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On the complete group classification of the reaction–diffusion equation with a delay

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Abstract

The reaction–diffusion delay differential equation

$$u_t(x,t) - u_{xx}(x,t) = g\big(x, u(x,t), u(x,t-\tau)\big)$$

arises in many applications in the sciences. Group analysis is applied in the study of this equation. A new definition of an equivalence Lie group for delay differential equations is given. As for the Lie group theory of differential equations, the determining equations for the equivalence and admitted Lie groups are constructed. The general solutions of the determining equations are obtained. The complete group classification of the reaction–diffusion equation with delay is presented in the manuscript and the invariant solutions of this equation are constructed.

Keywords: Delay differential equation; Lie group; Invariant solutions

1. Introduction

The manuscript is devoted to applications of group analysis to the reaction–diffusion equation with delay.[1] Delay differential equations appear in problems with delaying links where certain information processing is needed, for example, in population dynamics and bioscience problems, in control problems, electrical networks containing lossless transmission lines [1–6].

Group analysis is one of the methods for constructing exact solutions [7]. This method was developed for partial (ordinary as well) differential equations. An admitted Lie group plays the main role in this method. After obtaining an admitted Lie group one can use it for constructing invariant solutions. Recently a definition of an admitted Lie group for functional differential equations was given [8,9].[2]

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[1] The theory and applications of this equation can be found in [6].
[2] There are also other approaches [10,11]. The method developed in [8,9] has some similarities with the approach [12].

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For the sake of simplicity we give an introduction for constructing an admitted Lie group for a delay differential equation with a single independent variable

$$\Phi(t,x) \equiv x'(t) - F(t,x_t) = 0 \quad (t \in J). \tag{1}$$

Here[3] x_t denotes the function $x(t) \in D \subset R$, which is defined in the interval $[t-\tau, t]$ by

$$x_t(s) = x(t+s), \quad s \in [-\tau, 0],$$

D is an open set in R, J is an interval in R, F is a functional. For delay differential equations the functional F has the representation

$$F(t,x_t) \equiv f\big(t, x\big(g_1(t)\big), \ldots, x\big(g_m(t)\big)\big),$$

where $f:[t_0,\beta) \times D^m \to R^n$, and $g_j(t) \leqslant t$ for $t_0 \leqslant t \leqslant \beta$ for each $j = 1, \ldots, m$. The function g_1 is usually chosen to be the identity mapping. Here we consider the case where $m = 2$, $g_1(t) = t$, and $g_2(t) = t - \tau$.

The Cauchy problem for delay differential equations (1) is set as follows. The initial conditions are defined by a function $\psi : [-\tau, 0] \to D$,

$$x(t_0 + s) = \psi(s), \quad s \in [-\tau, 0]. \tag{2}$$

A continuous function $x(t)$, $t \in [t_0 - \tau, t_0 + \beta)$, is called a solution of the Cauchy problem (1), (2) if it is differentiable in the interval (t_0, β), satisfies Eqs. (1) in the interval $[t_0, \beta)$ and conditions (2) in the interval $[t_0 - \tau, t_0]$. The value $x'(t_0)$ is understood as the right-hand derivative. With some requirements[4] for the functional F one can guarantee the existence of the solution of the Cauchy problem (1), (2).

1.1. Admitted Lie group

Let there be given a one-parameter Lie group $G^1(X)$ of transformations

$$\bar{t} = f^t(t,x,a), \qquad \bar{x} = f^x(t,x,a) \tag{3}$$

with the generator

$$X = \xi(t,x)\partial_t + \eta(t,x)\partial_x.$$

Definition. A one-parameter Lie group G^1 of transformations (3) is a symmetry group admitted by Eq. (1) if G^1 satisfies the determining equation

$$(\overline{X}\Phi)\big(t, x(t)\big) = 0 \tag{4}$$

for any solution $x(t)$ of Eqs. (1).

Here the operator $\overline{X}$ is the prolongation of the canonical Lie–Bäcklund operator equivalent to the generator X given by

$$\overline{X} = \zeta^x \partial_x + \zeta^{x'} \partial_{x'} + \cdots,$$

where $\zeta^x = \eta - x'\xi$, $\zeta^{x'} = D_t \zeta^x$ and D_t is the total derivative with respect to t. The actions of the derivatives ∂_x and $\partial_{x'}$ are considered in terms of the Frechet derivatives.

Formally the determining equations (4) can be constructed similarly to those for integro-differential equations [9,13]. Assume that the Lie group $G^1(X)$ transforms a solution $x_0(t)$ of Eq. (1) into the solution $x_a(t)$ of the same equation. The transformed function $x_a(\bar{t})$ is

$$x_a(\bar{t}) = f^x\big(t, x_0(t), a\big)$$

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[3] The notations accepted in literature on functional differential equations are used.

[4] These requirements are similar to the conditions used in ordinary differential equations (see, for example, in [4]).

with the expression $t = \psi^t(\bar{t}; a)$ substituted, which is found from the relation $\bar{t} = f^t(t, x_0(t), a)$. Differentiating the equations $\Phi(t, x_a)$ with respect to the group parameter a and considering these equations for the value $a = 0$, that is,

$$\left(\frac{\partial}{\partial a}\Phi(t, x_a)\right)\Big|_{a=0} = 0, \tag{5}$$

one obtains the determining equations (4). Equation (5) can also be obtained by the action of the operator $\overline{X}$ on the delay differential equation. Thus the determining equations can be constructed without differentiation with respect to the parameter a. To illustrate this we consider the equation

$$\Phi(t, x) \equiv x'(t) - f\big(t, x(t), x(t-\tau)\big) = 0, \tag{6}$$

where $\tau > 0$. The action of the operator X on (6) gives

$$\zeta^{x'} - f_{,2}\zeta^x - f_{,3}\zeta^{\bar{x}} = 0,$$

where $f_{,i}$ $(i = 1, 2, 3)$ is the derivative of the function $f(x_1, x_2, x_3)$ with respect to x_i and

$$\zeta^x = \eta\big(t, x(t)\big) - x'(t)\xi\big(t, x(t)\big), \qquad \zeta^{\bar{x}} = \eta\big(t-\tau, x(t-\tau)\big) - x'(t-\tau)\xi\big(t-\tau, x(t-\tau)\big)$$

with $\zeta^{x'} = (D_t\zeta^x)(t, x(t), x'(t))$. Here $x(t)$ is an arbitrary solution of (6).

The main features of the determining equations in the given definition is that they must be satisfied for any solution of Eq. (1). This allows splitting the determining equations with respect to arbitrary elements. Since arbitrary elements of delay differential equations are similarly contained in the determining equations as for differential equations, the process of solving the determining equations for delay differential equations is similar to finding the solutions of the determining equations for differential equations.

Notice that the given definition is free from the requirement that the admitted Lie group should transform a solution into a solution, and also it can be applied when finding an equivalence group, contact and Lie–Bäcklund transformations for functional differential equations.

Group classification is already defined for partial differential equations. For partial differential equations it is known that each equation has an arbitrary function associated with it. For each function there is an admitted Lie group and equations can be classified according to the admitted Lie group which in turn is used for the construction of invariant solutions. We extend this notion to the treatment of delay differential equations. This is the first time that the complete Lie group classification of delay differential equations is being given. As a result we will include all calculations because there is a difference in the treatment of partial differential equations to that of delay differential equations.

This manuscript is devoted to the group classification of the reaction–diffusion equation with delay. For differential equations there is a theorem that if one finds the admitted Lie group it can be used to construct invariant solutions. However, for delay differential equations no such theorem exists. It appears that this can be applied to delay differential equations and this article confirms it. In this article the found admitted Lie groups are applied for the construction of invariant solutions of the reaction–diffusion equation with a delay.

The manuscript is organized as the following. Section 2 gives the new definition of an equivalence Lie group for functional differential equations. In Section 3 we discuss the reaction–diffusion equation with a delay. In Section 4 the equivalence Lie group for the reaction–diffusion equation with a delay is determined. In Sections 5–7 we determine the admitted Lie group and the various cases arising for group classification. In Section 8, a summary of the group classification is given and in Section 9 we construct the invariant solutions for the various cases arising.

2. Equivalence Lie group

A transformation of the independent and dependent variables, and arbitrary elements is called an equivalence transformation of a system of differential equations if it conserves a differential structure of the equations. If a set of equivalence transformations of partial differential equations composes a Lie group of transformations, then these transformations can be found by solving the determining equations. Conversely, any solution of the determining equations composes a Lie group of equivalence transformations of partial differential equations.

Since the Lie–Bäcklund representation of the determining equations of an equivalence Lie group for partial differential equations is nowhere written down, we present it here. Formally these equations can be obtained by differentiating with respect to the group parameter the transformed system of partial differential equations in which the transformed solution has been substituted.

