omega_\beta(\alpha) = \alpha - (\alpha, \check{\beta})\beta \in \alpha + \mathbb{Z}\beta \in R\alpha + R\beta \leq T$ and so $\omega_\alpha(\beta) \in T \cap \Phi$. So by (a), $T \cap \Phi$ is a subsystem of Φ. Clearly $T \cap \Phi$ is R-closed and so (d) holds.

(e) Follows from (d) applied to $T = R\Psi$. □

Lemma 3.1.13 [**creating root systems**] *Let $\Delta \subseteq E^\sharp$.*

(a) [**a**] *$\Sigma := \{\sigma \in \mathbb{Z}\Delta^\sharp \mid (\delta, \check{\sigma}) \in \mathbb{Z} \forall \delta \in \Delta\}$ is a weak roots system in $\mathbb{F}\Sigma$.*

(b) [**c**] *Suppose Δ is a pre-root system. Then $\langle\Delta\rangle$ is a weak root subsystem of Σ.*

(c) [**d**] *Suppose Δ is linearly independent pre-root system. Then $\langle\Delta\rangle$ is a root system in $\mathbb{F}\Delta$*

Proof: (a) Let $\alpha, \beta \in \Sigma$.

1° [**1**] $(\sigma, \check{\alpha})\beta \in \mathbb{Z}$.

Since $\alpha \in \Sigma \subseteq \mathbb{Z}\Delta$, $\alpha = \sum_{\delta\in\Delta} n_\delta d$ for some $n_\delta \in \mathbb{Z}$, almost all 0. Since $\beta \in \Sigma$, $(\delta, \check{\beta}) \in \mathbb{Z}$ for all $\delta \in \Delta$. So

$$(\alpha, \check{\beta}) = \sum_{\delta\in\Delta} n_\delta (\alpha, \check{\beta}) \in \mathbb{Z}.$$

2° [**2**] $\omega_\beta\alpha \in \Sigma$.

By (1°), $(\alpha, \check{\beta}) \in \mathbb{Z}$ and so $\omega_\beta(\alpha) = \alpha - (\alpha, \check{\beta})\beta \in \mathbb{Z}\Delta$. Let $\delta \in \Delta$. Then

$$(\delta, \omega_\beta(\alpha)\check{}\,) = (\delta, \omega_\beta(\check{\alpha})\,) = (\omega_\beta(\delta), \check{\alpha}\,) = (\delta - (\delta, \check{\beta}\,)\beta, \check{\alpha}\,) = (\delta, \check{\alpha}\,) - (\delta, \check{\beta}\,)(\beta, \check{\alpha}\,) \in \mathbb{Z}.$$

Note that (1°) and (2°) imply (a).

(b) Since Δ is a pre-root system $\Delta \subseteq \Sigma$. So by 3.1.12(b), Ψ is a weak root subsystem of Σ.

(c) By (b), Ψ a weak root system and $\Psi \subseteq \Sigma \subseteq \mathbb{Z}\Delta$. Let $n \in \mathbb{F}$ and $\alpha \in \Psi$ with $n\alpha \in \Psi$. We need to show that $n = \pm 1$. Since α is conjugate under $W(\Delta)$ to some element in Δ we may assume that $\alpha \in \Delta$. As $n\alpha \in \Sigma \subseteq \mathbb{Z}\Delta$, $n\alpha = \sum_{\delta\in\Delta} n_\delta \delta$ for some $n_\delta \in \mathbb{Z}$. Since Δ is linearly independent $n = n_\alpha \in \mathbb{Z}$.

By 3.1.9 $\check{\Psi} = \langle \check{\Delta} \rangle$.

Also $\check{n\alpha} = \frac{1}{n}\check{\alpha}$ and a symmetric result shows $\frac{1}{n} \in \mathbb{Z}$. Thus $n = \pm 1$. □

Definition 3.1.14 [**def:a string 2**] *Let $\alpha, \beta \in \Phi$. Then $\Delta = (\beta + \mathbb{F}\alpha) \cap \Phi$ is called the α-sring through β. Define a total ordering $\leq_\alpha$ on Δ by $\gamma \leq_\alpha \delta$ if $\delta - \gamma \in \mathbb{F}^{\geq 0}\alpha$. Let $\beta - r_{\alpha\beta}\alpha$ and $\beta + s_{\alpha\beta}\alpha$ be the minimal and maximal element in Δ with respect to $\leq_\alpha$.*

Lemma 3.1.15 [**a string**] *Suppose Φ is a root system, $\alpha, \beta \in \Phi$ and Δ is the α-string through β.*

(a) [**d**] *Suppose $\alpha \neq \pm\beta$. If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$ and if $(\alpha, \beta) > 0$, $\alpha - \beta \in \Phi$.*

(b) [**a**] *ω_α leaves Δ invariant and reverses the $\prec_\alpha$ ordering. So if $\Delta = \{\beta_0, \beta_1, \dots \beta_k\}$ with $\beta_0 <_\alpha \beta_1 <_\alpha \dots <_\alpha \beta_k$, then $\omega_\alpha(\beta_i) = \beta_{k-i}$.*

(c) [**b**] *If $\beta = \pm\alpha$ then $\Delta = \{\pm\alpha\}$. Otherwise*

$$\Delta = \{\beta + i\alpha \mid -r_{\alpha\beta} \leq i \leq s_{\alpha\beta}, i \in \mathbb{Z}\}$$

In particular, $r_{\alpha\beta}$ and $s_{\alpha\beta}$ are integers.

(d) [**c**] *$(\beta, \check{\alpha}) = r_{\alpha\beta} - s_{\alpha\beta}$.*

Proof: (a) Suppose that $(\alpha, \beta) < 0$. Without loss $(\alpha, \alpha) \geq (\beta, \beta)$. Then by 3.1.4 $(\beta, \check{\alpha}) = -1$. Thus $\beta + \alpha = \omega_\alpha(\beta) \in \Phi$. The second statement follows from the first applied to α and $-\beta$.

(b) Let $\delta \in \Delta$. Then $\omega_\alpha(\delta) = \delta + (\delta, \check{\alpha})\alpha \in \beta + \mathbb{F}\alpha$ and so $\omega_\alpha(\delta) \in \Delta$. If $\gamma \in \Delta$ with $\gamma \leq \delta$, then $\delta = \gamma + f\alpha$ for a nonnegative $f \in \mathbb{F}$. Thus $\omega_\alpha(\delta) = \omega_\alpha(\gamma) - f\alpha$ and so $\omega_\alpha(\delta) \leq \omega_\alpha(\gamma)$.

(c) The case $\beta = \pm\alpha$ is obvious. So suppose $\alpha \notin \Delta$. Without loss $\beta = \beta_0$. Then $r_{\alpha\beta} = 0$. Let $f \in \mathbb{F}$ with $0 \leq f \leq s_{\alpha\beta}$. We need to show that $\delta := \beta + f\alpha \in \Phi$ iff $f \in \mathbb{Z}$. Since $\omega_\alpha(\beta) = b_k$ we have $b_k = \beta + s_{\alpha\beta}\alpha$ and so $\omega_\alpha(\delta) = \beta + (s_{\alpha\beta} - f)\alpha$. So replacing δ by $\omega_\alpha(\delta)$ if necessary we may assume that $(\delta, \alpha) \leq 0$.

Pick $i \in \mathbb{N}$ maximal with $i \leq f$ and $\gamma := \beta + i\alpha \in \Phi$. Put $k = f - i$. Then $\delta = \gamma + k\alpha$ and $k \geq 0$. If $k = 0$ then $f \in \mathbb{Z}$ and $\delta \in \Phi$. So we may assume that $k > 0$. Then $(\gamma, \alpha) < (\delta, \alpha) \leq 0$ and so by (a) $\gamma + \alpha \in \Phi$. The maximality of i shows $i + 1 > f$ and so $k < 1$. It remains to show that $\delta \notin \Phi$. Suppose for a contradiction that $\delta \in \Phi$. Then

$(\delta, \check{\alpha}) = (\gamma, \check{\alpha}) + 2k$. As $0 < k < 1$ and both $(\delta, \check{\alpha})$ and $(\gamma, \check{\alpha})$ are integers this implies $k = \frac{1}{2}$. Hence $(\delta, \check{\gamma}) = 2 + \frac{1}{2}(\alpha, \check{\gamma})$. Thus $(\alpha, \check{\gamma})$ is even. Since $(\alpha, \check{\gamma}) < 0$ and we conclude from 3.1.4 that $(\alpha, \check{\gamma}) = -2$. Hence

$$\omega_\gamma(\alpha) = \alpha + 2\gamma = 2(\gamma + \frac{1}{2}\alpha) = 2\delta$$

and we obtained a contradiction to the definition of a roots system.

(d) Same proof as for 2.11.10(c). □

Definition 3.1.16 [**def rank**]

(a) [**a**] *The rank of Φ is the minimal size of subset Δ of Φ with $\Phi = \langle \Delta \rangle$.*

(b) [**b**] *Φ is called disconnected if it is the disjoint union of two proper perpendicular subsets. Otherwise, Φ is called connected.*

Let Φ be a connected rank two roots system and choose α, β with $\Phi = \langle \alpha, \beta \rangle$. If $\alpha \perp \beta$ then $\Phi = \{\pm\alpha\} \cup \{\pm\beta\}$ and Φ is disconnected. Also $\beta \neq \pm\alpha$ since otherwise $\Phi = \langle \alpha \rangle$ as rank 1. Using 3.1.4 and 3.1.12(c) one now easily obtains a complete list of connected rank 2 root systems. See Figure 3.1.

3.2 A base for root systems

Definition 3.2.1 [**def:base**]

(a) [**a**] *A subset Π of Φ is called base for Φ provided that Π is an $\mathbb{F}$-basis for E and $\Phi = \Phi^+ \cap \Phi^-$ where for $\Phi^+ = \mathbb{N}\Pi \cap \Phi$ and $\Phi^- = -\Phi^+$.*

(b) [**b**] *Let Π be a base for Φ. The elements of Π are called simple roots and the element of Φ^+ are called positive roots. For $e = \sum_{\alpha \in \Pi} f_\alpha \alpha$ define $\operatorname{ht} e = \sum_{\alpha \in Pi} f_\alpha$. $\operatorname{ht} e$ is called the height of e with respect to the base Π.*

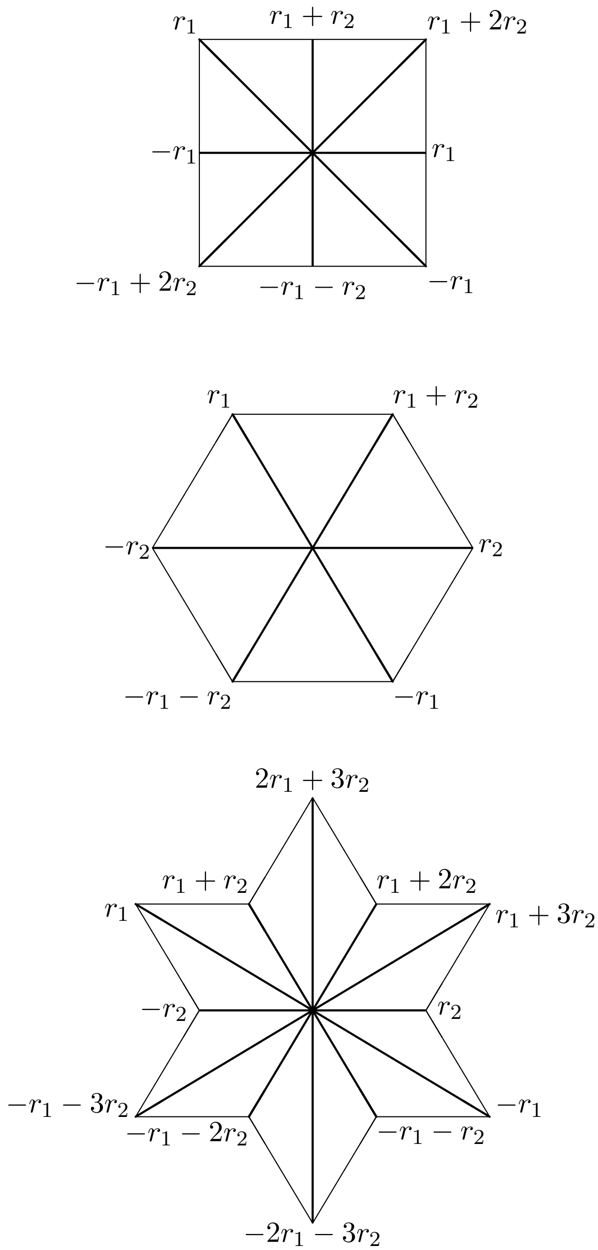
In this section we show that Φ has a base and that any two base are conjugate under W.

Lemma 3.2.2 [**no finite cover**] *Let V be an finite dimensional vector sapce over an infinite field $\mathbb{K}$ and let $\mathcal{H}$ a finite set of proper subspace of V. Then $V \neq \bigcup \mathcal{H}$.*

By induction $\dim V$. Each $H \in \mathcal{H}$ lies in a hyperplane $\tilde{H}$ of V. Since K is infinite there exists infintely many hyperplane in V. So we can choose a hyperplane W of V with $W \neq \tilde{H}$ for all $H \in \mathcal{H}$. Then $W \neq W \cap H$ and so by induction there exists $w \in W$ with $w \notin W \cap H$ for all $H \in \mathcal{H}$. Thus $w \notin \bigcup \mathcal{H}$. □

[**rank2**]

Figure 3.1: The connected Rank 2 Root Systems



Definition 3.2.3 [**def:regular**] *$e \in E$ is called regular if $(\alpha, e) \neq 0$ for all $\alpha \in \Phi$.*

Lemma 3.2.4 [**not perp**] *Let S be finite subset of $E \setminus \{0\}$. Then there exists $e \in E$ with $(s, e) \neq 0$ for all $s \in S$. In particular, there exist regular elements in E.*

Proof: By 3.2.2 $V \neq \bigcup_{s \in S} \alpha^\perp$. □

Lemma 3.2.5 [**s linear indep**] *Let S be a finite subset of V and $e \in E$. Suppose that*

$$(s, e) > 0 \quad \textit{and} \quad (s, t) \leq 0$$

for all $s \neq t \in S$ and $e \in E$. Then S is linearly independent.

