Group actions on manifolds

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1. Terminology and notation

1.1. Lie group actions.

Definition 1.1. An *action of a Lie group G on a manifold M* is a group homomorphism

$$G \to \mathrm{Diff}(M),\ \ g \mapsto \mathcal{A}_g$$

into the group of diffeomorphisms on M, such that the *action map*

$$G \times M \to M, \quad (g, m) \mapsto \mathcal{A}_g(m)$$

is smooth.

We will usually write $g.m$ rather than $\mathcal{A}_g(m)$. With this notation, $g_1.(g_2.m) = (g_1g_2).m$ and $e.m = m$.

Remarks 1.2. (a) One has similar definitions of group actions in other categories. For instance, an action of a topological group G on a topological space X to be a homomorphism $G \to \mathrm{Homeo}(X)$ such that the action map $G \times X \to X$ is continuous. An action of a (discrete) group G on a set S is simply a homomorphism into the permutation group of S.

(b) Some people call what we've just introduced a *left-action*, and define a *right-action* to be an *anti*-homomorphism $G \to \mathrm{Diff}(M)$. For such a right action $g \mapsto \mathcal{B}_g$ one would then write $m.g := \mathcal{B}_g(m)$; with this notation

$$(m.g_1).g_2 = m.(g_1g_2)$$

Any right action can be turned into a left action by setting $\mathcal{A}_g = \mathcal{B}_{g^{-1}}$. In this course, we will avoid working with right actions.

Examples 1.3. 1) An action of the (additive) Lie group $G = \mathbb{R}$ is the same thing as a global flow, while an action of the Lie algebra $G = S^1$ is sometimes called a periodic flow.

2) Let V be a finite-dimensional vector space. Then V (viewed as an Abelian group) acts on itself by translation. Also the general linear group $\mathrm{GL}(V)$ acts on V by its defining

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representation. The actions fit together to an action of the affine linear group, the semi-direct product

$$\mathrm{GL}(V) \ltimes V, \quad (g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1.v_2).$$

(In fact, the formula for the semi-direct product is most easily remembered from this action.) A group homomorphism $G \to \mathrm{GL}(V)$ (i.e. representation) defines a *linear action* of G on V, and more generally a group homomorphism $G \to \mathrm{GL}(V) \ltimes V$ is called an *affine action*.

3) Any Lie group G acts on itself by multiplication from the left, $L_g(a) = ga$, multiplication from the right $R_g(a) = ag^{-1}$, and also by the *adjoint* (=conjugation) action

$$\mathrm{Ad}_g(a) := L_g R_g(a) = gag^{-1}.$$

4) Given a G-action on M, and a submanifold $N \subset M$ that is G-invariant, one obtains an action on N by restriction. For example the rotation action on $\mathbb{R}^n$ restricts to an action on the unit sphere. Similarly, given a group homomorphism $H \to G$ one obtains an action of H by composition with $G\,\mathrm{Diff}(M)$.

5) Suppose $H \subset G$ is a closed subgroup, hence (by a theorem of Cartan) a Lie subgroup. Let G/H by the space of right cosets $\{aH\}$ with the quotient topology. By a well-known result from the theory of Lie groups, there is a unique smooth structure on G/H such that the quotient map $G \to G/H$ is smooth. Moreover, the left G-action on G descends to an action on G/H:

$$g.(aH) = (ga)H.$$

For a detailed proof, see e.g. Onishchik-Vinberg, [**26**, Theorem 3.1].

6) Lie group often arise as transformation groups preserving a certain structure. For instance, the Myers-Steenrod theorem asserts that the group $\mathrm{Diff}(M, g)$ of isometries of a Riemannian manifold is a Lie group, compact if M is compact. Similarly, if M is a complex (or even an almost complex) manifold, the group of diffeomorphisms preserving the (almost) complex structure is a Lie group, provided M is compact. (By contrast, the group of symplectomorphisms of a symplectic manifold $\mathrm{Diff}(M, \omega)$ is of course infinite-dimensional!) The general setting for this type of problem is explained in detail in Kobayashi's book on transformation groups. Let M be a manifold with a reduction of the structure group of TM to some subgroups $H \subset \mathrm{GL}(n, \mathbb{R})$, and let $G \subset \mathrm{Diff}(M)$ be the group of automorphisms for this reduction. Call H *elliptic* if its Lie algebra does not contain a rank 1 matrix, and *finite type* if the kth prolongation

$$\mathfrak{h}_k = \{t : S^{k+1}\mathbb{R}^n \to \mathbb{R}^n | \text{ for all } x_1, \ldots, x_k, \quad t(\cdot, x_1, \ldots, x_k) \in \mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R})\}$$

vanishes for k sufficiently large. For elliptic H, G is always a Lie group, and for finite type H, G is a Lie group provided M is compact.

7) The group of automorphisms $\mathrm{Aut}(G)$ of a Lie group G is itself a Lie group. It contains the subgroup $\mathrm{Int}(G)$ of *inner* automorphism, i.e. automorphism of the form $a \mapsto \mathrm{Ad}_g(a)$. The left-action of G on itself first together with the action of $\mathrm{Aut}(G)$ to an action of the semi-direct product, $\mathrm{Aut}(G) \ltimes G$, where the product is as follows:

$$(\sigma_1, g_1)(\sigma_2, g_2) = (\sigma_1\sigma_2, g_1\sigma_1(g_2)).$$

1.2. Lie algebra actions. Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on M, with bracket $[X, Y] = X \circ Y - Y \circ X$ where we view vector fields as derivations on the algebra $C^\infty(M)$ of smooth functions.

DEFINITION 1.4. An *action of a finite-dimensional Lie algebra* $\mathfrak{g}$ on M is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \xi_M$ such that the action map

$$\mathfrak{g} \times M \to TM, \quad (\xi, m) \mapsto \xi_M(m)$$

is smooth.

On the right hand side of this definition, vector fields are viewed as sections of the tangent bundle $TM \to M$.

EXAMPLES 1.5. Any Lie algebra representation $\mathfrak{g} \to \mathrm{gl}(V)$ may be viewed as a Lie algebra action. If (M, g) is a Riemannian manifold, the Lie algebra $\mathfrak{X}(M, g) = \{X \,|\, L_X(g) = 0\}$ of Killing vector fields is finite-dimensional (by Myers-Steenrod), and by definition acts on M.

If $\gamma : \mathbb{R} \to M$ is a smooth curve on M, we denote its tangent vector at $\gamma(0)$ by $\frac{d}{dt}|_{t=0}\gamma(t)$.

THEOREM 1.6. *Given an action of a Lie group G on a manifold M, one obtains an action of the corresponding Lie algebra $\mathfrak{g}$, by setting*

$$\xi_M(m) = \frac{d}{dt}|_{t=0} \exp(-t\xi).m$$

where $\exp : \mathfrak{g} \to G$ *is the exponential map for G. The vector field ξ_M is called the* generating vector field *corresponding to ξ.*

PROOF. Let us first note that if G acts on manifolds M_1, M_2, and if $F : M_1 \to M_2$ is a G-equivariant map, i.e.

$$F(g.m) = g.F(m) \quad \forall m \in M_1$$

then the vector fields ξ_{M_1}, ξ_{M_2} are F-related. since F takes integral curves for ξ_{M_1} to integral curves for ξ_{M_2}. Since Lie brackets of pairs of F-related vector fields are again F-related, it follows that if F is onto, and $\xi \mapsto \xi_{M_1}$ is a Lie algebra homomorphism, then so is $\xi \mapsto \xi_{M_2}$.

Consider the map

$$F : G \times M \to M, \quad (a, m) \mapsto a^{-1}.m$$

This map is onto and has the equivariance property,

$$F(ag^{-1}, m) = F(a, g.m)$$

This reduces the problem to the special case of G acting on itself by the action $g \mapsto R_g$. This action commutes with left-translations, i.e. each $l_h : G \to G$ is a G-equivariant map. Thus, it generating vector fields ξ_G are left-invariant. Since

$$\xi_G(e) = \frac{d}{dt}|_{t=0} \exp(-t\xi).e = \frac{d}{dt}|_{t=0} \exp(-t\xi)^{-1} = \frac{d}{dt}|_{t=0} \exp(t\xi) = \xi,$$

we conclude $\xi_G = \xi^L$, the left-invariant vector field defined by ξ. But $[\xi^L, \eta^L] = [\xi, \eta]^L$ by definition of the Lie bracket. $\square$

EXAMPLE 1.7. The generating vector field for the left action of G on itself is $-\xi^R$, and the generating vector field for the adjoint action is $\xi^L - \xi^R$. The generating vector fields for the action of an isometry group $\mathrm{Iso}(M, g)$ are the Killing vector fields $\mathfrak{X}(M, g)$.

REMARK 1.8. Many people omit the minus sign in the definition of the generating vector field ξ_M. But then $\xi \mapsto \xi_M$ is not a Lie algebra homomorphism but an anti-homomorphism. The minus sign is quite natural if we think of vector fields as derivations: If G acts on M, we get an action on the algebra $C^\infty(M)$ by $(g.f)(x) = f(g^{-1}.x)$, i.e. a group homomorphism $G \to \mathrm{Aut}(C^\infty(M))$. The generating vector fields is formally (ignoring that $\mathrm{Aut}(C^\infty(M))$ is infinite-dimensional) the induced map on Lie algebras.

Let us now consider the inverse problem: Try to integrate a given Lie algebra action to an action of the corresponding group! We will need the following Lemma:

LEMMA 1.9. *Let G be a connected Lie group, and $U \subset G$ an open neighborhood of the group unit $e \in G$. Then every $g \in G$ can be written as a finite product $g = g_1 \cdots g_N$ of elements $g_j \in U$.*

PROOF. We may assume that $g^{-1} \in U$ whenever $g \in U$. For each N, let $U^N = \{g_1 \cdots g_N | \, g_j \in U\}$. We have to show $\bigcup_{N=0}^\infty U^N = G$. Each U^N is open, hence their union is open as well. If $g \in G \backslash \bigcup_{N=0}^\infty U^N$, then $gU \in G \backslash \bigcup_{N=0}^\infty U^N$ (for if $gh \in \bigcup_{N=0}^\infty U^N$ with $h \in U$ we would have $g = (gh)h^{-1} \in \bigcup_{N=0}^\infty U^N$.) This shows that $G \backslash \bigcup_{N=0}^\infty U^N$ is also open. Since G is connected, it follows that the open and closed set $\bigcup_{N=0}^\infty U^N$ is all of G. ☐

COROLLARY 1.10. *An action of connected Lie group G on a manifold M is uniquely determined by its generating vector fields.*

THEOREM 1.11. *Suppose $\xi \mapsto \xi_M$ is a Lie algebra action of $\mathfrak{g}$ on a manifold M. Then this Lie algebra action integrates to an action of the simply connected Lie group G corresponding to $\mathfrak{g}$, if and only if each ξ_M is complete.*

PROOF. Every G-action on M decomposes $G \times M$ into submanifolds $\mathcal{L}_m = \{(g, g.m) | \, g \in G\}$. Note that each $\mathcal{L}_m$ projects diffeomorphically onto G. The action may be recovered from this foliation: Given (g, m) the leaf $\mathcal{L}_m$ contains a unique point having g as its first component, and then the second component is $g.m$.

The idea of proof, given a $\mathfrak{g}$-action, is to construct this foliation from an integrable distribution. Consider the Lie algebra action on $G \times M$, taking ξ to $(-\xi^R, \xi_M) \in \mathfrak{X}(G \times M)$. The subbundle of the tangent bundle spanned by the generating vector fields is a distribution of rank $\dim G$, which is integrable by Frobenius' theorem. Hence we obtain a foliation of $G \times M$, with leaves of dimension $\dim G$.

Let $\mathcal{L}_m \hookrightarrow G \times M$ be the unique leaf containing the point (e, m). Projection to the first factor induces a smooth map

$$\pi_m : \ \mathcal{L}_m \to G,$$

with tangent map taking $(-\xi^R, \xi_M)$ to $-\xi^R$. Since the tangent map is an isomorphism, the map $\mathcal{L}_m \to G$ is a local diffeomorphism (that is, every point in $\mathcal{L}_m$ has an open neighborhood over which the map is a diffeomorphism onto its image). We claim that π_m is a diffeomorphism. Since G is simply connected, it suffices to show that π_m is a covering map. Let $U_0 \subset \mathfrak{g}$ be a star-shaped open neighborhood of 0 over which the exponential map is a diffeomorphism, and $U = \exp(U_0)$. Given $(g, m') \in \mathcal{L}_m$ and $\xi \in U_0$, the curve $t \mapsto \exp(-t\xi)g$ is an integral curve of $-\xi_R$. Letting F_t^ξ be the flow of ξ_M, it follows that $t \mapsto (\exp(-t\xi)g, F_t^\xi(m))$ is an integral curve of $(-\xi^R, \xi_M)$, so it lies in the leaf $\mathcal{L}_m$. This shows that there exists an open neighborhood of (g, m') mapping diffeomorphically onto the right translate Ug. This proves that π_m is a covering map, ans also that π_m is onto.

Using that π_m is a diffeomorphism, we can now define the action by

$$g.m := \mathrm{pr}_2(\pi_m^{-1}(g))$$

where pr_2 denotes projection to the second factor. Concretely, the above argument shows that if we write $g = \prod g_i$ with $g_i = \exp(\xi_i)$ then $g.m = g_1.(g_2.\cdots g_N.m)\cdots)$ where each g_i acts by its time one flow. This description also shows directly that $\mathcal{A}_g(m) = g.m$ defines a group action. □

1.3. Terminology.

Definition 1.12. Let $G \to \mathrm{Diff}(M)$ be a group action.

(a) For any $m \in M$, the set $G.m := \{(g, m)\, g \in G\}$ is called the *orbit* of m. The subgroup $G_m = \{g \in G |\, g.m = m\}$ is called the *stabilizer* of m.
(b) The action is *free* if all stabilizer groups G_m are trivial.
(c) The action is *locally free* if all stabilizer groups G_m are discrete.
(d) The action is *effective* if the kernel of the homomorphism $G \to \mathrm{Diff}(M)$ defining the action is trivial.
(e) The action is *transitive* if $G.m = M$ for some (hence all) $m \in M$.

The space $M/G = \{G.m|\, m \in M\}$ is called the orbit space for the given action.

From the definition, it is clear that stabilizer subgroups are closed subgroups of G, hence Lie subgroups. For any $g \in G$, the stabilizers of a point m and of its translate $g.m$ are related by the adjoint action:

$$G_{g.m} = \mathrm{Ad}_g(G_m).$$

Hence, each to each orbit $\mathcal{O} = G.m$ there corresponds a conjugacy class of stabilizers. For subgroups H, H' we will write $H \sim H'$ if H is G-conjugate to H'. Clearly, this is an equivalence relation. We denote by (H) the equivalence class of H. We define a partial ordering on equivalence classes, by writing $(H) < (K)$ if H is G-conjugate to a subgroup of K.

Definition 1.13. For any subgroup $H \subset G$ we define,

$$\begin{aligned} M^H &= \{m \in M |\, H \subset G_m\} \\ M_H &= \{m \in M |\, H = G_m\} \\ M^{(H)} &= \{m \in M |\, (H) < (G_m)\} \\ M_{(H)} &= \{m \in M |\, (H) = (G_m)\}. \end{aligned}$$

The set M^H is the *fixed point set* of H, and $M_{(H)}$ set of points *of orbit type* (H).

Notice that the sets $M^{(H)}, M_{(H)}$ are both G-invariant. In fact they are the flow outs of M^H, M_H:

$$M_{(H)} = G.M_H, \quad M^{(H)} = G.M^H.$$

Examples 1.14. (a) The rotation action of $S^1 \subset \mathrm{SO}(3)$ on S^2 has fixed point set M^{S^1} consisting of the north and south poles. There are two orbit types: The trivial orbit type $H = e)$ and the orbit type of the fixed points, $H = G$.

(b) For the G-action on a homogeneous space G/H, there is only one orbit type equal to H (since there is only one orbit). The action of $N_G(H)/H$, however, has a much more interesting orbit type decomposition.
(c) Consider the action of $\mathrm{SO}(3)$ on itself by conjugation. Let $q : \mathrm{SO}(3) \to [0, \pi]$ be the map that associates to each $A \in \mathrm{SO}(3)$ the corresponding angle of rotation. Since each rotation is determined up to conjugacy by its angle, we may view $[0, \pi]$ as the orbit space for the conjugation action, with q as the quotient map. There are three different orbit type strata: The fixed point set $M^{\mathrm{SO}(3)} = q^{-1}(0)$, the set $M^{(S^1)} = q^{-1}((0, \pi))$ (where $S^1 = \mathrm{SO}(2)$ is the subgroup of rotations about the z-axis) and $M^{(\mathrm{O}(2))} = q^{-1}(\pi)$ consisting of rotations by π. Notice that $M^{(\mathrm{O}(2))}$ is a submanifold diffeomorphic to $\mathbb{R}P(2)$.
(d) Exercise: Study the orbit space decomposition for the conjugation action of the groups $\mathrm{O}(2)$, $\mathrm{SU}(3)$, $\mathrm{PU}(3)$.

Definition 1.15. Let $\mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action.

(a) For $m \in M$, the subalgebra $\mathfrak{g}_m = \{\xi \,|\, \xi_M(m) = 0\}$ is called the stabilizer algebra of m.
(b) The Lie algebra action is called free if all stabilizer algebras are trivial.
(c) The Lie algebra action is called effective if the kernel of the map $\mathfrak{g} \to \mathfrak{X}(M)$ is trivial.
(d) The Lie algebra action is called transitive if the map $\mathfrak{g} \to T_m M,\ \xi \to \xi_M(m)$ is onto for all $\mathfrak{m} \in M$.

Proposition 1.16. *Let $G \to \mathrm{Diff}(M)$ be a Lie group action, with corresponding Lie algebra action given by the generating vector fields ξ_M. Then the Lie algebra of G_m is the stabilizer algebra $\mathfrak{g}_m$.*

Proof. Let ξ be an element of the Lie algebra of G_m. Thus $\exp(-t\xi).m = m$ for all $t \in \mathbb{R}$. Taking the derivative at $t = 0$, we see that $\xi_M(m) = 0$. Thus $\xi \in \mathfrak{g}_m$, showing that the Lie algebra of G_m is contained in $\mathfrak{g}_m$. For the converse, one has to show that if $\xi \in \mathfrak{g}_m$ then $\exp(t\xi) \in G_m$ (where exp is the exponential map for G). But this follows since the action of $\exp(-t\xi)$ is the flow of ξ_M, which fixes m since $\xi_M(m) = 0$. $\square$

Note that the stabilizer algebras are slightly special: For a general subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the corresponding connected subgroup $H \subset G$ need not be a closed subgroup.

Proposition 1.17. *The G-action is locally free if and only if the $\mathfrak{g}$-action is free. If the G-action is transitive then so is the $\mathfrak{g}$-action. The converse holds if M is connected. If the G-action is effective then so is the $\mathfrak{g}$-action.*

Proof. If the $\mathfrak{g}$-action is transitive, it easily follows that the G-orbits must be open. Since M is connected, it must be a single G-orbit. All other statements are obvious. $\square$

1.4. Proper actions.

Definition 1.18. A G-action on a manifold M is called *proper* if the map

$$G \times M \to M \times M,\ (g.m) \mapsto (m, g.m)$$

is proper (pre-images of compact sets are compact).

For instance, the action of a group on itself by left or right multiplication is proper (because the map $G \times M \to M \times M$, $(g.m) \mapsto (m, g.m)$ is a diffeomorphism in that case). Actions of compact groups are always proper. Given a proper G-action, the induced action of a closed subgroup is also proper.

The "irrational flow" on a 2-torus is a non-proper $\mathbb{R}$-action.

LEMMA 1.19. *The stabilizer groups G_m for a proper group action are all compact.*

PROOF. The pre-image of the point $\{m\} \times \{m\} \in M \times M$ is $G_m \times \{m\}$. □

Thus for example the conjugation action of G on itself is not proper unless G is compact.

PROPOSITION 1.20. *The orbits $\mathcal{O}$ for a proper G-action on M are embedded, closed submanifolds, with*

$$T_m(\mathcal{O}) = \{\xi_M(m) \,|\, \xi \in \mathfrak{g}\}.$$

PROOF. We need to check the proposition near any given $m \in \mathcal{O}$. Consider the map $\phi : G \to M$ taking g to $g.m$. This map is G-equivariant for the left-action of G on itself, hence it has constant rank. Hence by the constant rank theorem its image is an immersed submanifold: for each $g \in G$ there exists an open neighborhood $U \subset G$ such that $\phi(U)$ is an embedded submanifold of M, with tangent space the image of the tangent map. In particular this applies to $g = e$, where the image of the tangent map $\mathfrak{g} \mapsto T_m M$ is clearly spanned by the generating vector fields. We claim that $W \cap \phi(G) = W \cap \phi(U)$ for $W \subset M$ a sufficiently small open neighborhood of m. Since $\phi(U) = \phi(Uh)$ for $h \in G_m$, it is no loss of generality to assume that U is invariant under right-translation by G_m:

$$UG_m = U.$$

Now if the claim was false, we could find a sequence of points $g_k.m \in \mathcal{O}$, converging to m, but with $g_k \notin U$. The sequence $(m, g_k.m) \in M \times M$ is contained in a compact set, hence by properness the sequence $(g_k, m) \in G \times M$ must be contained in a compact set. Hence, by passing to a subsequence we may assume that the sequence g_k converges: $g_k \to g_\infty$. Passing to the limit in $g_k.m \to m$, we see $g_\infty \in G_m$. Since U is open, this means $g_k \in U$ for large k, a contradiction. □

In immediate consequence of the fact that the orbits for a proper action are closed, is that the orbit space M/G is Hausdorff. Our main goal in this Section is the *cross-section* theorem, giving a local normal form for the action near any G-orbit. A slightly weaker version of the theorem may be stated as follows. Let $\mathcal{O} = G.m$ be an orbit, $T\mathcal{O} \subset TM|_{\mathcal{O}}$ its tangent bundle, and

$$\nu_{\mathcal{O}} = TM|_{\mathcal{O}}/T\mathcal{O}$$

the normal bundle. Both TM and $T\mathcal{O}$ carry actions of G by vector bundle automorphisms, hence $\nu_{\mathcal{O}}$ carries an induced G-action. The tubular neighborhood theorem says that there exists a diffeomorphism $\psi : \nu_{\mathcal{O}} \to U \subset M$ onto an open neighborhood of $\mathcal{O}$, such that ψ restricts to the identity map on $\mathcal{O}$. The main point of the cross-section theorem is that one may choose the map ψ to be *G-equivariant.* Since G acts transitively on $\mathcal{O}$, this then reduces the problem of studying the G-action near $\mathcal{O}$ to the study of the linear action of $H = G_m$ on the fiber $V = (\nu_{\mathcal{O}})|_m$, called the *slice representation.*

We will begin our discussion of the slice theorem by considering two extreme cases: (i) the action being *free*, and (ii) m being a fixed point for the G-action.

THEOREM 1.21. *If $G \to \mathrm{Diff}(M)$ is a proper, free action, the orbit space M/G admits a unique smooth structure such that the quotient map $\pi : M \to M/G$ is a submersion. It makes M into a* principal G-bundle *over M/G.*

We recall that *a principal H-bundle* is a H-manifold P together with a smooth map $\pi : P \to B$ onto another manifold, having the following *local triviality* property: For each $x \in B$, there exists an open neighborhood U of x and a H-equivariant diffeomorphism

$$\pi^{-1}(U) \to U \times H$$

where the H-action on the right hand side is $h.(x, h_1) = (x, h_1 h^{-1})$. One calls P the total space and B the base of the principal bundle. The maps $\pi^{-1}(U) \to U \times H$ are called *local trivializations.*

PROOF. Since the action is free, the infinitesimal action map gives an isomorphism $T_m(G.m) = \mathfrak{g}$ for all $m \in M$. Choose a submanifold $S \subset M$ with $m \in S$ and $\mathfrak{g} \oplus T_m S = T_m M$. The action map restricts to a smooth map $G \times S \to M$, with tangent map at (e, m) equal to the given splitting. By continuity, the tangent map stays invertible at (e, m') for $m' \in S$ sufficiently close to m. Thus, choosing S sufficiently small we may assume this is true for all points in S. By equivariance, it then follows that $G \times S \to M$ has invertible tangent map everywhere, thus it is a local diffeomorphism onto its image. In fact, choosing S smaller if necessary it becomes a diffeomorphism onto its image: Otherwise, we could choose a non-convergent sequence $(g_k, m_k) \in G \times S$ with $m_k \to m$ and $g_k.m_k = m$. Thus $g_k^{-1}.m = m_k \to m$. The sequence of points $(m, m_k) = (g_k.m_k, m)$ is contained in a compact set, hence by properness the sequence g_k is contained in a compact set. The equation $g_k^{-1}.m = m_k \to m$ shows that for any convergent subsequence of g_k, the limit stabilizes m, hence is e. Thus $g_k \to e$, contradicting the choice of sequence. The diffeomorphism $G \times S \to V \subset M$ identifies V/G with S, hence gives a manifold structure on V/G. This defines a smooth structure on M/G such that the quotient map is a submersion. [2] □

It is important to note that the Theorem becomes *false* if we consider principal bundles in the category of topological spaces. In the definition of a topological principal H-bundle, one replaces manifolds by topological spaces, smooth maps by continuous maps, and Lie groups by topological groups.

EXERCISE 1.22. Give an example of a compact topological space with a free action of $\mathbb{Z}_2$ that does not define a principal bundle. (There is an easy such example, using the indiscrete topology on a set. Give an example where the space has a more reasonable topology, e.g. with P a subspace of $\mathbb{R}^3$.)

Let M be a manifold with a proper free G-action. Given an action of a second group G_1 on M, such that the actions of G and G_1 commute, the quotient space inherits a natural G_1-action. (Smoothness is automatic by properties of quotient maps). For instance, let H be a closed subgroup of G. The restriction of the right action to H is proper, and commutes with the G-action from the left. Hence the induced action on G/H makes $G \to G/H$ into a G-equivariant principal H-bundle.