2.1. Lie–Bäcklund representation of determining equations for the equivalence Lie group

We consider a system of partial differential equations with the independent variable x, dependent variable u, and arbitrary element ϕ, which transfers a system of differential equations of the given class

$$F^k(x,u,p,\phi)=0 \quad (k=1,2,\dots,s) \tag{7}$$

to the system of equations of the same class. Here $(x,u)\in V\subset R^{n+m}$, and $\phi: V\to R^t$.

The problem of finding equivalent transformations consists of the construction a transformation of the space $R^{n+m+t}(x,u,\phi)$ that preserves the equations, while only changing their representative $\phi=\phi(x,u)$. Assume that a one-parameter Lie group of transformations of the space R^{n+m+t} with the group parameter a:

$$x'=f^x(x,u,\phi;a), \qquad u'=f^u(x,u,\phi;a), \qquad \phi'=f^\phi(x,u,\phi;a) \tag{8}$$

satisfies this property. The generator of this Lie group has the form

$$X^e=\xi\partial_x+\eta^u\partial_u+\eta^\phi\partial_\phi,$$

where the coordinates are

$$\xi^i=\xi^i(x,u,\phi), \quad \eta^{u^j}=\eta^{u^j}(x,u,\phi), \quad \eta^{\phi^k}=\eta^{\phi^k}(x,u,\phi) \quad (i=1,\dots,n;\ j=1,\dots,m;\ k=1,\dots,t).$$

The equivalent Lie–Bäcklund form of this generator is

$$\widehat{X}^e=\zeta^u\partial_u+\zeta^\phi\partial_\phi. \tag{9}$$

Here the coordinates are

$$\zeta^{u^j}=\eta^{u^j}-u_x^j\xi, \qquad \zeta^{\phi^k}=\eta^{\phi^k}-\xi D_x^e\phi^k,$$

where $D^e_{x_i}=\partial_{x_i}+u_{x_i}\partial_u+(\phi_u u_{x_i}+\phi_{x_i})\partial_\phi$.

Any solution $u_0(x)$ of system (7) with the functions $\phi(x,u)$ is transformed by (8) into the solution $u=u_a(x')$ of system (7) with the same functions F^k, and another (transformed) function $\phi_a(x,u)$. The function $\phi_a(x,u)$ is defined as follows. Solving the relations

$$x'=f^x\big(x,u,\phi(x,u);a\big), \qquad u'=f^u\big(x,u,\phi(x,u);a\big)$$

for (x,u), one obtains

$$x=g^x(x',u';a), \qquad u=g^u(x',u';a). \tag{10}$$

The transformed function is

$$\phi_a(x',u')=f^\phi\big(x,u,\phi(x,u);a\big),$$

where, instead of (x,u), one has to substitute in their place the expressions (10). Because of the definition of the function $\phi_a(x',u')$, there is the identity with respect to x and u,

$$\big(\phi_a\circ\big(f^x,f^u\big)\big)\big(x,u,\phi(x,u);a\big)=f^\phi\big(x,u,\phi(x,u);a\big).$$

The transformed solution $T_a(u)=u_a(x)$ is obtained by solving the relations

$$x'=f^x\big(x,u_o(x),\phi\big(x,u_o(x)\big);a\big)$$

for x and substituting this solution $x=\psi^x(x';a)$ into

$$u_a(x')=f^u\big(x,u_o(x),\phi\big(x,u_o(x)\big);a\big).$$

As for the function ϕ_a, there is the identity with respect to x

$$\big(u_a\circ f^x\big)\big(x,u_o(x),\phi\big(x,u_o(x)\big);a\big)=f^u\big(x,u_o(x),\phi\big(x,u_o(x)\big);a\big). \tag{11}$$

Formulae for transformations of the partial derivatives $p'_a=f^p(x,u,p,\phi,\dots,a)$ are obtained by differentiating (11) with respect to x'.

Since the transformed function $u_a(x')$ is a solution of system (7) with the transformed arbitrary element $\phi_a(x', u')$, the equations

$$F^k\big(x', u_a(x'), p'_a(x'), \phi_a\big(x', u_a(x')\big)\big) = 0 \quad (k = 1, 2, \ldots, s) \tag{12}$$

are satisfied for an arbitrary x'. Because of a one-to-one correspondence between x and x' one has

$$F^k\big(f^x\big(z(x), a\big), f^u\big(z(x), a\big), f^p\big(z_p(x), a\big), f^\phi\big(z(x)\big)\big) = 0 \quad (k = 1, 2, \ldots, s), \tag{13}$$

where $z(x) = (x, u_o(x), \phi(x, u_o(x)))$, $z_p(x) = (x, u_o(x), \phi(x, u_o(x)), p_o(x), \ldots)$.

Differentiating Eqs. (12) with respect to the group parameter a, and setting $a = 0$, one obtains the determining equations in Lie–Bäcklund form[5]:

$$\widetilde{X}^e F^k(x, u, p, \phi)|_{(S)} = 0 \quad (k = 1, 2, \ldots, s). \tag{14}$$

The prolonged operator for the equivalence Lie group

$$\widetilde{X}^e = \widehat{X}^e + \zeta^{u_x}\partial_{u_x} + \cdots$$

has the following coordinates. The coordinates related to the dependent functions are

$$\zeta^{u_\lambda} = D^e_\lambda \zeta^u, \quad D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi,$$

where λ takes the values x_i $(i = 1, 2, \ldots, n)$. The sign $|(S)$ means that the equations $\widetilde{X}^e F^k(x, u, p, \phi)$ are considered on any solution $u_o(x)$ of Eqs. (7).

The set of transformations, which is generated by one-parameter Lie groups corresponding to the generators X^e, is called an equivalence Lie group. This group is denoted by GS^e.

The determining equations (14) were obtained by using the existence of the solution of (7). After constructing (14) one can use a geometrical approach in which the equivalence group is defined by Eqs. (14) without the requirement of the existence a solution of (7). In this case the sign $|_{(S)}$ means that the equations $\widetilde{X}^e F^k(x, u, p, \phi)$ are considered on the manifold defined by Eqs. (7). The difference between these two approaches consists in defining the sign $|_{(S)}$. Note that the same difference between the geometrical approach and the others lies in the definitions for obtaining an admitted Lie group.

2.2. *Equivalence Lie group of delay differential equations*

For delay differential equations a notion of an equivalence Lie group is not defined. This section is devoted to a reasonable generalization of the definition of an equivalence Lie group for delay differential equations. The idea of the definition of an equivalence Lie group for delay differential equations is similar to the definition of an admitted Lie group.

Firstly determining equations are constructed. These equations are obtained on the basis that a Lie group of transformations of the independent and dependent variables, and arbitrary elements transforms a solution of the original system of equations into the solution of a system of equations which differs from the original system only by arbitrary elements. The arbitrary elements are functions and constants which are not specified in the system of equations. Differentiating with respect to the group parameter and assigning it to zero, one obtains the determining equations.

A solution of the determining equations gives the generator of a Lie group. This Lie group of transformations is called a potential equivalence Lie group. Notice that for partial differential equations by virtue of the inverse function theorem a potential equivalence Lie group simply becomes an equivalence Lie group.

Construction of a potential equivalence Lie group for the reaction–diffusion equation with a delay is given in the next section.

[5] In contrast, differentiating Eqs. (13) with respect to the group parameter a, and setting $a = 0$, one obtains the determining equations in classical form [9].

3. The reaction–diffusion equation with a delay

The manuscript is devoted to the study of group properties of the equation

$$u_t(t,x) = u_{xx}(t,x) + g\big(x, u(t,x), u(t-\tau,x)\big) \quad (t > t_o). \tag{15}$$

The theory of existence of solutions of Eq. (15) can be found in [6]. For example, initial conditions are

$$u(s,x) = \varphi(s,x) \quad (t_o - \tau \leqslant s \leqslant t_o),$$

where the function $\varphi(s,x)$ is an arbitrary function. Due to the arbitrariness of the function $\varphi(s,x)$ one can conclude that the values $u(t_o,x_o)$, $u(t_o-\tau,x_o)$, $u_x(t_o,x_o)$, $u_x(t_o-\tau,x_o)$, $u_{xx}(t_o,x_o)$, $u_{xx}(t_o-\tau,x_o)$ and some other derivatives are arbitrary. This will allow us to split the determining equations.

4. The equivalence Lie group of Eq. (15)

For the simplicity of the study let us introduce the new dependent variable v, which is related with u by the formula

$$v(t,x) = u(t-\tau,x). \tag{16}$$

Thus, Eq. (15) becomes the partial differential equations with two dependent variables

$$S \equiv u_t - (u_{xx} + g) = 0, \tag{17}$$

where the arbitrary element is $g = g(x,u,v)$. The generator of the Lie group will take the form

$$X^e = \xi\partial_x + \eta\partial_t + \zeta\partial_u + \zeta^v\partial_v + \zeta^g\partial_g,$$

where $\xi(t,x,u,v,g)$, $\eta(t,x,u,v,g)$, $\zeta(t,x,u,v,g)$, $\zeta^v(t,x,u,v,g)$, $\zeta^g(t,x,u,v,g)$.