Proof: Let $f_s \in \mathbb{F}$ with $\sum_{s \in S} f_s s = 0$. Let $S_+ = \{s \in S \mid f_s > 0\}$ and $S_- = S \setminus S_+$. Then

$$u := \sum_{s_+ \in S^+} f_{s_+} s_+ = \sum_{s_- \in S_-} (-f_{s_-}) s_-$$

and

$$0 \leq (u, u) = \sum_{s_+ \in S_+} \sum_{s_- \in S_-} (-f_{s_+} f_{s_-})(s_+, s_-) \leq 0.$$

Therefore $u = 0$ and $0 = (u, e) = \sum_{s_+ \in S_+} f_{s_+}(s_+, e) \geq 0$. Hence $f_{s_+} = 0$ for all $s_+ \in S_+$. By symmetry, $f_{s_-} = 0$ for all $s_- \in S_-$ and so S is linearly independent. □

Proposition 3.2.6 (Existence of Bases) [**existence of bases**] *Let $e \in E$ be regular. with $(\alpha, e) \neq 0$ for all $\alpha \in \Phi$. Put $\Phi_e^+ = \{\alpha \in \Phi \mid (\alpha, e) > 0\}$ and $\Pi_e = \Phi_e^+ \setminus (\Phi_e^+ + \Phi_e^+)$. Then Π_e is a base for Φ and $\Phi^+(\Pi_e) = \Phi_e^+$.*

Proof: Let $\alpha, \beta \in \Pi_e$. Since $\alpha = (\alpha - \beta) + \beta$ we have that $\alpha - \beta \notin \Phi_e^+$. Also $\beta = (\beta - \alpha) + \alpha$ and so $\beta - \alpha \notin \Phi_e^+$. So $\alpha - \beta \notin \Phi$ and by 3.1.15(a), $(\alpha, \beta) \leq 0$. Thus by 3.2.5 Π_e is linearly independent.

Let $\alpha \in \Phi_e^+$. We will show by induction on (α, e) that $\alpha \in \mathbb{N}\Pi$. If $\alpha \in \Pi_e$, this is obvious. So suppose $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Phi_e^+$. Then $(\alpha, e) = (\beta, e) + (\gamma, e)$, $(\beta, e) < (\alpha, e)$ and $(\gamma, e) < (\alpha, e)$. So by induction $\beta \in \mathbb{N}\Pi$, $\gamma \in \mathbb{N}\Pi$ and so also $\alpha \in \mathbb{N}\Pi$.

Hence $\Phi_e^+ = \mathbb{N}\Pi_e \cap \Phi = \Phi^+(\Pi_e)$. Thus $\Phi = \Phi_e^+ \cup \Phi_e^- = \Phi^+(\Pi_e) \cup \Phi^-(\Pi_e)$. In particular as Φ spans E, so does Π_e and Π_e is a base for Φ. □

3.3 Elementary Properties of Basis

Lemma 3.3.1 [**ch and sum**] *Let* Δ *be a linearly independent subset of* Φ *and* $e \in \langle \Delta \rangle$. *Write* $e = \sum_{\delta \in \Delta} n_\delta \delta$. *Then*

$$\check{\epsilon} = \sum_{\alpha \in \Delta} \frac{(\alpha, \alpha)}{(e, e)} n_\alpha \check{\alpha}.$$

and $\frac{(\alpha,\alpha)}{(e,e)} n_\alpha$ *is an integer.*

Proof: $\check{e} = \frac{2}{(e,e)} e = \sum_{\alpha \in \Delta} \frac{2(\alpha,\alpha)}{(e,e)(\alpha,\alpha)} n_\alpha \alpha = \sum_{\alpha \in \Delta} n_\alpha \check{\alpha}$.

By 3.1.9(c) a, $\check{e} \in \langle \check{\Delta} \rangle$ and hence by 3.1.13, $e \in \mathbb{Z}\check{\Delta}$. The linear independence of Δ now shows that $\frac{(\alpha,\alpha)}{(e,e)} n_\alpha$ is an integer. □

Lemma 3.3.2 [**basic base**] *Let* Π *be a base for* Φ.

(a) [**z**] $\check{\Pi}$ *is a base for* $\check{\Phi}$ *and* $(\Phi+)\check{} = (\check{P}hi)^+$.

(b) [**a**] *Let* $\alpha \neq \beta \in \Pi$. *Then* $\alpha - \beta \notin \Phi$ *and* $(\alpha, \beta) \leq 0$.

(c) [**b**] *Let* $\alpha \in \Phi$. *Then* $\operatorname{ht}\alpha$ *is an integer,* $\operatorname{ht}\alpha$ *is positive if and only if* α *is positive and* $\alpha \in \Pi$ *if and only if* $\operatorname{ht}\alpha = 1$.

(d) [**c**] *Let* $\alpha \in \Pi$. *Then* $\Phi^+ \setminus \{\alpha\}$ *is* ω_α *invariant.*

(e) [**d**] *Let* $\beta \in \Phi^+ \setminus \Pi$. *Then there exists* $\alpha \in \Pi$ *with* $(\alpha, \beta) > 0$. *For any such* α, *both* $\omega_\alpha(\beta)$ *and* $\beta - \alpha$ *are in* Φ^+ *and* $\operatorname{ht}(\omega_\alpha(\beta) \leq \operatorname{ht}(\beta - \alpha) = \operatorname{ht}\beta - 1$.

(f) [**e**] *Let* $\beta \in \Phi^+$. *Then there exists* $\alpha_1, \alpha_2, \ldots \alpha_k \in \Pi$ *such that* $\beta = \sum_{i=1}^k \alpha_i$ *and for all* $1 \leq j \leq k$, $\sum_{i=1}^j \alpha_i \in \Phi$.

(g) [**f**] *Let* $\beta \in \Phi^+$. *Then there exists* $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_k \in \Pi$ *such that if we inductively define* $\beta_0 = \alpha_0$ *and* $\beta_i = \omega_{\alpha_i}(\beta_{i-1})$ *then* $\beta = \beta_k$ *and* $\beta_i \in \Phi^+$ *for all* $0 \leq i \leq k$.

(h) [**g**] $\Phi = \langle \Pi \rangle$, $W = W(\Pi)$ *and each root is conjugate under* W *to some root in* Π.

(i) [**h**] *Put* $\delta_\Pi = \frac{1}{2} \sum \Phi^+$. *Then for all* $\alpha \in \Pi$, $\omega_\alpha(\delta) = \delta - \alpha$.

Proof: (a) This follows from 3.3.1

(b) Note that neither $\alpha - \beta$ nor $\beta - \alpha$ is in $\mathbb{N}\Pi$. So the definition of a base implies $\alpha - \beta \notin \Phi$. The second statement now follows from 3.1.15.

(c) Obvious.

(d) Let $\beta = \sum_{\gamma \in \Pi} n_\gamma \gamma$ and $i = (\beta, \check{\alpha})$.

$$\omega_\alpha(\beta) = \beta - i\alpha = (n_\alpha - i)\alpha + \sum_{\alpha \neq \gamma \in \Pi} n_\gamma \gamma.$$

Suppose that $\omega_\alpha(\beta) \in \Phi^-$. Then $n_\gamma = 0$ for all $\alpha \neq \gamma \in \Phi$ and so $\beta \in \mathbb{N}\alpha \cap \Phi = \{\alpha\}$ and $\beta \in \Pi$, contrary to the assumptions.

(e) $0 < (\beta, \beta) = \sum_{\alpha\in\Pi} n_\alpha(\beta, \alpha)$ and so $(\beta, \alpha) > 0$ for some $\alpha \in \Pi$. Suppose now that $\alpha \in \Pi$ with $i := (\beta, \check{\alpha}) > 0$. By (d), $\beta - i\alpha = \omega_\alpha(\beta) \in \Phi^+$. By 3.1.15(a), $\beta - \alpha \in \Phi$ and so $\beta - \alpha = \omega_\alpha(\beta) + (i-1)\alpha \in \Phi^+$.

(f) By induction on $\operatorname{ht}\beta$. If $\operatorname{ht}\beta = 1$, then $\beta \in \Pi$ and (f) holds with $k = 1$ and $\alpha_1 = \beta$. So suppose $\operatorname{ht} b > 1$ and thus $\beta \notin \Pi$. Choose α as in (e). By induction (f) holds for $\beta - \alpha$ and so also for β.

(g) By induction on $\operatorname{ht}\beta$. If $\operatorname{ht}\beta = 1$, then $\beta \in \Pi$ and (g) holds with $k = 0$ and $\alpha_o = \beta$. So suppose $\operatorname{ht} b > 1$ and thus $\beta \notin \Pi$. Choose α as in (e). By induction (f) holds for $\omega_\alpha(b)$ and so also for β.

(h) This follows from (g) and 3.1.12(c).

(i) By (d), ω_α fixes $\frac{1}{2}\sum_{\alpha\neq\beta\in\Phi^+}\beta$. Also $\omega_\alpha(\frac{1}{2}\alpha) = \frac{1}{2}\alpha - \alpha$ and so (i) holds. □

Lemma 3.3.3 [**bases equals chambers**] *Any bases is of the form Π_e for some regular element.*

Proof: Let Π be a base. Note that there exists a regular element e with $\Pi \subseteq \Phi_e^+$. . (Indeed choose $\alpha^* \in E, \alpha \in \Pi$ with $(\alpha^*, \beta) = \delta_{\alpha\beta}$ and put $e = \sum_{\alpha\in\pi}\alpha^*$.) Then $\Phi^+(\Pi) \subseteq \Phi_e^+$ and $\Phi^-(\Pi) \subseteq \Phi_e^-$. Hence $\Phi + (\Pi) = \Phi_e^+$ and by 3.3.2(e), $\Phi^+ \setminus (\Phi^+\Phi^+) = \Pi$. □

3.4 Weyl Chambers

Define two regular elements e and d to be equivalent if $\Phi_e^+ = \Phi_d^+$. The equivalence classes of this relation are called Weyl chambers. Note that there is natural 1-1 correspondence between Weyl chambers and bases for Φ. Also the equivalence relation is invariant under W and so W acts on the set Weyl of chambers. For a regular element e let $\mathfrak{C}(e)$ be the Weyl chamber containing e. For $\alpha \in \Phi$ let $P_\alpha(e) = \{d \in E \mid (\alpha, e)(\alpha, d) > 0\}$. Then

$$\mathfrak{C}(e) = \bigcap_{\alpha\in\Phi} P_\alpha(e)$$

Define $\overline{P}_\alpha(e) = \{d \in E \mid (\alpha, e)(\alpha, d) \geq 0$ and $\overline{\mathfrak{C}}(e) = \bigcap_{\alpha\in\Phi}\overline{P}_\alpha(e)$. $\overline{\mathfrak{C}}(e)$ is called a closed Weyl chamber. Topological, $P_\alpha(e)$ and so also $\mathfrak{C}(e)$ are open convex subsets of E. $\overline{P}_\alpha(e)$ and $\overline{\mathfrak{C}}(e)$ are their closures.

Definition 3.4.1 [**def:dominant**] *Given a base Π of Φ. Let $e, d \in E$. We say that is positive if $0 < e \in \mathbb{F}^{\geq 0}\Pi$. Define the relation $\prec$ on E by $d \prec e$ if $e - d$ is positive. e is called dominant if $(e, \check{\alpha}) \geq 0$ for all $\alpha \in \Pi$. e is strictly dominat if $(e, \check{\alpha}) > 0$ for all $\alpha \in \Pi$. Let $\overline{\mathfrak{C}}$ and $\mathfrak{C}$ be the set of dominant and strictly dominant elements in E.*

Let $w \in W$. If $w = \omega_\alpha$ for a simple root α, then w is called a simple reflection.

$l(w)$ is the minimal integer such that there simple reflections $\omega_1, \ldots, \omega_n$ with $w = \omega_n\omega_{n-1}\ldots\omega_1$. $n(w) = |\Phi^- \cap n(\Phi^+)|$.

Observe that $\overline{\mathfrak{C}} = \overline{\mathfrak{C}}(\delta_\Pi)$ and $\mathfrak{C} = \mathfrak{C}(\delta_\Pi)$.

Also $l(1) = 0$ and $l(\omega) = 1$ if and only if ω is a simple reflection; and $n(w)$ is the number of positive roots whose image under w is negative.

Proposition 3.4.2 [**existence of dominant**] *Let $e \in E$ and d be a element of maximal height in $W(e)$. Then d is dominant. Inparticular, there exists $w \in W$ with $w(e) \in \overline{\mathfrak{C}}$. If e is regular, then d is regular and $w(e) \in \mathfrak{C}$.*

Let $\alpha \in \Pi$. Then $\omega_\alpha(d) \in W(e)$ and $\omega_\alpha(d) = d-(d, \check{\alpha}\,)\alpha$ has height $d-(d, \check{\alpha}\,)$. The maximal choice of $\operatorname{ht} d$ implies that $(d, \check{\alpha}\,)0$. Thus $d \in \overline{P}(\alpha)$ and $d \in \overline{\mathfrak{C}}$. □

Lemma 3.4.3 [**reducing**] *Let $w \in W$ and $(\omega_1, \omega_2, \dots \omega_t)$ be a tuple of simple reflections with $w = \omega_t\omega_{t-1}\dots\omega_1$. Suppose that α a positive root with $w(\alpha)$ negative. Then there there exists $1 \leq i \leq s$ with*

$$w\omega_\alpha = \omega_t\omega_{t-1}\dots\omega_{s+1}\omega_{s-1}\omega_{s-2}\dots\omega_1.$$

Proof: Put $\rho_i = \omega_i\omega_{i-1}\dots\omega_1$ and $\beta_i = \rho_i(\alpha_0)$. Choose s minimal such that β_s is negative. Then β_{s-1} is positive and $\beta_s = \omega_s(\beta_{s-1})$ is negative. Since ω_s is a simple refections there exists $\delta \in \Pi$ with $\omega_s = \omega_\delta$ with $\delta \in \pi$. Since $\omega_\delta(\beta_{s-1})$ is negative, 3.3.2(d) implies have $\beta_{s-1} = \delta$. Thus

$$\omega_s = \omega_\delta = \omega_{\beta_{s-1}} = \omega_{\rho_{s-1}(\alpha)} = \rho_{s-1}\omega_\alpha\rho_{s-1}^{-1},$$

$\rho_s = \omega_s\rho_{s-1} = \rho_{s-1}\omega_\alpha$ and $\rho_s\omega_\alpha = \rho_{s-1}$. Multiplying the last equation with $\omega_t\omega_{t-1}\dots\omega_{s+1}$ from the left gives the lemma. □

Lemma 3.4.4 [**n(w)=l(w)**] *Let $w \in W$ and $\alpha \in \Pi$.*

(a) [**b**] *If $w(\alpha)$ is negative, then $l(w\omega_\alpha) = l(w) - 1$ and $n(w\omega_\alpha) = n(w) - 1$*

(b) [**c**] *If $w(\alpha)$ is positive, then $l(w\omega_\alpha) = l(w) + 1$ and $n(w\omega_\alpha) = n(w) + 1$.*

(c) [**a**] *$l(w) = n(w)$.*

Proof: (a) Let $t = l(w)$ and choose simple roots $\omega_1, \dots, \omega_t$ with $w = \omega_t\omega_{t-1}\dots\omega_1$. Then by 3.4.3 $l(w\omega_\alpha) \leq l - 1$. Since $w\omega_\alpha\omega_\alpha = w$, $l(w\omega_\alpha) \geq l(w) - 1$ and so the first statement in (c) hold

Let $\Sigma = \Phi^+ \setminus \{\alpha\}$. By 3.3.2(d) $\omega_\alpha(\Sigma) = \Sigma$. Hence also $w(\Sigma) \cap \Phi^- = (w\omega_\alpha)(\Sigma) \cap \Phi^-$. Now $w(\alpha) \in \Phi^-$ while $(w\omega_\alpha)(\alpha) \notin \Phi^-$. So also the second statement in (c) holds,

(b) We have $w\omega_\alpha(\alpha) = w(-\alpha)$ is negative. So (b) follows from (c) applied to $w\omega_\alpha$.