[2] Compatibility of the local manifold structures is automatic: For a given map $F : M \to N$, from a manifold M onto a set N, there is at most one manifold structure on N such that F is a submersion: A function on N is smooth if and only if its pull-back to M is smooth.

At another extreme, we now consider the case that a point $m \in M$ is fixed under the group action. For proper actions, this can only happen if the group is compact.

THEOREM 1.23. *Let M be a G-manifold, with G compact, and $m \in M^G$ a fixed point for the action. Then there exists a G-invariant open neighborhood U of m and a G-equivariant diffeomorphism $T_mM \cong U$ taking 0 to m.*

PROOF. Choose a G-invariant Riemannian metric on M. The exponential map $\exp_0$: $T_mM \to M$ for this metric is G-equivariant, and its differential at the identity is invertible. Hence it defines a G-equivariant diffeomorphism $B_\epsilon(0) \to U \subset M$ for $\epsilon > 0$ sufficiently small. But $B_\epsilon(0)$ is diffeomorphic to all of V, choosing a diffeomorphism preserving radial directions. □

PROPOSITION 1.24. *Let $G \times M \to M$ be a proper G-action, and $H \subset G$ a subgroup. Then the each component of the set M^H of H-fixed points is a (topologically) closed embedded submanifold of M.*

PROOF. We first observe that the fixed point set of H coincides with the fixed point set of its closure $\overline{H}$. Indeed, if $m \in M$ is H-fixed, and $h_i \in H$ a sequence converging to $h \in \overline{H}$, then $h.m = \lim_i h_i.m = m$, showing $\overline{H} \subset G_m$. Since G_m is compact, $\overline{H}$ is compact also. Thus we may assume H is a compact embedded Lie subgroup of G. The Lemma (with $G = H$) shows that a neighborhood of m in M is H-equivariantly modeled by a neighborhood of 0 in T_mM. In particular M^H corresponds to $(T_mM)^H$, which is a linear subspace of T_mM. □

To formulate the slice theorem, we need some more notation. Suppose $P \to B$ is a principal H-bundle. For any linear representation of H on a vector space V, the quotient $(P \times V)/H$ is naturally a vector bundle over $P/H = B$: Indeed, any local trivialization $\pi^{-1}(U) = U \times H$ of P gives rise to a fiberwise linear local trivialization

$$(\pi^{-1}(U) \times V)/H = (U \times H \times V)/H \cong U \times V$$

of $(P \times V)/H$. One calls this the *associated vector bundle* with fiber V, and writes

$$P \times_H V := (P \times V)/H \to B$$

If another group G acts on P by principal bundle automorphisms (i.e. the action of G commutes with the action of H), the associated bundle becomes a G-equivariant vector bundle.

A special case of this construction is the case $P = G$, with H a closed subgroup of G acting by the right action. In this case $G \times_H V$ is a G-equivariant vector bundle over the homogeneous space G/H. It is easy to see that any G-equivariant vector bundle over a homogeneous space is of this form. The following result is due to Koszul and Palais.

THEOREM 1.25 (Slice theorem for proper actions). *Let M be a manifold with proper action of G. Let $m \in M$, with stabilizer $H = G_m$, and denote by $V = T_mM/T_m(G.m)$ the* slice representation*. Then there exists a G-equivariant diffeomorphism $G \times_H V \to M$ taking $[(e, 0)]$ to m.*

PROOF. As above, we can choose an H-equivariant diffeomorphism $T_mM \to U \subset M$, taking 0 to m. Identifying V with a complement of $T_m(G.m)$ in T_mM (e.g. the orthogonal complement for an invariant inner product), we obtain an H-equivariant embedding $V \to M$, as a submanifold transversal to the orbit $G.m$. It extends to a G-equivariant map $G \times_H V \to M$.

As before, we see that the map $G \times_H V \to M$ is in fact a diffeomorphism over $G \times_H B_\epsilon(0)$ for ϵ sufficiently small. Choosing an H-equivariant diffeomorphism $V \cong B_\epsilon(0)$ the Slice Theorem is proved. $\square$

COROLLARY 1.26 (Partitions of unity). *If $G \times M \to M$ is a proper group action, and $M = \bigcup_\alpha U_\alpha$ a locally finite cover by invariant open sets, there exists an invariant partition of unity: I.e functions χ_α supported in U_α, with $0 \le \chi_\alpha \le 1$, and $\sum_\alpha \chi_\alpha = 1$.*

PROOF. Exercise. $\square$

COROLLARY 1.27 (Invariant Riemannian metrics). *If $G \times M \to M$ is a proper group action, there exists a G-invariant Riemannian metric on M.*

PROOF. Given an H-invariant inner product on V and an H-invariant inner product on $\mathfrak{g}$, one naturally constructs an invariant Riemannian metric on $G \times_H V$. This constructs the desired metric near any orbit, and globally by a partition of unity. $\square$

COROLLARY 1.28 (Equivariant tubular neighborhood theorem). *If $G \times M \to M$ is a proper group action, and $N \subset M$ a G-invariant embedded submanifold with normal bundle ν_N, there exists a G-equivariant diffeomorphism from ν_N to a neighborhood of N in M, restricting to the identity map from the zero section $N \subset \nu_N$ to $N \subset M$.*

PROOF. This follows from the standard proof of the tubular neighborhood theorem, by using a G-invariant Riemannian metric on M. $\square$

1.5. The orbit type decomposition. We will now describe the orbit type stratification of a manifold with a proper action. [3] We begin with some general definitions. A decomposition $X = \bigcup_i X_i$ of a topological space is called *locally finite* if each compact set in X meets only finitely many X_i. It satisfies the *frontier condition* if

$$X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}.$$

In this case define a partial ordering of the pieces X_i, and in fact of the indexing set, by setting $i \le j \Leftrightarrow X_i \subset \overline{X_j}$. The *depth* of a piece X_i is defined to be the largest k for which there exist pieces X_{i_j} with $i_k > i_{k-1} > i_1 \ge i$. The depth of a (finitely) decomposed space is the largest depth of any of its pieces.

If X is a decomposed space, then also the open cone over X,

$$\mathrm{cone}(X) := [0, \infty) \times X / (\{0\} \times X)$$

is a decomposed space, with pieces the tip of the cone together with all $\mathrm{cone}(X)_i = (0,1) \times X_i$. Clearly,

$$\mathrm{depth}(\mathrm{cone}(X)) = 1 + \mathrm{depth}(X)$$

Following Sjamaar-Lerman we define the notion of a stratified singular space[4] as follows:

[3] A good reference for this material is the paper by Sjamaar-Lerman [**28**]. For stratified spaces in general, see e.g. the book [**15**] by Goresky-MacPherson.

[4] There are other notions of a *stratified space*, the most common definition being due to Whitney. See Goresky-MacPherson [**15**] for this definition and Duistermaat-Kolk [**13**] for its application in the context of group actions.

DEFINITION 1.29. A depth k *stratification* of a topological space X is a locally finite decomposition $X = \bigcup_i X_i$ satisfying the frontier condition, with each X_i a smooth manifold (called the stratum), with the following property: For each $m \in X_i \subset X$ there exists an open neighborhood $U \subset X_i$ around m, and a stratified space L of depth at most $k-1$, together with a homeomorphism

$$U \times \mathrm{cone}(L) \to V \subset X$$

preserving the decompositions, and restricting to diffeomorphisms between strata.

Now let M be a manifold with a proper G-action, and $X = M/G$ the orbit space. (Recall that this is a singular space, in general). The decomposition $M = \bigcup_{(H)} M_{(H)}$ into orbit types induces a decomposition $X = \bigcup_{(H)} X_{(H)}$ where $X_{(H)} = M_{(H)}/G$. Decompose X further as

$$X = \bigcup_i X_i$$

where each X_i is a component of some $X_{(H_i)}$, and

$$M = \bigcup M_i$$

be the corresponding decomposition of M, where M_i is the pre-image of X_i. (Of course, if G is connected then the M_i will be connected.) We will call this the *orbit type decomposition* of M and of the orbit space X, respectively.

THEOREM 1.30 (Orbit type stratification). *The decompositions $M = \bigcup M_i$ and $X = \bigcup X_i$ are locally finite and satisfy the frontier condition. Each M_i is a smooth embedded submanifold of M, and $X_i = M_i/G$ inherits a unique manifold structure for which the quotient map is a submersion. With these manifold structures, the above decompositions are in fact stratifications.*

PROOF. Near any orbit $G.m \subset M_{(H)}$, with $G_m = H$, M is modeled by the associated bundle $G \times_H V$, with action

$$g_1.[(g,v)] = [(g_1 g, v)]$$

By definition of the associated bundle, we have $[(g_1 g, v)] = [g, v]$ if and only there exists $h \in H$ with $(g_1 g, v) = (gh^{-1}, h.v)$. This means $h \in H_v$ and $g_1 = gh^{-1}g^{-1}$. That is, the stabilizer group of $[(g,v)]$ is

$$G_{[(g,v)]} = \mathrm{Ad}_g(H_v),$$

where $H_v \subset H$ is the stabilizer of v. In particular, all stabilizer groups of points in the model are subconjugate to H. The stabilizer group is conjugate to H if and only if $H_v = H$. This shows

$$(G \times_H V)_{(H)} = G \times_H V^H = V^H \times G/H$$

which is a vector subbundle, in particular a submanifold. This shows that all M_i are embedded submanifolds. Furthermore,

$$(G \times_H V)_{(H)}/G = V^H$$

showing that the X_i are smooth manifolds also.

We next analyze how the strata fit together. Note that $G_{[(g,v)]} = G_{[(g,tv)]}$ for all $t \neq 0$. It follows that each orbit type stratum $(G \times_H V)_{(H_1)}$ for $(H_1) < (H)$ is invariant under scaling $(g,v) \mapsto (g,tv)$. In particular, the zero section G/H is in the closure of each orbit type stratum.

It follows that a component M_i of $M_{(H)}$ can meet the closure of some $M_{(H')}$ only if $(H') < (H)$, and in that case it is in fact contained in the closure.

Choose an H-invariant inner product on V and let W be the orthogonal complement of V^H in V. We have,

$$G \times_H V = V^H \times (G \times_H W) = V^H \times (G \times_H \operatorname{cone} S(W))$$

where $S(W) \subset W$ is the unit sphere bundle. Notice that all stabilizer groups for the action of H on $S(W)$ are proper subgroups of H. Thus, by induction, we may assume that the orbit type decomposition for the H-action on $S(W)$ gives a stratification: $S(W) = \bigcup_j S(W)_j$. By the above discussion, the orbit type decomposition for $G \times_H V$ is given by

$$(G \times_H V)_j = V^H \times (0, \infty) \times (G \times_H S(W)_j),$$

together with the stratum $V^H \times G/H$. This shows that the orbit type decomposition is a stratification. The orbit space $X = M/G$ is locally modeled by

$$(G \times_H V)/G = V/H = V^H \times W/H = V^H \times \operatorname{cone}(S(W)/H)$$

and induction shows that this is a stratified singular space. □

Each $M_{(H)}$ is naturally a fiber bundle over $X_{(H)}$, with fiber G/H. One can be more precise:

THEOREM 1.31. *For any $H \subset G$ there is a natural principal K-bundle over $P_{(H)} \to X_{(H)}$, with $K = N_G(H)/H$, such that*

$$M_{(H)} = P_{(H)} \times_K (G/H).$$

PROOF. Recall that $M_H = \{m \in M |\, G_m = H\}$. Clearly $G.M_H = M_{(H)}$. In the local model $G \times_H V$,

$$(G \times_H V)_H = N_G(H)/H \times V^H.$$

Thus M_H is a submanifold. If $m \in M_H$, then $g.m \in M_H$ if and only if $g \in N_G(H)$. The stabilizer for the action of $N_G(H)$ on M_H is exactly H everywhere. Thus $K = N_G(H)/H$ acts freely, with quotient $X_{(H)}$. Let $K = N_G(H)/H$ act on G/H by the action induced from the right multiplication, and let G act by left multiplication. Then

$$M_H \times_K G/H \to M_{(H)}, \quad [(m, gH)] \mapsto g.m$$

is well-defined, and is a diffeomorphism. □

1.6. Principal orbit type theorem.

THEOREM 1.32 (Principal orbit type theorem). *Let $G \times M \to M$ be a proper group action, with connected orbit space M/G. Among the conjugacy classes of stabilizer groups, there is a unique conjugacy class (H_{prin}) with the property that $(H_{\text{prin}}) < (H)$ for any other stabilizer group $H = G_m$. The corresponding orbit type stratum $M_{\text{prin}} := M_{(H_{\text{prin}})}$ is open and dense in M, and its quotient $X_{\text{prin}} = M_{(H_{\text{prin}})}/G \subset X$ is open, dense and connected.*

On calls H_{prin} (or any subgroup conjugate to it) a *principal stabilizer*, and $M_{\text{prin}}, X_{\text{prin}}$ the *principal stratum* of M, X. Note that if G is connected, then M_{prin} itself is connected.

PROOF. Note first that by definition of "depth",

$$\begin{aligned}\operatorname{depth}(M_i) > 0 \quad &\Leftrightarrow \quad \exists j \neq i : \; M_i \subset \overline{M_j} \\ &\Leftrightarrow \quad \exists j \neq i : \; M_i \cap \overline{M_j} \neq \emptyset \\ &\Leftrightarrow \quad \exists j \neq i : \; M_i \cap \overline{M_j} \cap M_j \neq \emptyset \\ &\Leftrightarrow \quad M_i \text{ is not closed.}\end{aligned}$$

Thus the orbit type strata of depth 0 are all open, and all other orbit type strata M_j are embedded submanifolds of positive codimension. It follows that the union of depth 0 orbit type strata is open and dense. We have to show that there exists a unique orbit type stratum of depth 0 (recall that $X_j = M_j/G$ are all connected by definition).

We use induction on the depth of the stratification. For any point $m \in M_j$ with $\operatorname{depth}(M_j) > 0$, consider the local model $G \times_H V = V^H \times G \times_H W$. where $H = G_m$ and $V = T_mM/T_m(G.m)$. By induction on the dimension of the depth of the stratification, the theorem applies for the H-action on $S(W)$. In particular, there exists a principal stabilizer $H_{\mathrm{prin}} \subset H$ for this action, with $S(W)_{(H_{\mathrm{prin}})}/H$ connected. Then H_{prin} is also a principal stabilizer for the G-action on $G \times_H V$, and

$$(G \times_H V)_{(H_{\mathrm{prin}})}/G = V_{(H_{\mathrm{prin}})}/H = V^H \times (0, \infty) \times S(W)_{(H_{\mathrm{prin}})}/H$$

is *connected*, open and dense. This shows that if M_i, M_i' have depth 0, and their closures intersect in $m \in M_j$, then the two must be equal. [5] □

In general, M_{prin} need not be connected even if M is connected: A counterexample is the action of $\mathbb{Z}_2$ on $\mathbb{R}$ generated by $t \mapsto -t$.

Besides the orbit type decomposition, one can also consider the decomposition into infinitesimal orbit types, by partitioning M into

$$M_{(\mathfrak{h})} = \{m \in M | \; \mathfrak{g}_m \sim \mathfrak{h}\}$$

(where $\sim$ denotes G-conjugacy of subalgebras of $\mathfrak{g}$). and then decomposing each $M_{(\mathfrak{h})}$ further into connected components. Thus $M_{(\mathfrak{h})}$ is the union of all $M_{(H)}$ where H ranges over subgroups having $\mathfrak{h}$ as its Lie algebra. Again, the local model shows that each $M_{(\mathfrak{h})}$ is a submanifold, in fact

$$(G \times_H V)_{(\mathfrak{h})} = G \times_H (V^{\mathfrak{h}})$$

where $V^{\mathfrak{h}}$ is the subspace fixed by $\mathfrak{h}$. If we decompose M into the $M_{(\mathfrak{h})}$'s and then decompose further into connected components M_i, we again obtain a stratification. We call these the *infinitesimal orbit type strata.* The same inductive argument as before shows that if M is connected, then there exists a unique open stratum, which we denote M_{reg} and call *regular elements*, following Duistermaat-Kolk [**13**][6]

THEOREM 1.33. *For any proper G-action on a connected manifold M, the set M_{reg} of regular elements is open, dense and connected.*

[5] Strictly speaking, the inductive argument only worked for finite depth stratifications. However, if $\operatorname{depth}(M) = \infty$ the same proof shows how to go from finite to infinite depth: Note that the H-actions on $S(W)$ always have finite depth, by compactness.

[6] The terminology "regular" is not entirely standard, in contrast to "principal".

PROOF. The complement is the union of all $M_{(\mathfrak{h})}$ with $\mathfrak{h}$ a non-minimal infinitesimal stabilizer. In the local model, $(G \times_H V)_{(\mathfrak{h})} = G \times_H V^{\mathfrak{h}}$. But $V^{\mathfrak{h}}$ has codimension at least two in V, since $\mathfrak{h}$ is the Lie algebra of a compact group, and non-trivial representations of compact groups are at least 2-dimensional. Hence $M_{(\mathfrak{h})}$ has codimension at least 2, so removing it doesn't disconnect M. □

1.7. Example: The adjoint action of G on its Lie algebra. Let G be a compact, connected Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint action. We would like to describe the orbit type decomposition for this action, as well as the orbit space. Let T be a maximal torus (i.e. a maximal *connected* abelian subgroup) in G, and $\mathfrak{t}$ its Lie algebra. We will need the following two facts from the theory of Lie groups:

a) Any two maximal tori are conjugate in G. (The standard proof of this is to show that at all regular points, the map $G/T \times T \to G$, $(gT, t) \mapsto gtg^{-1}$ is orientation-preserving. Hence it has positive mapping degree, which implies that it must by onto.)

b) If H is any torus in G, and g commutes with all elements of H, there exists a torus containing $H \cup \{g\}$. (To prove this, consider the closed subgroup B generated by H and g. It is easy to see that B is the direct product of a torus and a cyclic group. Hence there exists $x \in B$ such that the subgroup generated by x is dense in B. Choose a maximal torus T containing x, then also $B \subset T$ and we are done.)

Part b) shows in particular that maximal tori are maximal abelian. (The converse does not hold in general.) It also shows that all stabilizers G_ξ for the adjoint action are connected, and in fact that G_ξ is the union of all maximal tori for which the Lie algebra contains ξ. Indeed, if $g \in G_\xi$, then the torus H given as the closure of the 1-parameter subgroup $\exp(t\xi)$ fixes g, so there is a maximal torus containing $H \cup \{g\}$. (Conversely, any such torus lies in G_ξ.) Since all G_ξ are connected, orbit types and infinitesimal orbit types coincide.

For any maximal torus T, one can choose ξ such that the closure of $\exp(t\xi)$ equals T. Then $G_\xi = T$, since any other element stabilizing ξ would commute with T. Thus, the principal stabilizer for the adjoint action is (T). To determine the corresponding principal orbit type stratum, choose an invariant inner product on $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{g}^T \oplus \mathfrak{m} = \mathfrak{t} \oplus \mathfrak{m}$$

denote the orthogonal decomposition. [7] The subspace $\mathfrak{m}$ decomposes into a direct sum of irreducible representations of T. By representation theory of tori, any irreducible representation of T is equivalent to a representation of the form $\exp(\xi) \mapsto R_{2\pi\langle\alpha,\xi\rangle}$ where R_ϕ is the 2-dimensional rotation defined by ϕ. The element $\alpha \in \mathfrak{t}^*$ is a *root* and the corresponding 2-dimensional subspace $\mathfrak{m}_\alpha$ is called a root space.[8] We thus find that given elements $\xi \in \mathfrak{t}$, $\zeta \in \mathfrak{g}$, commute if and only if ζ has no component in any root space $\mathfrak{m}_\alpha$ with $\langle\alpha, \xi\rangle = 0$. We therefore find:

THEOREM 1.34. *For any $\xi \in \mathfrak{t}$, the infinitesimal stabilizer $\mathfrak{g}_\xi$ is the direct sum of $\mathfrak{t}$ together with all root space $\mathfrak{m}_\alpha$ such that $\langle\alpha, \xi\rangle = 0$. The stabilizer group G_ξ is the connected subgroup of G with Lie algebra $\mathfrak{g}_\xi$.*

The equations $\langle\alpha, \xi\rangle = 0$ subdivide $\mathfrak{t}$ into chambers, and each wall corresponds to a given orbit type. We know that the interior of each chamber corresponds to the principal orbit type,

[7] In fact, $\mathfrak{m}$ is the *unique* such complement, but this need not concern us here.

[8] The sign of α changes if one changes the orientation of $\mathbb{R}^2$.

i.e their union is $M_T \subset M_{(T)}$. Since the principal orbit type stratum is connected this easily implies that the *Weyl group* $N_G(T)/T$ acts transitively on the set of chambers.

Of course, there is much more to be said about this example, see e.g. Broecker-tom Dieck [**7**] or Duistermaat-Kolk [**13**].

2. Classifying bundles

2.1. Principal bundles. We recall the definition of principal bundles in the topological category.

DEFINITION 2.1. Let G be a topological group. A principal G-bundles is a topological space P, together with a continuous action of G satisfying the following *local triviality* condition: For any $x \in B = P/G$ there exists an open neighborhood U os x and a G-equivariant homeomorphism

$$\pi^{-1}(U) \to U \times G.$$

Here $\pi : P \to B$ is the quotient map and the action of G on $U \times G$ is given by $g.(y,h) = (y, hg^{-1})$. One calls P the total space and B the base of the principal bundle.

Thus, a principal G-bundle is the special case of a fiber bundle, where all fibers are principal homogeneous G-spaces (spaces with free transitive G-actions), and the local trivializations take the fibers to the "standard" principal homogeneous G-spaces, G itself.

A morphism of principal bundles $\pi : P \to B$ and $\pi' : P' \to B'$. is a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

where the upper horizontal map is G-equivariant.

EXAMPLES 2.2.

(a) The trivial bundle $P = B \times G$, with action $g.(y,h) = (y, hg^{-1})$. (It is standard to write principal bundle actions as multiplications from the right.)

In general, a principal bundle is isomorphic to the trivial bundle if and only if it admits a section $\sigma : B \to P$, $\pi \circ \sigma = \mathrm{id}_B$; in this case the isomorphism is given by

$$B \times G \to P,\ (x,g) \mapsto g^{-1}.\sigma(x).$$

(b) We had seen that if G is a Lie group acting freely and properly on a manifold M, then $P = M$ is a principal G-bundle over $B = M/G$. Important cases include: S^n as a principal $\mathbb{Z}_2$-bundle over $\mathbb{R}P^n$, and S^{2n+1} as a principal U(1)-bundle over $\mathbb{C}P^n$.

(c) More generally, for $k \le n$ one has the Stiefel manifold $\mathrm{St}_{\mathbb{R}}(k,n)$ of orthonormal k-frames in $\mathbb{R}^n$,

$$\mathrm{St}_{\mathbb{R}}(k,n) = \{(v_1, \ldots, v_k) \in \mathbb{R}^{kn} \,|\, v_i \cdot v_j = \delta_{ij}\}$$

as a principal $\mathrm{O}(k)$-bundle over the Grassmann manifold $\mathrm{Gr}_{\mathbb{R}}(k,n)$ of k-planes in $\mathbb{R}^n$. Indeed, the Stiefel manifold is a homogeneous space $\mathrm{St}_{\mathbb{R}}(k,n) = O(n)/O(n-k)$, where we think of $O(n-k)$ as the subgroup of $O(n)$ fixing $\mathbb{R}^k \subset \mathbb{R}^n$. (Alternatively, $\mathrm{St}_{\mathbb{R}}(k,n)$ may be thought of as the space of linear injections $\mathbb{R}^k \to \mathbb{R}^n$ preserving inner products.) The Grassmann manifold is a homogeneous space $\mathrm{Gr}_{\mathbb{R}}(k,n) = O(n)/(O(n-k) \times O(k))$. The quotient map takes $v_1, \ldots, v_k$ to the k-plane they span.

(d) Similarly we have a complex Stiefel manifold of unitary k-frames in $\mathbb{C}^n$, which is a principal $\mathrm{U}(k)$-bundle over the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$.

(e) If V is a real vector space of dimension k, the space of isomorphisms $V \to \mathbb{R}^k$ is a principal homogeneous space for $\mathrm{GL}(k,\mathbb{R})$. Thus is $\rho : E \to B$ is a real vector bundle of rank k, we obtain a principal $\mathrm{GL}(k,\mathbb{R})$-bundle $\mathrm{Fr}(E) \to B$ (called the *frame bundle*) with fibers $\pi^{-1}(x)$ the space of isomorphisms $\rho^{-1}(x) \to \mathbb{R}^k$. (One can think of this isomorphism as introducing a basis (frame) in $\rho^{-1}(x)$, hence the name.) Similarly, if E carries a fiberwise inner product, one has a bundle of orthonormal frames $\mathrm{Fr}_O(E) \to B$ which is a principal $\mathrm{O}(k)$ bundle.

(f) If $E \to B$ is a complex vector bundle of dimension k, one similarly defines the (complex) frame bundle $\mathrm{Fr}(E) \to B$ with structure group $\mathrm{GL}(k,\mathbb{C})$, and given fiberwise Hermitian inner products, one defines a unitary frame bundle $\mathrm{Fr}_{\mathrm{U}}(E) \to B$.

We will need two basic constructions with principal bundles $\pi : P \to B$.

2.2. Pull-backs. Let $\pi : P \to B$ be a principal G-bundle. If $f : X \to B$ is a continuous map, one defines a new principal bundle $f^*B \to X$ with fibers

$$(f^*P)_x = P_{f(x)}.$$

(A local trivialization of P over U gives a local trivialization of f^*P over $f^{-1}(U)$.) One has a commutative diagram,

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d] & P \arrow[d] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

which in fact defines f^*P up to isomorphism.

LEMMA 2.3. *Let X be a paracompact Hausdorff space (e.g. CW complex or a manifold). If $f_0, f_1 : X \to B$ are homotopic maps, the pull-back bundles f_0^*P and f_1^*P are isomorphic.*

The Lemma is essentially equivalent to the statement that for any principal bundle P over $X \times I$, the two pull-backs to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic. We will omit the proof, which can be found e.g. in Husemoller's book.

