Laura Menini · Antonio Tornambè

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Preface

The goal of this book is to present several concepts useful for the analysis of dynamical systems, and to illustrate, in the last two chapters, how they can be actually applied to improve the state of the art for two classical topics in nonlinear systems theory: the linearization of a nonlinear system by state immersion and the study of stability of equilibrium points.

The main reasoning that led us to writing this book is that some concepts that are already well developed in the literature become more important if presented together. Three of such concepts are homogeneity, symmetries (and orbital symmetries for continuous-time systems) and Lie algebras, which, in our opinion, can be better understood if symmetries are seen as a generalization of homogeneity, and Lie algebras (seen as generators of Lie groups) as a generalization of symmetries. Another very well known concept is that of first integral, that is particularly helpful for researchers working on Hamiltonian systems, or on stability of switched systems. In our opinion, similar attention should be paid to the generalization of first integrals represented by semi-invariants, which, in turn, have a special relation, that will be explored in the book, with orbital symmetries.

Nonlinear systems theory was traditionally developed for continuous-time systems, i.e., systems of ordinary differential equations. Only most recently, with the growth of the "digital world", the attention of many researchers is concentrated on discrete-time systems, i.e., systems of difference equations. For linear systems the similarity between continuous-time and discrete-time systems is nowadays well understood and, with some important exceptions, the study of both kinds of systems can be actually performed in parallel, obtaining very similar results. Since this is not so true for nonlinear systems, in this book we have made a special effort to extend some of the concepts that are standard and well known for continuous-time systems to discrete-time ones; in some cases, we report some results, already existing for discrete-time systems, but not so well known in the control literature, that turn out to be the analogous of well known results in continuous-time.

We have tried to be self-contained as much as possible, and sometimes we have reported not only the statements, but also the proofs of some very standard results, for two reasons: first because we would like the book to reach a wider audience, secondly because such derivations are often very similar to those that are needed to develop the less standard topics. Most of the material in the first six chapters of the book is not new, but, together with some new results, we sometimes propose an alternative derivation of some known result that we consider more useful to better understand the topic or its relationship with other results presented earlier.

Finally, we would like to apologize for the inevitable errors and omissions, especially in giving credit for the results presented in the book.

Rome, Italy

Laura Menini
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Chapter 1
Notation and Background

1.1 Notation

Symbols $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ represent the sets of real, complex and integer numbers, respectively. Given a set $\mathbb{A}$, with $\mathbb{A}$ being either $\mathbb{R}$ or $\mathbb{Z}$, symbols $\mathbb{A}^{<}$, $\mathbb{A}^{\leq}$, $\mathbb{A}^{>}$ and $\mathbb{A}^{\geq}$ denote the sets of all numbers $a \in \mathbb{A}$ such that $a < 0$, $a \leq 0$, $a > 0$ and $a \geq 0$, respectively; $\mathbb{A}^n$, with $\mathbb{A}$ denoting either one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}$, denotes the set of all vectors $a = [a_1 \ \cdots \ a_n]^\top$ (superscript $\top$ means transpose), with entries $a_i \in \mathbb{A}$; $\mathbb{A}^{n\times m}$ denotes the set of all $n \times m$ matrices

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ldots & \vdots \\ A_{n,m} & \cdots & A_{n,m} \end{bmatrix},$$

with entries $A_{i,j} \in \mathbb{A}$; E denotes the identity matrix: the ith column of E is denoted by e_i. Since some of the concepts that are introduced in the book are not defined on the whole $\mathbb{R}^n$, $\mathscr{U}$ denotes some (not necessarily, small) open and connected subset of $\mathbb{R}^n$; $\mathscr{U}$ need not contain the origin of $\mathbb{R}^n$; if necessary, this is explicitly assumed. It is worth pointing out that a set $\mathscr{U}$ of $\mathbb{R}^n$ is open if it contains a full neighborhood of x^o, for all $x^o \in \mathscr{U}$; this, in particular, implies that an open set $\mathscr{U}$ has always non-zero measure. Note that, in this book, a neighborhood of a point x^o contains x^o. Notation $h(x) : \mathscr{U} \to \mathbb{R}^m$ denotes a vector function $h(x)$ from $\mathscr{U}$ to $\mathbb{R}^m$; if it is not necessary to specify the domain $\mathscr{U}$ of the vector function, the simpler notation $h(x) \in \mathbb{R}^m$ is used, thus omitting that $x \in \mathscr{U}$; if no confusion can arise, the dependence of $h(x)$ on x is omitted. The image of $\mathscr{U}$ through h is denoted by $h(\mathscr{U})$. If $h(x) \in \mathbb{R}^n$, $n = 1$, $\frac{\partial h}{\partial x}$ (respectively, $\nabla h = (\frac{\partial h}{\partial x})^\top$) denotes the row (respectively, column) *gradient* of h; if $h(x) \in \mathbb{R}^n$, $n \geq 2$, $\frac{\partial h}{\partial x}$ is the *Jacobian matrix* of h. The *divergence* $\mathrm{div}(h)$ of $h(x) : \mathscr{U} \to \mathbb{R}^n$ is

$$\mathrm{div}(h) := \mathrm{trace}\left(\frac{\partial h}{\partial x}\right) = \sum_{i=1}^{n} \frac{\partial h_i}{\partial x_i},$$

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where h_i and x_i are the ith entries of h and x, respectively. A vector function $h(x) : \mathscr{U} \to \mathbb{R}^n$ is C^i at $x = x^o$, with $i \in \mathbb{Z}^{\geq}$, if the partial derivatives $\frac{\partial^i h(x)}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, $\sum_{k=1}^n j_k = i$, exist and are continuous at $x = x^o$. A vector function $h(x) : \mathscr{U} \to \mathbb{R}^n$ is C^∞ at $x = x^o$ if all partial derivatives $\frac{\partial^i h(x)}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, $\sum_{k=1}^n j_k = i$, exist and are continuous at $x = x^o$ for all $i \geq 0$; a C^∞-function is said to be *smooth*.

Both differential and difference equations are considered, with t denoting the independent variable that is called *time*; $t \in \mathbb{R}$ in case of differential equations and $t \in \mathbb{Z}$ in case of difference equations; the case $t \in \mathbb{R}$ is denoted as the *continuous-time* case and $t \in \mathbb{Z}$ is denoted as the *discrete-time* case; if such two cases can be jointly considered, the notation $t \in \mathbb{T}$, with either $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, is used.

1.2 Analytic and Meromorphic Functions

This section deals with some basic facts about analytic and meromorphic functions: the reader interested in a more extended exposition is referred to Sect. 1.1 of [35] or to [63].

A function $\alpha(x) : \mathbb{R}^n \to \mathbb{R}$ is *analytic at* $x^o \in \mathbb{R}^n$ if it admits a Taylor series expansion centered at $x = x^o$, which is convergent to $\alpha(x)$ for all $x \in \mathscr{B}$, with $\mathscr{B}$ being a neighborhood of x^o; α is *analytic on* $\mathscr{U}$, with $\mathscr{U}$ being some open and connected subset of $\mathbb{R}^n$, if α is *analytic* at each $x^o \in \mathscr{U}$; α is *analytic* on the whole $\mathbb{R}^n$ if it is analytic at each $x^o \in \mathbb{R}^n$.

Example 1.1 The function $\alpha(x) = \mathrm{e}^{-1/x^2}$ of $x \in \mathbb{R}$ is not analytic on $\mathbb{R}$; it is analytic on the open intervals $(-\infty, 0)$, $(0, +\infty)$, but not at $x = 0$, where it is only smooth. In particular, such a function is *flat* at $x = 0$, i.e., $\frac{\mathrm{d}^i \alpha(x)}{\mathrm{d}x^i}|_{x=0} = 0$, for all $i \in \mathbb{Z}^{\geq}$.

If $\alpha(x^o) = 0$, then $x^o \in \mathbb{R}^n$ is a *zero* of α. Given a function $\alpha(x) \in \mathbb{R}$ being analytic on a whole open and connected set $\mathscr{U}$ of $\mathbb{R}^n$, either $\alpha(x)$ is equal to zero for all $x \in \mathscr{U}$ or the set of the zeros of α in $\mathscr{U}$ has an empty interior (if $n = 1$, the zeros of α in $\mathscr{U}$ are isolated).

Example 1.2 Function $\alpha(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$ has an infinite (countable) number of zeros in any neighborhood of $x = 0$, and therefore, since $x = 0$ is a zero of $\alpha(x)$ and is not isolated, $\alpha(x)$ is not analytic at $x = 0$.

Given an open and connected $\mathscr{U} \subseteq \mathbb{R}^n$, the set $\mathscr{A}_n$ of all analytic functions $\alpha(x) : \mathscr{U} \to \mathbb{R}$, endowed with the usual operations of sum and product between functions, is a *ring*; denote by $\mathscr{K}_n$ the set of all functions $\alpha = \frac{a}{b}$, with $a, b \in \mathscr{A}_n$, with b that is not identically equal to zero; then, $\mathscr{K}_n$ is a *field* (the *quotient field* of the ring of analytic functions): $\alpha \in \mathscr{K}_n$ is called *meromorphic*. Actually, similarly to the field of rational functions, $\mathscr{K}_n$ is a field under the *equivalence relation* $\sim$ defined as follows: $\alpha_1, \alpha_2 \in \mathscr{K}_n$, $\alpha_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathscr{A}_n$, b_i not identically equal to zero, are *equivalent*, $\alpha_1 \sim \alpha_2$, if $a_1(x)b_2(x) = a_2(x)b_1(x), \forall x \in \mathscr{U}$; one can say that α_1 and α_2 *coincide* on $\mathscr{U}$. For instance, functions $\sin(x)$ and $\frac{1}{2}\frac{\sin(2x)}{\cos(x)}$ are equivalent (coincide) on the whole $\mathbb{R}$. Since a_1b_2 and a_2b_1 are analytic on $\mathscr{U}$, if α_1 and α_2 coincide on some open and connected $\mathscr{U}^* \subseteq \mathscr{U}$, then they coincide on the whole $\mathscr{U}$; e.g., $\alpha_1(x) = \mathrm{e}^{-1/x^2}$ and $\alpha_2(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0, \end{cases}$ coincide on $(0, +\infty)$, but they differ on $(-\infty, 0)$: at the boundary point $x = 0$, they are not analytic but only smooth.

The *zeros* and the *poles* of a meromorphic function $\alpha = \frac{a}{b}$, with $a, b \in \mathscr{A}_n$ and b not identically equal to zero, are the zeros of a and b, respectively. If $\alpha \in \mathscr{K}_n$, then there exists an open and connected subset $\mathscr{U}^*$ of $\mathscr{U}$ such that α is analytic on $\mathscr{U}^*$. The notations $\alpha = 0$ or $\alpha(x) = 0$ (respectively, $\alpha \neq 0$ or $\alpha(x) \neq 0$), for a meromorphic function α, denote a function α that is (respectively, that is not) equal to zero for all $x \in \mathscr{U}$; note that $\alpha(x^o) = 0$ means that $\alpha(x)$ is equal to zero at $x = x^o$.

Two vector functions $\alpha_1(x), \alpha_2(x) \in \mathbb{R}^n$, with entries in $\mathscr{K}_n$, are *co-linear* over $\mathscr{K}_n$, if there exists an element a of $\mathscr{K}_n$ such that $\alpha_1 = a\alpha_2$; a set of vector functions $\alpha_1(x), \ldots, \alpha_m(x) \in \mathbb{R}^n$, with entries in $\mathscr{K}_n$, are *linear independent* over $\mathscr{K}_n$ if there exist no $a_1, \ldots, a_m \in \mathscr{K}_n$, with $a_i \neq 0$ for at least one index i, such that $\sum_{i=1}^m a_i\alpha_i = 0$; otherwise, they are *linearly dependent* over $\mathscr{K}_n$. A matrix A, with entries in $\mathscr{K}_n$, has *generic rank* m, if there exists an $m \times m$ minor $\bar{A}$ of A such that $\det(\bar{A}) \neq 0$, and all its minors $\hat{A}$ of dimension $p \times p$, with $p > m$, are such that $\det(\hat{A}) = 0$. If the vector functions $\alpha_1(x), \ldots, \alpha_m(x) \in \mathbb{R}^n$, with entries in $\mathscr{K}_n$, are linearly independent, then the $n \times m$ matrix $[\alpha_1 \ \ldots \ \alpha_m]$ has generic rank m.

Property 1.1 Given $\alpha_1, \alpha_2 \in \mathscr{K}_n$, $\alpha_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathscr{A}_n$, $b_i \neq 0$, then:

(1.1.1) $\alpha_1\alpha_2 = \frac{a_1a_2}{b_1b_2} \in \mathscr{K}_n$;
(1.1.2) $\alpha_1\alpha_2 = 0$ if and only if either $\alpha_1 = 0$ or $\alpha_2 = 0$;
(1.1.3) $\alpha_1 + \alpha_2 = \frac{a_1b_2 + a_2b_1}{b_1b_2} \in \mathscr{K}_n$;
(1.1.4) $\frac{\partial \alpha_i}{\partial x_j} = \frac{b_i \frac{\partial a_i}{\partial x_j} - a_i \frac{\partial b_i}{\partial x_j}}{b_i^2} \in \mathscr{K}_n$;
(1.1.5) the equation $\alpha_1 \xi = \alpha_2$ in the unknown ξ, with $\alpha_1 \neq 0$, has a unique solution in $\mathscr{K}_n$ given by $\xi = \frac{\alpha_2}{\alpha_1}$.

The properties above need not hold when functions are not meromorphic; as for Property (1.1.2), let $a_1(x) = \begin{cases} \mathrm{e}^{1/x_1^2}, & \text{if } x_1 \ge 0, \\ 0, & \text{if } x_1 < 0, \end{cases}$ and $a_2(x) = \begin{cases} 0, & \text{if } x_1 \ge 0, \\ \mathrm{e}^{1/x_1^2}, & \text{if } x_1 < 0, \end{cases}$ with $x \in \mathbb{R}^2$; clearly, such functions are smooth on the whole $\mathbb{R}$, are not identically equal to zero, but their product a_1a_2 is identically equal to zero; similarly, $\alpha_1 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$ is not identically equal to zero, but $a_2\alpha_1$ is identically equal to zero. Let $\alpha_1 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}$; clearly, there exists no function a such that $\alpha_2 = a\alpha_1$, but $\det([\alpha_1 \ \alpha_2])$ and $a_2\alpha_1 + a_1\alpha_2$ are identically equal to zero.

Let $x \in \mathbb{R}$; if β is the anti-derivative (or indefinite integral) of $\alpha \in \mathscr{K}_n$, i.e., $\beta(x) = \int \alpha(x)\,\mathrm{d}x$, then β need not be meromorphic on $\mathscr{U}$, but it is certainly analytic on some open and connected set $\mathscr{U}^* \subseteq \mathscr{U}$. For instance, $\alpha(x) = \frac{1}{x}$ is meromorphic on the whole $\mathbb{R}$, but its anti-derivative $\beta(x) = \ln(|x|)$ is not, being analytic only on the intervals $(-\infty, 0)$, $(0, +\infty)$.

1.3 Differential and Difference Equations

Consider two vector functions $f(x), F(x) \in \mathbb{R}^n$ and the associated continuous-time (respectively, discrete-time) systems described by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(x(t)\big), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{1.1a}$$

$$x(t+1) = F\big(x(t)\big), \quad x \in \mathbb{R}^n, \ t \in \mathbb{Z}, \tag{1.1b}$$

where $x = [x_1 \ \dots \ x_n]^\top$ is the *state vector*; symbol $\Delta h(t)$ stands either for $\frac{\mathrm{d}h(t)}{\mathrm{d}t}$ in the continuous-time case (if $t \in \mathbb{R}$) or for $h(t+1)$ in the discrete-time case (if $t \in \mathbb{Z}$), for any scalar or vector function h; $\mathbb{T} = \mathbb{R}$ in the continuous-time case and $\mathbb{T} = \mathbb{Z}$ in the discrete-time case; for the sake of simplicity, it is assumed that all functions are meromorphic on some open and connected set $\mathscr{U}$ of $\mathbb{R}^n$ and, therefore, that they are analytic on $\mathscr{U}^*$, with $\mathscr{U}^*$ being some open and connected set of $\mathscr{U}$; note that $\mathscr{U}$ need not contain the origin of $\mathbb{R}^n$ and vector functions f and F need not satisfy $f(0) = 0$ and $F(0) = 0$. If 0 must belong to $\mathscr{U}$ and equalities $f(0) = 0$ and $F(0) = 0$ must hold, this is explicitly assumed. Under the above assumptions, systems (1.1a) and (1.1b) have unique *maximal solutions* [119] $x(t) = \Phi_f(t, x_0)$, $t \in \mathbb{R}$, t sufficiently close to 0 to avoid finite escape times, and $x(t) = \Psi_F(t, x_0)$, $t \in \mathbb{Z}$, t sufficiently close to 0, respectively, from the initial condition $x_0 \in \mathscr{U}^*$ at time $t = 0$; Φ_f and Ψ_F are the continuous-time [7] and discrete-time *flows* (briefly, the *CT-flow* and *DT-flow*) associated with f and F, respectively. If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *flow* is used instead of CT- and DT-flows.

Definition 1.1 Some meromorphic functions $h_i(x) : \mathscr{U} \to \mathbb{R}$, $i = 1, \dots, m, m \le n$, are *functionally dependent* [102] if there exists a meromorphic function $F(z_1, \dots, z_m) : \mathbb{R}^m \to \mathbb{R}$, which is not identically equal to zero, and an open and connected set $\mathscr{U}^* \subseteq \mathscr{U}$ such that $F(h_1(x), \dots, h_m(x)) = 0$ for all $x \in \mathscr{U}^*$; otherwise, they are called *functionally independent* [102].

Note that, when meromorphic functions are considered, the functional dependence and the functional independence are the only two possible cases; this is not true, if the considered functions are, for instance, only smooth.

For the proof of the following theorem, which is omitted, see the Notes at the end of Chap. 2 of [102].

Theorem 1.1 *Some analytic functions $h_i(x) : \mathscr{U} \to \mathbb{R}$, $i = 1, \dots, m$, $m \le n$, are functionally independent if and only if, letting $h = [h_1 \ \dots \ h_m]^\top$, the Jacobian matrix $\frac{\partial h}{\partial x}$ of h has full rank over the field $\mathscr{K}_n$ of meromorphic functions, i.e., $\frac{\partial h}{\partial x}$ has full rank for all x in some open and connected set $\mathscr{U}^* \subseteq \mathscr{U}$.*

Example 1.3 Take $h_1(x) = x_1$, $h_2(x) = x_1x_2$. Since $\frac{\partial h(x)}{\partial x} = \begin{bmatrix} 1 & 0 \\ x_2 & x_1 \end{bmatrix}$ and $\det(\frac{\partial h(x)}{\partial x})$ $= x_1$ is not identically equal to zero, h_1 and h_2 are functionally independent; note that, for h_1 and h_2 to be functionally independent, $\frac{\partial h}{\partial x}$ need not have full rank for all $x \in \mathbb{R}^2$.

Example 1.4 Take $h_1(x) = \frac{x_1}{x_2}$, $h_2(x) = \frac{x_2}{x_1+x_2}$. Since $\det(\frac{\partial h(x)}{\partial x}) = 0$, h_1 and h_2 are functionally dependent; as a matter of fact, taking $F(z_1, z_2) = z_2 + z_1z_2 - 1$, one can verify that $F(h_1(x), h_2(x)) = 0$ for all admissible $x \in \mathbb{R}^2$.

Consider a vector function $g(x) \in \mathbb{R}^n$ and the associated continuous-time system (from now on, the dependencies on times t, τ are omitted, if not necessary):

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = g(x), \quad x \in \mathbb{R}^n,\ \tau \in \mathbb{R}. \tag{1.2}$$

Since it is assumed that g is meromorphic on $\mathscr{U}$, g is analytic on some $\mathscr{U}^*$; therefore, system (1.2) has a unique maximal solution $x(\tau) = \Phi_g(\tau, x_0)$, $\tau \in \mathbb{R}$, τ sufficiently close to 0, from the initial condition $x_0 \in \mathscr{U}^*$ at time $\tau = 0$ (Φ_g is the CT-flow associated with g).

The *directional derivative* $L_f h \in \mathbb{R}$ of a scalar function h by f is $L_f h := \frac{\partial h}{\partial x} f$, where $\frac{\partial h}{\partial x}$ is the gradient of h ($L_f h$ is often called the *Lie derivative*, as in [100]); the *directional derivative* $L_f g \in \mathbb{R}^n$ of g by f is the vector having $L_f g_i$ as ith entry, with g_i being the ith entry of g, i.e., $L_f g := \frac{\partial g}{\partial x} f$, where $\frac{\partial g}{\partial x}$ is the Jacobian matrix of g; the *CT-Lie bracket* $[f, g] \in \mathbb{R}^n$ of f and g is [23, 69, 76, 100, 107]

$$[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = L_f g - L_g f,$$

and the *DT-Lie bracket* $\lfloor F, g \rfloor \in \mathbb{R}^n$ of F and g is [93]

$$\lfloor F, g \rfloor := g(F) - \frac{\partial F}{\partial x} g = g \circ F - L_g F,$$

where $\frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial F}{\partial x}$ are the Jacobian matrices of g, f and F, respectively, and $\circ$ denotes function composition. If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *Lie bracket* is used instead of CT and DT-Lie brackets.

For a given $\mathscr{U}$, consider the set $\mathscr{A}_n$ of all analytic functions $\alpha(x) : \mathscr{U} \to \mathbb{R}$. A derivation on $\mathscr{A}_n$ is defined formally below; for more details, the reader is referred to [40].

Definition 1.2 Consider a function $D(\alpha) : \mathscr{A}_n \to \mathscr{A}_n$. If D is linear, $D(a_1\alpha_1 + a_2\alpha_2) = a_1 D(\alpha_1) + a_2 D(\alpha_2)$ for all real constants $a_1, a_2 \in \mathbb{R}$ and for all $\alpha_1, \alpha_2 \in \mathscr{A}_n$, then D is called a *derivation* on $\mathscr{A}_n$ if it satisfies the *Leibniz rule*

$$D(\alpha_1\alpha_2) = D(\alpha_1)\alpha_2 + D(\alpha_2)\alpha_1, \quad \forall \alpha_1, \alpha_2 \in \mathscr{A}_n.$$

Theorem 1.2 *Consider a vector function $f(x) \in \mathbb{R}^n$, with entries $f_i \in \mathscr{A}_n$; then, the function D defined by $D(\alpha) = L_f\alpha,\ \forall\alpha \in \mathscr{A}_n$, is a derivation on $\mathscr{A}_n$.*

Proof The proof of both the linearity and the Leibniz rule are direct:

$$\begin{aligned} D(a_1\alpha_1 + a_2\alpha_2) &= L_f(a_1\alpha_1 + a_2\alpha_2) = a_1\frac{\partial\alpha_1}{\partial x}f + a_2\frac{\partial\alpha_2}{\partial x}f \\ &= a_1 L_f\alpha_1 + a_2 L_f\alpha_2 \end{aligned}$$

and

$$\begin{aligned} D(\alpha_1\alpha_2) = L_f(\alpha_1\alpha_2) &= \frac{\partial\alpha_1\alpha_2}{\partial x}f = \alpha_2\frac{\partial\alpha_1}{\partial x}f + \alpha_1\frac{\partial\alpha_2}{\partial x}f = \alpha_2 L_f\alpha_1 + \alpha_1 L_f\alpha_2 \\ &= D(\alpha_1)\alpha_2 + D(\alpha_2)\alpha_1. \end{aligned}$$

□

Theorem 1.3 *Let D be a derivation on $\mathscr{A}_n$. Then, there exists a vector function $f(x) \in \mathbb{R}^n$, with entries $f_i \in \mathscr{A}_n$, such that $D(\alpha) = L_f\alpha,\ \forall\alpha \in \mathscr{A}_n$.*

Proof Let $f_i(x) = D(x_i)$, $i = 1,\dots,n$, and $f = [f_1\ \dots\ f_n]^\top$; then, by Theorem 1.2, define the derivation $\hat{D}$ as $\hat{D}(\alpha) := L_f\alpha$, $\forall\alpha \in \mathscr{A}_n$. Clearly, also $\tilde{D}$ defined as $\tilde{D}(\alpha) := D(\alpha) - \hat{D}(\alpha)$, $\forall\alpha \in \mathscr{A}_n$, is a derivation. Now, it is shown that $\tilde{D}(p_m) = 0$, for any polynomial p_m of degree $m \geq 0$. First, it is shown than $\tilde{D}(p_m) = 0$ holds for $m = 0$ and $m = 1$. Then, under the induction assumption that $\tilde{D}(p_m) = 0$, for any polynomial p_m of degree m, it is shown that $\tilde{D}(p_{m+1}) = 0$, for any polynomial p_{m+1} of degree $m+1$. Consider the function α identically equal to 1; then, since $\alpha = \alpha\alpha$, by applying $\tilde{D}$ to such an equality, one concludes that

$$\tilde{D}(\alpha) = 2\alpha\tilde{D}(\alpha) = 2\tilde{D}(\alpha),$$

which shows that $\tilde{D}(\alpha) = \tilde{D}(1) = 0$; moreover, $\tilde{D}(c) = c\tilde{D}(1) = 0$, for any constant c. Since, by definition, $\hat{D}(x_i) = D(x_i)$, one has $\tilde{D}(x_i) = 0$, $i = 1,\dots,n$. Hence, $\tilde{D}(p_1) = 0$ for any polynomial p_1 of degree 1:

$$\tilde{D}(p_1) = \tilde{D}\left(c_0 + \sum_{i=1}^{n} c_i x_i\right) = c_0\tilde{D}(1) + \sum_{i=1}^{n} c_i\tilde{D}(x_i) = 0.$$

Any polynomial $p_{m+1} \in \mathscr{A}_n$ of degree $m+1$ can be rewritten as

$$p_{m+1}(x) - p_{m+1}(0) = \sum_{i=1}^{n} \beta_i(x)x_i,$$

where the β_i's are polynomials of degree lower than $m+1$; then, applying $\tilde{D}$ to such an equality, one concludes that

$$\tilde{D}(p_{m+1}) = \sum_{i=1}^{n}\big(x_i\tilde{D}(\beta_i) + \beta_i\tilde{D}(x_i)\big) = 0.$$

The proof is completed because the only analytic function whose Taylor expansion is zero is the function α that is identically equal to zero. □

Remark 1.1 For any scalar or vector function $\alpha(x)$ and for any pair of vector functions $f(x), g(x) \in \mathbb{R}^n$, with entries in $\mathscr{A}_n$, the following relation holds:

$$L_{[f,g]}\alpha = L_f L_g \alpha - L_g L_f \alpha,$$

which can be written in terms of operators as

$$L_{[f,g]} = L_f L_g - L_g L_f. \tag{1.3}$$

Such a relation can be proven by considering the operator $D(\alpha) := L_f L_g \alpha - L_g L_f \alpha$. Clearly, $D(a_1\alpha_1 + a_2\alpha_2) = a_1 D(\alpha_1) + a_2 D(\alpha_2)$, for all real constants $a_1, a_2 \in \mathbb{R}$ and for all $\alpha_1, \alpha_2 \in \mathscr{A}_n$, and

$$\begin{aligned}
D(\alpha_1\alpha_2) &= L_f L_g(\alpha_1\alpha_2) - L_g L_f(\alpha_1\alpha_2)\\
&= L_f(\alpha_1 L_g\alpha_2 + \alpha_2 L_g\alpha_1) - L_g(\alpha_1 L_f\alpha_2 + \alpha_2 L_f\alpha_1)\\
&= \alpha_1 L_f L_g\alpha_2 + \alpha_2 L_f L_g\alpha_1 + (L_f\alpha_1)(L_g\alpha_2) + (L_f\alpha_2)(L_g\alpha_1)\\
&\quad - \alpha_1 L_g L_f\alpha_2 - \alpha_2 L_g L_f\alpha_1 - (L_g\alpha_1)(L_f\alpha_2) - (L_g\alpha_2)(L_f\alpha_1)\\
&= \alpha_1(L_f L_g\alpha_2 - L_g L_f\alpha_2) + \alpha_2(L_f L_g\alpha_1 - L_g L_f\alpha_1)\\
&= \alpha_1 D(\alpha_2) + \alpha_2 D(\alpha_1);
\end{aligned}$$

hence, $D(\alpha)$ is a derivation and by Theorem 1.3 there exists $h(x) \in \mathbb{R}^n$, with entries in $\mathscr{A}_n$, such that $L_h\alpha = D(\alpha) = L_f L_g\alpha - L_g L_f\alpha$, for all $\alpha \in \mathscr{A}_n$. By the proof of Theorem 1.3, the entry h_i of h is given by

$$h_i(x) = D(x_i) = L_f L_g x_i - L_g L_f x_i = L_f g_i - L_g f_i,$$

which coincides with the ith entry of $[f, g]$. Therefore, $h = [f, g]$, thus showing (1.3).

The directional derivative $L_f h$ and the function composition $h \circ F$ are basic operations in the book. They have analogous meaning when applied, respectively, to continuous-time and discrete-time systems. Two of their properties are compared below (where $h_1(x), h_2(x) \in \mathbb{R}$):

$$\begin{aligned}
&L_f\left(\frac{h_1}{h_2}\right) = \frac{h_2 L_f h_1 - h_1 L_f h_2}{h_2^2}, &\quad &\left(\frac{h_1}{h_2}\right)\circ F = \frac{h_1\circ F}{h_2\circ F},\\
&L_f(h_1 h_2) = h_1 L_f h_2 + h_2 L_f h_1, &\quad &(h_1 h_2)\circ F = (h_1\circ F)(h_2\circ F).
\end{aligned}$$

Property 1.2 The CT-Lie bracket enjoys the following properties, with $f(x)$, $g(x)$, $h(x) \in \mathbb{R}^n$ [23, 69, 76, 100, 107]:

(1.2.1) $[f, g] = -[g, f]$ (*skew-symmetry*);
(1.2.2) $[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$ and $[h, \alpha f + \beta g] = \alpha[h, f] + \beta[h, g]$, with $\alpha, \beta \in \mathbb{R}$ being constants (*bi-linearity*);
(1.2.3) $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$ (*the Jacobi identity*).

Note that, in general, Properties 1.2 need not hold for the DT-Lie bracket (they hold for the DT-Lie bracket when f, g and h are linear functions of x).

The proof of Properties (1.2.1) and (1.2.2) can be done by direct substitution. As for the proof of the Jacobi identity (1.2.3), note that $L_f a = 0$, for any scalar function $a(x) \in \mathbb{R}$, if and only if $f = 0$, i.e., if and only if $L_f = 0$. Then, compute

$$\begin{aligned}
L_{[f,[g,h]]} &= L_f L_{[g,h]} - L_{[g,h]} L_f = L_f(L_g L_h - L_h L_g) - (L_g L_h - L_h L_g) L_f \\
&= L_f L_g L_h - L_f L_h L_g - L_g L_h L_f + L_h L_g L_f, \\
L_{[g,[h,f]]} &= L_g L_{[h,f]} - L_{[h,f]} L_g = L_g(L_h L_f - L_f L_h) - (L_h L_f - L_f L_h) L_g \\
&= L_g L_h L_f - L_g L_f L_h - L_h L_f L_g + L_f L_h L_g, \\
L_{[h,[f,g]]} &= L_h L_{[f,g]} - L_{[f,g]} L_h = L_h(L_f L_g - L_g L_f) - (L_f L_g - L_g L_f) L_h \\
&= L_h L_f L_g - L_h L_g L_f - L_f L_g L_h + L_g L_f L_h.
\end{aligned}$$

Sum $L_{[f,[g,h]]+[g,[h,f]]+[h,[f,g]]} = L_{[f,[g,h]]} + L_{[g,[h,f]]} + L_{[h,[f,g]]}$ is composed by 12 terms, each one appearing in the sum twice, with opposite signs, whence the sum is equal to 0.

Another useful property of the CT-Lie bracket is that, for any $\alpha(x) \in \mathbb{R}$ and $f(x), g(x) \in \mathbb{R}^n$, one has

$$[\alpha f, g] = \alpha[f, g] - (L_g \alpha) f. \tag{1.4}$$

Remark 1.2 A map $y = \varphi(x)$, with $\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ being analytic on $\mathscr{U}^*$, is a *diffeomorphism* on a neighborhood $\mathscr{B}_{x^o}$ of $x^o \in \mathscr{U}^*$ if $\det(\frac{\partial \varphi(x)}{\partial x}|_{x=x^o}) \neq 0$; if $\mathscr{U}^* = \mathbb{R}^n$, then $y = \varphi(x)$ is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ (briefly, a *global diffeomorphism* [122]) if $\det(\frac{\partial \varphi(x)}{\partial x}) \neq 0$, $\forall x \in \mathbb{R}^n$, and $y = \varphi(x)$ is a *proper map*, i.e., if the inverse image of any compact set is compact. Given a diffeomorphism $y = \varphi(x)$, a scalar function $h(x) \in \mathbb{R}$ and a vector function $f(x) \in \mathbb{R}^n$ if $\mathbb{T} = \mathbb{R}$ (respectively, $F(x) \in \mathbb{R}^n$ if $\mathbb{T} = \mathbb{Z}$), the *push-forward* of h by φ and the *push-forward* of f (respectively, F) by φ are (see, e.g., [86, 107]):

$$\begin{aligned}
\varphi_* h(y) &= h \circ \varphi^{-1}(y), \\
\varphi_* f(y) &= \left(\frac{\partial \varphi}{\partial x} f\right) \circ \varphi^{-1}(y), \quad \text{if } \mathbb{T} = \mathbb{R}, \\
\varphi_* F(y) &= \varphi \circ F \circ \varphi^{-1}(y), \quad \text{if } \mathbb{T} = \mathbb{Z}.
\end{aligned}$$

Given a scalar function $\tilde{h}(y) \in \mathbb{R}$ and a vector function $\tilde{f}(y) \in \mathbb{R}^n$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\tilde{F}(y) \in \mathbb{R}^n$ if $\mathbb{T} = \mathbb{Z}$), the *pull-back* of $\tilde{h}$ by φ and the *pull-back* of $\tilde{f}$

(respectively, $\tilde{F}$) by φ are

$$\varphi^* \tilde{h}(x) = \varphi_*^{-1} \tilde{h}(x) = \tilde{h} \circ \varphi(x),$$

$$\varphi^* \tilde{f}(x) = \varphi_*^{-1} \tilde{f}(x) = \left(\frac{\partial \varphi^{-1}}{\partial y} \tilde{f} \right) \circ \varphi(x), \quad \text{if } \mathbb{T} = \mathbb{R},$$

$$\varphi^* \tilde{F}(x) = \varphi_*^{-1} \tilde{F}(x) = \varphi^{-1} \circ \tilde{F} \circ \varphi(x), \quad \text{if } \mathbb{T} = \mathbb{Z}.$$

If no confusion can arise, the shorter notations $\tilde{h} = \varphi_* h$, $\tilde{f} = \varphi_* f$, $\tilde{F} = \varphi_* F$, $h = \varphi^* \tilde{h}$, $f = \varphi^* \tilde{f}$ and $F = \varphi^* \tilde{F}$ are used.

The following theorem shows how the directional derivative and the CT-Lie bracket behave under the action of a diffeomorphism [100, 107].

Theorem 1.4 *Let $f(x), g(x) \in \mathbb{R}^n$ and $\tilde{a}(y) \in \mathbb{R}$. Let $y = \varphi(x)$ be a diffeomorphism with inverse $x = \varphi^{-1}(y)$. Then,*

(1.4.1) $L_{\varphi_* f} \tilde{a} = \varphi_* L_f (\varphi^* \tilde{a}) = (L_f(\tilde{a} \circ \varphi)) \circ \varphi^{-1}$;
(1.4.2) $[\varphi_* f, \varphi_* g] = \varphi_* [f, g] = (\frac{\partial \varphi}{\partial x} [f, g]) \circ \varphi^{-1}$.

Proof The proof of Statement (1.4.1) of the theorem can be obtained by the following equalities:

$$L_{\varphi_* f} \tilde{a} = \frac{\partial \tilde{a}}{\partial y} \varphi_* f = \frac{\partial \tilde{a}}{\partial y} \left(\frac{\partial \varphi}{\partial x} f \right) \circ \varphi^{-1},$$

$$\big(L_f (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} = \left(\left(\frac{\partial \tilde{a}}{\partial y} \circ \varphi \right) \left(\frac{\partial \varphi}{\partial x} f \right) \right) \circ \varphi^{-1} = \frac{\partial \tilde{a}}{\partial y} \left(\frac{\partial \varphi}{\partial x} f \right) \circ \varphi^{-1}.$$

Statement (1.4.2) of the theorem is equivalent to $L_{[\varphi_* f, \varphi_* g]} \tilde{a} = L_{(\frac{\partial \varphi}{\partial x}[f,g]) \circ \varphi^{-1}} \tilde{a}$, for any $\tilde{a}(y) \in \mathbb{R}$. Hence, by (1.3), a repeated application of Statement (1.4.1) of the theorem yields:

$$\begin{aligned} L_{[\varphi_* f, \varphi_* g]} \tilde{a} &= L_{\varphi_* f} L_{\varphi_* g} \tilde{a} - L_{\varphi_* g} L_{\varphi_* f} \tilde{a} \\ &= L_{\varphi_* f} \big(\big(L_g (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} \big) - L_{\varphi_* g} \big(\big(L_f (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} \big) \\ &= \big(L_f L_g (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} - \big(L_g L_f (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} \\ &= \big(L_f L_g (\tilde{a} \circ \varphi) - L_g L_f (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} = \big(L_{[f,g]} (\tilde{a} \circ \varphi) \big) \circ \varphi^{-1} \\ &= L_{(\frac{\partial \varphi}{\partial x}[f,g]) \circ \varphi^{-1}} \tilde{a}. \end{aligned}$$

□

Since $L_f(\frac{\partial \varphi}{\partial x} g)$ is not, in general, equal to $\frac{\partial \varphi}{\partial x} L_f g$, then $L_{\varphi_* f}(\varphi_* g) \neq \varphi_* L_f g$, in general, namely the directional derivative of a vector function g along f is not invariant to diffeomorphisms, although by Statement (1.4.2) of Theorem 1.4, one concludes that

$$L_{\varphi_* f}(\varphi_* g) - L_{\varphi_* g}(\varphi_* f) = \varphi_* L_f g - \varphi_* L_g f.$$

Statement (1.4.2) of Theorem 1.4 is referred to as the *invariance of the CT-Lie bracket to diffeomorphisms*.

Definition 1.3 Given $g(x) \in \mathbb{R}^n$, the *continuous-time centralizer* $\mathscr{C}_C(g)$ (respectively, the *discrete-time centralizer* $\mathscr{C}_D(g)$) of g is the set of all $f(x) \in \mathbb{R}^n$ such that $[f, g] = -[g, f] = 0$ (respectively, of all $F \in \mathbb{R}^n$ such that $\lfloor F, g \rfloor = 0$). Given $B \in \mathbb{R}^{n\times n}$, the set of all Ax, $A \in \mathbb{R}^{n\times n}$, such that $[Ax, Bx] = \lfloor Ax, Bx \rfloor = 0$ is denoted by $\mathscr{L}_c(Bx)$ and it is called the *linear centralizer* of Bx.

By the skew-symmetry of the CT-Lie bracket (see Property (1.2.1)),

$$\begin{aligned} f \in \mathscr{C}_C(g) \quad &\Longleftrightarrow \quad g \in \mathscr{C}_C(f), \\ Ax \in \mathscr{L}_c(Bx) \quad &\Longleftrightarrow \quad Bx \in \mathscr{L}_c(Ax) \end{aligned}$$

(in general, however, $\mathscr{C}_C(g) \neq \mathscr{C}_C(f)$ and $\mathscr{L}_c(Bx) \neq \mathscr{L}_c(Ax)$).

If $f(x) = Ax$, $F(x) = Ax$ and $g(x) = Bx$, for some $A, B \in \mathbb{R}^{n\times n}$, then $[f(x), g(x)] = \lfloor F(x), g(x) \rfloor = (BA - AB)x$ and, therefore, $\mathscr{L}_c(Bx) \subset \mathscr{C}_C(Bx)$ and $\mathscr{L}_c(Bx) \subset \mathscr{C}_D(Bx)$.

One of the key concepts in this book is that of first integral, which is widely used in the continuous-time case (see, e.g., [56]), and has a natural generalization for the discrete-time case (see, e.g., [82]).

Definition 1.4 A *first integral* of the continuous-time system (1.1a) (respectively, of the discrete-time system (1.1b)) is a scalar function $I(x) : \mathscr{U}^* \to \mathbb{R}$, analytic on $\mathscr{U}^*$, such that $L_f I(x) = 0$ (respectively, $I(F(x)) = I \circ F(x) = I(x)$), $\forall x \in \mathscr{U}^*$, with $\mathscr{U}^*$ being an open and connected subset of $\mathscr{U}$; if I is a constant, then the first integral is said to be *trivial, non-trivial* otherwise. Note that $I(x)$ need not be defined on the whole $\mathscr{U}$.

The definition of first integral given in Definition 1.4 is strictly correlated with the definition of generalized first integral given in [84, 85].

Clearly, $L_f I(x) = 0$ is equivalent to $I \circ \Phi_f(t, x) = I(x)$ and $I \circ F(x) = I(x)$ is equivalent to $I \circ \Psi_F(t, x) = I(x)$, for all admissible $(t, x) \in \mathbb{R} \times \mathscr{U}$. For brevity, a first integral of system (1.1a) (respectively, (1.1b)) is also called a *CT-first integral* associated with f (a *DT-first integral* associated with F). Symbol $\mathscr{I}_C(f)$ (respectively, $\mathscr{I}_D(F)$) denotes the set of all first integrals of system (1.1a) (respectively, system (1.1b)). If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *first integral* is used instead of CT- and DT-first integrals.

Remark 1.3 In the continuous-time case, assume $f \neq [0 \; \dots \; 0]^\top$; given $n-1$ functionally independent CT-first integrals $I_1, \dots, I_{n-1} \in \mathscr{I}_C(f)$, any $I \in \mathscr{I}_C(f)$ can be expressed as $I = C(I_1, \dots, I_{n-1})$, with C being an arbitrary function. Note that there cannot be n functionally independent first integrals associated with $f \neq 0$; as a matter of fact, condition $L_f I = \frac{\partial I}{\partial x} f = 0$, with $I(x) \in \mathbb{R}^n$, implies that $\frac{\partial I}{\partial x}$ has generic rank less than n, whence the entries of I cannot be functionally independent. In the discrete-time case, set $\mathscr{I}_D(F)$ can be generated by m functionally independent first integrals $I_1, \dots, I_m \in \mathscr{I}_D(F)$, where, under the respective assumption $F \neq [1\ \dots\ 1]^\top$, m need not be equal to $n-1$; any $I \in \mathscr{I}_D(F)$ can be expressed as $I = C(I_1, \dots, I_m)$, with C being an arbitrary function. This, in particular, implies that, except for the case $f = 0$ (any I is a CT-first integral associated with $f = 0$), any scalar continuous-time system does not admit first integrals, whereas, under the assumption $F \neq 1$ (any I is a DT-first integral associated with $F = 1$), a scalar discrete-time system either admits no first integral or admits infinite functionally dependent first integrals. In the rest of the book, the two trivial cases $f = [0\ \dots\ 0]^\top$ if $\mathbb{T} = \mathbb{R}$ and $F = [1\ \dots\ 1]^\top$ if $\mathbb{T} = \mathbb{Z}$ are excluded.

Example 1.5 For any time-invariant mechanical system (subject to conservative forces, only), a first integral is given by the total energy I, which is defined as the sum of the kinetic and potential energies. As an example, $I(x) = \frac{1}{2}(m_1 x_3^2 + m_2 x_4^2) + \frac{1}{4}k(x_1 - x_2)^4$ is a first integral of the nonlinear mechanical system constituted by two point masses $m_1, m_2 > 0$, moving on a straight line and connected by a nonlinear spring characterized by an elastic energy $\frac{1}{4}k\xi^4$ corresponding to deformation ξ, whose equations of motion are given by (see Sect. 5.1)

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_3, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_4, \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\frac{k}{m_1}(x_1 - x_2)^3, \\
\frac{\mathrm{d}x_4}{\mathrm{d}t} &= \frac{k}{m_2}(x_1 - x_2)^3;
\end{aligned}$$

to be more precise,

$$L_f I(x) = \begin{bmatrix} k(x_1 - x_2)^3 & -k(x_1 - x_2)^3 & m_1 x_3 & m_2 x_4 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ -\frac{k}{m_1}(x_1 - x_2)^3 \\ \frac{k}{m_2}(x_1 - x_2)^3 \end{bmatrix} = 0.$$

As for the discrete-time Möbius-type system described by $x(t+1) = F(x(t))$, with $F(x) = \frac{a+bx}{-b+cx}$ and $a, b, c \in \mathbb{R}$, a first integral is given by $I(x) = (\frac{a+2bx-cx^2}{-b+cx})^2$, since

$$I \circ F(x) = \left(\frac{a + 2bF - cF^2}{-b + cF}\right)^2 \Bigg|_{F = \frac{a+bx}{-b+cx}} = \left(\frac{a + 2bx - cx^2}{-b + cx}\right)^2 = I(x).$$

Denote by Φ either one of the CT-flows Φ_f and Φ_g; then, the following relations hold whenever defined:

$$\Phi(0,x) = x, \tag{1.5a}$$

$$\Phi\big(t_1, \Phi(t_2,x)\big) = \Phi(t_1+t_2,x), \tag{1.5b}$$

$$\Phi\big(-t, \Phi(t,x)\big) = x. \tag{1.5c}$$

Thanks to the above properties, both Φ_f and Φ_g define a local *one-parameter group of transformations* [102], $y = \Phi(t,x)$, and f and g are the *infinitesimal generators* of the respective group: in particular, $x = \Phi(-t,y)$ is the inverse of $y = \Phi(t,x)$, for all admissible t, x, y. Given a local one-parameter group of transformations $\Phi(t,x)$, i.e., a vector function $\Phi(t,x) \in \mathbb{R}^n$ satisfying (1.5a)–(1.5c), there exists a vector function $f(x) \in \mathbb{R}^n$ such that $\Phi(t,x) = \Phi_f(t,x)$ for all admissible (t,x) in a neighborhood of the origin of $\mathbb{R} \times \mathbb{R}^n$; in particular, since $\frac{\partial \Phi_f(t,x)}{\partial t}|_{t=0} = f(\Phi_f(t,x))|_{t=0}$ and $\Phi_f(0,x) = x$, taking into account the uniqueness of $\Phi_f(t,x)$, the infinitesimal generator f of $\Phi(t,x)$ can be easily computed by $f(x) = \frac{\partial \Phi(t,x)}{\partial t}|_{t=0}$ (see [102]). As a matter of fact, letting $f(x) = \frac{\partial \Phi(t,x)}{\partial t}|_{t=0}$, one can compute

$$\begin{aligned}\frac{\partial \Phi(t,x)}{\partial t} &= \lim_{T\to 0^+} \frac{\Phi(t+T,x) - \Phi(t,x)}{T} = \lim_{T\to 0^+} \frac{\Phi(T,\Phi(t,x)) - \Phi(t,x)}{T} \\ &= \left(\lim_{T\to 0^+} \frac{\Phi(T,x) - \Phi(0,x)}{T}\right) \circ \Phi(t,x) = f(x) \circ \Phi(t,x),\end{aligned}$$

which, integrated from the initial condition $\Phi(0,x) = x$, yields $\Phi(t,x) = \Phi_f(t,x)$.

Finally, note that (1.5a), (1.5b) imply (1.5c), for small $|t|$.

Example 1.6 As an example of a local one-parameter group of transformations, take $\Phi(t,x) = [\frac{x_1}{1-tx_1} \ \ \frac{x_2}{1-tx_1}]^\top$ (see, Example 1.27(c) of [102]). Clearly, (1.5a) holds; (1.5b) can be checked by direct substitution

$$\begin{aligned}\Phi\big(t_1, \Phi(t_2,x)\big) &= \begin{bmatrix} \frac{\frac{x_1}{1-t_2x_1}}{1-t_1\frac{x_1}{1-t_2x_1}} & \frac{\frac{x_2}{1-t_2x_1}}{1-t_1\frac{x_1}{1-t_2x_1}} \end{bmatrix}^\top \\ &= \begin{bmatrix} \frac{x_1}{1-(t_1+t_2)x_1} & \frac{x_2}{1-(t_1+t_2)x_1} \end{bmatrix}^\top = \Phi(t_1+t_2,x).\end{aligned}$$

The infinitesimal generator of $\Phi(t,x)$ is

$$f(x) = \left.\frac{\partial \Phi(t,x)}{\partial t}\right|_{x=0} = \left.\begin{bmatrix} \frac{x_1^2}{(1-tx_1)^2} & \frac{x_1x_2}{(1-tx_1)^2} \end{bmatrix}^\top\right|_{t=0} = \begin{bmatrix} x_1^2 & x_1x_2 \end{bmatrix}^\top.$$

1.4 Differential Forms

In this section, some facts about the integration of differential forms are recalled; the reader interested in a more extensive treatment is referred to [35, 47, 107, 116].

If $I(x) \in \mathbb{R}$, then:

(1) the *differential* of I is $\mathrm{d}I := \frac{\partial I}{\partial x}\,\mathrm{d}x$, where $\mathrm{d}x = [\mathrm{d}x_1\ \dots\ \mathrm{d}x_n]^\top$;
(2) given a differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, with $f(x) \in \mathbb{R}^n$, the *directional differential* of I along the solutions of such a system is $\mathrm{d}I = (\frac{\partial I}{\partial x} f)\,\mathrm{d}t$, with $\frac{\partial I}{\partial x} f = L_f I$ being called the *value* of the differential of I on f (it coincides with the directional derivative of I by f).

A *one-form* is $\alpha = a^\top \mathrm{d}x = \sum_{i=1}^n a_i\,\mathrm{d}x_i$, with $a(x) \in \mathbb{R}^n$ being a vector function. The *value* of α on f is $\alpha(f) := a^\top f$; by abuse of notation, the row vector function $a^\top$ is also called a one-form. Let $\mathscr{F}_1$ be the set of all one-forms. Clearly, $\mathscr{F}_1$ is a vector space of dimension n over the field $\mathscr{K}_n$ of meromorphic functions.

A one-form α is *locally exact* [47, p. 67], if there exists a scalar function I such that $\mathrm{d}I = \alpha$ in $\mathscr{U}^*$, with $\mathscr{U}^*$ being some open and connected subset of $\mathscr{U}$ (the adverb *locally* is omitted in the following); such a scalar function is called a *first integral* of the one-form and is (differently from what happens for continuous-time and discrete-time systems, as discussed in Remark 1.3) locally unique, apart from the sum of an arbitrary constant. If $\mathrm{d}I = \alpha$ on the whole $\mathbb{R}^n$, then I is a global first integral of α.

Example 1.7 Consider the one-form $\alpha = (\frac{x_2}{x_1^2+x_2^2})\,\mathrm{d}x_1 + (-\frac{x_1}{x_1^2+x_2^2})\,\mathrm{d}x_2$. In a sufficiently small neighborhood of any point (x_1, x_2) such that $x_2 \neq 0$, a first integral of the one-form α is $I_1(x) = \arctan(\frac{x_1}{x_2})$; in a sufficiently small neighborhood of any point (x_1, x_2) such that $x_1 \neq 0$, a first integral of the one-form α is $I_2(x) = \arctan(-\frac{x_2}{x_1})$. Note that there exists no function I such that $\mathrm{d}I = \alpha$ holds on the whole $\mathbb{R}^2 - \{0\}$. However, it is worth pointing out that I_1 and I_2 are not functionally independent, since $I_1 = -\arctan(\frac{1}{\tan(I_2)})$.