Applying the algorithm described earlier to Eq. (17), one obtains the determining equation

$$\big(\zeta^{u_t} - \zeta^{u_{xx}} - \zeta^g + \xi D_x g + \eta D_t g\big)\big|_{(S)} = 0, \tag{18}$$

where

$$\zeta^{u_t} = D_t(\zeta - \xi u_c - \eta u_t), \qquad \zeta^{u_{xx}} = D_x^2(\zeta - \xi u_x - \eta u_t). \tag{19}$$

The determining equation related with Eq. (16) is

$$\begin{aligned}&\big\{\zeta^v\big(z(t,x)\big) - \zeta\big(z(t-\tau,x)\big) - v_t(t,x)\big(\xi\big(z(t,x)\big) - \xi\big(z(t-\tau,x)\big)\big)\\&\quad - v_x(t,x)\big(\eta\big(z(t,x)\big) - \eta\big(z(t-\tau,x)\big)\big)\big\}\big|_{(S)} = 0,\end{aligned} \tag{20}$$

where

$$z(t,x) = \big(t, x, u(t,x), v(t,x), g\big(x, u(t,x), v(t,x)\big)\big).$$

Substituting the coefficients (16) into (18) and replacing the derivatives

$$\begin{aligned}&u_{tt} = u_t g_u + g_v v_t + u_{xxt}, \qquad u_{xt} = v_x g_v + g_u u_x + u_{xxx} + g_x,\\&u_t = u_{xx} + g, \qquad v_t = v_{xx} + \bar g,\end{aligned}$$

found from (17), the determining equation (18) becomes

$$\begin{aligned}&-v_x^2 u_x\xi_{vv} - v_x^2 u_{xx}\eta_{vv} + v_x^2(-\eta_{vv}g - 2\eta_v g_v + \zeta_{vv}) - 2v_x u_x^2\xi_{uv} - 2v_x u_x u_{xx}\eta_{uv}\\&\quad + 2v_x u_x(-\eta_{uv}g - \eta_u g_v - \eta_v g_u - \xi_{xv} + \zeta_{uv}) - 2v_x u_{xx}(\eta_{xv} + \xi_v) - u_x^3\xi_{uu}\\&\quad + 2v_x(-\eta_{xv}g - \eta_v g_x - \eta_x g_v + \zeta_{xv}) - u_x^2 u_{xx}\eta_{uu} - 2u_x u_{xx}(\eta_{xu} + \xi_u)\\&\quad + u_x^2(-\eta_{uu}g - 2\eta_u g_u - 2\xi_{xu} + \zeta_{uu}) - 2v_x u_{xxx}\eta_v - 2u_x u_{xxx}\eta_u - 2u_{xxx}\eta_x\\&\quad + u_x(\bar g\xi_v - 2\eta_{xu}g - 2\eta_u g_x - 2\eta_x g_u + \xi_t + \xi_u g - \xi_{xx} + 2\zeta_{xu}) + u_{xx}(\bar g\eta_v + \eta_t + \eta_u g - \eta_{xx} - 2\xi_x)\\&\quad + \bar g(\eta_v g - \zeta_v) + \eta_t g + \eta_u g^2 - \eta_{xx}g - 2\eta_x g_x - \zeta_t - \zeta_u g + \zeta_{xx} + \zeta^g = 0.\end{aligned}$$

After splitting this equation with respect to $u_x, u_{xx}, v_x, u_{xxx}$ and $\bar{g}$ one obtains[6]

$$\eta_t g + \eta_u g^2 - \eta_{xx} g - 2\eta_x g_x - \zeta_t - \zeta_u g + \zeta_{xx} + \zeta^g = 0, \tag{21}$$

$$-2\eta_{xu} g - 2\eta_u g_x - 2\eta_x g_u + \xi_t + \xi_u g - \xi_{xx} + 2\zeta_{xu} = 0, \tag{22}$$

$$\eta_t + \eta_u g - \eta_{xx} - 2\xi_x = 0, \tag{23}$$

$$\eta_x = 0, \quad \eta_u = 0, \quad \eta_v = 0, \quad \eta_{xu} + \xi_u = 0, \quad \eta_{xv} + \xi_v = 0, \quad \xi_v = 0, \quad \eta_v g - \zeta_v = 0, \tag{24}$$

$$\xi_{uu} = 0, \quad \eta_{uu} = 0, \quad \eta_{vv} = 0, \quad \eta_{uv} = 0, \quad \xi_{vv} = 0, \quad \xi_{uv} = 0,$$

$$-\eta_{xv} g - \eta_v g_x - \eta_x g_v + \zeta_{xv} = 0, \quad -\eta_{vv} g - 2\eta_v g_v + \zeta_{vv} = 0,$$

$$-\eta_{uv} g - \eta_u g_v - \eta_v g_u - \xi_{xv} + \zeta_{uv} = 0, \quad -\eta_{uu} g - 2\eta_u g_u - 2\xi_{xu} + \zeta_{uu} = 0. \tag{25}$$

From (24) one obtains

$$\eta_x = 0, \quad \eta_u = 0, \quad \eta_v = 0, \quad \xi_u = 0, \quad \xi_v = 0, \quad \zeta_v = 0.$$

Differentiating (23) with respect to x, one gets $\xi_{xx} = 0$. Hence $\xi = \xi_1 x + \xi_0$, where $\xi_0 = \xi_0(t)$ and $\xi_1 = \xi_1(t)$, and then $\xi_1 = \eta_t/2$. The general solution of (25) is $\zeta = \zeta_1 u + \zeta_0$, where $\zeta_1 = \zeta_1(t,x)$, $\zeta_0 = \zeta_0(t,x)$. Solving Eq. (22), one obtains $\zeta_1 = -\xi_0' x/2 - \eta'' x^2/8 + \zeta_{10}$, where $\zeta_{10}(t)$.

For the sake of simplicity we study the case $g_x = 0$. The assumption that the function g does not depend on t and x gives the conditions

$$\zeta_t = 0, \quad \zeta_x = 0, \quad \zeta_t^g = 0, \quad \zeta_x^g = 0.$$

These equations give $\eta = 2k_3 t + k_4$, $\xi_0 = k_7$, $\zeta_{0x} = 0$. From Eq. (21) one finds

$$\zeta^g = \zeta_{0t} - 2k_3 g + g\zeta_{10}.$$

From the equation $\zeta_t^g = 0$ one obtains $\zeta_{0tt} = 0$, or

$$\zeta_0 = k_2 t + k_1.$$

Thus,

$$\xi = 2k_3 t + k_4, \quad \eta = k_3 x + k_6, \quad \zeta = k_1 + k_2 t + k_5 u$$

or

$$X^e = k_1 X_1^e + k_2 X_2^e + k_3 X_3^e + k_4 X_4^e + \zeta_{10} X_5^e + \xi_0 X_6^e + \zeta^v \partial_v,$$

where

$$X_1^e = \partial_u, \quad X_2^e = \partial_g + t\partial_u, \quad X_3^e = -2g\partial_g + 2t\partial_t + x\partial_x,$$

$$X_4^e = \partial_t, \quad X_5^e = g\partial_g + u\partial_u, \quad X_6^e = \partial_x.$$

Equation (20) becomes

$$\zeta^v\big(z(t,x)\big) = k_1 + k_2(t-\tau) + k_5 v(t,x).$$

This gives

$$\zeta^v(t,x,u,v) = k_1 + k_2(t-\tau) + k_5 v.$$

[6] One could also split with respect to g_u, g_v.

5. Admitted Lie group of Eq. (15)

The generator of a Lie group admitted by Eq. (15) is

$$X = \xi\partial_x + \eta\partial_t + \zeta\partial_u,$$

where ξ, η and ζ are functions of x, t and u.

According to the algorithm for constructing a determining equation of an admitted Lie group, one obtains

$$-\zeta^{u_t} + \zeta^{u_{xx}} + g_u\zeta^u + g_{\bar{u}}\zeta^{\bar{u}} = 0, \tag{26}$$

where

$$\zeta^u = \zeta - u_x\xi - u_t\eta, \qquad \zeta^{\bar{u}} = \bar{\zeta} - \bar{u}_x\bar{\xi} - \bar{u}_t\bar{\eta}, \qquad \zeta^{u_x} = D_x\zeta^u, \qquad \zeta^{u_{xx}} = D_x\zeta^{u_x}, \qquad \zeta^{u_t} = D_t\zeta^u.$$

Here the bar over a function $f(x,t)$ means $\bar{f}(x,t) = f(x,t-\tau)$. The determining equation has to be satisfied for any solution $u(x,t)$ of Eq. (15). Since the determining equation is considered on a solution of Eq. (15), the value $\bar{f}$ for a function $f(x,t,u)$ is defined as $\bar{f}(x,t) = f(x,t-\tau,u(x,t-\tau))$.