COROLLARY 2.4. *If $f : X \to Y$ is a homotopy equivalence between paracompact Hausdorff spaces, the pull-back map sets up a bijections between $\mathrm{Prin}_G(X)$ and $\mathrm{Prin}_G(Y)$. In particular, if X is a contractible paracompact Hausdorff space, then $\mathrm{Prin}_G(X)$ has only one element consisting of the trivial bundle.*

EXERCISE 2.5. Show that if $\pi : P \to B$ is any principal bundle, $\pi^*P \cong P \times G$.

2.3. Associated bundles. Let F be a topological space with a continuous G-action. Then the *associated bundle*

$$P \times_G F := (P \times F)/G$$

is a fiber bundle over $B = P/G$ with fiber F. Indeed, local trivializations $\pi^{-1}(U) \to U \times G$ of P give rise to local trivializations $\rho^{-1}(U) \to U \times F$ of the bundle $\rho : P \times_G F \to B$. If F is a vector space on which G acts linearly, the associated bundle is a vector bundle. If F is a principal homogeneous H-space on which G acts by morphisms of such spaces, the associated bundle is a principal H-bundle.

EXAMPLES 2.6.
Given a vector bundle $E \to B$, the associated bundle $\mathrm{Fr}(E) \times_{GL(k,\mathbb{R})} \mathbb{R}^k$ for the defining action of $\mathrm{GL}(k, \mathbb{R})$ recovers E.
Similarly, if $P \to B$ is any principal G-bundle, $P \times_G G = P$ for the left-action of G on itself.
If $P_i \to X_i$ are two principal G-bundles, one may view $P_1 \times_G P_2$ as a fiber bundle over B_1 with fiber P_2 or as a fiber bundle over B_2 with fiber P_1.

If $\rho : P \times_G F \to B$ is an associated bundle, then $\rho^* P \cong P \times F$ since the following diagram commutes:

$$\begin{tikzcd} P \times F \arrow[r] \arrow[d] & P \arrow[d] \\ P \times_G F \arrow[r, "\rho"'] & B \end{tikzcd}$$

2.4. Classifying G-bundles. We recall the following notions from topology: Given a sequence of spaces and inclusions $X^0 \subset X^1 \subset X^2 \subset \cdots$, the colimit $X^\infty = \mathrm{colim}_{n\to\infty} X^n$ is the union of the spaces X^n, with the *weak topology* where a set is open in X if its intersection with all X^n is open. This implies that a function on X is continuous if and only if its restriction to all X^n is continuous. A particular case of this construction is a CW-complex: Here X^0 is a discrete set, while X^n is obtained from X^{n-1} by attaching n-cells, by a family of attaching maps $\phi_\lambda : \partial D^n \to X^{n-1}$.

From now on, we will mostly deal with CW-complexes, or paracompact Hausdorff spaces that are homotopy equivalent to a CW-complex. For example, smooth manifolds have CW-complex structures. For compact topological manifolds, the existence of CW-complex structures is known, except in dimension 4 where this question is open. On the other hand, compact topological manifolds are homotopy equivalent to CW-complexes. See Hatcher [**18**, p. 529].

DEFINITION 2.7. A *classifying principal G-bundle* is a principal G-bundle $EG \to BG$, where the total space EG is contractible, and where BG is a paracompact Hausdorff space homotopy equivalent to a CW-complex.

EXAMPLES 2.8. (a) Let $S^\infty = \mathrm{colim}_{n\to\infty} S^n$ be the "infinite dimensional sphere". S^∞ is an example of an infinite-dimensional CW-complex, with two cells in each dimension (the upper and lower hemispheres). Each S^n is a principal $\mathbb{Z}_2$-bundle over $\mathbb{R}P^n$, and this makes S^∞ into a principal $\mathbb{Z}_2$-bundle over $\mathbb{R}P^\infty = \mathrm{colim}_{n\to\infty} \mathbb{R}P^n$. $\mathbb{R}P^\infty$ inherits a CW-structure from S^∞, with one cell in each dimension.

LEMMA 2.9. *The infinite sphere S^∞ is contractible.*

PROOF. (cf. Stöcker-Zieschang, "Algebraische Topologie, p.58) View S^∞ as the "unit sphere" $||x|| = (\sum x_i^2)^{1/2} = 1$ inside $\mathbb{R}^\infty = \mathrm{colim}_{n\to\infty} \mathbb{R}^n$. We denote by $f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ the shift operator

$$f(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

and by $x_\star = (1, 0, \ldots)$ our base point. Define a homotopy $h_t : S^\infty \to S^\infty$ as follows:

$$h_t(x) = \frac{(1-t)x + t\,f(x)}{||(1-t)x + t\,f(x)||}$$

This is well-defined: the enumerator is never 0 since $x, f(x)$ are always linearly independent. (To see that this is a homotopy, it suffices to note that the restriction of $h : S^\infty \times I \to S^\infty$ to each $S^k \times I$ is continuous.) The map h_1 takes S^∞ onto the subspace $A \subset S^\infty$ given by the vanishing of the first coordinate. Next, rotate everything back onto the basis vector $e_1 = (1, 0, 0, \ldots)$:

$$h'_t : A \to S^\infty, \ y \mapsto \frac{(1-t)y + te_1}{||(1-t)y + te_1||}$$

By concatenating these two homotopies, we obtain the desired retraction from S^∞ onto the base point. □

Thus $S^\infty \to \mathbb{R}P^\infty$ is an $E\mathbb{Z}_2 \to B\mathbb{Z}_2$.

(b) We can also view S^∞ as the colimit of odd-dimensional spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$. Let $\mathbb{C}P^\infty = \operatorname{colim}_{n\to\infty} \mathbb{C}P^n$. The principal U(1)-bundles $S^{2n+1} \to \mathbb{C}P^n$ define a principal U(1)-bundle $S^\infty \to \mathbb{C}P^\infty$ which gives a $E\,\mathrm{U}(1) \to B\,\mathrm{U}(1)$.

(c) The sequence of $\mathrm{O}(n)$-bundles $\mathrm{St}_\mathbb{R}(k, n) \to \mathrm{Gr}_\mathbb{R}(k, n)$ defines a principal $\mathrm{O}(n)$-bundle

$$\mathrm{St}_\mathbb{R}(k, \infty) = \operatorname{colim}_{n\to\infty} \mathrm{St}_\mathbb{R}(k, n) \to \mathrm{Gr}_\mathbb{R}(k, \infty) = \operatorname{colim}_{n\to\infty} \mathrm{Gr}_\mathbb{R}(k, n)$$

LEMMA 2.10. *The infinite-dimensional Stiefel manifold* $\mathrm{St}_\mathbb{R}(k, \infty)$ *is contractible.*

PROOF. A point in $\mathrm{St}_\mathbb{R}(k, \infty)$ is a k-tuple $v = (v_1, \ldots, v_k)$, where each $v_j \in \mathbb{R}^\infty$ and $v_i \cdot v_j = \delta_{ij}$. Let $f^{(k)} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the kth iteration of the shift operator introduced above. We define a homotopy by

$$h_t(v_1, \ldots, v_k) = \mathrm{Gram}\left(((1-t)v_1 + tf^{(k)}(v_1), \ldots, (1-t)v_k + tf^{(k)}(v_k))\right)$$

where Gram denotes Gram's orthogonalization procedure. To see this is well-defined we need to check that the vectors $(1-t)v_j + tf^{(k)}(v_j)$ are linearly independent for all t. A linear dependence $\sum_j \lambda_j((1-t)v_j + tf^{(k)}(v_j)) = 0$ would mean, however, that the vectors $\sum_j \lambda_j v_j$ and $f^{(k)}(\sum_j \lambda_j v_j)$ are proportional, which only happens if $\sum_j \lambda_j v_j = 0$, a contradiction. At the end of the homotopy h_t, the $\mathrm{St}_\mathbb{R}(k, \infty)$ has been moved into the subspace $\mathrm{St}_\mathbb{R}(k, \infty) \cap \mathrm{span}(e_1, \ldots, e_k)^\perp$. We define a homotopy h'_t from this subspace onto the frame $(e_1, \ldots, e_k)$ by letting

$$h'_t(w_1, \ldots, w_k) = \mathrm{Gram}\left(te_1 + (1-t)w_1, \ldots, te_k + (1-t)w_k\right).$$

Concatenation of these two homotopies gives the desired retraction from $\mathrm{Gr}(k, \infty)$ onto the frame $(e_1, \ldots, e_k)$. □

Thus $\mathrm{St}_\mathbb{R}(k, \infty) \to \mathrm{Gr}_\mathbb{R}(k, \infty)$ is a classifying $\mathrm{O}(n)$-bundle.

(d) Similarly, we have a classifying $\mathrm{U}(n)$-bundle,

$$\mathrm{St}_\mathbb{C}(k, \infty) = \operatorname{colim}_{n\to\infty} \mathrm{St}_\mathbb{C}(k, n) \to \mathrm{Gr}_\mathbb{C}(k, \infty) = \operatorname{colim}_{n\to\infty} \mathrm{Gr}_\mathbb{C}(k, n)$$

(e) Using Stiefel manifolds of not necessarily orthonormal frames, we similarly get classifying bundles for $\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C})$.

(f) Let $EG \to BG$ be a classifying G-bundle, where G is a Lie group. Let $H \subset G$ be a closed subgroup. Restricting the action to H, we can take $EH = EG$ with

$$BH = EG/H = EG \times_G (G/H).$$

(Our assumptions on G, H imply that this is a CW-complex. Of course, this holds under much more general assumptions.) In particular, we have constructed classifying bundles for all compact groups G, since any such group may be presented as a subgroup of $\mathrm{U}(n)$, and for all Lie groups admitting a faithful finite-dimensional representation.

(g) Let Σ be a compact connected 2-manifold (other than the 2-sphere or $\mathbb{R}P^2$) with base x_0, and let $G = \pi_1(\Sigma, x_0)$ be its fundamental group. Then the universal covering $\tilde{\Sigma} \to \Sigma$ defines a classifying bundle, $EG \to \Sigma = BG$ since the universal cover of such a surface is diffeomorphic to $\mathbb{R}^2$.

(h) The easiest examples of classifying bundles are: $E\mathbb{Z} = \mathbb{R}$ as a bundle over $B\mathbb{Z} = S^1$, and $E\mathbb{R} = \mathbb{R}$ as a bundle over $B\mathbb{R} = \mathrm{pt}$.

It is not immediately clear that classifying bundles exist in general. Let us however establish some properties of such bundles. One important property will be that for any "reasonable" topological space X, the space $\mathrm{Prin}_G(X)$ of isomorphism classes of principal G-bundles ober X is in 1-1 correspondence with the space $[X, BG]$ of homotopy classes of continuous maps $f : X \to BG$, by the pull-back construction. A sufficient condition will be that X is a CW-complex. In particular, manifolds are certainly allowed.

THEOREM 2.11. *Let $EG \to BG$ be a universal principal G-bundle, and $P \to X$ any principal G-bundle where the base X is a paracompact Hausdorff space homotopy equivalent to a CW-complex. Then the map assigning to each continuous map $f : X \to BG$ the pull-back bundle $P = f^*EG \to X$ sets up a bijection,*

$$\mathrm{Prin}_G(X) = [X, BG].$$

PROOF. We may assume that X is a CW-complex. Given any principal G-bundle $P \to X$, consider the associated bundle $P \times_G EG$. This is a fiber bundle with fibers EG. Since the fibers are contractible, and X is a CW-complex, one can one can construct a section $\sigma : X \to P \times_G EG$ of this fiber bundle, by induction. The induction starts by choosing pre-images in each point over the 0-skeleton $X^0 \subset X$. Having constructed the section over the $k-1$-skeleton, one wants to extends the section over the k-skeleton. For any characteristic map $\Phi : D^k \to X^k \subset X$, the pull-back $\Phi^*P \to D_k$ admits a trivialization $D^k \times G$, giving a trivialization of fiber bundles,

$$\Phi^*(P \times_G EG) \cong D^k \times EG.$$

The given section of $P \times_G EG$ over X_{k-1} amounts to a continuous map $S^{k-1} \to \partial D^k \to EG$, which we would like to extend to a map $D^k \to EG$. Equivalently, we need a map $S^{k-1} \times I \to EG$ equal to the given map on $S^{k-1} \times \{0\}$ and equal to a constant map on $S^{k-1} \times \{1\}$. Such a map is obtained by composing the map $S^{k-1} \times I \to EG \times I$ (equal to our given map on the first factor) with a contraction, $EG \times I \to EG$. This gives the desired section over X^k. Since a map from a CW-complex is continuous if and only if its restriction to all X^k is continuous, this gives the desired section. By a similar argument, one shows that any two sections σ_0, σ_1 are homotopic.

The section $\sigma : X \to P \times_G EG$ lifts uniquely to a G-equivariant section $\hat{\sigma} : P \to P \times EG$. (Indeed, for any $p \in P$ with base point $x \in X$, the fiber over $\sigma(x) \in P \times_G EG$ contains a unique point of the form (p, y), and this will be $\hat{\sigma}(p)$.) Thus we get a commutative diagram,

$$\begin{tikzcd} P \arrow[r, "\hat{\sigma}"'] \arrow[d] & P \times EG \arrow[r, "="'] \arrow[d] & EG \times P \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\sigma"'] & P \times_G EG \arrow[r, "="'] & EG \times_G P \arrow[r] & BG \end{tikzcd}$$

in which the upper horizontal maps are G-equivariant. The composition of the lower horizontal maps gives a map $f : X \to BG$, and the diagram shows $P \cong f^*EG$. Homotopic sections σ give rise to homotopic f's, and therefore isomorphic G-bundles. ☐

For any principal G-bundle $P \to X$, a map $f : X \to BG$ with $f^*EG \cong P$ is called a *classifying map* for P. We have shown that the choice of a classifying map is equivalent to the choice of a section of the bundle $P \times_G EG \to X$.

THEOREM 2.12. *If $EG \to BG$ and $E'G \to B'G$ are two classifying bundles, where $BG, B'G$ are paracompact Hausdorff spaces having the homotopy type of CW complexes, there exists a homotopy equivalence $B'G \to BG$ that is covered by a G-equivariant homotopy equivalence $E'G \to EG$. In this sense classifying bundles are unique up to homotopy equivalence.*

PROOF. We have a classifying map $f : B'G \to BG$ for $E'G \to B'G$ (viewing $EG \to BG$ as the classifying bundle) and also a classifying map $g : BG \to B'G$ (viewing $E'G \to B'G$ as the classifying bundle). The composition $f \circ g : BG \to BG$ is a classifying map for EG itself, so it must be homotopic to the identity. Similarly $g \circ f$ is homotopic to the identity. ☐

EXAMPLES 2.13. It is known that for any path connected space X with base point x_0, the fundamental group $\pi_1(X, x_0)$ is related to the set of homotopy classes of (not necessarily base point preserving) maps $[S^1, X]$ by

$$[S^1, X] = \pi_1(X, x_0)/\operatorname{Ad}(\pi_1(X, x_0)),$$

quotient by the conjugation action.[9] It follows that the space of principal G-bundles over S^1 is in 1-1 correspondence with the set of conjugacy classes in $\pi_1(BG)$. For $G = \mathbb{Z}_2$, we know that

$$\pi_1(\mathbb{R}P^\infty) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$$

which tells us that there are exactly two principal $\mathbb{Z}_2$-bundles over S^1, up to isomorphism: The trivial bundle and the non-trivial double covering. For $G = \mathrm{U}(1)$, we find $\pi_1(\mathbb{C}P^\infty) = \{1\}$, which means that any principal $\mathrm{U}(1)$-bundle over S^1 is trivial. Of course, these facts are easy to check directly, and in general the result $\mathrm{Prin}_G(X) = [X, BG]$ is hardly useful to actually determine the set $\mathrm{Prin}_G(X)$.

Consider $X = S^n$, $(n > 1)$, with its standard cover by open sets U, V given as the complement of north pole and south pole. Since $U, V \cong D^n$, and principal G-bundle over U, V is trivial. If G is path connected, these trivializations are moreover unique up to homotopy. On the intersection, the trivializations differ by a *transition map* $U \cap V \to G$. Homotopic

[9]A similar statement holds for higher homotopy groups: $[S^n, X] = \pi_n(X, x_0)/\pi_1(X, x_0)$. See e.g. Davis-Kirk, [**10**, Theorem 6.57].

transition maps give rise to isomorphic bundles, and conversely, any transition map defines a bundle. Since $U \cap V \simeq S^{n-1}$, it follows that

$$[S^{n-1}, G] = \mathrm{Prin}_G(S^n) = [S^n, BG].$$

In particular, G-bundles over S^2 are classified by $[S^1, G] = \pi_1(G)$ (using that the fundamental group of a topological group is abelian).

EXERCISE 2.14. Let Σ be a 2-dimensional CW-complex. Show (directly) that if G is simply connected, $\mathrm{Prin}_G(\Sigma)$ consist of only one element, given by the class of the trivial bundle. If G is furthermore a *semi-simple*, show that a similar statement holds true for 3-dimensional CW-complexes. (Use that the fact that $\pi_2(G)$ is trivial for such groups.) What does all this imply for the topology of BG? (Remark: Elements in $H_n(X)$ are represented by maps from n-dimensional simplicial complexes into X. See Hatcher [**18**, p.109].)

EXERCISE 2.15. Show that if X is a CW-complex, with X^2 its 2-skeleton, $\mathrm{Prin}_{\mathrm{U}(1)}(X) = \mathrm{Prin}_{\mathrm{U}(1)}(X^2)$. (Hint: For $n \geq 2$, any map $S^n \to \mathrm{U}(1)$ is homotopic to the constant map, since any such map can be lifted to the universal cover $\mathbb{R} \to \mathrm{U}(1)$ and $\mathbb{R}$ is contractible.)

For $G = \mathrm{GL}(n, \mathbb{R})$ (and similarly for $\mathrm{GL}(n, \mathbb{C})$), there is a more geometric way to see the classifying map, at least if X is compact Hausdorff. Let $P \to X$ be a principal $\mathrm{GL}(n, \mathbb{R})$-bundle and $E \to X$ the associated vector bundle. Since X is compact, E is isomorphic to a direct summand of a trivial vector bundle $X \times \mathbb{R}^n$, for n sufficiently large.[10] Thus, for each $x \in X$ the fiber E_x defines a k-plane in $\mathbb{R}^n$, i.e. a point in $\mathrm{Gr}_{\mathbb{R}}(k, n)$. This gives a map $X \to \mathrm{Gr}_{\mathbb{R}}(k, n) \subset \mathrm{Gr}_{\mathbb{R}}(k, \infty)$, with the property that E is the pull-back of the tautological k-plane bundle over X. In the particular case where $X = M$ is a k-dimensional embedded submanifold of $\mathbb{R}^n$, with tangent bundle $TM \subset M \times \mathbb{R}^n$, the resulting map $M \to \mathrm{Gr}_{\mathbb{R}}(k, n)$ is known as the *Gauss map*.

2.5. Characteristic classes. Fix a commutative coefficient ring R (usually $R = \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}_2$). Given $P \in \mathrm{Prin}_G(X)$ with classifying map $f : X \to BG$ one obtains a pull-back map (characteristic homomorphism)

$$c(P) = f^* : H^*(BG, R) \to H^*(X, R),$$

(depending only on the isomorphism P, since any two classifying maps are homotopic). The image of this map is the ring of *characteristic classes* of P. The construction is functorial: If $\tilde{F} : P \to P'$ is a morphism of principal bundles covering a map $F : X \to X'$ on the base, the diagram

$$\begin{tikzcd} & H^*(BG,R) \arrow[dl, "c(P')"'] \arrow[d, "c(P)"] \\ H^*(X',R) \arrow[r, "F^*"'] & H^*(X,R) \end{tikzcd}$$

[10]Proof: Using a partition of unity, one constructs sections $\sigma_1, \ldots, \sigma_n$ of E for which the images span the fiber E_x at any point x. These sections determine a surjective bundle map $X \times \mathbb{R}^n \to E$. We may identify E with the orthogonal complement of the fiberwise kernel of this map.

commutes. For instance, if $G = S^1$ so that $BG = \mathbb{C}P^\infty$, one knows that the cohomology ring $H^*(\mathbb{C}P^\infty, \mathbb{Z})$ is a polynomial ring freely over one degree 2 generator $\alpha \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$:

$$H^*(\mathbb{C}P^\infty, \mathbb{Z}) =< \alpha > .$$

(That is, $H^q(\mathbb{C}P^\infty, \mathbb{Z})$ vanishes for q odd, and for $q = 2r$ equals $\mathbb{Z}$ with generator α^r.) Hence, the ring of characteristic classes of a principal $\mathrm{U}(1)$-bundle $P = f^*EG$ over X is a polynomial ring in the *Chern class*,

$$c_1(P) := c(P)(\alpha) \in H^2(X, \mathbb{Z}).$$

It turns out that, in fact, the Chern class determines the $\mathrm{U}(1)$-bundle up to isomorphism.

The characteristic rings $H^*(BG, R)$ are known for many groups G (particularly Lie groups) and coefficient rings R. We quote some results without proof, see e.g. Milnor-Stasheff [**?**], or Bott-Tu [**6**].

(a) The cohomology ring of the infinite complex Grassmannian is

$$H^*(B\,\mathrm{U}(n), \mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n],$$

a free polynomial ring in generators $c_i \in H^{2i}(B\,\mathrm{U}(n))$, where c_i is called the ith Chern class.

(b) The cohomology ring for $\mathrm{SU}(n)$ looks very similar:

$$H^*(B\,\mathrm{SU}(n), \mathbb{Z}) = \mathbb{Z}[c_2, \dots, c_n]$$

i.e. it starts with a class $c_2 \in H^4(\mathrm{SU}(n), \mathbb{Z})$ in degree four.

(c) For $G = \mathrm{O}(n)$ one has, with $\mathbb{Z}_2$-coefficients,

$$H^*(B\,\mathrm{O}(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n]$$

where $w_i \in H^i(B\,\mathrm{O}(n), \mathbb{Z})$ is called the ith Stiefel-Whitney class.

(d) With $\mathbb{Q}$ coefficients

$$H^*(B\,\mathrm{O}(n), \mathbb{Q}) = \mathbb{Q}[p_1, \dots, p_{[n/2]}]$$

where $p_i \in H^{4i}(B\,\mathrm{O}(n), \mathbb{Q})$ is called the ith Pontrjagin class, and $[n/2]$ is the greatest integer $\leq n/2$.

(e) For $G = \mathrm{SO}(n)$, with n odd, the cohomology ring is generated by Pontrjagin classes as before.[11]. If n is even, it is generated by the Pontrjagin classes together with the *Euler class* $\mathrm{e} \in H^n(B\,\mathrm{SO}(n))$ subject to one relation, $e^2 = p_{n/2}$.

(f) For G a compact Lie group, and $R = \mathbb{R}$, the cohomology ring $H^*(BG)$ is isomorphic to the ring $(S\mathfrak{g}^*)^G$ of invariant polynomials on $\mathfrak{g}$, with degrees doubled. It is a classical fact that $(S\mathfrak{g}^*)^G$ itself is a polynomial ring with a finite set of generators. For instance, if $G = \mathrm{U}(n)$ one can take $A \mapsto \mathrm{tr}(A^k)$ $(1 \leq k \leq n)$ as generators.

[11]More precisely, we pull-back the Pontrjagin classes for $B\mathcal{O}(n)$ back under the classifying map for $E\,\mathrm{SO}(n) \times_{\mathrm{SO}(n)} \mathrm{O}(n)$

2.6. Equivariant cohomology. Classifying bundles may be used to define the equivariant cohomology of a G-space X, using the *Borel construction.* Fix a classifying bundle $EG \to BG$.

DEFINITION 2.16. For any G-space X, the equivariant cohomology ring $H^*_G(X, R)$ of X with coefficients in a commutative ring R is the cohomology ring $H^*(X_G, R)$ of the associated fiber bundle

$$X_G := EG \times_G X.$$

The space X_G is often called the *Borel construction.* Note that a different model for the classifying bundle gives a homotopy equivalent Borel construction, and hence the same equivariant cohomology ring. Let us describe a few basic properties of this construction:

2.6.1. *G-maps.* The Borel constructions is functorial with respect to G-maps. That is, if $f : X \to Y$ is a G-equivariant map of G-spaces, one gets a map of Borel constructions $X_G \to Y_G$, hence a ring homomorphism

$$f^* : H^*_G(Y) \to H^*_G(X).$$

If $f_0 \simeq f_1 : X \to Y$ are homotopic *through G-maps* then the induced maps in equivariant cohomology coincide.

Taking $Y = \mathrm{pt}$ to be the trivial G-space, it follows that there is a natural homomorphism

$$H^*_G(\mathrm{pt}) \to H^*_G(X).$$

It turns $H^*_G(X)$ into a module over the ring $H^*_G(\mathrm{pt})$. This is often a better point of view to think about $H^*_G(X)$, e.g. $H^*_G(X)$ is rarely finitely generated as an abelian group, but it often is as a $H^*_G(\mathrm{pt})$-module.

2.6.2. *Change of groups.* Suppose G is a Lie group and H a closed subgroup. Recall that $EG \to BG$ defines a classifying bundle $EH \to BH$ with $EH = EG$ (viewed as an H-space) and $BH = EG/H = EG \times_G (G/H)$. If X is a G-space, viewed as an H-space by restricting the action, we get a natural map of orbit spaces,

$$EH \times_H X \to EG \times_G X$$

taking H-orbits to G-orbits. This map $X_H \to X_G$ induces a map in cohomology, $H^*(X_G) \to H^*(X_H)$, i.e. a ring (and also a $H^*_G(\mathrm{pt})$-module) homomorphism

$$H^*_G(X) \to H^*_H(X).$$

In particular, there is a homomorphism $H^*_G(X) \to H^*(X)$ to ordinary cohomology.