The *wedge* product of two one-forms $\alpha = \sum_{i=1}^n a_i\,\mathrm{d}x_i$ and $\beta = \sum_{j=1}^n b_j\,\mathrm{d}x_j$, $a_i, b_j \in \mathscr{K}_n$, is denoted by $\alpha \wedge \beta$ and is defined by

$$\alpha \wedge \beta := \sum_{i=1}^n \sum_{j=1}^n (a_i b_j)\,\mathrm{d}x_i \wedge \mathrm{d}x_j, \tag{1.6}$$

where the wedge product $\wedge$ satisfies the following property (*skew-symmetry*):

$$\begin{cases} \mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i, & \text{if } i \neq j, \\ \mathrm{d}x_i \wedge \mathrm{d}x_j = 0, & \text{if } i = j. \end{cases}$$

By the skew-symmetry, summation (1.6) can be rewritten as

$$\alpha \wedge \beta = \sum_{i=1}^n \sum_{j=i+1}^n (a_i b_j - a_j b_i)\,\mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{1.7}$$

A summation such as (1.7) is called a *two-form*. A two-form is the formal sum

$$\gamma = \sum_{i=1}^{n} \sum_{j=i+1}^{n} c_{i,j}\, \mathrm{d}x_i \wedge \mathrm{d}x_j, \quad c_{i,j} \in \mathscr{K}_n; \tag{1.8}$$

let $\mathscr{F}_2$ be the set of all two-forms (1.8), which is a vector space of dimension $\frac{n(n-1)}{2}$ over the field $\mathscr{K}_n$ of meromorphic functions. For any two-form γ, there always exist two one-forms α, β such that $\gamma = \alpha \wedge \beta$.

Property 1.3 The elements of $\mathscr{F}_2$ satisfy the following properties ($\alpha, \alpha_1, \alpha_2, \alpha_3 \in \mathscr{F}_1$ and $a_1, a_2, a_3 \in \mathscr{K}_n$):

(1.3.1) $(a_1\alpha_1 + a_2\alpha_2) \wedge \alpha_3 = a_1\alpha_1 \wedge \alpha_3 + a_2\alpha_2 \wedge \alpha_3$ and $\alpha_1 \wedge (a_2\alpha_2 + a_3\alpha_3) = a_2\alpha_1 \wedge \alpha_2 + a_3\alpha_1 \wedge \alpha_3$ (*bi-linearity*);
(1.3.2) $\alpha \wedge \alpha = 0$ and $\alpha_1 \wedge \alpha_2 = -\alpha_2 \wedge \alpha_1$ (*skew-symmetry*).

Example 1.8 In $\mathbb{R}^2$, one finds that

$$\begin{aligned}\alpha \wedge \beta &= (a_1b_1)\, \mathrm{d}x_1 \wedge \mathrm{d}x_1 + (a_1b_2)\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 + (a_2b_1)\, \mathrm{d}x_2 \wedge \mathrm{d}x_1 \\ &\quad + (a_2b_2)\, \mathrm{d}x_2 \wedge \mathrm{d}x_2 \\ &= (a_1b_2 - a_2b_1)\, \mathrm{d}x_1 \wedge \mathrm{d}x_2.\end{aligned}$$

In $\mathbb{R}^3$, one finds that

$$\begin{aligned}\alpha \wedge \beta &= (a_1b_2 - a_2b_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (a_1b_3 - a_3b_1)\, \mathrm{d}x_1 \wedge \mathrm{d}x_3 \\ &\quad + (a_2b_3 - a_3b_2)\, \mathrm{d}x_2 \wedge \mathrm{d}x_3.\end{aligned}$$

In general, given some one-forms α, β, γ, $\delta, \ldots, \alpha \wedge \beta \wedge \gamma$ is a three-form, $\alpha \wedge \beta \wedge \gamma \wedge \delta$ is a four-form and so on, with the wedge product being associative.

A p-form is the formal summation

$$\gamma = \sum_{i_1=1}^{n} \sum_{i_2=i_1+1}^{n} \cdots \sum_{i_p=i_{p-1}+1}^{n} c_{i_1,i_2,\ldots,i_p}\, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_p}, \quad c_{i_1,i_2,\ldots,i_p} \in \mathscr{K}_n; \tag{1.9}$$

let $\mathscr{F}_p$ be the set of all p-forms (1.9), which is a vector space of dimension $\binom{n}{p}$ over the field $\mathscr{K}_n$ of meromorphic functions. For any p-form γ, there always exists p one-forms $\alpha_1, \ldots, \alpha_p$ such that $\gamma = \alpha_1 \wedge \cdots \wedge \alpha_p$.

Property 1.4 The elements of $\mathscr{F}_p$ satisfy the following properties ($\beta_i, \alpha_i \in \mathscr{F}_1$ and $b_i \in \mathscr{K}_n$):

(1.4.1) $(b_1\beta_1 + b_2\beta_2) \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = b_1\beta_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p + b_2\beta_2 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$, and any other similar property obtained by substituting any α_i in $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$ with $b_1\beta_1 + b_2\beta_2$;

(1.4.2) $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$;
(1.4.3) $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$ changes sign if any two α_i, α_j, $i \neq j$, are interchanged.

The *derivative* $\mathrm{d}\alpha$ of a one-form $\alpha = \sum_{i=1}^n a_i\,\mathrm{d}x_i$ is a two-form defined by

$$\mathrm{d}\alpha := \sum_{i=1}^n \mathrm{d}a_i \wedge \mathrm{d}x_i,$$

the derivative $\mathrm{d}\gamma$ of a two-form $\gamma = \sum_{i=1}^n \sum_{j=i+1}^n c_{i,j}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ is a three-form defined by

$$\mathrm{d}\gamma := \sum_{i=1}^n \sum_{j=i+1}^n \mathrm{d}c_{i,j} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j,$$

and so on.

The two-form $\mathrm{d}\alpha$ can be rewritten as

$$\mathrm{d}\alpha = \begin{bmatrix} c_{1,2} & \ldots & c_{1,n} & c_{2,3} & \ldots & c_{2,n} & \ldots & c_{n-1,n} \end{bmatrix} \begin{bmatrix} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ \vdots \\ \mathrm{d}x_1 \wedge \mathrm{d}x_n \\ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ \vdots \\ \mathrm{d}x_2 \wedge \mathrm{d}x_n \\ \vdots \\ \mathrm{d}x_{n-1} \wedge \mathrm{d}x_n \end{bmatrix}, \qquad (1.10)$$

with $c_{i,j}$'s being scalar functions; the transpose of the coefficient row vector in (1.10) having the $c_{i,j}$'s as entries is called the *curl* of the vector function a and is denoted by $\mathrm{curl}(a)$.

Example 1.9 In $\mathbb{R}^2$, the derivative of the one-form $\alpha = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2$ is

$$\begin{aligned} \mathrm{d}\alpha &= \left(\frac{\partial a_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial a_1}{\partial x_2}\mathrm{d}x_2\right) \wedge \mathrm{d}x_1 + \left(\frac{\partial a_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial a_2}{\partial x_2}\mathrm{d}x_2\right) \wedge \mathrm{d}x_2 \\ &= \frac{\partial a_1}{\partial x_2}\mathrm{d}x_2 \wedge \mathrm{d}x_1 + \frac{\partial a_2}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2. \end{aligned}$$

The curl of the vector function $a = [a_1\ a_2]^\top$ is the scalar $\mathrm{curl}(a) = \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}$. In $\mathbb{R}^3$, the derivative of the one-form $\alpha = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2 + a_3\,\mathrm{d}x_3$ is

$$\begin{aligned} \mathrm{d}\alpha = &\left(\frac{\partial a_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial a_1}{\partial x_2}\mathrm{d}x_2 + \frac{\partial a_1}{\partial x_3}\mathrm{d}x_3\right) \wedge \mathrm{d}x_1 \\ &+ \left(\frac{\partial a_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial a_2}{\partial x_2}\mathrm{d}x_2 + \frac{\partial a_2}{\partial x_3}\mathrm{d}x_3\right) \wedge \mathrm{d}x_2 \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\partial a_3}{\partial x_1}\mathrm{d}x_1+\frac{\partial a_3}{\partial x_2}\mathrm{d}x_2+\frac{\partial a_3}{\partial x_3}\mathrm{d}x_3\right)\wedge \mathrm{d}x_3\\
=&\left(\frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1\wedge \mathrm{d}x_2+\left(\frac{\partial a_3}{\partial x_1}-\frac{\partial a_1}{\partial x_3}\right)\mathrm{d}x_1\wedge \mathrm{d}x_3\\
&+\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)\mathrm{d}x_2\wedge \mathrm{d}x_3.
\end{aligned}
$$

The curl of the vector function $a = [a_1\ a_2\ a_3]^\top$ is the vector (note the special arrangement of the entries of $\mathrm{curl}(a)$ in the case $n = 3$ to be conform with the usual definition of curl in the vector calculus)

$$
\mathrm{curl}(a) = \begin{bmatrix} \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \end{bmatrix}.
$$

The proof of the following theorem is omitted: the necessity is given by the Poincaré Lemma (Lemma 2-15 of [116]) and the sufficiency is given by the converse Poincaré Lemma (Theorem 2.19 of [116]) (see also [47]).

Theorem 1.5 *A one-form α (respectively, a one-form $a^\top$) is locally exact if and only if* $\mathrm{d}\alpha = 0$ (*respectively*, $\mathrm{curl}(a) = 0$).

A one-form α is *closed* [47, page 67] if $\mathrm{d}\alpha = 0$; by Theorem 1.5, a one-form is locally exact if and only if it is closed.

Remark 1.4 Let $n = 2$ and assume that α is exact, namely assume the existence of a scalar function I such that

$$
\alpha = \mathrm{d}I = \frac{\partial I}{\partial x_1}\mathrm{d}x_1 + \frac{\partial I}{\partial x_2}\mathrm{d}x_2.
$$

Then,

$$
\mathrm{d}\alpha = \mathrm{d}\,\mathrm{d}I = \left(\frac{\partial}{\partial x_1}\frac{\partial I}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial I}{\partial x_1}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = 0.
$$

The proof of the following theorem, which is a version of the Frobenius Theorem, is omitted (see Proposition 2.4 of [116] for the necessity and the lemma at p. 96 of [47] for the sufficiency).

Theorem 1.6 *Let $\alpha \neq 0$ be a one-form. There exists $\omega(x) \in \mathbb{R}$, $\omega \neq 0$, such that $\frac{1}{\omega}\alpha$ is exact, namely such that $\mathrm{d}(\frac{1}{\omega}\alpha) = 0$ if and only if the following* condition of Frobenius *holds*:

$$
\mathrm{d}\alpha \wedge \alpha = 0. \tag{1.11}
$$

Remark 1.5 Assume $\mathrm{d}(\frac{1}{\omega}\alpha) = 0$, with $\omega \neq 0$. Since

$$\mathrm{d}\left(\frac{1}{\omega}\alpha\right) = -\frac{1}{\omega^2}\,\mathrm{d}\omega \wedge \alpha + \frac{1}{\omega}\,\mathrm{d}\alpha,$$

condition $\mathrm{d}(\frac{1}{\omega}\alpha) = 0$ implies $\mathrm{d}\alpha = \frac{1}{\omega}\,\mathrm{d}\omega \wedge \alpha$, whence $\mathrm{d}\alpha \wedge \alpha = \frac{1}{\omega}\,\mathrm{d}\omega \wedge \alpha \wedge \alpha = 0$. If (1.11) holds, the function $\omega(x) \in \mathbb{R}$, $\omega \neq 0$, such that $\mathrm{d}(\frac{1}{\omega}\alpha) = 0$ is called an *inverse integrating factor* of the one-form α. If (1.11) holds, then by the above reasoning there exists an exact one-form β such that $\mathrm{d}\alpha = \beta \wedge \alpha$ (in particular, $\beta = \frac{1}{\omega}\,\mathrm{d}\omega$), whence $\omega(x) = \mathrm{e}^{b(x)}c$, where $b(x) \in \mathbb{R}$ is the first integral of β, $\mathrm{d}b = \beta$, and $c \in \mathbb{R}$ is an arbitrary constant.

Example 1.10 Let $\alpha \neq 0$ be a one-form. In $\mathbb{R}^2$, one finds that

$$\begin{aligned}\mathrm{d}\alpha \wedge \alpha &= \left(\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2\right) \wedge (a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2)\\ &= a_1\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_1 + a_2\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_2\\ &= 0,\end{aligned}$$

which means that an inverse integrating factor always exists when $n = 2$. In $\mathbb{R}^3$, one finds that

$$\begin{aligned}\mathrm{d}\alpha \wedge \alpha = {}&\left(\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_3\right.\\ &\left. + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_3\right) \wedge (a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2 + a_3\,\mathrm{d}x_3)\\ = {}&\left(\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)a_3 - \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)a_2\right.\\ &\left. + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)a_1\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,\end{aligned}$$

which means that there exists an inverse integrating factor when $n = 3$ if and only if

$$\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)a_3 - \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)a_2 + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)a_1 = 0.$$

Example 1.11 Let $\alpha = -x_2\,\mathrm{d}x_1 + x_1\,\mathrm{d}x_2$ and compute $\mathrm{d}\alpha = 2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$. For any exact one-form $\beta = \frac{\partial b}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial b}{\partial x_2}\,\mathrm{d}x_2$, one computes $\beta \wedge \alpha = (\frac{\partial b}{\partial x_1}x_1 + \frac{\partial b}{\partial x_2}x_2)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$; equality $\mathrm{d}\alpha = \beta \wedge \alpha$ yields the partial differential equation $\frac{\partial b}{\partial x_1}x_1 + \frac{\partial b}{\partial x_2}x_2 = 2$. The *characteristic equation* associated with such a partial differential equation is

$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{db}{2}$. Two functionally independent first integrals of the characteristic equation are $I_1 = \ln(|\frac{x_1}{x_2}|)$ and $I_2 = \ln(|x_1|) - \frac{1}{2}b$. Therefore, all first integrals of the above partial differential equation are given by $I_2 = C(I_1)$, where C is an arbitrary function. In particular, choosing $C(I_1) = \frac{1}{2}I_1$, one computes $b(x) = \ln(|x_1 x_2|)$, which yields the inverse integrating factor $\omega(x) = x_1 x_2$ (choosing $c = 1$ as integration constant). With this choice, one obtains the exact one-form $\frac{1}{\omega}\alpha = -\frac{1}{x_1}\,dx_1 + \frac{1}{x_2}\,dx_2$, with the first integral $I = \ln(|\frac{x_2}{x_1}|)$.

Example 1.12 The one-form $\alpha = x_2\,dx_1 + x_3\,dx_2 + x_1\,dx_3$ does not admit any inverse integrating factor, because $d\alpha \wedge \alpha = -(x_1 + x_2 + x_3)\,dx_1 \wedge dx_2 \wedge dx_3$ is not identically equal to zero.

Theorem 1.7 *If ω is an inverse integrating factor of the one-form α and I is the corresponding first integral, i.e., if $\frac{\partial I}{\partial x} = \frac{1}{\omega}\alpha$, then $\hat{\omega} = \frac{\omega}{C(I)}$ is an inverse integrating factor of α, where $C \neq 0$ is an arbitrary function of I; in particular, the first integral of α corresponding to $\hat{\omega}$ is $\hat{I} = \int C(I)\,dI$, where $\int C(I)\,dI$ is the indefinite integral (the anti-derivative) of $C(I)$.*

Proof Clearly, $\frac{\partial \hat{I}}{\partial x} = C(I)\frac{\partial I}{\partial x} = C(I)\frac{1}{\omega}\alpha$. □

Remark 1.6 In this remark, assume that $x \in \mathbb{R}^3$. Three basic operations of vector calculus are the *gradient*, the *curl* and the *divergence*. Let $\nabla = [\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}\ \frac{\partial}{\partial x_3}]^\top$. The gradient of a scalar function $h(x) \in \mathbb{R}$, and the curl and divergence of a vector function $f(x) \in \mathbb{R}^3$ are, respectively, defined as follows:

$$\nabla h := \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_3} \end{bmatrix},$$

$$\nabla \times f := \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{bmatrix},$$

$$\nabla \cdot f := \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3},$$

where $\times$ and $\cdot$ are, respectively, the cross and scalar product. Using the language of differential forms, ∇h corresponds to the one-form $dh = \frac{\partial h}{\partial x_1}\,dx_1 + \frac{\partial h}{\partial x_2}\,dx_2 + \frac{\partial h}{\partial x_3}\,dx_3$, $\nabla \times f$ corresponds to the two-form $d\phi$, where ϕ is the one-form $\phi = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3$, i.e., $d\phi = (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\,dx_2 \wedge dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\,dx_3 \wedge dx_1 + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\,dx_1 \wedge dx_2$, and $\nabla \cdot f$ corresponds to the three-form $d\chi$, where χ is the two-form $\chi = f_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + f_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + f_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$, i.e., $\mathrm{d}\chi = \mathrm{d}f_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}f_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \mathrm{d}f_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 = (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3})\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. From vector calculus, it is known that the following properties hold in some open and connected domain of $\mathbb{R}^3$:

(1.6.1) if $\nabla h = 0$ for some scalar function h, then h is constant;
(1.6.2) if $\nabla \times f = 0$ for some vector function f, then there exists a scalar function h such that $f = \nabla h$;
(1.6.3) if $\nabla \cdot f = 0$ for some vector function f, then there exists a vector function g such that $f = \nabla \times g$;
(1.6.4) for any scalar function h, there exists a vector function f such that $\nabla \cdot f = h$.

In terms of differential forms, the above properties correspond to the following respective properties:

(1.6.1′) if $\mathrm{d}h = 0$ for some scalar function h, then h is constant;
(1.6.2′) if $\mathrm{d}\phi = 0$ for some one-form ϕ, then there exists a scalar function h such that $\mathrm{d}h = \phi$;
(1.6.3′) if $\mathrm{d}\chi = 0$ for some two-form χ, then there exists a one-form ϕ such that $\mathrm{d}\phi = \chi$;
(1.6.4′) for any scalar function h, there exists a two-form χ such that $\mathrm{d}\chi = h\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

1.5 The Cauchy–Kovalevskaya Theorem

In this section, some facts about the *Cauchy–Kovalevskaya Theorem* are recalled; the reader interested in a more extensive treatment is referred to [36, 103].

Consider m scalar functions $u_i(x) \in \mathbb{R}$, $i = 1, \dots, m$. Apart from a reordering of the entries x_i of x, consider the system of first order partial differential equations

$$\begin{cases} \frac{\partial u_1}{\partial x_1} = k_1(x, u_1, \dots, u_m, \frac{\partial u_1}{\partial x^a}, \dots, \frac{\partial u_m}{\partial x^a}), \\ \vdots \\ \frac{\partial u_m}{\partial x_1} = k_m(x, u_1, \dots, u_m, \frac{\partial u_1}{\partial x^a}, \dots, \frac{\partial u_m}{\partial x^a}), \end{cases} \tag{1.12}$$

where $k_i(\xi) : \mathbb{R}^{n(m+1)} \to \mathbb{R}$, $i = 1, \dots, m$, and $x^a = [x_2 \ \dots \ x_n]^\top$; system (1.12) is said to be in the *Kovalevskaya form*.

Assumption 1.1 Take a point $x^o = [x_1^o \ x_2^o \ \dots \ x_n^o]^\top$. Consider the Cauchy initial data

$$u_i(x_1^o, x_2, \dots, x_n) = h_i(x_2, \dots, x_n), \quad i = 1, \dots, m, \tag{1.13}$$

where the functions $h_i(x_2, \dots, x_n)$ are analytic at $[x_2 \ \dots \ x_n]^\top = [x_2^o \ \dots \ x_n^o]^\top$. Let $\Xi(x) = [x^\top \ u_1(x) \ \dots \ u_m(x) \ (\frac{\partial u_1(x)}{\partial x^a})^\top \ \dots \ (\frac{\partial u_m(x)}{\partial x^a})^\top]^\top$ and $\xi^o = \Xi(x^o)$. Let functions $k_i(\xi)$, $i = 1, \dots, m$, be analytic at $\xi = \xi^o$.

The Cauchy–Kovalevskaya Theorem can be stated as follows (the proof can be found in [36]).

Theorem 1.8 *Under Assumption 1.1, the Cauchy problem (1.12), (1.13) has a unique solution $u_1(x), \dots, u_m(x)$ in a neighborhood of x^o, which is analytic at $x = x^o$.*

Remark 1.7 If $n = 1$, system (1.12) reduces to a set of first order ordinary differential equations and Theorem 1.8 reduces to the classical Cauchy Theorem.

1.6 The Frobenius Theorem

Definition 1.5 Given m vector functions $g_1(x), \dots, g_m(x) \in \mathbb{R}^n$, with entries in $\mathscr{K}_n$, the *distribution* $\mathscr{D}$ spanned by $g_1, \dots, g_m$ over the field of meromorphic functions $\mathscr{K}_n$ is

$$\mathscr{D} = \operatorname{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\} = \left\{ g(x) \in \mathbb{R}^n : g = \sum_{i=1}^m \alpha_i g_i, \alpha_i \in \mathscr{K}_n \right\}.$$

The distribution $\operatorname{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\}$ is *involutive* if, for each pair $i, j \in \{1, \dots, m\}$, there exist m functions $c_{i,j;\ell} \in \mathscr{K}_n$, $\ell = 1, \dots, m$, such that

$$[g_i, g_j] = \sum_{\ell=1}^m c_{i,j;\ell}\, g_\ell.$$

The following theorem shows that the involutive property of a distribution $\mathscr{D}$ is independent of the basis $\{g_1, \dots, g_m\}$ chosen to represent $\mathscr{D}$.

Lemma 1.1 *A distribution $\mathscr{D} = \operatorname{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\}$ is involutive if and only if $[f, g] \in \mathscr{D}$, for all $f, g \in \mathscr{D}$.*

Proof Given an involutive distribution $\mathscr{D} = \operatorname{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\}$, if $f, g \in \mathscr{D}$, then $[f, g] \in \mathscr{D}$; as a matter of fact, letting $f = \sum_{i=1}^m \alpha_i g_i$ and $g = \sum_{j=1}^m \beta_j g_j$, for $\alpha_i, \beta_j \in \mathscr{K}_n$, one concludes that $[f, g]$ belongs to $\mathscr{D}$, as shown by the following equalities:

$$\begin{aligned}
[f, g] &= \sum_{i=1}^m \sum_{j=1}^m [\alpha_i g_i, \beta_j g_j] = \sum_{i=1}^m \sum_{i=1}^m \big(L_{\alpha_i g_i}(\beta_j g_j) - L_{\beta_j g_j}(\alpha_i g_i) \big) \\
&= \sum_{i=1}^m \sum_{i=1}^m \big(\alpha_i L_{g_i}(\beta_j g_j) - \beta_j L_{g_j}(\alpha_i g_i) \big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{i=1}^{m}\big(\alpha_i(\beta_j L_{g_i} g_j + g_j L_{g_i}\beta_j) - \beta_j(\alpha_i L_{g_j} g_i + g_i L_{g_j}\alpha_i)\big) \\
&= \sum_{i=1}^{m}\sum_{i=1}^{m}\big(\alpha_i\beta_j(L_{g_i} g_j - L_{g_j} g_i) + (\alpha_i L_{g_i}\beta_j) g_j - (\beta_j L_{g_j}\alpha_i) g_i\big) \\
&= \sum_{i=1}^{m}\sum_{i=1}^{m}\big(\alpha_i\beta_j[g_i, g_j] + (\alpha_i L_{g_i}\beta_j) g_j - (\beta_j L_{g_j}\alpha_i) g_i\big).
\end{aligned}
$$

Clearly, if $[f, g] \in \mathscr{D}$ for all $f, g \in \mathscr{D}$, then $[g_i, g_j] \in \mathscr{D}$, whence $[g_i, g_j] = \sum_{\ell=1}^{m} c_{i,j;\ell}\, g_\ell$, for some $c_{i,j;\ell} \in \mathscr{K}_n$. □

Lemma 1.2 *Let $y = \varphi(x)$ be a diffeomorphism. Let $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{g_1, \ldots, g_m\}$ and $\tilde{\mathscr{D}} = \mathrm{span}_{\mathscr{K}_n}\{\varphi_* g_1, \ldots, \varphi_* g_m\}$. Then,*

$$
f \in \mathscr{D} \quad \Longleftrightarrow \quad \varphi_* f \in \tilde{\mathscr{D}}.
$$

Proof If $f = \sum_{i=1}^{m} \alpha_i g_i$, then

$$
\varphi_* f = \left(\frac{\partial \varphi}{\partial x} f\right) \circ \varphi^{-1} = \sum_{i=1}^{m}\left(\alpha_i \frac{\partial \varphi}{\partial x} g_i\right) \circ \varphi^{-1} = \sum_{i=1}^{m} (\varphi_*\alpha_i)(\varphi_* g_i).
$$

The converse is similar. □

By Lemma 1.2, $\mathscr{D}$ is involutive if and only if $\tilde{\mathscr{D}}$ is involutive.

Definition 1.6 Let a distribution $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{g_1, \ldots, g_m\}$ be given, with $g_1, \ldots, g_m$ being linearly independent over $\mathscr{K}_n$. Let $G = [g_1 \ \ldots \ g_m]$. Point $x^o \in \mathbb{R}^n$ is *regular* for the distribution $\mathscr{D}$ if matrix $G(x)$ has constant rank m for all x in a neighborhood $\mathscr{U}^*$ of x^o.

By Lemma 1.2, if the domain of definition of the diffeomorphism $y = \varphi(x)$ contains the regular point x^o of $\mathscr{D}$, then $y^o = \varphi(x^o)$ is a regular point of $\tilde{\mathscr{D}}$.

The Frobenius Theorem can be stated as follows (for proof the reader is referred to [35, 69, 100, 107]), letting e_i denote the ith column of the identity matrix E.

Theorem 1.9 *Let a distribution $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{g_1, \ldots, g_m\}$ be given, with $g_1, \ldots, g_m$ being linearly independent over $\mathscr{K}_n$; let $x^o \in \mathbb{R}^n$ be a regular point of $\mathscr{D}$. There exists a diffeomorphism $y = \varphi(x)$, $\varphi(\cdot) : \mathscr{U}^* \to \mathbb{R}^n$, with $\mathscr{U}^*$ being some neighborhood of x^o, such that*

$$
\mathrm{span}_{\mathscr{K}_n}\{\varphi_* g_1, \ldots, \varphi_* g_m\} = \mathrm{span}_{\mathscr{K}_n}\{e_1, \ldots, e_m\} \tag{1.14}
$$

if and only if $\mathscr{D}$ is involutive.

By (1.14), any $\tilde{f} \in \text{span}_{\mathscr{K}_n}\{\varphi_* g_1, \dots, \varphi_* g_m\}$ has the last $n-m$ entries being equal to zero; this means that the last $n-m$ entries of $\varphi(x)$ are functionally independent first integrals of any $f \in \text{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\}$, and therefore they are joint functionally independent first integrals of $g_1, \dots, g_m$.

Let $h_i(x) \in \mathbb{R}^n$ be defined by the pull-back $h_i = \varphi^* e_i = (\frac{\partial \varphi}{\partial x})^{-1} e_i$, with φ being the diffeomorphism introduced in Theorem 1.9, under its assumptions; such h_i's are pairwise *commuting*, $[h_i, h_j] = 0$ (because $[e_i, e_j] = 0$), and $\mathscr{D} = \text{span}_{\mathscr{K}_n}\{g_1, \dots, g_m\} = \text{span}_{\mathscr{K}_n}\{h_1, \dots, h_m\}$. This means that any involutive distribution is spanned, about an arbitrary regular point, by pairwise commuting vector functions.

The reasoning in the following remark is used very often in the rest of the book.

Remark 1.8 Let $y = \varphi(x)$ be a diffeomorphism from $\mathscr{U}$ to $\mathbb{R}^n$, with $\mathscr{U}$ being an open and connected subset of $\mathbb{R}^n$ such that $\det(\frac{\partial \varphi(x)}{\partial x}) \neq 0$ for all x in $\mathscr{U}$. Let g_i be the ith column of $(\frac{\partial \varphi}{\partial x})^{-1}$; if φ has analytic entries on $\mathscr{U}$, then $(\frac{\partial \varphi}{\partial x})^{-1}$ has entries being meromorphic on $\mathscr{U}$, as well as its columns g_i. In particular, the following relation holds:

$$[g_i, g_j] = 0, \quad \forall i, j. \tag{1.15}$$

Vice versa, let $g_1(x), \dots, g_n(x) \in \mathbb{R}^n$ be n pairwise commuting meromorphic vector functions (i.e., such that (1.15) holds) such that

$$\det\bigl([g_1 \ \dots \ g_n]\bigr) \neq 0. \tag{1.16}$$

Then, the n rows of $[g_1 \ \dots \ g_n]^{-1}$ are exact one-forms, i.e., there exists an analytic diffeomorphism $y = \varphi(x)$ such that $\frac{\partial \varphi}{\partial x} = [g_1 \ \dots \ g_n]^{-1}$ locally. Moreover, such a diffeomorphism is *global* if and only if (1.15) and (1.16) hold for all $x \in \mathbb{R}^n$ (i.e., $\mathscr{U} = \mathbb{R}^n$) and the vector functions g_i are *complete* [39, 104], i.e., the CT-flow $\varPhi_{g_i}(t, x)$ associated with g_i is defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $i = 1, \dots, n$. It is worth pointing out that $y = \varphi(x)$ is the diffeomorphism that straightens jointly all vector functions g_i, i.e., $\varphi_* g_1 = e_1, \dots, \varphi_* g_n = e_n$, with e_i being the ith column of the $n \times n$ identity matrix E; in particular, for each $j = 1, \dots, n$, by construction

$$L_{g_i} \varphi_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases}$$

where φ_j is the jth entry of φ. As a consequence, the CT-flow associated with g_i is

$$\varPhi_{g_i}(t, x) = \varphi^{-1}\bigl(e_i t + \varphi(x)\bigr).$$

Example 1.13 Clearly, $y_1 = x_1$, $y_2 = x_2 + x_1^2$, with inverse $x_1 = y_1, x_2 = y_2 - y_1^2$, is a global diffeomorphism $y = \varphi(x)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then, from $(\frac{\partial \varphi(x)}{\partial x})^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ -2x_1 & 1 \end{smallmatrix}\right]$, the two vector functions $g_1(x) = \left[\begin{smallmatrix} 1 \\ -2x_1 \end{smallmatrix}\right]$ and $g_2(x) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ are found. Clearly, $[g_1, g_2] = 0$ for all x in $\mathbb{R}^2$ and the CT-flows $\varPhi_{g_1}(t, x) = \left[\begin{smallmatrix} t + x_1 \\ -t^2 - 2t x_1 + x_2 \end{smallmatrix}\right]$

and $\Phi_{g_2}(t,x) = \left[\begin{smallmatrix} x_1 \\ t+x_2 \end{smallmatrix}\right]$ associated with g_1 and g_2, respectively, are defined for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, i.e., g_1 and g_2 are complete.

Example 1.14 Let $g_1(x) = \left[\begin{smallmatrix} x_2+2 \\ 1 \end{smallmatrix}\right]$ and $g_2(x) = \left[\begin{smallmatrix} x_2+1 \\ 1 \end{smallmatrix}\right]$; then, $\det([g_1\ \ g_2]) = 1$ and $[g_1, g_2] = 0$ on the whole $\mathbb{R}^2$. Since g_1 and g_2 are complete (their CT-flows $\Phi_{g_1}(t,x) = \left[\begin{smallmatrix} 2t+x_1+\frac{1}{2}t^2+tx_2 \\ t+x_2 \end{smallmatrix}\right]$ and $\Phi_{g_2}(t,x) = \left[\begin{smallmatrix} x_1+\frac{1}{2}t^2+tx_2+t \\ t+x_2 \end{smallmatrix}\right]$ are defined for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$), the diffeomorphism $y = \varphi(x)$, which can be found by integrating the rows of $[g_1(x)\ g_2(x)]^{-1} = \left[\begin{smallmatrix} 1 & -x_2-1 \\ -1 & x_2+2 \end{smallmatrix}\right]$, is global: choosing zero integration constants, one finds the diffeomorphism $y_1 = -\frac{1}{2}x_2^2 + x_1 - x_2$, $y_2 = \frac{1}{2}x_2^2 - x_1 + 2x_2$ with inverse $x_1 = 2y_1 + y_2 + \frac{1}{2}(y_1 + y_2)^2$, $x_2 = y_1 + y_2$.

A useful result concerning the inverse $\varphi^{-1}(y)$ can be derived from the property

$$\Phi_{g_i}(\xi_i, \cdot) \circ \Phi_{g_j}(\xi_j, x) = \varphi^{-1}\big(e_i\xi_i + e_j\xi_j + \varphi(x)\big), \quad \forall i, j, \tag{1.17}$$

which implies that

$$\begin{aligned} \Phi_{g_1}(\xi_1, \cdot) \circ \Phi_{g_2}(\xi_2, \cdot) \circ \cdots \circ \Phi_{g_n}(\xi_n, x) &= \varphi^{-1}\big(e_1\xi_1 + e_2\xi_2 + \cdots + e_n\xi_n + \varphi(x)\big) \\ &= \varphi^{-1}\big(\xi + \varphi(x)\big), \end{aligned}$$

where $\xi = [\xi_1\ \ldots\ \xi_n]^\top$. Such an equality gives

$$\varphi^{-1}(y) = \Phi_{g_1}(\xi_1, \cdot) \circ \Phi_{g_2}(\xi_2, \cdot) \circ \cdots \circ \Phi_{g_n}(\xi_n, x)|_{\xi = y - \varphi(x)}. \tag{1.18}$$

Similarly, if $0 \in \mathscr{U}$, and $\varphi(0) = 0$, then

$$\Phi_{g_1}(\xi_1, \cdot) \circ \Phi_{g_2}(\xi_2, \cdot) \circ \cdots \circ \Phi_{g_n}(\xi_n, x)|_{\xi = y, x = 0} = \varphi^{-1}(y).$$

By (1.17), since $e_i\xi_i + e_j\xi_j = e_j\xi_j + e_i\xi_i$, one concludes that

$$\Phi_{g_i}(\xi_i, \cdot) \circ \Phi_{g_j}(\xi_j, x) = \Phi_{g_j}(\xi_j, \cdot) \circ \Phi_{g_i}(\xi_i, x), \tag{1.19}$$

which is a direct consequence of $[g_i, g_j] = 0$.

Example 1.15 Consider again the diffeomorphism of Example 1.14. Since

$$\Phi_{g_1}(\xi_1, \cdot) \circ \Phi_{g_2}(\xi_2, x) = \begin{bmatrix} 2\xi_1 + x_1 + \frac{1}{2}\xi_2^2 + \xi_2 x_2 + \xi_2 + \frac{1}{2}\xi_1^2 + \xi_1(\xi_2 + x_2) \\ \xi_1 + \xi_2 + x_2 \end{bmatrix},$$

letting

$$\xi_1 = y_1 - \left(-\frac{1}{2}x_2^2 + x_1 - x_2\right), \qquad \xi_2 = y_2 - \left(\frac{1}{2}x_2^2 - x_1 + 2x_2\right),$$

one concludes that

$$\varphi^{-1}(y) = \begin{bmatrix} 2y_1 + y_2 + \frac{1}{2}(y_2 + y_1)^2 \\ y_1 + y_2 \end{bmatrix}.$$

The following theorem follows from the above reasoning (see [69]).

Theorem 1.10 *Let $g_1, \dots, g_n$ be such that conditions (1.15) hold only for all $i, j \in \{1, \dots, m\}$, for some $m \le n$, and condition (1.16) holds; denoting by x^o any point of $\mathscr{U}$ such that* $\det([g_1(x^o)\ \dots\ g_n(x^o)]) \neq 0$, *then the diffeomorphism $y = \varphi(x)$, with $\varphi^{-1}(y)$ given by*

$$\varphi^{-1}(y) = \left[\Phi_{g_1}(\xi_1, \cdot) \circ \Phi_{g_2}(\xi_2, \cdot) \circ \dots \circ \Phi_{g_n}(\xi_n, x)\right]_{\xi=y, x=x^o}, \tag{1.20}$$

straightens $g_1, \dots, g_m$ (but, in general, not g_i, $i \ge m+1$) and satisfies $\varphi^{-1}(0) = x^o$.

Theorem 1.10 gives a procedure for the computation of a diffeomorphism $y = \varphi(x)$ that straightens jointly m pairwise commuting vector functions. It is worth pointing out that the last $n - m$ entries of $\varphi(x)$ are functionally independent first integrals of $g_1, \dots, g_m$. This procedure is detailed in the following example in the case $m = 1$.

Example 1.16 Consider $g_1(x) = [x_1\ 3x_2 + x_1^2]^\top$. The CT-flow $\Phi_{g_1}(t, x)$ associated with g_1 is

$$\Phi_{g_1}(t, x) = \begin{bmatrix} e^t x_1 \\ e^{3t} x_2 + (-e^{2t} + e^{3t}) x_1^2 \end{bmatrix}.$$

The vector function g_1 can be completed with $g_2(x) = [0\ 1]^\top$ in a neighborhood of any x such that $\det([g_1(x)\ g_2(x)]) \neq 0$, where

$$\det\left(\left[g_1(x)\ g_2(x)\right]\right) = \det\left(\begin{bmatrix} x_1 & 0 \\ 3x_2 + x_1^2 & 1 \end{bmatrix}\right) = x_1; \tag{1.21}$$

actually, g_1 and g_2 are not commuting (i.e., $[g_1, g_2] = [0\ -3]^\top \neq 0$). The CT-flow associated with g_2 is

$$\Phi_{g_2}(t, x) = \begin{bmatrix} x_1 \\ x_2 + t \end{bmatrix}.$$

Compute the composition of the two CT-flows at $x = x^o$:

$$\Phi_{g_1}(y_1, \cdot) \circ \Phi_{g_2}(y_2, x^o) = \begin{bmatrix} e^{y_1} x_1^o \\ e^{3y_1}(x_2^o + y_2) + (-e^{2y_1} + e^{3y_1})(x_1^o)^2 \end{bmatrix}.$$

Choosing $x_1^o = 1$ and $x_2^o = 0$ (by (1.21), one can choose any point such that $x_1^o \neq 0$), one obtains the diffeomorphism $x = \varphi^{-1}(y)$, with

$$\varphi^{-1}(y) = \begin{bmatrix} e^{y_1} \\ e^{3y_1} y_2 - e^{2y_1} + e^{3y_1} \end{bmatrix};$$

note that $\varphi^{-1}(0) = x^o = [1\ 0]^\top$.

A special case is when all g_i are linear, $g_i(x) = A_i x$, $i = 1, \dots, n$. Let $A_1, \dots, A_n \in \mathbb{R}^{n\times n}$ be such that $\det([A_1 x^o \ \dots \ A_n x^o]) \neq 0$, for some $x^o \in \mathbb{R}^n$. Then, the diffeomorphism that straightens $A_1 x$, about $x = x^o$, is $y = \varphi(x)$, with

$$\varphi^{-1}(y) = \mathrm{e}^{A_1 y_1} \mathrm{e}^{A_2 y_2} \cdots \mathrm{e}^{A_n y_n} x^o.$$

Example 1.17 Consider $g_1(x) = A_1 x$ and $g_2(x) = A_2 x$, with $A_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ (g_1 and g_2 are clearly not commuting, $[g_1(x), g_2(x)] = [x_2 \ 0]^\top \neq 0$). Since $\det([g_1(x) \ g_2(x)]) = -2x_2^2$, one can choose any point x^o such that $x_2^o \neq 0$: e.g., take $x^o = [0 \ 1]^\top$. Then, the diffeomorphism $x = \varphi^{-1}(y)$ is found,

$$\varphi^{-1}(y) = \mathrm{e}^{A_1 y_1} \mathrm{e}^{A_2 y_2} x^o = \mathrm{e}^{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right] y_1} \mathrm{e}^{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] y_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{y_1} y_2 + \mathrm{e}^{2y_1} - \mathrm{e}^{y_1} \\ \mathrm{e}^{2y_1} \end{bmatrix},$$

with inverse $y = \varphi(x)$ (with x in a neighborhood of x^o such that $x_2 > 0$),

$$\varphi(x) = \begin{bmatrix} \frac{1}{2} \ln(x_2) \\ 1 + \frac{x_1 - x_2}{\sqrt{x_2}} \end{bmatrix}.$$

Note that $L_{A_1 x} \varphi(x) = [1 \ 0]^\top$ and $\varphi(x^o) = 0$.

1.7 Semi-simple, Normal and Nilpotent Square Matrices

In this section, definitions and first standard properties of semi-simple, normal and nilpotent matrices are reported [52, 83]; some more results, crucial for the sequel of the book, are given in Sect. 2.1, where they are proven using results presented earlier.

Definition 1.7 A matrix $A \in \mathbb{R}^{n\times n}$ is *semi-simple* if it can be diagonalized over $\mathbb{C}$; A is *normal* if it commutes with its transpose under the matrix product, $AA^\top = A^\top A$; A is *nilpotent* if there exists an integer $k \in \mathbb{Z}^>$ such that $A^k = 0$.

Lemma 1.3 *If A is normal, then A is semi-simple.*

Proof By the Schur triangularization theorem (see Theorem 4.10.2 of [83]), for any matrix A there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ ($UU^{*\top} = E$) such that $UAU^{*\top} = T$ and $UA^\top U^{*\top} = T^{*\top}$, where $T \in \mathbb{C}^{n\times n}$ is triangular. Hence

$$UAA^\top U^{*\top} = UAU^{*\top} UA^\top U^{*\top} = TT^{*\top},$$
$$UA^\top AU^{*\top} = UA^\top U^{*\top} UAU^{*\top} = T^{*\top} T.$$

Now, since A and $A^\top$ are commuting, one concludes that $TT^{*\top} = T^{*\top} T$. Since T is triangular, this is possible if and only if T is diagonal (this can be proven by induction on the dimension of matrix T). □

By Lemma 1.3, a normal matrix is semi-simple, but the converse need not be true. Examples of normal matrices are the symmetric and skew-symmetric ones. A matrix A is nilpotent if and only if all its eigenvalues are equal to zero.

Example 1.18 Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be normal, i.e., $AA^\top - A^\top A = 0$. By solving the algebraic system that is found by equating to 0 the entries of $AA^\top - A^\top A$, one finds that there are two possible cases: $b = c$, for arbitrary $a, c, d \in \mathbb{R}$, i.e.,

$$A = \begin{bmatrix} a & c \\ c & d \end{bmatrix}, \tag{1.22}$$

and $a = d, b = -c$, for arbitrary $c, d \in \mathbb{R}$, i.e.,

$$A = \begin{bmatrix} d & -c \\ c & d \end{bmatrix}. \tag{1.23}$$

Matrix A given in (1.22) has eigenvalues $\frac{1}{2}d + \frac{1}{2}a + \frac{1}{2}\sqrt{(a-d)^2 + 4c^2}$, $\frac{1}{2}d + \frac{1}{2}a - \frac{1}{2}\sqrt{(a-d)^2 + 4c^2}$, which are always real for all $a, c, d \in \mathbb{R}$, whereas matrix A given in (1.23) has eigenvalues $d + \mathrm{i}c, d - \mathrm{i}c$, which are always non-real for all $c, d \in \mathbb{R}$, $c \neq 0$.

Lemma 1.4 *Let A, $B \in \mathbb{R}^{n\times n}$ be semi-simple and commuting, $AB = BA$. Then, A and B are jointly diagonalizable.*

Proof If both A and B have distinct eigenvalues, the proof of the theorem is particularly simple. Let v_i be eigenvector of matrix A with eigenvalue λ_i, $Av_i = \lambda_i v_i$. If $Bv_i = 0$, then v_i is eigenvector of matrix B with eigenvalue $\gamma_i = 0$. If $Bv_i \neq 0$, then

$$ABv_i = BAv_i = \lambda_i Bv_i,$$

which shows that Bv_i is eigenvector of matrix A with eigenvalue λ_i. Since the eigenvalues of A are distinct, v_i and Bv_i are necessarily co-linear over $\mathbb{C}$, i.e., there exists a number γ_i such that $Bv_i = \gamma_i v_i$, whence v_i is also eigenvector of B. From this, A and B are jointly diagonalized by $Q = [v_1\ v_2\ \ldots\ v_n]$, where the columns of Q are n linearly independent eigenvectors of A over $\mathbb{C}$ (whence, also linearly independent eigenvectors of B over $\mathbb{C}$).

Consider now the case of matrix A having repeated eigenvalues λ_i: let p_i be the algebraic multiplicity of the eigenvalue λ_i as root of the characteristic polynomial of A. Since A is semi-simple, let $Q \in \mathbb{R}^{n\times n}$ be such that $\tilde{A} = Q^{-1}AQ = \text{block\_diag}\{A_1, \ldots, A_p\}$, where $A_i = \lambda_i E_i$, E_i being the identity matrix of dimensions $p_i \times p_i$, and $\lambda_i \neq \lambda_j$ if $i \neq j$. Note that, letting $\tilde{B} = Q^{-1}BQ$, condition $AB = BA$ holds if and only if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. Since $AB = BA$, it can be easily verified that necessarily $\tilde{B} = \text{block\_diag}\{\tilde{B}_1, \ldots, \tilde{B}_p\}$, where $\tilde{B}_i$ is semi-simple and has the same dimensions as A_i. Each $\tilde{B}_i$, being semi-simple can be diagonalized by a transformation $\bar{Q}_i$; then, the transformation that jointly diagonalizes A and B is $\hat{Q} = Q\,\text{block\_diag}\{\bar{Q}_1, \ldots, \bar{Q}_p\}$. □

Example 1.19 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix};$$

A and B are commuting and semi-simple. Matrix A can be diagonalized by

$$Q = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q^{-1}AQ = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

but the transformed B is not diagonal (it is only block-diagonal)

$$Q^{-1}BQ = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Submatrix $\tilde{B}_1 = \left[\begin{smallmatrix} 5 & -4 \\ 3 & -2 \end{smallmatrix}\right]$ can be diagonalized by $\tilde{Q}_1 = \left[\begin{smallmatrix} 4 & -3 \\ 3 & -3 \end{smallmatrix}\right]$,

$$\tilde{Q}_1^{-1}\tilde{B}_1\tilde{Q}_1 = \begin{bmatrix} 4 & -3 \\ 3 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, A and B are jointly diagonalized by

$$Q_{\text{tot}} = Q \left[\begin{array}{cc} \tilde{Q}_1 & 0 \\ \hline 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 1.9 Let $B_1, \ldots, B_m$ be $m < n$ diagonal matrices, $B_i = \text{diag}\{b_{i,1}, b_{i,2}, \ldots, b_{i,n}\}$, whence semi-simple and pairwise commuting. Since such matrices are pairwise commuting, by the Frobenius Theorem 1.9, the m continuous-time linear systems $\frac{dx}{dt} = g_i(x) = B_i x$ share $n - m$ functionally independent first integrals. Define the matrix

$$B := \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}.$$

It can be seen that $I(x) = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, with k_i real, is a first integral associated with g_i if and only if

$$[k_1 \ \cdots \ k_n] \begin{bmatrix} b_{i,1} \\ \vdots \\ b_{i,n} \end{bmatrix} = 0;$$

this follows from

$$\begin{aligned} L_{g_i} I(x) &= \left[k_1 \left(x_1^{k_1-1} x_2^{k_2} \cdots x_n^{k_n} \right) \quad k_2 \left(x_1^{k_1} x_2^{k_2-1} \cdots x_n^{k_n} \right) \quad \ldots \quad k_n \left(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n-1} \right) \right] \\ &\quad \times \begin{bmatrix} b_{i,1} x_1 \\ b_{i,2} x_2 \\ \vdots \\ b_{i,n} x_n \end{bmatrix} \\ &= \left(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \right) [k_1 \ k_2 \ \ldots \ k_n] \begin{bmatrix} b_{i,1} \\ b_{i,2} \\ \vdots \\ b_{i,n} \end{bmatrix}. \end{aligned}$$

Hence, $I(x) = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ is a joint first integral associated with $g_1, \ldots, g_m$ if and only if vector $k = [k_1 \ \ldots \ k_n]^\top$ belongs to $\ker(B)$.

Example 1.20 Consider the diagonal matrices

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix B is given by

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

The kernel of B is spanned by $[1 \ -3 \ 1]^\top$, and therefore a joint first integral associated with both $B_1 x$ and $B_2 x$ is $I(x) = \frac{x_1 x_3}{x_2^3}$, as can be checked by

$$L_{B_1 x} I(x) = \begin{bmatrix} \frac{x_3}{x_2^3} & -3\frac{x_1 x_3}{x_2^4} & \frac{x_1}{x_2^3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

$$L_{B_2 x} I(x) = \begin{bmatrix} \frac{x_3}{x_2^3} & -3\frac{x_1 x_3}{x_2^4} & \frac{x_1}{x_2^3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Chapter 2
Analysis of Linear Systems

2.1 The Linear Centralizer and Linear Normalizer of a Square Matrix

Assume that systems (1.1a), (1.1b) are linear, i.e., $f(x) = Ax$ (respectively, $F(x) = Ax$),

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t), \quad t \in \mathbb{R}, \tag{2.1a}$$

$$x(t+1) = Ax(t), \quad t \in \mathbb{Z}, \tag{2.1b}$$

where $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ is said to be the *dynamic matrix* (which is assumed to be constant) of the linear system: a notation common to both (2.1a) and (2.1b) can be adopted:

$$\Delta x(t) = Ax(t), \quad t \in \mathbb{T},$$

where $\Delta x(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ if $\mathbb{T} = \mathbb{R}$ and $\Delta x(t) = x(t+1)$ if $\mathbb{T} = \mathbb{Z}$. Symbol $\mathscr{I}_C(Ax)$ (respectively, $\mathscr{I}_D(Ax)$) denotes the set of all first integrals of system (2.1a) (respectively, system (2.1b)).

Assume also that system (1.2) is linear, $g(x) = Bx$,

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = Bx = g(x), \tag{2.2}$$

where $B \in \mathbb{R}^{n \times n}$ is constant. As well known,

$$x = \mathrm{e}^{B\tau} y \tag{2.3}$$

is the unique solution of system (2.2) at time $\tau \in \mathbb{R}$, starting from the initial condition $x(0) = y$, with $y \in \mathbb{R}^n$.

A *one-parameter group of linear transformations* is given by $x = Q(\tau)y$, if $Q(\tau) \in \mathbb{R}^{n \times n}$ satisfies $Q(0) = E$, $Q(\tau_1)Q(\tau_2) = Q(\tau_1 + \tau_2)$ and $Q^{-1}(\tau) = Q(-\tau)$.