Substituting into the determining equation (26) the derivatives $u_t, u_{xt}, u_{tt}, \bar{u}_t$ found from Eq. (15) and its prolongations, we obtain

$$\begin{aligned}&\bar{g}g_{\bar{u}}(\eta-\bar{\eta}) - 2\bar{u}_x u_x\eta_u g_{\bar{u}} + \bar{u}_x g_{\bar{u}}(-2\eta_x + \xi - \bar{\xi}) + \bar{u}_{xx}g_{\bar{u}}(\eta-\bar{\eta}) - u_x^3\xi_{uu} - u_x^2 u_{xx}\eta_{uu}\\ &\quad + u_x^2(-\eta_{uu}g - 2\eta_u g_u - 2\xi_{xu} + \zeta_{uu}) - 2u_x u_{xx}(\eta_{xu} + \xi_u) - 2u_x u_{xxx}\eta_u\\ &\quad + u_x(-2\eta_{xu}g - 2\eta_u g_x - 2\eta_x g_u + \xi_t + \xi_u g - \xi_{xx} + 2\zeta_{xu}) + u_{xx}(\eta_t + \eta_u g - \eta_{xx} - 2\xi_x) - 2u_{xxx}\eta_x\\ &\quad + \eta_t g + \eta_u g^2 - \eta_{xx}g - 2\eta_x g_x + g_u\zeta + g_{\bar{u}}\zeta + g_x\xi - \zeta_t - \zeta_u g + \zeta_{xx} = 0,\end{aligned} \tag{27}$$

where $\bar{g} = g(u(x,t-\tau), u(x,t-2\tau))$. After splitting this equation with respect to the derivatives u_x, u_{xx}, u_{xxx}, $\bar{u}_x$, $\bar{u}_{xx}$ and using the property $g_{\bar{u}} \neq 0$, one obtains

$$\eta_t g + \eta_u g^2 - \eta_{xx}g - 2\eta_x g_x + g_u\zeta + g_{\bar{u}}\bar{\zeta} + g_x\xi - \zeta_t - \zeta_u g + \zeta_{xx} \tag{28}$$

$$\quad - 2\eta_{xu}g - 2\eta_u g_x - 2\eta_x g_u + \xi_t + \xi_u g - \xi_{xx} + 2\zeta_{xu} = 0, \tag{29}$$

$$-\eta_{uu}g - 2\eta_u g_u - 2\xi_{xu} + \zeta_{uu} = 0, \tag{30}$$

$$\eta_t + \eta_u g - \eta_{xx} - 2\xi_x = 0, \tag{31}$$

$$\eta_{xu} + \xi_u = 0, \tag{32}$$

$$\xi_{uu} = 0, \qquad \eta_{uu} = 0, \qquad \eta_x = 0, \qquad \eta_u = 0, \tag{33}$$

$$-2\eta_x + \xi - \bar{\xi} = 0, \tag{34}$$

$$\eta_u = 0, \tag{35}$$

$$\eta - \bar{\eta} = 0. \tag{36}$$

Hence,

$$\eta_u = 0, \qquad \eta_x = 0, \qquad \xi_u = 0,$$

and

$$\xi(x,t-\tau) = \xi(x,t), \qquad \eta(t-\tau) = \eta(t).$$

From Eq. (31) one gets $\xi = x\eta_t/2 + \xi_0$, where $\xi_0 = \xi_0(t)$. From Eq. (30) we obtain

$$\zeta = u\zeta_1 + \zeta_0,$$

where $\zeta_1 = \zeta_1(x,t)$ and $\zeta_0 = \zeta_0(x,t)$. Equation (29) becomes

$$\eta_{tt}x + 2\xi_{0t} + 4\zeta_{1x} = 0.$$

Integrating this equation with respect to x gives

$$\zeta_1 = -\eta_{tt}x^2/8 - x\xi_{0t}/2 + \zeta_{10},$$

where $\zeta_{10} = \zeta_{10}(t)$. Thus, from Eqs. (28)–(36), Eq. (28) is the only unsolved equation. This equation becomes

$$\begin{aligned}&8g_u\zeta_0 + 8g_{\bar u}\bar\zeta_0 + g_u u\big(-\eta_{tt}x^2 - 4\xi_{0t}x + 8\zeta_{10}\big) + g_{\bar u}\bar u\big(-\eta_{tt}x^2 - 4\xi_{0t}x + 8\bar\zeta_{10}\big) + 4g_x(\eta_t x + 2\xi_0)\\&\quad + g\big(\eta_{tt}x^2 + 8\eta_t + 4\xi_{0t}x - 8\zeta_{10}\big) + \eta_{ttt}ux^2 - 2\eta_{tt}u + 4\xi_{0tt}ux - 8\zeta_{0t} + 8\zeta_{0xx} - 8\zeta_{10t}u = 0.\end{aligned} \tag{37}$$

Differentiating (37) with respect to u and $\bar u$, one obtains

$$\begin{aligned}&8\zeta_0 g_{uu} + 8\bar\zeta_0 g_{u\bar u} + 8g_u\eta_t + \eta_{ttt}x^2 - \eta_{tt}g_{u\bar u}\bar u x^2 - \eta_{tt}g_{uu}ux^2 - 2\eta_{tt} + 4\eta_t g_{xu}x - 4g_{u\bar u}\xi_{0t}\bar u x\\&\quad + 8g_{u\bar u}\bar u\bar\zeta_{10} + 8g_{xu}\xi_0 - 4g_{uu}\xi_{0t}ux + 8g_{uu}u\zeta_{10} + 4\xi_{0tt}x - 8\zeta_{10t} = 0,\end{aligned} \tag{38}$$

$$\begin{aligned}&8\zeta_0 g_{u\bar u} + 8\bar\zeta_0 g_{\bar u\bar u} + 8g_{\bar u}(\eta_t - \zeta_{10} + \bar\zeta_{10}) - \eta_{tt}g_{u\bar u}ux^2 - \eta_{tt}g_{\bar u\bar u}\bar u x^2 + 4\eta_t g_{x\bar u}x\\&\quad - 4g_{u\bar u}\xi_{0t}ux + 8g_{u\bar u}u\zeta_{10} + 8g_{x\bar u}\xi_0 - 4g_{\bar u\bar u}\xi_{0t}\bar u x + 8g_{\bar u\bar u}\bar u\bar\zeta_{10} = 0.\end{aligned} \tag{39}$$

Equations (38) and (39) are linear algebraic equations with respect to ζ_0 and $\bar\zeta_0$. The determinant of the matrix of this linear system of equations is equal to

$$\Delta = g_{u\bar u}^2 - g_{uu}g_{\bar u\bar u}.$$

6. Case $\Delta \neq 0$

If $\Delta \neq 0$, then one can find ζ_0 and $\bar\zeta_0$ as follows:

$$\begin{aligned}\zeta_0 &= \big(4\eta_t\big(2(g_u g_{\bar u\bar u} - g_{\bar u}g_{u\bar u}) + x(g_{xu}g_{\bar u\bar u} - g_{u\bar u}g_{x\bar u})\big) + \eta_{ttt}g_{\bar u\bar u}x^2 + \eta_{tt}\big(ux^2\Delta - 2g_{\bar u\bar u}\big) + 4\xi_{0t}xu\Delta\\&\quad + 4\xi_{0tt}xg_{\bar u\bar u} + 8\xi_0(g_{xu}g_{\bar u\bar u} - g_{u\bar u}g_{x\bar u}) + 8g_{\bar u}g_{u\bar u}(\zeta_{10} - \bar\zeta_{10}) - 8u\Delta\bar\zeta_{10} - 8g_{\bar u\bar u}\zeta_{10t}\big)\big/(8\Delta),\\ \bar\zeta_0 &= \big(4\eta_t\big(2(g_{\bar u}g_{uu} - g_u g_{u\bar u}) + x(g_{uu}g_{x\bar u} - g_{u\bar u}g_{xu})\big) - \eta_{ttt}g_{u\bar u}x^2 + \eta_{tt}\big(\bar u x^2\Delta + 2g_{u\bar u}\big) + 4\xi_{0t}\bar u x\Delta - 4g_{u\bar u}\xi_{0tt}x\\&\quad + 8\xi_0(g_{uu}g_{x\bar u} - g_{u\bar u}g_{xu}) - 8g_{\bar u}g_{uu}(\zeta_{10} - \bar\zeta_{10}) - 8g_{u\bar u}^2\bar u\bar\zeta_{10} + 8g_{u\bar u}\zeta_{10t} + 8g_{uu}g_{\bar u\bar u}\bar u\bar\zeta_{10}\big)\big/(8\Delta).\end{aligned}$$