2.6.3. *Products.* In ordinary singular cohomology, the ring structure gives rise to a *cross product*

$$H^*(X_1) \otimes H^*(X_2) \to H^*(X_1 \times X_2), \quad [\alpha_1] \otimes [\alpha_2] \mapsto \mathrm{pr}_1^*[\alpha_1] \cup \mathrm{pr}_2^*[\alpha_2].$$

The Kuenneth theorem says that under favorable circumstances, e.g. if $R = \mathbb{R}$, this map is an *isomorphism.* More generally, if R is a principal ideal domain (e.g. $R = \mathbb{Z}$), the map is injective with cokernel given by torsion groups. (See e.g. Davis-Kirk [**10**, p. 56].) Similarly, in equivariant cohomology we have a cross product $H^*_G(X_1) \otimes H^*_G(X_2) \to H^*_G(X_1 \times X_2)$. This map, however, is rarely an isomorphism, even if the coefficient ring is $\mathbb{R}$. However, viewing $H^*_G(X)$ as a module over $H^*_G(\mathrm{pt})$ (as we should), we can also tensor over $H^*_G(\mathrm{pt})$ and get a cross product

$$H^*_G(X_1) \otimes_{H^*_G(\mathrm{pt})} H^*_G(X_2) \to H^*_G(X_1 \times X_2).$$

This has a much better chance of being an isomorphism (for coefficients $R = \mathbb{R}$), and often (but not always) it is. In general, the relationship between the two is given by a certain spectral sequence (see e.g. Hsiang, [**19**]).

2.6.4. *Equivariant cohomology of principal bundles.* In our construction of classifying maps, we essentially proved the following

PROPOSITION 2.17. *The equivariant cohomology ring of a principal G-bundle $P \to X$ (with X a paracompact Hausdorff space homotopy equivalent to a CW-complex) is the cohomology ring of the base X.*

PROOF. The associated bundle $P_G = EG \times_G P \to BG$ can also be viewed as a bundle $P \times_G EG \to X$. Since the fibers EG of this bundle are contractible, an argument similar to the proof of Theorem 2.11 shows that P_G retracts onto X. It follows that $H^*_G(P) = H^*(X)$. □

2.6.5. *Equivariant cohomology of homogeneous spaces.*

PROPOSITION 2.18. *Let G be a Lie groups and H a closed subgroup. Then*

$$H^*_G(G/H) = H^*_H(\mathrm{pt}) = H^*(BH).$$

The isomorphism is induced by the inclusion $eH \to G/H$. (Note that $H^*(BH)$ may be viewed as a $H^*(BG)$-module.)

PROOF. We observed above that EG, viewed a an H-space, is a model for EH with $BH = EG/H = EG \times_G (G/H)$. Thus

$$H^*_G(G/H) = H^*(EG \times_G (G/H)) = H^*(BH).$$

The inclusion of $\mathrm{pt} = eH$ into G/H gives a map

$$BH = EG \times_H (eH) \to EG \times_H (G/H) \to EG \times_G (G/H)$$

inducing maps in cohomology

$$H^*_G(G/H) \to H^*_H(G/H) \to H^*(BH).$$

□

2.6.6. *Mayer-Vietoris.* Suppose $X = U \cup V$ is an open cover of the G-space X by G-invariant open sets. Then we get an open cover $X_G = U_G \cup V_G$ of the Borel construction. By the usual Mayer-Vietoris-sequence for this cover, we get a Mayer-Vietoris-sequence in equivariant cohomology,

$$\cdots \to H^q_G(X) \to H^q_G(U) \oplus H^q_G(V) \to H^q_G(U \cap V) \to H^{q+1}_G(X) \to \cdots$$

More generally, for any open cover by G-equivariant open sets one has a spectral sequence for $H^*_G(X)$. In the manifold case, if G acts properly, we can always choose a cover by tubular neighborhoods of orbits. For example, if M/G is compact, one can use this to prove that $H^*_G(M)$ is a finitely generated $H^*_G(\mathrm{pt})$-module.

EXAMPLE 2.19. Let $\mathrm{U}(1)$ act on S^2 by rotation about the z-axis. We want to calculate $H^k_{\mathrm{U}(1)}(S^2)$ with coefficients $R = \mathbb{Z}$. Consider the open cover of S^2 given by the complement U of the fixed point set, and the complement V of the equator. Then $H^k_{\mathrm{U}(1)}(U) = 0$ in degree

$k > 0$, since U(1) acts freely on U and the quotient retracts onto a point. On the other hand $H^k_{\mathrm{U}(1)}(V) = H^k_{\mathrm{U}(1)}(\mathrm{pt}) \oplus H^k_{\mathrm{U}(1)}(\mathrm{pt})$. The Mayer-Vietoris sequence tells us therefore

$$H^k_{\mathrm{U}(1)}(S^2) = H^k_{\mathrm{U}(1)}(\mathrm{pt}) \oplus H^k_{\mathrm{U}(1)}(\mathrm{pt})$$

for $k > 0$.

2.6.7. *Equivariant characteristic classes.* The classifying bundle appears in two important constructions: Characteristic classes and equivariant cohomology. This can be combined, yielding *equivariant characteristic classes.* Namely, if $P \to X$ is a K-equivariant principal G-bundle, one obtains a principal G-bundle $P_K \to X_K$ over the Borel construction; the characteristic classes of this bundle live in $H^*(X_K) = H^*_K(X)$ and are called the equivariant characteristic classes.

3. Construction of EG by simplicial techniques

We will now explain a general construction of classifying bundles, using so-called (semi-)simplicial techniques. We begin with the case of discrete groups.

3.1. Construction of EG for discrete groups. Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard n-simplex,

$$\Delta^n = \{\sum_{i=0}^n t_i e_i | \sum_i t_i = 1\}.$$

Let G be a discrete group.[12]. Let $||EG||$ be the infinite-dimensional simplicial complex, with one standard n-simplex Δ^n for each ordered $n+1$-tuple $(g_0, \ldots, g_n) \subset G^{n+1}$ (repetitions of g_i are allowed). This simplex is denoted $[g_0, \ldots, g_n]$, with elements g_i as its vertices. Boundaries of simplices are identified in the obvious way. A free G-action is given on simplices by

$$g.[g_0, \ldots, g_n] = [g_0 g^{-1}, \ldots, g_n g^{-1}].$$

This action turns $||EG||$ into a principal G-bundle, as is easy to verify.

Lemma 3.1. *The space $||EG||$ is contractible.*

Proof. View the n-simplex $[g_0, \ldots, g_n]$ as the 0-face of the $n+1$-simplex $[e, g_0, \ldots, g_n]$. It is the closed face opposite to the vertex $[e]$. Given $x \in [g_0, \ldots, g_n]$, let $h_t(x) = (1-t)x + te$, using the linear structure in $[e, g_0, \ldots, g_n]$. This defines a homotopy h_t from $||EG||$ onto $[e]$. (It is not a strong deformation retraction, since $[e]$ is not fixed under the homotopy h_t.) $\square$

Thus $||EG|| \to ||BG|| = ||EG||/G$ is a model for the classifying bundle $EG \to BG$.

Remark 3.2. Note that this argument did not involve the group structure of G, thus it works for any set X, and the corresponding "free" simplicial complex with n-simplices parametrized by X^{n+1}.

There is a closely related model, defined as follows. Call an n-simplex $[g_0, \ldots, g_n]$ *degenerate* if $g_j = g_{j+1}$ for some j. There is a natural map from such a simplex onto the simplex

[12] For this case, the construction is nicely explained in Hatcher [**18**, p. 89]

$[g_0, \ldots, g_i, g_{i+2}, \ldots, g_n]$, collapsing the edge $[g_i g_{i+1}]$ onto a vertex $[g_i]$. Let $\sim$ denote the equivalence relation generated by such maps. Notice that

$$x \sim x' \Rightarrow h_t(x) \sim h_t(x')$$

Thus the homotopy h_t for $||EG||$ induces a retraction of $|EG|$ onto the simplex $[e]$. (Notice that this time, it is actually a strong deformation retract.) Furthermore, for $g \in G$ we have

$$x \sim x' \Leftrightarrow g.x \sim g.x'$$

hence the G-action on $|EG|$ is free. One may verify the local triviality condition, hence $|EG|$ is again a classifying bundle. The space $|EG|$ is a CW-complex, with one cell for each non-degenerate simplex.

EXAMPLE 3.3. Let $G = \mathbb{Z}_2 = \{e, c\}$. Then there are 2^{n+1} n-simplices $[g_0, \ldots, g_n]$ for each n, but only 2 non-degenerate ones: $[e, c, e, c, \ldots]$ and $[c, e, c, e, \ldots]$. To construct $|EG|$, one starts with two 0-simplices $[e], [c]$. Next one attaches two 1-simplices $[e, c]$ and $[c, e]$, obtaining S^1. One then attaches two 2-simplices $[e, c, e]$ and $[c, e, c]$. (Notice that one of the three edges of $[e, c, e]$ is the degenerate edge $[e, e]$, which gets mapped to $[e]$.) The resulting space is S^2 with its standard CW-complex structure. Iterating, one finds that $|EG|$ is just S^∞ with the usual CW-complex structure. The space $||EG||$ is much 'fatter' and does not have such a nice geometric interpretation.

3.2. Simplicial spaces. We may re-formulate the construction more systematically, as follows. For $n \geq 0$ denote $[n] := \{0, \ldots, n\}$. A map $f : [n] \to [m]$ is called increasing if $f(i+1) \geq f(i)$ for all i, and strictly increasing if $f(i+1) > f(i)$ for all i. One may think of $[n]$ as the vertices of an n-simplex. Any increasing map determines a map of simplices,

$$\Delta(f) : \Delta^n \to \Delta^m, \quad \sum_{i=0}^n t_i e_i \mapsto \sum_{i=0}^n t_i e_{f(i)}.$$

Under composition of increasing maps we have $\Delta(f_1 \circ f_2) = \Delta(f_1) \circ \Delta(f_2)$. The inclusions of the codimension one faces $\Delta^{n-1} \to \Delta^n$ correspond to the *face maps*

$$\partial^i : [n-1] \to [n], \quad \partial^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

The maps $\Delta^{n+1} \to \Delta^n$ collapsing the edge from e_i to e_{i+1} correspond to the *degeneracy maps*

$$\epsilon^i : [n+1] \to [n], \quad \epsilon^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

The face and degeneracy maps generate the set of all increasing maps. More precisely we have:

LEMMA 3.4. *Any increasing map $f : [n] \to [m]$ can be uniquely written as a composition $f = \partial^{i_k} \cdots \partial^{i_1} \epsilon^{j_1} \cdots \epsilon^{j_l}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$.*

PROOF. Suppose first that f is 1-1. Then f is uniquely described by its image, and clearly $f = \partial^{i_k} \cdots \partial^{i_1}$ where $i_1 < \cdots < i_k$ is the ordered list of indices that are not in the image. If $f : [n] \to [m]$ is not 1-1, let $j \leq n-1$ be the largest index with $f(j) = f(j+1)$. Then f may be uniquely written $f = f' \epsilon^j$, where $f'(j) < f'(j+1)$. Using induction, we eventually find $f = g \epsilon^{j_1} \cdots \epsilon^{j_l}$ where $j_1 < \cdots < j_l$ and g is 1-1. $\square$

Consider now a sequence of spaces $X_n := G^{n+1}$. Any increasing map $f : [n] \to [m]$ induces a continuous map

$$X(f) : X_m \to X_n, \quad (g_0, \dots, g_m) \mapsto (g_{f(0)}, \dots, g_{f(n)}).$$

Under composition of increasing maps, $X(f_1 \circ f_2) = X(f_2) \circ X(f_1)$. Our model $||EG||$ was defined as a quotient,

$$||X|| = \coprod_{n=0}^{\infty} X_n \times \Delta^n / \sim,$$

under the equivalence relations,

$$(x, \Delta(f)(y)) \sim (X(f)(x), y)$$

for every *strictly* increasing map $f : [n] \to [m]$. These are exactly the "natural" identifications made above. The model $|EG|$ is defined similarly,

$$|X| = \coprod_{n=0}^{\infty} X_n \times \Delta^n / \sim,$$

dividing out the relations for *all* increasing maps. In both cases, the topology is that of a colimit over the quotients $\coprod_{n=0}^{N} X_n \times \Delta^n / \sim$.

In this reformulation, the construction of a classifying bundle works for any topological group G. $||EG||$ is Milnor's model [**23**], while $|EG|$ is introduced in Segal's paper [**27**]. As pointed out by Segal, $|EG| \to |EG|/G$ may fail to be locally trivial, in general, but it is locally trivial if G is somewhat reasonable (e.g., a Lie group). If G is a Lie group, exactly the same argument as before gives a homotopy from $||EG||, |EG|$ onto $[e]$. There are many advantages to having such a universal construction. For instance, it is immediately clear that any group homomorphism $H \to G$ induces a map of classifying spaces.

The universal construction of the classifying bundle goes back to Milnor, [**23**]. The simplicial version of Milnor's construction was developed by Dold-Lashof, Segal, Stasheff, and Milgram.

Both the fat model $||EG||$ and the lean model $|EG|$ have their advantages and disadvantages. One advantage of the lean model is that the map

$$|E(G \times K)| \to |EG| \times |EK|,$$

induced by the two projections $G \times K \to G$ and $G \times K \to K$, is a homeomorphism provided one takes the "correct" product topology. [13] The analogue for the fat model is not true, even as sets. We'll return to this issue later.

EXERCISE 3.5. Convince yourself that the map $|E(G \times K)| \to |EG| \times |EK|$ is a bijection of sets.

The construction of the classifying bundle generalizes as follows:

[13] The product should be taken in the category of compactly generated spaces. This is similar the problem that while the direct product of two CW-complexes is again a CW-complex, the topology (given as a colimit) is slightly different from the product topology.

Definition 3.6. A simplicial space $X_\bullet$ is a collection of topological spaces $X_n,\ n = 0, 1, \ldots$, together with continuous maps $X(f) : X_m \to X_n$ for any increasing map $f : [n] \to [m]$, such that $X(f \circ g) = X(g) \circ X(f)$ under composition of such maps, and $X(\mathrm{id}) = \mathrm{id}$. A simplicial map $F_\bullet : X_\bullet \to X'_\bullet$ between simplicial spaces is a collection of continuous maps $F_n : X_n \to X'_n$ intertwining the maps $X(f), X'(f)$.

If we let ORD denote the category with objects the sets $[n]$ and morphisms the increasing maps $f : [n] \to [m]$, we may rephrase the definition as follows: A simplicial space is a contravariant functor from that category ORD into the category TOP of topological spaces. A simplicial map is a natural transformation between two such functors. Replacing TOP by other categories, one similarly defines simplicial sets, manifolds, groups, rings etc.

Definition 3.7. The geometric realization of a simplicial space $X_\bullet$ is the quotient space

$$|X| = \coprod_{n=0}^{\infty} \Delta^n \times X_n / \sim,$$

under the equivalence relations,

$$(\Delta(f)(y), x) \sim (y, X(f)(x))$$

for all $y \in \Delta^n$, $x \in X_m$, and any increasing map $f : [n] \to [m]$. One similarly defines the fat geometric realization $||X||$ by only dividing out the strictly increasing maps.

The maps $\partial_i := X(\partial^i) : X_n \to X_{n-1}$ and $\epsilon_i := X(\epsilon^i) : X_n \to X_{n+1}$ are called the face and degeneracy maps of the simplicial space.

Clearly, a simplicial map $F_\bullet : X_\bullet \to X_\bullet$ induces a continuous map $|F| : |X| \to |X'|$ between the geometric realizations, and similarly between the fat geometric realizations. That is, geometric realization is a functor from the category of simplicial spaces into the category of topological spaces.

3.3. Examples of simplicial spaces.

(a) Any topological space X may be viewed as a simplicial space, by taking all $X_n = X$ and all maps $X(f)$ the identity map. The geometric realization of this simplicial space is just X itself.

(b) Given a topological space X, one may define a simplicial space $E_n X = X^{n+1}$ with

$$(EX)(f) : (x_0, \ldots, x_m) \to (x_{f(0)}, \ldots, x_{f(n)})$$

for any increasing map $f : [n] \to [m]$. The same argument as for EG shows that the geometric realizations of this simplicial space is contractible. The diagonal embedding $X \to X^{n+1}$ gives a continuous map from X into the geometric realization, $X \to |EX|$. Note that if $X = G$ is a group, then $E_\bullet G$ is a simplicial group, and the structure as a principal G-bundle is given by the action of G as a (simplicial) subgroup from the right.

(c) Let X be a finite simplicial complex with N vertices. Choose an ordering on the set $\mathcal{V}$ of vertices, to identify $\mathcal{V} = [N]$. Let S_n be the set of all increasing maps $\phi : [n] \to [N]$ such that $\phi(0), \ldots, \phi(n)$ are the vertices of a (possibly degenerate) n-simplex in X, with

$$S(f) : S_m \to S_n, \quad S(f)(\phi) = \phi \circ \Delta(f).$$

(Note that $S_\bullet$ is a simplicial subset of $E_\bullet\mathcal{V}$.) The non-degenerate simplices in S_n are those given by strictly increasing maps $\phi: [n] \to [N]$, thus are in 1-1 correspondence with the (geometric) simplices in X. Using this fact, it is easy to see that

$$|S| = X.$$

In the important special case that $X = \Delta^k$ is the standard k-simplex, we write $S_n := \Delta_n[k]$. Thus $\Delta_n[k]$ is simply the set of morphisms $f: [n] \to [k]$.

(d) Let G be a topological group. Then $B_nG := G^n$ is a simplicial space, with face maps

$$\partial_i(h_1, \ldots, h_n) = \begin{cases} (h_2, \ldots, h_n) & \text{if} \quad i = 0 \\ (h_1, \ldots, h_i h_{i+1}, \ldots, h_n) & \text{if} \quad 0 < i < n \\ (h_1, \ldots, h_{n-1}) & \text{if} \quad i = n \end{cases}$$

and degeneracy maps

$$\epsilon_i(h_1, \ldots, h_n) = (h_1, \ldots, h_i, e, h_{i+1}, \ldots, h_n).$$

It is easy to check that the map $\pi_n : E_nG \to B_nG$ given by

$$(g_0, \ldots, g_n) \mapsto (g_0 g_1^{-1}, \ldots, g_{n-1} g_n^{-1})$$

is a simplicial map. This identifies $B_\bullet G$ as the base of the simplicial principal bundle makes $E_\bullet G \to E_\bullet G/G = B_\bullet G$.

(e) Generalizing this example, suppose X is a G-space. One may define a simplicial space $(X_G)_\bullet$ by letting

$$(X_G)_n = G^n \times X,$$

with face and degeneracy maps

$$\partial_i(h_1, \ldots, h_n, x) = \begin{cases} (h_2, \ldots, h_n, x) & \text{if} \quad i = 0 \\ (h_1, \ldots, h_i h_{i+1}, \ldots, h_n, x) & \text{if} \quad 0 < i < n \\ (h_1, \ldots, h_{n-1}, h_n.x) & \text{if} \quad i = n \end{cases}$$

$$\epsilon_i(h_1, \ldots, h_n, x) = (h_1, \ldots, h_i, e, h_{i+1}, \ldots, h_n, x).$$

This is naturally identified with $E_nG \times_G X$, with quotient map

$$(g_0, \ldots, g_n, x) \mapsto (g_0 g_1^{-1}, \ldots, g_{n-1} g_n^{-1}, g_n.x)$$

Hence the geometric realization of this simplicial space is the Borel construction X_G.

(f) Let X be a topological space, and $\mathcal{U} = \{U_a, a \in A\}$ an open cover of X indexed by an *ordered* set A. Given $a_0 \leq \cdots \leq a_n$ let

$$U_{a_0,\ldots,a_n} = U_{a_0} \cap \cdots U_{a_n}$$

$$A_n = \{(a_0, \ldots, a_n) \in A^{n+1} |\, a_0 \leq \cdots \leq a_n,\ U_{a_0,\ldots,a_n} \neq \emptyset\}.$$

Then $A_\bullet$ is a simplicial set, with face maps and degeneracy maps inherited from $E_\bullet A$. Its geometric realization is a simplicial complex, sometimes known as the *nerve* of the open cover. Define

$$\mathcal{U}_n X := \coprod_{(a_0,\ldots,a_n) \in A_n} U_{a_0,\ldots,a_n}$$

(disjoint union), with face maps induced by inclusions, and degeneracy maps the natural bijections $U_{a_0,\ldots,a_i,a_i,\ldots,a_n} \to U_{a_0,\ldots,a_n}$. There is a natural simplicial map $\mathcal{U}_n X \to X$ induced by the inclusions of open sets.

THEOREM 3.8. [**24**, Section 7] *If X is a paracompact Hausdorff space, the geometric realization*

$$|\mathcal{U}X| \to |X| = X$$

of the map $\mathcal{U}_n X \to X$ is a homotopy equivalence.

PROOF. We have to construct a homotopy inverse $f : X \to |\mathcal{U}X|$ to the given map $g : |\mathcal{U}X| \to X$. Choose a locally finite partition of unity χ_a subordinate to the cover U_a. Given $x \in X$, let $a_0 < \ldots < a_n$ be an ordered set of indices such that

$$\sum_{i=0}^{n} \chi_{a_i}(x) = 1, \quad \text{and } x \in U_{a_0,\ldots,a_n}.$$

Define $f(x) \in |\mathcal{U}X|$ to be the image of

$$\big(\sum_{i=0}^{n} \chi_{a_i}(x) e_i,\ x\big) \in \Delta^n \times U_{a_0,\ldots,a_n}.$$

It is easily checked that f is well-defined (i.e. independent of the choice of $a_0 < \ldots < a_n$ (note that we do allow $\chi_{a_i}(x) = 0$). Since the same collection of indices also works on a neighborhood of x, it is clear that f is continuous. We have $g(f(x)) = x$ by construction. The composition $f \circ g : |\mathcal{U}X| \to |\mathcal{U}X|$ is homotopic to the identity: The required homotopy is induced by the homotopies $I \times (\Delta^n \times U_{a_0,\ldots,a_n}) \to (\Delta^n \times U_{a_0,\ldots,a_n})$,

$$\big(t, (\sum_{i=0}^{n} s_i e_i, x)\big) \mapsto \big(\sum_{i=0}^{n}((1-t)s_i + t\chi_{a_i}(x))e_i, x)\big)$$

$\square$

REMARKS 3.9. (i) By essentially the same proof, the map $||\mathcal{U}X|| \to X$ is a homotopy equivalence as well.

(ii) It may be useful to visualize the result for an open cover by just two open sets U_0, U_1. The geometric realization is obtained from a disjoint union $U_0 \coprod U_1$ by gluing in a "cylinder" $I \times (U_0 \cap U_1)$. One obtains an inclusion $X \to |\mathcal{U}X|$, where the partition of unity shows how to embed X over the intersections $U_0 \cap U_1$. The linear retraction of the cylinder onto the image of X give the desired homotopy equivalence.

(iii) The result shows that all homotopy invariant topological invariants of X (in particular its homology/cohomology) may be studied (at least in principle) in terms of $\mathcal{U}_\bullet X$. This is particularly interesting if $\mathcal{U}$ is a *good* cover, i.e. if all $U_{a_0,\ldots,a_n}$ are contractible. In this case the topology of $\mathcal{U}_n X$ is trivial, and all information on the topology of $|\mathcal{U}X|$ lies in the face and degeneracy maps. Below this will lead us to a simplicial interpretation of *Čech cohomology.*

Classifying maps can be interpreted in the simplicial construction, as follows. Suppose G is a Lie group, $\pi : P \to X$ is a principal G-bundle over a paracompact Hausdorff space, and U_a is a trivializing open cover of X. That is, over each U_a there is a G-equivariant map $\phi_a : \pi^{-1}(U_a) \to G$. Suppose as before that we have chosen an ordering of the index set, and define a map

$$\psi : \pi^{-1}(U_{a_0,\ldots,a_n}) \to E_n G = G^{n+1}, \quad x \mapsto (\phi_{a_0}(x), \ldots, \phi_{a_n}(x)).$$

ψ is compatible with the face and degeneracy maps, hence it gives a simplicial map

$$(\pi^{-1}\mathcal{U})_\bullet P \to E_\bullet G$$

where $\pi^{-1}\mathcal{U}$ is the cover of P by set $\pi^{-1}(U_a)$. Since ψ is G-equivariant, it descends to a simplicial map

$$\mathcal{U}_\bullet X \to B_\bullet G.$$

The geometric realization of this map is the classifying map for P, composed with the map $X \simeq |\mathcal{U}X|$ for some partition of unity, is a classifying map for P.

Below we will mostly work with the *fat* geometric realization, which has simpler properties in a number of respects. There is, however, one important property of the (lean) geometric realization: It is well-behaved under products.

PROPOSITION 3.10 (Milnor,Segal). *Let $X_\bullet, X'_\bullet$ be simplicial spaces, and let $(X \times X')_\bullet$ be their direct product, i.e. $(X \times X')_n = X_n \times X'_n$. The natural map*

$$|(X \times X')_\bullet| \to |X_\bullet| \times |X'_\bullet|$$

induced by the two projections is a bijection of sets. It is a homeomorphism provided the product on the right hand side is taken in the category of "compactly generated spaces".

We indicate the main idea in an example (cf. Benson, p.25): Consider the simplicial space $\Delta_\bullet[1]$. Its non-degenerate simplices are $(0), (1), (01)$. Elements of $(\Delta[1] \times \Delta[1])_n$ are pairs of increasing sequences $(t_0 \ldots t_n, t'_0 \ldots t'_n)$ where each t_i, t'_i is 0 or 1. Thus $(\Delta[1] \times \Delta[1])_\bullet$ has

(1)	four 0-simplices	$(0,0), (0,1), (1,0), (1,1),$
(2)	five non-degenerate 1-simplices	$(00,01), (01,00), (01,01), (01,11), (11,01)$
(3)	two non-degenerate 2-simplices	$(001,011), (011,001).$

We hence see that the geometric realization is a square, could into two triangles along the diagonal from $(0,0)$ to $(1,1)$. Similarly, in the general case:

EXERCISE 3.11. Show that the geometric realization of $\Delta_\bullet[k] \times \Delta_\bullet[l]$ is the product $\Delta^k \times \Delta^l$, subdivided into $k+l$-simplices in a certain canonical way. (The fact that $\Delta^k \times \Delta^l$ admits a canonical subdivision enters the definition of the cross product in singular homology. See e.g. Hatcher [**18**, p. 277].)