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The family of linear transformations given in equation (2.3) qualifies as a one-parameter group of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$: $\mathrm{e}^{B0} = E$, $\mathrm{e}^{B\tau_1}\mathrm{e}^{B\tau_2} = \mathrm{e}^{B(\tau_1+\tau_2)}$ and $(\mathrm{e}^{B\tau})^{-1} = \mathrm{e}^{-B\tau}$. Given any one-parameter group of linear transformations $x = Q(\tau)y$, there exists a constant matrix $B \in \mathbb{R}^{n\times n}$ such that $Q(\tau) = \mathrm{e}^{B\tau}$. This matrix can be computed by $B = \frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau}|_{\tau=0}$. As a matter of fact, $Q(\tau) = \mathrm{e}^{B\tau}$ is obtained by integrating the following differential equation by the initial condition $Q(0) = E$:

$$
\begin{aligned}
\frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau} &= \lim_{T\to 0^+} \frac{Q(\tau+T) - Q(\tau)}{T} = \left(\lim_{T\to 0^+} \frac{Q(T) - E}{T}\right) Q(\tau) \\
&= \left(\frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=0}\right) Q(\tau) = BQ(\tau).
\end{aligned}
$$

Example 2.1 Consider the one-parameter family of rotations in $\mathbb{R}^2$ given by $Q(\tau) = \left[\begin{smallmatrix}\cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau)\end{smallmatrix}\right]$. Clearly, $Q(0) = E$ and

$$
Q(\tau_1)Q(\tau_2) = \begin{bmatrix}\cos(\tau_1+\tau_2) & -\sin(\tau_1+\tau_2) \\ \sin(\tau_1+\tau_2) & \cos(\tau_1+\tau_2)\end{bmatrix} = Q(\tau_1+\tau_2);
$$

therefore, $Q(\tau)$ is a one-parameter group of linear transformations. Then, $Q(\tau) = \mathrm{e}^{B\tau}$, with

$$
B = \frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=0} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}.
$$

Using (2.3) as a change of coordinates, one can rewrite systems (2.1a), (2.1b) in the new y-coordinates, as follows:

$$
\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \mathrm{e}^{-B\tau} A \mathrm{e}^{B\tau} y(t), \tag{2.4a}
$$

$$
y(t+1) = \mathrm{e}^{-B\tau} A \mathrm{e}^{B\tau} y(t). \tag{2.4b}
$$

From Theorem 1 of [37] (see, also, [52]), for $Q \in \mathbb{R}^{n\times n}$, the equation $Q = \mathrm{e}^{B\tau}$, with the requirement that B and τ are real, has a solution (not necessarily unique, also when τ is fixed) if and only if $\det(Q) \neq 0$ and each Jordan block of Q corresponding to an eigenvalue with negative real part occurs an even number of times. This shows that only a subset of the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be put into form (2.3), for some real τ. Moreover, since $\mathrm{e}^{B\tau} = E + B\tau + O(\tau^2)$, with E being the $n \times n$ identity matrix and $O(\tau^2)$ denoting second and higher order terms, for τ close to 0, transformation (2.3) is close to the *identity transformation* and (see the subsequent Sect. 6.2), for the transformed system (2.4a), (2.4b) one has

$$
\begin{aligned}
\mathrm{e}^{-B\tau} A \mathrm{e}^{B\tau} &= \bigl(E - B\tau + O(\tau^2)\bigr) A \bigl(E + B\tau + O(\tau^2)\bigr) \\
&= A - (BA - AB)\tau + O(\tau^2).
\end{aligned} \tag{2.5}
$$

Definition 2.1 The linear transformation (2.3) is a *linear symmetry* of systems (2.1a), (2.1b) and system (2.2) is its *infinitesimal generator* if

$$e^{-B\tau} A e^{B\tau} y = Ay, \quad \forall y \in \mathbb{R}^n, \ \forall \tau \in \mathbb{R}. \tag{2.6}$$

If (2.6) holds, by abuse of notation, also the infinitesimal generator (2.2) is called a *linear symmetry* of systems (2.1a), (2.1b); briefly, Bx is called a *linear symmetry* of Ax.

Remark 2.1 If (2.6) holds, then $e^{-B\tau} e^{At} e^{B\tau} = e^{At}$ (respectively, $e^{-B\tau} A^t e^{B\tau} = A^t$), $\forall \tau \in \mathbb{R}$, $\forall t \in \mathbb{T}$ ($t \geq 0$, in the discrete-time case if $\det(A) = 0$).

Definition 2.2 Given $x_0 \in \mathbb{R}^n$, the *orbit* of systems (2.1a), (2.1b) passing through x_0 is the set of the points x described by $x = e^{At} x_0$ if $\mathbb{T} = \mathbb{R}$ (respectively, $x = A^t x_0$ if $\mathbb{T} = \mathbb{Z}$), when $t \in \mathbb{T}$ varies from $-\infty$ to $+\infty$ (from 0 to $+\infty$, in the discrete-time case if $\det(A) = 0$).

The meaning of relation (2.6) is that any *orbit* $x(t) = e^{At} x_0$ (respectively, $x(t) = A^t x_0$) of systems (2.1a), (2.1b) is mapped into an *orbit* $y(t) = e^{At} y_0$ (respectively, $y(t) = A^t y_0$) of the same systems (2.1a), (2.1b) by the linear transformation (2.3) generated by system (2.2), $y = e^{-B\tau} x$ and $x_0 = e^{B\tau} y_0$, while preserving the time parameterization along the orbit:

$$y(t) = e^{-B\tau} x(t) = e^{-B\tau} e^{At} x_0 = e^{-B\tau} e^{At} e^{B\tau} y_0 = e^{At} y_0, \quad \text{if } \mathbb{T} = \mathbb{R},$$
$$y(t) = e^{-B\tau} x(t) = e^{-B\tau} A^t x_0 = e^{-B\tau} A^t e^{B\tau} y_0 = A^t y_0, \quad \text{if } \mathbb{T} = \mathbb{Z}.$$

Example 2.2 Let $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$; since $e^{At} = \left[\begin{smallmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{smallmatrix}\right]$, $A^t = \left[\begin{smallmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{smallmatrix}\right]$, $e^{B\tau} = e^{\tau}\left[\begin{smallmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{smallmatrix}\right]$, it is easy to check that $e^{-B\tau} e^{At} e^{B\tau} = e^{At}$ and $e^{-B\tau} A^t e^{B\tau} = A^t$.

The following definition and most of the following properties are standard (see, e.g., [13, 18, 34]).

Definition 2.3 (2.3.1) Given two square matrices $A, B \in \mathbb{R}^{n \times n}$, the *Lie bracket* (it is often called *matrix commutator*) of A and B is

$$[A, B] := BA - AB. \tag{2.7}$$

(2.3.2) The *linear centralizer* $\mathcal{L}_c(B)$ of B is the set of all matrices A such that $[A, B] = -[B, A] = 0$ (see, also, Definition 1.3 at p. 10).

Letting $g(x) = Bx$, $f(x) = Ax$ and $F(x) = Ax$, one finds that $[f(x), g(x)] = [A, B]x$, $\lfloor F(x), g(x) \rfloor = [A, B]x$; if $[A, B] = 0$, then A and B commute under the matrix product (briefly, A and B are *commuting*), and vice versa. In addition,

$$[A, B] = 0 \quad \Longleftrightarrow \quad \left[A^\top, B^\top\right] = 0.$$

Remark 2.2 If $[A, B] = 0$, then $e^{At}e^{B\tau} = e^{B\tau}e^{At} = e^{At+B\tau}$, for all $t, \tau \in \mathbb{R}$.

Theorem 2.1 *Relation* (2.6) *holds if and only if* $[A, B] = 0$, *i.e., if and only if* A *and* B *are commuting.*

Proof By (2.5), condition $[A, B] = 0$ is certainly necessary for relation (2.6) to hold. Since $e^{-B\tau}Ae^{B\tau}y|_{\tau=0} = Ay$, $\forall y \in \mathbb{R}^n$, equality (2.6) holds if and only if

$$\frac{\partial}{\partial \tau}\left(e^{-B\tau}Ae^{B\tau}y\right) = 0, \quad \forall y \in \mathbb{R}^n, \forall \tau \in \mathbb{R}. \tag{2.8}$$

In this way,

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(e^{-B\tau}Ae^{B\tau}y\right) &= -e^{-B\tau}BAe^{B\tau}y + e^{-B\tau}ABe^{B\tau}y = -e^{-B\tau}(BA - AB)e^{B\tau}y \\ &= -e^{-B\tau}[A, B]e^{B\tau}y.\end{aligned}$$

Since $e^{-B\tau}$ is invertible for all B and τ, equality (2.8) holds if and only if $[A, B] = 0$. □

Thanks to Theorem 2.1, the following definition is equivalent to Definition 2.1.

Definition 2.4 The linear transformation (2.3) is a *linear symmetry* of systems (2.1a), (2.1b) and system (2.2) is its *infinitesimal generator* if $[A, B] = 0$.

Remark 2.3 The Lie bracket of two square matrices enjoys the following properties, with $A, B, C \in \mathbb{R}^{n\times n}$ (which can be proven by simple substitution):

(2.3.1) $[A, B] = -[B, A]$ (*skew-symmetry*);
(2.3.2) $[\alpha B + \beta C, A] = \alpha[B, A] + \beta[C, A]$ and $[A, \alpha B + \beta C] = \alpha[A, B] + \beta[A, C]$, with $\alpha, \beta \in \mathbb{R}$ being constants (*bi-linearity*);
(2.3.3) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (*the Jacobi identity*).

By the skew-symmetry (Statement (2.3.1) of Remark 2.3),

$$A \in \mathcal{L}_c(B) \quad \Longleftrightarrow \quad B \in \mathcal{L}_c(A),$$

although, in general, $\mathcal{L}_c(B) \neq \mathcal{L}_c(A)$.

Another useful property is the *invariance of the matrix Lie bracket to linear transformations*, meaning the fact that, for any invertible $Q \in \mathbb{C}$, letting $\tilde{A} = Q^{-1}AQ$ and $\tilde{B} = Q^{-1}BQ$, one has

$$\left[\tilde{A}, \tilde{B}\right] = Q^{-1}[A, B]Q. \tag{2.9}$$

Some key facts about the linear centralizer of a square matrix are reviewed next, since such properties are of great importance in the sequel.

Lemma 2.1 *The linear centralizer* $\mathcal{L}_c(A)$ *of* $A \in \mathbb{R}^{n\times n}$ *is a finite dimensional vector space over* $\mathbb{R}$. *The dimension* r *of* $\mathcal{L}_c(A)$ *satisfies* $n \leq r \leq n^2$.

Proof It is easy to see that the bi-linearity (2.3.2) implies that $\mathscr{L}_c(A)$ is a vector space over $\mathbb{R}$: if $M_1, M_2 \in \mathscr{L}_c(A)$, then $\alpha_1 M_1 + \alpha_2 M_2 \in \mathscr{L}_c(A)$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ being constant. The fact that $\mathscr{L}_c(A)$ is finite dimensional is obvious, since $\mathscr{L}_c(A)$ is a subspace of $\mathbb{R}^{n\times n}$. As for its dimension r, the upper bound comes from the case $A = \alpha E$, with α being a (possibly zero) constant and E being the identity matrix; in such a case $\mathscr{L}_c(A) = \mathbb{R}^{n\times n}$, whence $r = n^2$. As for the lower bound, in view of (2.9), assume that $A = \text{block\_diag}\{J_1, \dots, J_p\}$ is in the Jordan form, with Jordan blocks J_i of dimension r_i; then, $J_i^0, \dots, J_i^{r_i-1}$ are linearly independent over $\mathbb{C}$. Therefore, in case of real eigenvalues, the n matrices

$$\begin{aligned}&\text{block\_diag}\{J_1^0, 0, \dots, 0\}, \dots, \text{block\_diag}\{J_1^{r_1-1}, 0, \dots, 0\}, \dots,\\ &\quad \text{block\_diag}\{0, \dots, 0, J_p^0\}, \dots, \text{block\_diag}\{0, \dots, 0, J_p^{r_p-1}\}\end{aligned} \tag{2.10}$$

commute with A and are linearly independent, whence $r \geq n$. In case of complex eigenvalues, a similar reasoning can be made by considering the real Jordan form (for a definition of the real Jordan form see the proof of the subsequent Lemma 2.5). □

As for the choice of a basis of $\mathscr{L}_c(A)$, it can be useful to take some of its elements in a simple way; therefore, note that the identity matrix $E = A^0$ can always be included in the basis of $\mathscr{L}_c(A)$, whereas A can be included except for the trivial case $A = 0$. More in general, if $A^0, A^1, \dots, A^{m-1}$, with $m \leq r$, are linearly independent over $\mathbb{R}$, then, with no loss of generality, one can assume that the first m elements of a basis of $\mathscr{L}_c(A)$ are $M_0 = E, M_1 = A^0, \dots, M_{m-1} = A^{m-1}$. A more powerful result is the subsequent Theorem 2.2, which is proven by means of the two lemmas below.

Lemma 2.2 *Let* $J = \text{block\_diag}\{J_1, \dots, J_p\}$ *be a Jordan matrix whose* p *Jordan blocks* J_i*, of dimension* r_i*, have distinct real eigenvalues* $\lambda_1, \dots, \lambda_p$*. Then, all the matrices commuting with* J *are of the form*

$$B = \text{block\_diag}\{B_1, \dots, B_p\}, \tag{2.11}$$

where $B_i \in \mathbb{R}^{r_i\times r_i}$ *and* $B_i J_i = J_i B_i$.

Proof The proof of the fact that matrix B in (2.11) commutes with J is trivial. To show the converse, assume that B commutes with J and partition it in blocks according to the dimensions r_i:

$$B = \begin{bmatrix} B_{1,1} & \dots & B_{1,p} \\ \vdots & \vdots & \vdots \\ B_{p,1} & \dots & B_{p,p} \end{bmatrix}, \quad B_{i,j} \in \mathbb{R}^{r_i\times r_j}.$$

By looking at the diagonal blocks of $BJ - JB$, it is easy to see that $BJ = JB$ implies, for each $i \in \{1, \dots, p\}$, that $B_{i,i} J_i = J_i B_{i,i}$, whence that matrices $B_{i,i}$ have the property of matrices B_i in (2.11). By looking at the off-diagonal blocks, it is easy to see that $BJ = JB$ implies that

$$B_{i,j} J_j = J_i B_{i,j}, \quad \forall i \neq j. \tag{2.12}$$

Consider the chain of generalized right eigenvectors of J_j, $v_1 = \bar{e}_1, \ldots, v_{r_j} = \bar{e}_{r_j}$ (with $\bar{e}_h$ being the hth column of the $r_j \times r_j$ identity matrix), that satisfy $J_j v_1 = \lambda_j v_1$, $J_j v_h = \lambda_j v_h + v_{h-1}$, $h = 2, \ldots, r_j$, and the chain of generalized left eigenvectors of J_i, namely $u_1^\top = \hat{e}_1^\top, \ldots, u_{r_i}^\top = \hat{e}_{r_i}^\top$ that satisfy $u_k^\top J_i = \lambda_i u_k^\top + u_{k+1}^\top$, $k = 1, \ldots, r_i - 1$, $u_{r_i}^\top J_i = \lambda_i u_{r_i}^\top$ (with $\hat{e}_k$ being the kth column of the $r_i \times r_i$ identity matrix). Equation (2.12) left multiplied by $u_{r_i}^\top$ and right multiplied by v_1 gives

$$\lambda_j [0 \ \ldots \ 0 \ 1] B_{i,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_i [0 \ \ldots \ 0 \ 1] B_{i,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.13}$$

which, since $\lambda_i \neq \lambda_j$, implies that the entry of the first column and of the last row of $B_{i,j}$ is zero. The equation similar to (2.13) obtained using v_2 instead of v_1 implies that the entry of the second column and of the last row of $B_{i,j}$ is zero. Using iteratively v_h with increasing h instead of v_1, one obtains that the last row of $B_{i,j}$ is zero. In the same manner, the equations similar to (2.13) obtained using $u_k^\top$ with decreasing k instead of $u_{r_i}^\top$ imply that the first column of $B_{i,j}$ is zero. The procedure can be repeated using in the proper order all the $u_k^\top$ and v_h to prove that $B_{i,j} = 0$. □

Lemma 2.3 *Let J be a Jordan block of dimension r relative to a real eigenvalue. Then, the dimension of $\mathscr{L}_c(J)$ is r and $\{J^0, J^1, \ldots, J^{r-1}\}$ is a basis of $\mathscr{L}_c(J)$.*

Proof The statement is equivalent to saying that the set of the matrices that commute with a Jordan block coincides with the set of all upper triangular *Toepliz matrices*, i.e., all upper triangular matrices such that, for a given $h \in \{0, \ldots, r-2\}$, the entries in position $(k, k+h)$, $k \in \{1, \ldots, r-h\}$, are equal. Assume that $A \in \mathbb{R}^{r \times r}$ commutes with J, i.e., $AJ = JA$. Taking into account the two entries in positions $(r-1, 1)$ and $(r, 2)$ of $AJ - JA$, one derives that, for them to be zero, it is necessary and sufficient that the entry $A_{r,1}$ of A is zero. Considering the three entries in positions $(r-2, 1)$, $(r-1, 2)$ and $(r, 3)$ of $AJ - JA$, one concludes that it is necessary and sufficient that $A_{r-1,1}$ and $A_{r,2}$ are zero. Then, iterating on h, which decreases from $r-2$ to 0, considering all the entries in positions $(k+h, k)$, $k \in \{1, \ldots, r-h\}$ of $AJ - JA$, one obtains that it is necessary and sufficient that all the entries $A_{i,j}$ such that $i - j = h + 1$ are zero. Therefore, A is upper triangular. Analogously, for each $h \in \{1, \ldots, r-1\}$, the entries in positions $(k, k+h)$, $k \in \{1, \ldots, r-h\}$, of $AJ - JA$, which must be zero, show that it is necessary and sufficient that all the entries $A_{i,j}$ such that $j - i = h - 1$ are equal, i.e., A is Toepliz. □

Results analogous to Lemmas 2.2 and 2.3 can be proven for the case of a matrix A having some complex eigenvalues, so that the following theorem holds in the general case.

Theorem 2.2 *Let $A \in \mathbb{R}^{n\times n}$. There exist linearly independent $M_0, \ldots, M_{n-1} \in \mathscr{L}_c(A)$ over $\mathbb{R}$, which are pairwise commuting, i.e., $[M_i, M_j] = 0$. If there are no Jordan blocks of A corresponding to the same eigenvalue, then $\mathscr{L}_c(A)$ has dimension n and $\{M_0, \ldots, M_{n-1}\}$ is a basis of $\mathscr{L}_c(A)$.*

Proof If A has only real eigenvalues, in view of (2.9), assume that A is in the Jordan form as in the proof of Lemma 2.1. It is easy to see that the matrices in (2.10), which are linearly independent and belong to $\mathscr{L}_c(A)$, are pairwise commuting, whence they constitute the set $\{M_0, \ldots, M_{n-1}\}$. To see that, when for each eigenvalue of A there is just one Jordan block, such a set is indeed a basis of $\mathscr{L}_c(A)$, in the case of real eigenvalues it suffices to consider Lemmas 2.2 and 2.3 to see that the n matrices in (2.10) actually generate the whole $\mathscr{L}_c(A)$. In case of complex eigenvalues, a similar reasoning can be made by considering the real Jordan form (see also the proof of the subsequent Lemma 2.5). □

Corollary 2.1 *If the Jordan form of A has not two Jordan blocks corresponding to the same eigenvalue, then $\{A^0, A^1, \ldots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$.*

Proof The proof follows from the proof of Theorem 2.2, taking into account that the minimal polynomial of A has degree n and, therefore, that $\{A^0, A^1, \ldots, A^{n-1}\}$ is a set of n linearly independent matrices that pairwise commute. □

Theorem 2.3 *Assume that $\{A^0, A^1, \ldots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$. Then, any pair $B_1, B_2 \in \mathscr{L}_c(A)$ is commuting.*

Proof Since $\{A^0, A^1, \ldots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$, B_1 and B_2 can be written as $B_1 = \sum_{i=0}^{n-1} a_{1,i} A^i$ and $B_2 = \sum_{j=0}^{n-1} a_{2,j} A^j$, for some constants $a_{1,i}, a_{2,i} \in \mathbb{R}$. Therefore, $[B_1, B_2] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{1,i} a_{2,j} [A^i, A^j] = 0$. □

Example 2.3 Let $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, with J_1 and J_2 being Jordan blocks of dimension two and three, with real eigenvalues λ_1 and λ_2, respectively. Then, the following matrices belong to $\mathscr{L}_c(J)$, are linearly independent over $\mathbb{R}$, and are commuting:

$$\left\{ \begin{bmatrix} J_1^0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_1^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & J_2^0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & J_2^1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & J_2^2 \end{bmatrix} \right\}.$$

Furthermore, if $\lambda_1 \neq \lambda_2$, then they constitute a basis of $\mathscr{L}_c(J)$.

Example 2.4 Consider matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix},$$

which is semi-simple with two coincident eigenvalues; $\mathscr{L}_c(A)$ has dimension $r = 5$ and one of its bases is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

There exist three linearly independent and commuting elements of $\mathscr{L}_c(A)$ over $\mathbb{R}$, which can be constructed from the Jordan form of A,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

as follows:

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

Such three matrices pairwise commute, but they do not constitute a basis of $\mathscr{L}_c(A)$.

A nice consequence of Theorem 2.2 is that if matrix A is semi-simple with distinct (possibly, complex) eigenvalues, then $\{A^0, A^1, \ldots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$.

Theorem 2.4 *Assume that A is semi-simple with distinct eigenvalues; let $Q \in \mathbb{C}^{n\times n}$, $\det(Q) \neq 0$, be such that $\tilde{A} = Q^{-1}AQ$ is diagonal. Then, $\tilde{B} = Q^{-1}BQ$ is diagonal for all $B \in \mathscr{L}_c(A)$; furthermore, any $B = Q\tilde{B}Q^{-1}$, with $\tilde{B}$ diagonal, is an element of $\mathscr{L}_c(A)$.*

Proof By (2.9), if $B \in \mathscr{L}_c(A)$, then $\tilde{B} \in \mathscr{L}_c(\tilde{A})$ for all $Q \in \mathbb{C}^{n\times n}$ such that $\det(Q) \neq 0$. If A is semi-simple with distinct eigenvalues, then $\{A^0, A^1, \ldots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$, whence a basis of $\mathscr{L}_c(\tilde{A})$ is $\{\tilde{A}^0, \tilde{A}^1, \ldots, \tilde{A}^{n-1}\}$. If $B \in \mathscr{L}_c(A)$, then there exist $\mu_0, \mu_1, \ldots, \mu_{n-1}$ such that $B = \sum_{i=0}^{n-1} \mu_i A^i$, whence $\tilde{B} =$

$\sum_{i=0}^{n-1} \mu_i \tilde{A}^i$, which implies that $\tilde{B}$ is diagonal. Vice versa, if $\tilde{B}$ is diagonal, then $\tilde{B} \in \mathscr{L}_c(\tilde{A})$, whence $B \in \mathscr{L}_c(A)$. □

If A is semi-simple, but with some coincident eigenvalues, and $[A, B] = 0$, then $\tilde{B} = Q^{-1}BQ$ need not be diagonal also if $\tilde{A} = Q^{-1}AQ$ is diagonal (see also Lemma 1.4 at p. 26).

Example 2.5 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; since A is semi-simple with distinct eigenvalues ($\pm$i), any $B \in \mathscr{L}_c(A)$ can be written as

$$B = \mu_0 A^0 + \mu_1 A^1 = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix}, \quad \mu_i \in \mathbb{R}.$$

Letting $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\mathrm{i} & -\frac{1}{2}\mathrm{i} \end{bmatrix}$, one has $\tilde{A} = Q^{-1}AQ = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}$. Therefore, Q jointly diagonalizes all elements of $\mathscr{L}_c(A)$:

$$\tilde{B} = \begin{bmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{bmatrix} \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\mathrm{i} & -\frac{1}{2}\mathrm{i} \end{bmatrix} = \begin{bmatrix} \mu_0 + \mathrm{i}\mu_1 & 0 \\ 0 & \mu_0 - \mathrm{i}\mu_1 \end{bmatrix}.$$

It is now possible to prove two lemmas that are very important in the sequel.

Lemma 2.4 *Any matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed as $A = A_s + A_n$, where $A_s \in \mathbb{R}^{n\times n}$ is semi-simple, $A_n \in \mathbb{R}^{n\times n}$ is nilpotent and A_s, A_n commute under the matrix product. Such matrices A_s and A_n can be expressed as polynomials in A, whence any matrix B that commutes under the matrix product with A also commutes with A_s and A_n.*

Proof It is sufficient to bring A into its complex Jordan form, $A = QJQ^{-1}$, where $\det(Q) \neq 0$ and $J = \text{block\_diag}\{J_1, \dots, J_p\}$, with J_i being a Jordan block with eigenvalue λ_i; if A has complex eigenvalues, then matrix Q has to be chosen so that its two block columns Q_i and Q_j containing two corresponding chains of generalized eigenvectors of A relative to λ_i and $\lambda_j = \lambda_i^*$, respectively, satisfy $Q_j = Q_i^*$. Then, in the new coordinates, letting

$$J_{i,s} = \text{diag}\{\lambda_i, \dots, \lambda_i\} \quad \text{and} \quad J_{i,n} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

one has $J_i = J_{i,s} + J_{i,n}$, with $J_{i,s}$ semi-simple and $J_{i,n}$ nilpotent. Let $A_s = QJ_sQ^{-1} = Q\,\text{block\_diag}\{J_{1,s}, \dots, J_{p,s}\}Q^{-1}$ and $A_n = QJ_nQ^{-1} = Q\,\text{block\_diag}\{J_{1,n}, \dots, J_{p,n}\}Q^{-1}$. Obviously, A_s is semi-simple and A_n is nilpotent. Moreover, in case of A having complex eigenvalues, thanks to the choice of Q required above, it is easy to verify that A_s and A_n are real. Now, to show that A_s and A_n can be written as polynomials in A, it is sufficient to show that J_s and J_n can be written as polynomials in J. Taking into account the expression of the kth power of a Jordan block, it is easy to see that if J_a and J_b are two Jordan blocks of dimensions $n_a \geq n_b$, relative to the same eigenvalue λ, then letting $J_{a,s} = \lambda E$ (where E has dimensions $n_a \times n_a$), $J_{a,n} = J_a - J_{a,s}$, $J_{b,s} = \lambda E$ (where E has dimensions $n_b \times n_b$) and $J_{b,n} = J_b - J_{b,s}$, the equations $J_{b,s} = p_s(J_b)$ and $J_{b,n} = p_n(J_b)$ hold necessarily, for any pair of polynomials $p_s(s)$ and $p_n(s)$ such that $J_{a,s} = p_s(J_a)$ and $J_{a,n} = p_n(J_a)$. Now, let $J_{\min}$ be a Jordan matrix with the same minimal polynomial of J, but without repeated eigenvalues (obtained by selecting from J just one Jordan block of highest dimension for each eigenvalue) and let D a diagonal matrix with the same diagonal as $J_{\min}$; let $N = J_{\min} - D$. The structures of J and $J_{\min}$ imply that if there exists a pair of polynomials $p_s(s)$ and $p_n(s)$ such that $D = p_s(J_{\min})$ and $N = p_n(J_{\min})$, then $J_s = p_s(J)$ and $J_n = p_n(J)$ hold necessarily. The existence of $p_s(s)$ and $p_n(s)$ is ensured by Theorem 2.2 with $A = J_{\min}$ and by Lemmas 2.2 and 2.3 (see also the beginning of the proof of Lemma 2.3). Hence, it is proven that A_s and A_n can be written as polynomials in A, and therefore that any matrix B that commutes under the matrix product with A also commutes with A_s and A_n. Clearly, if matrices A_s and A_n can be expressed as polynomials in A, then A_s, A_n commute under the matrix product. □

Remark 2.4 The decomposition $A = A_s + A_n$, with A_s being semi-simple and A_n being nilpotent is not unique; in general, there are many such decompositions with A_s and A_n that do not commute under the matrix product. But, as stated in [34, Lemma 14 at p. 104], if one requires that A_s and A_n commute, then such A_s and A_n are unique. As an example, take

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix};$$

clearly,

$$A_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is semi-simple,

$$A_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent and $A = A_s + A_n$, but

$$A_n A_s - A_s A_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not zero: it is easy to verify that neither A_s nor A_n can be expressed as a polynomial function of A. Bringing A in the Jordan form,

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

and then letting

$$A_s = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A_n = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

one concludes that $A = A_s + A_n$, $A_n A_s - A_s A_n = 0$; as a consequence, since

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2,$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2,$$

one finds that $A_s = 2A^0 - 2A^1 + A^2$ and $A_n = -2A^0 + 3A^1 - A^2$.

Lemma 2.5 *For any matrix A, there exists an invertible $Q \in \mathbb{R}^{n\times n}$, $\det(Q) \neq 0$, such that $\tilde{A} = Q^{-1}AQ$ can be uniquely decomposed as $\tilde{A} = \tilde{A}_{s,n} + \tilde{A}_n$, where $\tilde{A}_{s,n} \in \mathbb{R}^{n\times n}$ is normal, $\tilde{A}_n \in \mathbb{R}^{n\times n}$ is nilpotent and $\tilde{A}_n$, $\tilde{A}_n^\top$ and their powers $\tilde{A}_n^i$, $(\tilde{A}_n^\top)^i$, $i \in \mathbb{Z}^{\geq}$, commute with $\tilde{A}_{s,n}$ under the matrix product.*

Proof It is sufficient to proceed as in the proof of Lemma 2.4, but considering real Jordan blocks instead of (complex) Jordan blocks. In particular, choose Q so that $\tilde{A}$ is in the *real Jordan form*, i.e., a Jordan form in which each pair J_i, J_j of Jordan blocks of the same dimension $r_i = r_j$ corresponding to $\lambda_i = \alpha + i\beta$ and $\lambda_j = \lambda_i^* = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$, is substituted by a single block of dimension $2r_i$ that is the sum of a block-diagonal matrix whose r_i diagonal blocks are $\left[\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right]$ and a matrix having the only $2r_i - 2$ non-zero elements, which are equal to 1, in positions (k, h), where $h = k + 2$. □

Example 2.6 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}\mathrm{i} & \frac{1}{2}\mathrm{i} & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\mathrm{i} & 0 & 0 & 0 \\ 0 & 1-\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \mathrm{i} & \mathrm{i} & 1+\mathrm{i} \\ -1 & -\mathrm{i} & -\mathrm{i} & 1-\mathrm{i} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$A_s = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}\mathrm{i} & \frac{1}{2}\mathrm{i} & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\mathrm{i} & 0 & 0 & 0 \\ 0 & 1-\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \mathrm{i} & \mathrm{i} & 1+\mathrm{i} \\ -1 & -\mathrm{i} & -\mathrm{i} & 1-\mathrm{i} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is semi-simple, but it is not normal, and

$$A_n = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}\mathrm{i} & \frac{1}{2}\mathrm{i} & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & \mathrm{i} & \mathrm{i} & 1+\mathrm{i} \\ -1 & -\mathrm{i} & -\mathrm{i} & 1-\mathrm{i} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent, with $A = A_s + A_n$; by construction, such matrices A_s, A_n are commuting. Consider the real transformation represented by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

then,

$$\tilde{A} = Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

whence $\tilde{A} = \tilde{A}_{s,n} + \tilde{A}_n$, with

$$\tilde{A}_{s,n} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{A}_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

clearly, $\tilde{A}_{s,n}$ is normal, $\tilde{A}_n$ is nilpotent, and matrices $\tilde{A}_{s,n}, \tilde{A}_n$ (respectively, $\tilde{A}_{s,n}, \tilde{A}_n^\top$) are commuting.

The following definition extends the concept of linear symmetry to the concept of linear orbital symmetry, in the continuous-time case.

Definition 2.5 Assume $\mathbb{T} = \mathbb{R}$.

(2.5.1) The linear transformation (2.3) is a *linear orbital symmetry* of system (2.1a) and system (2.2) is its *infinitesimal generator* if

$$[A, B] = \mu A, \tag{2.14}$$

for some constant $\mu \in \mathbb{R}$. When (2.14) holds, by abuse of notation, also the infinitesimal generator (2.2) is called a *linear orbital symmetry* of system (2.1a); briefly, Bx is called a *linear orbital symmetry* of Ax.

(2.5.2) The *linear normalizer* $\mathscr{L}_n(A)$ of A is the set of all matrices B such that $[A, B] = \mu A$, for some constant $\mu \in \mathbb{R}$.

Remark 2.5 The linear orbital symmetry introduced in Definition 2.5 maps an orbit of system (2.1a) into an orbit of the same system, but the time parameterization along the orbit is not preserved when $\mu \neq 0$, as can be seen in the following. If B is a linear orbital symmetry of A, then

$$BA = AB + \mu A = A(B + \mu E),$$

with E being the $n \times n$ identity matrix; left multiplying such an equation by B,

$$B^2 A = BA(B + \mu E) = A(B + \mu E)^2,$$

and iterating such a process, one concludes that

$$B^h A = A(B+\mu E)^h, \quad \forall h \in \mathbb{Z}^{\geq}. \tag{2.15}$$

Hence,

$$\begin{aligned} e^{-B\tau} A &= \sum_{h=0}^{+\infty} \frac{(-\tau)^h}{h!} B^h A = \sum_{h=0}^{+\infty} \frac{(-\tau)^h}{h!} A(B+\mu E)^h \\ &= A e^{-(B+\mu E)\tau}, \end{aligned}$$

from which

$$e^{-B\tau} A e^{B\tau} = A e^{-\mu\tau}. \tag{2.16}$$

This shows that $\frac{dx}{dt} = Ax$ is transformed by the linear change of coordinates $x = e^{B\tau} y$ into $\frac{dy}{ds} = Ay$, where $\frac{ds}{dt} = e^{-\mu\tau}$, with s being the new time variable corresponding to the new time parameterization of the system thus transformed.

Example 2.7 Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$; B is a linear orbital symmetry of A, because $[A, B] = -2A$. Then, $e^{-B\tau} A e^{B\tau} = e^{2\tau} A$.

The following theorem, which can be easily proven by means of (2.9), shows that the concepts of linear symmetry and linear orbital symmetry do not depend on the particular coordinates chosen.

Theorem 2.5 *Let $\tilde{A} = Q^{-1} A Q$ and $\tilde{B} = Q^{-1} B Q$, with Q being an invertible square matrix. Then,*

$$[A, B] = \mu A \iff [\tilde{A}, \tilde{B}] = \mu \tilde{A},$$

namely

$$B \in \mathscr{L}_n(A) \iff \tilde{B} \in \mathscr{L}_n(\tilde{A}).$$

Theorem 2.6 *Let A be semi-simple with distinct eigenvalues. Then, $\mathscr{L}_n(A) = \mathscr{L}_c(A)$.*

Proof If $n = 1$, then $\mathscr{L}_n(A) = \mathscr{L}_c(A) = \mathbb{R}$, and the theorem holds. Assume $n \geq 2$. By definition $\mathscr{L}_c(A) \subseteq \mathscr{L}_n(A)$; hence, if $\mathscr{L}_n(A) \subseteq \mathscr{L}_c(A)$, then the theorem is proven. By Theorem 2.5, it can be assumed that

$$A = \text{block\_diag}\left\{\lambda_1, \ldots, \lambda_{n_r}, \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{n_c} & \beta_{n_c} \\ -\beta_{n_c} & \alpha_{n_c} \end{bmatrix}\right\},$$

with $n_r + 2n_c = n$, $\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j$ if $i \neq j$, and $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$, and $\alpha_i + i\beta_i \neq \alpha_j + i\beta_j$, if $i \neq j$. Under this assumption, it is now shown that $AM - MA = \mu A$

implies $\mu = 0$, whence that $M \in \mathscr{L}_n(A)$ implies $M \in \mathscr{L}_c(A)$. In the simpler case $n_c = 0$, when A has no complex eigenvalues, it is easily seen that all the diagonal elements of $AM - MA$ are structurally equal to zero; this, in view of the fact that at least one eigenvalue λ_i of A is not zero, implies that $\mu = 0$. As for the general case, if $n_r > 0$ and there is a real eigenvalue $\lambda_i \neq 0$ of A, then the equality $\mu = 0$ follows as above from the fact that the ith diagonal element of $AM - MA$ is zero. Otherwise (i.e., if $n_r = 0$, or $n_r = 1$ and $\lambda_1 = 0$), consider the 2×2 diagonal block of $AM - MA$ whose upper left element is in position $(n_r + 1, n_r + 1)$ and impose that it is equal to the same block of matrix μA, to obtain

$$\begin{bmatrix} \beta_1 L_2 & \beta_1 L_1 \\ \beta_1 L_1 & -\beta_1 L_2 \end{bmatrix} = \begin{bmatrix} \mu\alpha_1 & \mu\beta_1 \\ -\mu\beta_1 & \mu\alpha_1 \end{bmatrix},$$

where $L_1 = M_{n_r+2,n_r+2} - M_{n_r+1,n_r+1}$ and $L_2 = M_{n_r+2,n_r+1} - M_{n_r+1,n_r+2}$. Then, since $\beta_1 \neq 0$, the two equations $\beta_1 L_1 = \mu\beta_1$ and $\beta_1 L_1 = -\mu\beta_1$ imply $\mu = 0$. □

The following Theorem 2.7 shows that the linear normalizer $\mathscr{L}_n(A)$ and the linear centralizer $\mathscr{L}_c(A) \subseteq \mathscr{L}_n(A)$ of A are closed under the Lie bracket operation; in particular, since $B_1, B_2 \in \mathscr{L}_n(A)$ implies $[B_1, B_2] \in \mathscr{L}_c(A)$, by the analysis of the subsequent Sect. 6.2 and taking into account the subsequent Theorem 2.9, one concludes that $\mathscr{L}_n(A)$ is a Lie sub-algebra of the Lie algebra of matrices over $\mathbb{R}$, and $\mathscr{L}_c(A)$ is a Lie ideal of $\mathscr{L}_n(A)$.

Theorem 2.7 *If $B_1 x$ and $B_2 x$ are two linear orbital symmetries (possibly, linear symmetries) of Ax, then $[B_1, B_2]x$ is a linear symmetry of Ax.*

Proof From $[A, B_1] = \mu_1 A$ and $[A, B_2] = \mu_2 A$, with μ_1 and μ_2 being (possibly, equal to zero) real constants, one finds that (taking into account the Jacobi identity (2.3.3) reported in Remark 2.3):

$$\begin{aligned}\big[A, [B_1, B_2]\big] &= -\big[B_1, [B_2, A]\big] - \big[B_2, [A, B_1]\big] = [B_1, \mu_2 A] - [B_2, \mu_1 A] \\ &= -\mu_1\mu_2 A + \mu_1\mu_2 A = 0.\end{aligned}$$ □

In particular, the following theorem shows that, if A is semi-simple with distinct eigenvalues and B_1 and B_2 commute with A, then $[B_1, B_2] = 0$ (i.e., B_1 and B_2 are commuting).

Theorem 2.8 *Let A be semi-simple with distinct eigenvalues. Then, all elements of $\mathscr{L}_c(A)$ are commuting, i.e., if $B_1, B_2 \in \mathscr{L}_c(A)$, then $[B_1, B_2] = 0$.*

Proof By the invariance of the matrix Lie bracket to linear transformations, assume that A is diagonal, with distinct eigenvalues. Then, all $B \in \mathscr{L}_c(A)$ are diagonal, but two diagonal matrices B_1, B_2 are necessarily commuting. □

Example 2.8 Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 3 \end{smallmatrix}\right]$, which is semi-simple with distinct eigenvalues. A basis of $\mathscr{L}_c(A)$ is given by A^0 and A^1. Let B_1 and B_2 be two elements of $\mathscr{L}_c(A)$, then

$$B_1 = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 2a_2 \\ 0 & a_1 + 3a_2 \end{bmatrix},$$

$$B_2 = a_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} a_3 + a_4 & 2a_4 \\ 0 & a_3 + 3a_4 \end{bmatrix}.$$

Clearly, $[B_1, B_2] = 0$, for all $a_i \in \mathbb{R}$.

The following theorem shows that $\mathscr{L}_n(A)$ has the structure of a finite dimensional vector space over $\mathbb{R}$, similarly to $\mathscr{L}_c(A)$.

Theorem 2.9 *The linear normalizer $\mathscr{L}_n(A)$ of A is a finite dimensional vector space over $\mathbb{R}$ of dimension $n \leq r \leq n^2$.*

Proof If $B_1, B_2 \in \mathscr{L}_n(A)$, then there exist two constant $\mu_1, \mu_2 \in \mathbb{R}$ such that $[A, B_1] = \mu_1 A$ and $[A, B_2] = \mu_1 A$; by the bi-linearity of the Lie bracket operation, one finds that

$$[A, \alpha_1 B_1 + \alpha_2 B_2] = \alpha_1 [A, B_1] + \alpha_2 [A, B_2] = (\alpha_1 \mu_1 + \alpha_2 \mu_2) A, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R},$$

and, therefore, that $\alpha_1 B_1 + \alpha_2 B_2 \in \mathscr{L}_n(A)$. Since $\mathscr{L}_n(A) \subseteq \mathbb{R}^{n \times n}$, its dimension satisfies $r \leq n^2$. In addition, since $\mathscr{L}_c(A) \subseteq \mathscr{L}_n(A)$, the dimension r of $\mathscr{L}_n(A)$ satisfies $r \geq n$. □

Clearly, $\mathscr{L}_c(A) \subseteq \mathscr{L}_n(A)$. Determining the linear centralizer $\mathscr{L}_c(A)$ (respectively, the linear normalizer $\mathscr{L}_n(A)$) of A is equivalent to solving a set of n^2 algebraic (linear for each fixed μ) equations having the entries of B (respectively, the entries of B and the real number μ) as unknowns, as detailed in the following example.

Example 2.9 Consider matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ \alpha & \beta \end{smallmatrix}\right]$ with $\alpha, \beta \in \mathbb{R}$; such a matrix is not semi-simple when $\alpha = -\lambda^2$, $\beta = 2\lambda$ for some constant $\lambda \in \mathbb{R}$. Letting $B = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]$, from condition

$$0 = [A, B] = \begin{bmatrix} b_2\alpha - b_3 & b_1 + b_2\beta - b_4 \\ b_4\alpha - \alpha b_1 - \beta b_3 & b_3 - b_2\alpha \end{bmatrix},$$

one has the following system of four linear algebraic equations:

$$\begin{bmatrix} 0 & \alpha & -1 & 0 \\ -\alpha & 0 & -\beta & \alpha \\ 1 & \beta & 0 & -1 \\ 0 & -\alpha & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the rank of the coefficient matrix of the above system is equal to 2 for all $\alpha, \beta \in \mathbb{R}$, and matrices $M_0 = E$ and $M_1 = A$ are linearly independent over $\mathbb{R}$, one concludes that $\mathscr{L}_c(A) = \text{span}_{\mathbb{R}}\{E, A\}$. Consider now the case $\alpha = 0$, $\beta = 0$. The condition $[A, B] = \mu A$ leads to

$$b_1 = c_2 + c_3, \qquad b_2 = c_1, \qquad b_3 = 0, \qquad b_4 = c_2, \qquad \mu = c_3,$$

with $c_1, c_2, c_3 \in \mathbb{R}$ being arbitrary constants, which shows that $\mathscr{L}_n(A)$ has dimension 3. As basis of $\mathscr{L}_n(A)$, one can take the basis of $\mathscr{L}_c(A)$ (corresponding to the choice $c_3 = 0$) completed with the matrix B found by letting $c_1 = 0$, $c_2 = 0$ and $c_3 = 1$.

2.2 Darboux Polynomials and First Integrals

The following classical theorem, which is also used in other chapters, is stated and proven here because its statement and proof have strong similarities with the subsequent results. The theorem itself can be proven directly in the discrete-time case, and by an alternative simpler proof based on the Jordan form of A in the continuous-time case.

Theorem 2.10 *Let* $\varpi(t) = \det(\mathrm{e}^{At})$ *if* $\mathbb{T} = \mathbb{R}$ *(respectively,* $\varpi(t) = \det(A^t)$ *if* $\mathbb{T} = \mathbb{Z}$*); then* $\frac{\mathrm{d}\varpi(t)}{\mathrm{d}t} = \text{trace}(A)\varpi(t)$ *if* $\mathbb{T} = \mathbb{R}$ *(respectively,* $\varpi(t+1) = \det(A)\varpi(t)$ *if* $\mathbb{T} = \mathbb{Z}$*) and* $\varpi(0) = 1$.

Proof Clearly, $\varpi(0) = \det(E) = 1$ (also when $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$). Let $v_i(t) \in \mathbb{R}^n$ be the ith column of matrix e^{At} (respectively, A^t); then, $\Delta v_i = A v_i$ and $v_i(0) = e_i$, with e_i being the ith column of the $n \times n$ identity matrix E. The proof follows from the multi-linearity of the determinant in the continuous-time case:

$$\begin{aligned}
\Delta\varpi &= \Delta \det\big([v_1\ v_2\ \dots\ v_n]\big)\\
&= \det\big([\Delta v_1\ v_2\ \dots\ v_n]\big) + \det\big([v_1\ \Delta v_2\ \dots\ v_n]\big) + \dots + \det\big([v_1\ v_2\ \dots\ \Delta v_n]\big)\\
&= \det\big([A v_1\ v_2\ \dots\ v_n]\big) + \det\big([v_1\ A v_2\ \dots\ v_n]\big) + \dots + \det\big([v_1\ v_2\ \dots\ A v_n]\big)\\
&= \text{trace}(A) \det\big([v_1\ v_2\ \dots\ v_n]\big) = \text{trace}(A)\varpi, \quad \text{if } \mathbb{T} = \mathbb{R},
\end{aligned}$$

and by the following relationships in the discrete-time case:

$$\begin{aligned}
\Delta\varpi &= \Delta \det\big([v_1\ v_2\ \dots\ v_n]\big)\\
&= \det\big([\Delta v_1\ \Delta v_2\ \dots\ \Delta v_n]\big) = \det\big([A v_1\ A v_2\ \dots\ A v_n]\big)\\
&= \det(A) \det\big([v_1\ v_2\ \dots\ v_n]\big) = \det(A)\varpi, \quad \text{if } \mathbb{T} = \mathbb{Z}.
\end{aligned}$$ □

The concept of the Darboux polynomial extends the concept of polynomial first integral and the concept of left eigenfunction for linear systems. The definition of the

Darboux polynomial is given in the literature (see, e.g., [56, 88, 96]) for nonlinear systems but, in view of the importance of this concept, the special properties of Darboux polynomials for linear systems are studied in this section.

Definition 2.6 A scalar polynomial $\omega(x)$ is a *Darboux polynomial* (a *polynomial semi-invariant*) of systems (2.1a), (2.1b) if there exists a $\lambda \in \mathbb{R}$ such that the following relation holds for all $x \in \mathbb{R}^n$:

$$\Delta\omega = \lambda\omega, \tag{2.17}$$

with $\Delta\omega = L_{Ax}\omega$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\Delta\omega = \omega \circ Ax$ if $\mathbb{T} = \mathbb{Z}$). The number λ is called *characteristic value*.

In the subsequent Chaps. 3 and 4, the definition of the Darboux polynomial is given for nonlinear systems, by allowing λ to be a polynomial function of x. Here, a simple reasoning on the degree of polynomials and the linearity of systems (2.1a), (2.1b) imply that λ is necessarily constant. Therefore, there is no loss of generality in assuming λ constant as is done in Definition 2.6.

From Definition 2.6, a polynomial first integral is a Darboux polynomial with $\lambda = 0$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\lambda = 1$ if $\mathbb{T} = \mathbb{Z}$). It is worth pointing out that, if not empty, the set $\mathscr{I}_\omega$ of all $x \in \mathbb{R}^n$ such that $\omega(x) = 0$, with $\omega(x)$ being a Darboux polynomial of systems (2.1a), (2.1b), is an invariant subspace of systems (2.1a), (2.1b), i.e., if $x(0) \in \mathbb{R}^n$ is such that $\omega(x(0)) = 0$, then $\omega(x(t)) = 0$, $\forall t \in \mathbb{T}$ ($t \geq 0$ if $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$), along the solutions of systems (2.1a), (2.1b). To be more precise, by (2.17), $\omega(t) = \mathrm{e}^{\lambda t}\omega(0)$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\omega(t) = \lambda^t\omega(0)$ if $\mathbb{T} = \mathbb{Z}$), whence $\omega(t) = 0, \forall t \in \mathbb{T}$ ($t \geq 0$ if $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$), if and only if $\omega(0) = 0$. Clearly, the same is true for a polynomial first integral.

The following four theorems characterize the Darboux polynomials of systems (2.1a), (2.1b).

Theorem 2.11 *Let* $\mathbb{T} = \mathbb{R}$ *and* $M_i \in \mathscr{L}_n(A)$, $i = 1, \dots, n-1$. *Assume that the function* $\omega(x)$ *defined as follows*:

$$\omega(x) := \det\bigl(\Omega(x)\bigr), \quad \Omega(x) = [Ax\ M_1x\ \dots\ M_{n-1}x], \tag{2.18}$$

is not identically equal to zero. Then, the following properties hold:

(2.11.1) *relation* (2.17) *holds with* $\lambda = \mathrm{trace}(A)$, *i.e.*, ω *is a Darboux polynomial of system* (2.1a), *with characteristic value* $\lambda = \mathrm{trace}(A)$;

(2.11.2) *if the polynomial* ω *can be factorized as* $\omega = \prod_i \omega_i^{\nu_i}$, *with* ω_i*'s being coprime real polynomials and* $\nu_i \in \mathbb{Z}^>$, *then there exist constants* $\lambda_i \in \mathbb{R}$ *such that* $\sum_i \nu_i\lambda_i = \mathrm{trace}(A)$ *and such that* (2.17) *holds with* $\omega = \omega_i$ *and* $\lambda = \lambda_i$.

Proof Consider, first, Statement (2.11.1) of the theorem. By linear algebra, applying Δ to ω, one finds that

$$\Delta\omega = \det\bigl([A\Delta x\ M_1x\ \dots\ M_{n-1}x]\bigr) + \det\bigl([Ax\ M_1\Delta x\ \dots\ M_{n-1}x]\bigr)$$

$$+\cdots+\det\bigl([Ax\ M_1x\ \dots\ M_{n-1}\Delta x]\bigr);$$

since $\Delta x = Ax$ along the solutions of (2.1a), it follows that

$$\Delta\omega = \det\bigl([AAx\ M_1x\ \dots\ M_{n-1}x]\bigr) + \det\bigl([Ax\ M_1Ax\ \dots\ M_{n-1}x]\bigr)$$
$$+\cdots+\det\bigl([Ax\ M_1x\ \dots\ M_{n-1}Ax]\bigr);$$

$M_iA = AM_i + \mu_i A$, because $M_i \in \mathscr{L}_n(A)$, which implies

$$\begin{aligned}\Delta\omega = \det\bigl(&[AAx\ M_1x\ \dots\ M_{n-1}x]\bigr)\\ &+\det\bigl([Ax\ AM_1x+\mu_1Ax\ \dots\ M_{n-1}x]\bigr)\\ &+\cdots+\det\bigl([Ax\ M_1x\ \dots\ AM_{n-1}x+\mu_{n-1}Ax]\bigr),\end{aligned}$$

and, therefore, by linear algebra, one concludes that

$$\Delta\omega = \operatorname{trace}(A)\det\bigl([Ax\ M_1x\ \dots\ M_{n-1}x]\bigr) = \operatorname{trace}(A)\omega,$$

thus proving Statement (2.11.1) of the theorem. As for Statement (2.11.2) of the theorem, from $\omega = \prod_i \omega_i^{\nu_i}$, one finds that

$$\frac{L_{Ax}\omega}{\omega} = \sum_i \nu_i \frac{L_{Ax}\omega_i}{\omega_i}. \tag{2.19}$$

Since $L_{Ax}\omega_i = \frac{\partial\omega_i}{\partial x}Ax$, it follows that $\frac{L_{Ax}\omega_i}{\omega_i}$ is a proper rational function. Since all denominators of the rational functions $\frac{L_{Ax}\omega_i}{\omega_i}$ are co-prime, such functions do not have common poles (common roots of the denominators). Let x^o be a root of ω_j: then, $\omega_j(x^o) = 0$ and $\omega_i(x^o) \neq 0$, $\forall i \neq j$; since $\frac{L_{Ax}\omega}{\omega}$ is constant, $\sum_i \nu_i \frac{L_{Ax}\omega_i}{\omega_i}$ is constant only if $L_{Ax}\omega_j(x^o) = 0$; then, iterating through all roots of the polynomials ω_i, and taking into account that each $\frac{L_{Ax}\omega_i}{\omega_i}$ is proper, one concludes that each $\frac{L_{Ax}\omega_i}{\omega_i}$ is equal to a certain constant λ_i. Finally, equation (2.19) shows that $\sum_i \nu_i\lambda_i = \operatorname{trace}(A)$, taking into account that $\frac{L_{Ax}\omega}{\omega} = \operatorname{trace}(A)$ and $\frac{L_{Ax}\omega_i}{\omega_i} = \lambda_i$, thus proving Statement (2.11.2) of the theorem. □

Note that Theorem 2.11 extends Theorem 2.10 (in the case $\mathbb{T} = \mathbb{R}$), because if $x(0)$ is chosen so that $\omega(x(0)) = 1$, then $\omega(x(t))$ is just the function $\varpi(t)$ of Theorem 2.10.