Here we assume that[7]

$$g_x = 0.$$

Notice that in this case the kernel of admitted Lie algebras contains the shifts with respect to x and t:

$$X_1 = \partial_t, \qquad X_2 = \partial_x.$$

Splitting Eq. (37) with respect to x, one has

$$\eta_{tttt}\alpha_4 + \eta_{ttt}\alpha_3 + \alpha_2\eta_{tt} = 0, \qquad \xi_{0ttt}\beta_3 + \xi_{0tt}\beta_2 + \beta_1\xi_{0t} = 0, \tag{40}$$

$$\begin{aligned}&2\eta_{ttt}g_{\bar u\bar u} + 4\eta_t\big(g_u^2 g_{\bar u\bar u} - 2g_u g_{\bar u}g_{u\bar u} + g_{\bar u}^2 g_{uu} + g\Delta\big) - 5\eta_{tt}\alpha_3 + 4g_{\bar u}(g_u g_{u\bar u} - g_{\bar u}g_{uu})(\zeta_{10} - \bar\zeta_{10}) - 4g\Delta\zeta_{10}\\&\quad - 4g_u g_{\bar u\bar u}\zeta_{10t} + 4g_{\bar u}g_{u\bar u}4\bar\zeta_{10t} + 4g_{\bar u\bar u}\zeta_{10tt} = 0,\end{aligned} \tag{41}$$

where

$$\beta_3 = \alpha_4 = -g_{\bar u\bar u}, \qquad \beta_2 = \alpha_3 = -(g_{u\bar u}g_{\bar u} - g_u g_{\bar u\bar u}), \qquad \beta_1 = \alpha_2 = g\Delta.$$

The assumption $\eta_{tt}^2 + \xi_{0t}^2 \neq 0$ leads to a contradiction to the condition $\Delta \neq 0$. Since $\eta(t-\tau) = \eta(t)$ and $\xi(t-\tau) = \xi(t)$, one finds that $\eta(t)$ and $\xi(t)$ are constant. Hence, ζ_0, $\bar\zeta_0$ and Eq. (41) become

$$\zeta_0 = \alpha_1\zeta_{10t} + \beta_1\zeta_{10} + \gamma_1\bar\zeta_{10}, \qquad \bar\zeta_0 = \alpha_2\zeta_{10t} + \beta_2\zeta_{10} + \gamma_2\bar\zeta_{10},$$

$$\zeta_{10}\big(g_{\bar u}(g_{u\bar u}g_u - g_{uu}g_{\bar u}) - g\Delta\big) - \bar\zeta_{10}g_{\bar u}(g_{u\bar u}g_u - g_{uu}g_{\bar u}) + g_{\bar u}g_{u\bar u}\bar\zeta_{10t} - g_u g_{\bar u\bar u}\zeta_{10t} + g_{\bar u\bar u}\zeta_{10tt} = 0, \tag{42}$$

[7] The general case where $g_x \neq 0$ is complicated for solving.

where

$$\alpha_1 = -g_{\bar u\bar u}/\Delta, \qquad \beta_1 = -u + g_{u\bar u} g_{\bar u}/\Delta, \qquad \gamma_1 = -g_{\bar u} g_{u\bar u}/\Delta,$$
$$\alpha_2 = g_{u\bar u}/\Delta, \qquad \beta_2 = -g_{\bar u} g_{uu}/\Delta, \qquad \gamma_2 = -\bar u + g_{\bar u} g_{uu}/\Delta.$$

The case $\zeta_{10} = 0$ is the trivial case, which corresponds to the kernel of admitted Lie algebras. Extension of the kernel is possible if $\zeta_{10} \neq 0$. Firstly, we study the following two cases. In the first case we have,

$$\zeta_{10t} = -k_0\zeta_{10}, \qquad \bar\zeta_{10} = -k_1\zeta_{10} \quad (k_1 \neq 0), \tag{43}$$

and in the second case we have,

$$\zeta_{10t} = -h_2\zeta_{10} - h_1\bar\zeta_{10},$$
$$\beta_1 - h_2\alpha_1 = p_1, \qquad \gamma_1 - h_1\alpha_1 = p_2, \qquad \beta_2 - h_2\alpha_2 = p_3, \qquad \gamma_2 - h_1\alpha_2 = p_4. \tag{44}$$

Here k_i, h_i $(i = 1, 2)$, p_j $(j = 1, 2, 3, 4)$ are constant. Notice that the case where $h_1 = 0$ is reduced to (43). Hence, it is assumed that $h_1 \neq 0$.

6.1. Case (43)

In this case

$$\zeta_0 = (\beta_1 - \alpha_1 k_0 - \gamma_1 k_1)\zeta_{10}, \qquad \bar\zeta_0 = (\beta_2 - \alpha_2 k_0 - \gamma_2 k_1)\zeta_{10}.$$

Since ζ_0 and $\bar\zeta_0$ do not depend on u and $\bar u$, one obtains that

$$\beta_1 - k_0\alpha_1 - k_1\gamma_1 = C_o, \qquad \beta_2 - k_0\alpha_2 - k_1\gamma_2 = -C_o k_1.$$

Because $\Delta \neq 0$, the previous system of equations can be solved with respect to $g_{u\bar u}$ and g_{uu},

$$g_{u\bar u} = \big(g_{\bar u\bar u} k_1 (C_o + \bar u) + g_{\bar u}(k_1 + 1)\big)\big/(C_o + u),$$
$$g_{uu} = \big(k_1(\bar u + C_o) g_{u\bar u} - k_0\big)\big/(u + C_o). \tag{45}$$

Equation (42) becomes

$$-g_{\bar u} k_1 (C_o + \bar u) + g_u (C_o + u) - g + k_0 (u + C_o) = 0. \tag{46}$$

Any function $g(u, \bar u)$ satisfying (46) is also satisfies Eqs. (45). By virtue of the equivalence transformation corresponding to the generator X_1^e, the constant C_o is unessential since by shifting the dependent variable u it can be reduced to zero. The general solution of Eq. (46) is

$$g(u, \bar u) = u\big(-k_0 \ln(u) + \psi\big(\bar u u^{k_1}\big)\big), \tag{47}$$

where ψ is an arbitrary function. Notice also that the general solution of (43) is $\zeta_{10} = Ce^{-k_0 t}$, where $k_1 = -e^{k_0\tau}$. Thus, the extension of the kernel of admitted generators is

$$X = e^{-k_0 t} u \partial_u. \tag{48}$$

6.2. Case (44)

In this case

$$\zeta_0 = p_1\zeta_{10} + p_2\bar\zeta_{10}, \qquad \bar\zeta_0 = p_3\zeta_{10} + p_4\bar\zeta_{10}.$$

Since $h_1 \neq 0$ we have

$$\gamma_1 - h_1\alpha_1 - p_2 = 0, \qquad \beta_1 - h_2\alpha_1 - p_1 = 0, \qquad \gamma_2 - h_1\alpha_2 - p_4 = 0, \qquad \beta_2 - h_2\alpha_2 - p_3 = 0.$$

These equations can be solved with respect to g_{uu}, $g_{u\bar{u}}$, $g_{\bar{u}\bar{u}}$ and $g_{\bar{u}}$ to give

$$g_{uu} = \frac{(h_1 p_3 - h_2 p_4 - h_2\bar{u})}{p_1 p_4 + p_1\bar{u} - p_2 p_3 + p_4 u + u\bar{u}}, \qquad g_{u\bar{u}} = \frac{g_{\bar{u}}(p_3 + p_4 + \bar{u})}{p_1 p_4 + p_1\bar{u} - p_2 p_3 + p_4 u + u\bar{u}},$$
$$g_{\bar{u}\bar{u}} = \frac{g_{\bar{u}}^2(p_1 + p_2 + u)((p_1 + u)h_1 - h_2 p_2)(p_4 + \bar{u} + p_3)}{(h_1 p_1 + h_1 u - h_2 p_2)(p_1 p_4 + p_1\bar{u} - p_2 p_3 + p_4 u + u\bar{u})}, \qquad g_{\bar{u}} = \frac{-h_1 p_1 - h_1 u + h_2 p_2}{p_3 + p_4 + \bar{u}}.$$

Considering $(g_{\bar{u}})_u - g_{u\bar{u}} = 0$ and $(g_{\bar{u}})_{\bar{u}} - g_{\bar{u}\bar{u}} = 0$ one obtains

$$h_1(p_1 + p_2 + u)p_3 - h_2(p_3 - p_4 - \bar{u})p_2 = 0,$$
$$\big(h_1(u + p_1) - h_2 p_2\big)\big((u + p_1 + p_2)p_3 + p_2(\bar{u} + p_3 + p_4)\big) = 0.$$