REMARK 3.12. One can use this to give a very clean proof of the contractibility of the space $|E_\bullet X|$ onto a base point $x_* \in X \subset |E_\bullet X|$. Thinking of I as the geometric realization of $\Delta_\bullet[1]$, we would like to obtain the retraction as the geometric realization of a simplicial map

$$\Delta_\bullet[1] \times E_\bullet X \to E_\bullet X.$$

Recalling that $\Delta_\bullet[1]$ consists of increasing maps $\phi : [n] \to [1]$, the map is defined as follows:

$$(\phi, (x_0, \ldots, x_n)) = (x'_0, \ldots, x'_n)$$

where $x'_i = x_i$ if $\phi(i) = 0$, $x'_i = x_*$ if $\phi(i) = 1$. It is straightforward to check that this is a simplicial map. An inclusion $\{1\} \hookrightarrow I$ is obtained by geometric realization of the simplicial maps

$$\{1\}_\bullet \to \Delta_\bullet[1], \ 1 \mapsto (1, \ldots, 1)$$

and similarly for $\{0\}$. The geometric realizations of the restricted maps

$$\{1\}_\bullet \times E_\bullet X \to E_\bullet X,\ \ ((1,\ldots,1),(x_0,\ldots,x_n)) \mapsto (x_*,\ldots,x_*),$$
$$\{0\}_\bullet \times E_\bullet X \to E_\bullet X,\ \ ((0,\ldots,0),(x_0,\ldots,x_n)) \mapsto (x_0,\ldots,x_n)$$

are the constant map and the identity map, respectively.

3.4. The homology and cohomology of simplicial spaces. Our goal is to develop techniques for calculating the equivariant cohomology of a G-manifold M, particularly the cohomology of BG. Any simplicial space $X_\bullet$ defines a double complex

$$(C_\bullet(X_\bullet), \mathrm{d}, \delta),$$

where $\mathrm{d} : C_q(X_p) \to C_{q-1}(X_p)$ is the usual boundary map and $\delta : C_q(X_p) \to C_q(X_{p-1})$ is defined in terms of the face maps as

$$\delta = \sum_{i=0}^{p} (-1)^i (\partial_i)_*.$$

Clearly, the two differentials d, δ commute. Define the associated total complex $(C_\bullet(X), D)$, where

$$C_k(X) := \bigoplus_{p+q=k} C_q(X_p)$$

and $D = \mathrm{d} + (-1)^q \delta$ on $C_q(X_p)$. (The sign guarantees that D squares to 0.) We define a homomorphism $\psi : C_q(X_p) \to C_{p+q}(||X||)$

$$C_q(X_p) \to C_{p+q}(\Delta^p \times X_p) \to C_{p+q}(||X||),$$

where the first map is $(-1)^{pq}$ times cross-product with the identity map id_{Δ^p} (viewed as a singular chain $\mathrm{id}_{\Delta^p} \in C_p(\Delta^p)$), and the second map is push-forward under the quotient map $\Delta^p \times X_p \to ||X||$. Summing over all $p + q = k$, this gives a group homomorphism

$$\psi : C_k(X) \to C_k(||X||). \tag{4}$$

ψ is natural with respect to morphisms of simplicial spaces $X_\bullet \to Y_\bullet$.

THEOREM 3.13. [**24**, Theorem 4.2] *The map* (4) *is a chain homotopy equivalence, and hence induces an isomorphism in homology. That is, the singular homology of $\|X\|$ may be computed as the homology of the total complex associated to the double complex $(C_q(X_p), d, \delta)$.*

SKETCH OF PROOF. A detailed proof can be found in the Bott-Mostow-Perchik article. Here are some of the ideas involved. Let $\pi : \coprod_{n=0}^\infty \Delta^n \times X_n \to \|X\|$ be the quotient map. For any $\alpha \in C_q(X_p)$, we have

$$\begin{aligned}
\psi(D\alpha) &= \psi(\mathrm{d}\alpha) + (-1)^q \psi(\delta\alpha)\\
&= (-1)^{p(q-1)} \pi_*(\mathrm{id}_{\Delta^p} \times \mathrm{d}\alpha) + (-1)^q \psi(\delta\alpha)\\
&= (-1)^{pq} \mathrm{d}\pi_*(\mathrm{id}_{\Delta^p} \times \alpha) - (-1)^{pq} \pi_*(\mathrm{d}\,\mathrm{id}_{\Delta^p} \times \alpha) + (-1)^q \psi(\delta\alpha)\\
&= \mathrm{d}\psi(\alpha) - (-1)^{pq} \sum_{j=0}^{p} (-1)^j \big(\pi_*(\partial^j \times \alpha) - \pi_*(\mathrm{id}_{\Delta^{p-1}} \times (\partial_j)_*\alpha)\big)
\end{aligned}$$

The sum is zero due to the identifications given by π. This shows that $\psi(D\alpha) = \mathrm{d}\psi(\alpha)$.

The proof that (4) is an isomorphism in homology is similar to the proof that the simplicial homology of a simplicial complex equals is singular homology. Recall that the topology on $||X||$ was defined by taking a colimit of spaces $||X||_{(N)} = \pi(\coprod_{n=0}^{N}(\Delta^n \times X_n)$. It may be shown (cf. Bott-Mostow-Perchik) that any compact set in $||X||$, and in particular the image of any singular chain, is contained in some $||X||_{(N)}$ with N sufficiently large. This defines natural filtrations of the chain complexes $C_\bullet(\|X\|)$ and $C_\bullet(X)$. One obtains a map between the spectral sequences (cf. infra) associated to these filtrations, and the main point of the proof is now to show that these spectral sequences coincide, already at the E_1 stage. □

If R is any abelian coefficient group (typically $R = \mathbb{R}, \mathbb{Z}, \mathbb{Z}_2$), we can consider homology with coefficients in R, and the theorem shows that the homology groups of $||X||$ with coefficients in R can be computed from a double complex $C_p(X_q; R)$.

Dually there is a double complex of singular cochains,

$$(C^\bullet(X_\bullet; R), \mathrm{d}, \delta),$$

where d is the usual coboundary map (dual to the boundary map, which also denote by d) and

$$\delta = \sum_{i=0}^{q+1}(-1)^i(\partial_i)^*$$

(dual to the map δ for the chain complex). Let $C^k(X; R)$ be the total complex,

$$C^k(X; R) = \bigoplus_{p+q=k} C^q(X_p; R).$$

By dualizing the maps from homology, we see that there is a natural homotopy equivalence

$$C^k(\|X\|; R) \to C^k(X; R).$$

Hence there is a canonical isomorphism between the cohomology of the total complex and the cohomology of the fat geometric realization, $||X_\bullet||$.

EXAMPLE 3.14. The *group homology* $\tilde{H}_\bullet(G)$ of a discrete group G is defined as the homology of a chain complex $\tilde{C}_\bullet G$, where $\tilde{C}_k(G)$ is the free Abelian group generated by elements of G^k, and the differential is given as

$$\delta(h_1, \ldots, h_k) = (h_2, \ldots, h_k) + \sum_{i=1}^{k-1}(-1)^i(h_1, \ldots, h_i h_{i+1}, \ldots, h_k) + (-1)^k(h_1, \ldots, h_{k-1}).$$

Note that if we view elements of G as 0-simplices in G, this is just the complex $(C_0(B_kG), \delta)$ inside the double complex $C_p(B_qG)$. Dually, we have the cochain complex $\tilde{C}^k(G) = \mathrm{Hom}(C_k, \mathbb{Z})$ and the corresponding group cohomology $\tilde{H}^k(G)$. Elements of $\tilde{C}^k(G)$ are functions $\phi : G^k \to \mathbb{Z}$, and the dual differential (once again denoted δ) is

$$(\delta\phi)(h_1, \ldots, h_{k+1}) = \phi(h_2, \ldots, h_{k+1}) + \sum_{i=1}^{k}(-1)^i\phi(h_1, \ldots, h_i h_{i+1}, \ldots, h_{k+1}) + (-1)^{k+1}\phi(h_1, \ldots, h_k).$$

THEOREM 3.15. *The inclusion $\tilde{C}^k(G) \to C^0(B_kG)$ induces an isomorphism from the group cohomology $\tilde{H}^\bullet(G)$ of G to the cohomology $H^\bullet(BG)$ of the classifying space BG.*

PROOF. This will be "immediate" once we have the spectral sequence set-up, but we can easily give a direct proof. Indeed, since G is discrete the d-cohomology is trivial.

We first show that the map $\tilde{H}^k(G) \to H^k(BG)$ is surjective. Let $\alpha \in Z^k(BG)$ represent a class in $H^k(BG)$. Write $\alpha = \alpha^{0,k} + \alpha^{1,k-1} + \cdots$. Since $D\alpha = 0$, we have in particular $\mathrm{d}\alpha^{0,k} = 0$. Since the d-cohomology is trivial, we can write $\alpha^{0,k} = \mathrm{d}\beta^{0,k-1}$. Replacing α with $\alpha - D\beta^{0,k-1}$, and denoting the new form by α again, we achieve $\alpha^{0,k} = 0$. The new form has $\mathrm{d}\alpha^{1,k-1} = 0$, so again we may subtract a D-coboundary to achieve $\alpha^{1,k-1} = 0$. Iterating, we find that α is D-cohomologous to a form in $\tilde{C}^k(G) = C^0(B_kG)$, closed under both d and δ. By a similar argument, one shows that the map $\tilde{H}^k(G) \to H^k(BG)$ is injective. $\square$

EXAMPLE 3.16. Let X be a paracompact Hausdorff space, with cover $\mathcal{U} = \{U|\, a \in A\}$ where A is an ordered set. Given a coefficient ring R consider the subcomplex

$$\check{C}^k(X, \mathcal{U}, R) \subset C^0(\mathcal{U}_k X, R),$$

consisting of *locally constant* functions, in other words the kernel of $\mathrm{d} : C^0(\mathcal{U}_k X, R) \to C^1(\mathcal{U}_k X, R)$. If $\mathcal{U}$ is "good", that is, each non-empty intersection $U_{a_0 \ldots a_k}$ is contractible, elements of $C^0(\mathcal{U}_k X, R)$ are in fact constant functions, thus are collections of elements $f_{a_0,\ldots,a_k} \in R$, one for each non-empty intersection $U_{a_0,\ldots,a_k}$ with $a_0 \le \cdots \le a_k$, and the differential reads,

$$(\delta f)_{a_0 \cdots a_{k+1}} = \sum_{i=0}^{k+1} (-1)^i f_{a_0 \cdots \hat{a}_i \cdots a_k}.$$

The cohomology of this complex is called the Čech cohomology of X with respect to the cover $\mathcal{U}$.

THEOREM 3.17 (Isomorphism between Čech and singular cohomology). *Suppose the cover is "good", that is, each non-empty intersection $U_{a_0 \ldots a_k}$ is contractible. Then $\check{H}^k(X, \mathcal{U}, R)$ is canonically isomorphic to the singular cohomology $H^k(X, R)$. More precisely, the inclusion map $\check{C}^k(X, \mathcal{U}, R) \to C^k(\mathcal{U}X, R)$ gives an isomorphism*

$$\check{H}^k(X, \mathcal{U}, R) \to H^k(||\mathcal{U}X||, R) = H^k(X, R).$$

PROOF. This follows since the d-cohomology of the double complex $C^q(\mathcal{U}_p X)$ is trivial in positive degree, as for the previous theorem. $\square$

EXAMPLE 3.18. Let X be a simplicial complex, with a given ordering on its set of vertices, and $S_\bullet$ the corresponding simplicial set. Then $|S_\bullet| = X$, while $\|S_\bullet\|$ is homotopy equivalent to X. In this case, $(C_0(S_\bullet), \delta)$ may be identified with the simplicial chain complex. Let $C^k_\Delta(X) := C^0(S_k)$ denote the simplicial cochain complex. Again, we easily see that the inclusion into the double complex gives an isomorphism $H^k_\Delta(X) = H^k(||S||) = H^k(X)$.

4. Spectral sequences

Suppose $(C^{\bullet,\bullet}, \mathrm{d}, \delta)$ is a bi-complex: That is, $C^{\bullet,\bullet}$ is a bi-graded R-module for some coefficient ring R, and d, δ are two commuting differentials with

$$\mathrm{d} : C^{p,q} \to C^{p,q+1}, \quad \delta : C^{p,q} \to C^{p+1,q}.$$

One can then introduce the total complex

$$C^k = \bigoplus_{p+q=k} C^{p,q}$$

with differential $D = \mathrm{d} + (-1)^q\delta$. Our goal is to compute the cohomology of the complex $(C^\bullet, D)$. Cocycles for the differential D are elements of the form

$$\alpha = \alpha^{0,k} + \alpha^{1,k-1} + \cdots + \alpha^{k,0}$$

satisfying a system of equations,

$$\begin{aligned} \mathrm{d}\alpha^{0,k} &= 0 \\ \mathrm{d}\alpha^{1,k-1} &= \pm\delta\alpha^{0,k} \\ &\cdots \\ 0 &= \delta\alpha^{k,0}. \end{aligned} \tag{5}$$

It is convenient to picture this system of equations in an array of boxes, labeled by (p,q), with p increasing in horizontal direction and q increasing in vertical direction.

4.1. The idea of a spectral sequence. It seems natural to find solutions by induction. (In a sense, a spectral sequence will be similar to a "power series" ansatz for solving an ordinary differential equation.) Let us consider a solutions having their first non-trivial term in the p,q position, and extending downwards. That is, consider the system

$$\begin{aligned} \mathrm{d}\alpha^{p,q} &= 0 \\ \mathrm{d}\alpha^{p+1,q-1} &= \pm\delta\alpha^{p,q} \\ &\cdots \\ 0 &= \delta\alpha^{k,0}. \end{aligned} \tag{6}$$

The first equation says that $\alpha^{p,q}$ is a d-cocycle. This leads us to consider the d-cohomology $E_1 := H(C, \mathrm{d})$, with bigrading

$$E_1^{p,q} := \frac{\ker \mathrm{d} \cap C^{p,q}}{\operatorname{im} \mathrm{d} \cap C^{p,q}}.$$

The second equation has a solution if and only if $\delta\alpha^{p,q}$ is exact. Put differently, the class of $\alpha^{p,q}$ in $E_1^{p,q}$ should be closed under the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ which is induced by δ. This leads us to consider

$$E_2^{p,q} := \frac{\ker \mathrm{d}_1 \cap E_1^{p,q}}{\operatorname{im} \mathrm{d}_1 \cap E_1^{p,q}}.$$

Classes in $E_2^{p,q}$ are represented $\alpha^{p,q}$ admitting an extension to an element $\alpha^{p,q} + \alpha^{p+1,q-1}$ solving the first two equations. Iterating this idea will lead to the concept of spectral sequence, (E_r, d_r) with E_{r+1} the cohomology of E_r. Before we explain how to continue the sequence, let us explain the abstract notion of a spectral sequence and what it actually computes.

4.2. What does the spectral sequence compute?

DEFINITION 4.1. A spectral sequence is a sequence of bigraded differential complexes

$$(E_r^{p,q}, d_r),\ r = 0, 1, 2, \ldots,$$

where d_r raises the total degree by 1 and the p-degree by r, with

$$E_{r+1}^{p,q} = \frac{\ker \mathrm{d}_r \cap E_r^{p,q}}{\operatorname{im} \mathrm{d}_r \cap E_r^{p,q}}.$$

We will only consider *first quadrant* bigraded spectral sequences, i.e. $E_r^{p,q} = 0$ unless $p, q \geq 0$. In this case

$$\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

is zero for r sufficiently large (namely, $r > q + 1$). Hence the spectral sequence *stabilizes*: $E_{r+1}^{p,q} = E_r^{p,q}$ for r suffiently large. The limiting groups are denoted $E_\infty^{p,q}$.

Consider now a (first quadrant) double complex $C^{p,q}$ as above. Define the *horizontal filtration* of $C^{\bullet,\bullet}$ by direct sums $F^i(C) := \bigoplus_{i \geq p} \bigoplus_{q \geq 0} C^{i,q}$. Note that $D : F^i(C) \to F^i(C)$, hence we obtain a filtration of the D-cohomology $H^k = H^k(C, D)$:

$$F^0(H^k) \supset F^1(H^k) \supset F^2(H^k) \supset \cdots,$$

where $F^p(H^{p+q})$ is the subspace represented by D-closed zig-zags starting in the (p, q)-position and extending downwards.

THEOREM 4.2. *There is a bigraded spectral sequence $E_r^{p,q}$ with E_2-term $E_2^{p,q} = H_\delta(H_d(C^{p,q}))$, such that*

$$E_\infty^{p,q} = F^p H^{q+p} / F^{p+1} H^{q+p}.$$

One writes (abusing notation)

$$E_2^{p,q} \Rightarrow H^{p+q}(C).$$

REMARK 4.3. One could also switch the role of d and δ and gets another spectral sequence computing the same cohomology.

The result may seem a little disappointing at first sight: Of course, in reality we would rather get $H^k = H^{p+q}$ itself, rather than the associated graded group of some filtration of H^{p+q}.

REMARKS 4.4. (cf. Davis-Kirk [**10**, p.240]) Suppose V is an R-module with a decreasing filtration $F^0(V) \supset F^1(V) \supset \cdots$, and let $\mathrm{Gr}^\bullet(V)$ be the associated graded module,

$$\mathrm{Gr}^j(V) = F^j(V)/F^{j+1}(V).$$

Suppose the filtration is finite, i.e. $F^j(V) = 0$ for j sufficiently large.

(a) If $\mathrm{Gr}^\bullet(V) = 0$ then $V = 0$. More generally, suppose we are given a morphism $A : V' \to V''$ of filtered R-modules, and consider the induced map $\mathrm{Gr}(A) : \mathrm{Gr}^\bullet(V') \to \mathrm{Gr}^\bullet(V'')$. If $\mathrm{Gr}(A)$ is an isomorphism then so is A. (This follows from the first part, by considering induced filtration on the kernel and cokernel of A.)

(b) If R is a field, and $\dim V < \infty$ then $\dim V = \sum_i \dim \mathrm{Gr}^i(V)$. Thus $\mathrm{Gr}^\bullet(V)$ carries almost the same information as V in this case.

(c) In general, one cannot recover V from $\mathrm{Gr}^\bullet(V)$. For example, let $R = \mathbb{Z}$. Consider the following two filtered $\mathbb{Z}$-modules,
$$\mathbb{Z} \supset 2\mathbb{Z} \supset 0, \quad \mathbb{Z} \oplus \mathbb{Z}_2 \supset \mathbb{Z} \supset 0.$$
In both cases the associated graded module is $\mathbb{Z}_2 \oplus \mathbb{Z}$, but the groups are non-isomorphic. Similarly
$$\mathbb{Z}_4 \supset \mathbb{Z}_2 \supset 0, \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \supset \mathbb{Z}_2 \supset 0.$$

Thus, if our coefficient ring is a field, the spectral sequence really is going to compute the cohomology. In general, we can still say that for any morphism of double complexes $C^{\bullet,\bullet} \to (C')^{\bullet,\bullet}$, and the induced map in cohomology is an isomorphism if one has $E_r = (E')_r$ for r sufficiently large.

Remark 4.5. More generally, there are spectral sequences for the cohomology of chain complexes $(C^\bullet, \mathrm{d})$ with a given filtration $F^0(C^\bullet) \supset F^1(C^\bullet) \supset \cdots$ by subcomplexes. In our case, the filtration came from a bigrading.

4.3. How does one construct the spectral sequence? The spectral sequence of a double complex $C^{\bullet,\bullet}$, with filtration $F^p(C)$ as defined as above, may be defined as follows. Let
$$Z_r^{p,q} := \{a \in F^pC^{p+q} |\, Da \in F^{p+r}C^{p+q+1}\}$$
Elements of $Z_r^{p,q}$ are represented by zig-zags of length r, starting in the (p,q) position and solving the first r of our system of equations. Define a submodule
$$B_r^{p,q} := Z_{r-1}^{p+1,q-1} + D(Z_{r-1}^{p-r+1,q+r-2}).$$
Here $Z_{r-1}^{p+1,q-1} \subset Z_r^{p,q}$ may be viewed as those zig-zags for which the first term $\alpha^{p,q}$ is zero. Note that $D(Z_{r-1}^{p-r+1,q+r-2}) \subset C^{p,q}$ since D of any element in Z_r is just $\pm\delta$ of the tail of the zig-zag, and the tail of an element in $Z_{r-1}^{p-r+1,q+r-2}$ sits in the $(p-1,q)$ position. Let
$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}.$$
If $a \in Z_r^{p,q}$, its differential Da is obviously contained in $Z_r^{p+r,q-r+1}$. Since
$$DB_r^{p,q} = DZ_{r-1}^{p+1,q-1} \subset B_r^{p+r,q-r+1}$$
the class of Da in $E_r^{p+r,q-r+1}$ depends only on the class of a in $E_r^{p,q}$. This defines
$$\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$
Since d_r is induced from D, it is immediate that d_r squares to 0, so d_r is a differential. Its kernel d_r is represented by zig-zag's $a \in Z_r^{p,q}$ of length r that can be extended to a zig-zag $Z_{r+1}^{p,q}$ of lengthe $r+1$. It is straightforward (but slightly tedious) to check that indeed, E_{r+1} is the cohomology for E_r. By construction, $E_\infty^{p,q}$ is represented by elements in the p,q-position that may be extended down to a D-cocycle.

As a first example, let us re-examine the isomorphism between Čech cohomology and singular cohomology. Given a good cover $\mathcal{U}$, view the Čech complex $\check{C}^p(X,\mathcal{U};R)$ as the zeroth row of a double complex $C^{p,q}$. Let $\tilde{C}^{p,q} = C^q(\mathcal{U}^pX;R)$. The inclusion map $C^{p,q} \to \tilde{C}^{p,q}$ induces maps between the spectral sequences. Already at the E_1 stage the two spectral sequences coincide. It follows that the map in cohomology $H(C,D) \to H(\tilde{C},D)$ is an isomorphism. But $H(C,D) = H(\check{C}^\bullet(X,\mathcal{U};R),\delta)$ while $H(\tilde{C},D) = H(||\mathcal{U}X||;R) = H(X;R)$.

4.4. The simplicial de Rham theorem. As an example of spectral sequence techniques, we will now proof the simplicial analogue of de Rham's isomorphism between singular cohomology (with coefficients in $\mathbb{R}$) and de Rham cohomology. Let us recall briefly that the proof of this isomorphism uses an intermediate complex of smooth singular chains $C_k^{sm}(X) \subset C_k(X)$. One proves that the inclusion map is a chain homotopy equivalence, hence dually the map $C^k(X, \mathbb{R}) \to \mathrm{Hom}(C_k^{sm}(X), \mathbb{R})$ is a cochain homotopy equivalence. Integration over smooth chains defines a map $\Omega^k(X) \to \mathrm{Hom}(C_k^{sm}(X), \mathbb{R})$, and the essence of de Rham's theorem is that this is a cochain homotopy equivalence as well.

Suppose now $X_\bullet$ is a simplicial manifold, and define a double complex of differential forms,

$$(\Omega^\bullet(X_\bullet), \mathrm{d}, \delta),$$

where again δ is an alternating sum of pull-backs. Let $\Omega^k(X)$ be the associated total complex.

THEOREM 4.6 (Simplicial de Rham theorem). *The cohomology of the complex $\Omega^k(X)$ is canonically isomorphic to the singular cohomology of $\|X\|$ with coefficients in $\mathbb{R}$.*

PROOF. Define a map

$$\Omega^p(X_q) \to \mathrm{Hom}(C_p^{\mathrm{sm}}(X_q), \mathbb{R})$$

by integration over chains. This is a homomorphism of double complexes, and by the usual de Rham theorem the induced map in d-cohomology is an isomorphism. Hence, the associated spectral sequences coincide already at the E_1-term. It follows that the above map induces an isomorphism of cohomology groups for the total complex. Similarly, the maps

$$C^p(X_q, \mathbb{R}) = \mathrm{Hom}(C_p(X_q), \mathbb{R}) \to \mathrm{Hom}(C_p^{\mathrm{sm}}(X_q), \mathbb{R})$$

give an isomorphism for the total cohomology. $\square$

4.5. The cohomology $H^i(BG)$ for $i \leq 4$. We will now give a systematic computation of $H^i(BG)$ $(0 \leq i \leq 4)$ for a compact connected simple Lie group G. We will quickly need some basic facts about the cohomology of Lie groups, so let us briefly review those facts. We introduce the following notation. For $\xi \in \mathfrak{g}$ let ξ^L denote the left-invariant vector field on G equal to ξ at e, and ξ^R the right-invariant vector field. Let $\Omega(G)^L, \Omega(G)^R$ be the spaces of left-/right- invariant differential forms and $\Omega(G)^{L\times R}$ the space of bi-invariant differential forms. The space $\Omega(G)^L$ is generated by the components of the left Maurer-Cartan form $\theta^L \in \Omega^1(G)^L \otimes \mathfrak{g}$ which is defined by $\iota(\xi^L)\theta^L = \xi$. Similarly let θ^R be the right-Maurer-Cartan form. These satisfy structure equations,

$$\mathrm{d}\theta^L + \tfrac{1}{2}[\theta^L, \theta^L] = 0, \quad \mathrm{d}\theta^R - \tfrac{1}{2}[\theta^R, \theta^R] = 0.$$

Note also that $[\theta^L, [\theta^L, \theta^L]] = 0$ by the Jacobi identity (plug in left-invariant vector fields to see this.) Both inclusions

$$\Omega(G)^{L\times R} \to \Omega(G)^L \to \Omega(G)$$

induce isomorphisms in cohomology: Indeed, if $[\alpha] \in H(G)$, then $[l_g^*\alpha] = l_g^*[\alpha] = [\alpha]$ by homotopy invariance, since G is connected. Thus, the average of α under the left action is cohomologous to α. Similarly, if we also average under the right-action we get a bi-invariant form cohomologous to α.