The next theorem is the analogous of Theorem 2.11 for discrete-time systems, but two important differences have to be stressed: in the discrete-time case, matrices M_i are required to commute with A and a statement analogous to Statement (2.11.2) of Theorem 2.11 does not hold.

Theorem 2.12 *Let $\mathbb{T} = \mathbb{Z}$ and $M_i \in \mathscr{L}_c(A)$, $i = 1, \dots, n-1$. Assume that the function $\omega(x)$ defined as follows*:

$$\omega(x) := \det\bigl(\Omega(x)\bigr), \quad \Omega(x) = [Ax\ M_1x\ \dots\ M_{n-1}x],$$

is not identically equal to zero. Then, relation (2.17) *holds with* $\lambda = \det(A)$, *i.e.*, ω *is a Darboux polynomial of system* (2.1*b*), *with characteristic value* $\lambda = \det(A)$.

Proof Taking into account that $M_i A = AM_i$, one has

$$\begin{aligned}\Delta\omega &= \det\big(\Omega(\Delta x)\big) = \det\big(\Omega(Ax)\big) = \det\big([AAx\ M_1Ax\ \dots\ M_{n-1}Ax]\big)\\ &= \det\big([AAx\ AM_1x\ \dots\ AM_{n-1}x]\big) = \det(A)\det\big(\Omega(x)\big) = \det(A)\omega(x),\end{aligned}$$

as to be proven. □

By Theorem 2.6, if matrix A is semi-simple and has distinct eigenvalues, then $\mathscr{L}_n(A) = \mathscr{L}_c(A)$; for this reason, the continuous-time and discrete-time cases are considered jointly in the next theorem.

Theorem 2.13 *Consider jointly both cases* $\mathbb{T} = \mathbb{R}$ *and* $\mathbb{T} = \mathbb{Z}$. *Let* A *be semi-simple with distinct eigenvalues, let* n_r *be the number of real eigenvalues of* A, *let* n_c *be the number of pairs of complex conjugate eigenvalues of* A (*in particular*, $n_r + 2n_c = n$), *and let the eigenvalues of* A *be ordered as*

$$\big\{\lambda_1, \dots, \lambda_{n_r}, \lambda_{n_r+1}, \lambda^*_{n_r+1}, \dots, \lambda_{n_r+n_c}, \lambda^*_{n_r+n_c}\big\}.$$

Then, the polynomial $\omega(x)$ *defined in Theorems* 2.11 *and* 2.12 *can be factorized as follows*:

$$\omega(x) = k\,\hat{\omega}_1(x)\cdots\hat{\omega}_{n_r+n_c}(x), \tag{2.20}$$

where $k \in \mathbb{R}$ *is a constant, and*

$$\hat{\omega}_i(x) = \begin{cases} u_i^\top x, & i = 1, \dots, n_r,\\ (u_i^\top x)(u_i^\top x)^*, & i = n_{r+1}, \dots, n_r + n_c,\end{cases}$$

with $u_i^\top$ *being the left eigenvector relative to eigenvalue* λ_i, $i = 1, \dots, n_r + n_c$.

Proof Let $\Omega(x)$ be defined as in Theorems 2.11 and 2.12. Since one of the bases of $\mathscr{L}_c(A)$ is given by $\{E, A, \dots, A^{n-1}\}$, one has $\Omega(x) = \bar{\Omega}(x)T$ for some invertible matrix $T \in \mathbb{R}^{n\times n}$, where

$$\bar{\Omega}(x) = \big[Ex\ Ax\ \dots\ A^{n-1}x\big]. \tag{2.21}$$

For $i = 1, \dots, n_r + n_c$, one has

$$\begin{aligned}u_i^\top\bar{\Omega}(x) &= \big[u_i^\top Ex\ u_i^\top Ax\ \dots\ u_i^\top A^{n-1}x\big]\\ &= \big[\big(u_i^\top x\big)\ \lambda_i\big(u_i^\top x\big)\ \dots\ \lambda_i^{n-1}\big(u_i^\top x\big)\big]\\ &= \big(u_i^\top x\big)\big[1\ \lambda_i\ \dots\ \lambda_i^{n-1}\big];\end{aligned}$$

hence, defining the (possibly, complex) matrices

$$U := \begin{bmatrix} u_1^\top \\ \vdots \\ u_{n_r}^\top \\ u_{n_r+1}^\top \\ \vdots \\ u_{n_r+n_c}^\top \\ u_{n_r+1}^{*\top} \\ \vdots \\ u_{n_r+n_c}^{*\top} \end{bmatrix}, \qquad V := \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r} & \dots & \lambda_{n_r}^{n-1} \\ 1 & \lambda_{n_r+1} & \dots & \lambda_{n_r+1}^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r+n_c} & \dots & \lambda_{n_r+n_c}^{n-1} \\ 1 & \lambda_{n_r+1}^* & \dots & (\lambda_{n_r+1}^*)^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r+n_c}^* & \dots & (\lambda_{n_r+n_c}^*)^{n-1} \end{bmatrix},$$

one obtains

$$\begin{aligned} U\Omega(x) &= U\bar{\Omega}(x)T \\ &= \operatorname{diag}\{u_1^\top x, \dots, u_{n_r}^\top x, u_{n_r+1}^\top x, \dots, u_{n_r+n_c}^\top x, u_{n_r+1}^{*\top} x, \dots, u_{n_r+n_c}^{*\top} x\} VT; \end{aligned}$$

taking the determinant of both sides of the above equation, one proves the theorem with $k = \frac{\det(V)\det(T)}{\det(U)}$, where $\det(V) \neq 0$ because the eigenvalues of A are distinct. □

The following theorem shows, also in the case that matrix A has repeated eigenvalues, that a Darboux polynomial can be computed by starting from every left eigenvector of matrix A.

Theorem 2.14 *Let $u^\top$ be a left eigenvector of A, i.e., $u^\top A = \lambda u^\top$. Then,*

$$\omega(x) = \begin{cases} u^\top x, & \text{if } \lambda \in \mathbb{R}, \\ (u^\top x)(u^\top x)^*, & \text{if } \lambda \notin \mathbb{R}, \end{cases}$$

is a Darboux polynomial of systems (2.1a), (2.1b).

Proof If $\lambda \in \mathbb{R}$, then

$$\begin{aligned} \Delta\omega(x) = \Delta u^\top x = u^\top \Delta x = u^\top A x = \lambda u^\top x \\ = \lambda\omega(x), \end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value λ. Consider now the case $\lambda \notin \mathbb{R}$. When $\mathbb{T} = \mathbb{Z}$,

$$\begin{aligned} \Delta\omega(x) &= \Delta(u^\top x)\Delta(u^{*\top} x) = (u^\top \Delta x)(u^{*\top}\Delta x) = (u^\top A x)(u^{*\top} A x) \\ &= (\lambda u^\top x)(\lambda^* u^{*\top} x) = \lambda\lambda^*(u^\top x)(u^{*\top} x) \\ &= |\lambda|^2\omega(x), \end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value $|\lambda|^2$. When $\mathbb{T}=\mathbb{R}$,

$$\begin{aligned}\Delta\omega(x) &= \big(u^{*\top}x\big)\Delta\big(u^{\top}x\big)+\big(u^{\top}x\big)\Delta\big(u^{*\top}x\big)\\ &= \big(u^{*\top}x\big)\big(u^{\top}\Delta x\big)+\big(u^{\top}x\big)\big(u^{*\top}\Delta x\big)\\ &= \big(u^{*\top}x\big)\big(u^{\top}Ax\big)+\big(u^{\top}x\big)\big(u^{*\top}Ax\big)\\ &= \lambda\big(u^{*\top}x\big)\big(u^{\top}x\big)+\lambda^*\big(u^{\top}x\big)\big(u^{*\top}x\big)\\ &= \big(\lambda+\lambda^*\big)\omega(x)=2\operatorname{Re}(\lambda)\omega(x),\end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value $2\operatorname{Re}(\lambda)$. □

Remark 2.6 Let ω_1 and ω_2 be two Darboux polynomials of systems (2.1a), (2.1b) with characteristic values λ_1 and λ_2, respectively: $\Delta\omega_i=\lambda_i\omega_i$, $i=1,2$. Then, $\omega=\omega_1^{\nu_1}\omega_2^{\nu_2}$, with $\nu_i\in\mathbb{Z}^{\geq}$, is still a Darboux polynomial of systems (2.1a), (2.1b), with characteristic value $\lambda=\nu_1\lambda_1+\nu_2\lambda_2$ if $\mathbb{T}=\mathbb{R}$ ($\lambda=\lambda_1^{\nu_1}\lambda_2^{\nu_2}$ if $\mathbb{T}=\mathbb{Z}$). To be more precise, if $\mathbb{T}=\mathbb{R}$, then

$$\begin{aligned}\Delta\omega &= \nu_1\omega_1^{\nu_1-1}\omega_2^{\nu_2}\Delta\omega_1+\nu_2\omega_1^{\nu_1}\omega_2^{\nu_2-1}\Delta\omega_2=(\nu_1\lambda_1+\nu_2\lambda_2)\omega_1^{\nu_1}\omega_2^{\nu_2}\\ &= (\nu_1\lambda_1+\nu_2\lambda_2)\omega,\end{aligned}$$

whereas, if $\mathbb{T}=\mathbb{Z}$, then

$$\Delta\omega=(\Delta\omega_1)^{\nu_1}(\Delta\omega_2)^{\nu_2}=\lambda_1^{\nu_1}\lambda_2^{\nu 2}\omega_1^{\nu_1}\omega_2^{\nu_2}=\lambda_1^{\nu_1}\lambda_2^{\nu_2}\omega.$$

The second part of the next remark suggests a practical way for the computation of first integrals for linear systems.

Remark 2.7 Following the same reasoning of the proof of Statement (2.11.2) of Theorem 2.11, one can demonstrate the following claims. If $\mathbb{T}=\mathbb{R}$ and I is a rational first integral of system (2.1a), then I can be factorized as $I=\prod_i\omega_i^{\nu_i}$, with ω_i being Darboux polynomials of system (2.1a) and ν_i being (positive or negative) integers. Conversely, both in the continuous-time and discrete-time cases, if $\omega_1,\omega_2,\ldots$ are Darboux polynomials of systems (2.1a), (2.1b), $\Delta\omega_i=\lambda_i\omega_i$, such that $\sum_i\nu_i\lambda_i=0$ (respectively, $\prod_i\lambda_i^{\nu_i}=1$), with ν_i being either positive or negative integers, then $I=\prod_i\omega_i^{\nu_i}$ is a rational first integral of systems (2.1a), (2.1b).

Example 2.10 Constants $\lambda_i\in\mathbb{R}$ appearing in Statement (2.11.2) of Theorem 2.11 need not be eigenvalues of matrix A. For instance, if $A=\left[\begin{smallmatrix}1&1\\-1&1\end{smallmatrix}\right]$, then

$$\omega(x)=\det\big([Ax\ x]\big)=\det\left(\begin{bmatrix}x_1+x_2 & x_1\\ -x_1+x_2 & x_2\end{bmatrix}\right)=x_1^2+x_2^2$$

satisfies

$$\Delta\omega=\begin{cases}[2x_1\ 2x_2]\left[\begin{smallmatrix}x_1+x_2\\-x_1+x_2\end{smallmatrix}\right]=2(x_1^2+x_2^2), & \text{if } \mathbb{T}=\mathbb{R},\\ (F_1^2+F_2^2)|_{F_1=x_1+x_2,F_2=x_2-x_1}=2(x_1^2+x_2^2), & \text{if } \mathbb{T}=\mathbb{Z},\end{cases}$$

with 2 that is not eigenvalue of A (where $\det(A) = \text{trace}(A) = 2$).

In the next example, the concept of irreducible polynomial is needed. A polynomial with real coefficients is said to be *irreducible over* $\mathbb{R}$ if it is not constant and cannot be rewritten as the product of two non-constant polynomials with real coefficients.

Example 2.11 Consider matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1\lambda_2\lambda_3 & -\lambda_1\lambda_3 - \lambda_1\lambda_2 - \lambda_2\lambda_3 & \lambda_1 + \lambda_3 + \lambda_2 \end{bmatrix},$$

with $\lambda_1, \lambda_2, \lambda_3$ being scalars. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then such a matrix has three real eigenvalues λ_1, λ_2 and λ_3 with respective real left eigenvectors:

$$u_1 = \begin{bmatrix} \lambda_2\lambda_3 \\ -\lambda_3 - \lambda_2 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} \lambda_1\lambda_3 \\ -\lambda_1 - \lambda_3 \\ 1 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} \lambda_1\lambda_2 \\ -\lambda_1 - \lambda_2 \\ 1 \end{bmatrix},$$

which are linearly independent over $\mathbb{R}$ when $\lambda_i \neq \lambda_j$, $i \neq j$. Two linear symmetries of Ax, such that $\det(\Omega) \neq 0$, are given by Ex and A^2x, thus yielding

$$\begin{aligned} \omega(x) &= \det\big(\Omega(x)\big) = \det\big(\big[Ax \; Ex \; A^2x\big]\big) \\ &= \big(\lambda_2\lambda_3x_1 - (\lambda_3 + \lambda_2)x_2 + x_3\big)\big(\lambda_1\lambda_3x_1 - (\lambda_1 + \lambda_3)x_2 + x_3\big) \\ &\quad \times \big(\lambda_1\lambda_2x_1 - (\lambda_1 + \lambda_2)x_2 + x_3\big) \\ &= \big(u_1^\top x\big)\big(u_2^\top x\big)\big(u_3^\top x\big). \end{aligned}$$

Note that if $\lambda_1 = \alpha + \mathrm{i}\beta$ and $\lambda_2 = \alpha - \mathrm{i}\beta$, for some constants $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, then the two complex factors $(u_1^\top x)(u_2^\top x)$ lead to a real Darboux polynomial associated with Ax, irreducible over $\mathbb{R}$,

$$\begin{aligned} \omega_4 &= \big(u_1^\top x\big)\big(u_2^\top x\big) \\ &= \big(2\lambda_3\alpha + \lambda_3^2 + \alpha^2 + \beta^2\big)x_2^2 + \big(-2\lambda_3^2\alpha - 2\lambda_3\beta^2 - 2\lambda_3\alpha^2\big)x_1x_2 \\ &\quad + (-2\lambda_3 - 2\alpha)x_2x_3 + \big(\lambda_3^2\alpha^2 + \lambda_3^2\beta^2\big)x_1^2 + 2\lambda_3\alpha x_1x_3 + x_3^2, \end{aligned}$$

such that $\Delta\omega_4 = \lambda\omega_4$, where $\lambda = \lambda_1 + \lambda_2 = 2\alpha$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\lambda = \lambda_1\lambda_2 = \alpha^2 + \beta^2$ if $\mathbb{T} = \mathbb{Z}$). Moreover, note that the same factor $(u_i^\top x)$ appears more than once as a factor of $\omega(x)$ in case of multiple eigenvalues.

Example 2.12 Consider again matrix A of Example 2.11 with $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Clearly, Ex and Mx are two linear symmetries of Ax. Let

$$\omega(x) = \det\big([Ax\ Ex\ Mx]\big) = \det\left(\begin{bmatrix} x_2 & x_1 & x_1 \\ x_3 & x_2 & 3x_2 \\ 0 & x_3 & 5x_3 \end{bmatrix}\right) = -2x_3\big(2x_1x_3 - x_2^2\big);$$

let $\omega_1(x) = x_3$ and $\omega_2(x) = 2x_1x_3 - x_2^2$. Clearly, $\omega_1(x) = [0\ 0\ 1]x$, where $[0\ 0\ 1]$ is a left eigenvector of A with eigenvalue $\lambda = 0$, is a Darboux polynomial associated with Ax, both in the continuous-time and discrete-time cases. Furthermore, since

$$L_{Ax}\omega_2(x) = [2x_3\ -2x_2\ 2x_1]\begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = 0,$$

ω_2 is another Darboux polynomial of the continuous-time system (not corresponding to any left eigenvector), whereas since

$$\omega_2 \circ Ax = \big(2F_1F_3 - F_2^2\big)\big|_{F_1=x_2, F_2=x_3, F_3=0} = -x_3^2,$$

the factor ω_2 of ω is not a Darboux polynomial of the discrete-time system.

Remark 2.8 Example 2.12 shows that, although there is a strong relationship between left eigenvectors and Darboux polynomials associated with Ax, there may exist Darboux polynomials of (2.1a), (2.1b) that are not generated by left eigenvectors.

Theorem 2.15 *Assume that* $\{A^0, A^1, \dots, A^{n-1}\}$ *is a basis of* $\mathscr{L}_c(A)$. *Let* $\Omega(x) = [A^0x\ A^1x\ \dots\ A^{n-1}x]$ *and* $\omega = \det(\Omega)$. *Then, all linear systems* $\Delta x = Bx$, *with* $B \in \mathscr{L}_c(A)$, $B \neq 0$, *have* ω *as a Darboux polynomial.*

Proof If $B \in \mathscr{L}_c(A)$, $B \neq 0$, then $B = \sum_{i=0}^{n-1} \mu_i A^i$, with $\mu_j \neq 0$ for at least one index j. Let $\hat{\Omega}(x) = [Bx\ A^1x\ \dots\ A^{n-1}x]$ and $\hat{\omega} = \det(\hat{\Omega})$; assume that $\hat{\omega} \neq 0$, otherwise define $\hat{\Omega}$ as the matrix obtained from Ω by replacing one of the last $n-1$ columns with Bx. By construction, $\hat{\omega}$ is a Darboux polynomial of $\Delta x = Bx$, and in addition

$$\begin{aligned}\hat{\omega}(x) &= \det\big(\big[Bx\ A^1x\ \dots\ A^{n-1}x\big]\big) = \det\left(\left[\sum_{i=0}^{n-1}\mu_i A^i x\ A^1x\ \dots\ A^{n-1}x\right]\right) \\ &= \det\big(\big[\mu_0 A^0x\ A^1x\ \dots\ A^{n-1}x\big]\big) = \mu_0 \det\big(\big[A^0x\ A^1x\ \dots\ A^{n-1}x\big]\big) \\ &= \mu_0\omega(x),\end{aligned}$$

where $\mu_0 \neq 0$ by $\hat{\omega} \neq 0$. □

By Corollary 2.1, recall that if either A is semi-simple with distinct eigenvalues or if A is a Jordan block, then $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathscr{L}_c(A)$.

Example 2.13 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; a basis of $\mathscr{L}_c(A)$ is $\{A^0, A^1\}$; hence, any $B \in \mathscr{L}_c(A)$ can be written as $B = \mu_0 A^0 + \mu_1 A^1 = \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix}$, for constant $\mu_0, \mu_1 \in \mathbb{R}$. Then, letting $\Omega(x) = [A^0 x \ A^1 x] = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}$, one computes $\omega(x) = \det(\Omega(x)) = -(x_1^2 + x_2^2)$. Under the assumption that $\mu_0 \neq 0$, letting

$$\hat{\Omega}(x) = \begin{bmatrix} Bx \ A^1 x \end{bmatrix} = \begin{bmatrix} \mu_0 x_1 + \mu_1 x_2 & x_2 \\ -\mu_1 x_1 + \mu_0 x_2 & -x_1 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_0(x_1^2 + x_2^2) \neq 0$; similarly, if $\mu_1 \neq 0$, letting

$$\hat{\Omega}(x) = \begin{bmatrix} A^0 x \ Bx \end{bmatrix} = \begin{bmatrix} x_1 & \mu_0 x_1 + \mu_1 x_2 \\ x_2 & -\mu_1 x_1 + \mu_0 x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_1(x_1^2 + x_2^2) \neq 0$. This shows that all systems having a dynamic matrix in $\mathscr{L}_c(A)$ share the same Darboux polynomial $x_1^2 + x_2^2$.

Example 2.14 Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; A is not semi-simple. The linear centralizer $\mathscr{L}_c(A)$ has dimension two and $\mathscr{L}_c(A) = \text{span}_{\mathbb{R}}\{A^0, A^1\}$. Any $B \in \mathscr{L}_c(A)$ can be expressed as $B = \mu_0 A^0 + \mu_1 A^1 = \begin{bmatrix} \mu_0+\mu_1 & \mu_1 \\ -\mu_1 & \mu_0-\mu_1 \end{bmatrix}$, for constant $\mu_0, \mu_1 \in \mathbb{R}$. Then, by $\Omega(x) = [A^0 x \ A^1 x] = \begin{bmatrix} x_1 & x_1+x_2 \\ x_2 & -x_1-x_2 \end{bmatrix}$, one computes $\omega(x) = \det(\Omega(x)) = -(x_1 + x_2)^2$. Under the assumption that $\mu_0 \neq 0$, letting

$$\hat{\Omega}(x) = \begin{bmatrix} Bx \ A^1 x \end{bmatrix} = \begin{bmatrix} (\mu_0 + \mu_1)x_1 + \mu_1 x_2 & x_1 + x_2 \\ -\mu_1 x_1 + (\mu_0 - \mu_1)x_2 & -x_1 - x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_0(x_1 + x_2)^2 \neq 0$; similarly, under the assumption that $\mu_1 \neq 0$, letting

$$\hat{\Omega}(x) = \begin{bmatrix} A^0 x \ Bx \end{bmatrix} = \begin{bmatrix} x_1 & (\mu_0 + \mu_1)x_1 + \mu_1 x_2 \\ x_2 & -\mu_1 x_1 + (\mu_0 - \mu_1)x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_1(x_1 + x_2)^2 \neq 0$. This shows how all systems having a dynamic matrix belonging to $\mathscr{L}_c(A)$ have $(x_1 + x_2)^2$ as Darboux polynomial ($x_1 + x_2$ in the continuous-time case).

Remark 2.9 Assume that the Jordan form of A has not two Jordan blocks corresponding to the same eigenvalue, i.e., that $A^0, A^1, \dots, A^{n-1}$ are linearly independent over $\mathbb{R}$; since $[A^i, A^j] = 0$, by the analysis carried out in Sect. 1.6, one concludes that the rows of $\Omega^{-1}(x) = [A^0 x \ A^1 x \ \dots \ A^{n-1} x]^{-1}$ are exact one-forms. Then, the diffeomorphism $y = \varphi(x)$ such that $\frac{\partial \varphi}{\partial x} = \Omega^{-1}$ satisfies $L_{A^i x}\varphi = e_{i+1}$, with e_i being the ith column of the $n \times n$ identity matrix E. This can be useful to compute $n - 1$ independent first integrals of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$, when A is not semi-simple, as illustrated in the following example.

Example 2.15 Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};$$

then,

$$\Omega(x) = \left[A^0 x \; A^1 x \; A^2 x\right] = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \\ x_3 & 0 & 0 \end{bmatrix}.$$

The rows of Ω^{-1} are exact one-forms and yield, by integration, the diffeomorphism $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} \ln(|x_3|) \\ \frac{x_2}{x_3} \\ \frac{x_1}{x_3} - \frac{1}{2}\frac{x_2^2}{x_3^2} \end{bmatrix};$$

it is not difficult to see that the first and last entry of $\varphi(x)$ are functionally independent first integrals of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$.

Chapter 3
Analysis of Continuous-Time Nonlinear Systems

3.1 Semi-invariants and Darboux Polynomials of Continuous-Time Nonlinear Systems

The semi-invariants (using the name given in [118]) are widely studied in the literature under various names, such as: second integrals, special integrals (polynomials), eigenpolynomials, Darboux polynomials (curves), algebraic invariant curves (manifolds), particular algebraic solutions; an introductory reference is Sect. 2.5 in [56] in case of polynomial semi-invariants, with polynomial characteristic function. The concept of semi-invariant dates back to Darboux [38] (see also [30, 105]).

Definition 3.1 A *semi-invariant* of system (1.1a) is a meromorphic function $\omega(x) \in \mathbb{R}$ such that

$$L_f \omega = \lambda \omega, \tag{3.1}$$

with $\lambda(x) \in \mathbb{R}$ being meromorphic and such that there is no zero/pole cancelation between λ and ω; if ω and λ are polynomial in x, then ω is said to be a *Darboux polynomial*; λ is called the *characteristic function* (respectively, the *characteristic polynomial)* of the semi-invariant (respectively, of the Darboux polynomial). If λ is constant, it is called the *characteristic value*.

A semi-invariant (respectively, a Darboux polynomial) of system (1.1a) is also called a CT-semi-invariant (respectively, a CT-Darboux polynomial) associated with f. If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *semi-invariant* is used instead of CT-semi-invariant.

Clearly, if not empty, set $\mathscr{I}_\omega = \{x \in \mathscr{U} : \omega(x) = 0\}$ is invariant, i.e., if $x(0) \in \mathscr{I}_\omega$, then $x(t) \in \mathscr{I}_\omega$ for all real t belonging to some interval $[0, T)$. From Definition 3.1, a first integral associated with f is a semi-invariant associated with f, with characteristic value $\lambda = 0$.

Remark 3.1 For any $\alpha(x) \in \mathbb{R}$, function $\omega = \mathrm{e}^\alpha$ satisfies (3.1) with $\lambda = L_f \alpha$:

$$L_f \omega = L_f \mathrm{e}^\alpha = (L_f \alpha)\mathrm{e}^\alpha = \lambda \omega,$$

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but in such a case set $\mathscr{I}_{e^\alpha}$ is empty. Definition 3.1 could be amended to exclude such trivial semi-invariants (including the constant ones), by requiring that set $\mathscr{I}_\omega$ is not empty, but this would be paid by a more cumbersome exposition; moreover, such a change would not drop out other trivial semi-invariants. As a matter of fact, if ω is a semi-invariant of system (1.1a), with characteristic function λ, then ωe^α is a semi-invariant of system (1.1a) for any analytic scalar function α, with characteristic function $\lambda + L_f\alpha$. To be more precise,

$$L_f\big(\omega e^\alpha\big) = e^\alpha (L_f\omega) + \omega\big(L_f e^\alpha\big) = e^\alpha \lambda\omega + \omega e^\alpha (L_f\alpha) = (\lambda + L_f\alpha)\omega e^\alpha;$$

functions ωe^α are trivial extensions of the semi-invariant ω.

For simplicity, the following theorem considers the Darboux polynomials associated with polynomial f, although some of such properties hold for semi-invariants and non-polynomial f too, subject to the necessary amendments.

Theorem 3.1 *Assume that f is polynomial.*

(3.1.1) *If $I = \frac{\omega_1}{\omega_2}$ is a first integral of system* (1.1a), *with ω_1 and ω_2 being co-prime polynomials, then ω_1 and ω_2 are Darboux polynomials of system* (1.1a), *with the same characteristic polynomial $\lambda_1 = \lambda_2$.*

(3.1.2) *Let ω, ω_1 and ω_2 be Darboux polynomials of system* (1.1a) *with respective characteristic polynomials λ, λ_1 and λ_2; then, all irreducible factors of ω are Darboux polynomials of system* (1.1a), *and the product $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system* (1.1a) *for arbitrary constants $n_1, n_2 \in \mathbb{Z}^{\geq}$, with characteristic polynomial $n_1\lambda_1 + n_2\lambda_2$.*

Proof As for Statement (3.1.1) of the theorem, since I is a first integral of system (1.1a), it follows that

$$0 = L_f I = \frac{\omega_2 L_f\omega_1 - \omega_1 L_f\omega_2}{\omega_2^2}.$$

Then, taking into account that ω_1 and ω_2 are co-prime and that $\omega_2 L_f\omega_1 = \omega_1 L_f\omega_2$, one concludes that ω_1 is a factor of $L_f\omega_1$ and ω_2 is a factor of $L_f\omega_2$, with $\lambda_1 = \frac{L_f\omega_1}{\omega_1}$ and $\lambda_2 = \frac{L_f\omega_2}{\omega_2}$ being the respective characteristic polynomials; substituting these expressions in $\omega_2 L_f\omega_1 = \omega_1 L_f\omega_2$, one concludes that $\omega_1\omega_2(\lambda_1 - \lambda_2) = 0$, which shows that $(\lambda_1 - \lambda_2) = 0$, because $\omega_1\omega_2$ is not the zero function. As for Statement (3.1.2) of the theorem, in order to show that $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system (1.1a), compute

$$\begin{aligned} L_f\big(\omega_1^{n_1}\omega_2^{n_2}\big) &= \omega_2^{n_2} L_f\omega_1^{n_1} + \omega_1^{n_1} L_f\omega_2^{n_2} \\ &= n_1\omega_1^{n_1-1}\omega_2^{n_2} L_f\omega_1 + n_2\omega_1^{n_1}\omega_2^{n_2-1} L_f\omega_2 = (n_1\lambda_1 + n_2\lambda_2)\omega_1^{n_1}\omega_2^{n_2}. \end{aligned}$$

In order to show that all irreducible factors of ω are Darboux polynomials of system (1.1a), let $\omega = \omega_1^{n_1}\omega_2$, with ω_1 being irreducible and pair ω_1, ω_2 being co-prime.

Then,

$$L_f\omega = L_f\left(\omega_1^{n_1}\omega_2\right) = n_1\omega_1^{n_1-1}\omega_2 L_f\omega_1 + \omega_1^{n_1} L_f\omega_2,$$

which implies (because $L_f\omega = \lambda\omega$)

$$n_1\omega_1^{n_1-1}\omega_2 L_f\omega_1 + \omega_1^{n_1} L_f\omega_2 = \lambda\omega_1^{n_1}\omega_2.$$

Hence, $\omega_1^{n_1}$ divides $n_1\omega_1^{n_1-1}\omega_2 L_f\omega_1 + \omega_1^{n_1} L_f\omega_2$; since ω_1 and ω_2 are co-prime, ω_1 must divide $L_f\omega_1$, where $\frac{L_f\omega_1}{\omega_1}$ is the characteristic polynomial of ω_1. □

If $\hat{I}(x) \in \mathbb{R}$ satisfies $L_f\hat{I} = 1$, then $I(t,x) = \hat{I}(x) - t$ is a *time-varying first integral* associated with f, since

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x} f = -1 + L_f\hat{I} = 0.$$

The following theorem shows that the knowledge of a semi-invariant with a constant characteristic value leads to a time-varying first integral.

Theorem 3.2 *It ω is a semi-invariant of system* (1.1a) *with a constant characteristic value $\lambda \neq 0$, then $\hat{I} = \frac{1}{\lambda}\ln(|\omega|)$ satisfies $L_f\hat{I} = 1$.*

Proof The theorem is proven by $L_f\hat{I} = L_f(\frac{1}{\lambda}\ln(|\omega|)) = \frac{1}{\lambda}\frac{1}{\omega}L_f\omega = 1$. □

Definition 3.2 Assume that $f(x)$ is analytic at $x = x^o$, with $x^o \in \mathscr{U}$; the point x^o is *regular* for $f(x) \in \mathbb{R}^n$ if $f(x^o) \neq 0$, *singular* if $f(x^o) = 0$.

The following theorem is known as the *flow box theorem* and is a particular case of Theorem 1.10 at p. 24; it gives the conditions for the local *straightening* of the flow of system (1.1a).

Theorem 3.3 *Assume that $f(x)$ is analytic at $x = x^o$, with $x^o \in \mathscr{U}$. Around any regular point $x^o \in \mathscr{U}$ of $f(x) \in \mathbb{R}^n$, there exists an open and connected subset $\mathscr{U}^*$ of $\mathscr{U}$, containing x^o, and an analytic diffeomorphism $y = \varphi(x)$, $\varphi(\cdot) : \mathscr{U}^* \to \mathbb{R}^n$, such that $L_f\varphi = e_1$, where e_1 is the first column of the $n \times n$ identity matrix E.*

Proof Since $f(x^o) \neq 0$, apart from a reordering of the entries x_i of x, assume that $f_1(x^o) \neq 0$, so that $\det([f(x^o)\ e_2\ \ldots\ e_n]) \neq 0$. Hence, relation $\varphi_* f = e_1$, which is equivalent to $L_f\varphi = e_1$, can be rewritten in the Kovalevskaya form (1.12):

$$\begin{cases} \frac{\partial\varphi_1}{\partial x_1} = \frac{1}{f_1}\left(1 - \frac{\partial\varphi_1}{\partial x_2} f_2 - \cdots - \frac{\partial\varphi_1}{\partial x_n} f_n\right), \\ \frac{\partial\varphi_2}{\partial x_1} = -\frac{1}{f_1}\left(\frac{\partial\varphi_2}{\partial x_2} f_2 + \cdots + \frac{\partial\varphi_2}{\partial x_n} f_n\right), \\ \quad\vdots \\ \frac{\partial\varphi_n}{\partial x_1} = -\frac{1}{f_1}\left(\frac{\partial\varphi_n}{\partial x_2} f_2 + \cdots + \frac{\partial\varphi_n}{\partial x_n} f_n\right), \end{cases} \tag{3.2}$$

with the right-hand sides being analytic in a neighborhood of x^o, when φ is analytic in a neighborhood of x^o. The Cauchy–Kovalevskaya Theorem 1.8 at p. 20 guarantees that such a system, with the Cauchy initial data

$$\varphi\big(x_1^o, x_2, \dots, x_n\big) = \big[0\ x_2 - x_2^o\ \dots\ x_n - x_n^o\big]^\top, \tag{3.3}$$

has a unique solution in a neighborhood of x^o, being analytic at $x = x^o$; by the first of (3.2) computed at $x = x^o$ and by the chosen Cauchy initial data, $y = \varphi(x)$ is a diffeomorphism in a neighborhood of x^o. The solution of such a Cauchy problem is

$$\varphi^{-1}(y_1, y_2, \dots, y_n) = \Phi_f\left(y_1, x^o + \sum_{i=2}^{n} y_i e_i\right), \tag{3.4}$$

which satisfies $\varphi^{-1}(0, y_2, \dots, y_n) = \Phi_f(0, x^o + \sum_{i=2}^n y_i e_i) = x^o + \sum_{i=2}^n y_i e_i$, i.e., the Cauchy initial data (3.3). In addition, condition $\varphi_* f(y) = e_1$, is equivalent to $\Phi_{\varphi_* f}(t, y) = y + te_1$ and, by $y = \varphi(x)$, one has

$$\begin{aligned}\Phi_{\varphi_* f}(t, y) &= \varphi\big(\Phi_f\big(t, \varphi^{-1}(y)\big)\big) = \varphi\left(\Phi_f\left(t, \Phi_f\left(y_1, x^o + \sum_{i=2}^{n} y_i e_i\right)\right)\right)\\ &= \varphi\left(\Phi_f\left(t + y_1, x^o + \sum_{i=2}^{n} y_i e_i\right)\right)\\ &= \varphi\big(\varphi^{-1}(t + y_1, y_2, \dots, y_n)\big) = y + te_1.\end{aligned}$$

Notice that $x = \varphi^{-1}(y)$ is actually a diffeomorphism about $y = 0$; in particular, taking into account that $\frac{\partial \Phi_f(t,x)}{\partial t}|_{t=0} = f(x)$ and $\frac{\partial \Phi_f(t,x)}{\partial x}|_{t=0} = E$,

$$\begin{aligned}\frac{\partial \varphi^{-1}(y)}{\partial y}\bigg|_{y=0} &= \left[\frac{\partial \Phi_f(t, x)}{\partial t}\ \frac{\partial \Phi_f(t, x)}{\partial x} e_2\ \dots\ \frac{\partial \Phi_f(t, x)}{\partial x} e_n\right]\bigg|_{\substack{t=y_1,\\ x=x^o+\sum_{i=2}^n y_i e_i}}\bigg|_{y=0}\\ &= \big[f(x^o)\ e_2\ \dots\ e_n\big],\end{aligned}$$

which has full rank by the assumption $f_1(x^o) \neq 0$. □

Remark 3.2 If $f_j(x^o) \neq 0$ instead of $f_1(x^o) \neq 0$, then formula (3.4) becomes

$$\varphi^{-1}(y_1, y_2, \dots, y_n) = \Phi_f\left(y_j, x^o + \sum_{i=1, i\neq j}^{n} y_i e_i\right).$$

It is noted that such an analytic diffeomorphism $y = \varphi(x)$ can also be computed by (1.20), as detailed in Examples 1.16 at p. 24 and 1.17 at p. 25.

Remark 3.3 Actually, for any $n \geq 1$, the flow box Theorem 3.3 still holds if $f(x)$ is C^1 at the considered regular point x^o, with the resulting diffeomorphism being C^1

at x^o. When $n = 1$, f only needs to be continuous at a regular point x^o, $f(x^o) \neq 0$. As a matter of fact, it is sufficient to define

$$\varphi(x) := \int \frac{1}{f(x)}\,\mathrm{d}x + c, \tag{3.5}$$

where c is such that $\varphi(x^o) = 0$; since $f(x)$ is continuous and satisfies $f(x) \neq 0$ for all x in a neighborhood $\mathscr{B}$ of x^o, $y = \varphi(x)$ is a C^1-diffeomorphism on $\mathscr{B}$, for which $\varphi_* f(y) = 1$. As an example, consider $f(x) = \begin{cases} 1, & \text{if } x < 1, \\ x, & \text{if } x \geq 1. \end{cases}$ By (3.5), one computes the diffeomorphism $y = \varphi(x)$, about $x^o = 1$, with $\varphi(x) = \begin{cases} x-1, & \text{if } x < 1, \\ \ln(x), & \text{if } x \geq 1, \end{cases}$ being C^1 at $x^o = 1$, for which $\varphi_* f(y) = 1$.

The important point to note here is that, by the flow box Theorem 3.3, locally about regular points, any system is diffeomorphic to any other system having the same dimension. To this end, consider two systems $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ and $\frac{\mathrm{d}\xi}{\mathrm{d}t} = h(\xi)$, $x, \xi \in \mathbb{R}^n$, with f and h being arbitrary. System $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ (respectively, $\frac{\mathrm{d}\xi}{\mathrm{d}t} = f(\xi)$) can be transformed by some diffeomorphism $y = \varphi(x)$ (respectively, $y = \hat{\varphi}(\xi)$), in a neighborhood of any regular x^o (respectively, ξ^o), into $\frac{\mathrm{d}y}{\mathrm{d}t} = e_1$; hence, $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ can be transformed into $\frac{\mathrm{d}\xi}{\mathrm{d}t} = h(\xi)$ by $\xi = \hat{\varphi}^{-1} \circ \varphi(x)$. As an example, consider $f(x) = x$ and $h(\xi) = -\xi$; the rectifying diffeomorphisms are $\varphi(x) = \ln(x)$ and $\hat{\varphi}(\xi) = \ln(\frac{1}{\xi})$, taking any $x^o > 0$ and $\xi^o > 0$; then, $\frac{\mathrm{d}x}{\mathrm{d}t} = x$ is transformed into $\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\xi$, by $\xi = \hat{\varphi}^{-1} \circ \varphi(x) = \frac{1}{\mathrm{e}^y}|_{y=\ln(x)} = \frac{1}{x}$. Note that such a diffeomorphism is not defined at the singular point $x = 0$.

Remark 3.4 By the flow box Theorem 3.3, the entries $I_1 = \varphi_2, \ldots, I_{n-1} = \varphi_n$ of φ are $n-1$ functionally independent first integrals of system (1.1a); by Remark 1.3 at p. 10, any first integral of system (1.1a) can be expressed as $I = C(\varphi_2, \ldots, \varphi_n)$, where C is an arbitrary function of the arguments.

Example 3.1 Consider system (1.1a) with $f = g_1$ and $g_1(x) = [x_1 \; 3x_2 + x_1^2]^\top$ given in Example 1.16 at p. 24. The CT-flow $\Phi_f(t, x)$ associated with f is

$$\Phi_f(t, x) = \begin{bmatrix} \mathrm{e}^t x_1 \\ \mathrm{e}^{3t} x_2 + (-\mathrm{e}^{2t} + \mathrm{e}^{3t}) x_1^2 \end{bmatrix}.$$

Then, letting $x^o = [1 \; 0]^\top$, one has

$$\varphi^{-1}(y) = \begin{bmatrix} \mathrm{e}^t x_1 \\ \mathrm{e}^{3t} x_2 + (-\mathrm{e}^{2t} + \mathrm{e}^{3t}) x_1^2 \end{bmatrix}_{t=y_1, x_1=1, x_2=y_2} = \begin{bmatrix} \mathrm{e}^{y_1} \\ \mathrm{e}^{3y_1} y_2 - \mathrm{e}^{2y_1} + \mathrm{e}^{3y_1} \end{bmatrix},$$

which coincides with the diffeomorphism found in Example 1.16 at p. 24. By inverting (in a neighborhood of $x = x^o$) the diffeomorphism $x_1 = \mathrm{e}^{y_1}$, $x_2 = \mathrm{e}^{3y_1} y_2 - \mathrm{e}^{2y_1} + \mathrm{e}^{3y_1}$, one finds $y_1 = \ln(x_1)$, $y_2 = \frac{x_2 + x_1^2 - x_1^3}{x_1^3}$. Clearly, $I_1(x) = \frac{x_2 + x_1^2 - x_1^3}{x_1^3}$ is a first integral associated with f, and any other first integral I associated with f can be expressed as $I = C(I_1)$, with $C(\cdot)$ being an arbitrary function.

3.2 Symmetries and Orbital Symmetries of Continuous-Time Nonlinear Systems

The concept of (orbital) symmetry of a differential equation was introduced by S. Lie [81] in the second half of the 19th century, as an attempt of generalizing the theory of Galois, and it was primarily used for the solution in closed form of differential equations admitting given orbital symmetries. In [80], S. Lie proven that a planar system, described by a pair of first order time-invariant differential equations, (or, equivalently, one time-varying differential equation) admits an inverse integrating factor, whence by quadrature a (non-trivial) first integral, if and only if it admits a (non-trivial) orbital symmetry. Modern reference on the subject can be found in many books, among which [5, 20–22, 34, 67, 102, 111].

One of the oldest applications of symmetries is based on fixing a (possibly, simple) vector function and then look for all systems admitting the given vector function as orbital symmetry (with a simple vector function, one can also generate very cumbersome systems admitting it as orbital symmetry): this was, for instance, used by [73] for tabularizing classes of differential equations for which the solution can be written in closed form by quadrature, and it is now used in symbolic algebraic manipulation languages (see, e.g., [28]) for the automatic generation of the solutions of differential equations.

For any $g(x) \in \mathbb{R}^n$ and for any *admissible* τ (to be considered as a constant parameter),

$$x = \Phi_g(\tau, y) \tag{3.6}$$

qualifies as a local analytic diffeomorphism; system (1.1a) is transformed, according to such a diffeomorphism, as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g. \tag{3.7}$$

Since $\Phi_g(\tau, y) = Ey + g(y)\tau + O(\tau^2)$, with E being the $n \times n$ identity matrix and $O(\tau^2)$ denoting second and higher order terms, for τ close to 0, (3.6) is close to the identity transformation; moreover, by a Taylor series expansion with respect to τ, it can be seen that

$$\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g = f - [f, g]\tau + O\big(\tau^2\big) = f + [g, f]\tau + O\big(\tau^2\big). \tag{3.8}$$

In particular, a possible and alternative definition of the CT-Lie bracket is [23]

$$[g, f] := \lim_{\tau \to 0} \frac{(\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g - f}{\tau}. \tag{3.9}$$

By (3.9), $[g, f]$ can be interpreted as the "derivative" of f along g (by some authors, it is indicated by $L_g f$, but not in this book); the reader is advised not to confuse the Lie bracket $[g, f] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$ with the directional derivative $L_g f = \frac{\partial f}{\partial x} g$.

Definition 3.3 The diffeomorphism (3.6) is a *symmetry* of system (1.1a) and system (1.2) is its *infinitesimal generator* if

$$\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g(\tau, y) = f(y), \quad \forall(\tau, y) \in \mathscr{V}, \tag{3.10}$$

with $\mathscr{V}$ being an open and connected subset of $\mathbb{R} \times \mathbb{R}^n$ including $\{0\} \times \mathscr{U}$. If (3.10) holds, by abuse of notation, also the infinitesimal generator (1.2) is called a *symmetry* of system (1.1a); briefly, g is called a *CT-symmetry* of f.

If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *symmetry* is used instead of *CT-symmetry*.

Theorem 3.4 *Vector function g is a symmetry of f if and only if* $[f, g] = 0$.

Proof By (3.8), condition $[f, g] = 0$ is certainly necessary. Since $\Phi_g(0, y) = y$, one finds that $\frac{\partial \Phi_g}{\partial y}|_{\tau=0} = I$. Since

$$\left.\left(\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g(\tau, y)\right)\right|_{\tau=0} = f(y), \tag{3.11}$$

equality (3.10) holds if and only if

$$\frac{\partial}{\partial \tau}\left(\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g(\tau, y)\right) = 0. \tag{3.12}$$

In this way,

$$\frac{\partial}{\partial \tau}\frac{\partial \Phi_g}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Phi_g}{\partial \tau} = \frac{\partial g(\Phi_g)}{\partial y} = \left.\frac{\partial g}{\partial x}\right|_{x=\Phi_g}\frac{\partial \Phi_g}{\partial y}.$$

If $X(\tau)$ is a square invertible matrix, then from $XX^{-1} = I$, it follows that $\frac{\partial X}{\partial \tau} X^{-1} + X\frac{\partial X^{-1}}{\partial \tau} = 0$, which implies

$$\frac{\partial X^{-1}}{\partial \tau} = -X^{-1}\frac{\partial X}{\partial \tau} X^{-1}. \tag{3.13}$$

Hence, if $\frac{\partial}{\partial \tau} X = JX$, for some square matrix J, then one concludes that $\frac{\partial X^{-1}}{\partial \tau} = -X^{-1}J$. This shows that $\frac{\partial}{\partial \tau}(\frac{\partial \Phi_g}{\partial y})^{-1} = -(\frac{\partial \Phi_g}{\partial y})^{-1}\frac{\partial g}{\partial x}|_{x=\Phi_g}$. Thus,

$$\frac{\partial}{\partial \tau}\left(\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g(\tau, y)\right)$$

$$= -\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1}\left(\frac{\partial g}{\partial x} f\right)\bigg|_{x=\Phi_g} + \left(\frac{\partial \Phi_g}{\partial y}\right)^{-1}\frac{\partial f}{\partial x}\bigg|_{x=\Phi_g}\frac{\partial \Phi_g}{\partial \tau}$$

$$= -\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1}\left(\frac{\partial g}{\partial x} f\right)\bigg|_{x=\Phi_g} + \left(\frac{\partial \Phi_g}{\partial y}\right)^{-1}\left(\frac{\partial f}{\partial x} g\right)\bigg|_{x=\Phi_g}$$

$$= -\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1}[f, g]\bigg|_{x=\Phi_g},$$

whence (3.12) holds if and only if $[f, g] = 0$. □

By Definition 1.3 at p. 10 and by Theorem 3.4, the set of all symmetries g of f is given by the *centralizer* $\mathscr{C}_C(f)$ of f. Similarly to Theorem 3.4, by computations wholly similar to those yielding (1.19), it is possible to show that

$$[f, g] = 0 \quad \Longleftrightarrow \quad \Phi_f(t, \cdot) \circ \Phi_g(\tau, x) = \Phi_g(\tau, \cdot) \circ \Phi_f(t, x).$$

Remark 3.5 If $[f, g] = 0$, with $g(x) = Bx$ for some $B \in \mathbb{R}^{n\times n}$, then (3.10) becomes $\mathrm{e}^{-B\tau} f(\mathrm{e}^{B\tau} y) = f(y)$, which implies $f(\mathrm{e}^{B\tau} y) = \mathrm{e}^{B\tau} f(y)$.

Thanks to Theorem 3.4, the following definition is equivalent to Definition 3.3.

Definition 3.4 The diffeomorphism (3.6) is a *symmetry* of system (1.1a) and system (1.2) is its *infinitesimal generator* if $[f, g] = 0$.

The following definition extends the concept of symmetry to the concept of orbital symmetry.

Definition 3.5 The diffeomorphism (3.6) is an *orbital symmetry* of system (1.1a) and system (1.2) is its *infinitesimal generator* if $[f, g] = \mu f$, with μ being a meromorphic scalar function. The *normalizer* $\mathscr{N}_C(f)$ of f is the set of all g such that $[f, g] = \mu f$, for some $\mu(x) \in \mathbb{R}$.

The following theorem shows that the normalizer $\mathscr{N}_C(f)$ and the centralizer $\mathscr{C}_C(f)$ of f are closed under the Lie bracket operation, i.e., $g_1, g_2 \in \mathscr{N}_C(f)$ implies $[g_1, g_2] \in \mathscr{N}_C(f)$ and $g_1, g_2 \in \mathscr{C}_C(f)$ implies $[g_1, g_2] \in \mathscr{C}_C(f)$.

Theorem 3.5 *If g_1 and g_2 are two orbital symmetries (respectively, symmetries) of f, then $[g_1, g_2]$ is an orbital symmetry (respectively, symmetry) of f.*

Proof From $[f, g_1] = \mu_1 f$ and $[f, g_2] = \mu_2 f$, it follows that (taking into account the Jacobi identity, given in Property (1.2.3)):

$$\begin{aligned}\big[f, [g_1, g_2]\big] &= -\big[g_1, [g_2, f]\big] - \big[g_2, [f, g_1]\big] = [g_1, \mu_2 f] - [g_2, \mu_1 f] \\ &= \mu_2[g_1, f] + (L_{g_1}\mu_2) f - \mu_1[g_2, f] + (L_{g_2}\mu_1) f\end{aligned}$$

$$= (L_{g_1}\mu_2 + L_{g_2}\mu_1)f.$$

In particular, if $\mu_1 = \mu_2 = 0$, then $[f, [g_1, g_2]] = 0$. □

Theorem 3.6 *Let J be a first integral of* (1.2) (*i.e.*, $L_g J = 0$) *such that $L_f J \neq 0$. If g is an orbital symmetry of f, then g is a symmetry of*

$$\hat{f} := \frac{1}{L_f J} f.$$

Proof Compute $[\hat{f}, g] = \frac{1}{L_f J}[f, g] - f L_g(\frac{1}{L_f J})$. Then, $L_{[f,g]}J = L_f L_g J - L_g L_f J = -L_g(L_f J)$; taking into account that $[f, g] = \mu f$, one finds that $L_{[f,g]}J = \mu L_f J$ and, therefore, that $L_g(L_f J) = -\mu(L_f J)$; finally, since $L_g(\frac{1}{L_f J}) = -\frac{1}{(L_f J)^2} L_g(L_f J)$, one concludes that $L_g(\frac{1}{L_f J}) = \mu \frac{1}{L_f J}$, which implies

$$[\hat{f}, g] = \frac{1}{L_f J}[f, g] - f L_g\left(\frac{1}{L_f J}\right) = \frac{1}{L_f J}\mu f - f \mu \frac{1}{L_f J} = 0,$$

as to be shown. □

Remark 3.6 Since g is a symmetry of $\hat{f} = \frac{1}{L_f J} f$, for any τ for which $\Phi_g(\tau, y)$ is defined, $x = \Phi_g(\tau, y)$ maps any orbit of $\frac{\mathrm{d}x}{\mathrm{d}s} = \hat{f}$ into the same orbit, while preserving the time parameterization. Furthermore, since $\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{L_f J} f$ leads to $\frac{\mathrm{d}x}{\mathrm{d}t} = f$, with $\frac{\mathrm{d}s}{\mathrm{d}t} = L_f J$, it is easy to see that $\frac{\mathrm{d}x}{\mathrm{d}s} = \hat{f}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = f$ have the same orbits (except for the possible equilibrium points of $\hat{f}$ that do not coincide with those of f), but with different time parameterizations; this shows that $x = \Phi_g(\tau, y)$ maps any orbit of $\frac{\mathrm{d}x}{\mathrm{d}t} = f$ into the same orbit (except for the possible equilibrium points of $\hat{f}$ that do not coincide with those of f), but with a different time parameterization.