Since $h_1 \neq 0$, one finds $p_3 = 0$ and $p_2 = 0$. Because $\bar{\zeta}_0 = \bar{\zeta}_{10} p_4$ and $\zeta_0 = \zeta_{10} p_1$, one obtains $p_4 = p_1$. Hence, $g_{\bar{u}} + h_1(p_1 + u)/(p_1 + \bar{u}) = 0$, and Eq. (42) becomes

$$g_u = \frac{g}{(p_1 + u)} - h_2.$$

The general solution for the function g is

$$g = -(p_1 + u)\big(h_2 \ln(p_1 + u) + h_1 \ln(p_1 + \bar{u}) + k_3\big).$$

As before the constant p_1 is unessential and can be reduced to zero, that is,

$$g = -u\big(h_2 \ln(u) + h_1 \ln(\bar{u}) + k_3\big). \tag{49}$$

In this case an extension of the kernel of admitted generators is

$$X = q(t)u\partial_u, \tag{50}$$

where the function $q(t)$ is a solution of the delay differential equation

$$q'(t) + h_2 q(t) + h_1 q(t - \tau) = 0. \tag{51}$$

6.3. Case $\alpha_{1u} \neq 0$

Differentiating ζ_0 with respect to u and $\bar{u}$, one obtains

$$\zeta_{10t} + \frac{\beta_{1u}}{\alpha_{1u}}\zeta_{10} + \frac{\gamma_{1u}}{\alpha_{1u}}\bar{\zeta}_{10} = 0. \tag{52}$$

Differentiating the previous equation with respect to u and $\bar{u}$, it gives

$$\left(\frac{\beta_{1u}}{\alpha_{1u}}\right)_u \zeta_{10} + \left(\frac{\gamma_{1u}}{\alpha_{1u}}\right)_u \bar{\zeta}_{10} = 0,$$
$$\left(\frac{\beta_{1u}}{\alpha_{1u}}\right)_{\bar{u}} \zeta_{10} + \left(\frac{\gamma_{1u}}{\alpha_{1u}}\right)_{\bar{u}} \bar{\zeta}_{10} = 0.$$

If $\frac{\beta_{1u}}{\alpha_{1u}}$ or $\frac{\gamma_{1u}}{\alpha_{1u}}$ is not constant, then the above system of equations leads to (43). Hence, one needs to study the case

$$\frac{\beta_{1u}}{\alpha_{1u}} = h_2, \qquad \frac{\gamma_{1u}}{\alpha_{1u}} = h_1,$$

where h_1 and h_2 are constant.

Substituting the derivative ζ_{10t} into the equations

$$\zeta_{0\bar{u}} = 0, \qquad \bar{\zeta}_{0u} = 0, \qquad \bar{\zeta}_{0\bar{u}} = 0,$$

one has

$$(\beta_{1\bar{u}} - h_2\alpha_{1\bar{u}})\zeta_{10} + (\gamma_{1\bar{u}} - h_1\alpha_{1\bar{u}})\bar{\zeta}_{10} = 0,$$
$$(\beta_{2u} - h_2\alpha_{2u})\zeta_{10} + (\gamma_{2u} - h_1\alpha_{2u})\bar{\zeta}_{10} = 0,$$
$$(\beta_{2\bar{u}} - h_2\alpha_{2\bar{u}})\zeta_{10} + (\gamma_{2\bar{u}} - h_1\alpha_{2\bar{u}})\bar{\zeta}_{10} = 0.$$

If one of the coefficients of this linear system of equations (with respect to ζ_{10}, $\bar{\zeta}_{10}$) is not equal to zero, then this leads to (43). If all of the coefficients are equal to zero, then this leads to (44).

A similar result is obtained for $(\alpha_{1\bar{u}})^2 + (\alpha_{2u})^2 + (\alpha_{2\bar{u}})^2 \neq 0$. Hence, to proceed one needs to study the only case $\alpha_1 = \text{const}$, $\alpha_2 = \text{const}$.

6.4. *Case* $\alpha_1 = \text{const}$, $\alpha_2 = \text{const}$

Assume that $\alpha_1 = p_1$, $\alpha_2 = p_2$, where p_1 and p_2 are constant. Notice that because of $\Delta \neq 0$, one has $p_1^2 + p_2^2 \neq 0$.

We first consider the case for which $p_1 \neq 0$. From the equations $\alpha_1 = p_1$ and $\alpha_2 = p_2$ we find the derivatives $g_{u\bar{u}}$ and g_{uu} as

$$g_{u\bar{u}} = -g_{\bar{u}\bar{u}} p_2/p_1, \qquad g_{uu} = \big(g_{\bar{u}\bar{u}} p_2^2 + p_1\big)/p_1^2. \tag{53}$$

The first equation can be integrated to give

$$g_u = -p_2 g_{\bar{u}}/p_1 + \beta,$$

where $\beta = \beta(u)$ is an arbitrary function of the integration. Substituting this into the second equation of (53) and integrating it, one obtains

$$\beta = u/p_1 + C_o,$$

where C_o is constant. Notice that in this case $\Delta = g_{\bar{u}\bar{u}}/p_1 \neq 0$. Differentiating ζ_0 with respect to u and $\bar{u}$, one has

$$g_{\bar{u}\bar{u}} p_2^2(\zeta_{10} - \bar{\zeta}_{10}) + p_1\zeta_{10} = 0, \qquad (\zeta_{10} - \bar{\zeta}_{10}) g_{\bar{u}\bar{u}} p_2 = 0.$$

This gives a contradiction to $\zeta_{10} \neq 0$.

If $p_1 = 0$, then $p_2 \neq 0$ and from the equations $\alpha_1 = p_1$ and $\alpha_2 = p_2$ one finds

$$g_{\bar{u}\bar{u}} = 0, \qquad g_{u\bar{u}} = 1/p_2.$$

The equation $\zeta_{0u} = 0$ also gives the contradiction $\zeta_{10} = 0$.

7. Case $\Delta = 0$

7.1. *Case* $g_{\bar{u}\bar{u}} \neq 0$

Let $g_{\bar{u}\bar{u}} \neq 0$. In this case the general solution of the equation $\Delta = 0$ is

$$g_u = \phi(g_{\bar{u}}),$$

where ϕ is an arbitrary function of the integration.

Excluding ζ_0 and $\bar{\zeta}_0$ from (38) and (39), one finds

$$8 g_{\bar{u}}\phi'(-\eta_t + \zeta_{10} - \bar{\zeta}_{10}) + \eta_{ttt}x^2 - 2\eta_{tt} + 8\eta_t\phi + 4\xi_{0tt}x - 8\zeta_{10t} = 0. \tag{54}$$

Splitting this equation with respect to x, one has

$$\eta_{ttt} = 0, \qquad \xi_{0tt} = 0.$$

Hence,

$$\eta = a_2t^2 + a_1t + a_0, \qquad \xi_0 = b_1t + b_0,$$

where a_0, a_1, a_2, b_1, b_0 are constants. Since $\xi_0(t) = \xi_0(t - \tau)$ and $\eta(t) = \eta(t - \tau)$, one gets $\xi_0 = b_0$ and $a_2 = a_1 = 0$. Notice that for the existence of a nontrivial extension of the kernel of admitted Lie groups one needs that $\zeta_0^2 + \zeta_{10}^2 \neq 0$.

Equation (54) becomes

$$g_{\bar u}\phi'(\zeta_{10}-\bar\zeta_{10})-\zeta_{10t}=0. \tag{55}$$

If $\phi'=0$, then $\bar\zeta_{10}=\zeta_{10}$ are constant. In this case Eq. (39) is

$$\bar\zeta_0+\bar u\bar\zeta_{10}=0.$$

This leads to $\zeta_0=0$ and $\zeta_{10}=0$, which means that there is no any extension of the kernel of admitted Lie groups. Hence, for extension of the kernel one needs to study $\phi'\neq 0$.

Differentiating (54) with respect to $\bar u$, one obtains

$$(g_{\bar u}\phi')_{\bar u}(\zeta_{10}-\bar\zeta_{10})=0. \tag{56}$$

Let $\bar\zeta_{10}-\zeta_{10}=0$. Equation (54) gives that ζ_{10} is constant. Equations (38) and (39) are reduced to the equation

$$\phi'=-\frac{\bar\zeta_0+\bar u\zeta_{10}}{\zeta_0+u\zeta_{10}}. \tag{57}$$

Differentiating this equation with respect to t and x, one gets

$$\phi'\zeta_{0t}+\bar\zeta_{0t}=0,\qquad \phi'\zeta_{0x}+\bar\zeta_{0x}. \tag{58}$$

Assume that $\phi''\neq 0$, then from (58) one obtains that ζ_0 is also constant. By virtue of the inverse function theorem, from (57), one has

$$g_{\bar u}=h\left(\frac{\bar\zeta_0+\bar u\zeta_{10}}{\zeta_0+u\zeta_{10}}\right),$$

where h is the inverse function of the function ϕ'. Because $g_{\bar u\bar u}\neq 0$, the constant $\zeta_{10}\neq 0$. Using the equivalence transformation corresponding to the generators X_1^e and X_5^e, one can account $\zeta_0=0$, $\zeta_{10}=1$. Since $g_u=\phi(g_{\bar u})$, integrating these equations, one finds

$$g(u,\bar u)=uh\left(\frac{\bar u}{u}\right)+k_1u+k_0, \tag{59}$$

where k_0 and k_1 are constants of the integration.