LEMMA 4.7 (Koszul). *The de Rham differential on $\Omega(G)^L$ is given by the formula*

$$d|_{\Omega(G)^L} = \tfrac{1}{2}\sum_a \langle e^a, \theta^L\rangle\, L_{e_a^L}.$$

PROOF. Since both sides are graded derivations, it suffices to check on θ^L. We have

$$\begin{aligned}
\tfrac{1}{2}\sum_a \langle e^a, \theta^L\rangle\, L_{e_a^L}\theta^L &= \tfrac{1}{2}\sum_a \langle e^a, \theta^L\rangle \iota_{e_a^L}\mathrm{d}\theta^L \\
&= -\frac{1}{4}\sum_a \langle e^a, \theta^L\rangle \iota_{e_a^L}[\theta^L, \theta^L] \\
&= -\tfrac{1}{2}\sum_a \langle e^a, \theta^L\rangle [e_a, \theta^L] \\
&= -\tfrac{1}{2}[\theta^L, \theta^L] = \mathrm{d}\theta^L.
\end{aligned}$$

□

Since the left-invariant vector fields generate right-translations, this formula shows that d is in fact 0 on $\Omega(G)^{L\times R}$! Using $\Omega(G)^{L\times R} = (\wedge\mathfrak{g}^*)^G$ this proves,

PROPOSITION 4.8. *The de Rham cohomology of a compact connected Lie group is isomorphic to $(\wedge\mathfrak{g}^*)^G$. Every cohomology class has a unique bi-invariant representative.*

The structure of the algebra $(\wedge\mathfrak{g}^*)^G$ is completely known, by the Hopf-Koszul-Samelson theorem (see e.g. Greub-Halperin-Vanstone [**16**]). Some basic facts are easy to figure out by hand, however. For instance, if $p \in (S^m\mathfrak{g}^*)^G$ is an invariant polynomial on $\mathfrak{g}$, one has a corresponding bi-invariant form on G:

$$\alpha^p = \theta^L \cdot p'([\theta^L, \theta^L])$$

here $p' \in S^{m-1}(\mathfrak{g}^*) \otimes \mathfrak{g}^*$ is the gradient of p, defined by $\xi \cdot p'(\zeta) = \frac{d}{\mathrm{d}t}|_{t=0} p(\zeta + t\xi)$. (We leave it as an exercise that this form is indeed bi-invariant). In particular, if B is an invariant inner product on $\mathfrak{g}$, the polynomial $p(\xi) = B(\xi, \xi)$ defines a form of degree 3, which we prefer to normalize as follows:

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]).$$

It is easy to check that if G is simple, $H^i(G) = 0$ for $i = 1, 2$ while $H^3(G) = \mathbb{R}$ with generator this form η.

Now consider the double complex for BG, i.e. $C^{p,q} = \Omega^q(G^p)$. The E_1-term is $H^q(G^p)$. Note this is 0 for $p+q \le 4$ and $q > 0$, except for the term $H^3(G) = \mathbb{R}$ which we just computed. (Draw a picture of the E_1-term!) To compute the E_2-term, we have calculate the image of $[\eta]$ under $\delta : H^3(G) \to H^3(G^2) = \mathbb{R} \oplus \mathbb{R}$. By the explicit formulas for the face maps,

$$\delta : \Omega^q(G) \to \Omega^q(G^2), \quad \delta\alpha = \mathrm{pr}_1^*\,\alpha + \mathrm{pr}_2^*\,\alpha - \mu^*\alpha$$

where $\mu : G \times G \to G$ is group multiplication and pr_i are the two projections. For η, one may verify that

$$\delta\eta = \mathrm{d}\Big(\tfrac{1}{2} B(\mathrm{pr}_1^*\,\theta^L, \mathrm{pr}_2^*\,\theta^R)\Big).$$

Thus $\mathrm{d}_1[\eta] = [\delta\eta] = 0$, and $E_2^{1,3} = E_1^{p,q} = \mathbb{R}$. Also, the $d_1 = \delta$-cohomology of the row $p = 0$ is trivial. Thus for $p+q \le 4$, $E_2^{1,3} = E_2^{0,0} = \mathbb{R}$, and $E_2^{p,q} = 0$ otherwise. Note that the differentials

d_2 from the remaining entries $E_2^{p,q}$ with $p+q \le 4$ are all zero, and similarly for $\mathrm{d}_3, \mathrm{d}_4, \ldots$. We conclude that $E_2^{p,q} = E_\infty^{p,q}$ for $p+q \le 4$. We have thus shown (real coefficients)

$$H^1(BG) = H^2(BG) = H^3(BG) = 0, \quad H^4(BG) = \mathbb{R}.$$

We can be more precise: Let $\beta = \frac{1}{2} B(\mathrm{pr}_1^* \theta^L, \mathrm{pr}_2^* \theta^R) \in \Omega^2(G^2)$. Then $\delta\beta = 0$ (as one verifies by direct calculation) and therefore $\eta + \beta \in \bigoplus_{p+q=4} \Omega^q(G^p)$ is a D-cocycle representing a generator of $H^4(BG)$.

REMARK 4.9. If G is simple and simply connected, it is known that $H^i(G, \mathbb{Z})$ has no torsion in degree $i \le 4$. Therefore, the above argument also works with $\mathbb{Z}$ coefficients, using singular cochains, and one finds that $H^4(BG, \mathbb{Z}) = \mathbb{Z}$ while $H^i(BG, \mathbb{Z}) = 0$ for $1 \le i \le 3$.

4.6. Product structures. Recall that the *front p-face* of an n-simplex $p \ge n$, is the simplex spanned by the first $p+1$ vertices $e_0, \ldots, e_p$, while the *back p-face* is spanned by the last $p+1$ vertices $e_{n-p}, \ldots, e_n$. We recall that these enter the definition of the cup product on singular cochains: Given cochains $\alpha \in C^q(X), \beta \in C^{q'}(X)$, one defines the value of $\alpha \cup \beta \in C^{q+q'}(X)$ on a singular $q+q'$-simplex $\sigma : \Delta^{q+q'} \to X$ to be the value of α on the front q-simplex, $\Delta^q \to \Delta^{q+q'} \to X$, times the value of β on the back q'-simplex $\Delta^{q'} \to \Delta^{q+q'} \to X$.

The front-face and back-face correspond to the following two morphisms in the category Ord:

$$\phi_p^n : [p] \to [n], \ i \mapsto i, \quad \psi_p^n : [p] \to [n], \ i \mapsto i + n - p.$$

If $X_\bullet$ is a simplicial space, we obtain corresponding maps

$$X(\phi_p^n), \, X(\psi_p^n) : \, X_n \to X_p.$$

We can use these to define a product structure on the double complex of singular cochains $C^\bullet(X_\bullet)$, by composition,

$$C^q(X_p) \otimes C^{q'}(X_{p'}) \to C^q(X_{p+p'}) \otimes C^{q'}(X_{p+p'}) \to C^{q+q'}(X_{p+p'}).$$

Here the first map is $(-1)^{q'p}$ times pull-back under the map $X(\phi_p^{p+p'}) \times X(\phi_{p'}^{p+p'})$, while the second map is the usual cup product. The sign $(-1)^{q'p}$ is necessary in order that D becomes a derivation for the product structure. More precisely we have:

PROPOSITION 4.10. *Both differentials d, δ on the double complex are (graded) derivations with respect to the product structure. That is, if $\alpha \in C^q(X_p)$ and $\beta \in C^{q'}(X_{p'})$, we have*

$$\begin{aligned} d(\alpha\beta) &= (d\alpha)\beta + (-1)^{q+p}\alpha(d\beta) \\ \delta(\alpha\beta) &= (-1)^{q'}(\delta\alpha)\beta + (-1)^p \alpha(\delta\beta) \\ D(\alpha\beta) &= (D\alpha)\beta + (-1)^{q+p}\alpha(D\beta). \end{aligned}$$

Since D is a derivation, the product structure descends to the cohomology $H^\bullet(C^\bullet(X_\bullet), D))$. By a similar formula, we have a ring structure on the simplicial de Rham complex.

THEOREM 4.11. *The isomorphism $H^\bullet(C^\bullet(X_\bullet), D) \to H^\bullet(||X||)$ is an isomorphism of graded rings.*

See e.g. Dupont, [**14**].

In general, if a double complex $(C^{p,q}, \mathrm{d}, \delta)$ has a product structure relative to which d, δ satisfy

$$\begin{aligned} \mathrm{d}(\alpha\beta) &= (\mathrm{d}\alpha)\beta + (-1)^{q+p}\alpha(\mathrm{d}\beta) \\ \delta(\alpha\beta) &= (-1)^{q'}(\delta\alpha)\beta + (-1)^{p}\alpha(\delta\beta) \end{aligned}$$

for $\alpha \in C^{p,q}, \beta \in C^{p',q'}$, the differential D becomes a derivation for the total complex. Furthermore, each $\mathrm{d}_r : E_r \to E_r$ in the spectral sequence becomes a graded derivation, for the product structure induced from E_{r-1}. Passing to the limit, we obtain a product structure on E_∞. On the other hand, the product structure on the cohomology of the total complex is compatible with the filtration, hence it descends to a product structure on the associated graded group, which we saw is E_∞. The two product structures on E_∞ coincide.

The upshot is: The total cohomology of the double complex $\Omega^p(X_q)$, with the ring structure just introduced, is isomorphic to the cohomology algebra of $||X||$ as a ring.

5. The Chern-Weil construction

5.1. Connections and curvature on principal bundles. Let G be a Lie group and $\pi : P \to B$ be a (smooth) principal G-bundle. A differential form α on P is called *horizontal* if $\iota_{\xi_P}\alpha = 0$ for all $\xi \in \mathfrak{g}$, and *basic* if it is both invariant and horizontal. It is well-known (and easy to see, using local trivializations) that the pull-back map $\pi^* : \Omega(B) \to \Omega(P)$ with image the basic forms. That is, any basic form *descends* to a unique form on B. More generally, if V is a G-represenations, a V-valued form $\alpha \in \Omega(P) \otimes V$ is called basic if it is in $(\Omega(P)_{\mathrm{hor}} \otimes V)^G$. Basic V-valued forms descend to forms with values in the associated vector bundle $P \times_G V$.

Let $VP \subset TP$ be the vertical subbundle, i.e. $V_pP = \ker(\mathrm{d}_p\pi)$ for $p \in P$. There is an exact sequence of vector bundles,

$$0 \to VP \to TP \to \pi^*TB \to 0.$$

A connection on P is a G-equivariant splitting of this sequence, i.e. a G-equivariant surjective bundle homomorphism $TP \to VP$, restricting to the identity on VP. This can be reformulated as follows: The bundle VP is trivialized by the generating vector fields for the G-action on P:

$$P \times \mathfrak{g} \to VP, \quad (p, \xi) \mapsto \xi_P(p).$$

Thus, a connection on P is a G-equivariant bundle map $TP \to P \times \mathfrak{g}$ taking ξ_P to ξ. It is thus given by a *connection 1-form* $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ with defining properties,

$$g^*\theta = \mathrm{Ad}_g\, \theta, \quad \iota_{\xi_P}\theta = \xi.$$

Any two connections differ by a form in $\Omega^1(P) \otimes \mathfrak{g}$ that is *invariant* (i.e. lies in $(\Omega^1(P) \otimes \mathfrak{g})^G$ and *horizontal* (i.e. annihilated by all ι_{ξ_P}.) That is, the space of connections is an affine space, with underlying vector space the horizontal and invariant $\mathfrak{g}$-valued 1-forms. Given a connection, one may define the corresponding horizontal bundle $HP = \ker\theta$, and the splitting identifies $HP = \pi^*TB$.

The curvature of a connection θ is defined as

$$F^\theta = \mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta].$$

It is an invariant and horizontal $\mathfrak{g}$-valued 2-form. To remove any possible ambiguities, let us write out the curvature in terms of a basis e_a of $\mathfrak{g}$: Introduce structure constants f^a_{bc} of $\mathfrak{g}$ by $[e_b, e_c] = \sum_a f^a_{bc} e_a$, and write $\theta = \sum_a \theta^a e_a$. Then $F^\theta = \sum_a (F^\theta)^a e_a$ where $(F^\theta)^a$ are 2-forms,

$$(F^\theta)^a = \mathrm{d}\theta^a + \tfrac{1}{2} \sum_{bc} f^a_{bc} \theta^b \wedge \theta^c.$$

One of its basic properties is the *Bianchi identity*

$$\mathrm{d}^\theta F^\theta := (\mathrm{d} + [\theta, \cdot]) F^\theta = 0.$$

(The proof relies on the fact that $[\theta, [\theta, \theta]] = 0$, which in turn follows from the Jacobi identity.) There are many geometric interpretations of the curvature – for example, it measures the failure of the horizontal lift $\mathrm{Lift} : \mathfrak{X}(B) \to \mathfrak{X}(P)$ to be a Lie algebra homomorphism.

We are interested in the role of F^θ in the Chern-Weil construction of characteristic classes on B. If $p \in S^m \mathfrak{g}^*$ is a polynomial on $\mathfrak{g}$, we may form $\tilde{c}^\theta(p) := p(F^\theta) \in \Omega^{2m}(P)$. More accurately, we may view the curvature as a map $\mathfrak{g}^* \to \Omega^2(P)$, as such it extends to an algebra homomorphism $\tilde{c}^\theta : S\mathfrak{g}^* \to \Omega^{\mathrm{even}}(P)$. Note that the image of c^θ lies in the space of forms on P that are horizontal and invariant, i.e. *basic*. The space $\Omega(P)_{\mathrm{basic}}$ of basic forms is isomorphic to $\Omega(B)$, by pull-back. Hence, $\tilde{c}^\theta$ descends to an algebra homomorphism

$$c^\theta : S\mathfrak{g}^* \to \Omega^{\mathrm{even}}(B).$$

Theorem 5.1 (Chern-Weil construction). *If $p \in (S\mathfrak{g}^*)^G$ is an invariant polynomial, the form $c^\theta(p)$ is closed. Its cohomology class does not depend on the choice of θ.*

We will postpone the (not very difficult) proof, since we will prove a more general result further down. The cohomology classes $[c^\theta(p)]$ are called the characteristic classes of the principal bundle P. Indeed, we will see that they are exactly the characteristic classes (for real coefficients) obtained from the classifying map for P.

5.2. $\mathfrak{g}$-differential algebras. Cartan's idea [**8, 9**] was to introduce an algebraic model for the space of differential forms on the classifying bundle $EG \to BG$, and to re-phrase the Chern-Weil construction in those terms. (We will show how his model is related to the simplicial model discussed earlier.)

Definition 5.2. A differential graded algebra is an graded algebra $\mathcal{A} = \bigoplus_{i=0}^\infty \mathcal{A}^i$ with a differential d of degree $+1$, such that d is a derivation. It is called a $\mathfrak{g}$-differential algebra if, in addition, there are derivations L_ξ of degree 0 and ι_ξ of degree -1, for all $\xi \in \mathfrak{g}$, satisfying the relations of contractions, Lie derivative and differential on a manifold with a $\mathfrak{g}$-action:

$$[\iota_\xi, \iota_{\xi'}] = 0,\ \ [L_\xi, \iota_{\xi'}] = \iota_{[\xi,\xi']_\mathfrak{g}}$$
$$[\mathrm{d}, L_\xi] = 0, \quad [L_\xi, L_{\xi'}] = L_{[\xi,\xi']_\mathfrak{g}},$$
$$[\mathrm{d}, \mathrm{d}] = 0, \quad [\mathrm{d}, \iota_\xi] = L_\xi$$

(using graded commutators).

There is an obvious notion of homomorphism of $\mathfrak{g}$-differential algebras. Sometimes one also considers $\mathfrak{g}$-differential spaces (i.e. one doesn't require algebra structures.) A first example of a $\mathfrak{g}$-differential algebra is the algebra of differential forms on a manifold with a $\mathfrak{g}$-action.

DEFINITION 5.3. A connection on a $\mathfrak{g}$-differential algebra is an element $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$ satisfying $\iota_\xi \theta = \xi$ and $L_\xi \theta = -[\xi, \theta]_\mathfrak{g}$. The curvature of the connection θ is the element $F^\theta \in \mathcal{A}^2 \otimes \mathfrak{g}$ defined as

$$F^\theta = \mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta].$$

Note that the condition $L_\xi \theta = -[\xi, \theta]_\mathfrak{g}$ is the infinitesimal version of the G-invariance condition for a principal connection. It is equivalent to the global condition if G is connected. An example is therefore the connection on the space of differential forms on a principal G-bundle, if G is connected. More generally, if P is a manifold with a Lie algebra action of $\mathfrak{g}$, the existence of a connection on $\Omega(P)$ implies that the action is *locally free* (and e.g. for $\mathfrak{g}$ compact the converse holds true). As in the case of principal bundles, if a connection exists, the space of connections is an affine space with underlying vector space the space $(\mathcal{A}^1_{\mathrm{hor}} \otimes \mathfrak{g})_{\mathrm{inv}}$.

We will often take the equivalent point of view that a connection is an equivariant map $\theta : \mathfrak{g}^* \to \mathcal{A}^1$ with $\iota_\xi \theta(\mu) = \langle \mu, \xi \rangle$, and the curvature is an equivariant map $F^\theta : \mathfrak{g}^* \to \mathcal{A}^2$.

DEFINITION 5.4. Let $\mathcal{A}$ be a $\mathfrak{g}$-dga. One defines subalgebras of horizontal, invariant and basic elements by

$$\mathcal{A}_{\mathrm{hor}} = \bigcap_\xi \ker(\iota_\xi), \quad \mathcal{A}_{\mathrm{inv}} = \bigcap_\xi \ker(L_\xi), \quad \mathcal{A}_{\mathrm{basic}} = \mathcal{A}_{\mathrm{hor}} \cap \mathcal{A}_{\mathrm{inv}}.$$

As in the case of principal bundles, F^θ takes values in horizontal elements. If we extend F^θ to an algebra homomorphism $S\mathfrak{g}^* \to \mathcal{A}^{\mathrm{even}}$, $p \mapsto p(F^\theta)$ then $p(F^\theta)$ is basic provided $p \in (S\mathfrak{g}^*)^{\mathrm{inv}}$.

For any principal bundle $\pi : P \to B$, with G connected, a form on P is basic if and only if it is the pull-back of a form on the base: $\pi^* : \Omega(B) \to \Omega(P)_{\mathrm{basic}}$ is an isomorphism.

LEMMA 5.5. *The basic subalgebra $\mathcal{A}_{\mathrm{basic}}$ is invariant under d. Its cohomology is called the basic cohomology of $\mathcal{A}$.*

PROOF. Follows from $[\iota_\xi, \mathrm{d}] = L_\xi$ and $[L_\xi, \mathrm{d}] = 0$. □

Note that by contrast, the subalgebra of horizontal elements is not d-invariant.

5.3. The Weil algebra. The Weil algebra $W\mathfrak{g}$ is most quickly defined in terms of generators and relations:

DEFINITION 5.6. The Weil algebra $W\mathfrak{g}$ is the commutative graded algebra, freely generated by elements $\mu \in \mathfrak{g}^*$ of degree 1 and elements $\overline{\mu} \in \mathfrak{g}^*$ of degree 2 (linear over the two copies of $\mathfrak{g}^*$), with contractions, Lie derivatives and differential given on degree 1 generators by

$$\iota_\xi \mu = \langle \mu, \xi \rangle, \quad L_\xi \mu = -\operatorname{ad}^*_\xi \mu, \quad \mathrm{d}\mu = \overline{\mu}.$$

The canonical connection on $W\mathfrak{g}$ is given by $\theta^W : \mathfrak{g}^* \to W\mathfrak{g}, \ \mu \mapsto \mu$.

This is well-defined: The relations among contractions, Lie derivatives and differential force us to put

$$\iota_\xi \overline{\mu} = \iota_\xi \mathrm{d}\mu = L_\xi \mu = -\operatorname{ad}^*_\xi \mu$$

as well as $L_\xi \overline{\mu} = -\overline{\operatorname{ad}^*_\xi \mu}$ and $\mathrm{d}\overline{\mu} = 0$. It is easily checked that these definitions are consistent, essentially because $W\mathfrak{g}$ is free over the given set of generators.

Remark 5.7. Note that the construction of an algebra and a differential makes sense for $\mathfrak{g}^*$ replaced with any vector space V, and defines a commutative graded differential algebra. This is usually called the *Koszul algebra.*

As an algebra, $W\mathfrak{g}$ is simply the tensor product of $S\mathfrak{g}^*$ (corresponding to the degree 2 generators) and $\wedge\mathfrak{g}^*$ (corresponding to the degree 1 generators). We may also view $W\mathfrak{g}$ as a symmetric algebra (in the graded sense) over the graded vector space $\mathfrak{g}^* \oplus \mathfrak{g}^*$, where the first copy corresponds to degree 1 generators and the second copy to degree 2 generators.

Proposition 5.8 (Acyclicity). *There exists a canonical homotopy operator $h : W\mathfrak{g} \to W\mathfrak{g}$ with $[\mathfrak{h}, d] = \mathrm{id} - \Pi$, where $\Pi : W\mathfrak{g} \to W\mathfrak{g}$ is projection onto $W^0\mathfrak{g} = \mathbb{R}$.*

Proof. Let σ be the degree -1 derivation given on generators by $\sigma\mu = 0$ and $\sigma\overline{\mu} = \mu$. The commutator $[\sigma, \mathrm{d}]$ is a derivation of degree 0, equal to the identity on generators. It hence extends to the *Euler operator* on $W\mathfrak{g}$, given on a product of generators simply by multiplication by the number of generators in that product. In particular, it is invertible on $W^+\mathfrak{g} = \ker \Pi$. It is easy to check that $[\sigma, \mathrm{d}]$ commutes with σ and d. The homotopy operator is defined by $h = 0$ on $W^0\mathfrak{g}$ and $h \circ [\sigma, \mathrm{d}] = \sigma$ on $W^+\mathfrak{g}$. □

Proposition 5.9 (Universal property). *If $\mathcal{A}$ is any $\mathfrak{g}$-dga with connection θ, there is a unique homomorphism of $\mathfrak{g}$-dga's $c^\theta : W\mathfrak{g} \to \mathcal{A}$ such that the following diagram commutes,*

$$\begin{tikzcd} W\mathfrak{g} \arrow[r, "c^\theta"'] & \mathcal{A} . \\ \mathfrak{g}^* \arrow[u] \arrow[ur] & \end{tikzcd}$$

Proof. The homomorphism takes μ to $\theta(\mu)$ and $\overline{\mu}$ to $\mathrm{d}\theta(\mu)$. It is straightforward to check that this has the correct properties. □

These two properties show clearly that we should think of $W\mathfrak{g}$ as the algebraic analogue of $\Omega(EG)$, with c^θ the analogue to pull-back under a classifying map $\Omega(EG) \to \Omega(P)$, and h the algebraic analogue of the homotopy operator for a contraction of EG to a base point.

Proposition 5.10. *Let $\mathcal{A}$ be a locally free $\mathfrak{g}$-dga, and θ_0 and θ_1 two connections on $\mathfrak{g}$. Let $c_0, c_1 : W\mathfrak{g} \to \mathcal{A}$ be the two characteristic homomorphisms defined by θ_0, θ_1. There exists an operator $h : \mathcal{W}\mathfrak{g} \to \mathcal{A}$ of degree -1 with the following properties,*

$$[h, d] = c_0 - c_1, \quad [h, \iota_\xi] = 0, \quad [h, L_\xi] = 0.$$

Proof. Consider the connection θ on $\mathcal{A} \otimes \Omega([0,1])$ given by $\theta = (1-t)\theta_0 + t\theta_1$, where t is the coordinate on $[0,1]$. It defines a characteristic homomorphism

$$c : W\mathfrak{g} \to \mathcal{A} \otimes \Omega([0,1])$$

that pulls back to c_0, c_1 at $t = 0, t = 1$. Let h be $\pm$ the composition of c with fiber integration over $[0,1]$. For the appropriate choice of sign, Stokes' theorem gives $[h, \mathrm{d}] = c_0 - c_1$. The identities $[h, \iota_\xi] = 0, \quad [h, L_\xi] = 0$ hold because the $\mathfrak{g}$-action has no component in the M-direction. □

In general, given two homomorphisms of $\mathfrak{g}$-differential spaces, $c_0, c_1 : \mathcal{A} \to \mathcal{A}'$, we define a $\mathfrak{g}$-chain homotopy to be an operator $h : \mathcal{A} \to \mathcal{A}'$ of degree -1 having the properties in this Proposition.