Theorem 3.7 *If g is an orbital symmetry of f, then g is an orbital symmetry of $\tilde{f} = \alpha f$, for any arbitrary $\alpha(x) \in \mathbb{R}$, $\alpha \neq 0$.*

Proof If $[f, g] = \mu f$, then $[\alpha f, g] = \alpha[f, g] - f L_g\alpha = (\alpha\mu - L_g\alpha) f = \frac{\alpha\mu - L_g\alpha}{\alpha}\alpha f$. □

Remark 3.7 By the flow box Theorem 3.3, about any regular point of g, there are local coordinates such that $g = e_1$, with e_1 being the first column of the $n \times n$ identity matrix. Consider first the case $n = 2$ and $g = [1\ \ 0]^\top$. Let f have g as symmetry; then, the equalities

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = [f, g] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \end{bmatrix},$$

imply $\frac{\partial f_1}{\partial x_1} = 0$ and $\frac{\partial f_2}{\partial x_1} = 0$, namely f has g as symmetry if and only if

$$f = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

with α and β being arbitrary functions of x_2. If f has g as orbital symmetry, then condition $[f, g] = \mu f$ implies $-\frac{\partial f_1}{\partial x_1} = \mu f_1$ and $-\frac{\partial f_2}{\partial x_1} = f_2$, namely f has g as orbital symmetry if and only if

$$f = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \alpha,$$

with α being an arbitrary function of x_1 and x_2, and β being an arbitrary function of x_2 (then, $\mu = -\frac{1}{\alpha}\frac{\partial \alpha}{\partial x_1}$). In the general case, assume $g = e_1$, with e_1 being the first column of the $n \times n$ identity matrix E; similarly, it is easy to show that

(3.7.1) f has g as symmetry if and only if

$$f = \begin{bmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix},$$

with α and β_i being arbitrary functions of $x_2, \dots, x_n$;

(3.7.2) f has g as orbital symmetry if and only if

$$f = \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \alpha,$$

with α being an arbitrary function of $x_1, x_2, \dots, x_n$ and β_i being an arbitrary function of $x_2, \dots, x_n$, $i = 1, \dots, n-1$.

Theorem 3.8 *Let $y = \varphi(x)$ be an analytic diffeomorphism on $\mathscr{U}$. Then, $\varphi_* g$ is an orbital symmetry (respectively, a symmetry) of $\varphi_* f$ if and only if g is an orbital symmetry (respectively, a symmetry) of f:*

$$[f, g] = \mu f \quad \Longleftrightarrow \quad [\varphi_* f, \varphi_* g] = (\varphi_* \mu)(\varphi_* f).$$

Proof Follows from the invariance of the Lie bracket to diffeomorphisms:

$$\left[\left(\frac{\partial \varphi}{\partial x} f\right) \circ \varphi^{-1}, \left(\frac{\partial \varphi}{\partial x} g\right) \circ \varphi^{-1}\right] = \left(\frac{\partial \varphi}{\partial x}[f, g]\right) \circ \varphi^{-1}.$$

□

Theorem 3.9 *Let g be given. Let $J_0, J_1, \dots, J_{n-1}$ be functionally independent and such that $L_g J_0 = 1$ and $L_g J_i = 0, i = 1, \dots, n-1$. Let $J = [J_0 \; J_1 \; \dots \; J_{n-1}]^\top$. Then,* (3.9.1) *$f$ has g as symmetry if and only if*

$$f = \left(\frac{\partial J}{\partial x}\right)^{-1} \begin{bmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}, \tag{3.14}$$

with α and β_i's being arbitrary functions of $J_1, \ldots, J_{n-1}$;

(3.9.2) *f has g as orbital symmetry if and only if*

$$f = \left(\frac{\partial J}{\partial x}\right)^{-1} \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \alpha,$$

with α being an arbitrary function of x and β_i being an arbitrary function of $J_1, \ldots, J_{n-1}$, $i = 1, \ldots, n-1$.

Proof The proof follows easily from Remark 3.7 and Theorem 3.8. □

Remark 3.8 By the analysis of the subsequent Sect. 6.3, the centralizer $\mathscr{C}_C(g)$ of g is a Lie algebra over the field $\mathscr{I}_C(g)$ of the meromorphic functions of $J_1, \ldots, J_{n-1}$; Statement (3.9.1) of Theorem 3.9 shows that $\mathscr{C}_C(g)$ is a vector space over $\mathscr{I}_C(g)$ spanned by the columns $g_1, \ldots, g_n$ of $(\frac{\partial J}{\partial x})^{-1}$ (which, by construction, satisfy $[g_i, g_j] = 0$), with coefficients being arbitrary meromorphic functions of $J_1, \ldots, J_{n-1}$, whereas Theorem 3.5 shows that $\mathscr{C}_C(g)$ is closed under the Lie bracket.

Example 3.2 Let $g(x) = [x_1 \ \ -x_2]^\top$. Then, letting $J_0(x) = \frac{1}{2}\ln(|\frac{x_1}{x_2}|)$ and $J_1(x) = x_1 x_2$, one verifies that $L_g J_0 = 1$ and $L_g J_1 = 0$. Then, all f having g as symmetry are given by (3.14), with $J(x) = [\frac{1}{2}\ln(|\frac{x_1}{x_2}|) \ x_1 x_2]^\top$,

$$f(x) = \begin{bmatrix} x_1 & \frac{1}{2x_2} \\ -x_2 & \frac{1}{2x_1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_1\alpha + \frac{1}{2x_2}\beta \\ -x_2\alpha + \frac{1}{2x_1}\beta \end{bmatrix}, \tag{3.15}$$

where α and β are arbitrary functions of J_1. Similarly, all f having g as orbital symmetry are given by

$$f(x) = \begin{bmatrix} x_1 & \frac{1}{2x_2} \\ -x_2 & \frac{1}{2x_1} \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \alpha,$$

with α and β being arbitrary functions of x and J_1, respectively.

The following theorem characterizes the centralizer $\mathscr{C}_C(Bx)$, which is constituted by all f such that $[f(x), Bx] = 0$; for a more deep analysis see [34].

Theorem 3.10 *Assume* $g(x) = Bx$. *Let* $\{M_0, \ldots, M_{r-1}\}$ *be a basis of* $\mathscr{L}_c(B)$. *Hence*, g *is a symmetry of* f *if and only if*

$$f(x) = \mu_0 M_0 x + \mu_1 M_1 x + \cdots + \mu_{r-1} M_{r-1} x, \tag{3.16}$$

where $\mu_i \in \mathscr{I}_C(Bx)$, $i = 0, \ldots, r-1$.

Proof By Remark 3.5, $g(x) = Bx$ is a symmetry of f if and only if $f(\mathrm{e}^{Bt}x) = \mathrm{e}^{Bt} f(x)$. Then, the f given in (3.16) has g as symmetry, since

$$\begin{aligned} f\big(\mathrm{e}^{Bt}x\big) &= \mu_0\big(\mathrm{e}^{Bt}x\big)M_0\mathrm{e}^{Bt}x + \mu_1\big(\mathrm{e}^{Bt}x\big)M_1\mathrm{e}^{Bt}x + \cdots + \mu_{r-1}\big(\mathrm{e}^{Bt}x\big)M_{r-1}\mathrm{e}^{Bt}x \\ &= \mu_0(x)\mathrm{e}^{Bt}M_0x + \mu_1(x)\mathrm{e}^{Bt}M_1x + \cdots + \mu_{r-1}(x)\mathrm{e}^{Bt}M_{r-1}x \\ &= \mathrm{e}^{Bt} f(x). \end{aligned}$$

As for the necessity, note that $r \geq n$. By Theorem 2.2 at p. 35, there exist $N_i \in \mathscr{L}_c(B)$, $i = 1, \ldots, n-1$, such that $B, N_1, \ldots, N_{n-1}$ are linearly independent over $\mathbb{R}$ and pairwise commuting; then, letting $\Omega(x) = [Bx\ N_1x\ \ldots\ N_{n-1}x]$, one concludes that the rows of Ω^{-1} are exact one-forms, i.e., there exists a $J(x) \in \mathbb{R}^n$ such that $\frac{\partial J}{\partial x} = \Omega^{-1}$. From this, (3.14) can be rewritten as

$$f(x) = \alpha Bx + \beta_1 N_1 x + \cdots + \beta_{n-1} N_{n-1} x,$$

thus proving the theorem. □

As for the computation of all first integrals of Bx, the case when B is semi-simple is solved by Remarks 1.9 at p. 27 and 2.7 at p. 50, whereas, when B is not semi-simple, the computations can be carried out as suggested in Remark 2.9 at p. 53 and Example 2.15 at p. 54.

The linear centralizer $\mathscr{L}_c(Bx)$ is found by taking a linear combination, with constant parameters, of $M_0x, \ldots, M_{r-1}x$, where $\{M_0, \ldots, M_{r-1}\}$ is any basis of $\mathscr{L}_c(B)$, whereas the centralizer $\mathscr{C}_C(Bx)$ is found by taking a linear combination, with coefficients belonging to $\mathscr{I}_C(Bx)$, of $M_0x, \ldots, M_{r-1}x$, whence $\mathscr{L}_c(Bx) \subset \mathscr{C}_C(Bx)$. The following pictorial symbols can be used to represent $\mathscr{L}_c(Bx)$ and $\mathscr{C}_C(Bx)$: $\mathscr{L}_c(Bx) = \mathbb{R} \otimes \mathscr{L}_c(B)$ and $\mathscr{C}_C(Bx) = \mathscr{I}_C(Bx) \otimes \mathscr{L}_c(B)$.

The representation $\mathscr{C}_C(Bx) = \mathscr{I}_C(Bx) \otimes \mathscr{L}_c(B)$ is somewhat redundant if $r > n$, because it gives any element of $\mathscr{C}_C(Bx)$ as linear combination of r linear symmetries of Bx, with coefficients in $\mathscr{I}_C(Bx)$, whereas, by (3.14), it is known that it is possible to express any element of $\mathscr{C}_C(Bx)$ as linear combination of just n symmetries of Bx with coefficients in $\mathscr{I}_C(Bx)$, where such n symmetries are given by the columns of $(\frac{\partial J}{\partial x})^{-1}$, with the first one being trivial as it coincides with g.

Theorem 3.11 *For a given* $g(x) \in \mathbb{R}^n$, *let* $h_1, \ldots, h_n \in \mathscr{C}_C(g)$ *be such that matrix* $[h_1\ \ldots\ h_n]$ *has generic rank equal to* n. *Then, any* $h \in \mathscr{C}_C(g)$ *can be rewritten as*

$$h(x) = \sum_{i=1}^{n} \mu_i(x) h_i(x), \tag{3.17}$$

with $\mu_i \in \mathscr{I}_C(g)$, $i = 1, \dots, n$, *and, conversely, any* h *of the form* (3.17), *with* $\mu_i \in \mathscr{I}_C(g)$, $i = 1, \dots, n$, *belongs to* $\mathscr{C}_C(g)$.

Proof Since $\operatorname{rank}_{\mathscr{K}_n}([h_1 \ \dots \ h_n]) = n$, it is clear that for any $h \in \mathscr{C}_C(g)$ there exist n scalar functions $\mu_1, \dots, \mu_n$ such that (3.17) holds. To prove that $h \in \mathscr{C}_C(g)$ implies that such functions μ_i are first integrals of g, property (1.2.2) (bi-linearity) and equation (1.4) are used to write:

$$[h, g] = \mu_1[h_1, g] + \cdots + \mu_n[h_n, g] - (L_g\mu_1)h_1 - \cdots - (L_g\mu_n)h_n. \tag{3.18}$$

The first terms are zero because $[h_i, g] = 0$, whence $h \in \mathscr{C}_C(g)$ implies

$$[h_1 \ \dots \ h_n]\begin{bmatrix} L_g\mu_1 \\ \vdots \\ L_g\mu_n \end{bmatrix} = 0;$$

since $\operatorname{rank}_{\mathscr{K}_n}([h_1 \ \dots \ h_n]) = n$, the above equation implies that $L_g\mu_1 = 0, \dots, L_g\mu_n = 0$, namely that $\mu_i \in \mathscr{I}_C(g)$. On the other hand, formula (3.18) clearly implies that any h of form (3.17), with $\mu_i \in \mathscr{I}_C(g)$, $i = 1, \dots, n$, is a symmetry of g. □

Theorem 3.11 implies that any n independent symmetries $h_1, \dots, h_n \in \mathscr{C}_C(Bx)$ can be taken as basis to generate, by linear combination with coefficients in $\mathscr{I}_C(g)$, the whole $\mathscr{C}_C(Bx)$, as well as its subset $\mathscr{L}_c(Bx)$, as illustrated in the following example.

Remark 3.9 Take

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

one has $\mathscr{L}_c(B) = \operatorname{span}_{\mathbb{R}}\{M_0, \dots, M_4\}$, $\mathscr{L}_c(Bx) = \operatorname{span}_{\mathbb{R}}\{M_0x, \dots, M_4x\}$ and $\mathscr{C}_C(Bx) = \operatorname{span}_{\mathscr{I}_C(Bx)}\{M_0x, \dots, M_4x\}$, where

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathscr{I}_C(Bx)$ is the set of arbitrary functions of $J_1 = \frac{x_1}{x_2}$ and $J_2 = \frac{x_1^2}{x_3}$. Since Bx, M_1x, and M_2x are linearly independent over the field of meromorphic functions,

$$\Omega(x) = \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & 0 & x_1 \\ 2x_3 & 0 & 0 \end{bmatrix}, \quad \det\bigl(\Omega(x)\bigr) = 2x_1x_2x_3 \neq 0,$$

Theorem 3.11 implies that $\mathscr{C}_C(Bx) = \operatorname{span}_{\mathscr{I}_C(Bx)}\{Bx, M_1x, M_2x\}$. Moreover, since $\mathscr{L}_c(Bx) \subset \mathscr{C}_C(Bx)$, then any $Ax \in \mathscr{L}_c(Bx)$ can be obtained by taking linear combination of Bx, M_1x, and M_2x, with coefficients in $\mathscr{I}_C(Bx)$. For instance, $M_0x = \Omega(x)(\Omega^{-1}(x)M_0x)$, with the entries of $\Omega^{-1}(x)M_0x$ belonging to $\mathscr{I}_C(Bx)$,

$$\Omega^{-1}(x)M_0x = \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & 0 & x_1 \\ 2x_3 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{x_1}{x_2} \\ 0 \end{bmatrix},$$

namely $M_0x = \frac{x_1}{x_2}M_1x$.

Remark 3.10 The set of all f such that $[f(x), Bx] = \mu(x)f(x)$ (which should not be confused with the normalizer $\mathscr{N}_C(Bx)$ of Bx, which is the set of all g such that $[Bx, g(x)] = \mu(x)Bx)$ can be easily constructed by multiplying any $f \in \mathscr{C}_C(Bx)$ for an arbitrary function α. To be more precise, by Theorem 3.6, if f is such that $[f(x), Bx] = \mu(x)f(x)$, then $\hat{f} = \frac{1}{L_f J} f \in \mathscr{C}_C(Bx)$, with $J \in \mathscr{I}_C(Bx)$ such that $L_f J \neq 0$. Conversely, by Theorem 3.7, if $f \in \mathscr{C}_C(Bx)$, then $\hat{f} = \alpha f$ satisfies $[\hat{f}(x), Bx] = \mu(x)\hat{f}(x)$ for some $\mu(x) \in \mathbb{R}$.

Example 3.3 Consider again the vector function g introduced in Example 3.2. Since $g(x) = Bx$, with $B = \operatorname{diag}\{1, -1\}$ being semi-simple, a basis of $\mathscr{L}_c(B)$ is given by B^0 and B^1; set $\mathscr{I}_C(Bx)$ is constituted by all functions of $J_1(x) = x_1x_2$. Then, any f having g as symmetry can be rewritten as

$$f(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\mu_0 + \mu_1)x_1 \\ (\mu_0 - \mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of J_1. Formula (3.15) is thus recovered by taking $\mu_0(J_1) = \frac{1}{2}\frac{\beta(J_1)}{J_1}$ and $\mu_1(J_1) = \alpha(J_1)$.

Let $B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$ be $m < n$ linearly independent and pairwise commuting matrices, $[B_i, B_j] = 0$. Let $\mathscr{L}_c(B_1, \ldots, B_m)$ be the *linear centralizer* of $\{B_1, \ldots, B_m\}$, i.e., the set of all matrices A commuting with B_i, $[A, B_i] = 0$, $i = 1, \ldots, m$; clearly, $\mathscr{L}_c(B_1, \ldots, B_m)$ is a vector space over $\mathbb{R}$; let $\{M_0, \ldots, M_{\bar{r}-1}\}$ be a basis of such a linear centralizer $\mathscr{L}_c(B_1, \ldots, B_m)$. Let $\mathscr{I}_C(B_1x, \ldots, B_mx)$ be the set of all joint first integrals associated with $g_1(x) = B_1x, \ldots, g_m(x) = B_mx$: namely, $\mathscr{I}_C(B_1x, \ldots, B_mx)$ is the set of all $J(x)$ such that $L_{g_i}J = 0$, $i = 1, \ldots, m$. Since $[B_i, B_j] = 0$, by the Frobenius Theorem 1.9 at p. 21, there exist $n - m$ functionally independent functions $J_1(x), \ldots, J_{n-m}(x) \in \mathbb{R}$ such that any $J \in \mathscr{I}_C(B_1x, \ldots, B_mx)$ can be expressed by $J = C(J_1, \ldots, J_{n-m})$, where C is an arbitrary function.

The proof of the following theorem is omitted since it is similar to the proof of Theorem 3.10.

Theorem 3.12 *Let* $B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$ *be* $m < n$ *linearly independent and pairwise commuting matrices*, $[B_i, B_j] = 0$. *Then, the set of all* $f(x) \in \mathbb{R}^n$ *having* $g_1(x) =$

$B_1x, \ldots, g_m(x) = B_m x$ as symmetries is parameterized by

$$f(x) = \mu_0 M_0 x + \mu_1 M_1 x + \cdots + \mu_{\bar{r}-1} M_{\bar{r}-1} x,$$

where $\{M_0, \ldots, M_{\bar{r}-1}\}$ is a basis of $\mathscr{L}_c(B_1, \ldots, B_m)$ and $\mu_i \in \mathscr{I}_C(B_1x, \ldots, B_m x)$, $i = 0, \ldots, \bar{r}-1$.

Example 3.4 Let

$$B_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

which are clearly linearly independent over $\mathbb{R}$ and pairwise commuting. A basis of $\mathscr{L}_c(B_1, B_2)$ is

$$\left\{ M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

To compute the joint first integral associated with both B_1x and B_2x, take any element of $\mathscr{L}_c(B_1, B_2)$ being linearly independent of B_1 and B_2; for instance, take M_0. Since B_1, B_2 and M_0 are pairwise commuting, then the rows of $[B_1x\ B_2x\ M_0x]^{-1}$ are exact one-forms; hence, the first integral of the last row of $[B_1x\ B_2x\ M_0x]^{-1}$ is a joint first integral associated with both B_1x and B_2x, thus obtaining $\ln(|\frac{x_1 x_2^{5/7}}{\sqrt[7]{x_3}}|)$; therefore, set $\mathscr{I}_C(B_1x, B_2x)$ is constituted by the arbitrary functions of $J(x) = \frac{x_1 x_2^{5/7}}{\sqrt[7]{x_3}}$. Finally, all vector functions $f(x)$ having both B_1x and B_2x as symmetries are parameterized by

$$f(x) = \mu_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} \mu_0 x_1 \\ \mu_1 x_2 \\ \mu_2 x_3 \end{bmatrix},$$

where μ_0, μ_1, μ_2 are arbitrary functions of J.

The following theorem shows that, if $[f, g] = \mu f$, then the knowledge of a first integral associated with f yields another (possibly, trivial) first integral associated with f, and the knowledge of a first integral associated with g yields a (possibly, trivial) semi-invariant associated with g.

Theorem 3.13 *Assume that g is an orbital symmetry of f,* $[f, g] = \mu f$.

(3.13.1) *If I is a first integral of system* (1.1a)*, then* $L_g I$ *is again a first integral of system* (1.1a).

(3.13.2) *If J is a first integral of system* (1.2)*, then* $L_f J$ *is a semi-invariant of system* (1.2)*, with characteristic function* $-\mu$*, provided that there is no zero/pole cancelation between* $L_f J$ *and* μ.

Proof Since $L_{[f,g]} = L_f L_g - L_g L_f$ and $L_{\mu f} = \mu L_f$, by $[f, g] = \mu f$, it follows that $L_f L_g - L_g L_f = \mu L_f$. Let I be a first integral of system (1.1a), i.e., $L_f I = 0$; then, $L_f L_g I - L_g L_f I = \mu L_f I$ implies that $L_f L_g I = 0$. Let J be a first integral of system (1.2), i.e., $L_g J = 0$; then, $L_f L_g J - L_g L_f J = \mu L_f J$ implies $L_g(L_f J) = -\mu(L_f J)$. □

3.3 Continuous-Time Homogeneous Nonlinear Systems

The easiest standard concept of homogeneity defines a scalar function $h(x)$ to be homogeneous of degree m if $h(\alpha x) = \alpha^m h(x)$, for a scalar α. In this way, a polynomial of x is homogeneous if all its terms are monomials of the same order; homogeneity allows for example to recognize which terms of a polynomial are dominant for small x, and, consequently, to derive approximations about the origin. A natural extension of this concept is that of homogeneity with respect to a dilation [9, 55], and a further extension is that of homogeneity with respect to a vector function [64, 74, 75]. Both such extensions have been used to study the stability of equilibrium points by means of the concept of *stability in the first approximation* [11, 64, 106], which extends the well known method based on the study of the linearized system. Most of the introductory definitions and results of this section, and some further material, can be found in [9, 11, 64, 74, 75].

Definition 3.6 Given a vector of real numbers $w = [w_1 \ \dots \ w_n]^\top$ ($w_1, \dots, w_n$ are called *weights*), a *dilation* $\delta_\varepsilon^w x$ is defined as $\delta_\varepsilon^w x := [\varepsilon^{w_1} x_1 \ \dots \ \varepsilon^{w_n} x_n]^\top$, with $\delta_\varepsilon^w \in \mathbb{R}^{n\times n}$, $\delta_\varepsilon^w = \mathrm{diag}\{\varepsilon^{w_1}, \dots, \varepsilon^{w_n}\}$, for any $\varepsilon \in \mathbb{R}$ such that ε^{w_i} is defined for all $i \in \{1, \dots, n\}$. A function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $m \in \mathbb{R}$ *with respect to dilation* $\delta_\varepsilon^w x$ if:

$$h\big(\delta_\varepsilon^w x\big) = \varepsilon^m h(x), \quad \text{whenever defined.} \tag{3.19}$$

If all w_i are rational numbers, negative integers or positive integers, then $\delta_\varepsilon^w x$ is referred to as a *rational*, *negative integer* or *positive integer dilation*, respectively. If (3.19) holds for all $x \in \mathbb{R}^n$ and for all $\varepsilon \in \mathbb{R}$, then h is said to be *homogeneous on the whole* $\mathbb{R}^n$ with respect to $\delta_\varepsilon^w x$. If all weights w_i are equal to 1, then the dilation is said to be *standard*.

Since $\delta_{\varepsilon^{-1}}^w = \delta_\varepsilon^{-w}$, the case of a negative integer dilation can always be reduced to the case of a positive integer dilation, when (3.19) holds for all $\varepsilon \in \mathbb{R}$.

For positive ε, letting $\tau = \ln(\varepsilon)$ and $B_w = \mathrm{diag}\{w_1, \dots, w_n\}$, one has $\delta_\varepsilon^w = \mathrm{e}^{B_w \tau}$.

Given a positive integer dilation $\delta_\varepsilon^w x$, any function $h(x) \in \mathbb{R}$, analytic on a neighborhood of the origin of $\mathbb{R}^n$, can be expanded in an infinite series $h = \sum_{i=0}^{+\infty} h^{[i]}$, with $h^{[i]}$ being polynomial and homogeneous of degree i with respect to $\delta_\varepsilon^w x$: this can be done by expanding $h(\delta_\varepsilon^w x)$ in Taylor series with respect to ε, about $\varepsilon = 0$, $h(\delta_\varepsilon^w x) = \sum_{i=0}^{+\infty} h^{[i]}(x)\varepsilon^i$, and then formally letting $\varepsilon = 1$; this can be certainly done if x is taken in a sufficiently small neighborhood of the origin of $\mathbb{R}^n$, because $h(\delta_\varepsilon^w x)$ is a function of ε analytic at $\varepsilon = 0$, for any x in a sufficiently small neighborhood of the origin of $\mathbb{R}^n$, if $w_i > 0$, $i = 1, \dots, n$. If $h(\delta_\varepsilon^w x) = \sum_{i=i^*}^{+\infty} h^{[i]}(x)\varepsilon^i$, for some $i^* \geq 0$, then $h^{[i^*]}(x)$ is called the *first approximation* of $h(x)$ with respect to $\delta_\varepsilon^w x$.

Example 3.5 Let $w = [1\ 2]^\top$ and consider the function $h(x) = x_2 \sin(x_1)$; then, $h(\delta_\varepsilon^w) = \varepsilon^2 x_2 \sin(\varepsilon x_1) = x_1 x_2 \varepsilon^3 + (-\frac{1}{6} x_1^3 x_2)\varepsilon^5 + O(\varepsilon^7)$, whence one concludes that $h^{[0]}(x) = 0$, $h^{[1]}(x) = 0$, $h^{[2]}(x) = 0$, $h^{[3]}(x) = x_1 x_2$, $h^{[4]}(x) = 0$, $h^{[5]}(x) = -\frac{1}{6} x_1^3 x_2$ and $h^{[6]}(x) = 0$. The first approximation of h with respect to $\delta_\varepsilon^w x$ is $h^{[3]}(x) = x_1 x_2$.

Definition 3.7 Given a dilation $\delta_\varepsilon^w x$ and a number $m \in \mathbb{R}$, the vector function $f(x) = [f_1(x)\ \dots\ f_n(x)]^\top : \mathbb{R}^n \to \mathbb{R}^n$ is *homogeneous of degree m with respect to $\delta_\varepsilon^w x$* if f_i is homogeneous of degree $w_i - m$ with respect to $\delta_\varepsilon^w x$, namely if:

$$f_i\left(\delta_\varepsilon^w x\right) = \varepsilon^{w_i - m} f_i(x), \quad \text{whenever defined, } i = 1, \dots, n. \tag{3.20}$$

Note that for $w_i - m$ to be positive for $i = 1, \dots, n$, it is necessary and sufficient that $m < \min\{w_1, \dots, w_n\}$. Similarly to the scalar case, the vector function f can be expanded with respect to $\delta_\varepsilon^w x$ by expanding each entry f_j of f with respect to $\delta_\varepsilon^w x$; then, collecting all terms according to their degree of homogeneity with respect to $\delta_\varepsilon^w x$, one concludes that $f = \sum_{i=-\infty}^{i^*} f^{[i]}$, where $f^{[i]}$ is homogeneous of degree i with respect to $\delta_\varepsilon^w x$; hence, $f^{[i^*]}$ is called the *first approximation* of f with respect to $\delta_\varepsilon^w x$.

The following example shows that, given a positive integer dilation, one can construct a homogeneous system on the whole $\mathbb{R}^n$ (i.e., a system described by a vector function f homogeneous on the whole $\mathbb{R}^n$): note that, if a function is analytic and homogeneous on the whole $\mathbb{R}^n$ with respect to a positive integer dilation, then it is necessarily a polynomial. The following example also illustrates that the same does not hold if the dilation is not positive integer.

Example 3.6 Consider the positive integer dilation $\delta_\varepsilon^w x$, with $w = [1\ 2]^\top$. Let $\mathscr{P}_i$ be the set of all scalar functions being analytic and homogeneous of degree i, with respect to $\delta_\varepsilon^w x$, on the whole $\mathbb{R}^2$ (the letter $\mathscr{P}$ is used because such functions are polynomials): $\mathscr{P}_0 = c_0$, $\mathscr{P}_1 = \{c_1 x_1\}$, $\mathscr{P}_2 = \{c_1 x_1^2 + c_2 x_2\}$, $\mathscr{P}_3 = \{c_1 x_1^3 + c_2 x_1 x_2\}$, $\mathscr{P}_4 = \{c_1 x_1^4 + c_2 x_1^2 x_2 + c_3 x_2^2\}$, and so on, with the constants $c_i \in \mathbb{R}$ being arbitrary. Let $f^{[i]} = [f_1^{[i]}\ f_2^{[i]}]^\top$ be a vector function analytic and homogeneous of degree i on the whole $\mathbb{R}^2$; then, $f_1^{[i]}$ is homogeneous of degree $1 - i$ and $f_2^{[i]}$ is homogeneous of degree $2-i$. Hence,

$$f^{[0]} = \begin{bmatrix} a_1 x_1 \\ b_1 x_1^2 + b_2 x_2 \end{bmatrix}, \qquad f^{[-1]} = \begin{bmatrix} a_1 x_1^2 + a_2 x_2 \\ b_1 x_1^3 + b_2 x_1 x_2 \end{bmatrix},$$

$$f^{[-2]} = \begin{bmatrix} a_1 x_1^3 + a_2 x_1 x_2 \\ b_1 x_1^4 + b_2 x_1^2 x_2 + b_3 x_2^2 \end{bmatrix},$$

and so on, with constants $a_i, b_i \in \mathbb{R}$ being arbitrary. Note that $h_1(x) = (x_1 + x_2^2)^{1/2}$ is homogeneous of degree 1 with respect to $\delta_\varepsilon^w x$, with $w = [2\ 1]^\top$, but it is not analytic at $x = 0$; $h_2(x) = \sin(x_1 x_2)$ is analytic and homogeneous of degree 0, with respect to $\delta_\varepsilon^w x$, with $w = [-1\ 1]^\top$, on the whole $\mathbb{R}^2$, but the two weights have not the same sign; $h_3(x) = x_2 \sin(x_1)$ is analytic and homogeneous of degree 1, with respect to $\delta_\varepsilon^w x$, with $w = [0\ 1]^\top$, on the whole $\mathbb{R}^2$, but one of the two weights is equal to zero.

Example 3.7 Consider the integer dilation $\delta_\varepsilon^w x$, with $w = [w_1\ w_2]^\top = [-1\ 1]^\top$. Since $w_1 + w_2 = 0$, all monomials of degree -1 are $x_1^{h+1} x_2^h$ and all monomials of degree 1 are $x_1^h x_2^{h+1}$, for $h \in \mathbb{Z}^{\geq}$. If $f^{[0]} = [f_1^{[0]}\ f_2^{[0]}]^\top$ is analytic and homogeneous of degree 0, with respect to $\delta_\varepsilon^w x$, on the whole $\mathbb{R}^2$, then $f_1^{[0]}$ has degree -1,

$$f_1^{[0]}(x) = \sum_{h=0}^{+\infty} a_h x_1^{h+1} x_2^h = x_1 \sum_{h=0}^{+\infty} a_h (x_1 x_2)^h = x_1 \alpha(x_1 x_2),$$

and $f_2^{[0]}$ has degree 1,

$$f_2^{[0]}(x) = \sum_{h=0}^{+\infty} b_h x_1^h x_2^{h+1} = x_2 \sum_{h=0}^{+\infty} b_h x_1^h x_2^h = x_2 \beta(x_1 x_2),$$

with α, β being arbitrary analytic functions. Similarly, if $f^{[-1]} = [f_1^{[-1]}\ f_2^{[-1]}]^\top$ is analytic and homogeneous of degree -1, on the whole $\mathbb{R}^2$, then

$$f_1^{[-1]}(x) = \gamma(x_1 x_2), \qquad f_2^{[-1]}(x) = x_2^2 \delta(x_1 x_2),$$

with $\gamma(\cdot), \delta(\cdot)$ being arbitrary analytic functions of the argument.

Theorem 3.14 *Let $h(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}^n$ be homogeneous of degree m with respect to $\delta_\varepsilon^w x$. Then,*

$$L_{g^w} h = m h, \quad \textit{whenever defined}, \tag{3.21a}$$

$$[f, g^w] = m f, \quad \textit{whenever defined}, \tag{3.21b}$$

where $g^w(x) := B_w x$ and $B_w := \operatorname{diag}\{w_1, \ldots, w_n\}$.

Proof Since h is homogeneous of degree m with respect to $\delta_\varepsilon^w x$,

$$h\big(\varepsilon^{w_1}x_1,\dots,\varepsilon^{w_n}x_n\big)=\varepsilon^m h(x_1,\dots,x_n),$$

taking the derivative with respect to ε of the above equation

$$\frac{\partial h(x)}{\partial x_1}\bigg|_{x=\delta_\varepsilon^w x} w_1\varepsilon^{w_1-1}x_1+\dots+\frac{\partial h(x)}{\partial x_n}\bigg|_{x=\delta_\varepsilon^w x} w_n\varepsilon^{w_n-1}x_n=m\varepsilon^{m-1}h(x)$$

and letting $\varepsilon=1$, one concludes that

$$\frac{\partial h(x)}{\partial x_1}w_1x_1+\dots+\frac{\partial h(x)}{\partial x_n}w_nx_n=mh(x),$$

namely $L_{g^w}h=mh$. Since f is homogeneous of degree m with respect to $\delta_\varepsilon^w x$, the ith entry f_i of f is homogeneous of degree w_i-m, namely $L_{g^w}f_i=(w_i-m)f_i$, which implies $L_{g^w}f=\operatorname{diag}\{w_1-m,\dots,w_n-m\}f=(B_w-mE)f$, where $B_w=\operatorname{diag}\{w_1,\dots,w_n\}$; since $L_f g^w=B_w f$, one concludes that

$$[f,g^w]=L_f g^w-L_{g^w}f=B_w f-(B_w-mE)f=mf. \qquad \square$$

The vector function $g^w(x)=B_w x$ is called the *Euler vector function* associated with the dilation $\delta_\varepsilon^w x$; note that $e^{B_w\ln(\varepsilon)}=\delta_\varepsilon^w$.

Example 3.8 Consider again the functions h_1, h_2 and h_3, introduced in Example 3.6; then, letting $g_1^w(x)=[2x_1\ x_2]^\top$, $g_2^w(x)=[-x_1\ x_2]^\top$ and $g_3^w(x)=[0\ x_2]^\top$, it is easily checked that $L_{g_1^w}h_1=h_1$, $L_{g_2^w}h_2=0$ and $L_{g_3^w}h_3=h_3$. Consider the vector function $f^{[-1]}$ introduced in Example 3.7; then, letting $g^w(x)=[-x_1\ x_2]^\top$, it is easily checked that $[f^{[-1]},g^w]=-f^{[-1]}$.

By (3.21b), g^w is a symmetry (respectively, an orbital symmetry, but not a symmetry) of f, if f is homogeneous of degree $m=0$ (respectively, $m\neq 0$) with respect to $\delta_\varepsilon^w x$. Let a dilation $\delta_\varepsilon^w x$ be given; if the corresponding g^w is an orbital symmetry of f, then kg^w is an orbital symmetry of f for any number $k\neq 0$. If f is homogeneous of degree m with respect to $\delta_\varepsilon^w x$, i.e., if $[f,g^w]=mf$, then $[f,kg^w]=kmf$. Therefore, if $m\neq 0$, one can take $k=-\frac{1}{m}$, so that $[f,kg^w]=-f$. Similarly, if $L_{g^w}h=mh$, then $L_{kg^w}h=kmh$; therefore, also in this case, if $m\neq 0$, one can take $k=-\frac{1}{m}$, so that $L_{kg^w}h=-h$. This reasoning implies that by rescaling the vector of weights, one could just consider the two cases $m=0$ and $m=-1$.

Let an analytic diffeomorphism $y=\varphi(x)$ be given. Then, by the invariance of the Lie bracket to diffeomorphisms, $[f,g^w]=mf$ if and only if $[\varphi_* f,\varphi_* g^w]=m(\varphi_* f)$. This justifies the following general definition of homogeneity.

Definition 3.8 Let $f(x), g(x)\in\mathbb{R}^n$ and $h(x)\in\mathbb{R}$. Scalar function h is homogeneous of degree $m\in\mathbb{R}$ with respect to g if $L_g h=mh$. Vector function f is homogeneous of degree $m\in\mathbb{R}$ with respect to g if $[f,g]=mf$.

Note that $[f, g] = mf$ implies $L_g f_i = (r_i - m) f_i$ when $g(x) = [w_1 x_1 \ \dots \ w_n x_n]^\top$, but this implication need not hold for a general g.

Theorem 3.15 *Let $f(x), g(x) \in \mathbb{R}^n$ and $h(x) \in \mathbb{R}$; let $x = \Phi_g(\tau, y)$ be the flow associated with g. Then,*

$$L_g h = mh \quad \Longleftrightarrow \quad h(x) \circ \Phi_g(\tau, y) = \mathrm{e}^{m\tau} h(y),$$

$$[f, g] = mf \quad \Longleftrightarrow \quad \left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f(x) \circ \Phi_g(\tau, y) = \mathrm{e}^{-m\tau} f(y),$$

$$\left.\begin{array}{l} L_g h = m_1 h \\ [f, g] = m_2 f \end{array}\right\} \quad \Rightarrow \quad L_g L_f h = (m_1 - m_2) L_f h.$$

Proof Clearly, $h \circ \Phi_g = \mathrm{e}^{m\tau} h$ holds if and only if $h = \mathrm{e}^{-m\tau} h \circ \Phi_g$ holds. Such a relation is satisfied for $\tau = 0$; since the left-hand side of $h = \mathrm{e}^{-m\tau} h \circ \Phi_g$ is independent of τ, then such a relation holds if and only if $\frac{\partial}{\partial \tau}(\mathrm{e}^{-m\tau} h \circ \Phi_g) = 0$, for all admissible $\tau \in \mathbb{R}$. Now,

$$\begin{aligned}
\frac{\partial}{\partial \tau}\big(\mathrm{e}^{-m\tau} h \circ \Phi_g\big) &= -m\mathrm{e}^{-m\tau} h \circ \Phi_g + \mathrm{e}^{-m\tau} \left.\frac{\partial h}{\partial x}\right|_{x=\Phi_g} \frac{\partial \Phi_g}{\partial \tau} \\
&= -m\mathrm{e}^{-m\tau} h \circ \Phi_g + \mathrm{e}^{-m\tau} \left(\frac{\partial h}{\partial x} g\right) \circ \Phi_g \\
&= \mathrm{e}^{-m\tau} (-mh + L_g h) \circ \Phi_g,
\end{aligned}$$

whence condition $L_g h = mh$ is equivalent to condition $\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{-m\tau} h \circ \Phi_g) = 0$, for all admissible $\tau \in \mathbb{R}$.

Similarly, since $f = \mathrm{e}^{m\tau} (\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g$ and such a relation is satisfied for $\tau = 0$, one has to show that $\frac{\partial}{\partial \tau}(\mathrm{e}^{m\tau} (\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g) = 0$. By The proof of Theorem 3.4,

$$\frac{\partial}{\partial \tau}\left(\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g\right) = -\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} [f, g] \circ \Phi_g,$$

and therefore

$$\frac{\partial}{\partial \tau}\left(\mathrm{e}^{m\tau}\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g\right) = \mathrm{e}^{m\tau}\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} \big(mf - [f, g]\big) \circ \Phi_g,$$

which is identically equal to zero if and only if $[f, g] = mf$.

Finally, since $L_f L_g - L_g L_f = L_{[f,g]}$, it follows that

$$L_g L_f h = L_f L_g h - L_{[f,g]} h = m_1 L_f h - m_2 L_f h = (m_1 - m_2) L_f h. \qquad \square$$

Remark 3.11 If g is a symmetry of f, then f is homogeneous of degree 0 with respect to g and vice versa.

The following theorem characterizes all scalar functions h homogeneous of degree m with respect to a vector function g.

Theorem 3.16 *Let $x_e = [x^\top \ h]^\top$ and $g_e = [g^\top \ mh]^\top$. Consider the set $\mathscr{I}_C(g)$ of all first integrals $J(x)$ associated with g and the set $\mathscr{I}_C(g_e)$ of all first integrals $J(x_e)$ associated with g_e. Let $J_1, \dots, J_{n-1}$ be $n-1$ functionally independent elements of $\mathscr{I}_C(g)$ and let $J_n(x, h) \in \mathscr{I}_C(g_e)$ be such that $J_1, \dots, J_n$ are functionally independent as functions of x_e. Then, all functions h, being homogeneous of degree m with respect to $\delta_\varepsilon^w x$, are given by the solution in h of*

$$J_n(x, h) = C\big(J_1(x), \dots, J_{n-1}(x)\big), \tag{3.22}$$

where C is an arbitrary function of the arguments.

Proof The proof is based on the method of the *characteristic equation* for solving partial differential equations (see, [46]). Condition $L_g h = mh$ yields the partial differential equation

$$\frac{\partial h}{\partial x_1} g_1 + \cdots + \frac{\partial h}{\partial x_n} g_n = mh, \tag{3.23}$$

where g_i is the ith entry of g. The characteristic equation associated with (3.23) is

$$\frac{\mathrm{d}x_1}{g_1} = \frac{\mathrm{d}x_2}{g_2} = \cdots = \frac{\mathrm{d}x_n}{g_n} = \frac{\mathrm{d}h}{mh}. \tag{3.24}$$

To solve (3.23), one writes the characteristic equation

$$\frac{\mathrm{d}x_1}{g_1} = \frac{\mathrm{d}x_2}{g_2} = \cdots = \frac{\mathrm{d}x_n}{g_n} \tag{3.25}$$

of the homogeneous equation of (3.23), that is,

$$\frac{\partial h}{\partial x_1} g_1 + \cdots + \frac{\partial h}{\partial x_n} g_n = 0. \tag{3.26}$$

The set of solutions of (3.26) is just $\mathscr{I}_C(g)$; such functions are also called first integrals of (3.25), and each of them can be written as a function of $J_1(x), \dots, J_{n-1}(x)$. All solutions of (3.23) are found by equating one particular first integral of g_e, being not trivial and not belonging to $\mathscr{I}_C(g)$, namely $J_n(x, h)$, and equating it to the general first integral of (3.25). In this way, (3.22) is obtained. Note also that, with the notation introduced here, the set of first integrals of (3.24) is just $\mathscr{I}_C(g_e)$. □

Example 3.9 Suppose that one wants to find all $h(x) \in \mathbb{R}$, $x \in \mathbb{R}^2$, being homogeneous of degree 3 with respect to the dilation $\delta_\varepsilon^w x$, with $w = [1 \ 2]^\top$. Then, $g^w(x) = B_w x$ and $g_e^w(x_e) = B_{w,e} x_e$, where $B_w = \mathrm{diag}\{1, 2\}$ and $B_{w,e} = \mathrm{diag}\{1, 2, 3\}$. An element of $\mathscr{I}_C(g^w)$, which is not constant, is $I_1(x) = \frac{x_1^2}{x_2}$, whereas an element I_2 of

$\mathscr{I}_C(g_e^w)$, being functionally independent of I_1, is $I_2(x,h) = \frac{h}{x_1^3}$. Then, all functions h homogeneous of degree 3 with respect to $\delta_\varepsilon^w x$ are found by solving $I_2 = C(I_1)$, where $C(\cdot)$ is an arbitrary function of the argument; in particular, from $\frac{h}{x_1^3} = C(\frac{x_1^2}{x_2})$, it follows that $h(x) = x_1^3 C(\frac{x_1^2}{x_2})$. If, in addition, one imposes that h is analytic on the whole $\mathbb{R}^2$, then $C(I_1) = a + b\frac{1}{I_1}$, namely $h(x) = x_1^3(a + b\frac{x_2}{x_1^2}) = ax_1^3 + bx_1x_2$, with $a, b \in \mathbb{R}$ being arbitrary constants.

Example 3.10 Suppose that one wants to find all $h(x) \in \mathbb{R}$, $x \in \mathbb{R}^2$, being homogeneous of degree 4 with respect to $g(x) = [x_1 \; x_2^3]^\top$: note that there exists no diffeomorphism $y = \varphi(x)$ analytic at $x = 0$ such that $\varphi_* g$ is linear (this will be clear in the subsequent Sect. 3.12). Let $g_e(x_e) = [x_1 \; x_2^3 \; 4h]^\top$. Clearly, $I_1(x) = \ln(|x_1|) + \frac{1}{2x_2^2}$, which is not constant, is an element of $\mathscr{I}_C(g^w)$ and $I_2(x,h) = \frac{h}{x_1^4}$ is and element of $\mathscr{I}_C(g_e^w)$, being functionally independent of I_1. Then, all functions h homogeneous of degree 4 with respect to g are found by solving $I_2 = C(I_1)$, where $C(\cdot)$ is an arbitrary function of the argument; in particular, from $\frac{h}{x_1^4} = C(\ln(|x_1|) + \frac{1}{2x_2^2})$, it follows that $h(x) = x_1^4 C(\ln(|x_1|) + \frac{1}{2x_2^2})$.

Theorem 3.17 [92] *Let $g(x) \in \mathbb{R}^n$. Let $J_0, J_1, \ldots, J_{n-1}$ be functionally independent and such that $L_g J_0 = 1$ and $L_g J_i = 0, i = 1, \ldots, n-1$. Let $J = [J_0 \; J_1 \; \ldots \; J_{n-1}]^\top$. Then, $f(x) \in \mathbb{R}^n$ is homogeneous of degree m with respect to g if and only if*

$$f = \left(\frac{\partial J}{\partial x}\right)^{-1} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \mathrm{e}^{-mJ_0}, \tag{3.27}$$

with β_i's being arbitrary functions of $J_1, \ldots, J_{n-1}$.

Proof In the local coordinates $y = \varphi(x)$, with $\varphi = [J_0 \; J_1 \; \ldots \; J_{n-1}]^\top$, one has $\varphi_* g = e_1$. Then, letting $\tilde{f} = \varphi_* f$, equality $[\varphi_* f, \varphi_* g] = m(\varphi_* f)$ reduces to the following set of partial differential equations:

$$-\frac{\partial \tilde{f}_i}{\partial y_1} = m\tilde{f}_i, \quad i = 1, \ldots, n,$$

with solution $\tilde{f}_i(y) = \mathrm{e}^{-my_1} C_i(y_2, \ldots, y_n)$, where C_i is an arbitrary function of the arguments. Then, (3.27) is found by the pull-back $f = \varphi^* \tilde{f}$. □

The proof of the following corollary is similar to the proof of Theorem 3.10.

Corollary 3.1 *Assume $g(x) = Bx$. Let $\{M_0, \ldots, M_{r-1}\}$ be a basis of $\mathscr{L}_c(B)$. Hence, f is homogeneous of degree m with respect to g if and only if*

$$f(x) = \mathrm{e}^{-m\eta}(\mu_0 M_0 x + \cdots + \mu_{r-1} M_{r-1} x),$$

where $\mu_i \in \mathscr{I}_C(g)$, $i = 0, \ldots, r-1$, and η is a scalar function such that $L_g \eta = 1$.

Example 3.11 Let $B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$; then, $\mathscr{L}_c(B) = \mathrm{span}_{\mathbb{R}}\{B, E\}$. Let $\Omega(x) = [Bx \;\; Ex]$; by integrating the rows of

$$\Omega^{-1}(x) = \begin{bmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 0 & \frac{1}{x_2} \end{bmatrix},$$

one concludes that $\eta(x) = \frac{x_1}{x_2}$ satisfies $L_{Bx}\eta = 1$ and that all continuous-time first integrals associated with Bx are given by the arbitrary functions of x_2. Then, all f being homogeneous of degree m with respect to $g(x) = Bx$ are given by

$$\begin{aligned} f(x) &= \mathrm{e}^{-m\frac{x_1}{x_2}} \left(\mu_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} \mathrm{e}^{-m\frac{x_1}{x_2}} \mu_1 x_1 + \mathrm{e}^{-m\frac{x_1}{x_2}} \mu_0 x_2 \\ \mathrm{e}^{-m\frac{x_1}{x_2}} \mu_1 x_2 \end{bmatrix}, \end{aligned}$$

where $\mu_0, \mu_1 \in \mathscr{I}_C(Bx)$, i.e., μ_0 and μ_1 are arbitrary functions of x_2.

Remark 3.12 Let $h(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}^n$; let $\Phi_g(\tau, x)$ be the flow associated with g. The following statements are equivalent:

(3.12.1) h is homogeneous of degree 0 with respect to g;
(3.12.2) h is a first integral associated with g;
(3.12.3) $L_g h = 0$;
(3.12.4) $h \circ \Phi_g = h$.

Similarly, the following statements are equivalent:

(3.12.5) h is homogeneous of degree m with respect to g;
(3.12.6) h is a semi-invariant associated with g, with a characteristic value $\lambda = m$;
(3.12.7) $L_g h = mh$;
(3.12.8) $h \circ \Phi_g = \mathrm{e}^{m\tau} h$.

Remark 3.13 Let $f(x), g(x) \in \mathbb{R}^n$; let Φ_f and Φ_g be the flows associated with f and g, respectively. The following statements are equivalent:

(3.13.1) f is homogeneous of degree 0 with respect to g;
(3.13.2) $[f, g] = 0$;
(3.13.3) $(\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g = f$;
(3.13.4) $\Phi_f(t, \cdot) \circ \Phi_g(\tau, x) = \Phi_g(\tau, \cdot) \circ \Phi_f(t, x)$.

Similarly, the following statements are equivalent:

(3.13.5) f is homogeneous of degree m with respect to g;
(3.13.6) $[f, g] = mf$;
(3.13.7) $(\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g = \mathrm{e}^{-m\tau} f$;
(3.13.8) $\Phi_f(t, \cdot) \circ \Phi_g(\tau, x) = \Phi_g(\tau, \cdot) \circ \Phi_f(t\mathrm{e}^{-m\tau}, x)$.

As for a scalar function $h^{[m]}(x) \in \mathbb{R}$, if $L_g h^{[m]} = mh^{[m]}$, letting $\tau = \ln(\varepsilon)$, $\varepsilon > 0$, in (3.12.8), one finds that $h^{[m]} \circ \Phi_g(\ln(\varepsilon), x) = \varepsilon^m h^{[m]}(x)$. Therefore, for a not necessarily homogeneous $h(x) \in \mathbb{R}$, under the assumption that $h \circ \Phi_g(\ln(\varepsilon), x)$ admits a convergent Laurent series expansion with respect to ε, one can write

$$h \circ \Phi_g\big(\ln(\varepsilon), x\big) = \sum_{m=m^*}^{+\infty} \varepsilon^m h^{[m]}(x),$$

where $h^{[m]}$ is homogeneous of degree m with respect to g. This means that $h = \sum_{m=m^*}^{+\infty} h^{[m]}$ is the homogeneous series expansion of h with respect to g; in particular, $h^{[m^*]}$ is the *first approximation* of h with respect to g.