Equation (37) becomes

$$g_uu+g_{\bar u}\bar u-g=0. \tag{60}$$

Substituting in this equation the function g, one finds that $k_0=0$. The extension of the kernel of admitted Lie algebras is given by the generator

$$X=u\partial_u.$$

Let $\phi''=0$ or

$$g_u=k_1g_{\bar u}-k_0, \tag{61}$$

where k_0 and $k_1\neq 0$ are constant. Equation (57) gives $\zeta_{10}=0$ and

$$\bar\zeta_0=-k_1\zeta_0. \tag{62}$$

Equation (37) becomes

$$\zeta_{0t}=\zeta_{0xx}-k_0\zeta_0.$$

If there exists a solution $q(t,x)$ of the partial differential equation

$$q_t=q_{xx}-k_0q, \tag{63}$$

satisfying the condition

$$q(t-\tau,x) = -k_1 q(t,x), \tag{64}$$

then the extension of the kernel is given by the generator

$$X = q(t,x)\partial_u.$$

Let $\bar\zeta_{10} - \zeta_{10} \neq 0$. Equation (56) leads to $(g_{\bar u}\phi')_{\bar u} = 0$ or $\phi = k_1 \ln(g_{\bar u}) + k_0$, where k_0 and $k_1 \neq 0$ are constants of the integration. Equations (54) and (39) give

$$\bar\zeta_{10} = \zeta_{10} - \zeta_{10t}/k_1, \tag{65}$$

$$g_{\bar u\bar u} = \frac{g_{\bar u}^2(\zeta_{10} - \bar\zeta_{10})}{k_1(\zeta_0 + u\zeta_{10}) + g_{\bar u}(\bar\zeta_0 + \bar u\bar\zeta_{10})}. \tag{66}$$

Differentiating (66) with respect to x and t, one obtains

$$g_{\bar u}\bar\zeta_{0x} + k_1\zeta_{0x} = 0, \tag{67}$$

and

$$\begin{aligned} &g_{\bar u}\big(\bar u(\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10}) + \bar\zeta_{0t}(\bar\zeta_{10} - \zeta_{10}) - \bar\zeta_0(\bar\zeta_{10t} - \zeta_{10t})\big) \\ &\quad + k_1\big(u(\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10}) + \zeta_{0t}(\bar\zeta_{10} - \zeta_{10}) - \zeta_0(\bar\zeta_{10t} - \zeta_{10t})\big) = 0. \end{aligned} \tag{68}$$

Since $g_{\bar u\bar u} \neq 0$, Eq. (67) gives that $\zeta_0 = \zeta_0(t)$.

Notice that

$$\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10} = \frac{1}{k_1}\big(\zeta_{10}\zeta_{10tt} - \zeta_{10t}^2\big).$$

Assume that $\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10} \neq 0$. Equation (67) can be solved with respect to $g_{\bar u}$,

$$g_{\bar u} = -k_1\frac{u+b}{\bar u + c}, \tag{69}$$

where

$$b = \frac{\bar\zeta_{0t}(\bar\zeta_{10} - \zeta_{10}) - \bar\zeta_0(\bar\zeta_{10t} - \zeta_{10t})}{(\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10})}, \qquad c = \frac{\zeta_{0t}(\bar\zeta_{10} - \zeta_{10}) - \zeta_0(\bar\zeta_{10t} - \zeta_{10t})}{(\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10})}.$$

Differentiating (69) with respect to t, we find that b and s are constants. Substituting the derivatives $g_{\bar u}$ and $g_{\bar u\bar u}$, found from (69), into Eq. (66), one obtains

$$\bar u(\zeta_0 - b\zeta_{10}) - u(\bar\zeta_0 - c\bar\zeta_{10}) + c\zeta_0 - b\bar\zeta_0 + cb(\bar\zeta_{10} - \zeta_{10}) = 0.$$

This leads to $\zeta_0 = b\zeta_{10}$ and $\bar\zeta_0 = c\bar\zeta_{10}$. Because $\bar\zeta_{10} = \zeta_{10}(t-\tau)$, $\bar\zeta_0 = \zeta_0(t-\tau)$, and $\bar\zeta_{10} - \zeta_{10} \neq 0$, one gets $c = b$. By virtue of the equivalence transformations corresponding to the generator X_1^e, one can account that $b = 0$. Integrating the found derivatives g_u and $g_{\bar u}$, one finds

$$g(u,\bar u) = \lambda u - k_1\ln(\bar u/u) + \gamma, \tag{70}$$

where λ and γ are constant. Substituting (70) into (37), one obtains $\gamma = 0$. The extension of the kernel of admitted Lie algebras is given by the generator

$$X = q(t)u\partial_u, \tag{71}$$

where $q(t)$ is a solution of the delay differential equation (65):

$$q'(t) = k_1\big(q(t) - q(t-\tau)\big). \tag{72}$$

Assume that $\bar\zeta_{10}\zeta_{10t} - \bar\zeta_{10t}\zeta_{10} = 0$. As was noticed the function $\zeta_{10}(t)$ has to satisfy the equation

$$\zeta_{10}\zeta_{10tt} - \zeta_{10t}^2 = 0.$$

Hence, $\zeta_{10}(t) = Ce^{\lambda t}$, where C and λ are constant such that $C\lambda \neq 0$. In this case

$$\bar{\zeta}_{10} = k\zeta_{10},$$

where $k = e^{-\lambda\tau}$. Hence, Eq. (68) becomes

$$g_{\bar{u}}(\bar{\zeta}_{0t}\zeta_{10} - \bar{\zeta}_0\zeta_{10t}) + k_1(\zeta_{0t}\zeta_{10} - \zeta_0\zeta_{10t}) = 0.$$

Since $g_{\bar{u}\bar{u}} \neq 0$, one obtains that

$$\zeta_0 = \alpha\zeta_{10},$$

where α is constant. Without loss of generality one can account that $\alpha = 0$. Equation (37) becomes

$$ug_u + k\bar{u}g_{\bar{u}} = g + \lambda u.$$

The general solution of this equation is

$$g(u, \bar{u}) = \lambda u \ln(u) + u\psi(\bar{u}u^{-k}). \tag{73}$$

By virtue of the relation $g_u = k_1 g_{\bar{u}} + k_0$, the function $\psi(z)$ has to satisfy the ordinary differential equation

$$k_1 \ln(\psi') + kz\psi' = \psi + \lambda - k_0. \tag{74}$$

The extension of the kernel of admitted Lie algebras is given by the generator

$$X = e^{\lambda t} u\partial_u. \tag{75}$$

7.2. Case $g_{\bar{u}\bar{u}} = 0$

Assuming that $g_{\bar{u}\bar{u}} = 0$, one has

$$g(u, \bar{u}) = k_1\bar{u} + h(u),$$

where $k_1 \neq 0$ is a constant. Hence Eq. (39) gives

$$\bar{\zeta}_{10} = \zeta_{10} - \eta_t.$$

Furthermore, if we let $g_{uu} = h'' \neq 0$, we can define from (38)

$$\zeta_0 = u(x^2\eta_{tt} + 4x\xi_{0t} - 8\zeta_{10})/8 + (2\eta_{tt} - 8\eta_t h' + 8\zeta_{10t} - \eta_{ttt}x^2 - 4\xi_{0tt}x)/(8h'').$$

Since $\zeta_{0u} = 0$, then

$$(\eta_{ttt}h''' + \eta_{tt}h''^2)x^2 + 4(h'''\xi_{0tt} + h''^2\xi_{0t})x - 2\eta_{tt}h''' + 8\eta_t h''' h' - 8\eta_t h''^2 - 8h'''\zeta_{10t} - 8h''^2\zeta_{10} = 0.$$

The last equation can be split with respect to x so that

$$\eta_{ttt}(h'''/h''^2) + \eta_{tt} = 0,$$
$$\xi_{0tt}(h'''/h''^2) + \xi_{0t} = 0,$$

and

$$(-\eta_{tt}h''' + 4\eta_t h''' h' - 4\eta_t h''^2)/(4h''^2) - (\zeta_{10t}(h'''/h''^2) + \zeta_{10}) = 0. \tag{76}$$

Differentiating the first and the second equations with respect to u, one obtains

$$\eta_{ttt}(h'''/h''^2)' = 0, \qquad \xi_{0tt}(h'''/h''^2)' = 0.$$

Notice that if $(h'''/h''^2)' \neq 0$, then $\eta_{tt} = 0$, $\xi_{0t} = 0$, and because of $\eta(t-\tau) = \eta(t)$, $\xi(t-\tau) = \xi(t)$, one obtains that $\eta = \text{const}$ and $\xi = \text{const}$. In this case (76) gives

$$\zeta_{10} = 0,$$

which corresponds to the kernel of admitted Lie algebras. Thus, one needs to study the case $(h'''/h''^2)' = 0$ or $h''' = Kh''^2$ with some constant K. This case also leads to the same result, that is, there is no extension of the kernel. In fact, differentiating (76) with respect to u, one obtains $\eta_t K = 0$. If $K = 0$, then $\eta_{tt} = 0$, which also gives that $\eta = \text{const}$. This leads to $\bar{\zeta}_{10} = \zeta_{10}$. Similar analysis of the equations $\xi_{0tt} K + \xi_{0t} = 0$ and $\xi(t - \tau) = \xi(t)$ gives that $\xi_0 = \text{const}$. The function $\zeta_{10}(t)$ has to satisfy the equations

$$\zeta_{10t} K + \zeta_{10} = 0, \qquad \bar{\zeta}_{10} = \zeta_{10}.$$

The general solution of these equations is $\zeta_{10} = 0$. Thus, the case $g_{uu} \neq 0$ does not give extensions of the kernel.