5.4. The algebraic Chern-Weil construction. To determine the basic subcomplex, it is convenient to replace the degree 2 generators $\overline{\mu}$ by the curvatures $\hat{\mu} = F^\theta(\overline{\mu})$. If e^a denotes a basis of $\mathfrak{g}^*$ dual to the basis e_a of $\mathfrak{g}$ introduced above, we have

$$\hat{\mu} = \mu_a\big(\overline{e^a} + \tfrac{1}{2}\sum_{bc} f^a_{bc} e^b \wedge e^c\big),$$

so these are again generators, and we get another isomorphism $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ where now $S\mathfrak{g}^*$ is generated by the curvatures $\hat{\mu} = F^{\theta^W}(\mu)$, rather than $\overline{\mu}$. It is immediate from this description that $(W\mathfrak{g})_{\rm hor}$ is the subalgebra $S\mathfrak{g}^*$ generated by the curvatures, and therefore

$$(W\mathfrak{g})_{\rm basic} = (S\mathfrak{g}^*)_{\rm inv}.$$

PROPOSITION 5.11. *In terms of the isomorphism $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where $S\mathfrak{g}^*$ is generated by the curvatures $\hat{\mu}$, we have $\iota^W_\xi = 1 \otimes \iota^{\wedge\mathfrak{g}^*}_\xi$ and the Weil differential is*

$$d^W = e^a(L^{S\mathfrak{g}^*}_{e_a} + \tfrac{1}{2}L^{\wedge\mathfrak{g}^*}_{e_a}) + \widehat{e^a}\,\iota^{\wedge\mathfrak{g}^*}_{e_a}$$

PROOF. We use that if $\mathcal{A}$ is a commutative graded algebra, then the space $\mathrm{Der}(A) := \bigoplus_i \mathrm{Der}^i(A)$ of graded derivations of $\mathcal{A}$ is a left-$\mathcal{A}$-module. In particular the right hand side of the formula for d^W defines a derivation, since $(L^{S\mathfrak{g}^*}_{e_a}, L^{\wedge\mathfrak{g}^*}_{e_a}, \iota^{\wedge\mathfrak{g}^*}_{e_a}$ are all derivations of $W\mathfrak{g}$. Two derivations agree if and only if they agree on generators. On generators e^a, the right hand side gives

$$\begin{aligned}(e^b(L^{S\mathfrak{g}^*}_{e_b} + \tfrac{1}{2}L^{\wedge\mathfrak{g}^*}_{e_b}) + \widehat{e^b}\iota^{\wedge\mathfrak{g}^*}_{e_b})e^a &= \tfrac{1}{2}e^b L^{\wedge\mathfrak{g}^*}_{e_b} e^a + \widehat{e^b}(\iota^{\wedge\mathfrak{g}^*}_{e_b})e^a\\ &= -\tfrac{1}{2}f^a_{bc}e^b e^c + \widehat{e^a}\\ &= \overline{e^a}.\end{aligned}$$

Similarly, one verifies $(e^b(L^{S\mathfrak{g}^*}_{e_b} + \tfrac{1}{2}L^{\wedge\mathfrak{g}^*}_{e_b}) + \widehat{e^b}\iota^{\wedge\mathfrak{g}^*}_{e_b})\overline{e^a}$ by direct computation. $\square$

From this formula for the Weil differential, we see that the restricted differential on $(W\mathfrak{g})_{\rm basic}$ is in fact 0. Thus $H_{\rm basic}(W\mathfrak{g}) = H((W\mathfrak{g})_{\rm basic}) = (S\mathfrak{g}^*)_{\rm inv}$.

To summarize, we see that if $\mathcal{A}$ is a $\mathfrak{g}$-dga with connection, the characteristic homomorphism $c^\theta : W\mathfrak{g} \to \mathcal{A}$ restricts to a chain map

$$c^\theta : (S\mathfrak{g}^*)^{\mathfrak{g}} \to \mathcal{A}_{\rm basic}$$

(where $(S\mathfrak{g}^*)^{\mathfrak{g}}$ carries the 0 differential), and that the two chain maps defined by two connections are related by a chain homotopy $c^\theta : (S\mathfrak{g}^*)^{\mathfrak{g}} \to \mathcal{A}_{\rm basic}$ of degree -1. Hence one obtains an algebra homomorphism

$$(S\mathfrak{g}^*)^{\mathfrak{g}} \to H_{\rm basic}(\mathcal{A})$$

independent of the choice of connection. This is Cartan's algebraic analogue to the Chern Weil construction.

PROPOSITION 5.12. *Let A be a locally free commutative $\mathfrak{g}$-dga. Then the mapping*

$$\mathcal{A} \to \mathcal{A} \otimes W\mathfrak{g},\ \ \alpha \mapsto \alpha \otimes 1$$

is a $\mathfrak{g}$-chain homotopy equivalence. Given a connection θ, a homotopy inverse is given by the map

$$\mathcal{A} \otimes W\mathfrak{g} \to \mathcal{A},\quad (\alpha \otimes w) \mapsto \alpha\, c^\theta(w).$$

PROOF. We have to show that the composition

$$W\mathfrak{g}\otimes\mathcal{A}\to\mathcal{A}\to W\mathfrak{g}\otimes\mathcal{A},\quad w\otimes\alpha\mapsto(c^\theta w)\alpha\mapsto 1\otimes(c^\theta w)\alpha$$

is $\mathfrak{g}$-homotopic to the identity. For this, it suffices to show that the two maps

$$\begin{aligned} W\mathfrak{g}&\to W\mathfrak{g}\otimes\mathcal{A}, \qquad && w\mapsto 1\otimes c^\theta w\\ W\mathfrak{g}&\to W\mathfrak{g}\otimes\mathcal{A}, \qquad && w\mapsto w\otimes 1\end{aligned}$$

are $\mathfrak{g}$-chain homotopic. But this follows, since the first map is the characteristic homomorphism for the connection $1\otimes\theta$ on $\tilde{W}\mathfrak{g}\otimes\mathcal{A}$, while the second map is the characteristic homomorphism for the connection $\theta^W\otimes 1$. □

5.5. The Weil model of equivariant cohomology. Recall that if M is a G-manifold, we defined the equivariant cohomology ring of M to be the cohomology ring of the borel construction $M_G=EG\times_G M$. If EG were a finite-dimensional principal bundle, and thus M_G were a manifold, this would be the cohomology of the de Rham complex

$$\Omega(M_G)\cong(\Omega(EG\times M))_{\text{basic}}=(\Omega(EG)\otimes\Omega(M))_{\text{basic}}.$$

Thinking of $W\mathfrak{g}$ as an algebraic model for $\Omega(EG)$, and of a $\mathfrak{g}$-dga as the algebraic counterpart of $\Omega(M)$, this motivated the following definition:

DEFINITION 5.13. The equivariant cohomology of a $\mathfrak{g}$-dga $\mathcal{A}$ is the basic cohomology of $W\mathfrak{g}\otimes\mathcal{A}$:

$$H_\mathfrak{g}(\mathcal{A})=H_{\text{basic}}(W\mathfrak{g}\otimes\mathcal{A})$$

This is known as the *Weil model* of equivariant cohomology, especially in the case $\mathcal{A}=\Omega(M)$ for a G-manifold M. In this case we will write $H_\mathfrak{g}(M):=H_\mathfrak{g}(\Omega(M))$. We will show later that (for G a compact connected Lie group) it is equivalent to the Borel model, i.e. that $H_\mathfrak{g}(M)\cong H_G(M):=H(M_G)$. Taking $\mathcal{A}=\mathbb{R}=\Omega(\text{pt})$ to be the trivial $\mathfrak{g}$-dga,

$$H_\mathfrak{g}(\text{pt})=H_{\text{basic}}(W\mathfrak{g})=(S\mathfrak{g}^*)_{\text{inv}}.$$

Any homomorphism of $\mathfrak{g}$-dga's $\mathcal{A}_1\to\mathcal{A}_2$ induces an algebra homomorphism $H_\mathfrak{g}(\mathcal{A}_1)\to H_\mathfrak{g}(\mathcal{A}_2)$; in particular (taking $\mathcal{A}_1=\mathbb{R}$ and $\mathcal{A}_2=\mathcal{A}$), $H_\mathfrak{g}(\mathcal{A})$ is a module over $(S\mathfrak{g}^*)_{\text{inv}}$. For any $\mathfrak{g}$-dga, the inclusion map $\mathcal{A}\to W\mathfrak{g}\otimes\mathcal{A}$, $\alpha\mapsto 1\otimes\alpha$ is a homomorphism of $\mathfrak{g}$-dga's, hence it induces an algebra homomorphism $H_{\text{basic}}(\mathcal{A})\to H_\mathfrak{g}(\mathcal{A})$. Above we proved that if $\mathcal{A}$ is locally free and commutative then the map $\mathcal{A}\to W\mathfrak{g}\otimes\mathcal{A}$ is a $\mathfrak{g}$-homotopy equivalence. That is,

PROPOSITION 5.14. *If $\mathcal{A}$ is a locally free $\mathfrak{g}$-dga, the map $\mathcal{A}\to W\mathfrak{g}\otimes\mathcal{A}$, $\alpha\mapsto 1\otimes\alpha$ induces an algebra isomorphism $H_{\text{basic}}(\mathcal{A})=H_\mathfrak{g}(\mathcal{A})$.*

Thus, if P is a principal G-bundle, the pull-back map $\Omega(B)\to\Omega(P)_{\text{basic}}\subset\Omega(P)$ induces an isomorphism $H_\mathfrak{g}(\Omega(P))=H(\Omega(B))\equiv H(B)$. This provides further evidence for the conjecture $H_\mathfrak{g}(M)=H_G(M)$, since indeed

$$H_G(P)=H(EG\times_G P)=H(P\times_G EG)=H(B)$$

(using that $P\times_G EG\to B$ is a fiber bundle with contractible fibers EG.)

5.6. The Cartan model of equivariant cohomology. The Weil model of equivariant cohomology has the advantage of a good conceptual explanation, $W\mathfrak{g}$ playing the role of differential forms on EG. For computational purposes, it is usually more convenient to work with an equivalent model known as the *Cartan model.* Let $\mathcal{A}$ be a $\mathfrak{g}$-dga (we usually have in mind the algebra of differential forms on a G-manifold $\mathcal{A} = \Omega(M)$). The first step in calculating $H_{\mathfrak{g}}(\mathcal{A})$ is to determine the basic subcomplex $(W\mathfrak{g}\otimes\mathcal{A})_{\mathrm{basic}}$. Identify $W\mathfrak{g} = S\mathfrak{g}^*\otimes\wedge\mathfrak{g}^*$, where $S\mathfrak{g}^*$ is the symmetric algebra generated by the curvature variables. To simplify notation, we will denote the degree 2 generators by $v^a := \widehat{e^a}$. We have

$$W\mathfrak{g} = (S\mathfrak{g}^*\otimes\wedge^+\mathfrak{g}^*)\oplus S\mathfrak{g}^*,$$

where $\wedge^+\mathfrak{g}^* = \bigoplus_{i>0}\wedge^i\mathfrak{g}^*$, hence a projection $W\mathfrak{g}\to S\mathfrak{g}^*$ ("setting the connection variables equal to 0"). Extend to a projection

$$W\mathfrak{g}\otimes\mathcal{A}\to S\mathfrak{g}^*\otimes\mathcal{A}.$$

THEOREM 5.15 (Cartan). *The projection $W\mathfrak{g}\otimes\mathcal{A}\to S\mathfrak{g}^*\otimes\mathcal{A}$ restricts to an algebra isomorphism*

$$(W\mathfrak{g}\otimes\mathcal{A})_{\mathrm{basic}}\to(S\mathfrak{g}^*\otimes\mathcal{A})_{\mathrm{inv}}.$$

This isomorphism takes the differential $d^W + d^{\mathcal{A}}$ to the equivariant differential

$$d_{\mathfrak{g}} = d - v^a\otimes\iota^{\mathcal{A}}_a.$$

(where $d = d^{\mathcal{A}}$).

PROOF. We prove this result using an elegant trick due to Kalkman [**20**]. Consider the following derivation on $W\mathfrak{g}\otimes\mathcal{A}$,

$$\psi = e^r\otimes\iota^{\mathcal{A}}_r.$$

Note that ψ has degree 0 and is $\mathfrak{g}$-equivariant. Hence $\exp(\psi)$ is a $\mathfrak{g}$-equivariant algebra automorphism. Let us compute

$$\exp(\psi)\circ(\iota^W_a+\iota^{\mathcal{A}}_a)\circ\exp(-\psi) = \mathrm{Ad}(\exp\psi)(\iota^W_a+\iota^{\mathcal{A}}_a) = \exp(\mathrm{ad}_\psi)(\iota^W_a+\iota^{\mathcal{A}}_a).$$

Write $\exp(\mathrm{ad}_\psi) = \sum_{j=0}^\infty\frac{1}{j!}\mathrm{ad}^j_\psi$. We find,

$$\begin{aligned}\mathrm{ad}_\psi\,\iota_a &= \mathrm{ad}_\psi(\iota^W_a+\iota^{\mathcal{A}}_a)\\ &= -\iota^{\mathcal{A}}_a\\ \mathrm{ad}^2_\psi\,\iota_a &= 0.\end{aligned}$$

Thus $\exp(\psi)$ takes $\iota^W_a+\iota^{\mathcal{A}}_a$ to ι^W_a. Thus,

$$\exp(\psi):\ (W\mathfrak{g}\otimes\mathcal{A})_{\mathrm{basic}}\to((W\mathfrak{g})_{\mathrm{hor}}\otimes\mathcal{A})_{\mathrm{inv}} = (S\mathfrak{g}^*\otimes\mathcal{A})_{\mathrm{inv}}.$$

On the other hand, observe that the automorphism $\exp(\psi)$ does not change the projection of an element onto $S\mathfrak{g}^*\otimes\mathcal{A}$, hence on $(W\mathfrak{g}\otimes\mathcal{A})_{\mathrm{hor}}$ it coincides with that projection. It remains to work out the induced differential. Consider the formula for the Weil differential,

$$\mathrm{d}^W = e^a(L^{S\mathfrak{g}^*}_{e_a}+\tfrac12 L^{\wedge\mathfrak{g}^*}_{e_a}) + v^a\,\iota^{\wedge\mathfrak{g}^*}_{e_a}.$$

To compute the induced differential, we must consider

$$\begin{aligned}\mathrm{d}^W+\mathrm{d}^{\mathcal{A}} &= e^a(L^{S\mathfrak{g}^*}_{e_a}+\tfrac12 L^{\wedge\mathfrak{g}^*}_{e_a}) + v^a\,\iota^{\wedge\mathfrak{g}^*}_{e_a}+\mathrm{d}^{A}\\ &= e^a(L^{S\mathfrak{g}^*}_{e_a}+\tfrac12 L^{\wedge\mathfrak{g}^*}_{e_a}) + v^a\,(\iota^{\wedge\mathfrak{g}^*}_{e_a}+\iota^{\mathcal{A}}_{e_a}) - v^a\,\iota^{\mathcal{A}}_{e_a}+\mathrm{d}^{A}.\end{aligned}$$

on an element of $(W\mathfrak{g}\otimes\mathcal{A})_{\mathrm{basic}}$, followed by projection to $S\mathfrak{g}^*\otimes\mathcal{A}$. The term involving $(\iota_{e_a}^{\wedge\mathfrak{g}^*}+\iota_{e_a}^{\mathcal{A}})$ disappears on horizontal elements, while the terms involving $e^a(L_{e_a}^{S\mathfrak{g}^*}$ and $e^a L_{e_a}^{\wedge\mathfrak{g}^*}$ disappear after projection onto $S\mathfrak{g}^*\otimes\mathcal{A}$. Hence the differential on $(S\mathfrak{g}^*\otimes\mathcal{A})_{\mathrm{inv}}$ is induced from the term $-v^a\,\iota_{e_a}^{\mathcal{A}}+\mathrm{d}^A$, which commutes with the projection. □

If we identify $S\mathfrak{g}^*$ with polynomials on $\mathfrak{g}$, the algebra $(S\mathfrak{g}^*\otimes\mathcal{A})_{\mathrm{inv}}$ becomes the algebra of $\mathfrak{g}$-equivariant polynomial maps $\alpha:\mathfrak{g}\to\mathcal{A}$. In terms of these identifications, the differential reads

$$(\mathrm{d}_{\mathfrak{g}}\alpha)(\xi)=\mathrm{d}(\alpha(\xi))-\iota_\xi\alpha(\xi).$$

In the case $\mathcal{A}=\Omega(M)$, one often calls

$$\Omega_G(M):=(S\mathfrak{g}^*\otimes\Omega(M))^G$$

the complex G-equivariant differential forms, and $\mathrm{d}_G=\mathrm{d}-\iota(\xi_M)$ the equivariant differential. Recall that the generators of the symmetric algebra have degree 2, hence the grading on $\Omega_G(M)$ is given by

$$\Omega_G^k(M)=\bigoplus_{2i+j=k}(S^i\mathfrak{g}^*\otimes\Omega^j(M))^G.$$

Let us verify that after all these computations, the differential still squares to 0:

$$\begin{aligned}\mathrm{d}_G^2\alpha(\xi)&=\mathrm{d}(\mathrm{d}_G\alpha(\xi))-\iota_\xi\mathrm{d}_G\alpha)(\xi)\\&=\mathrm{d}\mathrm{d}\alpha(\xi)-\mathrm{d}\iota_\xi\alpha(\xi)-\iota_\xi\mathrm{d}\alpha(\xi)-\iota_\xi\iota_\xi\alpha(\xi)\\&=-L_\xi\alpha(\xi)\\&=-\alpha([\xi,\xi])=0.\end{aligned}$$

It is also interesting to re-examine the proof that $H_{\mathfrak{g}}(\mathcal{A})=H_{\mathrm{basic}}(\mathcal{A})$ for any commutative locally free $\mathfrak{g}$-dga. Suppose $\theta=\theta^a e_a\in\mathcal{A}^1\otimes\mathfrak{g}$ is a connection on $\mathcal{A}$. *In the following Lemma and its proof we do not use the summation convention.*

LEMMA 5.16. *The operator $P_{\mathrm{hor}}^\theta:=\prod_a\iota_{e_a}\theta^a=\prod_a(1-\theta^a\iota_{e_a})$ is a projection operator onto the space $\mathcal{A}_{\mathrm{hor}}$ of horizontal elements.*

PROOF. Note that the operators $\iota_{e_a}\theta^a$ ($a=1,\ldots,\dim\mathfrak{g}$) are a family of pairwise commuting projection operators. Hence their product is again a projection operator. For any $\alpha\in\mathcal{A}$ we have

$$\iota_r P_{\mathrm{hor}}^\theta\alpha=\iota_r\prod_a\iota_{e_a}\theta^a=0,$$

since ι_r commutes with terms $a\neq r$ and its product with the terms $a=r$ is zero. If α is horizontal, we have

$$P_{\mathrm{hor}}^\theta\alpha=\prod_a(1-\theta^a\iota_{e_a})\alpha=\alpha,$$

showing that P_{hor} is projection onto $\mathcal{A}_{\mathrm{hor}}$. □

Note that the horizontal projection operator may also be written,

$$P_{\mathrm{hor}}=\exp(-y^a\iota_{e_a})|_{y^a=\theta^a}$$

where y^a are degree 1 variable corresponding to e^a, and the notation indicates that we first apply the operator, and then set $y^a=\theta^a$ in the resulting expression. Let $F^\theta:\mathfrak{g}^*\to\mathcal{A}^2$ be the curvature of θ. Recall that for $p \in S\mathfrak{g}^*$, we defined $p(F^\theta)$ as the image of p under the algebra homomorphism $S\mathfrak{g}^* \to \mathcal{A}^{\text{even}}$ defined by F^θ. ("Plugging in the curvature for the variable ξ".) Similarly, for $\alpha = \sum_I p_I \otimes \alpha_I \in S\mathfrak{g}^* \otimes \mathcal{A}$ we define $\alpha(F^\theta) = \sum_I p_I(F^\theta)\alpha_I$.

THEOREM 5.17 (Cartan). *Let $\mathcal{A}$ be a locally free $\mathfrak{g}$-dga, and θ a connection. Define a map*

$$\operatorname{Car}^\theta : (S\mathfrak{g}^* \otimes \mathcal{A})_{\text{inv}} \to \mathcal{A}, \ \alpha \mapsto P_{\text{hor}}(\alpha(F^\theta)).$$

Then $\operatorname{Car}^\theta$ *is an algebra homomorphism taking values in* $\mathcal{A}_{\text{basic}}$. *Furthermore,* $\operatorname{Car}^\theta$ *is a chain map, and induces an isomorphism in cohomology. In fact, the projection $\alpha \mapsto 1 \otimes \operatorname{Car}^\theta(\alpha)$ is a projection, chain homotopic to the identity, by an explicit homotopy operator.*

Note that the Cartan map extends the Chern-Weil homomorphism $(S\mathfrak{g}^*)_{\text{inv}} \to \mathcal{A}_{\text{basic}}$: For an invariant polynomial $p \in (S\mathfrak{g}^*)_{\text{inv}}$, $p(F^\theta)$ is already basic and so the horizontal projection operator can be omitted.

PROOF. We will prove this result by comparing with the Weil model. Identify $(S\mathfrak{g}^* \otimes \mathcal{A})_{\text{inv}}$ with $(W\mathfrak{g} \otimes \mathcal{A})_{\text{basic}}$. We had shown above that the map $W\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A}$ taking $w \otimes \alpha$ to $1 \otimes c^\theta(w)\alpha$ is a $\mathfrak{g}$-chain homotopy inverse to the map $\mathcal{A} \to W\mathfrak{g} \otimes \mathcal{A}$, $\alpha \mapsto 1 \otimes \alpha$. Given $\alpha \in (S\mathfrak{g}^* \otimes \mathcal{A})_{\text{inv}}$, the corresponding element of $(W\mathfrak{g} \otimes \mathcal{A})_{\text{basic}}$ is obtained by applying the $\exp(-e^a \otimes \iota_a^{\mathcal{A}})\alpha$. The element $c^\theta(w)\alpha$ is exactly

$$\exp(-y^a \otimes \iota_a^{\mathcal{A}})|_{y^a = \theta^a} \alpha(F^\theta) = \operatorname{Car}^\theta(\alpha).$$

□

This Theorem (at least the first part) is contained in Cartan's paper "La transgression dans un groupe de Lie", Théorème 4 (p.64). Since Cartan's proof was a little cryptic, the result has been re-proved several times. See e.g. Kumar-Vergne [**21**, p. 171–176], Duistermaat [**11**, p. 227–234], Guillemin-Sternberg, [**17**, p. 53–59], Nicolaescu, [**25**, p. 17–38]. (The argument in Nicolaescu's paper seems more or less identical to the one presented here.)

5.7. Examples of equivariant differential forms. Cartan's model of equivariant cohomology is very popular in differential geometry, particularly symplectic geometry.

Suppose M is a G-manifold. We will study equivariant forms in low degrees. Since $\Omega^1_G(M) = \Omega^1(M)_{\text{inv}}$, an equivariant 1-form on M is simply an invariant 1-form. Since

$$\Omega^2_G(M) = \Omega^2(M)_{\text{inv}} \oplus (\mathfrak{g}^* \otimes \Omega^0(M))_{\text{inv}}$$

an equivariant 2-form is a sum $\omega + \Psi$, where ω is an invariant 2-form and $\Psi : M \to \mathfrak{g}^*$ an equivariant function, viewed as an element of $(\mathfrak{g}^* \otimes \Omega^0(M))_{\text{inv}}$. The equivariant 2-form is equivariantly closed if and only if

$$0 = \mathrm{d}_G(\omega + \Psi)(\xi) = \mathrm{d}\omega - \iota_{\xi_M}\omega + \mathrm{d}\langle \Psi, \xi \rangle,$$

which gives two equations $\mathrm{d}\omega = 0$ and $\iota_{\xi_M}\omega = \mathrm{d}\langle \Psi, \xi \rangle$. Thus ω should be closed and invariant, while the second condition is the familiar moment map condition from symplectic geometry, with $-\Psi$ as the moment map! (The minus sign is a matter of convention.) Indeed, if ω is not only closed but also non-degenerate, the map Ψ determines the generating vector fields by this equation. The equivariant 2-form is exact if there exists an invariant 1-form ν with $\omega = \mathrm{d}\nu$ and $\Psi = -\iota(\xi_M)\nu$.

An equivariant 3-form is an element of

$$\Omega^3_G(M) = \Omega^3(M)_{\mathrm{inv}} \oplus (\mathfrak{g}^* \otimes \Omega^1(M))_{\mathrm{inv}}$$

It therefore has the form $\eta + \beta(\xi)$, where η is an invariant 3-form and β is an equivariant map from $\mathfrak{g}$ to $\Omega^1(M)$. The equivariant 3-form is closed if and only if

$$\mathrm{d}\eta = 0,\ \iota(\xi_M)\eta = \mathrm{d}\beta(\xi),\quad \iota(\xi_M)\beta = 0.$$

For example, if $M = G$ where G acts by conjugation, and B is an invariant inner product on $\mathfrak{g}$ (possibly indefinite), one may check that

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]),\quad \beta(\xi) = \tfrac{1}{2} B(\theta^L + \theta^R, \xi)$$

defines an equivariant 3-cocycle.

5.8. Equivariant formality. In general, the map $H_G(M) \to H(M)$ induced by the chain map $\Omega_G(M) \to \Omega(M)^G$, $\alpha(\xi) \mapsto \alpha(0)$ need not be surjective: Not every cohomology class admits an equivariant extension. One defines,

Definition 5.18. A compact G-manifold M is called equivariantly formal if the map $H_G(M) \to H(M)$ is onto.

There are many equivalent conditions for a G-manifold to be equivariantly formal. For example, M is equivariantly formal if and only if $H_G(M) = (S\mathfrak{g}^*)^G \otimes H(M)$ as graded vector spaces, or equivalently if and only if $H_G(M)$ is a free $S\mathfrak{g}^*$-module.

Examples 5.19. (a) If (M, ω) is a compact symplectic manifold, with a symplectic G-action admitting a moment map, then M is equivariantly formal. (This result is due (independently) to Ginzburg and Kirwan, and is proved using Morse theory.
(b) The conjugation action of a compact Lie group on itself is equivariantly formal. (There are explicit generators of the cohomology, and explicit equivariant extensions. See e.g. Jeffrey's paper "group cohomology construction ...")
(c) The left-action of a compact Lie group G on itself, and more generally free G-actions are *not* equivariantly formal.
(d) Suppose G is compact and simply connected, and that M is connected. Then one may show that the map $H^i_G(M) \to H^i(M)$ is an isomorphism in degree $i \leq 2$, while in degree $i = 3$ there is a short exact sequence,

$$0 \to H^3_G(M) \to H^3(M) \to H^3(G.x) \to 0$$

for any $x \in M$. (The cohomology groups $H^3(G.x)$ are all isomorphic.) Hence, if the action is equivariantly formal one must have $H^3(G.x) = 0$.

(No time for proofs..)

5.9. The Künneth formula. (No time ...See Kumar-Vergne [**21**].)

6. Equivalence between the simplicial and Weil model

6.1. A non-commutative version of the Weil algebra. As explained above, the Weil algebra $W\mathfrak{g}$ is the universal commutative locally free $\mathfrak{g}$-dga. It seems natural to ask if there exists a similar universal object if one drops commutativity. This is indeed the case: Simply omit "commutative" from the definition of $W\mathfrak{g}$.