Remark 3.14 It is worth pointing out that such an expansion with respect to a general g may fail to exist. For instance, if $h(x) = \sin(x_1 + x_2)$ and $g = [x_2\ 0]^\top$, since $\Phi_g(\tau, x) = [x_1 + \tau x_2\ x_2]^\top$, it is easy to check that

$$h \circ \Phi_g\big(\ln(\varepsilon), x\big) = \sin\big(x_1 + \big(1 + \ln(\varepsilon)\big)x_2\big)$$

does not admit a convergent Laurent series expansion with respect to ε.

Similarly, for a vector function $f^{[m]}(x) \in \mathbb{R}^n$, if $[f^{[m]}, g] = mf^{[m]}$, letting $\tau = \ln(\varepsilon)$, $\varepsilon > 0$, in (3.13.7), one finds that

$$\left(\frac{\partial \Phi_g(\ln(\varepsilon), x)}{\partial x}\right)^{-1} f^{[m]} \circ \Phi_g\big(\ln(\varepsilon), x\big) = \varepsilon^{-m} f^{[m]}(x).$$

Therefore, for a not necessarily homogeneous vector function $f(x) \in \mathbb{R}^n$, under the assumption that each entry of $(\frac{\partial \Phi_g(\ln(\varepsilon), x)}{\partial x})^{-1} f \circ \Phi_g(\ln(\varepsilon), x)$ admits a convergent Laurent series expansion with respect to ε, one can write

$$\left(\frac{\partial \Phi_g(\ln(\varepsilon), x)}{\partial x}\right)^{-1} f \circ \Phi_g\big(\ln(\varepsilon), x\big) = \sum_{m=-\infty}^{m^*} \varepsilon^{-m} f^{[m]},$$

where $f^{[m]}$ is homogeneous of degree m with respect to g. This means that $f = \sum_{m=-\infty}^{m^*} f^{[m]}$ is the homogeneous series expansion of f with respect to g; in particular, $f^{[m^*]}$ is the *first approximation* of f with respect to g.

Example 3.12 Let $f(x) = [\sin(x_1 + x_2)\ x_1 \cos(x_2)]^\top$ and $g(x) = [x_1\ 2x_2]^\top$. Since $\Phi_g(\tau, x) = [\mathrm{e}^\tau x_1\ \mathrm{e}^{2\tau} x_2]^\top$, one computes

$$\left(\frac{\partial \Phi_g(\ln(\varepsilon), x)}{\partial x}\right)^{-1} f \circ \Phi_g\big(\ln(\varepsilon), x\big)$$
$$= \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \varepsilon^{-1} + \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \varepsilon + \begin{bmatrix} -\frac{1}{6}x_1^3 \\ 0 \end{bmatrix} \varepsilon^2 + \begin{bmatrix} -\frac{1}{2}x_1^2 x_2 \\ -\frac{1}{2}x_1 x_2^2 \end{bmatrix} \varepsilon^3 + O\big(\varepsilon^4\big),$$

which implies the homogeneous series expansion $f(x) = \sum_{m=-\infty}^{1} f^{[m]}(x)$ given by

$$f^{[1]}(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \qquad f^{[0]}(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad f^{[-1]}(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix},$$
$$f^{[-2]}(x) = \begin{bmatrix} -\frac{1}{6}x_1^3 \\ 0 \end{bmatrix}, \qquad f^{[-3]}(x) = \begin{bmatrix} -\frac{1}{2}x_1^2 x_2 \\ -\frac{1}{2}x_1 x_2^2 \end{bmatrix}, \ldots.$$

Obviously, such homogeneous terms can also be computed by letting $\sin(x_1 + x_2) = (x_1 + x_2) - \frac{1}{6}(x_1 + x_2)^3 + \cdots$ and $x_1 \cos(x_2) = x_1 - \frac{1}{2}x_1 x_2^2 + \cdots$ and considering that the first scalar entry of $f^{[i]}$ must be homogeneous of degree $1 - i$ and the second one must be homogeneous of degree $2 - i$ with respect to $\delta_\varepsilon^w x$, with $w = [1\ 2]^\top$.

3.4 Characteristic Solutions of Continuous-Time Homogeneous Nonlinear Systems

Characteristic solutions are a generalization of the concept of eigensolutions for a linear system. This sections follows the spirit of Sect. 17 of [60], where the characteristic solutions are defined for a continuous-time nonlinear system being homogeneous with respect to the standard dilation.

Assume that $f(x) \in \mathbb{R}^n$ is homogeneous of degree $m \neq 0$ with respect to $g(x) \in \mathbb{R}^n$, i.e., $[f, g] = mf$; this assumption implies that $(\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g(\ln(\varepsilon), y) = \varepsilon^{-m} f(y)$ and $\Phi_f(t, \cdot) \circ \Phi_g(\ln(\varepsilon), y) = \Phi_g(\ln(\varepsilon), \cdot) \circ \Phi_{\varepsilon^{-m} f}(t, y)$. The changes of coordinates $x = \Phi_g(\ln(\varepsilon), y)$, $s = \varepsilon^{-m} t$, $\mathrm{d}s = \varepsilon^{-m}\, \mathrm{d}t$ transform the equation $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ into $\frac{\mathrm{d}y}{\mathrm{d}s} = f(y)$. This implies that a change of the time scale can be compensated by a transformation on the state space,

$$\Phi_f\big(\varepsilon^{-m} t, y\big) = \Phi_{\varepsilon^{-m} f}(t, y) = \Phi_g\big(-\ln(\varepsilon), \cdot\big) \circ \Phi_f(t, \cdot) \circ \Phi_g\big(\ln(\varepsilon), y\big).$$

Let $J(x) \in \mathbb{R}$ be homogeneous of degree $k > 0$ with respect to g: assume that $J(x)$ is a positive definite function, $J(x) > 0, \forall x \neq 0$, $J(0) = 0$. Define the state immersion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $\rho^k = J(x)$ and by the diffeomorphism

$y = \Phi_g(-\ln(\rho), x)$: ρ is taken non-negative. By the homogeneity of $J(x)$ with respect to g,

$$J(y) = J\big(\Phi_g\big(-\ln(\rho), x\big)\big) = \rho^{-k} J(x) = 1,$$

which implies that $J(y) = 1$, for all $\rho \neq 0$; this shows that the dynamics of the homogeneous system have been projected on the hyper-surface $J(y) = 1$ by the above immersion $x \to (\rho, y)$. Taking the derivative of ρ^k along f (i.e., the derivative with respect to t), one finds

$$k\rho^{k-1}\frac{\mathrm{d}\rho}{\mathrm{d}t} = L_f J(x);$$

since $J(x)$ is homogeneous of degree k and f is homogeneous of degree m with respect to g, $L_f J$ is homogeneous of degree $k - m$ with respect to g, which implies that $L_f J(x) = L_f J \circ \Phi_g(\ln(\rho), y) = \rho^{k-m} L_f J(y)$, whence that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{k}\rho^{1-m} L_f J(y). \tag{3.28}$$

Furthermore, taking into account that $y = \Phi_g(-\ln(\rho), x)$, one concludes that

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial \Phi_g(\tau, x)}{\partial \tau}\bigg|_{\tau=-\ln(\rho)}\left(-\frac{1}{\rho}\right)\frac{1}{k}\rho^{1-m} L_f J(y) + \frac{\partial \Phi_g(-\ln(\rho), x)}{\partial x} f(x) \\
&= g \circ \Phi_g\big(-\ln(\rho), x\big)\left(-\frac{1}{\rho}\right)\frac{1}{k}\rho^{1-m} L_f J(y) + \rho^{-m} f \circ \Phi_g\big(-\ln(\rho), x\big) \\
&= \rho^{-m}\left(f(y) - \frac{L_f J(y)}{k} g(y)\right).
\end{aligned} \tag{3.29}$$

Form (3.29), it follows that

$$\begin{aligned}
\frac{\mathrm{d}J(y)}{\mathrm{d}t} &= \frac{\partial J(y)}{\partial y}\rho^{-m}\left(f(y) - \frac{L_f J(y)}{k} g(y)\right) \\
&= \rho^{-m}\left(L_f J(y) - \frac{1}{k}\big(L_f J(y)\big)\big(L_g J(y)\big)\right) \\
&= \rho^{-m} L_f J(y)\big(1 - J(y)\big),
\end{aligned}$$

which shows that the set of points characterized by $J(y) = 1$ is invariant, as expected.

Assume that there exists a non-zero real solution y_0 of the equation $f(y) = \frac{L_f J(y)}{k} g(y)$. Clearly, $y(t) = y_0$ is a solution of (3.29). Then, from (3.28), one has

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = a\rho^{1-m}, \quad a := \frac{L_f J(y_0)}{k},$$

which, if $m \in \mathbb{Z}$, $m \neq 0$, yields

$$\rho^m(t) - \rho^m(0) = \frac{mL_f J(y_0)}{k} t,$$

namely (taking ρ non-negative)

$$\rho(t) = \left(\rho^m(0) + \frac{mL_f J(y_0)}{k} t\right)^{1/m}.$$

If $m = 0$, then $\rho(t) = \mathrm{e}^{at}\rho(0)$. If $m \in \mathbb{Z}$, $m \neq 0$, the solution of the original system corresponding to $[\rho(t)\ y_0]^\top$ is called *characteristic solution* and it is given by

$$x(t) = \Phi_g\left(\ln\left(\rho^m(0) + \frac{mL_f J(y_0)}{k} t\right)^{1/m}, y_0\right);$$

the initial condition satisfies $x(0) = \Phi_g(\ln(\rho(0)), y_0) = \rho(0) y_0$, where $\rho^k(0) = J(x(0))$, i.e., $\frac{x_0}{\sqrt[k]{J(x_0)}} = y_0$.

For a given real constant $a \neq 0$ and for an integer $m \in \mathbb{Z}$, $m \neq 0$, assuming that $\rho(0) > 0$, the behavior of the solution $\rho(t) = (\rho^m(0) + mat)^{1/m}$ of $\frac{\mathrm{d}\rho}{\mathrm{d}t} = a\rho^{1-m}$ depends on the values of a and m. If $a < 0$ and $m < 0$, the quantity $\rho^m(0) + mat$ is always positive for a non-negative t, and therefore $\rho(t)$ asymptotically goes to 0 as $t \to +\infty$; since ρ is a positive definite function of x, this means that $x(t)$ tends to the origin. If $a > 0$ and $m < 0$, the quantity $\rho^m(0) + mat$ is equal to 0 at $t^* = -\frac{\rho^m(0)}{ma} > 0$, which is a *finite escape time* ($\lim_{t \to t^*} \rho(t) = +\infty$), after which the solution of the differential equation cannot be continued. If $m > 0$, there is no finite escape time; if $a < 0$, there exists a finite time $t^* = -\frac{\rho^m(0)}{ma} > 0$ such that $\rho(t^*) = 0$, but from that time the solution is no longer unique if m is odd ($\rho(t) = -(-ma(t - t^*))^{1/m}$ and $\rho(t) = 0$, $t \geq t^*$, are two different solutions starting from $\rho(t^*) = 0$) or the solution is $\rho(t) = 0$, $t \geq t^*$, if m is even; if $a > 0$, $\rho(t)$ asymptotically goes to $+\infty$ as $t \to +\infty$.

Example 3.13 If $g(x) = x$, $x \in \mathbb{R}^n$, then $J(x) = x^\top x$, $k = 2$, $L_f J(x) = 2x^\top f(x)$, $\rho(t) = (\rho^m(0) + m(y_0^\top f(y_0))t)^{1/m}$, $f(y) - \frac{L_f J(y)}{k} g(y) = f(y) - (y^\top f(y))y$. If for example $f(x) = [2x_1x_2\ x_2^2 - x_1^2]^\top$, $m = -1$, then the non-zero solutions of

$$0 = f(y) - \big(y^\top f(y)\big)y = \begin{bmatrix} 2y_1y_2 - y_1^3y_2 - y_1y_2^3 \\ y_2^2 - y_1^2 - y_1^2y_2^2 - y_2^4 \end{bmatrix}$$

are $y_0 = [0\ \pm 1]^\top$. Taking into account that $y_0^\top f(y_0) = \pm 1$, these yield the characteristic solutions $\rho(t) = \frac{1}{\frac{1}{\rho(0)} \mp t}$: the first solution, from $x(0) = [0\ x_{2,0}]^\top$ with $x_{2,0} > 0$, has a finite escape time at $t = \rho(0)$, whereas the second solution, from $x(0) = [0\ x_{2,0}]^\top$ with $x_{2,0} < 0$, asymptotically goes to 0 as $t \to +\infty$, because $\rho(t)$ tends to zero and $\rho = \|x\|^2$.

Remark 3.15 For a linear system $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$, it is easy to see that the eigensolutions $x(t) = \mathrm{e}^{\lambda t} v$, with $x(0) = v$ being an eigenvector of A relative to the eigenvalue λ, are characteristic solutions of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$ corresponding to $g(x) = x$ and $J(x) = x^\top x$ (with $m = 0$ and $k = 2$).

3.5 Reduction of Continuous-Time Nonlinear Systems

The knowledge of a symmetry allows to reduce by one the dimension of the state vector of a continuous-time nonlinear system [67]; this section shows how this can be achieved.

Let $f(x), g(x) \in \mathbb{R}^n$ be such that $[f, g] = 0$. Then, by the analysis of Sect. 1.6, about any regular point of the distribution $\mathrm{span}_{\mathscr{K}_n}\{f, g\}$, there exist n functionally independent functions $I_1, I_2, \ldots, I_n$ such that

$$\begin{cases} L_f I_1 = 1, \\ L_f I_i = 0, \quad i \neq 1, \end{cases} \quad \text{and} \quad \begin{cases} L_g I_2 = 1, \\ L_g I_i = 0, \quad i \neq 2. \end{cases} \tag{3.30}$$

As a matter of fact, by the Frobenius Theorem 1.9 at p. 21, there exists a diffeomorphism $y = \varphi(x)$ such that $\varphi_* f = e_1$ and $\varphi_* g = e_2$; then, letting $I_i = \varphi_i$, $i = 1, 2$, (3.30) hold. Set $\mathscr{I}_C(g)$ is constituted by all J that are arbitrary functions of $I_1, I_3, \ldots, I_n$ (see Remark 1.3 at p. 10). Let $J_1, \ldots, J_{n-1}$ be functionally independent elements of $\mathscr{I}_C(g)$; then, $J_i = C_i(I_1, I_3, \ldots, I_n)$, where the C_i's are functionally independent functions of $I_1, I_3, \ldots, I_n$. Hence,

$$L_f J_i = \frac{\partial C_i}{\partial I_1} L_f I_1 + \sum_{k=3}^{n} \frac{\partial C_i}{\partial I_k} L_f I_k = \frac{\partial C_i}{\partial I_1}$$

is an arbitrary function of $I_1, I_3, \ldots, I_n$. Therefore, by the *projection* $\mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\xi_i = C_i(I_1(x), I_3(x), \ldots, I_n(x)) = \breve{C}_i(x)$, $i = 1, \ldots, n-1$, one can write a nonlinear system of reduced dimension $n-1$. The reduced system does not describe wholly the original system, but it can be useful to study it. As an example, if the reduced system has an equilibrium, this corresponds to an invariant set along which the original dynamics are of order one. This analysis, illustrated in the following example, is generalized in Sect. 3.17.

Example 3.14 Let $f(x) = [x_1 \; 2x_2 + x_1^3 \; 4x_3 - x_1^3]^\top$; since such an f is homogeneous of degree 0 with respect to a dilation with weights $w_1 = 1$, $w_2 = 3$ and $w_3 = 3$, a simple symmetry of f is $g(x) = [x_1 \; 3x_2 \; 3x_3]^\top$. Two functionally independent first integrals associated with g are $J_1(x) = \frac{x_1^3}{x_2}$ and $J_2(x) = \frac{x_1^3}{x_3}$. Then, taking $\xi_1 = \frac{x_1^3}{x_2}$ and

$\xi_2 = \frac{x_1^3}{x_3}$ as state variables in the projected space, since

$$L_f\xi_1 = \begin{bmatrix} 3\frac{x_1^2}{x_2} & -\frac{x_1^3}{x_2^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 2x_2 + x_1^3 \\ 4x_3 - x_1^3 \end{bmatrix} = \frac{x_1^3}{x_2} - \frac{x_1^6}{x_2^2} = \xi_1 - \xi_1^2,$$

$$L_f\xi_2 = \begin{bmatrix} 3\frac{x_1^2}{x_3} & 0 & -\frac{x_1^3}{x_3^2} \end{bmatrix} \begin{bmatrix} x_1 \\ 2x_2 + x_1^3 \\ 4x_3 - x_1^3 \end{bmatrix} = -\frac{x_1^3}{x_3} + \frac{x_1^6}{x_3^2} = -\xi_2 + \xi_2^2,$$

one has the planar reduced system $\frac{\mathrm{d}\xi}{\mathrm{d}t} = \hat{f}_r(\xi)$, with $\hat{f}_r(\xi) = [\xi_1 - \xi_1^2 \;\; -\xi_2 + \xi_2^2]^\top$. Since $\xi_1 = 1$, $\xi_2 = 1$ is an equilibrium of $\frac{\mathrm{d}\xi}{\mathrm{d}t} = \hat{f}_r(\xi)$, the original system has the curve $\{x_2 = x_1^3, x_3 = x_1^3\}$ as invariant set, along which the dynamics are described by $\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1$. A symmetry $\hat{g}_r$ of $\hat{f}_r$ is $\hat{g}_r(\xi) = [\xi_1 - \xi_1^2 \;\; 0]^\top$; a first integral associated with g_r is $J_3(\xi) = \xi_2$. Clearly, taking $\eta = \xi_2$ as state variable in the projected space, one obtains the scalar reduced system $\frac{\mathrm{d}\eta}{\mathrm{d}t} = \tilde{f}_r(\eta)$, with $\tilde{f}_r(\eta) = -\eta + \eta^2$.

Remark 3.16 If g is an orbital symmetry of f, a reduced system is found by taking, as state variables in the projected space, $n - 1$ functionally independent first integrals associated with g, but an additional change in the independent variable t may be necessary, as shown in the following example (see Statement (3.13.2) of Theorem 3.13).

Example 3.15 Let $f(x) = [-x_1^3 + x_2 \;\; -x_1^2x_2 + x_3 \;\; -x_1^2x_3]^\top$. Since f is homogeneous of degree $m = -2$, with respect to the dilation with weights $w_1 = 1$, $w_2 = 3$ and $w_3 = 5$, a simple orbital symmetry of f is $g(x) = [x_1 \; 3x_2 \; 5x_3]^\top$. Two functionally independent first integrals associated with g are $J_1(x) = \frac{x_1^3}{x_2}$ and $J_2(x) = \frac{x_1^5}{x_3}$. Taking $\xi_1 = \frac{x_1^3}{x_2}$ and $\xi_2 = \frac{x_1^5}{x_3}$ as state variables in the projected space, since

$$L_f\xi_1 = x_1^2\left(3 - 2\frac{x_1^3}{x_2} - \frac{x_1x_3}{x_2^2}\right) = x_1^2\left(-2\xi_1 + 3 - \frac{\xi_1^2}{\xi_2}\right),$$

$$L_f\xi_2 = x_1^2\left(-4\frac{x_1^5}{x_3} + 5\frac{x_1^2x_2}{x_3}\right) = x_1^2\left(-4\xi_2 + 5\frac{\xi_2}{\xi_1}\right),$$

one has the planar reduced system $\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = f_r(\xi)$, where $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{x_1^2}$ and

$$f_r(\xi) = \begin{bmatrix} -2\xi_1 + 3 - \frac{\xi_1^2}{\xi_2} \\ -4\xi_2 + 5\frac{\xi_2}{\xi_1} \end{bmatrix}.$$

Since $\xi_1 = \frac{5}{4}$, $\xi_2 = \frac{25}{8}$ is an equilibrium of $\frac{\mathrm{d}\xi}{\mathrm{d}t} = \hat{f}_r(\xi)$, the original system has the curve $\{x_2 = \frac{4}{5}x_1^3, x_3 = \frac{8}{25}x_1^3\}$ as invariant set, along which the dynamics are simply described by $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{5}x_1^3$.

3.6 Continuous-Time Nonlinear Planar Systems

This section collects several results relating symmetries and semi-invariants for planar systems (i.e., $x \in \mathbb{R}^2$). This section is based on some derivations in [117, 118] and extends some results in [88, 96]). Some of the results described here are extended to the general case $n > 2$ in the remainder of the book.

Theorem 3.18 *Let*

$$\Omega = [f \ g], \qquad \omega = \det(\Omega).$$

Under the assumption that $\omega \neq 0$, *the following equation holds*:

$$[f, g] = \left(\operatorname{div}(g) - \frac{1}{\omega} L_g\omega\right) f + \left(-\operatorname{div}(f) + \frac{1}{\omega} L_f\omega\right) g. \tag{3.31}$$

Proof Since Ω is invertible over the field of meromorphic functions, any meromorphic vector function, whence also $[f, g]$, can be expressed as a linear combination of f and g, with the coefficients being meromorphic functions of x; in particular, one finds that

$$[f, g] = [f \ g] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},$$

where, by the *Cramer rules*, $\alpha_1 = \frac{\omega_1}{\omega}$ and $\alpha_2 = \frac{\omega_2}{\omega}$, with $\omega_1 = \det([[f, g] \ g])$ and $\omega_2 = \det([f \ [f, g]])$. It is easy to see that

$$\begin{aligned} \omega_1 &= \det\bigl([L_f g - L_g f \ g]\bigr) = \det\bigl([L_f g \ g]\bigr) - \det\bigl([L_g f \ g]\bigr), \\ \omega_2 &= \det\bigl([f \ L_f g - L_g f]\bigr) = \det\bigl([f \ L_f g]\bigr) - \det\bigl([f \ L_g f]\bigr), \end{aligned}$$

and that

$$\begin{aligned} L_g\omega &= L_g \det\bigl([f \ g]\bigr) = \det\bigl([L_g f \ g]\bigr) + \det\bigl([f \ L_g g]\bigr), \\ L_f\omega &= L_f \det\bigl([f \ g]\bigr) = \det\bigl([L_f f \ g]\bigr) + \det\bigl([f \ L_f g]\bigr). \end{aligned}$$

Then,

$$\begin{aligned} \omega_1 + L_g\omega &= \det\bigl([L_f g \ g]\bigr) + \det\bigl([f \ L_g g]\bigr) = \det\left(\begin{bmatrix} \frac{\partial g}{\partial x} f & g \end{bmatrix}\right) + \det\left(\begin{bmatrix} f & \frac{\partial g}{\partial x} g \end{bmatrix}\right) \\ &= \operatorname{trace}\left(\frac{\partial g}{\partial x}\right) \det\bigl([f \ g]\bigr) = \operatorname{div}(g)\omega, \end{aligned}$$

which implies $\alpha_1 = \frac{\omega_1}{\omega} = \mathrm{div}(g) - \frac{1}{\omega} L_g \omega$, and

$$\begin{aligned}\omega_2 - L_f\omega &= -\det\big([f\ L_g f]\big) - \det\big([L_f f\ g]\big)\\ &= -\det\left(\begin{bmatrix} f & \frac{\partial f}{\partial x} g\end{bmatrix}\right) - \det\left(\begin{bmatrix} \frac{\partial f}{\partial x} f & g\end{bmatrix}\right) = -\,\mathrm{trace}\left(\frac{\partial f}{\partial x}\right)\det\big([f\ g]\big)\\ &= -\,\mathrm{div}(f)\omega,\end{aligned}$$

which implies $\alpha_2 = \frac{\omega_2}{\omega} = -\,\mathrm{div}(f) + \frac{1}{\omega} L_f \omega$. □

By the analysis of Sect. 1.4, a one-form

$$[\beta_1\ \beta_2], \tag{3.32}$$

with $\beta_1(x), \beta_2(x) \in \mathbb{R}$, is *exact* if $\frac{\partial \beta_1}{\partial x_2} = \frac{\partial \beta_2}{\partial x_1}$. A function ω is an *inverse integrating factor* of the one-form (3.32) if the one-form $\frac{1}{\omega}[\beta_1\ \beta_2]$ is exact.

Let $f = [f_1\ f_2]^\top$ and let ω be the inverse integrating factor (which certainly exists by the analysis of Example 1.10) of the one-form $[f_2\ -f_1]$, namely assume that one-form

$$\frac{1}{\omega}[f_2\ -f_1] \tag{3.33}$$

is exact. Then, there exists a first integral I of system (1.1a) such that

$$\frac{\partial I}{\partial x_1} = \frac{f_2}{\omega}, \qquad \frac{\partial I}{\partial x_2} = -\frac{f_1}{\omega}; \tag{3.34}$$

note that, when ω is known, such a first integral can be computed by integration,

$$I(x_1, x_2) = \int \frac{f_2(x_1, x_2)}{\omega(x_1, x_2)}\,\mathrm{d}x_1 + C(x_2), \tag{3.35a}$$

$$\frac{\mathrm{d}C(x_2)}{\mathrm{d}x_2} = -\frac{f_1(x_1, x_2)}{\omega(x_1, x_2)} - \int \frac{\partial}{\partial x_2}\left(\frac{f_2(x_1, x_2)}{\omega(x_1, x_2)}\right)\mathrm{d}x_1. \tag{3.35b}$$

It should be noted that, despite its apparent form, the right-hand side of (3.35b) does not depend on x_1, as one can easily check by differentiating such a quantity with respect to x_1, on the basis of the subsequent relation (3.36a),

$$-\frac{\partial}{\partial x_1}\left(\frac{f_1(x_1, x_2)}{\omega(x_1, x_2)}\right) - \frac{\partial}{\partial x_2}\left(\frac{f_2(x_1, x_2)}{\omega(x_1, x_2)}\right) = -\,\mathrm{div}\left(\frac{1}{\omega} f\right) = 0,$$

and therefore a single integration of (3.35b) gives $C(x_2)$.

Definition 3.9 A function $\omega(x) \in \mathbb{R}$, $\omega \neq 0$, is an *inverse integrating factor* of system (1.1a) (briefly, associated with f) if the one-form (3.33) is exact.

Theorem 3.19 *A function $\omega \neq 0$ is an inverse integrating factor of system* (1.1a) *if and only if one of the following two equivalent conditions holds*:

$$\operatorname{div}\left(\frac{1}{\omega} f\right) = 0, \tag{3.36a}$$

$$\operatorname{div}(f) = \frac{1}{\omega} L_f \omega. \tag{3.36b}$$

Proof As for the proof of (3.36a), if (3.33) is exact, then $\frac{\partial(\frac{1}{\omega} f_2)}{\partial x_2} = \frac{\partial(-\frac{1}{\omega} f_1)}{\partial x_1}$, which implies $\frac{\partial(\frac{1}{\omega} f_1)}{\partial x_1} + \frac{\partial(\frac{1}{\omega} f_2)}{\partial x_2} = 0$ and, therefore, $\operatorname{div}(\frac{1}{\omega} f) = 0$. Taking into account (3.36a) and the following relations

$$\begin{aligned}
\operatorname{div}\left(\frac{1}{\omega} f\right) &= \frac{\partial}{\partial x_1}\left(\frac{1}{\omega} f_1\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\omega} f_2\right) \\
&= -\frac{1}{\omega^2}\frac{\partial \omega}{\partial x_1} f_1 + \frac{1}{\omega}\frac{\partial f_1}{\partial x_1} - \frac{1}{\omega^2}\frac{\partial \omega}{\partial x_2} f_2 + \frac{1}{\omega}\frac{\partial f_2}{\partial x_2} = \frac{1}{\omega}\operatorname{div}(f) - \frac{1}{\omega^2}\frac{\partial \omega}{\partial x} f \\
&= \frac{1}{\omega}\operatorname{div}(f) - \frac{1}{\omega^2} L_f \omega = \frac{1}{\omega}\left(\operatorname{div}(f) - \frac{1}{\omega} L_f \omega\right),
\end{aligned}$$

relation (3.36b) is proven. □

Theorem 3.20 *If g is an orbital symmetry of f, i.e., $[f, g] = \mu f$, and*

$$\omega = \det\bigl([f \ \ g]\bigr) \tag{3.37}$$

is not identically equal to zero, then ω is an inverse integrating factor of system (1.1a), *namely* $\operatorname{div}(\frac{1}{\omega} f) = 0$. *If* $\operatorname{div}(\frac{1}{\omega} f) = 0$ *for some $\omega \neq 0$, then any g such that* (3.37) *holds is an orbital symmetry of f.*

Proof If $[f, g] = \mu f$ and $\omega \neq 0$, then from (3.31):

$$\operatorname{div}(f) = \frac{1}{\omega} L_f \omega, \tag{3.38}$$

and therefore, by (3.36b), ω is an inverse integrating factor of system (1.1a). Conversely, if ω is an inverse integrating factor of system (1.1a), then (3.38) holds, and therefore, by (3.31), $[f, g] = \mu f$ holds with $\mu = \operatorname{div}(g) - \frac{1}{\omega} L_g \omega$. □

Theorem 3.21 *If ω and $\hat{\omega}$ are two inverse integrating factors of system* (1.1a), *then $I = \frac{\omega}{\hat{\omega}}$ is a (possibly, trivial) first integral of system* (1.1a).

Proof If ω and $\hat{\omega}$ are two inverse integrating factors, then

$$L_f \omega = \omega \operatorname{div}(f), \qquad L_f \hat{\omega} = \hat{\omega} \operatorname{div}(f).$$

Then, $L_f I = \frac{\hat{\omega} L_f \omega - \omega L_f \hat{\omega}}{\hat{\omega}^2} = \frac{\hat{\omega}\omega \operatorname{div}(f) - \omega\hat{\omega} \operatorname{div}(f)}{\hat{\omega}^2} = 0$. □

As a consequence of the above theorem, if ω is an inverse integrating factor of system (1.1a) and I is any (non-trivial) first integral of system (1.1a), all inverse integrating factors $\hat{\omega}$ of system (1.1a) are parameterized by:

$$\hat{\omega} = \omega C(I),$$

where C is an arbitrary function, $C \neq 0$.

If ω is an inverse integrating factor and I is a first integral of system (1.1a), then all orbital symmetries $g = [g_1 \ g_2]^\top$ of f are given by the solutions g_1, g_2 of

$$\omega C(I) = f_1 g_2 - f_2 g_1,$$

which exist provided that $f \neq 0$; for instance, if $f_1 \neq 0$, then

$$g = \begin{bmatrix} g_1 & \frac{\omega C(I) + f_2 g_1}{f_1} \end{bmatrix}^\top \tag{3.39}$$

parameterizes all (non-trivial) orbital symmetries of f, with g_1 being an arbitrary function (a similar expression can be found if $f_2 \neq 0$); an orbital symmetry g of f is *trivial* if $\det([f \ g]) = 0$ (e.g., f is a trivial orbital symmetry of f). The orbital symmetry g resulting from (3.39) by letting $C(I) = 0$ is trivial, because the resulting $g = \frac{g_1}{f_1} f$ is co-linear with f over $\mathscr{K}_n$.

Remark 3.17 If ω is an inverse integrating factor of system (1.1a) and $\mathrm{div}(f) \neq 0$, then an orbital symmetry g of f can be computed by

$$g = \frac{1}{\mathrm{div}(f)} \begin{bmatrix} -\frac{\partial \omega}{\partial x_2} \\ \frac{\partial \omega}{\partial x_1} \end{bmatrix}; \tag{3.40}$$

as a matter of fact,

$$\begin{aligned} \det\big([f \ g]\big) &= \det\left(\begin{bmatrix} f_1 & -\frac{1}{\mathrm{div}(f)} \frac{\partial \omega}{\partial x_2} \\ f_2 & \frac{1}{\mathrm{div}(f)} \frac{\partial \omega}{\partial x_1} \end{bmatrix}\right) = \frac{1}{\mathrm{div}(f)} \frac{\partial \omega}{\partial x_1} f_1 + \frac{1}{\mathrm{div}(f)} \frac{\partial \omega}{\partial x_2} f_2 \\ &= \frac{1}{\mathrm{div}(f)} L_f \omega; \end{aligned}$$

since ω is an inverse integrating factor, one finds that $\mathrm{div}(f) = \frac{1}{\omega} L_f \omega$, which implies that $\det([f \ g]) = \omega$. Note that, thanks to the choice made in (3.40), ω is a first integral associated with g, $L_g \omega = 0$; hence, one can think as ω and g to be associated to each other (see [93]).

Theorem 3.22 *Let g be an orbital symmetry of f, i.e., $[f, g] = \mu f$, and let $\omega = \det([f \ g]) \neq 0$; then, $L_f \omega = \mathrm{div}(f)\omega$. Thus, if there are no zero/pole cancelations between ω and* $\mathrm{div}(f)$, *then ω is a semi-invariant associated with f, with characteristic function* $\mathrm{div}(f)$; *if f and g are polynomial, then ω is a Darboux polynomial associated with f, as well as its irreducible factors.*

Proof From (3.31), one concludes that $-\operatorname{div}(f) + \frac{1}{\omega} L_f \omega = 0$. □

When ω is not polynomial, all factors of ω as meromorphic function are candidates to be semi-invariants.

Remark 3.18 Some classes of nonlinear planar systems having a simple orbital symmetry, whence such that an inverse integrating factor can be easily computed, are pointed out. Specific examples for these classes are given later.

(3.18.1) f is homogeneous of degree m, but non necessarily analytic at $x = 0$, with respect a dilation $\delta_\varepsilon^w x$, with weights w_1 and w_2; $g(x) = [w_1 x_1 \ w_2 x_2]^\top$ is an orbital symmetry of f and the corresponding inverse integrating factor is

$$\omega(x) = \det\left(\begin{bmatrix} f_1(x) & w_1 x_1 \\ f_2(x) & w_2 x_2 \end{bmatrix}\right) = w_2 x_2 f_1(x) - w_1 x_1 f_2(x).$$

(3.18.2) f has the form $f(x) = \mu_0 E x + \mu_1 A x$, with $\mu_0, \mu_1 \in \mathscr{I}_C(Ax)$, for some $A \in \mathbb{R}^{2\times 2}$; $g(x) = Ax$ is a symmetry of f and the corresponding inverse integrating factor is

$$\omega(x) = \det\bigl([\mu_0 E x + \mu_1 A x \ \ A x]\bigr) = \det\bigl([\mu_0 E x \ \ A x]\bigr).$$

(3.18.3) f is *area-preserving* (i.e., $\operatorname{div}(f) = 0$); in this case, $\omega = 1$ is an inverse integrating factor of f and an orbital symmetry of f is $g = \frac{1}{f_1^2 + f_2^2}[-f_2 \ f_1]^\top$, if $f_1^2 + f_2^2 \neq 0$.

(3.18.4) f satisfies the conditions $\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}$ and $\frac{\partial f_1}{\partial x_2} = -\frac{\partial f_2}{\partial x_1}$; if $f_1^2 + f_2^2 \neq 0$, then $g = [-f_2 \ f_1]^\top$ is an orbital symmetry of f and the corresponding inverse integrating factor is

$$\omega = \det\left(\begin{bmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{bmatrix}\right) = f_1^2 + f_2^2.$$

(3.18.5) $f(x) = [a(x_1)b(x_2) \ c(x_1)d(x_2)]^\top$, with $a, b, d \neq 0$ (the variables are *separable*); an orbital symmetry of f is $g = [0 \ \frac{d}{b}]^\top$, with the corresponding inverse integrating factor

$$\omega(x) = \det\left(\begin{bmatrix} a(x_1)b(x_2) & 0 \\ c(x_1)d(x_2) & \frac{d(x_2)}{b(x_2)} \end{bmatrix}\right) = a(x_1)d(x_2).$$

As a matter of fact, the one-form $\frac{1}{\omega}[f_2 \ -f_1] = [\frac{c(x_1)}{a(x_1)} \ \frac{b(x_2)}{d(x_2)}]$ is exact.

(3.18.6) f admits a known first integral I. Since $[\frac{\partial I}{\partial x_1} \ \frac{\partial I}{\partial x_2}] = \frac{1}{\omega}[f_2 \ -f_1]$, let either $\omega = \frac{f_2}{\frac{\partial I}{\partial x_1}}$ if $f_2 \neq 0$ (i.e., if $\frac{\partial I}{\partial x_1} \neq 0$) or $\omega = -\frac{f_1}{\frac{\partial I}{\partial x_2}}$ if $f_1 \neq 0$ (i.e., if $\frac{\partial I}{\partial x_2} \neq 0$); the respective orbital symmetries are either $g = [\frac{\omega}{f_2} \ 0]^\top$ if $f_2 \neq 0$ or $g = [0 \ -\frac{\omega}{f_1}]^\top$ if $f_1 \neq 0$.

Remark 3.19 If $\omega = \det([f \;\; g]) \neq 0$ and f is homogeneous of degree m with respect to g, then by (3.31), in addition to $L_f\omega = \mathrm{div}(f)\omega$, one finds that $\mathrm{div}(g) - \frac{1}{\omega}L_g\omega = m$, i.e., $L_g\omega = (\mathrm{div}(g) - m)\omega$. In particular, if $g(x) = B_w x$, with $B_w = \mathrm{diag}\{w_1, w_2\}$, then ω is homogeneous of degree $w_1 + w_2 - m$; furthermore, if the degree of f_1 is $w_1 - m$, then the degree of $\frac{\partial f_1}{\partial x_1}$ is $w_1 - m - w_1 = -m$, as well as the degree of $\frac{\partial f_2}{\partial x_2}$, which shows that $\mathrm{div}(f)$ is homogeneous of degree $-m$ with respect to $g(x) = B_w x$.

Example 3.16 Consider $f(x) = [x_2 - x_1^3 \; -x_1^2 x_2]^\top$. Since f is homogeneous of degree -2 with respect to the integer dilation $\delta_\varepsilon^w x$, with $w = [1\;3]^\top$, $g(x) = [x_1 \; 3x_2]^\top$ is an orbital symmetry of f. Then,

$$\omega(x) = \det\left(\begin{bmatrix} x_2 - x_1^3 & x_1 \\ -x_1^2 x_2 & 3x_2 \end{bmatrix}\right) = x_2\big(3x_2 - 2x_1^3\big).$$

The factors $\omega_1(x) = x_2$ and $\omega_2(x) = 3x_2 - 2x_1^3$ of ω are Darboux polynomials associated with f, with respective characteristic polynomials $\lambda_1(x) = -x_1^2$ and $\lambda_2(x) = -3x_1^2$.

Example 3.17 Take $g(x) = Bx$, with $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$; then, $\mathscr{L}_c(B) = \mathrm{span}_{\mathbb{R}}\{B, E\}$, and set $\mathscr{I}_C(Bx)$ is constituted by all functions $J = C(x_1^2 + x_2^2)$ of $x_1^2 + x_2^2$. Then, any element of $\mathscr{C}_C(Bx)$ can be expressed as $f(x) = \mu_0 Bx + \mu_1 Ex = \left[\begin{smallmatrix} \mu_1 x_1 + \mu_0 x_2 \\ -\mu_0 x_1 + \mu_1 x_2 \end{smallmatrix}\right]$, with $\mu_0, \mu_1 \in \mathscr{I}_C(Bx)$. An inverse integrating factor associated with f is then given by

$$\omega(x) = \det\big([f(x) \; g(x)]\big) = \det\left(\begin{bmatrix} \mu_1 x_1 + \mu_0 x_2 & x_2 \\ -\mu_0 x_1 + \mu_1 x_2 & -x_1 \end{bmatrix}\right) = -\big(x_1^2 + x_2^2\big)\mu_1,$$

which shows that $\omega_1(x) = x_1^2 + x_2^2$ is a semi-invariant associated with f, for all $\mu_0, \mu_1 \in \mathscr{I}_C(Bx)$, $\mu_1 \neq 0$, with characteristic function $\lambda = 2\mu_1$, provided that there is no zero/pole cancelation between μ_1 and $x_1^2 + x_2^2$; it is observed that μ_1 is a function of ω_1. In particular, the choice $\mu_1 = 1 - \omega_1$ leads to $L_g\omega_1 = 2(1 - \omega_1)\omega_1$; from this, $\omega_1 = 0$ and $\omega_1 = 1$ are two algebraic invariant curves for any μ_0: $\omega_1 = 0$ is unstable and $\omega_1 = 1$ is asymptotically stable. It is worth pointing out that this is true for any choice of μ_0, even not differentiable at $x = 0$. A well-known system (e.g., see (3.719) of [20]) is found by taking $\mu_1 = 1 - \omega_1 = 1 - (x_1^2 + x_2^2)$ and $\mu_0 = \sqrt{\omega_1} = \sqrt{x_1^2 + x_2^2}$:

$$f(x) = \begin{bmatrix} x_1(1 - (x_1^2 + x_2^2)) + x_2\sqrt{x_1^2 + x_2^2} \\ -x_1\sqrt{x_1^2 + x_2^2} + x_2(1 - (x_1^2 + x_2^2)) \end{bmatrix}.$$

3.7 Parameterization of Continuous-Time Nonlinear Planar Systems Having a Given Orbital Symmetry

In this section, a sort of partial classification of planar systems is reported, by studying systems that have an orbital symmetry with a given structure [90]. For such systems, using the results in Sect. 3.6, it is easy to find semi-invariants.

Consider two vector functions $f(x), g(x) \in \mathbb{R}^2$ and let $x^o \in \mathbb{R}^2$ be a regular point of g, i.e., $g(x^o) \neq 0$; by the flow box Theorem 3.3, in a neighborhood of x^o, there exists a diffeomorphism $y = J(x)$, $J = [J_0\ J_1]^\top$, such that the vector function g is straightened in such coordinates: $J_* g = (\frac{\partial J}{\partial x} g) \circ J^{-1} = e_1$, with e_1 being the first column of the 2×2 identity matrix.

By Theorems 3.9 and 3.17, the set of all $f^{[m]}$ such that $[f^{[m]}, g] = m f^{[m]}$, with $m \in \mathbb{Z}$, is parameterized by

$$f^{[m]} = \left(\frac{\partial J}{\partial x}\right)^{-1} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} \mathrm{e}^{-mJ_0}, \tag{3.41}$$

where C_1, C_2 are arbitrary scalar functions of J_1, whereas the set of all f such that $[f, g] = \mu f$, for some scalar function μ, is parameterized by

$$f = \left(\frac{\partial J}{\partial x}\right)^{-1} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} C_0(J_0, J_1), \tag{3.42}$$

where C_0, C_1, C_2 are arbitrary scalar functions of the arguments.

Case 1: orbital symmetry $g = [P(x_1) + Q(x_2)\ \ 0]^\top$ Assume that there exists a point $x^o \in \mathscr{U}$ such that $g(x^o) \neq 0$. Then, there exists an open and connected subset $\mathscr{U}^* \subset \mathscr{U}$ such that $P(x_1) + Q(x_2) \neq 0$ in $\mathscr{U}^*$. A diffeomorphism $y = J(x)$ straightening the vector function g is given by:

$$J_0 = \int \frac{1}{P(x_1) + Q(x_2)} \mathrm{d}x_1, \qquad J_1 = x_2.$$

In particular, taking into account that

$$\left(\frac{\partial J}{\partial x}\right)^{-1} = \begin{bmatrix} P+Q & (P+Q)\frac{\partial Q}{\partial x_2} \int \frac{1}{(P+Q)^2} \mathrm{d}x_1 \\ 0 & 1 \end{bmatrix},$$

all vector functions $f^{[m]}$ being homogeneous of order m with respect to g are parameterized by:

$$f^{[m]} = \begin{bmatrix} P+Q & (P+Q)\frac{\partial Q}{\partial x_2} \int \frac{1}{(P+Q)^2} \mathrm{d}x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \mathrm{e}^{-m \int \frac{1}{P+Q} \mathrm{d}x_1},$$

whereas the set of all f having g as orbital symmetry is parameterized by

$$f = \begin{bmatrix} P+Q & (P+Q)\frac{\partial Q}{\partial x_2} \int \frac{1}{(P+Q)^2} \mathrm{d}x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} C_0,$$

where C_1 and C_2 are arbitrary functions of x_2 and C_0 of x_1, x_2. As for the system described by the above f, to compute the semi-invariants, let

$$\omega = \det\big([f \; g]\big) = -(P+Q)C_2C_0,$$

which shows that any system belonging to this class has $\omega_1 = P(x_1) + Q(x_2)$, $\omega_2 = C_2(x_2)$ and $\omega_3 = C_0(x_1, x_2)$ as candidates to be semi-invariants, as well as their possible factors. Such functions are actually semi-invariants if there are not zero-pole cancelations among them and $\operatorname{div}(f)$.

Example 3.18 Consider the case $P(x_1) = ax_1$ and $Q(x_2) = bx_2$. All vector functions $f^{[m]}$ being homogeneous of order m with respect to g are parameterized by

$$f^{[m]}(x) = \begin{bmatrix} ax_1 + bx_2 & -\frac{b}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1(x_2) \\ C_2(x_2) \end{bmatrix} (ax_1 + bx_2)^{-\frac{m}{a}},$$

where C_1 and C_2 are arbitrary functions of x_2; in particular, for $m = 0$, one has

$$f^{[0]}(x) = \begin{bmatrix} (ax_1 + bx_2)C_1(x_2) - \frac{b}{a}C_2(x_2) \\ C_2(x_2) \end{bmatrix}. \tag{3.43}$$

All vector functions f having g as orbital symmetry are parameterized by

$$f(x) = \begin{bmatrix} ax_1 + bx_2 & -\frac{b}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1(x_2) \\ C_2(x_2) \end{bmatrix} C_0(x_1, x_2),$$

where C_0 is an arbitrary function of x_1, x_2. Then,

$$\omega(x) = -(ax_1 + bx_2)C_0C_2,$$

which shows that any system belonging to the considered class has $\omega_1(x) = ax_1 + bx_2$, $\omega_2(x) = C_2(x_2)$ and $\omega_3(x) = C_0(x_1, x_2)$ as candidates to be semi-invariants, as well as their possible factors.

Remark 3.20 The proposed technique allows to parameterize all vector functions f having g as orbital symmetry, around one of its regular points. Often, it can be used also to find such vector functions f defined at one of the singular points x^s of g. As a matter of fact, being g analytic, if g is not identically equal to zero, in any neighborhood of x^s there are regular points x^o of g where the technique can be applied. Some of the vector functions f computed about x^o can be well defined also at x^s. This happens, e.g., in Example 3.18 when $m = 0$: the vector functions $f^{[0]}$ given in (3.43) are analytic at $x = 0$ if C_1 and C_2 are chosen as analytic at $x = 0$.

Case 2: orbital symmetry $g = [P(x_1) \; Q(x_1)]^\top$ Assume that both functions $P(x_1)$ and $Q(x_1)$ are not identically equal to zero. This implies that there exists a point x^o such that $g(x^o) \neq 0$. In a neighborhood of x^o, a diffeomorphism $y = J(x)$ straightening g is

$$J_0(x) = \int \frac{1}{P(x_1)}\,\mathrm{d}x_1, \qquad J_1(x) = x_2 - \int \frac{Q(x_1)}{P(x_1)}\,\mathrm{d}x_1.$$

In particular, taking into account that

$$\left(\frac{\partial J(x)}{\partial x}\right)^{-1} = \begin{bmatrix} P(x_1) & 0 \\ Q(x_1) & 1 \end{bmatrix},$$

all vector functions $f^{[m]}$ being homogeneous of order m with respect to g are parameterized by:

$$f^{[m]} = \begin{bmatrix} P(x_1) & 0 \\ Q(x_1) & 1 \end{bmatrix} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} \mathrm{e}^{-m \int \frac{1}{P(x_1)}\,\mathrm{d}x_1},$$

where $C_1(J_1)$ and $C_2(J_1)$ are arbitrary functions of J_1, whereas the set of all vector functions f having g as orbital symmetry is parameterized by

$$f = \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} C_0 = \begin{bmatrix} PC_0C_1 \\ QC_0C_1 + C_0C_2 \end{bmatrix},$$

where C_0 is an arbitrary function of x_1, x_2. Then,

$$\omega = \det\left(\begin{bmatrix} PC_0C_1 & P \\ QC_0C_1 + C_0C_2 & Q \end{bmatrix}\right) = -PC_0C_2,$$

which shows that any system in this class has $\omega_1(x) = C_2(x_2 - \int \frac{Q(x_1)}{P(x_1)}\,\mathrm{d}x_1)$, $\omega_2(x) = P(x_1)$ and $\omega_3(x) = C_0(x_1, x_2)$ as candidates to be semi-invariants, as well as their possible factors.

Case 3: orbital symmetry $g = [P(x_1)\;\; Q(x_2)]^\top$ Assume that both functions $P(x_1)$ and $Q(x_2)$ are not identically equal to zero. This implies that there exists a point x^o such that $g(x^o) \neq 0$. In a neighborhood of x^o, a diffeomorphism $y = J(x)$ straightening g is

$$J_0(x) = \int \frac{1}{P(x_1)}\,\mathrm{d}x_1, \qquad J_1(x) = \int \frac{1}{P(x_1)}\,\mathrm{d}x_1 - \int \frac{1}{Q(x_2)}\,\mathrm{d}x_2.$$

In particular, taking into account that

$$\left(\frac{\partial J(x)}{\partial x}\right)^{-1} = \begin{bmatrix} P(x_1) & 0 \\ Q(x_2) & -Q(x_2) \end{bmatrix},$$

all vector functions $f^{[m]}$ being homogeneous of order m with respect to g are parameterized by:

$$f^{[m]} = \begin{bmatrix} P(x_1) & 0 \\ Q(x_2) & -Q(x_2) \end{bmatrix} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} \mathrm{e}^{-mJ_0},$$

whereas the set of all f having g as orbital symmetry is parameterized by

$$f = \begin{bmatrix} P(x_1) & 0 \\ Q(x_2) & -Q(x_2) \end{bmatrix} \begin{bmatrix} C_1(J_1) \\ C_2(J_1) \end{bmatrix} C_0(x_1, x_2) = \begin{bmatrix} PC_0C_1 \\ QC_0(C_1 - C_2) \end{bmatrix},$$

where C_0, C_1 and C_2 are arbitrary functions of the arguments. Then,

$$\omega = \det\left(\begin{bmatrix} PC_0C_1 & P \\ QC_0(C_1 - C_2) & Q \end{bmatrix} \right) = PQC_0C_2,$$

which shows that any system belonging to this class has $\omega_1(x) = P(x_1)$, $\omega_2(x) = Q(x_2)$, $\omega_3(x) = C_2(J_1)$ and $\omega_4(x) = C_0(x_1, x_2)$ as candidates to be semi-invariants, as well as their possible factors.