(c) Since $l(1) = n(1)$ this follows from (a) and (b) and induction on $l(w)$ □

Theorem 3.4.5 [**transitivity on bases**] *(a)* [**a**] *Let $w \in W$ and $e \in \overline{\mathfrak{C}}$ with $w(e) \in \overline{\mathfrak{C}}$. Then $w(e) = e$ and $w \in W(\Pi \cap e^{\perp})$. If, in addition, $e \in \mathfrak{C}$, then $w = 1$.*

(b) [**b**] *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be Weyl chambers. Then there exists a unique $w \in W$ with $w(\mathfrak{D}) = \mathfrak{D}'$.*

(c) [**c**] *Let Π and Π' be bases for Φ. Then there exists a unqiue $w \in W$ with $w(\Pi) = \Pi'$.*

(d) [**d**] *$|W|$ is the number of Weyl chambers.*

(e) [**e**] *There exists a unique element $w_0 \in W$ with $n(w_0)$ maximal. Moreover $n(w_0) = l(w_0) = |\Phi^+|$, $w_0(\Pi) = -\Pi$, $w_0(\Phi^+) = \Phi^-$ and $w_0^2 = 1$.*

Proof: (a) If $e \in \mathfrak{C}$, then $\Pi \cap e^{\perp} = 0$. So it suffices to proof first statement. If $l(w) = 0$, $w = 1$ and (a) holds. So suppose $l(w) > 0$ and pick a simple root α with $l(w\omega_\alpha) < l(w)$. Then by 3.4.4, $w(\alpha)$ is negative. As both e and $w(e)$ are in $\overline{\mathfrak{C}}$ we have

$$0 \leq (e, \alpha) = (w(e), w(\alpha)) \leq 0$$

Thus $\alpha \in e^{\perp}$, $w\omega_\alpha(e) = w(e)$ and the results follows by induction on $l(w)$.

(b) Without loss $\mathfrak{D}' = \mathfrak{C}$. Pick $d \in \mathfrak{D}$. Then by 3.4.2 there exists $w \in W$ with $w(e) \in \mathfrak{C}$. Then $w(\mathfrak{D}) = \mathfrak{C}$. Let w' be any element of W with $w'(\mathfrak{D} = \mathfrak{C}$. Then $w'w^{-1})(w(e)) = w'(e) \in \mathcal{C}$ and so by (a) appled to $''w = w'w''$ and $''e = w(e)''$ we have $w'w^{-1} = 1$ and so $w' = w$.

(c) and (d) follow immediately from (b).

(e). Note that $n(w) \leq |\Phi^-| = |\Phi^+|$ for all $w \in W$. Also $n(w) = |\Phi^+|$ if and only if $w(\Phi^+) = \Phi^-$ and so if and only if $w(\Pi) = -\Pi$. By (c) such an w exists and is unique. Also $w^2(\Pi) = \Pi$ and so $w^2 = 1$. Thus (e) holds. □

Definition 3.4.6 [**def:obtuse**] *A subset S of $E \setminus \{0\}$ is called acute (obtuse) if $(s, t) \geq 0$ ($(s, t) \leq 0$) for all $s \neq t \in E$.*

Lemma 3.4.7 [**easy base**] *Let Δ be a linear independent obtuse preroot system in E. Then Δ is base for the root system $\langle\Delta\rangle$ in $\mathbb{F}\Delta$.*

Proof: Put $\Phi = \langle\Delta\rangle$. Then by 3.1.13 Φ is a root sytem.

Let $\alpha \in \Phi$ and write $\alpha = \sum_{\beta\in\Delta} n_\beta \beta$ with $n_\beta \in \mathbb{Z}$. We need to show that the non-zero n_β all have the same sign. Suppose not and choose such an α with $\sum_{\beta\in\Psi} |n_\beta|$ minimal. Since $(\alpha, \alpha$ is positive there exists $\delta \in \Delta$ with $n_\delta(\alpha, \delta) \geq 0$. Replacing α by $-\alpha$ if necessary we may assume that $n_\delta > 0$. Then also (α, δ) is positive. Note that $\alpha \notin \mathbb{F}\delta$ and so by 3.1.15(a), $\alpha - \delta$ is a root. Now

$$\alpha - \delta = (n_\delta - 1)\delta + \sum_{\delta\neq\beta\in\Delta} n_\beta \beta$$

By the minimal choice of α, the non zero coefficents of $\alpha - \delta$ are either all positive or all negative. Let $\delta \neq \beta \in \Delta$. If $n_\beta > 0$ then $n_\gamma > 0$ for all $\gamma \trianglelefteq \Delta$, contrary to our assumptions. Hence n_β
*leq*0. Thus also $n_\delta - 1 \leq 0$. But $n_\delta - 1 \geq 0$ and so $n_\delta - 1 = 0$ and $n_\delta = 1$. Since $(\beta, \delta) \leq 0$ this implies that

$$(\alpha - \delta, \delta) = \sum_{\delta \neq \beta \in \Delta} (\beta, \delta) \geq 0.$$

So $(\alpha, \check{\delta}) \geq (\delta, \check{\delta}) = 2$. Note also that $\alpha \neq \delta$ and so 3.1.4 implies that $(\delta, \delta) < (\alpha, \alpha)$. On the otherhand by 3.3.1 $\frac{(\delta,\delta)}{(\alpha,\alpha)} n_\delta$ is an integer. Since $n_\delta = 1$, this implies $(\delta, \delta) \geq (\alpha, \alpha)$, a contradiction. □

3.5 Orbits and Connected Components

Definition 3.5.1 [**def:coxeter graph**] *Let* $\Sigma \subseteq \Phi$*.*

$$\Gamma(\Sigma) = \{(\alpha, \beta, i) \mid \alpha, \beta \in \Phi, (\alpha, \check{\beta}) \neq 0, i \in \mathbb{Z}_+, i \leq |(\alpha, \check{\beta})|\}$$

We view $\Gamma(\Sigma)$ *as a multiple edged directed graph on* Σ*, namely each* $(\alpha, \beta, i) \in \Gamma(\Sigma)$ *is an edge from* α *to* β*. So if* $(\alpha, \check{\beta})) = 0$*, then there exists no edge from* α *to* β*, and if* $(\alpha\beta, \check{\neq})0$*, then there exists* $|(\alpha, \check{\beta})|$ *edges from* α *to* β*.* $\Gamma(\Phi)$ *is called the Coxeter graph of* Φ*.* $\Gamma(\Pi)$ *is called the Dynkin diagram of* Φ*. For* $S \subseteq E$ *let* $\Gamma^0(S)$ *be the undirect graph without multiply edges, where* s, t *are adjacent if and only if* $(s, t) \neq 0$*.*

Note that for $\Sigma \subseteq \Phi$, $\Gamma(\Sigma)$ and $\Gamma^0(\Sigma)$ have the same connected component.

Lemma 3.5.2 [**connected components**] *Let* Φ *be a root system.*

(a) [**a**] *Let* $\alpha, \beta \in \Phi$ *with* $\alpha \not\perp \beta$*. Then* α, β *and* $\omega_\alpha(\beta)$ *are in the same connected component (with respect to the coxeter graph* $\Gamma(\Phi)$*).*

(b) [**b**] *Let* $\mathcal{D}$ *be the set of connected components of* Φ *. Then* $E = \bigoplus_{\Lambda \in \mathcal{D}} \mathbb{F}\Lambda$ *and* $W = \bigtimes_{\Lambda \in \mathcal{D}} W(\Lambda)$*.*

(c) [**c**] *Let* $\tilde{\Delta}$ *be a connected component of* Φ*. Then* $\tilde{\Delta}$ *is invariant under* W*,* $\tilde{\Delta}$ *is a subsystem of* Φ*,* $\tilde{\Delta} \cap \Pi$ *is a base for* $\tilde{\Delta}$*,* $\tilde{\Delta} = \langle \tilde{D} \cap \Pi \rangle$ *and* $\tilde{\Delta} \cap \Pi$ *is connected.*

(d) [**d**] *The map* $\Delta \to \langle \Delta \rangle$ *is one 1-1 correspondence between the connected components if* Π *and the connected components of* Φ*.*

(e) [**e**] Φ *is connected iff* Π *is connected.*

Proof: Since $\alpha \not\perp \beta$ we also have $\alpha \not\perp \omega_\alpha(\beta)$ and so (a) holds.

Let Δ be a connected component of Π. Also let $\tilde{\Delta}$ the connected component of Φ containing Δ . We claim that $\tilde{\Delta}$ is W-invariant. For this let $\alpha \in \Phi$ and $\beta \in \tilde{\Delta}$. If $\alpha \perp \beta$ then $\omega_\alpha(\beta) = \beta \in \tilde{\Delta}$. If $\alpha \not\perp \beta$, then by $\omega_\alpha(\beta)$, $\omega_\alpha(\beta)$ and β are in the same connected component of $\Gamma(\Phi)$ and again $\omega_\alpha(\beta) \in \tilde{D}elta$. So $\tilde{\Delta}$ is invarinat under all $\omega_\alpha, \alpha \in \Phi$ and so also under W. Thus $\langle \Delta \rangle = \bigcup \Delta^{W(\Delta)} \subseteq \tilde{\Delta}$.

Put $\Sigma = \Pi \setminus \Delta$. Then $\Sigma \perp \Delta$. Hence$W(\Delta)$ centralizes $W(\Sigma)] = 1$, $W = W(\Delta)W(\Sigma)$. In particular, $\langle \Delta \rangle \perp < \Sigma \rangle$ and $\langle \Delta \rangle$ and $\langle \Sigma \rangle$ are $W(\Delta)W(\Sigma) = W$ invariant

Thus

$$\Phi = \bigcup \Pi^W = \bigcup \Delta^W \cup \bigcup \Sigma^W = \langle \Delta \rangle \cup < \Sigma \rangle$$

Since $\tilde{\Delta}$ is connected, this implies $\tilde{\Delta} \subseteq \langle \Delta \rangle$ and $\tilde{\Delta} \cap \langle \Sigma \rangle = \emptyset$. Hence $\tilde{\Delta} = \langle \Delta \rangle$ and so by 3.4.7 Δ is a base for $\tilde{\Delta}$. Moreover $\tilde{D} \cap \Pi = \Delta \cup (\tilde{\Delta} \cap \Sigma) = \Delta$ and so (c) holds.

Note that $W(\Delta) \cap W(\Sigma)$ centralizes $\mathbb{F}\Delta + \mathbb{F}\Sigma = E$ and so $W(\Delta) \cap W(\Sigma) = 1$ and $W = W(\delta) \times W(\Sigma)$. An easy induction proof now shows that (b) holds.

(d) and (e) follow easily from from (c). □

Lemma 3.5.3 [**z closed**] *Let Φ be a root system and $\Psi \subseteq \Phi$. Then Ψ is $\mathbb{Z}$-closed iff $-\Psi \subseteq \Psi$ and $\alpha + \beta \in \Psi$ for all $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$.*

Proof: One direction id obvious. For the other suppose that $-\alpha \in \Psi$ for all $\alpha \in \Phi$ and $\alpha + \beta \in \Psi$ for all $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$. Let $\alpha \in \langle \Psi \rangle_{\mathbb{Z}}$. Then $\alpha = \sum_{\beta \in \Psi} n_\beta \beta$ with $n_\beta \in \mathbb{Z}$. Since $n_\beta \beta = (-n_\beta)(-\beta)$ we may assume that $n_\beta \geq 0$ for all $\beta \in \Phi$.