7.3. *Case $g_{\bar{u}\bar{u}} = 0$ and $g_{uu} = 0$*

We extend our study to the case of a linear function

$$g(u, \bar{u}) = k_1 \bar{u} + k_2 u + k, \tag{77}$$

where $k_1 \neq 0$. In this case (38) becomes

$$\eta_{ttt} x^2 + 4\xi_{0tt} x - 2(\eta_{tt} - 4\eta_t k_2 + 4\zeta_{10t}) = 0.$$

Splitting this equation with respect to x, one finds

$$\eta_{ttt} = 0, \qquad \xi_{0tt} = 0, \qquad \eta_{tt} - 4\eta_t k_2 + 4\zeta_{10t} = 0.$$

By virtue of $\eta(t - \tau) = \eta(t)$ and $\xi(t - \tau) = \xi(t)$ the values η, ξ and ζ_{10} are constant. Equation (37) becomes

$$\zeta_{0t} = \zeta_{0xx} + k_2 \zeta_0 + k_1 \bar{\zeta}_0 - \zeta_{10} k.$$

If $k_2 + k_1 \neq 0$, then by using the equivalence transformation related with the generator X_1^e the constant k_0 can be reduced to zero. In this case the extension of the kernel is given by the generators $X = u\partial_u$ and $X_q = q(t, x)\partial_u$, where the function $q(t, x)$ satisfies the delay partial differential equation

$$q_t(t, x) = q_{xx}(t, x) + k_2 q(t, x) + k_1 q(t - \tau, x). \tag{78}$$

If $k_2 + k_1 = 0$, then introducing $q = \zeta_0 - \zeta_{10} k x^2/2$, one gets that the extension of the kernel is given by the generators X_q and

$$X = (2u + kx^2)\partial_u. \tag{79}$$

8. Summary of the group classification

Case 1. Combining (47) and (73),

$$g(u, \bar{u}) = u\left(-k_0 \ln(u) + \psi\left(\bar{u} u^{k_1}\right)\right), \tag{80}$$

where ψ is an arbitrary function, $k_1 = -e^{k_0 \tau}$, and

$$X = e^{-k_0 t} u \partial_u. \tag{81}$$

Case 2. Combining (49) and (70),

$$g = -u\left(h_2 \ln(u) + h_1 \ln(\bar{u}) + k_3\right) \tag{82}$$

and

$$X = q(t) u \partial_u, \tag{83}$$

where the function $q(t)$ is a solution of the delay differential equation

$$q'(t) + h_2 q(t) + h_1 q(t - \tau) = 0. \tag{84}$$

A particular solution of Eq. (84) is $q = e^{-k_0 t}$, where $k_0 = h_2 + h_1 e^{k_0 \tau}$.

Case 3. The general solution of Eq. (61) is

$$g(u,\bar{u}) = -k_0u + \psi(\bar{u} + k_1u) \quad (k_1 \neq 0), \tag{85}$$

where ψ is an arbitrary function. The extension of the kernel is given by the generator

$$X = q(t,x)\partial_u, \tag{86}$$

where $q(t,x)$ is a solution of equation

$$q_t = q_{xx} - k_0q, \tag{87}$$

satisfying the condition

$$q(t-\tau,x) = -k_1q(t,x). \tag{88}$$

For particular cases of k_0 and k_1 the problem (87), (88) has a solution. For example, let $k_1 = -1$, then a solution can be sought in the form $q = q(x)$, where

$$q''(x) - k_0q(x) = 0.$$

If τ, k_0 and k_1 are related by the formula $k_1 = -e^{k_0\tau}$, then a particular solution of the problem (87) and (88) is $q = e^{-k_0t}$.

Case 4. If the function $g(u,\bar{u})$ is a linear function

$$g(u,\bar{u}) = k_1\bar{u} + k_2u + k \quad (k_1 \neq 0), \tag{89}$$

then the extension of the kernel of admitted generators consists of the generators

$$X_q = q(t,x)\partial_u, \tag{90}$$

where the function $q(t,x)$ satisfies the reaction–diffusion equation with a delay

$$q_t(t,x) = q_{xx}(t,x) + k_2q(t,x) + k_1q(t-\tau,x), \tag{91}$$

and one more generator, which depends on the value of the constants k_1 and k_2. If $k_2 + k_1 \neq 0$, then one can account that $k = 0$, and the additional generator is

$$X = u\partial_u. \tag{92}$$

If $k_2 + k_1 = 0$, then

$$X = \left(2u + kx^2\right)\partial_u. \tag{93}$$

A particular solution of Eq. (91) is $q = e^{-k_0t}$, where $k_0 = -(k_1e^{k_0\tau} + k_2)$.

9. Invariant solutions

Invariant solutions can be sought for a subalgebra of an admitted Lie algebra. Essentially different invariant solutions are obtained on the base of an optimal system of admitted subalgebras. The set of all generators nonequivalent with respect to automorphisms composes an optimal system of one-dimensional subalgebras [7]. This set is used for constructing nonequivalent invariant solutions. Equivalence of invariant solutions is considered with respect to an admitted Lie algebra.

Apart from automorphisms for constructing the optimal system of subalgebras one has to use involutions. Equations (15) possess the involution E corresponding to the change $x \to -x$.

9.1. Optimal system of subalgebras

Let us consider the algebra $L_3 = \{X_1, X_2, X_3\}$, with the table of commutators

	X_1	X_2	X_3
X_1	0	0	$-k_0X_3$
X_2	0	0	0
X_3	k_0X_3	0	0

Such algebras are admitted by Eq. (15) with the function $g(u, \bar{u})$ in (80), (82), (85) and (89). The generator X_2 composes a center of the algebra L_3.

The coordinates (x_1, x_2, x_3) of the generator

$$X = x_1X_1 + x_2X_2 + x_3X_3$$

are simplified [7] by the automorphisms A_1 and A_3, which are defined by the table of commutators

$$A_1:\quad x_3' = x_3e^{-k_0a_1}, \qquad A_3:\quad x_3' = x_3 + k_0x_1a_2.$$

Here only changed coordinates are presented.

The optimal system of subalgebras of the algebra L_3 with $k_0 \neq 0$ consists of the subalgebras

$$H_1 = X_3 + \alpha X_2, \qquad H_2 = X_1 + \alpha X_2, \qquad H_3 = X_2,$$

where α is an arbitrary constant.

Representations of the invariant solutions corresponding to the subalgebras H_2 and H_3 are

$$u = \varphi(x - \alpha t), \qquad u = \varphi(t),$$

respectively. It is obvious that these representations reduce the number of the independent variables.

9.2. Invariant solutions with respect to H_1

Case 1. For the function (80) the generator $X_3 = q(t)\partial_u$ and the representation of an invariant solution is

$$u = e^{\beta x q(t)}\varphi(t),$$

where $\beta = 1/\alpha$ and $q(t) = e^{-k_0t}$. The reduced equation is

$$\varphi'(t) = \varphi(t)\big(\beta^2q^2 - k_0\ln\big(\varphi(t)\big) + \psi\big(\varphi(t-\tau)\varphi^{k_1}(t)\big)\big). \tag{94}$$

Case 2. For the function (82) the generator X_3 and the representation of an invariant solution is the same as in the previous case. The reduced equation is

$$\varphi'(t) = \varphi(t)\big(\beta^2q^2 - h_2\ln\big(\varphi(t)\big) - h_1\ln\big(\varphi(t-\tau)\big) + k_3\big). \tag{95}$$

Case 3. For the function (85) in the case, where $k_1 = -e^{k_0\tau}$, the generator $X_3 = e^{-k_0t}\partial_u$, and the representation of an invariant solution is $u = \beta xe^{-k_0t} + \varphi(t)$. The reduced equation is

$$\varphi'(t) = -k_0\varphi(t) + \psi\big(\varphi(t-\tau) + k_1\varphi(t)\big). \tag{96}$$

Case 4. If the function $g(u, \bar{u})$ is as given in (89) and the generator $X_3 = e^{-k_0t}\partial_u$, then the invariant solution is $u = \beta xe^{-k_0t} + \varphi(t)$, where the function $\varphi(t)$ satisfies the reduced equation

$$\varphi'(t) = k_1\varphi(t-\tau) + k_2\varphi(t) + k. \tag{97}$$

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