Case 4: orbital symmetry $g = [\frac{Q(x_2)}{P(x_1)}\ 0]^\top$ Assume that both functions $P(x_1)$ and $Q(x_2)$ are not identically equal to zero (note that such a g includes the case of a linear, non-zero and nilpotent $g(x) = [x_2\ 0]^\top$). A diffeomorphism $y = J(x)$ straightening g is

$$J_0(x) = \frac{1}{Q(x_2)} \int P(x_1)\,\mathrm{d}x_1, \qquad J_1(x) = x_2.$$

In particular, taking into account that

$$\left(\frac{\partial J(x)}{\partial x}\right)^{-1} = \begin{bmatrix} \frac{Q(x_2)}{P(x_1)} & \frac{1}{Q(x_2)P(x_1)} \frac{\partial Q_2(x_2)}{\partial x_2} \int P(x_1)\,\mathrm{d}x_1 \\ 0 & 1 \end{bmatrix},$$

all vector functions $f^{[m]}$ being homogeneous of order m with respect to g are parameterized by:

$$f^{[m]} = \begin{bmatrix} \frac{Q}{P} & \frac{1}{QP} \frac{\partial Q_2}{\partial x_2} \int P\,\mathrm{d}x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \mathrm{e}^{-mJ_0},$$

whereas the set of all f having g as orbital symmetry is parameterized by

$$f = \begin{bmatrix} \frac{Q}{P} & \frac{1}{QP} \frac{\partial Q_2}{\partial x_2} \int P\,\mathrm{d}x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} C_0 = \begin{bmatrix} \frac{Q}{P} C_1 C_0 + \frac{1}{QP} \frac{\partial Q_2}{\partial x_2} \int P\,\mathrm{d}x_1 C_2 C_0 \\ C_2 C_0 \end{bmatrix},$$

where C_1 and C_2 are arbitrary functions of x_2 and C_0 of x_1, x_2.

Remark 3.21 One of the motivations for the study carried out above is that it can be useful for creating systems that have a desired semi-invariant, or a desired invariant set.

3.8 The Inverse Jacobi Last Multiplier

Now, the analysis of Sect. 3.6 is extended to the general case $f(x) \in \mathbb{R}^n$. In particular, next theorem generalizes Theorem 3.20.

Theorem 3.23 *Let $g_1(x), \ldots, g_{n-1}(x) \in \mathbb{R}^n$ be $n-1$ orbital symmetries of f, $[f, g_i] = \mu_i f$. Let*

$$\Omega = \det\big([f \; g_1 \; g_2 \; \ldots \; g_{n-1}]\big), \qquad \omega = \det(\Omega),$$

and assume that ω is not identically equal to zero. Then,

$$L_f \omega = \operatorname{div}(f)\omega. \tag{3.44}$$

Thus, if there is no zero/pole cancelation between ω and $\operatorname{div}(f)$, then ω is a semi-invariant associated with f, with characteristic function $\operatorname{div}(f)$; if f and g are polynomial, then ω is a Darboux polynomial associated with f, as well as its irreducible factors.

Proof First, it is noted that

$$L_f \omega = \det\big([L_f f \; g_1 \; g_2 \; \ldots \; g_{n-1}]\big) + \sum_{i=1}^{n-1} \det\big([f \; g_1 \; \ldots \; L_f g_i \; \ldots \; g_{n-1}]\big).$$

From $[f, g_i] = \mu_i f$ (i.e., from $L_f g_i - L_{g_i} f = \mu_i f$), it follows that $L_f g_i = L_{g_i} f + \mu_i f$, from which

$$L_f \omega = \det\big([L_f f \; g_1 \; g_2 \; \ldots \; g_{n-1}]\big) + \sum_{i=1}^{n-1} \det\big([f \; g_1 \; \ldots \; L_{g_i} f \; \ldots \; g_{n-1}]\big)$$
$$+ \sum_{i=1}^{n-1} \det\big([f \; g_1 \; \ldots \; \mu_i f \; \ldots \; g_{n-1}]\big).$$

In this way, taking into account that $\det([f \; g_1 \; \ldots \; \mu_i f \; \ldots \; g_{n-1}]) = 0$ for any i, one concludes that

$$L_f \omega = \det\big([L_f f \; g_1 \; g_2 \; \ldots \; g_{n-1}]\big) + \sum_{i=1}^{n-1} \det\big([f \; g_1 \; \ldots \; L_{g_i} f \; \ldots \; g_{n-1}]\big)$$
$$= \det\left(\begin{bmatrix} \frac{\partial f}{\partial x} f & g_1 & g_2 & \ldots & g_{n-1} \end{bmatrix}\right)$$
$$+ \sum_{i=1}^{n-1} \det\left(\begin{bmatrix} f & g_1 & \ldots & \frac{\partial f}{\partial x} g_i & \ldots & g_{n-1} \end{bmatrix}\right)$$

$$= \operatorname{trace}\left(\frac{\partial f}{\partial x}\right)\det\bigl([f\ g_1\ g_2\ \dots\ g_{n-1}]\bigr) = \operatorname{div}(f)\omega,$$

as to be shown. □

Note that the vector functions $g_1, g_2, \dots, g_{n-1}$ need not be commuting, i.e., condition $[g_i, g_j] = 0$ is not required in Theorem 3.23. When ω is not polynomial, all factors of ω as meromorphic function are candidates to be semi-invariants.

The following definition extends the concept of the inverse integrating factor to the concept of the inverse Jacobi last multiplier.

Definition 3.10 A function $\omega(x) \in \mathbb{R}$, $\omega \neq 0$, is an *inverse Jacobi last multiplier* of system (1.1a) (briefly, associated with $f(x)$) if $\operatorname{div}(\frac{1}{\omega} f) = 0$.

An inverse Jacobi last multiplier is an inverse integrating factor when $n = 2$.

Remark 3.22 Since $\operatorname{div}(\frac{1}{\omega} f) = \frac{1}{\omega}\operatorname{div}(f) - \frac{1}{\omega^2}\frac{\partial \omega}{\partial x} f = \frac{1}{\omega}\operatorname{div}(f) - \frac{1}{\omega^2} L_f \omega$, from relation $L_f\omega = \operatorname{div}(f)\omega$, under the assumptions and notation of Theorem 3.23, one finds that $\operatorname{div}(\frac{1}{\omega} f) = 0$ if $\omega = \det(\Omega)$; therefore, if different from zero, $\det(\Omega)$ is an inverse Jacobi last multiplier associated with f. Let ω_1 and ω_2 be two inverse Jacobi last multipliers associated with f; then, $I = \frac{\omega_1}{\omega_2}$ is a first integral of system (1.1a), since:

$$L_f I = \frac{\omega_2 L_f \omega_1 - \omega_1 L_f \omega_2}{\omega_2^2} = \frac{-\omega_2\omega_1 \operatorname{div}(f) + \omega_1\omega_2 \operatorname{div}(f)}{\omega_2^2} = 0.$$

The following theorem shows how an inverse Jacobi last multiplier is modified by a state diffeomorphism.

Theorem 3.24 *Let $y = \varphi(x)$ be a diffeomorphism. Under the assumptions and notations of Theorem 3.23, let $\tilde{\Omega} = [\varphi_* f\ \varphi_* g_1\ \dots\ \varphi_* g_{n-1}]$ and $\tilde{\omega} = \det(\tilde{\Omega})$. Then, $\tilde{\omega} \circ \varphi = \det(\frac{\partial \varphi}{\partial x})\omega$, i.e., $\tilde{\omega} = \varphi_*(\det(\frac{\partial \varphi}{\partial x})\omega)$.*

Proof The proof is based on the following computations:

$$\begin{aligned}
\tilde{\omega} &= \det\bigl([\varphi_* f\ \varphi_* g_1\ \dots\ \varphi_* g_{n-1}]\bigr) \\
&= \det\left(\left[\left(\frac{\partial \varphi}{\partial x} f\right)\circ\varphi^{-1}\ \left(\frac{\partial \varphi}{\partial x} g_1\right)\circ\varphi^{-1}\ \dots\ \left(\frac{\partial \varphi}{\partial x} g_{n-1}\right)\circ\varphi^{-1}\right]\right) \\
&= \det\left(\frac{\partial \varphi}{\partial x}\right)\det\bigl([f\ g_1\ \dots\ g_{n-1}]\bigr)\circ\varphi^{-1}.
\end{aligned}$$

□

The following theorem shows how a first integral associated with f can be computed from an inverse Jacobi last multiplier (see [56]).

Theorem 3.25 *Let $I_1, I_2, \ldots, I_{n-2}$ be $n-2$ functionally independent first integrals of system* (1.1a). *Then, the knowledge of an inverse Jacobi last multiplier allows the computation of another first integral I_{n-1} of system* (1.1a), *being functionally independent of $I_1, I_2, \ldots, I_{n-2}$.*

Proof The equations $y_1 = I_1, \ldots, y_{n-2} = I_{n-2}$ constitute a partial diffeomorphism that can be completed by defining two extra variables $y_{n-1} := J_{n-1}(x)$ and $y_n := J_n(x)$, so that

$$y = \begin{bmatrix} I_1(x) \\ \vdots \\ I_{n-2}(x) \\ J_{n-1}(x) \\ J_n(x) \end{bmatrix} =: \varphi(x),$$

qualifies as a diffeomorphism in an open and connected domain; apart from a reordering of the state variables, it is always possible to assume that $y_{n-1} = x_{n-1}$ and $y_n = x_n$. In these new coordinates, the push-forward of f takes the form

$$\tilde{f}(y) = \varphi_* f(y) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{f}_{n-1}(y_1, \ldots, y_{n-2}, y_{n-1}, y_n) \\ \tilde{f}_n(y_1, \ldots, y_{n-2}, y_{n-1}, y_n) \end{bmatrix},$$

for some functions $\tilde{f}_{n-1}$ and $\tilde{f}_n$. By Theorem 3.24, if ω is an inverse Jacobi last multiplier associated with f, then $\tilde{\omega} = \varphi_*(\omega \det(\frac{\partial \varphi}{\partial x}))$ is an inverse Jacobi last multiplier associated with $\tilde{f}$. Since, $y_1, \ldots, y_{n-2}$ are constants, $y_1 = c_1, \ldots, y_{n-2} = c_{n-2}$, one concludes that $\tilde{\omega}_0 = \tilde{\omega}(c_1, \ldots, c_{n-2}, y_{n-1}, y_n)$ is an inverse Jacobi last multiplier of the following planar system (therefore, it is an inverse integrating factor):

$$\begin{aligned} \frac{\mathrm{d}y_{n-1}}{\mathrm{d}t} &= \tilde{f}_{n-1}(c_1, \ldots, c_{n-2}, y_{n-1}, y_n), \\ \frac{\mathrm{d}y_n}{\mathrm{d}t} &= \tilde{f}_n(c_1, \ldots, c_{n-2}, y_{n-1}, y_n); \end{aligned}$$

hence, the one-form $\frac{1}{\tilde{\omega}_0}[\tilde{f}_n \quad -\tilde{f}_{n-1}]$ is exact and its first integral $\tilde{I}_{n-1}(y_{n-1}, y_n)$ can be computed by integration since $\tilde{\omega}_0$ is known; moreover, since $\tilde{I}_{n-1}$ does not depend on $y_1, \ldots, y_{n-2}$, its pull-back $I_{n-1} = \varphi^* \tilde{I}_{n-1}$ is a first integral of the original system being functionally independent of the other first integrals $I_1, \ldots, I_{n-2}$. □

Example 3.19 Consider $f(x) = [x_2 - ax_1^3 \;\; x_3 - ax_1^2 x_2 \;\; -ax_1^2 x_3]^\top$. It is easy to see that $\omega_1(x) = x_3$ and $\omega_2(x) = -2x_1x_3 + x_2^2$ are two functionally independent Darboux polynomials associated with f, with respective characteristic polynomials

$\lambda_1(x) = -ax_1^2$ and $\lambda_2(x) = -2ax_1^2$. Since $\lambda_2 = 2\lambda_1$, $I_1(x) = \frac{\omega_2(x)}{\omega_1^2(x)} = \frac{-2x_1x_3+x_2^2}{x_3^2}$ is a first integral of system (1.1a); moreover, by the reasoning at the beginning of Remark 3.22, since $\operatorname{div}(f(x)) = -5ax_1^2 = 5\lambda_1(x)$, $\omega(x) = \omega_1^5(x) = x_3^5$ is an inverse Jacobi last multiplier associated with f. Let $y_1 = I_1(x)$, $y_2 = x_2$ and $y_3 = x_3$, so that

$$\varphi^{-1}(y) = \begin{bmatrix} \frac{1}{2}\frac{y_2^2 - y_1 y_3^2}{y_3} \\ y_2 \\ y_3 \end{bmatrix}.$$

Clearly,

$$\tilde{\omega}_0(y_2, y_3) = \tilde{\omega}(c_1, y_2, y_3) = \left(\omega(x)\frac{\partial I_1(x)}{\partial x_1}\right)\bigg|_{x=\varphi^{-1}(y)} = -2y_3^4$$

is an inverse integrating factor of the reduced system

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = \left(y_3 - ax_1^2 y_2\right)\Big|_{x_1 = -\frac{1}{2}\frac{c_1 y_3^2 - y_2^2}{y_3}} = y_3 - \frac{1}{4}a\frac{(c_1 y_3^2 - y_2^2)^2}{y_3^2}y_2,$$

$$\frac{\mathrm{d}y_3}{\mathrm{d}t} = \left(-ax_1^2 y_3\right)\Big|_{x_1 = -\frac{1}{2}\frac{c_1 y_3^2 - y_2^2}{y_3}} = -\frac{1}{4}a\frac{(c_1 y_3^2 - y_2^2)^2}{y_3},$$

whence the one-form (in the variables y_2, y_3)

$$\left[\frac{a}{8y_3^5}\left(c_1 y_3^2 - y_2^2\right)^2 \quad \frac{1}{2y_3^3} - \frac{a}{8y_3^6}\left(c_1 y_3^2 - y_2^2\right)^2 y_2\right]$$

is exact, and its first integral is

$$\tilde{I}_2(y) = \frac{ay_2}{120}\frac{3y_2^4 - 10c_1 y_2^2 y_3^2 + 15c_1^2 y_3^4}{y_3^5} - \frac{1}{4y_3^2}.$$

The new first integral $I_2(x)$ for the original system is computed as follows:

$$\begin{aligned} I_2(x) &= \tilde{I}_2(y)|_{c_1 = I_1(x), y_2 = x_2, y_3 = x_3} \\ &= \frac{1}{60}\frac{4ax_2^5 - 20ax_1x_2^3x_3 + 30ax_1^2x_2x_3^2 - 15x_3^3}{x_3^5}. \end{aligned}$$

In particular, this shows that $\omega_3(x) = 4ax_2^5 - 20ax_1x_2^3x_3 + 30ax_1^2x_2x_3^2 - 15x_3^3$ is another Darboux polynomial of the considered system: $L_f\omega_3(x) = -5ax_1^2\omega_3(x)$.

Example 3.20 Consider $f(x) = [x_2 \; x_3 - bx_1^5 \; -bx_1^4x_2]^\top$. It is easy to see that

$$g_1(x) = \begin{bmatrix} x_1 \\ 3x_2 \\ 5x_3 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} 0 \\ \frac{1}{x_2} \\ 0 \end{bmatrix}$$

are two orbital symmetries of f. The determinant ω of matrix

$$\Omega(x) = \begin{bmatrix} f(x) & g_1(x) & g_2(x) \end{bmatrix} = \begin{bmatrix} x_2 & x_1 & 0 \\ x_3 - bx_1^5 & 3x_2 & \frac{1}{x_2} \\ -bx_1^4 x_2 & 5x_3 & 0 \end{bmatrix}$$

is $\omega(x) = \det(\Omega(x)) = -5x_3 - bx_1^5$. This is clearly a Darboux polynomial associated with f; actually, since $\operatorname{div}(f) = 0$, one concludes that $I_1 = \omega$ is a first integral of system (1.1a). Let $y_1 = I_1(x)$, $y_2 = x_1$ and $y_3 = x_2$, so that

$$\varphi^{-1}(y) = \begin{bmatrix} y_2 \\ y_3 \\ -\frac{1}{5}y_1 - \frac{b}{5}y_2^3 \end{bmatrix}.$$

Clearly,

$$\tilde{\omega}_0(y_2, y_3) = \tilde{\omega}(c_1, y_2, y_3) = \left(\omega(x) \det\left(\frac{\partial \varphi(x)}{\partial x}\right)\right)\bigg|_{x=\varphi^{-1}(c_1, y_2, y_3)} = -5c_1,$$

is an inverse integrating factor of the reduced system

$$\begin{aligned} \frac{\mathrm{d}y_2}{\mathrm{d}t} &= x_2|_{x=\varphi^{-1}(c_1,y_2,y_3)} = y_3, \\ \frac{\mathrm{d}y_3}{\mathrm{d}t} &= \left(x_3 - bx_1^5\right)\big|_{x=\varphi^{-1}(c_1,y_2,y_3)} = -\frac{c_1}{5} - \frac{6}{5}by_2^5. \end{aligned}$$

Since $\tilde{\omega}_0$ is constant, it is not necessary to use it; as a matter of fact, the one-form

$$\begin{bmatrix} -\frac{c_1}{5} - \frac{6}{5}by_2^5 & -y_3 \end{bmatrix}$$

is exact, and its first integral is

$$\tilde{I}_2(y) = -\frac{b}{5}y_2^6 - \frac{1}{2}y_3^2 - \frac{c_1}{5}y_2.$$

The new first integral $I_2(x)$ for the original system is computed as follows:

$$I_2(x) = \tilde{I}_2(y)|_{c_1=I_1(x), y_2=x_1, y_3=x_2} = x_1x_3 - \frac{1}{2}x_2^2.$$

3.9 Matrix Integrating Factors

Definition 3.11 A skew-symmetric matrix function $\Sigma(x) \in \mathbb{R}^{n\times n}$, $\Sigma^\top = -\Sigma$, for which there exists $I(x) \in \mathbb{R}$ such that

$$\frac{\partial I}{\partial x} = f^\top \Sigma \tag{3.45}$$

is called a *matrix integrating factor* associated with f.

As shown in the subsequent Sect. 5.5, given a non-trivial first integral I associated with f, there exists a skew-symmetric matrix function $S(x) \in \mathbb{R}^{n \times n}$ such that

$$f^\top = -\frac{\partial I}{\partial x} S.$$

If $\det(S(x)) \neq 0$, then the one-form

$$\frac{\partial I}{\partial x} = -f^\top S^{-1} \tag{3.46}$$

is exact, whence $\Sigma = -S^{-1}$ is a matrix integrating factor.

Example 3.21 If $n = 2$, then any f can be written as

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial I}{\partial x_1} \\ \frac{\partial I}{\partial x_2} \end{bmatrix}, \tag{3.47}$$

where ω is an inverse integrating factor associated with f and I is the corresponding first integral. From (3.47), if $\omega \neq 0$, one obtains the one-form (3.33),

$$\begin{bmatrix} \frac{\partial I}{\partial x_1} & \frac{\partial I}{\partial x_2} \end{bmatrix} = [f_1 \; f_2] \begin{bmatrix} 0 & -\frac{1}{\omega} \\ \frac{1}{\omega} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} f_2 & -\frac{1}{\omega} f_1 \end{bmatrix}.$$

If a skew-symmetric matrix has odd dimension, then it has necessarily zero determinant, whence (3.46) does not hold. Nevertheless, matrix integrating factors can exist also in this case, as shown in the following example.

Example 3.22 Consider the continuous-time system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -3x_2x_3,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 3x_1x_3,$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = -x_1x_2.$$

Multiplying the first equation by x_1, the second by $2x_2$ and the third by $3x_3$, and summing the results, one obtains $x_1 \frac{\mathrm{d}x_1}{\mathrm{d}t} + 2x_2 \frac{\mathrm{d}x_2}{\mathrm{d}t} + 3x_3 \frac{\mathrm{d}x_3}{\mathrm{d}t} = 0$, which can be easily integrated, thus obtaining the first integral $I_1 = \frac{1}{2}x_1^2 + x_2^2 + \frac{3}{2}x_3^2$. Similarly, multiplying the first equation by x_1 and the second equation by x_2, and summing the results, one obtains $x_1 \frac{\mathrm{d}x_1}{\mathrm{d}t} + x_2 \frac{\mathrm{d}x_2}{\mathrm{d}t} = 0$, which can be easily integrated, thus obtaining the first integral $I_2 = \frac{1}{2}(x_1^2 + x_2^2)$. In particular, for each first integral, the corresponding matrix integrating factor can be found by requiring that (3.45) holds:

$$[x_1\ 2x_2\ 3x_3] = [-3x_2x_3\ 3x_1x_3\ -x_1x_2]\begin{bmatrix} 0 & -\frac{2}{3x_3} & -\frac{1}{x_2} \\ \frac{2}{3x_3} & 0 & 0 \\ \frac{1}{x_2} & 0 & 0 \end{bmatrix},$$

$$[x_1\ x_2\ 0] = [-3x_2x_3\ 3x_1x_3\ -x_1x_2]\begin{bmatrix} 0 & -\frac{1}{3x_3} & 0 \\ \frac{1}{3x_3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.10 Lax Pairs for Continuous-Time Nonlinear Systems

Lax pairs are a powerful tool for the computation of first integrals of continuous time nonlinear systems; in view of the connection with first integrals, many physical systems admit a Lax pair representation (see, for instance, [49]). Most of the material in this section is adapted from [56], but the extension in Remark 3.25, which generalizes Lax pairs to connect them to semi-invariants, is from [94]. The notation in this section is somewhat different from the one in the rest of the book, e.g., matrices A and B are not constant here.

Let a vector function $f(x) \in \mathbb{R}^n$ be given. Given a matrix function $A(x) \in \mathbb{R}^{\nu \times \nu}$, with entries $A_{i,j}(x)$, define the symbol $L_f A$ as the matrix function having $L_f A_{i,j}$ as entries.

Definition 3.12 Given a vector function $f(x) \in \mathbb{R}^n$, a *CT-Lax pair* (briefly, a Lax pair if no confusion can arise) associated with $f(x)$ is an ordered pair of matrix functions (A, B), with $A(x), B(x) \in \mathbb{R}^{\nu \times \nu}$, $\nu^2 \geq n$, such that

$$L_f A = [A, B], \tag{3.48}$$

where $[A, B]$ is the Lie bracket of matrices A, B, $[A, B] = BA - AB$.

Theorem 3.26 *Let (A, B) be a Lax pair associated with a given f. Then, for any $k \in \mathbb{Z}^{\geq}$, $I = \text{trace}(A^k)$ is a first integral associated with f.*

Proof It is worth pointing out that $L_f\,\text{trace}(A) = \text{trace}(L_f A)$, $\text{trace}(AB) = \text{trace}(BA)$ and that $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$. Hence,

$$\begin{aligned} L_f\,\text{trace}\big(A^k\big) &= \text{trace}\big((L_f A)A^{k-1} + A(L_f A)A^{k-2} + A^2(L_f A)A^{k-3} + \cdots \\ &\quad + A^{k-1}(L_f A)\big) = \text{trace}\big(BA^k - ABA^{k-1} + ABA^{k-1} - A^2BA^{k-2} \\ &\quad + A^2BA^{k-2} - A^3BA^{k-3} + \cdots + A^{k-1}BA - A^{k-1}AB\big) \\ &= \text{trace}\big(BA^k - A^kB\big) = \text{trace}\big(BA^k\big) - \text{trace}\big(A^kB\big) = 0, \end{aligned}$$

as to be proven. □

Since $\mathrm{trace}(A_1A_2) = \mathrm{trace}(A_2A_1)$, for any pair $A_1, A_2 \in \mathbb{R}^{\nu\times\nu}$, if $A = C\Lambda C^{-1}$, then $\mathrm{trace}(A) = \mathrm{trace}(C\Lambda C^{-1}) = \mathrm{trace}(\Lambda C^{-1}C) = \mathrm{trace}(\Lambda)$. Denoting by Λ the Jordan form of A, this implies that $\mathrm{trace}(A^k) = \mathrm{trace}(\Lambda^k) = \sum_{i=1}^{\nu}\lambda_i^k$, where λ_i is eigenvalue of A. Since the functions $\alpha_k(\lambda_1,\dots,\lambda_\nu) = \sum_{i=1}^{\nu}\lambda_i^k$, $k = 1,\dots,\nu$, are functionally independent as functions of $\lambda_1,\dots,\lambda_\nu$, the eigenvalues of A, as well as the coefficients of the characteristic polynomial of A, as well as $\det(A) = \prod_i \lambda_i$, are first integrals associated with f. This, in particular, shows that at most ν of $n-1$ functionally independent first integrals associated with f can be computed from the knowledge of A.

Remark 3.23 For given $A(x), B(x) \in \mathbb{R}^{\nu\times\nu}$ and an unknown $f(x) \in \mathbb{R}^n$, (3.48) is a set of ν^2 algebraic equations in the n unknown entries of f. If such a system has a unique solution f, then (A, B) is called a *regular Lax pair* associated with the vector function f thus identified. For instance, take $\nu = 2$ and $n = 3$,

$$A(x) = \begin{bmatrix} x_1 & x_2 \\ 1 & x_3 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} 1+x_3 & x_1^2 \\ \frac{1+x_3-x_2}{x_1-x_3} & x_2 \end{bmatrix};$$

then,

$$L_f A(x) = \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix},$$

$$\big[A(x), B(x)\big] = \begin{bmatrix} \frac{-x_2-x_2x_3-x_1^2x_3+x_1^3+x_2^2}{x_1-x_3} & x_2+x_2x_3+x_1^2x_3-x_1^3-x_2^2 \\ 0 & -\frac{-x_2-x_2x_3-x_1^2x_3+x_1^3+x_2^2}{x_1-x_3} \end{bmatrix},$$

from which (A, B) is a regular Lax pair associated with

$$f(x) = \begin{bmatrix} \frac{-x_2-x_2x_3-x_1^2x_3+x_1^3+x_2^2}{x_1-x_3} \\ x_2+x_2x_3+x_1^2x_3-x_1^3-x_2^2 \\ \frac{x_2+x_2x_3+x_1^2x_3-x_1^3-x_2^2}{x_1-x_3} \end{bmatrix}.$$

Hence, $I_1(x) = \mathrm{trace}(A(x)) = x_1 + x_3$ and $I_2(x) = \mathrm{trace}(A^2(x)) = x_1^2 + 2x_2 + x_3^2$ are two functionally independent first integrals associated with f. Clearly, the coefficients of the characteristic polynomial of A and the determinant of A are first integrals associated with f, and they can be written as functions of I_1 and I_2:

$$p_A(\lambda) = \lambda^2 - (x_1+x_3)\lambda + x_1x_3 - x_2 = \lambda^2 - I_1\lambda + \frac{1}{2}I_1^2 - \frac{1}{2}I_2,$$

$$\det(A) = \frac{1}{2}I_1^2 - \frac{1}{2}I_2.$$

Theorem 3.27 *Let (A, B) be a Lax pair associated with a given f. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a polynomial scalar function of the argument. Then, $(\alpha(A), B)$ is a Lax pair associated with f.*

Proof First, it is shown how (A^k, B) is a Lax pair associated with f, for any $k \in \mathbb{Z}^{\geq}$,

$$\begin{aligned}
L_f A^k &= (L_f A)A^{k-1} + A(L_f A)A^{k-2} + A^2(L_f A)A^{k-3} + \cdots + A^{k-1}(L_f A) \\
&= (BA - AB)A^{k-1} + A(BA - AB)A^{k-2} + A^2(BA - AB)A^{k-3} + \cdots \\
&\quad + A^{k-1}(BA - AB) \\
&= BA^k - ABA^{k-1} + ABA^{k-1} - A^2BA^{k-2} + A^2BA^{k-2} - A^3BA^{k-3} + \cdots \\
&\quad + A^{k-1}BA - A^kB \\
&= BA^k - A^kB = \left[A^k, B\right].
\end{aligned}$$

Clearly, if (A, B) is a Lax pair associated with f, then (aA, B) is a Lax pair associated with f, for any constant $a \in \mathbb{R}$. Finally, if (A_1, B) and (A_2, B) are two Lax pairs associated with f, then $(A_1 + A_2, B)$ is a Lax pair associated with f,

$$[A_1 + A_2, B] = [A_1, B] + [A_2, B] = L_f A_1 + L_f A_2 = L_f(A_1 + A_2). \qquad \square$$

Theorem 3.28 *Let (A, B_1) be a Lax pair associated with a given f. Then, (A, B_2) is a Lax pair associated with f if and only if $[A, B_1 - B_2] = 0$.*

Proof If (A, B_1) and (A, B_2) are two Lax pairs associated with f, then

$$L_f A = [A, B_1], \qquad L_f A = [A, B_2]$$
$$\downarrow$$
$$[A, B_1] = [A, B_2].$$

Vice versa, if $L_f A = [A, B_1]$ and $[A, B_1] = [A, B_2]$, then $L_f A = [A, B_2]$. $\square$

Example 3.23 Let $f(x) = [x_1(x_2 - x_3)\ x_2(x_3 - x_1)\ x_3(x_1 - x_2)]^\top$. Take

$$A(x) = \begin{bmatrix} 0 & 1 & x_1 \\ x_2 & 0 & 1 \\ 1 & x_3 & 0 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} x_1 + x_2 & 0 & 1 \\ 1 & x_2 + x_3 & 0 \\ 0 & 1 & x_1 + x_3 \end{bmatrix}.$$

Since both $L_f A(x)$ and $[A(x), B(x)]$ are equal to the matrix

$$\begin{bmatrix} 0 & 0 & x_1(x_2 - x_3) \\ x_2(x_3 - x_1) & 0 & 0 \\ 0 & x_3(x_1 - x_2) & 0 \end{bmatrix},$$

one concludes that (A, B) is a Lax pair associated with f. Now, since

$$\operatorname{trace}\big(A(x)\big) = \operatorname{trace}\left(\begin{bmatrix} 0 & 1 & x_1 \\ x_2 & 0 & 1 \\ 1 & x_3 & 0 \end{bmatrix}\right) = 0,$$

$$\mathrm{trace}\big(A^2(x)\big) = \mathrm{trace}\left(\begin{bmatrix} x_1+x_2 & x_1x_3 & 1 \\ 1 & x_2+x_3 & x_1x_2 \\ x_2x_3 & 1 & x_1+x_3 \end{bmatrix}\right) = 2(x_1+x_2+x_3),$$

$$\begin{aligned}\mathrm{trace}\big(A^3(x)\big) &= \mathrm{trace}\left(\begin{bmatrix} 1+x_1x_2x_3 & x_1+x_2+x_3 & x_1^2+x_1x_2+x_1x_3 \\ x_2^2+x_2x_3+x_1x_2 & 1+x_1x_2x_3 & x_1+x_2+x_3 \\ x_1+x_2+x_3 & x_2x_3+x_1x_3+x_3^2 & 1+x_1x_2x_3 \end{bmatrix}\right) \\ &= 3(1+x_1x_2x_3),\end{aligned}$$

$I_1(x) = x_1 + x_2 + x_3$ and $I_2(x) = 1 + x_1x_2x_3$ are two first integrals associated with f. Clearly, the coefficients of the characteristic polynomial of A and the determinant of A are first integrals associated with f,

$$\begin{aligned} p_A(\lambda) &= \lambda^3 - (x_1+x_2+x_3)\lambda - (1+x_1x_2x_3) = \lambda^3 - I_1\lambda - I_2, \\ \det\big(A(x)\big) &= 1 + x_1x_2x_3 = I_2(x). \end{aligned}$$

Theorem 3.29 *Let (A, B) be a Lax pair associated with a given f. Then, for any matrix $M(x) \in \mathbb{R}^{\nu\times\nu}$ invertible over the field of meromorphic functions, pair $(\tilde{A}, \tilde{B})$, with*

$$\tilde{A} = MAM^{-1}, \qquad \tilde{B} = (L_f M)M^{-1} + MBM^{-1}, \tag{3.49}$$

is a Lax pair associated with f.

Proof Taking into account that $L_f M = (\tilde{B} - MBM^{-1})M$, one concludes that (see relation (3.13))

$$\begin{aligned} L_f\tilde{A} &= (L_f M)AM^{-1} + M(L_f A)M^{-1} - MAM^{-1}(L_f M)M^{-1} \\ &= \big(\tilde{B} - MBM^{-1}\big)MAM^{-1} + M(BA - AB)M^{-1} \\ &\quad - MAM^{-1}\big(\tilde{B} - MBM^{-1}\big) \\ &= \tilde{B}\tilde{A} - \tilde{A}\tilde{B} - MBAM^{-1} + MBAM^{-1} - MABM^{-1} + MABM^{-1} \\ &= \big[\tilde{A}, \tilde{B}\big]. \end{aligned}$$ □

Theorem 3.30 *Let $I_1, \dots, I_m$ be $m < n$ functionally independent first integrals associated with a given f. Let $M(x) \in \mathbb{R}^{n\times n}$ be invertible over the field of meromorphic functions. Then,*

$$A = M\varLambda M^{-1}, \qquad B = (L_f M)M^{-1}, \tag{3.50}$$

where $\varLambda = \mathrm{diag}\{I_1, \dots, I_m, c_{m+1}, \dots, c_n\}$ and the c_i's are arbitrary constants, is a Lax pair associated with f.

Proof The proof follows from Theorem 3.29, taking into account that $(\varLambda, 0)$ is a Lax pair associated with f, since $L_f\varLambda = \mathrm{diag}\{L_f I_1, \dots, L_f I_m, 0, \dots, 0\} = 0$. □

The Lax pair given in (3.50) is not regular, but a regular one can be obtained about any regular point, as shown in the following theorem.

Theorem 3.31 *About any regular point x^o of f, $f(x^o) \neq 0$, there exists a regular Lax pair $A(x), B(x) \in \mathbb{R}^{n\times n}$ associated with f.*

Proof About any regular point x^o, $f(x^o) \neq 0$, by the flow box Theorem 3.3, there exist n functionally independent functions $I_0, I_1, \ldots, I_{n-1}$ such that $L_f I_0 = 1$ and $L_f I_i = 0$, $i = 1, \ldots, n-1$. Define

$$A := \begin{bmatrix} I_1 & I_0 & \ldots & 0 & 0 \\ 0 & I_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

and $B := \operatorname{diag}\{I_0^{-1}, 0, \ldots, 0\}$. Since

$$[A, B] = \begin{bmatrix} I_1 I_0^{-1} & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} - \begin{bmatrix} I_1 I_0^{-1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = L_f A,$$

one concludes that (A, B) is a Lax pair associated with f. Since the equations $L_f I_0 = 1$ and $L_f I_i = 0$, $i = 1, \ldots, n-1$, uniquely define the vector function f about x^o, then pair (A, B) is regular. Any pair $(\tilde{A}, \tilde{B})$ obtained from (A, B) by (3.49) is a regular Lax pair associated with f. □

Example 3.24 Take $I_0(x) = x_1$, $I_2(x) = x_2 + x_1^2$. Equations $L_f I_0(x) = f_1 = 1$ and $L_f I_2(x) = 2x_1 f_1 + f_2 = 0$, define uniquely the vector function $f(x) = [1 \;\; -2x_1]^\top$, and

$$A(x) = \begin{bmatrix} x_2 + x_1^2 & x_1 \\ 0 & 0 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

is a regular pair associated with f.

From $A = M \Lambda M^{-1}$, letting $\hat{A}(t) = A(x(t))$ and $\hat{M}(t) = M(x(t))$, and taking into account that Λ is constant along any solution $x(t)$ of the system, $\Lambda(x(t)) =$

$\Lambda(x(0))$, one has

$$\hat{A}(t) = \hat{N}(t)\hat{A}(0)\hat{N}^{-1}(t),$$

where $\hat{N}(t) = \hat{M}(t)\hat{M}^{-1}(0)$.

Theorem 3.30 may be particularly helpful to generate polynomial systems having polynomial first integrals and a polynomial Lax pair, as shown in the following example.

Example 3.25 Take two functionally independent polynomials of $x \in \mathbb{R}^3$, $I_1 = x_1x_2$ and $I_2 = x_1^2 + x_3^2$. Take a simple polynomial matrix $M(x)$, with polynomial inverse,

$$M(x) = \begin{bmatrix} 1 & 0 & x_2 \\ x_1 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M^{-1}(x) = \begin{bmatrix} 1 & 0 & -x_2 \\ -x_1 & 1 & -x_3 + x_1x_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute $L_f M(x)$ with respect to a vector function f having arbitrary entries f_1, f_2 and f_3,

$$L_f M(x) = \begin{bmatrix} 0 & 0 & f_2 \\ f_1 & 0 & f_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let (any constant value is acceptable as the $(3, 3)$-entry of Λ)

$$\Lambda(x) = \begin{bmatrix} x_1x_2 & 0 & 0 \\ 0 & x_1^2 + x_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

then,

$$\begin{aligned} &A(x) \\ &\quad = M(x)\Lambda(x)M^{-1}(x) \\ &\quad = \begin{bmatrix} x_1x_2 & 0 & -x_1x_2^2 + x_2 \\ x_1^2x_2 - x_1^3 - x_1x_3^2 & x_1^2 + x_3^2 & -x_1^2x_2^2 - x_1^2x_3 + x_1^3x_2 - x_3^3 + x_1x_2x_3^2 + x_3 \\ 0 & 0 & 1 \end{bmatrix}, \\ &B(x) = \big(L_f M(x)\big)M^{-1}(x) = \begin{bmatrix} 0 & 0 & f_2 \\ f_1 & 0 & -f_1x_2 + f_3 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

By imposing the equality $L_f A = [A, B]$, one obtains the following algebraic system in the unknowns f_1, f_2, f_3:

$$\begin{aligned} 0 &= x_2 f_1 + x_1 f_2, \\ 0 &= -x_1(-x_2 + 2x_1) f_1 + x_1^2 f_2 - 2x_1x_3 f_3, \\ 0 &= 2x_1 f_1 + 2x_3 f_3, \end{aligned}$$

$$0 = -x_2^2 f_1 - x_1 x_2 f_2,$$
$$0 = x_1\big(-x_2^2 - 2x_3 + 2x_1x_2\big) f_1 - x_1^2 x_2 f_2 + 2x_3(-x_3 + x_1x_2) f_3,$$

with solution $f_1(x) = -\frac{x_3}{x_1} f_3(x)$, $f_2(x) = \frac{x_3}{x_1^2} x_2 f_3(x)$, where f_3 is an arbitrary function of x. Letting $f_3(x) = x_1^2$, one verifies that I_1 and I_2 are two functionally independent first integrals associated with $f(x) = [-x_3x_1 \;\; x_2x_3 \;\; x_1^2]^\top$ and that (A, B) is a Lax pair associated with f. Clearly, $\text{trace}(A) = 1 + I_1 + I_2$ and $\text{trace}(A^2) = 1 + I_1^2 + I_2^2$ are two functionally independent polynomial first integrals associated with the polynomial f.

Remark 3.24 Let $\mathfrak{X}$ be the set of all $g(x) \in \mathbb{R}^n$ with entries in $\mathscr{K}_n$. Two linear operators L_1 and L_2 from $\mathfrak{X}$ to $\mathfrak{X}$ are said to be *compatible* if $L_2L_1g = L_1L_2g$, for all $g \in \mathfrak{X}$. Given two matrix functions $A(x), B(x) \in \mathbb{R}^{n\times n}$ and a vector function $f(x) \in \mathbb{R}^n$, with entries in $\mathscr{K}_n$, consider the two linear operators $L_1 = A$ and $L_2 = L_f - B$ that associate to each $g \in \mathfrak{X}$ the vector functions $L_1g, L_2g \in \mathfrak{X}$ defined as $L_1g := Ag$ and $L_2g := L_f g - Bg$. Clearly, L_1 and L_2 are compatible if and only if (A, B) is a Lax pair associated with f. As a matter of fact,

$$L_2L_1g = L_f(Ag) - BAg = (L_fA)g + AL_fg - BAg,$$
$$L_1L_2g = AL_fg - ABg,$$

now, by the arbitrariness of g, the compatibility condition $L_2L_1g = L_1L_2g$ is satisfied if and only if $(L_fA) + AL_f - BA = AL_f - AB$, i.e., if and only if $L_fA = [A, B]$, as to be shown.

Remark 3.25 Once a Lax pair (A, B) of the vector function f has been identified, some of the first integrals associated with f can be computed, as well as (by possible factorization) some of the semi-invariants associated with f. The concept of the Lax pair can be generalized for the direct computation of semi-invariants. A *generalized Lax pair* associated with f is an ordered pair (A, B) of matrix functions $A(x), B(x) \in \mathbb{R}^{\nu\times\nu}$ such that $L_fA - [A, B]$ and A are co-linear over the field of meromorphic functions, i.e., such that

$$L_fA = \alpha A + [A, B],$$

for some meromorphic scalar function $\alpha(x) \in \mathbb{R}$. In such a case, for any $k \in \mathbb{Z}^{\geq}$, if $\text{trace}(A^k)$ and α have not zero/pole in common, then $\omega = \text{trace}(A^k)$ is a semi-invariant associated with f, with characteristic function $k\alpha$. As a matter of fact,

$$L_f\omega = \text{trace}\big(L_fA^k\big) = k\,\text{trace}\big(A^{k-1}L_fA\big) = k\,\text{trace}\big(\alpha A^k + A^{k-1}BA - A^kB\big)$$
$$= k\alpha\,\text{trace}\big(A^k\big) = k\alpha\omega.$$

If $M(x) \in \mathbb{R}^{\nu\times\nu}$ is invertible over the field of meromorphic functions and (A, B) is a generalized Lax pair associated with f, then the pair $(\tilde{A}, \tilde{B})$ given in (3.49) is a generalized Lax pair associated with f, for the same function α. Define the diagonal matrix $\Lambda := \text{diag}\{\omega_1, \dots, \omega_m, 0, \dots, 0\}$, with the ω_i's being semi-invariants associated with f, with the same characteristic function $\lambda_i = \alpha$. Clearly, $(\Lambda, 0)$ is a generalized Lax pair associated with f, $L_f \Lambda = \alpha\Lambda$. Therefore, $A = M\Lambda M^{-1}$ and $B = (L_f M)M^{-1}$ is a generalized Lax pair associated with f, for any matrix $M(x) \in \mathbb{R}^{n\times n}$ being invertible over the field of meromorphic functions.

Example 3.26 Consider the vector function

$$f(x) = \begin{bmatrix} x_1 + x_1^2 + x_2x_4 - x_2x_3 \\ x_2 + x_1x_3 + x_2^2 - x_1x_2 - x_1x_4 - x_1^2 \\ x_3 + x_2^2 - x_1^2 \\ -x_2x_4 + x_2x_3 + x_4 - x_2^2 \end{bmatrix}.$$

A generalized Lax pair associated with f is (A, B),with

$$A(x) = \begin{bmatrix} x_3 & x_1 \\ x_2 & x_4 + x_1 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix},$$

which satisfy $L_f A = A + [A, B]$. Then, $\omega_1(x) = \text{trace}(A(x)) = x_3 + x_4 + x_1$ and $\omega_2(x) = \text{trace}(A^2(x)) = x_3^2 + 2x_1x_2 + x_4^2 + 2x_1x_4 + x_1^2$ are two Darboux polynomials associated with f, with characteristic values $\lambda_1 = 1$ and $\lambda_2 = 2$.

3.11 A "Computational" Result for the Darboux Polynomials of Continuous-Time Nonlinear Systems

Aim of this section is to give an algorithm [53, 91] for the computation of the Darboux polynomials associated with a given f, which, for the sake of simplicity, is assumed to be polynomial; note that this algorithm can be adapted to cover the computation of semi-invariants associated with f, when f is not polynomial.

Assume that ω is a Darboux polynomial associated with f, with characteristic polynomial λ, i.e., $L_f\omega = \lambda\omega$. Assume, in addition, that ω is a linear combination with real and constant coefficients c_i of some functionally independent polynomials $p_1, p_2, \dots, p_k$, for some $k > 0$, $\omega = \sum_{i=1}^{k} c_i p_i$. Consider the square $k \times k$ matrix

$$\Gamma = \begin{bmatrix} p_1 & p_2 & \dots & p_k \\ L_f p_1 & L_f p_2 & \dots & L_f p_k \\ \vdots & \vdots & \vdots & \vdots \\ L_f^{k-1} p_1 & L_f^{k-1} p_2 & \dots & L_f^{k-1} p_k \end{bmatrix}, \tag{3.51}$$

where $L_f^1 p_j = L_f p_j$ and $L_f^{i+1} p_j = L_f L_f^i p_j$.

Theorem 3.32 [91] *Under the above positions, if* $\det(\Gamma) \neq 0$, *then* ω *is a factor of* $\det(\Gamma)$.

Proof Assume $\omega = \sum_{i=1}^{k} c_i p_i$, for $c_i \in \mathbb{R}$; with no loss of generality, apart from a reordering of polynomials p_i, assume that $c_k \neq 0$. First, note that if ω is a Darboux polynomial associated with f, with characteristic polynomial λ, i.e., $L_f\omega = \lambda\omega$, then for any $i \in \mathbb{Z}^{>}$, $L_f^i\omega = \lambda_i\omega$, for some polynomial λ_i, with $\lambda_1 = \lambda$. This fact can be proven by induction: the base step, for $i = 1$, is trivial, whereas the induction step is proven as follows:

$$L_f^{i+1}\omega = L_f\big(L_f^i\omega\big) = L_f(\lambda_i\omega) = \omega L_f\lambda_i + \lambda_i L_f\omega = (L_f\lambda_i + \lambda_i\lambda_1)\omega = \lambda_{i+1}\omega,$$

where $\lambda_{i+1} = L_f\lambda_i + \lambda_i\lambda_1$. Note that if λ is constant, then $\lambda_i = \lambda^i$, and if $\lambda = 0$, then $\lambda_i = 0$, $i \geq 1$. Since $\omega = \sum_{i=1}^{k} c_i p_i$, it follows that $L_f^j\omega = \sum_{i=1}^{k} c_i L_f^j p_i$, which implies that

$$\begin{aligned}
&\Gamma \cdot \begin{bmatrix} 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{k-1} \\ 0 & 0 & \dots & 0 & c_k \end{bmatrix} \\
&= \begin{bmatrix} p_1 & p_2 & \dots & \sum_{i=1}^{k} c_i p_i \\ L_f p_1 & L_f p_2 & \dots & \sum_{i=1}^{k} c_i L_f p_i \\ \vdots & \vdots & \vdots & \vdots \\ L_f^{k-1} p_1 & L_f^{k-1} p_2 & \dots & \sum_{i=1}^{k} c_i L_f^{k-1} p_i \end{bmatrix} \\
&= \begin{bmatrix} p_1 & p_2 & \dots & \omega \\ L_f p_1 & L_f p_2 & \dots & L_f\omega \\ \vdots & \vdots & \vdots & \vdots \\ L_f^{k-1} p_1 & L_f^{k-1} p_2 & \dots & L_f^{k-1}\omega \end{bmatrix} \\
&= \begin{bmatrix} p_1 & p_2 & \dots & \omega \\ L_f p_1 & L_f p_2 & \dots & \lambda_1\omega \\ \vdots & \vdots & \vdots & \vdots \\ L_f^{k-1} p_1 & L_f^{k-1} p_2 & \dots & \lambda_{k-1}\omega \end{bmatrix} \\
&=: \tilde{\Gamma},
\end{aligned}$$

whence $\det(\Gamma) = \frac{1}{c_k}\det(\tilde{\Gamma})$, from which the theorem follows. □

Remark 3.26 When $\det(\Gamma) \neq 0$, Theorem 3.32 guarantees that if a Darboux polynomial ω, associated with f, is a linear combination with constant coefficients of $p_1, \dots, p_k$, then ω is a factor of $\det(\Gamma)$. But in the application of the theorem, all factors of $\det(\Gamma)$ or of the determinants of its minors, not only those that are linear combinations of $p_1, \dots, p_k$, are good candidates to be Darboux polynomials associated with f, because Γ could be a minor of another matrix $\check{\Gamma}$ found with an enlarged choice of the polynomials $p_1, \ldots, p_{\check{k}}$.

Remark 3.27 When $\det(\Gamma) = 0$, Theorem 3.32 cannot be applied: in such a case, good candidates to be Darboux polynomials associated with f are the factors of the determinants of minors of Γ that are not zero. As a matter of fact, one typical reason for $\det(\Gamma)$ to be identically equal to zero is that two or more different linear combinations, with constant coefficients, of some polynomials $p_1, \ldots, p_k$ are Darboux polynomials associated with f, with the same characteristic polynomial.

Remark 3.28 In [11, 64, 106], it is shown that the first homogeneous approximation $f^{[m]}$ of f with respect to a given g ($g(0) = 0$ and g being analytic on a neighborhood of the origin) can be used, under some technical assumption, for the stability analysis of the origin, thus giving an extension of the Lyapunov Theorem of stability in the first approximation [60]. By simply extending the reasoning used in [56] when the linear part of g is semi-simple and has real eigenvalues (being all positive or all negative) and there are no resonant terms in g, one concludes that if ω is a Darboux polynomial associated with f, then the first approximation $\omega^{[m]}$ of ω with respect to g is a Darboux polynomial, homogeneous with respect to g, associated with $f^{[m]}$. In this way, if there are enough homogeneous Darboux polynomials $\omega_i^{[m]}$ associated with the homogeneous $f^{[m]}$ to construct a Lyapunov function V, with negative definite derivative for such a first approximation ($L_{f^{[m]}} V < 0$), then one has proven the asymptotic stability of the origin of $\frac{\mathrm{d}x}{\mathrm{d}t} = f^{[m]}$, which, under some technical conditions, implies the asymptotic stability of the origin of $\frac{\mathrm{d}x}{\mathrm{d}t} = f$.

Example 3.27 Consider $f(x) = [-x_1 \; x_1^2 - 4x_2 \; -6x_3 + 2x_1^3 + 3x_1x_2]^\top$. Such an f is homogeneous of degree 0 with respect to $g(x) = [x_1 \; 2x_2 \; 3x_3]^\top$, i.e., $[f, g] = 0$. Clearly, $\omega_1(x) = x_1$ is a Darboux polynomial associated with f, homogeneous of degree 1 with respect to g, with characteristic value $\lambda_1 = -1$, since

$$L_f \omega_1(x) = -x_1 = \lambda_1 \omega_1(x).$$

Consider the Darboux polynomials associated with f, being homogeneous of degree 2 with respect to g; these polynomials are generated by taking as basis the set of all monomials of degree 2, $p_1(x) = x_1^2$ and $p_2(x) = x_2$:

$$\Gamma(x) = \begin{bmatrix} x_1^2 & x_2 \\ -2x_1^2 & x_1^2 - 4x_2 \end{bmatrix}, \qquad \det\bigl(\Gamma(x)\bigr) = x_1^2\bigl(x_1^2 - 2x_2\bigr);$$

in particular, $\omega_2(x) = x_1^2 - 2x_2$ is a Darboux polynomial associated with f, homogeneous of degree 2 with respect to g, with characteristic value $\lambda_2 = -4$, since

$$L_f \omega_2(x) = -4\bigl(x_1^2 - 2x_2\bigr) = \lambda_2 \omega_2(x).$$

Finally, consider the Darboux polynomials associated with f, being homogeneous of degree 3 with respect to g; these polynomials are generated by taking as basis the set of all monomials of degree 3, $p_1(x) = x_1^3$, $p_2(x) = x_1 x_2$ and $p_3(x) = x_3$:

$$\Gamma(x) = \begin{bmatrix} x_1^3 & x_1x_2 & x_3 \\ -3x_1^3 & -5x_1x_2 + x_1^3 & -6x_3 + 2x_1^3 + 3x_1x_2 \\ 9x_1^3 & 25x_1x_2 - 8x_1^3 & -15x_1^3 - 33x_1x_2 + 36x_3 \end{bmatrix},$$

$$\det\bigl(\Gamma(x)\bigr) = x_1^4\bigl(-x_1^2 + 2x_2\bigr)\bigl(9x_1x_2 - x_1^3 - 3x_3\bigr);$$

in particular, $\omega_3(x) = 9x_1x_2 - x_1^3 - 3x_3$ is a Darboux polynomial associated with f, with characteristic value $\lambda_3 = -6$, since

$$L_f\omega_3(x) = -6\bigl(9x_1x_2 - x_1^3 - 3x_3\bigr) = \lambda_3\omega_3(x).$$

When f is not polynomial and homogeneous, it could be difficult to guess a priori the structure of a semi-invariant: this is the case, for instance, when ω is polynomial but its degree depends on some unknown parameters. Nevertheless, the procedure previously outlined endowed with some additional tricks, can be still successfully applied, as shown in the following example.