Since $\sum_{\beta \in \Psi} n_\beta (\alpha, \beta\,) = (\alpha, \alpha\,) > 0$ there exists $\delta \in \Psi$ with $n_\delta(\alpha, \delta\,) > 0$. Thus $n_\delta \geq 1$ and $(\alpha, \delta\,) > 0$. If $\alpha = \pm\delta$, then $\alpha \in \Psi$. If $\alpha \neq \pm\delta$ then by 3.1.15(a), $\alpha - \beta \in \Phi$. Thus $\alpha - \beta \in \langle \Psi \rangle_{\mathbb{Z}}$ and by induction on $\sum_{\beta \in \Psi} n_\beta$ we conclude that $\alpha - \beta \in \Psi$. Thus $\alpha = (\alpha - \beta) + \beta \in \Psi$. □

Lemma 3.5.4 [**root lengths**] *Let Φ be a connected root system, $\mathcal{L}(\Phi) = \{(\alpha, \alpha\,) \mid \alpha \in \Phi\}$. Let $r \in \mathcal{L}(\Phi)$ and put $\Phi_r = \{\alpha \in \Phi \mid (\alpha, \alpha\,) = r\}$.*

(a) [**a**] *$E = \mathbb{F}\Phi_r$ and $\Phi = \langle \Phi_r \rangle_{\mathbb{Q}}$*

(b) [**b**] *If r is minimal in $\mathcal{L}(\Phi)$, then $\Phi = \langle \Phi_r \rangle_{\mathbb{Z}} \leq \mathbb{Z}\Phi_r$.*

(c) [**c**] *W acts transitively on Φ_r.*

(d) [**e**] *If r is maximal in $\mathcal{L}(\Phi)$, then $\Phi_r = \langle Phi_r \rangle_{\mathbb{Z}}$ is $\mathbb{Z}$-closed.*

(e) [**d**] *$|\mathcal{L}(\Phi)| \leq 2$.*

Proof: Let Σ be an orbit for W on Φ_r and let $\alpha \in \Phi$. Suppose that $\alpha \not\perp \sigma$ for some $\sigma \in \Sigma$. Then

$$(*) \qquad \alpha = \frac{1}{(\sigma, \check{\alpha})}(\sigma - \omega_\alpha(\sigma)) \in \mathbb{Q}\Sigma$$

Thus $\Phi = (\Phi \cap (\mathbb{Q}\Sigma)^\perp) \cup \langle \Sigma \rangle_\mathbb{Q}$ and since Φ is connected, $\Phi = \langle \Sigma \rangle_\mathbb{Q} \subseteq < \Phi_r \rangle_\mathbb{Q} \subseteq \Phi$ and so (a) holds. In particular, $\Sigma^\perp \cap \Phi = \emptyset$. If r is minimal in $\mathcal{L}(\Phi)$, then $(\alpha, \alpha) \geq (\sigma, \sigma)$. So by 3.1.4 either $\alpha = \pm\sigma$ or $(\sigma, \check{\alpha}) = \pm 1$. From (*) we get in any case that $\alpha \in \mathbb{Z}\Sigma$ and so (b) holds.

Suppose now that $\alpha \in \Phi_r$. Then either $\alpha = \pm\sigma$ or $\langle \alpha, \sigma \rangle$ is a root system of type A_2. In either case α and σ are conjugate in $W(\langle \alpha, \sigma \rangle$. Thus (c) holds.

Suppose that r is maximal. Let $\alpha, \beta \in \Phi_r$ with $\alpha + \beta \in \Phi$. Then $\alpha \neq \pm b$ and so since $(\alpha, \alpha) = (\beta, \beta)$ 3.1.4 implies $(\alpha, \check{\beta}) \geq -1$. Thus

$$\begin{aligned} (\alpha+\beta, \alpha+\beta) = \quad & (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta) & = (\alpha, \alpha) + (\alpha, \check{\beta})(\beta, \beta) + (\beta, \beta) \\ = \quad & (2 + s\alpha\beta)(\beta, \beta) & \geq \quad (\beta, \beta) \end{aligned}$$

So $\alpha + \beta \in \Phi_r$. (c) now follows from 3.5.3.

Let $s, l \in \mathcal{L}(\Phi)$ with $s < l$. Then by (a) we can choose $\beta \in \Phi s$ and $\alpha \in \Phi_l$ with $\beta \not\perp \alpha$. Then by 3.1.4, $\frac{l}{s} = \frac{(\alpha,\alpha)}{(\beta,\beta)} \in \{2, 3\}$. If $|l(\Phi)| > 2$, we can choose $s < l < t \in \lambda(\Phi)$. But then $\frac{t}{s} = \frac{l}{s}\frac{t}{l}$ is not a prime, a contradiction and (e) holds. □

Lemma 3.5.5 [**dominant**] *Let Φ be a root system in E and $\tilde{E}$ an Euclidean F space with $E \leq \tilde{E}$. Let Δ be an orbit for W on $\tilde{E}$ and $e \in \overline{\mathfrak{C}}$. Then*

(a) [**a**] *Δ contains a unique dominant member d.*

(b) [**b**] *$b \prec d$ and for all $b \in \Delta$.*

(c) [**c**] *$(e, b) \leq (e, d)$ for all $b \in \Delta$.*

(d) [**d**] *Let $\Pi \cap e^\perp$ is a basis for $\Phi \cap e^\perp$.*

(e) [**e**] *Let $b \in \Delta$ with $(e, b) = (e, d)$. Then there exists $w \in W(\Pi \cap e^\perp)$ with $w(d) = b$.*

Proof:

(a) Let e and d be dominant in Δ. Then $d = w(e)$ for some $w \in W$ and 3.4.5(a) implies that $d = e$.

(b) Choose $a \in \Delta$ such that $b \prec a$ and a is maximal in Δ with respect to $\prec$. We claim that a is dominant. Otherwise there exists $\beta \in \Pi$ with $(a, \check{\beta}) < 0$. Then $a \prec a - (a, \check{\beta})a =$

$\omega_\beta(a) \in \Delta$, a contradiction to the maximality of a. Thus a is dominant and so by (a) $a = d$ and (b) holds.

(c) By (b) $d - b \in \mathbb{N}\Pi$ and since e is dominant, $(e, d - b\,) \geq 0$.

(d) Let $\beta \in \Phi^+ \cap e^\perp$ we need to show that $\beta \in \mathbb{N}(\Pi \cap e^\perp)$. If $\operatorname{ht}\beta = 1$, then $\beta \in \Pi \cap e^\perp$. So suppose $\operatorname{ht}\beta > 1$, that is $\beta \notin \Phi$. Then by 3.3.2(e) there exists $\alpha \in \Pi$ with $\delta = \beta - \alpha \in \Phi^+$. Since $e \in \overline{\mathfrak{C}}$ both $(e, \alpha\,)$ and $(e, \delta\,)$ are non-negative. Since $0 = (e, \beta\,) = (e, \alpha\,) + (e, \delta\,)$ we conclude that both α and δ are in $e^\perp$. $\operatorname{ht}\delta < \operatorname{ht}\beta$ and so by induction on $\operatorname{ht}\beta$, $\delta \in \mathbb{N}(\Pi \cap e^\perp)$.

(e) If b is dominant, then by (a) $b = d$ and we are done. done. So suppose that b is not dominant and and choose exists $\alpha \in \Pi$ with $(b, \check{\alpha}\,) < 0$. Then $c := \omega_\alpha(b) \in \Psi$, $(e, c\,) = (e, b\,) - (\beta, \check{\alpha}\,)(e, \alpha\,) \geq (e, c\,)$. On the other hand, by (c) $(e, c\,) \leq (e, d\,)$. This implies $(e, c\,) = (e, d\,)$ and $(\epsilon, \alpha\,) = 0$. So $\alpha \in \Phi \cap e^\perp$ and by induction on $\operatorname{ht} b$, $c = w(d)$ for some $w \in W(\Pi \cap e^\perp)$. Hence $b = \omega_\alpha(c) = (\omega_\alpha w)(d)$ and (e) holds. □

3.6 Cramer's Rule and Dual Bases

Lemma 3.6.1 (Cramer's Rule) [**cramer rule**] *Let I a finite set, R a commutative ring with 1, $A : I \times I \to R$ be $I \times I$-matrix over R. Define $(i, j) \in I \times I$ to be an edge if $a_{ij} \neq 0$. Let $S(i, j)$ be the set of all direct paths $s = (i_0, i_1, \ldots, i_n)$ for i to j, where the i_k are pairwise distinct Put $|s| = n$, $m(s) = \prod_{k=1}^n a_{i_{k-1} i_k}$ and $I - s = I \setminus \{i_0, i_1, \ldots i_k\}$. For $J \subseteq I$ let A_J be the restriction of A to $I - J \times I - J$. Define*

$$b_{ij} = \sum_{s \in S(i,j)} (-1)^{|s|} m(s) \det A_{I-s}.$$

and $B = (b_{ij})$. Then $AB = \det(A)\operatorname{Id}_I$, where Id_I is the $I \times I$ identity matrix.

Proof: Let $i, j \in I$ and define the matrix $D = D^{ij}$ by $d_{kl} = a_{kl}$ if $k \neq j$ and $d_{jl} = \delta_{il}$. We will show that $b_{ij} = \det D$. For $K \subseteq I$ and $\sigma \in \operatorname{Sym}(K)$ define

$$a(\sigma) = \operatorname{sgn}(\sigma) \prod_{k \in I} a_{k\sigma(k)}.$$

Similarly define $d(\sigma)$. Then

$$\det D = \sum_{\pi \in \operatorname{Sym}(\Pi)} d(\pi)$$

We investigate $d(\pi)$ for $\pi \in \operatorname{Sym}(\pi)$. If $\pi(j) \neq i$, then $d_{j\pi j} = \delta_{i\pi j} = 0$ and so also $d(\pi) = 0$.

Suppose $\pi(j) = i$. Let $n \in \mathbb{N}$ be minimal with $\pi^{n+1}(i) = i$. For $0 \leq k \leq n$, put $i_k = \pi^k(i)$ and $s = (i_0, i_1, \ldots, i_n)$. The i_k are pairwise distinct, $i_0 = i$ and $i_n = j$. If (i_{k-1}, i_k) not an edge for some $1 \leq k \leq n$, then $d_{i_{k-1}\pi(i_{k-1})} = a_{i_{k-1} i_k} = 0$ and so also $d(\pi) = 0$.

Suppose s is a string and view s as a cycle in $\operatorname{Sym}(\{i_0, \ldots, i_n\})$. Then $\pi = s\sigma$ for a unique $\sigma \in \operatorname{Sym}(I - s)$. Now $d(\pi) = d(s)d(\sigma)$, $\operatorname{sgn} s = (-1)^n = (-1)^{|s}|$ and $d_{ji} = 1$. Thus $d(s) = (-1)^{|s|} m(s)$, $d(\sigma) = a(\sigma)$ and so

$$d(\pi) = (-1)^{|s|} m(s) a(\sigma).$$

It follows that

$$\begin{aligned}\det D &= \sum_{\pi\in \mathrm{Sym}(I)} d(\pi) \\ &= \sum_{s\in S(i,j)} \sum_{\sigma\in \mathrm{Sym}(I-s)} (-1)^{|s|} m(s) a(\sigma) \\ &= \sum_{s\in S(i,j)} (-1)^{|s|} m(s) \det A_{I-s}\end{aligned}$$

Thus indeed $b_{ij} = \det D^{ij}$.

Note that $\sum_{j\in J} a_{ij} b_{jk} = \sum_{j\in J} a_{ij} \det D^{jk}$ is the determinant of the matrix E^{ik} obtained from A by replacing row k of A by row i. Now $\det E^{ik} = \delta_{ik} \det(A)$ and so $AB = \det A\, \mathrm{Id}_I$. □

Lemma 3.6.2 [**dual basis**] *Let $\mathcal{B}$ be a basis for E. For $b \in \mathcal{B}$ define $b^* \in E$ by $(b, a) = \delta_{ba}$. Put $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$ and let $A(\mathcal{B})$ be the $I \times I$ matrix $((a, b))$.*

(a) [**a**] *Then $d = \sum_{b\in\mathcal{B}} (d, b) b^* = \sum_{b\in\mathcal{B}} (d, b^*) b$.*

(b) [**b**] $A(\mathcal{B}^*) = A(\mathcal{B})^{-1}$.

(c) [**c**] $\det A(\mathcal{B}) > 0$.

(d) [**d**] *Suppose that $\mathcal{B}$ is obtuse. Then $\mathcal{B}^*$ is acute and, for $a, b \in \mathcal{B}$, $(a^*, b^*) > 0$ if and only if a and b lie in the same connected component of the $\perp$-graph on $\mathcal{B}$.*

(a) Let $d = \sum_{b\in\mathcal{B}} f_b b$ and let $a \in \mathcal{B}$. Then $(d, a^*) = f_a$. Also $\mathcal{B}^{**} = \mathcal{B}$ and so (a) holds.

(b) Let $a \in \mathcal{B}$. Then by (a),

$$a = \sum_{b\in\mathcal{B}} (a, b) b^* = \sum_{b\in\mathcal{B}} \sum_{d\in\mathcal{B}} (a, b)(b^*, d^*) d$$

and so (b) holds.

(c) Let $\mathcal{E}$ be an orthogonal basis for E and let D be the $\mathcal{B} \times \mathcal{E}$ matrix defined by $b = \sum d_{be} e$. Then $A(\mathcal{B}) = D A(\mathcal{E}) D^{\mathrm{T}}$ and so $\det A(\mathcal{B}) = (\det D)^2 \det A(\mathcal{E})$. Since $A(\mathcal{E})$ is a diagonal matrix with positive diagonal elements, $\det A(\mathcal{E})$ is positive and so (c) holds.

(d) Let $A = A(\mathcal{B})$. From (b) and 3.6.1 we have

$$(a^*, b^*) = \sum_{s\in S(a,b)} (-1)^{|s|} m(s) \det A_{\mathcal{B}-s}.$$

Since $\mathcal{A}$ is obtuse $m(s)$ is the product of $|s|$ negative elements. Hence $(-1)^{|s|} m(s)$ is positive. By (c) also $\det A_{I-s}$ is positive. Hence (a^*, b^*) is non-negative and $(a^*, b^*) = 0$ if and only if $S(a, b) = \emptyset$. So (d) holds. □

3.7 Minimal Weights

Throughout this section Φ is root system. We call a root α long (short) if $(\alpha, \alpha) \geq (\beta, \beta)$ ($(\alpha, \alpha) \leq (\beta, \beta)$) for all $\beta \in \Phi$, which are on the same connected component of Φ as α. Note that if Φ has only one root length then all roots are long and short. Φ_{l} and Φ_{s} denotes the sets of long and short roots in Φ, respectively.

Definition 3.7.1 [**def:weights for phi**] *Let $\lambda \in E$. We say that λ is an integral weight of Φ if $(\lambda, \alpha) \in \mathbb{Z}$. For $\alpha \in \Pi$ define $\alpha^* \in E$ by $(\alpha^*, \beta) = \delta_{\alpha\beta}$ for all $\beta \in \Pi$. For $e = \sum_{\alpha\in\Pi} f_\alpha \alpha$ put $e^* = \sum_{\alpha\in\Pi} f_\alpha \alpha^*$.*

$\breve{\Lambda} = \breve{\Lambda}(\Phi)$ is the set of integral weights and $\Pi^* = \{\alpha^* \mid \alpha \in \Pi\}$. $\breve{\Lambda}^+$ is the set of dominant integral weights.