Thus, we let $\tilde{W}\mathfrak{g}$ be the $\mathfrak{g}$-dga which is freely generated by degree 1 elements $\mu \in \mathfrak{g}^*$ and degree 2 elements $\overline{\mu}$. Thus, while $W\mathfrak{g}$ was the symmetric algebra over the graded vector space $E_{\mathfrak{g}^*}$ with $E^1_{\mathfrak{g}^*} = \mathfrak{g}^*$, $E^2_{\mathfrak{g}^*} = \mathfrak{g}^*$, $E^i_{\mathfrak{g}^*} = 0$ otherwise, $\tilde{W}\mathfrak{g}$ is the *tensor algebra*:

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}), \quad \tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}).$$

The contractions, Lie derivatives and differential are defined on degree 1 generators, by the exact same formulas as for $W\mathfrak{g}$:

$$\iota_\xi\mu = \langle \mu, \xi \rangle, \quad L_\xi\mu = -\operatorname{ad}^*_\xi \mu, \quad \mathrm{d}\mu = \overline{\mu},$$

and again the formulas on degree 2 generators are determined by the relations. It is immediate that these formulas extend to derivations of $\tilde{W}\mathfrak{g}$, and that $\mathfrak{g}^* \to \tilde{W}\mathfrak{g}$, $\mu \mapsto \mu$ is a connection.

Some essential properties of $W\mathfrak{g}$ carry over to $\tilde{W}\mathfrak{g}$:

THEOREM 6.1 (Acyclicity of $\tilde{W}\mathfrak{g}$). *The inclusion $\mathbb{R} \to \tilde{W}\mathfrak{g}$ as multiples of the identity induces an isomorphism in cohomology. In fact, there is a canonically defined homotopy operator $h : \tilde{W}^\bullet\mathfrak{g} \to \tilde{W}^{\bullet-1}\mathfrak{g}$ with the property $[h, d] = \mathrm{id} - \Pi$ where Π is projection onto the degree 0 part.*

THEOREM 6.2 (Locally free $\mathfrak{g}$-dga's). *Suppose $\mathcal{A}$ is a locally free $\mathfrak{g}$-dga.*

(a) *For any connection θ on $\mathcal{A}$, there exists a unique homomorphism of $\mathfrak{g}$-dga's $c^\theta : \tilde{W}\mathfrak{g} \to \mathcal{A}$ (called the characteristic homomorphism) such that the diagram*

$$\begin{tikzcd} \tilde{W}\mathfrak{g} \arrow[r, "c^\theta"'] & \mathcal{A} \\ \mathfrak{g}^* \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutes.

(b) *If θ_0, θ_1 are two connections on $\mathcal{A}$, there is a canonically defined $\mathfrak{g}$-homotopy operator $h : \tilde{W}^i\mathfrak{g} \to \mathcal{A}^{i-1}$, i.e. a linear map such that $[h, \iota_\xi] = 0$, $[h, L_\xi] = 0$ and*

$$[h, d] = c^{\theta_1} - c^{\theta_0}.$$

(c) *The inclusion map $\mathcal{A} \to \tilde{W}\mathfrak{g} \otimes \mathcal{A}$, $\alpha \mapsto 1 \otimes \alpha$ is a $\mathfrak{g}$-homotopy equivalence. For any connection θ, a $\mathfrak{g}$-homotopy inverse is given by the map*

$$\tilde{W}\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A}, \quad w \otimes \alpha \mapsto (c^\theta w)\alpha.$$

In particular the connection on $W\mathfrak{g}$ defines a characteristic map

$$\tilde{W}\mathfrak{g} \to W\mathfrak{g},$$

which is simply the quotient map from the tensor algebra to the symmetric algebra.

The proofs of the two theorems are similar to the commutative setting.

6.2. Uniqueness property of Weil algebras. There does not seem to be a simple description of the horizontal or basic subalgebras of $\tilde{W}\mathfrak{g}$. We will however prove the following:

THEOREM 6.3. *Suppose* $W = \bigoplus_{i \geq 0} W^i$ *is a* $\mathfrak{g}$*-dga with connection, such that there exists a linear operator* $h : W \to W$ *of degree* -1 *with* $[\mathfrak{h}, d] = \mathrm{id} - \Pi$ *where* Π *is a projection onto* $\mathbb{R} \subset W$*). Assume* $[h, L_\xi] = 0$*. Then the characteristic map*

$$\tilde{W}\mathfrak{g} \to W$$

induces an isomorphism in basic cohomology. More generally, for any $\mathfrak{g}$*-dga* $\mathcal{B}$ *the map*

$$\tilde{W}\mathfrak{g} \otimes \mathcal{B} \to W \otimes \mathcal{B}$$

induces an isomorphism in basic cohomology. If W *is commutative, similar statements hold for* $W\mathfrak{g}$*.*

REMARKS 6.4. (a) The theorem is analogous to the result that if $EG \to BG$ is a classifying bundle, and $E' \to B'$ is another principal G-bundle with contractible total space, then $E' \to B'$ is also a classifying bundle and the classifying map $E' \to EG$ is a G-equivariant homotopy equivalence. Unfortunately, the analogy is not perfect: We would prefer a stronger statement that $\tilde{W}\mathfrak{g} \to W$ (resp. $W\mathfrak{g} \to W$ if W is commutative) is a $\mathfrak{g}$-homotopy equivalence.

(b) An immediate consequence of this theorem is that $H_{\mathrm{basic}}(\tilde{W}\mathfrak{g}) = H_{\mathrm{basic}}(W\mathfrak{g}) = (S\mathfrak{g}^*)_{\mathrm{inv}}$. It shows furthermore that in the definition of equivariant cohomology, the Weil algebra $W\mathfrak{g}$ may be replaced by any other locally free $\mathfrak{g}$-dga with trivial cohomology.

Theorem 6.3 will easily follow from the following Lemma:

LEMMA 6.5. *Let* W *be as in Theorem 6.3, any* $\mathcal{A}$ *any locally free* $\mathfrak{g}$*-dga. Then the inclusion* $\mathcal{A} \hookrightarrow W \otimes A$, $\alpha \mapsto 1 \otimes \alpha$ *induces an isomorphism in basic cohomology.*

PROOF OF THEOREM 6.3. The characteristic map $c^\theta : \tilde{W}\mathfrak{g} \to W$ can be written as a composition of two maps,

$$\tilde{W}\mathfrak{g} \to \tilde{W}\mathfrak{g} \otimes W \to W,$$

where the first map is given by $z \mapsto z \otimes 1$ and the second map is $z \otimes w \mapsto c^\theta(z)w$. The second map is a $\mathfrak{g}$-chain homotopy equivalence by part (b) of Theorem 6.2. Hence the map obtained by tensoring with the identity map for any $\mathfrak{g}$-dga $\mathcal{B}$ is a $\mathfrak{g}$-chain homotopy equivalence as well, and in particular induces an isomorphism in basic cohomology. Lemma 6.5 applied to $\mathcal{A} = \tilde{W}\mathfrak{g} \otimes \mathcal{B}$ shows that the first induces an isomorphism in basic cohomology as well. □

We now turn to the proof of Lemma 6.5. Note that this is slightly weaker than the corresponding statement for $\tilde{W}\mathfrak{g}$.

PROOF. The proof is modeled after Guillemin-Sternberg, [**17**][page 46] The idea is to apply $h \otimes 1$ to $W \otimes \mathcal{A}$ to show that the W factor does not change the basic cohomology. This does not directly work, however, since h need not commute with the contraction operators ι^W_ξ. To get around this difficulty we use the Kalkman trick: Let θ be a connection on $\mathcal{A}$, and let ψ be the nilpotent degree 0 operator $\psi = \theta^a \iota^W_a$. (Note that this need not be a derivation, since θ^a are elements of a non-commutative algebra.) Then $\exp\psi$ is $\mathfrak{g}$-equivariant and intertwines $\iota^W_\xi + \iota^{\mathcal{A}}_\xi$ with $\iota^{\mathcal{A}}_a$. Thus, after applying $\exp\psi$ the operator h no longer interferes with contractions on W.

Unfortunately, h is no longer a homotopy operator since ψ changes the differential! Fortunately, the change of $\mathrm{d}^W + \mathrm{d}^{\mathcal{A}}$ can be controlled. Introduce a filtration

$$W_{-1} \subset W_0 \subset W_1 \subset \cdots$$

on W by setting $W_{-1} = \mathbb{R}$ and

$$W_i = \bigoplus_{j \leq i} W^j$$

for $i \geq 0$. Then ι_ξ^W lowers the filtration degree by 1, L_ξ^W preserves it, d^W raises it by 1, and h lowers it by -1. Let $W \otimes \mathcal{A}$ be equipped with the filtration induced from the filtration on W (the grading on $\mathcal{A}$ plays no role). It is easily checked that the twisted differential

$$D := \mathrm{Ad}(\exp \psi)(\mathrm{d}^W + \mathrm{d}^{\mathcal{A}})$$

has the form $D = \mathrm{d}^W + \cdots$ where the dots indicate additional terms that *preserve* the filtration degree. Suppose now that $\alpha \in (W \otimes \mathcal{A}_{\mathrm{hor}})_{\mathrm{inv}}$ is D-closed and has filtration degree $N \geq 0$. Then $\alpha - D(h \otimes 1)\alpha$ is cohomologous to α, lies in $(W \otimes \mathcal{A}_{\mathrm{hor}})_{\mathrm{inv}}$, and has filtration degree $N - 1$. Indeed, the equation $[\mathrm{d}, h] = \mathrm{id} - \Pi$ implies

$$[D, h \otimes 1] = \mathrm{id} - \Pi \otimes 1 + \cdots ,$$

where the dots lower the filtration degree by at least 1. Iterating, it follows that

$$(\mathrm{id} - D(h \otimes 1))^{N+1} \alpha$$

is cohomologous to α and has filtration degree -1, thus lies in $\mathcal{A}_{\mathrm{basic}}$. This shows that the map $H_{\mathrm{basic}}(W \otimes \mathcal{A}) \to H_{\mathrm{basic}}(\mathcal{A})$ is onto. By a similar argument, given $\alpha \in \mathcal{A}_{\mathrm{basic}}$ with $\alpha = D\beta$ for some $\beta \in (W \otimes \mathcal{A})_{\mathrm{basic}}$, we may add a cocycle to β to obtain an element of $(W_0 \otimes \mathcal{A})_{\mathrm{basic}}$. Hence the map is also injective. $\square$

6.3. Equivalence of simplicial and Weil model of equivariant cohomology. Another example comes from the simplicial model for the classifying bundle: Recall that we defined a non-commutative product structure on the double complex

$$C^{p,q} = \Omega^q(G^{p+1}).$$

The principal G-action on $E_\bullet G = G^{\bullet+1}$ is given by the diagonal action from the right, and this defines Lie derivatives and contractions

$$\iota_\xi : C^{p,q} \to C^{p,q-1}, \quad L_\xi : C^{p,q} \to C^{p,q}.$$

Letting $W^i = \bigoplus_{p+q=i} C^{p,q}$, this defines a $\mathfrak{g}$-dga. A connection 1-form is given by the left-invariant Maurer-Cartan form $\theta^L \in \Omega^1(G \otimes \mathfrak{g}$, viewed as an element

$$\theta^L \in C^{0,1} \otimes \mathfrak{g}.$$

Finally, the D-cohomology is acyclic, since it is isomorphic to the cohomology of EG. (One may also directly construct a homotopy operator.) The basic subcomplex of $\Omega^\bullet(E_\bullet G)$ is isomorphic to $\Omega^\bullet(B_\bullet G)$. It follows that the characteristic maps

$$W\mathfrak{g} \leftarrow \tilde{W}^\bullet \mathfrak{g} \to C^\bullet(G^{\bullet+1})$$

induce isomorphisms in basic cohomology, and more generally, if M is a G-manifold,

$$\begin{tikzcd} & H_{\mathrm{basic}}(\tilde{W}\mathfrak{g} \otimes \Omega(M)) \arrow[dl] \arrow[dr] & \\ H_{\mathrm{basic}}(W\mathfrak{g} \otimes \Omega(M)) & & H^*(EG \times_G M) \end{tikzcd} .$$

This proves that the Weil/Cartan model does indeed compute the equivariant cohomology of M, defined in terms of the Borel model.

We should point out that there is, in fact, a *canonical* homomorphism of $\mathfrak{g}$-differential spaces (not of algebras) $W\mathfrak{g} \to C^\bullet(G^{\bullet+1})$. Indeed, we have

THEOREM 6.6. *The symmetrization map $W\mathfrak{g} = S(E_{\mathfrak{g}^*}) \to \tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*})$ is a homomorphism of $\mathfrak{g}$-differential spaces, i.e. it intertwines d, ι_ξ, L_ξ. For any $\mathfrak{g}$-dga $\mathcal{A}$, the induced map in basic cohomology*

$$H_{\mathrm{basic}}(W\mathfrak{g} \otimes \mathcal{A}) \to H_{\mathrm{basic}}(\tilde{W}\mathfrak{g} \otimes \mathcal{A})$$

is inverse to the algebra isomorphism $H_{\mathrm{basic}}(\tilde{W}\mathfrak{g}\otimes\mathcal{A}) \to H_{\mathrm{basic}}(W\mathfrak{g}\otimes\mathcal{A})$ induced by $\tilde{W}\mathfrak{g} \to W\mathfrak{g}$.

PROOF. Since $W\mathfrak{g} \to \tilde{W}\mathfrak{g} \to W\mathfrak{g}$ (symmetrization followed by the quotient map) is the identity, and we already know that $H_{\mathrm{basic}}(\tilde{W}\mathfrak{g} \otimes \mathcal{A}) \to H_{\mathrm{basic}}(W\mathfrak{g} \otimes \mathcal{A})$ is an isomorphism, it's enough to show that $W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ intertwines $\iota_\xi, L_\xi, \mathrm{d}$. This in fact follows from a more general statement: *If E is any graded vector space, and $A\,\mathrm{End}(E)$ any endomorphism, then A extends to derivations of both the symmetric algebra $S(E)$ and the tensor algebra $T(E)$, and the symmetrization map intertwines the two derivations.* This directly applies to d, L_ξ since these are both induced from endomorphisms of $E_{\mathfrak{g}^*}$. For ι_ξ, one may apply a small trick and replace $E_{\mathfrak{g}^*}$ with $E_{\mathfrak{g}^*} \oplus \mathbb{R}$, and consider the contraction operators defined by

$$\tilde{\iota}_\xi(\mu) = \langle \mu, \xi \rangle\, c,\ \ \tilde{\iota}_\xi(\overline{\mu}) = -\operatorname{ad}^*_\xi \mu$$

where c is a generator of $\mathbb{R}$. The statement above applies to $A = \tilde{\iota}_\xi$, and taking the quotient by the ideal generated by $c - 1$ we find that $W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ intertwines the contractions as well. $\square$

Thus, in particular we have a canonical map

$$(S\mathfrak{g}^*)_{\mathrm{inv}} \subset (W\mathfrak{g})_{\mathrm{basic}} \to (\tilde{W}\mathfrak{g})_{\mathrm{basic}} \to \Omega^\bullet(G^{\bullet+1})_{\mathrm{basic}} = \Omega^\bullet(G^\bullet).$$

For example, if one is given an inner product on $\mathfrak{g}$, the corresponding quadratic polynomial $\xi \mapsto ||\xi||^2$ gives rise to an element of $\Omega^\bullet(G^\bullet)$ of total degree 4. One may check (with some effort) that this is the element of $\Omega^3(G) \oplus \Omega^2(G^2)$ described earlier.

7. Localization

Let M be a compact oriented manifold. The integration map $\int : \Omega(M) \to \mathbb{R}$ extends to a map from the Cartan complex,

$$\int : \Omega_G(M) \to \Omega_G(\mathrm{pt}) = (S\mathfrak{g}^*)^G.$$

By Stokes' theorem, the integral vanishes on equivariant coboundaries, since $(\mathrm{d}_G\beta)(\xi)$ equals $\mathrm{d}(\beta(\xi))$ up to terms of lower differential form degree. Hence it induces an integration map in cohomology,

$$\int : H_G(M) \to H_G(\mathrm{pt}) = (S\mathfrak{g}^*)^G.$$

The localization formula of Berline-Vergne [**4, 3, 2**] and Atiyah-Bott [**1**] gives an explicit expression for the integral of any equivariant cocycle in terms of fixed point data, provided G is a compact Lie group. The formula generalizes the Duistermaat-Heckman formula [**12**] from symplectic geometry, and also Bott's formulas for characteristic numbers [**5**]. Berline-Vergne's proof used differential-geometric ideas, while Atiyah-Bott's proof was more topological in nature. The proof given below is essentially Berline-Vergne's proof, except that we use "real blow-ups" to replace their limiting arguments.

7.1. Statement of the localization formula. In fact, as pointed out in Berline-Vergne's paper [**4**], the localization formula holds in a wider context than that of equivariant cohomology. Indeed, for *fixed* $\xi \in \mathfrak{g}$ consider any differential form $\alpha \in \Omega(M)$ such that α is annihilated by the derivation $\mathrm{d}_\xi = \mathrm{d} - \iota_{\xi_M}$ on $\Omega(M)$. For example, if $\beta \in \Omega_G(M)$ is an equivariant cocycle, then $\alpha := \beta(\xi)$ is annihilated by d_ξ.

EXAMPLE 7.1. Let $\omega + \Phi$ be a closed equivariant 2-form on M. Then $\alpha := \exp(\omega + \langle \Phi, \xi\rangle)$ is d_ξ-closed. It does not strictly speaking define an equivariant differential form, however, it is not polynomial in ξ. This is the setting for the Duistermaat-Heckman theorem.

The fixed point formula expresses the integral $\int_M \alpha$ as a sum over the zeroes of the vector field ξ_M. Note that the zeroes of ξ_M are also the fixed point sets for the action of the torus generated by ξ (i.e. of $\overline{\{\exp(t\xi)|,\, t \in \mathbb{R}\}}$), and in particular are smooth embedded submanifolds. To simplify the discussion, we will first assume that the set of zeroes is isolated, i.e. 0-dimensional.

For any zero $x \in \xi_M^{-1}(0)$, let $A_x(\xi) : T_xM \to T_xM$ denote the infinitesimal action of ξ. That is,

$$A_x(\xi)(v) := \frac{d}{dt}|_{t=0} \exp(t\xi)_* v.$$

Choose a G-invariant Riemannian metric on M, then A_x is skew-adjoint for such a Riemannian metric. Using the orientation on M, we may therefore define the Pfaffian, [14]

$$\det{}^{1/2}(A_x(\xi)).$$

THEOREM 7.2. *Let G be a compact Lie group, and M a compact, oriented G-manifold. Suppose that the vector field generated by $\xi \in \mathfrak{g}$ has isolated zeroes. Then for all forms $\alpha \in \Omega^*(M)$ such that $d_\xi\alpha = 0$, one has the integration formula*

$$\int_M \alpha_{[\dim M]} = \sum_{\xi_M(x)=0} \frac{\alpha_{[0]}(x)}{\det^{1/2}(A_x(\xi))}.$$

[14]Recall that for any vector space V with a given inner product (possibly indefinite), there is a canonical isomorphism $\mathrm{o}(V) \cong \wedge^2(V)$, $A \mapsto \lambda(A)$ between skew-symmetric matrices and the second exterior power of V. Suppose $\dim V$ is even. The Pfaffian $\det^{1/2} : \mathrm{o}(V) \to \mathbb{R}$ is a distinguished choice of square root, where for A invertible, the sign is characterized by the condition that $\frac{\lambda(A)^n}{\det^{1/2}(A)}$ is a volume form compatible with the orientation on V. The Pfaffian changes sign with any change of orientation.

Here $\alpha_{[0]} \in C^\infty(M)$ is the form degree 0 *part of* α.

7.2. Proof of the Localization formula. In the proof we will use the notion of *real blow-ups.* (We learned about this from lecture notes of Richard Melrose, see e.g. [**22**]. The concept is also briefly discussed in Duistermaat-Kolk [**13**], page 125.) Consider first the case of a real vector space V. Let

$$S(V) = V\backslash\{0\}/\mathbb{R}_{>0}$$

be its sphere, thought of as the space of rays based at 0. Define $\hat{V}$ as the subset of $V \times S(V)$,

$$\hat{V} := \{(v, x) \in V \times S(V) |\, v \text{ lies on the ray parametrized by } x\}.$$

Then $\hat{V}$ is a manifold with boundary. (In fact, if one introduces an inner product on V then $\hat{V} = S(V) \times \mathbb{R}_{\geq 0}$). There is a natural smooth map $\pi : \hat{V} \to V$ which is a diffeomorphism away from $S(V)$. If M is a manifold and $m \in M$, one can define its blow-up $\pi : \hat{M} \to M$ by using a coordinate chart based at m. Just as in the complex category, one shows that this is independent of the choice of chart (although this is actually not important for our purposes).

Suppose now that M is a G-space as above. Let $\pi : \hat{M} \to M$ be the manifold with boundary obtained by real blow-up at all the zeroes of ξ_M. It follows from the construction that the vector field ξ_M on M lifts to a vector field $\xi_{\hat{M}}$ on $\hat{M}$ *with no zeroes.* Choose a $\xi_{\hat{M}}$-invariant Riemannian metric g on $\hat{M}$, and define

$$\theta := \frac{g(\xi_{\hat{M}}, \cdot)}{g(\xi_{\hat{M}}, \xi_{\hat{M}})} \in \Omega^1(\hat{M}).$$

Then θ satisfies $\iota(\xi_{\hat{M}})\theta = 1$ and $\mathrm{d}_\xi^2\theta = L_{\xi_M}\theta = 0$. Therefore

$$\gamma := \frac{\theta}{\mathrm{d}_\xi\theta} = \frac{\theta}{\mathrm{d}\theta - 1} = -\theta \wedge \sum_j (\mathrm{d}\theta)^j$$

is a well-defined form satisfying $\mathrm{d}_\xi\gamma = 1$. The key idea of Berline-Vergne is to use this form for partial integration:

$$\begin{aligned}
\int_M \alpha &= \int_{\hat{M}} \pi^*\alpha \\
&= \int_{\hat{M}} \pi^*\alpha \wedge \mathrm{d}_\xi\gamma \\
&= \int_{\hat{M}} \mathrm{d}_\xi(\pi^*\alpha \wedge \gamma) \\
&= \int_{\hat{M}} \mathrm{d}(\pi^*\alpha \wedge \gamma) \\
&= \sum_{p \in M^T} \int_{S(T_pM)} \pi^*\alpha \wedge \gamma \\
&= \sum_{p \in M^T} \alpha_{[0]}(p) \int_{S(T_pM)} \gamma
\end{aligned}$$

Thus, to complete the proof we have to carry out the remaining integral over the sphere. We will do this by a trick, defining a d_ξ-closed form α where we can actually compute the integral by hand.

For any zero $x \in M$, choose a decomposition $T_xM = \bigoplus V_i$, where each V_i is a 2-dimensional subspace invariant under $A_x(\xi)$. Choose orientations on V_i such that the product orientation is the given orientation on T_xM. Then the Pfaffian of $A_x(\xi)$ is the product of the Pfaffians for the restrictions to V_i. On each V_i, introduce polar coordinates r_i, ϕ_i compatible with the orientation. Given $\epsilon > 0$ let $\chi \in C^\infty(\mathbb{R}_{\geq 0})$ be a cut-off function, with $\chi(r) = 1$ for $r \leq \epsilon$ and $\sigma = 0$ for $\epsilon \geq 2$. Let

$$\alpha = (2\pi)^{-n} \prod_{j=1}^{n} \mathrm{d}_\xi(\chi(r_j)\mathrm{d}\phi_j) = (2\pi)^{-n} \prod_{j=1}^{n} \big(\chi(r_j)\iota_\xi \mathrm{d}\phi_j - \chi'(r_j)\mathrm{d}r_j \wedge \mathrm{d}\phi_j\big).$$

Note that this form is well-defined (even though the coordinates are not globally well-defined), compactly supported and d_ξ-closed. Its integral is equal to

$$\int_{T_pM} \alpha = \prod_{j=1}^{n} (-\chi'(r_j)\mathrm{d}r_j) = 1.$$

On the other hand

$$\alpha_{[0]}(0) = (2\pi)^{-n} \prod_{j=1}^{n} \big(\iota_\xi \mathrm{d}\phi_j\big).$$

which (as one easily verifies) is just the Pfaffian of $A_x(\xi)$. Choosing ϵ sufficiently small, we can consider α as a form on M, vanishing at all the other fixed points. Applying the localization formula we find

$$1 = \int_M \alpha = \det{}^{1/2}(A_x(\xi)) \int_{S(T_pM)} \gamma,$$

thus

$$\int_{S(T_pM)} \gamma = \det{}^{-1/2}(A_x(\xi)).$$

Q.E.D.

The above discussion extends to non-isolated fixed points, in this case the Pfaffian $\det^{1/2}(A_x(\xi))$ is replaced by the equivariant Euler class of the normal bundle of the fixed point manifold.

7.3. The Duistermaat-Heckman formula. One often applies the Duistermaat-Heckman theorem in order to compute Liouville volumes of symplectic manifolds with Hamiltonian group action. Consider for example a Hamiltonian $S^1 = \mathbb{R}/\mathbb{Z}$-action on a symplectic manifold (M, ω), with isolated fixed points. That is, the action is defined by a Hamiltonian $H \in C^\infty(M)$ with periodic flow, of period 1. Then

$$\int_M e^{tH}\frac{\omega^n}{n!} = \frac{1}{t^n} \sum_{p \in M^{S^1}} \frac{e^{tH(p)}}{\prod_j a_j(p)}.$$

where $a_j(p)$ are the weights for the S^1 actions at the fixed points. Notice by the way that the individual terms on the right hand side are singular for $t = 0$. This implies very subtle relationships between the weight, for example one must have

$$\sum_{p \in M^{S^1}} \frac{H(p)^k}{\prod_j a_j(p)} = 0$$

for all $k < n$. For the volume one reads off,

$$\mathrm{Vol}(M) = \frac{1}{n!} \sum_{p \in M^{S^1}} \frac{H(p)^n}{\prod_j a_j(p)}.$$

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