Lemma 3.7.2 [**z basis**]

(a) [**a**] $\check{\Phi} \subset \breve{\Lambda}$.

(b) [**b**] *Let $e \in E$. Then $e = \sum_{\alpha\in\Pi} (e, \alpha)\alpha^*$. In particular, $\overline{\mathfrak{C}} = \mathbb{F}^{\geq 0}\, \Pi^*$.*

(c) [**c**] *Π^* is a $\mathbb{Z}$-basis for $\breve{\Lambda}$*

(d) [**d**] *$\overline{\mathfrak{C}}^\sharp$ is acute and, if Φ is connected, strictly acute.*

(e) [**e**] *Let $e \in \overline{\mathfrak{C}}^\sharp$ then $e = \sum_{\alpha\in\Pi} (e, \alpha^*)\alpha$, $(e, a^*) \geq 0$ and if Φ is connected, $(e, \alpha^*) > 0$.*

Proof: (a) follows directly from the definition of a root system.

(b) Follows from 3.6.2(a).

(c) Since Π is a base for Φ, every $\beta \in \Phi$ is a integral linear combination of Π. This implies that each α^* for α in Π is an integral weight. (c) now follows from (b).

(d) and (e) follows easily from 3.6.2 □

Lemma 3.7.3 [**along min**] *Let Φ be a connected root system.*

(a) [**a**] *Φ_{l} has contains unique dominant root α_{l} and Φ_{s} has a unique dominant root α_{s}.*

(b) [**b**] *If $\alpha \in \Phi$ with $\alpha \neq \alpha_{\mathrm{l}}$ then there exists $\beta \in \Phi^+$ with $\alpha + \beta \in \Phi$.*

(c) [**c**] *Let $e \in \overline{\mathfrak{C}}^\sharp$ and $\alpha \in \Phi$. Then $-\alpha_{\mathrm{l}} \prec \alpha \prec \alpha_{\mathrm{l}}$ and $-(e, \alpha_{\mathrm{l}}) \leq (e, \alpha) \leq (e, \alpha_{\mathrm{l}})$.*

Proof: By 3.5.4(c) W is transitive on Φ_l. So (a) follows from 3.5.5.

For (b) suppose first that α is not dominant. Then there exists $\beta \in \Pi$ with $(\alpha, \beta) < 0$ and so by 3.1.15(b), $\alpha + \beta \in \Phi$. Suppose next that α is dominant. Since $\alpha \neq \alpha_{\mathrm{l}}$ we conclude that Φ has two root lengths and $\alpha = \alpha_{\mathrm{s}}$. By 3.7.2(d) $(\alpha_{\mathrm{l}}, \alpha) > 0$ and so by 3.1.4 $(\alpha, \check{\alpha}_{\mathrm{l}}) = 1$. Thus $\beta := \alpha_{\mathrm{l}} - \alpha$ is a roots and $(\beta, \check{\alpha}_{\mathrm{l}}) = 2 - 1 = 1 > 0$. Since α_{l} is dominant this implies $\beta \in \Phi^+$ and so (b) holds.

(c) Note that by (b), α_l is the unique element of maximal height in Φ. From (b) and induction on $\operatorname{ht}\alpha_l - \operatorname{ht}\alpha$, $\alpha \prec \alpha_l$ and so $\alpha_l = \alpha + \phi$ for some $\phi \in \mathbb{N}\Pi$. Since $(e, \phi) > 0$, $(\epsilon, \alpha) \leq (e, \alpha_l)$. Note that this results also holds for the base $-\Pi$ and so (c) is proved. □

Definition 3.7.4 [**def:minimal**] *$\lambda \in \check{\Lambda}$ is called minimial if $(\lambda, \alpha)) \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$.*

Proposition 3.7.5 [**minimal 1**] *Let $0 \neq \lambda \in \check{\Lambda}^+$ Then the following are eqiuvalent*

(a) [**a**] *λ is minimal.*

(b) [**b**] *$(\lambda, \alpha_l) = 1$.*

(c) [**c**] *$\lambda = \beta^*$ for some $\beta \in \Pi$ where $n_\beta = 1$ is defined by $\alpha_l = \sum_{\delta \in \Pi} n_\delta \delta$.*

Proof: (a)$\Longrightarrow$ (b): Since λ is dominant and minimal, $(\lambda, \alpha_l) \in \{0, 1\}$. If $(\lambda, \alpha_l) = 0$, then 3.7.3 implies $\lambda = 0$.

(b)$\Longrightarrow$ (a): Suppose $(\lambda, \alpha_l) = 1$. Then by 3.7.3(c) shows that λ is minimal.

(b)$\Longleftrightarrow$ (c): Note that $(\lambda, \alpha_l) = \sum_{\delta \in \Pi} n_\delta (\lambda, \delta)$.

By 3.7.2(e) each n_δ is a positive integer. So we see that $(\lambda, \alpha_l) = 1$ iff the following holds:

There exists a unique $\beta \in \Pi$ with $(\lambda, \beta) \neq 0$; and for this β, $(\lambda, \beta) = 1 = n_\beta$.

Note that this is equivalent to (c). □

Definition 3.7.6 [**def:affine**]

(a) [**a**] *$\Pi^\circ = \Pi \cup \{-\alpha_l\}$. $\Gamma(\Pi^\circ)$ is called the affine diagram of Φ.*

(b) [**b**] *w_Π is the unique element in W with $w_\Pi(\Pi) = -\Pi$ (and so $(-w_\Pi)(\Pi) = \Pi$).*

For an example let $I = \{0, 1, \dots n\}$ and let E_0 be the euclidean $\mathbb{F}$-space with orthonormal basis $(e_i \mid i \in I)$. For $0 \neq i \in I$ put $\alpha_i = e_{i-1} - e_i$. Put $\Pi = \{\alpha_i \mid 1 \leq i \leq n$ and $\Phi = \langle \Pi \rangle$. Note that $(\alpha_i, \alpha_j) = 2$ if $i = j$, -1 if $|i - j| = 1$ and 0 if $|i - j| > 1$. In particular, $\check{\alpha}_i = \alpha_i$ and Π is a linear independent pre-root system. Thus Φ is a root system. Note that $\omega_{\alpha_i}(e_i) = e_{i-1}$, $\omega_{\alpha_i}(e_{i-1}) = e_i$ and $\omega_{\alpha_i}(e_j) = e_j$ if $j \neq i, i-1$. Hence if we view $\mathrm{Sym}(I)$ as a subgroup of $GL(E_0)$, then ω_{a_i} is the cycle $(i-1, i)$ in $\mathrm{Sym}(I)$ and $W(\Pi) = \mathrm{Sym}(I)$. Thus the definition of $\langle \Pi \rangle$ implies that $\Phi = \{e_i - e_j \mid i \neq j \in I\}$. Let $e = -\sum_{i \in I} i e_i$. Then $(e, \alpha_i) = 1$ for all $i \in I$ and so e is a regular and dominant. Hence $\Phi^+ = \{\alpha \in \Phi \mid (e, \alpha) > 0\} = \{e_i - e_j \mid i < j \in I\}$. Let $\alpha = e_0 - e_n$. Suppose that $n > 1$. Then $(\alpha, \alpha_1) = (\alpha, \alpha_n) = 1$ and $(\alpha, \alpha_i) = 0$ for $1 < i < n$. If $n = 1$ then $(\alpha, \alpha_1) = 2$. In any case α is dominant, $\alpha = \alpha_l$ and so $\Pi^\circ = \Pi \cup \{-\alpha\}$. Note that $\Gamma(\Pi)$ is a string of lenght

n with only single bonds. If $n > 1$ then $\Gamma(\Pi^\circ)$ is circle of length $n+1$ with only single bonds and if $n = 1$, then $\Gamma(\Pi^\circ)$ consist of two vertices with a double bond.

Let $w \in \mathrm{Sym}(I)$ be defined by $w(i) = n - i$. Then $w(\alpha_i) = e_{n-(i-1)} - e_{n-i} = -\alpha_{n+1-i}$. Thus $w(\Pi) = -\Pi$ and so $w_\Pi = w$. Note that $-w$ induces the unique non-trivial graph automorphism on $\Gamma(\Pi)$.

Lemma 3.7.7 [**phi-sigma invariant**] *Let $\Sigma \subseteq \Phi$. Then $\Phi^+ \setminus \langle\Sigma\rangle$ is invarinant under $W(\Sigma)$.*

Proof: By definition, $\langle\Sigma\rangle$ is invarinat under $W(\Sigma)$. Hence also $\Phi \setminus \langle\Sigma\rangle$ is $W(\Sigma)$-invariant. Let $\alpha \in \Phi^+ \setminus \langle Sigma\rangle$ and $\sigma \in \Sigma$. Then $\alpha \neq \sigma$ and so by 3.3.2(d), $\omega_\sigma(\alpha) \in \Phi^+$. Thus $\Phi^+ \setminus \langle\Sigma\rangle$ is ω_σ-invariant and so also $W(\Sigma)$ invariant. □

Proposition 3.7.8 [**minimal 2**] *Let $\beta \in \Pi$ and $\Sigma = \Pi - \beta$. Then the following are equivalent.*

(a) [**a**] *β^* is minimal.*

(b) [**b**] *β is long and $W(\Sigma)$ act transitively on $\Phi_{\mathrm{l}}^+ \setminus \langle\Sigma\rangle$.*

(c) [**c**] *$w_\Sigma(\beta) = \alpha_{\mathrm{l}}$.*

(d) [**d**] *Π° is invariant under $-w_\Sigma$*

(e) [**e**] *There exists a graph automorphism σ of $\Gamma(\Pi^\circ))$ with $\sigma(\beta) = -\alpha_{\mathrm{l}}$.*

(f) [**f**] *$\Gamma(\Pi^\circ - \beta)$ and $\Gamma(\Pi)$ are isomorphic graphs.*

(g) [**g**] *$\Phi = \langle\Pi^\circ - \beta\rangle$.*

Proof: (a)$\Longrightarrow$ (b): By 3.7.5 $n_\beta = 1$. By 3.3.1 $\frac{(\beta,\beta)}{(\alpha_{\mathrm{l}},\alpha_{\mathrm{l}})} n_\beta$ is an integer and so $(\beta,\beta) = (\alpha_{\mathrm{l}},\alpha_{\mathrm{l}})$ and β is long.

Let $\delta \in \Phi_{\mathrm{l}}^+ \setminus \Sigma$. By 3.5.5(d), $\langle\Sigma\rangle = \Phi \cap \beta^{*\perp}$ and so $(\beta^*,\delta) \neq$. Since β^* is minimal, $(\beta^*,\delta) = 1 = (\beta^*,\alpha_{\mathrm{l}})$. (b) now follows from 3.5.5(e).

(b)$\Longrightarrow$ (c): Since Π is obtuse and $w_\Sigma(\Sigma) = -\Sigma$, $w_\Sigma(\beta)$ is dominant on Σ. Since also α_{l} is dominant on Σ and since $w_\Sigma(\beta)$ and α_{l} are conjugate under $W(\Sigma$, 3.5.5(a) (applied to $\langle\Sigma\rangle$ in place of Φ) implies $w_\Sigma(\beta) = \alpha_{\mathrm{l}}$.

(c)$\Longrightarrow$ (d): We have $-w_\Sigma(\Sigma) = \Sigma$ and $-w_\Sigma(\beta) = -\alpha_{\mathrm{l}}$. Since $-w_\Sigma$ has order 2, $-w_\Sigma(-\alpha_{\mathrm{l}}) = \beta$ and so (d) holds.

(d)$\Longrightarrow$ (e): By 3.7.7 $w_\Sigma(\beta) \in \Phi^+$ and so $-w_\Sigma(\beta) \neq \beta$. Since $-w_\Sigma$ leaves Π° and Σ invariant we conclude $-w_\Sigma(\beta) = -\alpha_{\mathrm{l}}$. Also $-w_\Sigma$ is an isometry and so w_Σ induces a graph automorphism on $\Gamma(\Phi^\circ)$. So (e) holds with $\sigma = (-w_\Sigma\mid_{\Phi^\circ}$.

(e)$\Longrightarrow$ (f): Obvious.

(f)$\Longrightarrow$ (g): By (f) $\langle \Pi^\circ - \beta \rangle$ is a subroot system of Φ isomorphic to $\langle \Pi \rangle = \Phi$. As Φ is finite, (g) holds.

(g)$\Longrightarrow$ (a): We have $\beta \in \langle \Pi^\circ - \beta \rangle$ and so $\alpha = n\alpha_l + \sigma$ for some $n \in \mathbb{Z}$ and $\sigma \in \mathbb{Z}\Sigma \in b^{*\perp}$. Thus $1 = (\beta^*, \beta) = n(b^*, \alpha_l)$ and so $(b^*, \alpha_l) = 1$. Hence by 3.7.5, β^* is minimal. □

For $\Phi = A_n$ we have that Π° is circle of lenght $n+1$. Hence for all $\alpha \in \Pi$, $\Pi^\circ - \alpha$ is a string of length n and so isomorphic to Π. Thus each α^* for $\alpha \in \Pi$ is a minimal weight. Also 0 is a minimal weight and hence A_n has $n+1$ minimal weights.

Definition 3.7.9 [**def:cartan matrix**] *C and E are the $\Pi \times \Pi$ matrix defined by $c_{\alpha\beta} = (\check{\alpha}, \beta)$ and $e_{\alpha\beta} = \frac{(\alpha^*, \beta^*)(\beta,\beta)}{2}$. C is called the Cartan matrix of Φ. Put $e_\alpha := e_{\alpha\alpha}$.*

Lemma 3.7.10 [**basic cartan matrix**]

(a) [**a**] $\check{\alpha} = \sum_{\beta\in\Pi} c_{\alpha\beta}\beta^*$.

(b) [**b**] $\alpha^* = \sum_{\beta\in\Pi} e_{\alpha\beta}\check{\beta}$.

(c) [**c**] $E = C^{-1}$.

Proof: (a) follows from 3.6.2(a) applied to $\mathcal{B} = \Pi$.

(b) By 3.6.2(a), $\alpha^* = \sum_{\beta\in\Pi} (\alpha^*, \beta^*)\beta = \sum_{\beta\in\Pi} (\alpha^*, \beta^*)\frac{(\beta,\beta)}{2}\check{\beta} = \sum_{\beta\in\Pi} e_{\alpha\beta}\check{\beta}$.

(c) Follows easily from (a) and (b). □

Proposition 3.7.11 [**decomposing pi**] *Suppose Φ is connected. Let $\alpha \in \Pi$ be long and let Δ be the set of neighbors of α in $\Gamma(\Pi)$. Put $\Sigma = \Pi - \alpha$ and let $\{\tilde{\sigma} \mid \sigma \in \Sigma\}$ the basis of $\mathbb{F}\Sigma$ dual to Σ. For $\delta \in \Delta$ define $\tilde{e}_\delta = \frac{(\delta,\delta)(\tilde{\delta},\tilde{\delta})}{2}$ and $r_\delta = \frac{(\alpha,\alpha)}{(\delta,\delta)}$. Then*

(a) [**a**] *Each connected component of Σ contains exactly one element of Δ.*

(b) [**b**] *$\tilde{\delta}$ is a minimal dominant weight for $\langle \Sigma \rangle$.*

(c) [**c**] $\check{\alpha} = \frac{\alpha^*}{e_\alpha} - \sum_{\delta\in\mathrm{D}} \tilde{\delta}$.

(d) [**d**] $\frac{1}{e_\alpha} + \sum_{\delta\in\Delta} r_\delta \tilde{e}_\delta = 2$.

Proof: Let $\mathcal{D}$ be the set of connected components of Σ. Note that

$$E = \mathbb{F}\alpha^* \oplus \mathbb{F}\Sigma = \mathbb{F}\alpha^* \oplus \bigoplus_{D\in\mathcal{D}} \mathbb{F}D.$$

and so

$$(*) \qquad \check{a} = m\alpha^* - \sum_{D\in\mathcal{D}} \lambda_D$$

for some $m \in \mathbb{F}$ and $\lambda_D \in \mathbb{F}D$. Let $\beta \in \langle D\rangle$. The $(\lambda_D, \beta) = -(\beta, \check{\alpha})$. Since Π is linearly independent, $\beta \notin \mathbb{F}\alpha$ and since α is long we conclude that $(\beta, \check{\alpha}) \in \{-1, 0, 1\}$. Thus λ_D is a minimal weight for $\langle D\rangle$. Since Π is obtuse, λ_D is dominant for D. From 3.7.5 we conclude that $\lambda_D = \tilde{\delta}$ for some $\delta \in D$. Then clearly δ is the unique element of Δ contained in D and so (a) and (b) hold.

Note that $1 = (\alpha^*, \alpha) = (\alpha^*, \check{\alpha})\frac{(\alpha,\alpha)}{2}$ and so by (*), $1 = m(\alpha^*, \alpha^*)\frac{(\alpha,\alpha)}{2} = me_\alpha$. Thus $m = \frac{1}{e_\alpha}$ and (c) follows from (*).

Note that

$$(\check{\alpha}, \check{\alpha}) = \frac{4}{(\alpha, \alpha)},$$

$$(\frac{\alpha^*}{e_\alpha}, \frac{\alpha^*}{e_\alpha}) = \frac{1}{e_\alpha}\frac{(\alpha^*, \alpha^*)}{e_\alpha} = \frac{1}{e_\alpha}\frac{2(\alpha^*, \alpha^*)}{(\alpha^*, \alpha^*)(\alpha, \alpha)} = \frac{2}{(\alpha, \alpha)}\frac{1}{e_\alpha}$$

and

$$(\tilde{\delta}, \tilde{\delta}) = \frac{2\tilde{e}_\delta}{(\delta, \delta)} = \frac{2}{(\alpha, \alpha)} r_\delta \tilde{e}_\delta$$

Computing the squared lengths of both sides in (c) we now obtain

$$\frac{4}{(\alpha, \alpha)} = \frac{2}{(\alpha, \alpha)}\frac{1}{e_\alpha} + \sum_{\delta \in \Delta} \frac{2}{(\alpha, \alpha)} r_\delta \tilde{e}_\delta$$

Multiplying with $\frac{(\alpha,\alpha)}{2}$ we get

$$2 = \frac{1}{e_\alpha} + \sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta.$$

Thus (d) holds. □

Proposition 3.7.12 [**composing pi**] *Let I be a finite set. For $i \in i$ let E_i an euclidean $\mathbb{F}$-space, Φ_i a connected root system in E_i with base Π_i and $\delta_i \in \Pi_i$. Let $\{\tilde{\delta} \mid \delta \in \Pi_i\}$ be the basis dual to Π_i in E_i and put $\tilde{e}_i = \frac{(\delta_i,\delta_i)(\tilde{\delta}_i,\tilde{\delta}_i)}{2}$. Also let l in $\mathbb{F}$ be positive. Suppose that for all $i \in I$*

- *(i)* [**a**] *$\tilde{\delta}_i$ is a minimal dominant weight for Φ_i.*
- *(ii)* [**b**] *$r_i := \frac{l}{(\delta_i,\delta_i)}$ is an integer.*
- *(iii)* [**c**] *$\sum_{i\in I} r_i\tilde{e}_i < 2$.*

Define $e \in \mathbb{F}$ by $\frac{1}{e} + \sum_{i\in I} r_i\tilde{e}_i = 2$. Choose a one dimensional euclidean $\mathbb{F}$ space X and $x \in X$ with $(x, x) = \frac{2e}{l}$. Put $E = X \oplus \bigoplus_{i\in I} E_i$. Put $\alpha = \frac{l}{2}(\frac{x}{e} - \sum_{i\in I} \tilde{\delta}_i)$, $\Pi = \{\alpha\} \cup \bigcup_{i\in I} \Pi_i$ and $\Phi = \langle \Pi\rangle$. Then Φ is a root system with base Π, α is a long root with $(\alpha, \alpha) = l$, $e_\alpha = e$, $\alpha^* = x$, $\{\Pi_i \mid i \in I\}$ is the set of connected components of $\Pi - \alpha$ and, for $i \in I$, $(\delta_i, \check{\alpha}) = -1$, and δ_i is the unique neighbor of α in Π_i.

Proof: A straight forward calculation shows that $(\alpha, \alpha) = l$ and so

$$\check{\alpha} = \frac{x}{e} - \sum_{i \in I} \tilde{\delta}_i.$$

Hence $(\delta_i, \check{\alpha}) = -1$ and $(\delta, \check{\alpha}) = 0$ for all other $d \in \Pi - \alpha$. Also $(\alpha, \check{d}_i) = r_i(\delta_i, \check{\alpha}) = -r_i$ is a negative integer. Hence Π is a linearly independent, obtuse pre-root system and so by 3.1.13 Φ is a root system.$x \perp \Pi - \alpha$ and $(x, \alpha) = \frac{l}{2e}(x, x) = 1$. So $x = \alpha^*$ and $e_\alpha = \frac{(\alpha,\alpha)(\alpha^*,\alpha^*)}{2} = \frac{l\frac{2e}{l}}{2} = e$. □

Lemma 3.7.13 [**echa**] *Let $((\check{\alpha})^* \mid \alpha \in \Pi)$ be the basis for E dual to $\check{\Pi}$. Then $(\check{\alpha})^* = \frac{(\alpha,\alpha)}{2}\alpha^*$ and $e_{\check{\alpha}} = e_\alpha$.*

Proof: Let $r := \frac{(\alpha,\alpha)}{2}$. Then $r\check{\alpha} = \alpha$. Clearly $\check{\beta} \perp r\alpha^*$ for all $\alpha \neq \beta \in \Pi$. Also

$$(r\alpha^*, \check{\alpha}) = (\alpha^*, r\check{\alpha}) = (\alpha^*, \check{\alpha}) = 1$$

and so $(\check{\alpha})^* = r\alpha^*$.

$$\begin{aligned} 2e_{\check{a}} &= (\check{a}, \check{a}) \cdot ((\check{a})^*, (\check{\alpha})^*) &&= (\check{a}, \check{a}) \cdot (r\alpha^*, r\alpha^*) \\ &= (r\check{a}, r\check{a}) \cdot (\alpha^*, \alpha^*) &&= (\alpha, \alpha) \cdot (\alpha^*, \alpha^*) &&= 2e_\alpha. \end{aligned}$$

So $e_{\check{a}} = e_\alpha$ and the lemma is proved. □

Lemma 3.7.14 [**pi a tree**] *Let Φ be a connected root system.*

(a) [**a**] *$\Gamma^0(\Pi)$ is a tree.*

(b) [**z**] *Let $\alpha \in \Pi_l$. Then $e_\alpha \geq \frac{1}{2}$ with equality iff $\Pi = \{\alpha\}$.*

(c) [**b**] *Suppose $\alpha \in \Pi_l$ with $e_\alpha < 1$. Then $\Phi \cong A_n$, α is an end-node of Π and $e_\alpha = \frac{n}{n+1}$.*

(d) [**c**] *Exactly one of the following holds:*

1. [**a**] *$\check{\alpha}_l = \beta^*$ for a long root $\beta \in \Pi$.*
2. [**b**] *$\Phi \cong A_n$ and $\check{\alpha}_l = \beta_1^* + \beta_n^*$ where β_1 and β_n are the end nodes of Π (with $\check{\alpha}_l = 2\beta_1^*$ if $|\Pi| = 1$).*
3. [**c**] *$\Gamma(\Pi^\circ) =$*  *and $\check{\alpha}_l = \beta^*$ for the short end-node β of Π.*

(e) [**y**] *If $\Phi \ncong A_n$ then α_l is an end-node of Π° and $\Gamma(\Pi^\circ)$ is a tree.*

(f) [**d**] *Suppose $\Phi \not\cong A_n$ and $\beta \in \Pi$ such that β^* is a minimal weight. Then β is an end-node of Π° and Π.*

Proof: (a) Let $\alpha \in \Pi$ be long. By induction each connected component of $\Pi - \alpha$ is a tree. Also by 3.7.11, α is joint to exactly one vertex from each connected component of $\Pi - \alpha$. Thus also Π is tree.

By 3.7.13 $e_\alpha = e_{\check{\alpha}}$. So (b) and (c) are true for (α, Φ) iff they are true for $(\check{\alpha}, \check{\Phi})$. So for (b) and (c) we assume without loss that α is long.

(b) By 3.7.11(d) $\frac{1}{e_\alpha} + \sum_{\delta\in\Delta} r_\delta \tilde{e}_\delta = 2$. So $e_\alpha \geq \frac{1}{2}$ with equality iff $\Delta = \emptyset$. Since Π is connected (b) holds.

(c) If $\Delta = \emptyset$, (c) holds with $n = 1$. Suppose that $|\Delta| > 0$. Then $e_\alpha < 1$ implies $\sum_{\delta\in\Delta} r_\delta \tilde{e}_\delta < 1$. Thus $\Delta = \{\delta\}$, $r_\delta = 1$ and $\tilde{e}_\delta < 1$. So by induction on Π, $\langle \Pi - \alpha\rangle \cong A_m$, δ is an end-node of $\Pi - \alpha$ and $e_\delta = \frac{m}{m+1}$. Thus $\Phi \cong A_{m+1}$ and $\frac{1}{e_\alpha} = 2 - \frac{m}{m+1} = \frac{m+2}{m+1}$ and (c) is proved.

(d) Since $\check{\alpha}_l$ is a dominant intergral weight $\check{\alpha}_l = \sum_{i=1}^k \beta_i^*$ for some $\beta_i \in \Pi$. Since $2 = (\alpha_l, \check{\alpha}_l) = 2 = \sum_{i=1}^l (\beta_i^*, \alpha_l)$ and $((b_i^*, \alpha_l)$ is a positive integer we get, $k \leq 2$.

If $k = 2$, then $(\beta_i^*, \alpha_l) = 1$, β_i^* is a minimal weight and so by 3.7.8(b), β_i^* is long. Also by 3.7.2(d), $(\beta_1^*, \beta_2^*) > 0$ and so $(\check{\alpha}_l, \check{\alpha}_l) > (\beta_1, \beta_1) + (\beta_2, \beta_2)$. Since β_i is long, $(\alpha_l, \alpha_l) = (\beta_i, \beta_i)$ and multiplication with $\frac{(\alpha_l,\alpha_l)}{2}$ gives $2 > e_{\beta_1} + e_{\beta_2}$. So $e_{\beta_i} < 1$ for at least one i. By (c), $\Phi \cong A_n$. For $\Phi = A_n$ we have $\alpha_l = e_0 - e_n$ and (d:2) holds in this case.

So suppose $k = 1$ and put $\beta = \beta_1$. If β is long, (d:1) holds. So suppose that β is not long. Put $r = \frac{(\alpha_l,\alpha_l)}{(\beta,\beta)}$. By 3.7.13,

$$(\check{\beta})^* = \frac{(\beta,\beta)}{2}\beta^* = \frac{(\beta,\beta)}{2}\check{\alpha}_l = \frac{1}{r}\alpha_\lambda.$$

Hence $\alpha_l = r(\check{\beta})^*$. Since $(\check{\beta})^*$ is an integral weight on $\check{\Phi}$ we conclude that r divides $(\alpha_l, \check{\alpha})$ for all $\alpha \in \Phi$. Choosing $\alpha = \alpha_l$ we see that $r = 2$. If $\alpha \in \Phi_l$ with $\alpha \neq \pm\alpha_l$ we get $\alpha \perp \alpha_l$. Let δ be a long root of minimal distance from β in $\Gamma^0(\Pi)$. Let Σ be the set of vertices of the path from β to δ. By 3.5.4 a we have $\Sigma \subseteq \mathbb{F}(\langle\Sigma\rangle_l)$ and so there exists a long root $\epsilon \in \Sigma^+$ with $\alpha_l \not\perp \epsilon$. Then $\alpha_l = \epsilon \in \mathbb{F}\Sigma$. Suppose $\rho \in \Pi \setminus \Sigma$. Then $\alpha_l \in \mathbb{F}\Sigma$ implies $(\rho^*, \alpha_l) = 0$, a contradiction to 3.7.2(d). Thus $\Sigma = \Pi$ and (d) holds.

(e) By (d), $\check{\alpha}_l = \beta^*$ for some $\beta \in \Pi$. Thus β is the unique neighbor of $-\alpha_l$ in $\Gamma^0(\Pi^\circ$. By (a), $\Gamma^0(\Pi)$ is a tree and so (e) holds.

(f) By (e), $-\alpha_l$ is an end-node of Π°. Hence by 3.7.8(e), also β is an end-node of Φ°.

□

Lemma 3.7.15 [**w pi**] *Let Φ be a connected root system with $|\Pi| > 1$. Put $\Sigma = \Pi \cap \alpha_l^\perp$ and let $\alpha \in \Pi \setminus \Sigma$.*

(a) [**a**] $w_\Pi = \omega_{\alpha_l} w_\Sigma = \omega_{\alpha_l} w_\Sigma$

(b) [**b**] $\alpha_l = (-w_\Pi)(\alpha) + w_\Sigma(\alpha)$.

(c) [**c**] $\Pi \setminus \Sigma = \{\alpha, (-w_\Pi)(\alpha)\}$.

(d) [**d**] $(-w_\Pi) \mid_\Sigma = (-w_\Sigma) \mid_\Sigma$.

(e) [**e**] *Each connected component of* $\Gamma(\Sigma)$ *is invariantt under* $-w_\Pi$.

Proof: (a) Let $\beta \in \Phi^+$ and put $\delta = w_\Sigma(\beta)$. We claim that $(\omega_{\alpha_l} w_\Sigma)(\beta) = \omega_\alpha(\delta) \in \Phi^-$. Since $\Sigma \perp \alpha_l$ we have $w_\Sigma(\alpha_l) = \alpha_l$. Since w_Σ is an isometry,

$$(*) \quad (\beta, \alpha_l\,) = (w_\Sigma \delta, w_\Sigma(\alpha_l)\,) = (\delta, \alpha_l\,)$$

Suppose first that $\beta \perp \alpha_l$. By 3.5.5(d), $\Phi \cap \alpha_l^\perp =< \Pi \cap \alpha_l^\perp\rangle = \langle\Sigma\rangle$ and so $\beta \in \langle\Sigma\rangle$. Thus by definition of w_Σ, $\delta = w_\Sigma(\beta) \in \Phi^-$. By (*), $\delta \perp \alpha_l$ and so $\omega_{\alpha_l}(\delta) = \delta \in \Phi^-$.

Suppose next that $(\beta\alpha_l, >)0$. Then since ω_{α_l} is an isomoetry and has order two

$$(\omega_{\alpha_l}(\delta), \alpha_l\,) = (\delta, \omega_{\alpha_l}(\alpha_l) = -(\delta, (\,)\alpha_l\,) = -(\beta, \alpha_l\,) < 0$$

and again $\omega_{\alpha_l}(\delta) \in \Phi^-$.

This proves the claim and so $w_\Pi = \omega_{\alpha_l} w_\Sigma$. Taking the inverse on both sides of this equation gives $w_\Pi = w_\Sigma \omega_{\alpha_l}$.

(b) Since $\Pi \neq \{\alpha\}$, 3.7.2(e) implies $\alpha_l \neq \alpha$ and so $(\alpha, \check{\alpha}_l\,) = 1$. Thus $w_{\alpha_l}(\alpha) = \alpha - \alpha_l$. Also $w_\Sigma(\alpha_l) = -\alpha_l$ and so by (a) $w_\Pi(\alpha) = w_\Sigma(w_{\alpha_l}(\alpha)) = w_\Sigma(\alpha) - \alpha_l$.

(c) Let $\Pi' = \Sigma \cup \{\alpha, (-w_\Pi)(\alpha)\}$. Note that $w_\Sigma(\alpha) \leq \langle\alpha, \Sigma\rangle \leq \mathbb{F}\Pi'$. So by (a) also $\alpha_l \leq \mathbb{F}\Pi'$. Suppose that $\beta \in \Pi \setminus \Pi'$, then $(\alpha_l, \beta^*\,) = 0$, a contradiction to 3.7.2(e).

(d) Since ω_{α_l} acts trivially on Σ, this follows from (a).

(e) Let $\mathcal{D}$ be the set of connected componenent of Σ. Then $-w_\Sigma = -\prod_{\Delta \in \mathcal{D}} w_\Delta$ fixes each $\Delta \in \mathcal{D}$. So (e) follows from (d). □

Proposition 3.7.16 [**decomposing affine**] *Suppose that* $\check{\alpha}_l = \alpha^*$ *for a long root* α. *Retain the notation from 3.7.11 and for* $\delta \in \Delta$ *let* Π_δ *be the connected component of* Σ *containing* δ.

(a) [**a**] $e_\alpha = 2$.

(b) [**b**] $-w_\Sigma(\delta) = \delta$ *for all* $\delta \in \Delta$.

(c) [**c**] $\sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta = \frac{3}{2}$.

(d) [**d**] *One of the following holds.*

1. [**a**] $|\Delta| = 3$, δ *is long and* $\Pi_\delta = \{\delta\}$ *for all* $\delta \in \Delta$.
2. [**b**] $\Delta = \{\delta, \epsilon\}$, δ *and* ϵ *are long,* $\Pi_\delta = \{\delta\}$ *and* $\tilde{e}_\epsilon = 1$.
3. [**c**] $\Delta = \{\delta, \epsilon\}$, $\Pi_\delta = \{\delta\}$, $\Pi_\epsilon = \{\epsilon\}$, δ *is long and* $r_\epsilon = 2$.
4. [**d**] $\Delta = \{\delta\}$, δ *is long and* $\tilde{e}_\delta = \frac{3}{2}$.

5. [**e**] $\Delta = \{\delta\}$, $\Pi_\delta = \{\delta\}$ *and* $r_\delta = 3$.

Proof: (a) $\epsilon_\alpha = \frac{(\alpha,\alpha)(\alpha^*,\alpha^*)}{2} = \frac{(\alpha_l,\alpha_l)(\check{\alpha}_l,\check{\alpha}_l)}{2} = 2$.

(b) Note that $(-w_\Pi)$ fixes α_l and Π and so also α and Δ. By 3.7.15(e), w_Π also fixes Π_δ and so $\Pi\delta \cap \Delta = \{\delta\}$. Thus $(-w_\Pi)(\delta) = \delta$ and (b) follows from 3.7.15(d).

(c) Follows (a) and 3.7.11(d).

(d) Let $\delta \in \Delta$. If $\tilde{e}_\delta < 1$, then by 3.7.14(c) $\langle \Pi_d \rangle \cong A_n$ and δ is an end-node in Π_δ. Thus (b) implies that $n = 1$ and so $\langle \Pi_\delta \rangle = \{d\}$. (d) now follows easily from (d).

Proposition 3.7.17 [**composing affine**] *Retain the assumptions and notations of 3.7.12. Suppose in addition that for all* $i \in I$,

(iii') [**a**] $\sum_{i\in I} r_i \tilde{e}_i = \frac{3}{2}$

(iv) [**b**] $-w_i\Pi_i(\delta_i) = \delta_i$.

Then $\alpha_l = \alpha + w(\alpha)$ *and* $\check{a}_l = \alpha^*$.

Proof: Put $\lambda = \sum_{i\in I} \tilde{\delta}_i$ and $w = \prod_{i\in I} w_{\Pi_i}$. Since $-w_{\Pi_i}$ normalizes Π_i and by (iv) fixes δ_i we have $-w_{\Pi_i}(\tilde{\delta}_i) = \tilde{\delta}_i$. Thus $w(\lambda) = -\lambda$. From (iii') we have $e = 2$ and $\check{\alpha} = \frac{1}{2}x - \lambda$. Hence $\check{\alpha} + w(\check{\alpha}) = x = \alpha^*$. Since $(x, x) = \frac{4}{l} = (\check{\alpha}, \check{\alpha})$ we see that $\check{x} = \alpha + w(\alpha)$. By 3.7.12, α is long and so also and x is a long root. Since $x = \alpha^*$ is dominant and α_l is the unique dominant long root, $\check{x} = \alpha_l$. □

Lemma 3.7.18 [**l-m**] *Let* λ *and* μ *dominant minimal integral weights on* Φ. *Then also* $\lambda - \mu$ *is minimal.*

Proof: Let $\alpha \in \Phi^+$. Then $(\lambda, \alpha) \in \{0, 1\}$ and $(\mu, \alpha) \in \{0, 1\}$ and so $(\lambda - \mu, \alpha) \in \{-1, 0, 1\}$. □

Lemma 3.7.19 [**basic min**]

(a) [**a**] *Let* $a, b \in E$ *with* $a \check{\preceq} b$. *Then* $a + \mathbb{Z}\check{\Phi} = b + \mathbb{Z}\check{\Phi}$.

(b) [**b**] *If* W *acts trivially on* $\check{\Lambda}/\mathbb{Z}\check{\Phi}$.

(c) [**c**] *Let* $e \in \overline{\mathfrak{C}}$. *Then* $\{b \in \overline{\mathfrak{C}} \mid b \check{\preceq} e\}$ *is finite.*

(d) [**d**] *Every coset of* $\mathbb{Z}\check{\Phi}$ *in* $\check{\Lambda}$ *contains a dominant integral weight which is minimal in* $\check{\Lambda}^+$ *wit respect to* $\check{\preceq}$.

Proof: (a) By definition of $\check{\prec}$, $b-a \in \mathbb{N}\check{\Pi} \leq \mathbb{Z}\check{\Phi}$.

(b). Let $\lambda \in \check{\Lambda}$ and $\alpha \in \Phi$. Then $\omega_\alpha(\lambda) = \alpha - (\lambda, \alpha)\check{\alpha} \in \lambda + \mathbb{Z}\check{\Phi}$.

(c) Let $b \in \overline{\mathfrak{C}}$ with $b\check{\prec}e$. Then $e - b \in \mathbb{N}\check{\Pi}$ and $e + b \in \overline{\mathfrak{C}}$. Thus $(e+b, e-b) \geq 0$ and $(b,b) \leq (e,e)$. By 3.1.6, $\mathbb{Z}\check{\Pi}$ is discret Hence also $b + \mathbb{Z}\Pi$ is discret. Therefore $\{b \in \overline{\mathfrak{C}} \mid b\check{\prec}e\}$ is discret and bounded and so by 3.1.7 finite.

(d). Let $\lambda \in \check{\Lambda}$. Then $w(\lambda)$ is dominant for some $w \in W$. By (c) there exists $b \prec w(\lambda)$ such that b is $\check{\prec}$-minimal in $\check{\Lambda}^+$. Then by (a) and (b), $b, w(\lambda)$ and λ all lie in the same coset of $\mathbb{Z}\check{\Phi}$. So (d) holds.

Lemma 3.7.20 [**min equal min**]

(a) [**b**] *Let $\lambda \in \mathbb{Z}\check{\Phi}$ be minimal. Then $\lambda = 0$.*

(b) [**a**] *Let $\lambda \in \check{\Lambda}^+$. Then λ is $\check{\prec}$- minimal if and only if λ is minimal.*

(c) [**c**] *Every coset of $\mathbb{Z}\check{\Phi}$ in $\check{\Lambda}$ contains a unique dominant minimal weight.*

Proof: Without loss Φ is connected.

(a) By 3.4.2 there exists $w \in W$ such that $w(\lambda$ dominant. Then also $w(\lambda)$ is minimal and we may assume that λ is dominant. Let $\lambda = \sum_{a\in\Pi} n_\alpha \check{\alpha}$ with $n_\alpha \in \mathbb{Z}$. Suppose that $\lambda \neq 0$. Let $a \in \Pi$. By 3.7.2(e), $(\lambda, \alpha^*) > 0$. So also $n_\alpha = \frac{2}{\alpha\alpha}(\lambda, \alpha^*) > 0$. Also $(\lambda, \alpha_l) = 1$ and since $(\check{\alpha}, \alpha_l) \in \mathbb{N}$ we conclude that there existts a unique $\alpha \in \Pi$ with $(\check{\alpha}, \alpha_l) \neq 0$. Moreover, $n_\alpha = 1 = (\check{\alpha}, \alpha_{\mathrm{l}})$. and α is long. As α is long $-1 \leq (\beta, \check{\alpha}) \leq 1$ for all $\pm\alpha \neq p \in \Phi$. Also Π is obtuse and so $-\check{\alpha}$ is a dominant minimal weight on $\Sigma := \Pi - \alpha$. Hence also $-\omega_\Sigma(-\check{\alpha}) = \omega_\Sigma(\check{\alpha})$ is a dominant minimal weight on Σ. Since $n_\alpha = 1$ we have $\lambda - \check{\alpha} \in \mathbb{Z}\check{\Sigma}$. By 3.7.19(b), α and and $\omega_\Sigma(\check{\alpha})$ lie in the same coset of $\mathbb{Z}\check{\Sigma}$. Thus $\lambda - \omega_\sigma(\check{\alpha}) \in \mathbb{Z}\check{\Sigma}$. By 3.7.18 $\lambda - \omega_\Sigma(\check{\alpha})$ is a minimal weight on Σ. Thus by induction $\lambda - \omega_\Sigma(\check{\alpha}) = 0$. So $\lambda = \omega_\Sigma(\check{\alpha})$. Thus $\check{\alpha}$ is a mimimal weight a contradiction to $(\alpha, \check{\alpha}) = 2$.

(b) and (c): We frist show that

(**) If $\lambda \in \check{\Lambda}^+$ is $\check{\prec}$- minimal then λ is minimal.

For this it suffices to show that $(\lambda, \alpha_{\mathrm{l}}) \leq 1$. Choose a long root δ of minimal height with $(\lambda, \delta) = (\lambda, \alpha)_\lambda$. Since λ is $\check{\prec}$-minimal, $\lambda - \check{\delta}$ is not dominant and so there exists $\beta \in \Pi$ with $(\lambda - \check{d}, \beta) < 0$. So

$$(*). \qquad (\lambda, \beta) < (\beta, \check{\delta}).$$

Suppose that $\delta \neq \beta$. Then since δ is long, $(\beta, \check{\delta}) = 1$ and so $(\lambda, \beta) = 0$. Hence $(\lambda, \omega_\beta(\delta)) = (\lambda, \delta) = (\lambda, \alpha)$ and $\omega_\beta(\delta)$ is a positive long root of smaller height than δ, a contradiction to the choice of δ. Hence $\delta = \beta$. So by (*) $(\lambda, \alpha_{\mathrm{l}}) = (\lambda, \delta) < (\delta, \check{\delta}) = 2$. Hence λ is minimal.

Next we show that

(***) Every coset of $\mathbb{Z}\check{\Phi}$ in $\check{\Lambda}$ contains at most one minimal dominant weight.

For this let λ and μ be minimal dominant weights in the same coset of $\mathbb{Z}\check{\Phi}$. Then $\lambda - \mu \in \mathbb{Z}\check{\Phi}$ and by 3.7.18, $\lambda - \mu$ is minimal. So by (a), $\lambda - \mu = 0$ and $\lambda = \mu$.

Now let λ be any dominant minimal weight in $\check{\Lambda}$. By 3.7.19(d), $\lambda + \mathbb{Z}\check{\Phi}$ contains a $\check{\prec}$-minimal element μ. By (**) μ is minimal and by (***) $\lambda = \mu$. Thus (b) holds.

(c) follows from (b), 3.7.19(d) and (***). □

Definition 3.7.21 [**o ab**]

(a) [**a**] *For a path* $p = (\alpha_0, \alpha_1, \dots \alpha_n)$ *in* $\Gamma^0(\Phi)$ *define* $s(p) = \prod_{i=1}^n |(\check{\alpha}_{i-1}, \alpha_i)|$.

(b) [**b**] *If* $\alpha, \beta \in \Pi$ *ie in the same connected component of* $\Gamma^(\Phi)$*, then* $\overline{\alpha\beta}$ *denotes the unique path in* $\Gamma^0(\Pi)$ *from* α *to* β.

(c) [**c**] $\det \Pi$ *is the number of minimal dominant weights for* Φ.

Lemma 3.7.22 [**basic det pi**]

(a) [**a**] $\det \Pi = |\check{\Lambda}/\mathbb{Z}\check{\Phi}| = \det C$.

(b) [**b**] *Let* $\alpha, \beta \in \Pi$ *If* α *and* β *are in the same connected componenent of* $\Gamma^0(\Pi)$*, then* $e_{\alpha\beta} = s(\overline{\alpha\beta})\frac{\det(\Pi - \overline{\alpha\beta})}{\det \Pi}$. *Otherwise* $e_{\alpha\beta} = 0$.

Proof: (a) By 3.7.20(c), $\det \Pi = |\check{\Lambda}/\mathbb{Z}\check{\Phi}|$.

Define $T \in \mathrm{End}_{\mathbb{Z}}(\check{\Lambda})$ by $T(\alpha^*) = \check{\alpha} = \sum_{\beta \in \Pi} c_{\alpha\beta}\beta^*$. Then $T(\check{\Lambda}) = \mathbb{Z}(\check{\Phi})$ and so

$$|\check{\Lambda}/\mathbb{Z}\check{\Phi}| = |\det T| = \det C$$

Thus (a) holds.

By 3.7.10(c) $E = C^{-1}$. Let $\alpha, \beta \in \Pi$. Then there either exists no path or exactly one path from α to β in $\Gamma^0(\Pi)$. In the first case 3.6.1 implies $e_{\alpha\beta} = 0$. In the second let $\overline{\alpha\beta} = (\alpha_0, \alpha_1 \dots, \alpha_n)$. Then since Π is obtuse,

$$(-1)^n \prod_{i=1}^n c_{\alpha_{i-1}\alpha_i} = \prod_{i=1} |(\check{a}_{i-1}, \check{\alpha}_i)| = s(\overline{\alpha\beta}).$$

Thus by 3.6.1

$$e_{\alpha\beta} = s(\overline{\alpha\beta})\frac{\det C(\Pi - \overline{\alpha\beta})}{\det C(\Pi)}.$$

(b) now follows from (a). □

3.8 The classification of root system

In the section we determine all the connected roots systems up to isomorphism. We also determine the affine diagrams, the action of $-w_\Pi$ on Π and the minimal weights. we combine all thus information in what we call the labeled affine diagram:

Recall that the non-zero minimal weights are all of the from α^* for some root $\alpha \in \Pi$. We will label such an α with $\det(\Pi - \alpha$. We also label $-\alpha_l$ with $\det \Pi$. We use a filled node to distinguish $-\alpha_l$ from the remaining vertices for Π°. We also draw a dotted line betweeen any two distinct elements of Π which are interchanged by $-w_\Pi$.

Theorem 3.8.1 [**labeled affine**] *The labeled affine diagrams of the connected root systems are exactly as listed in Figure 3.8.*

Proof: By induction we assume that labeled affine diagrams of rank smaller than n are exactly as in Figure 3.8.

Suppose we know the affine diagrams for the rank connected roots sytems. Then 3.7.8[f] gives us $\det \Pi$ and all $\alpha \in \Pi$ such that α^* is minimal. From 3.7.15 and induction we obtain the action of $-w_\Pi$ on Π. Also by induction we can compute $\det(\Pi - \alpha)$.

So it remains to determine the affine diagrams.

In case 3.7.14(d:2),d:2 we see that the $\Pi^\circ = A_n^\circ$ or $\Pi^\circ = C_n^\circ$.

So suppose that $\alpha_l = \alpha^*$ for a long root α.

We now consider the different case of 3.7.16(d).

In case d:1 $\Pi^\circ \cong D_4^\circ$.

In case d:2 Π_ϵ is a connected rank$n-2$ root system, $\tilde{e}_\epsilon = 1$ and $w_{\Pi_\epsilon}(\epsilon) = \epsilon$. Note that by 3.7.22(b), $\tilde{e}_\epsilon = \frac{\det(\Pi_\epsilon - \epsilon)}{\det \Pi_\epsilon}$ and so e_ϵ can be computed from the labeled affine diagram of Π_ϵ.

Suppose that $\Pi_\epsilon = A_{n-2}$. Then since w_{Π_ϵ} fixes ϵ, we get $n-2 = 2k+1$ and

$$1 = \tilde{e}_\epsilon = \frac{(\det \Pi(A_k))^2}{\det \Pi(A_{2k+1})} = \frac{(k+1)^2}{2k+2} = \frac{k+1}{2}.$$

Thus $k = 1$, $n = 5$ and $\Pi^\circ = D_5^\circ$.

If $\Pi_\epsilon = B_{n-2}$, then ϵ is the long end-node and $\Pi^\circ = B_n^\circ$ for $n \geq 5$.

If $\Pi_\epsilon = C_{n-2}$, then again ϵ is the long end-node, $\frac{n-2}{2} = 1$ and so $n = 4$ and $\Pi^\circ = B_4^\circ$.

If $\Pi_\epsilon = D_{n-2}$ then either ϵ is the left end-node or $n-2 = 4$. In any case $\Pi^\circ = D_n^\circ$.

According to Figure 3.8 no other possibilities occur in the current case.

In case d:3 $\Pi^\circ = B_3^\circ$.

In case d:4 Π_δ is a connected rank$n-1$ root system, $\tilde{e}_\delta = \frac{3}{2}$ and $w_{\Pi_\delta}(\delta) = \delta$.

If $\Pi_\delta = A_{n-1}$. $n-1 = 2k+1$ and $\frac{3}{2} = \tilde{e}_\delta = \frac{(k+1)^2}{2k+2} = \frac{k+1}{2}$. Thus $k = 2$, $n = 6$ and $\Pi^\circ = E_6^\circ$.

If $\Pi_\delta = C_{n-1}$, then ϵ is the long end-node, $\frac{n-1}{2} = \frac{3}{2}$ and so $n = 4$ and $\Pi^\circ = F_4^\circ$.

If $\Pi_\delta = D_{n-1}$ then ϵ is one of the right end-nodes end-node and $\frac{n-1}{4} = \frac{3}{2}$ so $n = 7$ and $\Pi^\circ = E_7^\circ$.

If $\Pi_\delta = E_7$, then δ is the right end-node and $\Pi^\circ = E_8^\circ$.

According to Figure 3.8 no other possibilities occurs in the current case.

In case d:5 $\Pi^\circ = G_2^\circ$.

Finally we remark that 3.7.17 ensures that all the root systems encounter actually do exit. □

Figure 3.2: The labeled affine diagrams

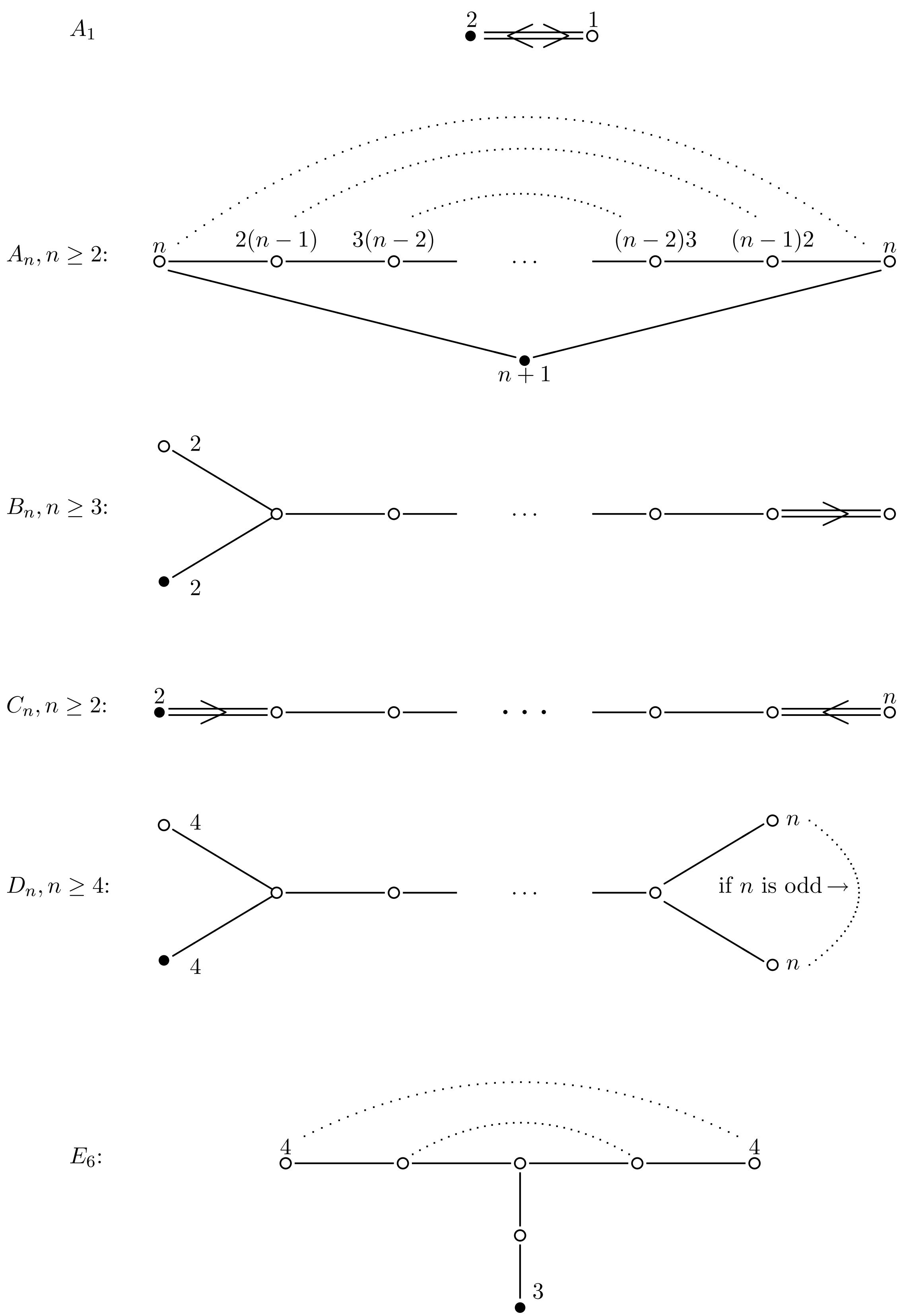
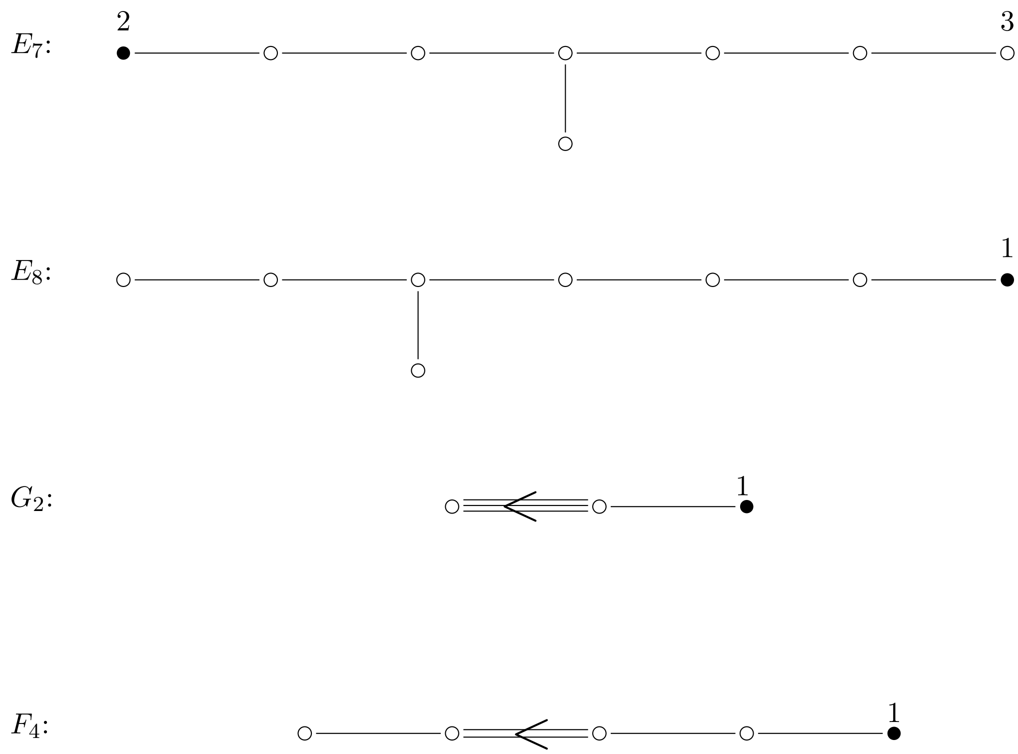


Figure 3.2: The labeled affine diagrams



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