Example 3.28 Consider the planar system, described by $f(x) = [x_1x_2 \; -ax_2^2]^\top$, for some unknown constant $a \in \mathbb{Z}$, $a \geq 1$. It is easy to see that such a system admits the monomial first integral $I(x) = x_1^a x_2$; since the degree $a+1$ of such a monomial depends on the parameter a, it seems difficult to a priori guess the basis $p_1, \dots, p_k$ so that $x_1^a x_2$ is a factor of $\det(\Gamma)$: nevertheless, a simple choice of the basis and an additional trick allows to also find such a first integral, by the procedure previously outlined. Consider the Darboux polynomials associated with f that are functions of $p_1(x) = x_1$ and $p_2(x) = x_2$:

$$\Gamma(x) = \begin{bmatrix} x_1 & x_2 \\ x_1x_2 & -ax_2^2 \end{bmatrix}, \qquad \det\bigl(\Gamma(x)\bigr) = -(a+1)x_1x_2^2;$$

this yields two Darboux polynomials associated with f: $\omega_1(x) = x_1$ with characteristic polynomial $\lambda_1(x) = x_2$ and $\omega_2(x) = x_2$ with characteristic polynomial $\lambda_2(x) = -ax_2$. If ω_1 and ω_2 are Darboux polynomials associated with f, then $\omega_3 = \omega_1^{k_1}\omega_2^{k_2}$ is also a Darboux polynomial associated with f, for any constant k_1, k_2, with characteristic polynomial $k_1\lambda_1 + k_2\lambda_2$ (see Statement (3.1.2) of Theorem 3.1); then, imposing that $\lambda_1 k_1 + k_2\lambda_2 = 0$, one finds the condition $k_1 - ak_2 = 0$, whence, taking $k_1 = a$ and $k_2 = 1$, one concludes that I is a first integral associated with f.

3.12 The Poincaré–Dulac Normal Form of Continuous-Time Nonlinear Systems

The concept of the Poincaré–Dulac normal form arises when one tries to find a diffeomorphism that linearizes a given $f(x) \in \mathbb{R}^n$, having a linear part characterized by a semi-simple matrix A. When the given f can be linearized, its Poincaré–Dulac normal form is linear, otherwise its Poincaré–Dulac normal form is as close to be linear as possible, in a sense that will be clear later. When the matrix A of the linear part of f is not semi-simple, similar results are obtained using the Belitskii normal form described in Sect. 3.14. Among the numerous works dealing with the Poincaré–Dulac normal form, the reader is referred to [5, 14, 16, 25, 31, 32, 34, 45, 50, 58, 113, 117].

Throughout this section, assume that $f(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $f(0) = 0$. The *linear part* of f is Ax, with $A = \frac{\partial f(x)}{\partial x}|_{x=0}$. If not otherwise specified, assume throughout this section that A is *semi-simple*.

Definition 3.13 Vector function $f(x) = Ax + h(x)$, with A being semi-simple, $h(x)$ being analytic at $x = 0$, $h(0) = 0$, and having linear part equal to zero, is in the *Poincaré–Dulac normal form* if

$$\big[h(x), Ax\big] = 0. \tag{3.52}$$

Remark 3.29 The Poincaré–Dulac normal form is often introduced under the assumption that the linear part Ax of f is characterized by A being *normal*, instead of simply *semi-simple*. The two definitions coincide, apart from a linear transformation, because, by Lemma 2.5 at p. 39, any semi-simple matrix can be rendered normal by a linear transformation, and any normal matrix is certainly semi-simple. Let $f(x) = A_s x + h_s(x)$, with A_s being semi-simple; let $x = Qy$, $\det(Q) \neq 0$, be a linear transformation such that $\tilde{A}_{s,n} = Q^{-1} A_s Q$ is normal, and let $\tilde{h}_{s,n}(y) = Q^{-1} h_s(Qy)$. By Statement (1.4.2) of Theorem 1.4, the following relation holds:

$$\big[h_s(x), A_s x\big] = 0 \quad \Longleftrightarrow \quad \big[\tilde{h}_{s,n}(y), \tilde{A}_{s,n} y\big] = 0.$$

Case $A = 0$ is trivial, because any h satisfies (3.52) for such an A, whence the Poincaré–Dulac normal form of a system with a zero linear part does not give any insight about its properties.

Remark 3.30 Since $[Ax, Ax] = 0$, for any $A \in \mathbb{R}^{n\times n}$, from (3.52) and by the bilinearity of the Lie bracket (see Property (1.2.2)), it follows that $[f(x), Ax] = 0$. Hence, the following statements are equivalent:

(3.30.1) f is analytic at $x = 0$, $f(0) = 0$, and it is in the Poincaré–Dulac normal form;

(3.30.2) Ax is a symmetry of f (conversely, f is a symmetry of Ax), with f being analytic at $x = 0$, $f(0) = 0$, and f having Ax as linear part;

(3.30.3) f is analytic at $x = 0$, $f(0) = 0$, has linear part Ax and it is homogeneous of degree 0 with respect to Ax;

(3.30.4) f is analytic at $x = 0$, $f(0) = 0$, has linear part Ax and belongs to $\mathscr{C}_C(Ax)$.

Given $A \in \mathbb{R}^{n\times n}$, let $\{M_0, \ldots, M_{r-1}\}$ be a basis of $\mathscr{L}_c(A)$. By Theorem 3.10, all $f \in \mathscr{C}_C(Ax)$ are parameterized by

$$f(x) = \mu_0 M_0 x + \mu_1 M_1 x + \cdots + \mu_{r-1} M_{r-1} x, \tag{3.53}$$

with $\mu_i \in \mathscr{I}_C(Ax)$, $i = 0, \ldots, r-1$. By Statement (3.30.4) of Remark 3.30, given the linear part Ax of f, the set of all f being in the Poincaré–Dulac normal form can be found from (3.53) by requiring that the resulting f is analytic at $x = 0$, $f(0) = 0$, and has Ax as linear part. Note that, when A is semi-simple, the computation of the first integrals associated with Ax can be made as in Remark 2.9 at p. 53 or, if A is diagonal, as in Remark 1.9 at p. 27.

Another interpretation can be given of the Poincaré–Dulac normal form, thus yielding the notion of *resonance*. Thanks to the invariance of the Lie bracket to diffeomorphisms, assume that $A \in \mathbb{R}^{n\times n}$ is diagonal (possibly, complex); $h(x)$ is the linear combination (possibly, infinite) of terms

$$\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k, \tag{3.54}$$

where e_k is the kth column of the $n \times n$ identity matrix E and $n_1, n_2, \ldots, n_n \in \mathbb{Z}^{\geq}$ are such that $n_1 + n_2 + \cdots + n_n \geq 2$. Since A is diagonal and therefore semi-simple (each eigenvector e_k of A is mapped by A into a vector co-linear with e_k over $\mathbb{C}$, i.e., $\lambda_k e_k$), the operator $[\cdot, Ax]$ is linear and semi-simple too, in the sense that each term $(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k$ is mapped by the operator $[\cdot, Ax]$ into a term co-linear with $(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k$ over $\mathbb{C}$:

$$\begin{aligned}
&\left[\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k, Ax\right] \\
&\quad = A e_k \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) - e_k \left[n_1 \left(x_1^{n_1-1} x_2^{n_2} \cdots x_n^{n_n}\right) \ \ldots \ n_n \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n-1}\right)\right] Ax \\
&\quad = e_k \left(\lambda_k - (n_1\lambda_1 + n_2\lambda_2 + \cdots + n_n\lambda_n)\right)\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right).
\end{aligned}$$

Then, condition $[h(x), Ax] = 0$ is equivalent to $[(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k, Ax] = 0$ for each $(n_1, \ldots, n_n, k)$, and condition $[(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k, Ax] = 0$ holds if and only if the following *continuous-time resonance condition* (briefly, *resonance condition* if no confusion can arise between the continuous-time and discrete-time cases) among the eigenvalues of A holds:

$$n_1\lambda_1 + n_2\lambda_2 + \cdots + n_n\lambda_n = \lambda_k, \quad n_i \in \mathbb{Z}^{\geq}, \sum_{i=1}^{n} n_i \geq 2. \tag{3.55}$$

If (3.55) holds, then term (3.54) is called *resonant*; note that such a resonant term need not appear into the linear combination constituting $h(x)$ (it depends on the value of its coefficient into the linear combination constituting h).

If A is not diagonal, but only semi-simple, the *resonant terms* h are those belonging to the kernel of the linear operator $[\cdot, Ax]$.

Remark 3.31 Resonance condition (3.55) is equivalent to the condition

$$L_{Ax}\left(x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}\right) = \lambda_k\left(x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}\right), \tag{3.56}$$

i.e., term $x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}e_k$ is resonant if and only if the monomial $x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}$ is homogeneous of degree λ_k with respect to Ax. A monomial $x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}$ being homogeneous of degree λ with respect to Ax, with λ being eigenvalue of A, is said to be *resonant*.

Remark 3.32 There are three possible cases of Poincaré–Dulac normal forms: the number of n-plets $(n_1, \dots, n_n)$ such that (3.55) holds for some $k \in \{1, \dots, n\}$ is equal to zero, it is finite, it is infinite. If f is in the Poincaré–Dulac normal form and there are no resonances among the eigenvalues of its linear part, then f is necessarily linear: the Poincaré–Dulac normal form of such an f coincides with its linear part. Of course, the absence of resonances among the eigenvalues of the linear part of f is not necessary for f to be linear: it is sufficient that the corresponding resonant term "is not present" in the Poincaré–Dulac normal form, where the term "is not present" is better specified in the subsequent Example 3.33. If there is an n-plet $(m_1, \dots, m_n)$ such that

$$m_1\lambda_1 + \cdots + m_k\lambda_k + \cdots + m_n\lambda_n = 0, \quad m_i \in \mathbb{Z}^{\geq}, \sum_{i=1}^{n} m_i \geq 1, \tag{3.57}$$

then there is an infinite number of resonances among $\{\lambda_1, \dots, \lambda_n\}$; to be more precise, since the following relation (which is not a resonance condition) clearly holds:

$$(0)\lambda_1 + \cdots + (1)\lambda_k + \cdots + (0)\lambda_n = \lambda_k,$$

one has

$$(\ell m_1)\lambda_1 + \cdots + (\ell m_k + 1)\lambda_k + \cdots + (\ell m_n)\lambda_n = \lambda_k, \tag{3.58}$$

which is a resonance condition for each $\ell \in \mathbb{Z}$, $\ell \geq 1$. Vice versa, if (3.55) holds with $n_k \geq 1$, then (3.57) holds with $m_i = n_i$, $i \neq k$, and $m_k = n_k - 1$, whence there is an infinite number of resonances among $\{\lambda_1, \dots, \lambda_n\}$.

It is worth pointing out that the resonance condition (3.55) implies that

$$\frac{x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}}{x_k} \in \mathscr{I}_C(Ax);$$

as a matter of fact, letting $\omega_1(x) = x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}$ and $\omega_2(x) = x_k$, one finds that condition (3.55) is equivalent to $L_{Ax}\omega_1 = \lambda_k\omega_1$, and (since A is diagonal) $L_{Ax}\omega_2 = \lambda_k\omega_2$: ω_1 and ω_2 are two Darboux polynomials associated with Ax, with the same characteristic value. Then, $\frac{\omega_1}{\omega_2}$ is a first integral associated with Ax, whence

$$x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n} e_k = \frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k} [0 \ \dots \ e_k \ \dots \ 0] \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix}.$$

Since the coefficient matrix $\bar{M}_{k-1} := [0 \ \dots \ e_k \ \dots \ 0]$ commutes with matrix A and the coefficient $\frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k}$ is a first integral of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$, one concludes that $h(x) = \sum_{i=0}^{n-1} \mu_i \bar{M}_i x$, with $\bar{M}_0, \bar{M}_1, \dots, \bar{M}_{n-1}$ belonging to the linear centralizer $\mathscr{L}_c(A)$ of A and the coefficients μ_i being first integrals of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$.

Example 3.29 Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$; A is semi-simple with distinct eigenvalues; a basis of $\mathscr{L}_c(A)$ is given by $\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right\}$. Set $\mathscr{I}_C(Ax)$ is constituted by arbitrary functions of $I_1(x) = \frac{(x_1 - x_2)^2}{x_2}$. The set of all $f \in \mathscr{C}_C(Ax)$ is given by

$$f(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(\mu_0 - \mu_1) + x_2 \mu_1 \\ x_2 \mu_0 \end{bmatrix},$$

with μ_0, μ_1 being arbitrary functions of I_1. In order that such an f is in the Poincaré–Dulac normal form, it is now enough to impose that f is analytic at $x = 0$, $f(0) = 0$, and has Ax as linear part. This holds if and only if $\mu_0 = 2 + aI_1$, $\mu_1 = 1 + aI_1$, with constant $a \in \mathbb{R}$ being arbitrary. Such a Poincaré–Dulac normal form can be deduced in an alternative way. Diagonalize A with the linear transformation $x = Qy$,

$$Q = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \tilde{A} = Q^{-1} A Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

There is only one resonance between the two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ of $\tilde{A}$, $\lambda_2 = (2)\lambda_1 + (0)\lambda_2$, which yields the resonant term $y_1^2 y_2^0 e_2 = y_1^2 [0 \ 1]^\top$. Then, in the y-coordinates, the Poincaré–Dulac normal form is characterized by

$$\tilde{f}(y) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + b y_1^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where constant $b \in \mathbb{R}$ is arbitrary. Then, by the pull-back to the original coordinates, one finds that

$$\begin{aligned} f(x) = Q \tilde{f} \circ \left(Q^{-1} x\right) &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ 2y_2 + b y_1^2 \end{bmatrix}_{y_1 = -x_1 + x_2, y_2 = x_2} \\ &= \begin{bmatrix} x_1 + x_2 + b(-x_1 + x_2)^2 \\ 2x_2 + b(-x_1 + x_2)^2 \end{bmatrix}, \end{aligned}$$

and such an f coincides with the one above if $b = a$.

Example 3.30 Let

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix};$$

the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -1$, whence there are infinite resonances among the eigenvalues of A,

$$\lambda_1 = (n_1)\lambda_1 + (n_2)\lambda_2 + (n_1 + 2n_2 - 1)\lambda_3, \quad \forall n_i \in \mathbb{Z}^{\geq}, 2n_1 + 3n_2 - 1 \geq 2,$$
$$\lambda_2 = (n_1)\lambda_1 + (n_2)\lambda_2 + (n_1 + 2n_2 - 2)\lambda_3, \quad \forall n_i \in \mathbb{Z}^{\geq}, 2n_1 + 3n_2 - 2 \geq 2,$$
$$\lambda_3 = (n_1)\lambda_1 + (n_2)\lambda_2 + (n_1 + 2n_2 + 1)\lambda_3, \quad \forall n_i \in \mathbb{Z}^{\geq}, 2n_1 + 3n_2 + 1 \geq 2.$$

Since A is semi-simple with distinct eigenvalues, a basis of the linear centralizer $\mathscr{L}_c(A)$ is $\{A^0, A^1, A^2\}$. Set $\mathscr{I}_C(Ax)$ is constituted by all arbitrary functions of $I_1(x) = (x_1 - 2x_2)(x_1 - x_2)$ and $I_2(x) = (x_1 - 2x_2)^2(x_1 - \frac{4}{5}x_2 + \frac{3}{5}x_3)$. From this, all f, $f(0) = 0$, having linear part Ax and being in the Poincaré–Dulac normal form are parameterized by $f(x) = Ax + h(x)$, with $h(x) = \mu_0 A^0 x + \mu_1 A^1 x + \mu_2 A^2 x$, where μ_0, μ_1 and μ_2 are arbitrary functions of I_1, I_2, such that h is analytic at $x = 0$, $h(0) = 0$, and the linear part of $h(x)$ is zero:

$$f(x) = \begin{bmatrix} (3 + \mu_0 + 3\mu_1 + \mu_2)x_1 + (-4\mu_1 - 4)x_2 \\ (2 + 2\mu_1)x_1 + (-3 + \mu_0 - 3\mu_1 + \mu_2)x_2 \\ (1 + \mu_1 + 5\mu_2)x_1 - 4\mu_2 x_2 + (2 + \mu_0 + 2\mu_1 + 4\mu_2)x_3 \end{bmatrix}.$$

Remark 3.33 Assume that $A \in \mathbb{R}^{n \times n}$ is diagonal, with integer and positive eigenvalues; by this assumption, the number of resonances among the eigenvalues of A is finite. Then, denoting by $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n} e_k$ a resonant term, one concludes that a nonlinear system having Ax as linear part and being in the Poincaré–Dulac normal form can always be linearized by a finite dimensional state immersion, by taking as additional state variables just the resonant monomials $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ (see [51, 87, 89]). As a matter of fact, since $L_{Ax}(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) = \lambda_k (x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n})$ by the resonance condition (3.55), and $[f(x), Ax] = 0$, because f is in the Poincaré–Dulac normal form, one concludes that the monomial $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ is homogeneous of degree λ_k with respect to Ax and f is polynomial and homogeneous of degree 0 with respect to Ax; by Theorem 3.15, this implies that $L_f(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n})$ is polynomial and homogeneous of degree λ_k with respect to Ax (i.e., $L_{Ax} L_f(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) = \lambda_k L_f(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n})$), whence it is a linear combination of all resonant monomials of degree λ_k (any monomial of degree λ_k with respect to A is resonant!), as to be shown.

Example 3.31 Let $A = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, with $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$; since, in this case, the only resonances are given by

$$\lambda_2 = (2)\lambda_1 + (0)\lambda_2 + (0)\lambda_3 \quad \Rightarrow \quad e_2 x_1^2,$$

$$\lambda_3 = (4)\lambda_1 + (0)\lambda_2 + (0)\lambda_3 \quad \Rightarrow \quad e_3 x_1^4,$$
$$\lambda_3 = (2)\lambda_1 + (1)\lambda_2 + (0)\lambda_3 \quad \Rightarrow \quad e_3 x_1^2 x_2,$$
$$\lambda_3 = (0)\lambda_1 + (2)\lambda_2 + (0)\lambda_3 \quad \Rightarrow \quad e_3 x_2^2,$$

all f, $f(0) = 0$, with linear part Ax and being in the Poincaré–Dulac normal form are given by

$$f(x) = \begin{bmatrix} x_1 \\ 2x_2 + a_1 x_1^2 \\ 4x_3 + a_2 x_1^4 + a_3 x_1^2 x_2 + a_4 x_2^2 \end{bmatrix},$$

with constants $a_i \in \mathbb{R}$ being arbitrary; any such an f can be linearized by taking as additional state variables $x_4 = x_1^2$, $x_5 = x_1^4$, $x_6 = x_1^2 x_2$ and $x_7 = x_2^2$. To be more precise, the dynamics of x_4 are described by $L_f x_4 = L_f x_1^2 = 2x_1^2 = 2x_4$, the dynamics of x_5 are described by $L_f x_5 = L_f x_1^4 = 4x_1^4 = 4x_5$, the dynamics of x_6 are described by $L_f x_6 = L_f x_1^2 x_2 = a_1 x_1^4 + 4x_1^2 x_2 = a_1 x_5 + 4x_6$, and the dynamics of x_7 are described by $L_f x_7 = L_f x_2^2 = 2a_1 x_1^2 x_2 + 4x_2^2 = 2a_1 x_6 + 4x_7$. Then, collecting such dynamics, one has the extended linear system $\frac{\mathrm{d}x_e}{\mathrm{d}t} = A_e x_e$, with

$$A_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a_1 & 4 \end{bmatrix}, \qquad x_e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

Note that, under the assumption that the real numbers a_i are non-zero, the Jordan form of A_e is

$$J_e = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right],$$

namely, although the original A is semi-simple, the state immersion has generated in A_e Jordan blocks of dimension greater than 1 (A_e is not semi-simple), and this justifies the name *resonance* used to represent this phenomenon. It is worth pointing out that if some a_i is equal to zero, i.e., if some resonant term is missing in the Poincaré–Dulac normal form, then the Jordan form of A_e may differ from the above reported J_e. For instance, if $a_1 = 0$, the Jordan form of A_e is

$$\bar{J}_e = \left[\begin{array}{c|c|c|c|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right].$$

More details about the linearization by state immersion are reported in Chap. 7.

Definition 3.14 A diffeomorphism $y = \varphi(x)$ is *near-identity* if it is analytic on a neighborhood of the origin of $\mathbb{R}^n$, $\varphi(0) = 0$, and $\frac{\partial \varphi(x)}{\partial x}|_{y=0} = E$.

A diffeomorphism $y = \varphi(x)$ is near-identity if and only if its inverse $x = \varphi^{-1}(y)$ is near-identity.

Let $g(y) \in \mathbb{R}^n$ be expanded as $g(y) = By + \sum_{h=2}^{+\infty} g_h(y)$, where the entries of $g_h(y) \in \mathbb{R}^n$ are homogeneous of degree h with respect to the standard dilation, i.e., $[g_h(y), y] = (1-h)g_h(y)$. As well known, the flow associated with g can be expanded in Taylor series about $\tau = 0$ as $\Phi_g(\tau, y) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} L_g^i y$, where $L_g^0 f = f$ and $L_g^{i+1} f = L_g L_g^i f$, $i \in \mathbb{Z}^{\geq}$, $f(y) \in \mathbb{R}^n$. Clearly, $y = \Phi_g(-\tau, x) = \sum_{i=0}^{+\infty} \frac{(-\tau)^i}{i!} L_g^i x$ is the inverse of $x = \Phi_g(\tau, y)$. Since $\mathrm{e}^{B\tau} y$ is the linear part of $\Phi_g(\tau, y)$ (see Lemma 2 of [93]), if $B = 0$, then $y = \Phi_g(-\tau, x)$ is near-identity. By statement (a) of Proposition 6.1 of [57], for any formal near-identity diffeomorphism $y = \varphi(x)$ and for any arbitrary $\tau \in \mathbb{R}^{>}$, there exists a formal $g(x)$ such that $\varphi(x) = \Phi_g(-\tau, x)$, which need not be unique; g can be called the *logarithm* of φ (see also [93]).

Remark 3.34 If $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, since the eigenvalues of A are coincident positive real numbers, then (see [37]) $A = \mathrm{e}^B$ has an uncountable set of solutions; for instance, $B = \left[\begin{smallmatrix} -2\frac{ac+bd}{ad-bc}h\pi & 2\frac{a^2+b^2}{ad-bc}h\pi \\ -2\frac{c^2+d^2}{ad-bc}h\pi & 2\frac{ac+bd}{ad-bc}h\pi \end{smallmatrix}\right]$ is solution of $A = \mathrm{e}^B$ for any integer h (including $h = 0$) and for any reals a, b, c, d such that $ad - bc \neq 0$. Hence, $g(x) = -\frac{1}{\tau} Bx$ is solution of $x = \Phi_g(-\tau, x)$, for any $\tau > 0$.

Assume that $B = 0$, $g(y) = \sum_{h=2}^{+\infty} g_h(y)$. Since the entries of $L_{g_h}^i y$ are homogeneous of degree i with respect to the standard dilation, i.e., $[L_{g_h}^i y, y] = (1-i) L_{g_h}^i y$, the entries of $L_g^i y$ have degree greater than or equal to $i \geq 2$. The Taylor expansion of $\Phi_g(\tau, y)$ (respectively, $\Phi_g(-\tau, x)$) with respect to y (respectively, x), up to order $m \geq 2$, coincides with the Taylor expansion of $\sum_{i=0}^{m} \frac{\tau^i}{i!} L_g^i y$ (respectively, $\sum_{i=0}^{m} \frac{(-\tau)^i}{i!} L_g^i x$); actually, if the Taylor expansion is limited to order m, then g can be substituted with $\sum_{h=2}^{m} g_h$.

Example 3.32 Consider the near-identity diffeomorphism $y = \varphi(x)$, with $\varphi(x) = [x_1 \;\; x_2 + x_1 x_2 + x_2^2]^\top$. The objective is to find $g(x) \in \mathbb{R}^2$ such that the Taylor expansions of $\varphi(x)$ and of $\Phi_g(-1, x)$ coincide up to order 3 and, consequently, that the Taylor expansions of $\varphi^{-1}(y)$ and of $\Phi_g(1, y)$ coincide up to order 3. Let $g = g_2 + g_3$, with

$$g_2(x) = \begin{bmatrix} a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 \\ a_4 x_1^2 + a_5 x_1 x_2 + a_6 x_2^2 \end{bmatrix},$$

$$g_3(x) = \begin{bmatrix} b_1 x_1^3 + b_2 x_1^2 x_2 + b_3 x_1 x_2^2 + b_4 x_2^3 \\ b_5 x_1^3 + b_6 x_1^2 x_2 + b_7 x_1 x_2^2 + b_8 x_2^3 \end{bmatrix}.$$

By imposing that the Taylor expansion up to order 3 of

$$\sum_{i=0}^{3} \frac{(-1)^i}{i!} L_g^i x = x - g_2(x) - g_3(x) + \frac{1}{2} L_{g_2+g_3}\big(g_2(x) + g_3(x)\big)$$

coincides with the Taylor expansion up to order 3 of $\varphi(x)$, the coefficients a_i of g_2 and b_i of g_3 can be determined uniquely,

$$g_2(x) = \begin{bmatrix} 0 \\ -x_1 x_2 - x_2^2 \end{bmatrix}, \qquad g_3(x) = \begin{bmatrix} 0 \\ \frac{1}{2} x_1^2 x_2 + \frac{3}{2} x_1 x_2^2 + x_2^3 \end{bmatrix}.$$

The following result, known as the *Poincaré–Dulac Theorem* (see [5, 34]), gives sufficient conditions for transforming a continuous-time nonlinear system into its Poincaré–Dulac normal form and, in particular, for linearizing a continuous-time nonlinear system by a near-identity diffeomorphism.

Theorem 3.33 *Let $f(x) \in \mathbb{R}^n$ be analytic at $x = 0$, $f(0) = 0$, and $A = \frac{\partial f(x)}{\partial x}|_{x=0}$ be semi-simple. If the eigenvalues of A belong to the Poincaré domain (i.e., the convex hull of the n points $\lambda_1, \ldots, \lambda_n$ in the complex plane does not contain the origin of $\mathbb{C}$), then there exists a near-identity diffeomorphism $y = \varphi(x)$ such that $\varphi_* f(y) = Ay + \tilde{h}(y)$, with $\tilde{h}(y) \in \mathbb{R}^n$, $\tilde{h}(0) = 0$, $\tilde{h}$ having zero linear part, such that $[\tilde{h}(y), Ay] = 0$; in particular, if there are no resonances among the eigenvalues λ_i of A (i.e., condition (3.55) does not hold), then $\varphi_* f$ is linear, $\varphi_* f(y) = Ay$.*

By the proof of the Poincaré–Dulac Theorem 3.33, which is omitted for space reasons, any f with a semi-simple linear part can be *formally* transformed into its normal form through a *formal* series; in many cases, some convergence conditions like the belonging to the Poincaré domain used in Theorem 3.33 (e.g, the Siegel criterion, the Pliss criterion, the Bruno criterion and the BMW-C theory: see [50]) guarantee that such a transformation is analytic at $x = 0$. When the series is not convergent, by the Borel Lemma [62], there exists a C^∞-transformation such that the transformed f differs from its normal form for a vector function being flat at $x = 0$ (see [16] and the subsequent Theorem 3.34), i.e., by a vector function that is

C^∞ in a neighborhood of the origin and has all derivatives equal to zero at $x = 0$; this also means that, for any arbitrarily large integer $m > 0$, there exists a polynomial diffeomorphism such that the transformed f differs from its normal form for terms of order higher than m.

Although the proof of the Poincaré–Dulac Theorem 3.33 is not reported, it is worth pointing out that the near-identity diffeomorphism $y = \varphi(x)$, when it exists, can always be made as the composition

$$\varphi = \varphi_2 \circ \varphi_3 \circ \varphi_4 \circ \cdots,$$

where φ_k is a near-identity diffeomorphism satisfying

$$\varphi_k(x) = x + \psi_k(x),$$

with the entries $\psi_{k,i}(x) \in \mathbb{R}$ of $\psi_k(x) \in \mathbb{R}^n$ being homogeneous of degree k with respect to the standard dilation (namely, $L_x\psi_{k,i} = k\psi_{k,i}$ and $[\psi_k(x), x] = (1 - k)\psi_k(x)$). Let $f(x) = Ax + \sum_{h=2}^{+\infty} f_h(x)$, with the entries $f_{h,i}(x) \in \mathbb{R}$ of $f_h(x) \in \mathbb{R}^n$ being homogeneous of degree h with respect to the standard dilation (namely, $L_x f_{h,i} = hf_{h,i}$ and $[f_h(x), x] = (1 - h) f_h(x)$). Consider the near-identity diffeomorphism $y = \varphi_k(x)$. Hence, taking into account that $\varphi_k^{-1}(y) = y - \psi_k(y) + \cdots$ (where $\cdots$ denotes terms of degree higher than k), one concludes that

$$\varphi_{k*} f(y) = Ay + \sum_{h=2}^{k-1} f_h(y) + f_k(y) - \big[\psi_k(y), Ay\big] + \cdots,$$

where $\cdots$ denotes terms of degree higher than k: the terms $Ax + \sum_{h=2}^{k-1} f_h(x)$ of $f(x)$ having degree less than k are not affected by the near-identity diffeomorphism $y = \varphi_k(x)$, the term $f_k(x)$ becomes $f_k(y) - [\psi_k(y), Ay]$, and the terms of $f(x)$ having degree greater than k are modified, but their push-forward is not reported. Letting

$$f_k(y) = \sum_{n_1,n_2,\ldots,n_n,k} a_{n_1,n_2,\ldots,n_n,k}\big(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\big)e_k,$$

$$\psi_k(y) = \sum_{n_1,n_2,\ldots,n_n,k} b_{n_1,n_2,\ldots,n_n,k}\big(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\big)e_k,$$

since

$$\big[\psi_k(y), Ay\big] = \sum_{n_1,n_2,\ldots,n_n,k} b_{n_1,n_2,\ldots,n_n,k}\big(\lambda_k - (n_1\lambda_1 + \cdots + n_n\lambda_n)\big)\big(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\big)e_k,$$

each non-resonant term $a_{n_1,n_2,\ldots,n_n,k}(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n})e_k$ of $f_k(y)$ can be removed by taking

$$b_{n_1,n_2,\ldots,n_n,k} = \frac{a_{n_1,n_2,\ldots,n_n,k}}{\lambda_k - (n_1\lambda_1 + n_2\lambda_2 + \cdots + n_n\lambda_n)},$$

whereas if the term $a_{n_1,n_2,\ldots,n_n,k}(x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n})e_k$ of $f_k(y)$ is resonant with respect to the eigenvalues of A, then it cannot be removed by the near-identity diffeomorphism $y=\varphi_k(x)$.

The following Example 3.33 shows that, although a vector function f does not contain resonant terms, there may exist near-identity diffeomorphisms $y=\varphi(x)$ such that $\varphi_* f(y)$ contains resonant terms; in particular, this show that the lack of resonant terms in f does not guarantee that such an f can be linearized. On the contrary, the subsequent Example 3.34 shows that the lack of resonant terms in f is not necessary for f to be linearizable.

Example 3.33 Consider $f(x)=[-x_1+x_2^3\; x_2+x_1^4x_2]^\top$; the linear part of f is Ax, with $A=\mathrm{diag}\{-1,1\}$, and the Poincaré–Dulac normal form associated with A is $Ax+h(x)$, with $h(x)=[(\mu_0-\mu_1)x_1\; (\mu_0+\mu_1)x_2]^\top$, where μ_0,μ_1 are arbitrary functions of x_1x_2 such that $h(x)$ is analytic at $x=0$ and has zero linear part. Clearly, the two nonlinear terms $x_2^3e_1$ and $x_1^4x_2e_2$ appearing in f are not resonant: so one could say that “no resonant term is present in f”, which is not true. As a matter of fact, consider the near-identity diffeomorphism $y=\varphi(x)$, $\varphi(x)=[x_1-\frac{1}{4}x_2^3\; x_2]^\top$, which is chosen with the aim of eliminating the nonlinear term $x_2^3e_1$ of lower degree appearing in f. The resulting push-forward of f is

$$\varphi_* f(y)=\begin{bmatrix}-y_1-\frac{3}{4}y_2^3(y_1+\frac{1}{4}y_2^3)^4\\ y_2+(y_1+\frac{1}{4}y_2^3)^4y_2\end{bmatrix};$$

now, $\varphi_* f(y)$ contains two resonant terms, $-\frac{3}{4}y_2^3y_1^4e_1$ and $y_1^3y_2^4e_2$.

Example 3.34 Consider

$$f(x)=\begin{bmatrix}x_1\\ \frac{2x_1^3+x_1^2+4x_1x_2+3x_2}{1+x_1}\end{bmatrix}=\begin{bmatrix}x_1\\3x_2\end{bmatrix}+\begin{bmatrix}0\\x_1^2+x_1x_2\end{bmatrix}+\begin{bmatrix}0\\x_1^3-x_1^2x_2\end{bmatrix}+\cdots;$$

by the Taylor expansion of f it is easy to check that $A=\mathrm{diag}\{1,3\}$ is the dynamic matrix of the linear part Ax of f and that the resonant term $x_1^3e_2$ is present into f. Nevertheless, $\varphi_* f(y)=Ay$, for the near-identity diffeomorphism $y=\varphi(x)$, with $\varphi(x)=[x_1\; \frac{x_2+x_1^2}{1+x_1}]^\top$ and $\varphi^{-1}(y)=[y_1\; y_2-y_1^2+y_1y_2]^\top$. The resonant term $x_1^3e_2$ does not imply that f cannot be linearized, because it disappears when the first transformation φ_2 is applied in order to eliminate the second order terms.

Remark 3.35 The two examples above have clarified that the transformation computed to eliminate nonlinearities of order k may introduce or remove resonant terms of order higher than k. Assume that matrix A of the linear part Ax of f is diagonal. If either all monomials contained in $f(x)-Ax$ are resonant or there are no non-resonant monomials in $f(x)-Ax$ of degree lower than a resonant one, then there exists no near-identity diffeomorphism $y=\varphi(x)$ such that $\varphi_* f(y)=Ay$. For instance, $f(x)=[x_1\; 2x_2+x_1^2]^\top$ and $f(x)=[x_1+x_1x_2^2\; 2x_2+x_1^2]^\top$ are not linearizable by a near-identity diffeomorphism.

Remark 3.36 Let $f(x) = A_s x + h_s(x)$, with A_s being semi-simple, $h_s(0) = 0$, h_s having zero linear part. Let $x = Q\xi$, $\det(Q) \neq 0$, be a linear transformation such that $\tilde{A}_{s,n} = Q^{-1}A_s Q$ is normal, and let $\tilde{f}(\xi) = Q^{-1}f(Q\xi) = \tilde{A}_{s,n}\xi + \tilde{h}_{s,n}(\xi)$. Assume that $\xi = \tilde{\phi}(\eta)$ is a near-identity diffeomorphism such that $\hat{f}(\eta) = (\frac{\partial\tilde{\phi}}{\partial\eta})^{-1}\tilde{f} \circ \tilde{\phi}(\eta)$ is in the Poincaré–Dulac normal form, $\hat{f}(\eta) = \tilde{A}_{s,n}\eta + \hat{h}_{s,n}(\eta)$, where $[\hat{h}_{s,n}(\eta), \tilde{A}_{s,n}\eta] = 0$. Then, $\bar{f}(y) = Q\hat{f}(Q^{-1}y)$ is in the Poincaré–Dulac normal form, $\bar{f}(y) = A_s y + \bar{h}_s(y)$, because $[\bar{h}_s(y), A_s y] = 0$, by the invariance of the Lie bracket to diffeomorphisms. Hence, $f(x) = A_s x + h_s(x)$ can be directly transformed into $\bar{f}(y) = A_s y + \bar{h}_s(y)$, with $[\bar{h}_s(y), A_s y] = 0$, by the near-identity diffeomorphism $x = \phi(y)$, where $\phi(y) = Q\tilde{\phi}(Q^{-1}y)$. This can be represented by the following commutative diagram:

$$
\begin{array}{ccc}
f(x) = A_s x + h_s(x) & \xrightarrow{\xi = Q^{-1}x} & \tilde{f}(\xi) = \tilde{A}_{s,n}\xi + \tilde{h}_{s,n}(\xi) \\
x=\phi(y) \Big\downarrow & & \Big\downarrow \xi=\tilde{\phi}(\eta) \\
\begin{cases} \bar{f}(y) = A_s y + \bar{h}_s(y) \\ [\bar{h}_s(y), A_s y] = 0 \end{cases} & \xleftarrow[y = Q\eta]{} & \begin{cases} \hat{f}(\eta) = \tilde{A}_{s,n}\eta + \hat{h}_{s,n}(\eta) \\ [\hat{h}_{s,n}(\eta), \tilde{A}_{s,n}\eta] = 0 \end{cases}
\end{array}
$$

Theorem 3.34 *For any given vector function $f(x) \in \mathbb{R}^n$ being C^∞ in a neighborhood of the origin of $\mathbb{R}^n$, with linear part Ax (A being semi-simple), there exists a diffeomorphism $y = \varphi(x)$ being C^∞ in a neighborhood of the origin of $\mathbb{R}^n$, with $\varphi(0) = 0$ and $\frac{\partial\varphi(x)}{\partial x}|_{x=0} = E$, such that the push-forward of f takes the form $\varphi_* f(y) = Ay + h(y) + \alpha(y)$, where $Ay + h(y)$ is in the Poincaré–Dulac normal form and $\alpha(y)$ is flat.*

Combining the Poincaré–Dulac Theorem 3.33 with the concept of symmetry, one can prove the following theorem, which gives necessary and sufficient conditions for the linearization of f.

Theorem 3.35 *Assume that $f(x) \in \mathbb{R}^n$ is analytic at $x = 0$ with $f(0) = 0$ and with linear part Ax, where A need not be semi-simple. There exists a near-identity diffeomorphism $y = \varphi(x)$ such that the push-forward of f takes the form $\varphi_* f(y) = Ay$ if and only if there exists a $g(x) \in \mathbb{R}^n$, analytic at $x = 0$, $g(0) = 0$, such that $[f, g] = 0$, and the linear part of g is x.*

Proof If $\tilde{f}(y) = Ay$, then $\tilde{g}(y) = y$ satisfies $[\tilde{f}, \tilde{g}] = 0$. Hence, by the pull-back of $\tilde{g}$, one obtains $g(x) = \varphi^*\tilde{g}(x) = (\frac{\partial\varphi}{\partial x})^{-1}\varphi(x)$ that is analytic at $x = 0$, satisfies $g(0) = 0$, and has x as linear part. Furthermore, by the invariance of the Lie bracket to diffeomorphisms, $[f, g] = 0$. Conversely, for any g being analytic at $x = 0$, $g(0) = 0$, and with linear part x, the Poincaré–Dulac Theorem 3.33 implies the existence of a near-identity diffeomorphism $y = \varphi(x)$ such that the push-forward of g

takes the form $\varphi_* g(y) = y$, by virtue of the absence of resonances among the eigenvalues of the linear part of g. If $[f, g] = 0$, then $[\varphi_* f(y), \varphi_* g(y)] = [\varphi_* f(y), y] = 0$; condition $[\varphi_* f(y), y] = 0$ implies that $\varphi_* f$ is homogeneous of degree 0 with respect to the standard dilation (i.e., the dilation with all weights being equal to 1); since $\varphi_* f$ is analytic at $y = 0$ and $\varphi_* f(0) = 0$, $\varphi_* f$ is necessarily linear. □

Remark 3.37 Let $g(x) \in \mathbb{R}^n$ be analytic at $x = 0$, $g(0) = 0$, with linear part x. Hence, by the Poincaré–Dulac Theorem 3.33, there exists a near-identity diffeomorphism $y = \varphi(x)$ such that $\varphi_* g(y) = y$. By statement (a) of Proposition 6.1 of [57], such a near-identity diffeomorphism can be formally computed by expressing $\varphi(x) = \Phi_h(-\tau, x)$, for some $\tau \in \mathbb{R}^>$ and $h(x) \in \mathbb{R}^n$ being analytic at $x = 0$, $h(0) = 0$, with zero linear part. With no loss of generality, let $\tau = 1$. Hence, one has

$$\Phi_{h*} g = g + [h, g] + \frac{1}{2}\big[h, [h, g]\big] + \frac{1}{3!}\big[h, \big[h, [h, g]\big]\big] + \cdots \tag{3.59}$$

Letting $h(y) = h_2(y) + h_3(y) + h_4(y) + \cdots$, $g(y) = y + g_2(y) + g_3(y) + g_4(y) + \cdots$, and $\tilde{g} = \Phi_{h*} g$, where the entries of h_i and g_i are polynomial and homogeneous of degree i with respect to the standard dilation, $[h_i, y] = (1 - i)h_i$ and $[g_i, y] = (1 - i)g_i$, one finds that $\tilde{g}(y) = y + \tilde{g}_2(y) + \tilde{g}_3(y) + \tilde{g}_4(y) + \cdots$, where

$$\begin{aligned}
\tilde{g}_2 &= g_2 + [h_2, y],\\
\tilde{g}_3 &= g_3 + [h_2, g_2] + [h_3, y] + \frac{1}{2}\big[h_2, [h_2, y]\big],\\
\tilde{g}_4 &= g_4 + [h_2, g_3] + [h_3, g_2] + [h_4, y] + \frac{1}{2}\big[h_2, [h_2, g_2]\big]\\
&\quad + \frac{1}{2}\big[h_2, [h_3, y]\big] + \frac{1}{2}\big[h_3, [h_2, y]\big] + \frac{1}{3!}\big[h_2, \big[h_2, [h_2, y]\big]\big], \ldots
\end{aligned}$$

Therefore, one can obtain $\tilde{g}(y) = y$ by letting $\tilde{g}_i = 0$, $i \in \mathbb{Z}$, $i \geq 2$. Taking into account that $[h_i(y), y] = (1 - i)h_i(y)$, the equations $\tilde{g}_i = 0$, $i \in \mathbb{Z}$, $i \geq 2$, can be solved uniquely in h_i, $i \in \mathbb{Z}$, $i \geq 2$: at the first step, one computes uniquely

$$h_2 = g_2;$$

at the second step, using the knowledge of h_2 and some simplifications, one computes uniquely

$$h_3 = \frac{1}{2} g_3;$$

then, using the knowledge of h_2 and h_3 and some simplifications, one computes uniquely

$$h_4 = \frac{1}{3} g_4 + \frac{1}{12}[g_2, g_3];$$

proceeding in the same manner, one obtains

$$
\begin{aligned}
h_5 &= \frac{1}{4}g_5 + \frac{1}{12}[g_2, g_4],\\
h_6 &= \frac{1}{5}g_6 + \frac{3}{40}[g_2, g_5] + \frac{1}{60}[g_3, g_4] - \frac{1}{240}\big[g_3, [g_2, g_3]\big] + \frac{1}{360}\big[g_2, [g_2, g_4]\big]\\
&\quad - \frac{1}{720}\big[g_2, \big[g_2, [g_2, g_3]\big]\big],\\
h_7 &= \frac{1}{6}g_7 + \frac{1}{15}[g_2, g_6] + \frac{1}{48}[g_3, g_5] + \frac{1}{180}\big[g_2, [g_3, g_4]\big] - \frac{1}{144}\big[g_3, [g_2, g_4]\big]\\
&\quad + \frac{1}{240}\big[g_2, [g_2, g_5]\big] - \frac{1}{720}\big[g_2, \big[g_3, [g_2, g_3]\big]\big] - \frac{1}{720}\big[g_2, \big[g_2, [g_2, g_4]\big]\big].
\end{aligned}
$$

If the desired order of approximation is higher than 7, the same computations can be performed by increasing the order of approximation of h and g. It is worth pointing out that if $g(x) = x + g_k(x)$, for some $k \in \mathbb{Z}$, $k \geq 2$, then the unique solution of the above equations is $h(x) = \frac{1}{k-1}g_k(x)$.

Example 3.35 Consider $g(x) = x + g_2(x)$, where $g_2(x) = [x_2^2 \; x_1^2]^\top$. Then, letting $h(x) = g_2(x) = [x_2^2 \; x_1^2]^\top$, one finds that $\Phi_{h*}g(y) = y$, where

$$
\begin{aligned}
y &= \Phi_{h*}(-1, x) = x - h(x) + \frac{1}{2}L_h h(x) - \frac{1}{3!}L_h^2 h(x) + O\big(x^5\big)\\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2x_1^2x_2 \\ 2x_1x_2^2 \end{bmatrix} - \frac{1}{3!}\begin{bmatrix} 4x_1x_2^3 + 2x_1^4 \\ 2x_2^4 + 4x_1^3x_2 \end{bmatrix} + O\big(x^5\big)\\
x &= \Phi_{h*}(1, y) = y + h(y) + \frac{1}{2}L_h h(y) + \frac{1}{3!}L_h^2 h(y) + O\big(y^5\big)\\
&= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_2^2 \\ y_1^2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2y_1^2y_2 \\ 2y_1y_2^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 4y_1y_2^3 + 2y_1^4 \\ 2y_2^4 + 4y_1^3y_2 \end{bmatrix} + O\big(y^5\big).
\end{aligned}
$$

The following theorem gives necessary and sufficient conditions for the transformation of f into its Poincaré–Dulac normal form when the linear part of f is semi-simple; it can be seen as an extension of Theorem 3.35.

Theorem 3.36 *Assume that $f(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $f(0) = 0$, and with linear part Ax, where A is semi-simple. There exists a near-identity diffeomorphism $y = \varphi(x)$ such that the push-forward $\varphi_* f$ of f is in the Poincaré–Dulac normal form if and only if there exist $g_1(x) \in \mathbb{R}^n$ and $g_2(x) \in \mathbb{R}^n$, analytic at $x = 0$, $g_1(0) = 0$ and $g_2(0) = 0$, such that $[f, g_1] = 0$ and $[g_1, g_2] = 0$, with the linear part of g_1 being Ax and the linear part of g_2 being x.*

Proof If $\tilde{f}(y) = Ay + \tilde{h}$, where $\tilde{h}$ satisfies $[\tilde{h}(y), Ay] = 0$, then $\tilde{g}_1(y) = Ay$ and $\tilde{g}_2(y) = y$ satisfy $[\tilde{f}, \tilde{g}_1] = 0$ and $[\tilde{g}_1, \tilde{g}_2] = 0$. Hence, by the pull-backs of $\tilde{g}_1$

and $\tilde{g}_2$, one concludes that $g_1 = \varphi^* \tilde{g}_1$ and $g_2 = \varphi^* \tilde{g}_2$ are analytic at $x = 0$, satisfy $g_1(0) = 0$ and $g_2(0) = 0$, and have Ax and x as linear part, respectively. Moreover, by the invariance of the Lie bracket to diffeomorphisms, $[f, g_1] = 0$ and $[g_1, g_2] = 0$. Conversely, for any g_2 being analytic at $x = 0$, $g_2(0) = 0$, and with linear part x, the Poincaré–Dulac Theorem 3.33 implies the existence of a near-identity diffeomorphism $y = \varphi(x)$ such that in these coordinates $\varphi_* g_2(y) = y$. If $[g_1, g_2] = 0$, then $[\varphi_* g_1(y), \varphi_* g_2(y)] = [\varphi_* g_1(y), y] = 0$; condition $[\varphi_* g_1(y), y] = 0$ implies that $\varphi_* g_1$ is homogeneous of degree 0 with respect to the standard dilation; since $\varphi_* g_1$ is analytic at $y = 0$, it is necessarily linear, $\varphi_* g_1(y) = Ay$. Similarly, if $[f, g_1] = 0$, then $[\varphi_* f(y), \varphi_* g_1(y)] = [\varphi_* f(y), Ay] = 0$; condition $[\varphi_* f(y), Ay] = 0$ implies that $\varphi_* f$ is homogeneous of degree 0 with respect to the Ay; since $\varphi_* f$ is analytic at $y = 0$, $\varphi_* f(0) = 0$, and has Ay as linear part, condition $[\varphi_* f(y), Ay] = 0$ implies that $\varphi_* f$ is necessarily in the Poincaré–Dulac normal form. □

Remark 3.38 Assume that g is analytic at $x = 0$, $g(0) = 0$, and has x as linear part. By the proof of Theorem 3.35, $g = (\frac{\partial \varphi}{\partial x})^{-1}\varphi$. Then, all $f \in \mathscr{C}_C(g)$ being analytic at $x = 0$ are jointly linearized by $y = \varphi(x)$. Assume that g_2 is analytic at $x = 0$, $g_2(0) = 0$, and has x as linear part. By the proof of Theorem 3.36, $g_2 = (\frac{\partial \varphi}{\partial x})^{-1}\varphi$. Let $g_1 \in \mathscr{C}_C(g_2)$ be analytic at $x = 0$, with linear part Ax and A being semi-simple; then, all $f \in \mathscr{C}_C(g_1)$ being analytic at $x = 0$ and having the same linear part Ax as g_1 are jointly transformed in the Poincaré–Dulac normal form by $y = \varphi(x)$.

The next result, valid for scalar systems, can be seen as a corollary of Theorem 3.35, but has a constructive proof.

Corollary 3.2 *Assume that $f \neq 0$. Under the assumptions of Theorem 3.35, if $n = 1$, then $f(x) = \lambda x + k(x)$, with $\lambda = \frac{\partial f(x)}{\partial x}|_{x=0}$ and $k(x)$ denoting second and higher order terms, can be linearized by a near-identity diffeomorphism $y = \varphi(x)$ (i.e., $\varphi_* f(y) = \lambda y$) if and only if $\lambda \neq 0$.*

Proof If f can be linearized by a near-identity diffeomorphism $y = \varphi(x)$, then the push-forward of f takes the form $\varphi_* f(y) = \lambda y$. Since case $\varphi_* f = 0$ has been excluded by the assumption $f \neq 0$, one finds that $\lambda \neq 0$. The sufficiency follows from Theorem 3.35 by taking $g = \frac{1}{\lambda} f$. In this simple case, a more concrete sufficiency proof can be provided. The linearizing diffeomorphism can be recast as $y = x + x\vartheta(x)$, where $\vartheta(0) = 0$. Then, letting $f(x) = \lambda x + x^2 \bar{k}(x)$, $\bar{k}(x)$ analytic at $x = 0$, by $\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y$, one obtains

$$\left(1 + \vartheta(x) + x\frac{\mathrm{d}\vartheta(x)}{\mathrm{d}x}\right)\bigl(\lambda x + x^2 \bar{k}(x)\bigr) = \lambda\bigl(x + x\vartheta(x)\bigr),$$

which can be rewritten as the following Cauchy problem:

$$\frac{\mathrm{d}\vartheta(x)}{\mathrm{d}x} = -\frac{(1 + \vartheta(x))\bar{k}(x)}{\lambda + x\bar{k}(x)}, \quad \vartheta(0) = 0, \tag{3.60}$$

which by well known existence results (see, e.g., the Cauchy–Kovalevskaya Theorem 1.8 at p. 20) has a local solution $\vartheta(x)$ analytic at $x = 0$. □

Example 3.36 Let $f(x) = \sin(x)$. Then, since $\sin(x) = x - \frac{1}{6}x^3 + O(x^4)$, one finds that $\lambda = 1$ and $\bar{k}(x) = \frac{\sin(x)-x}{x^2} = -\frac{1}{6}x + O(x^2)$. Relation (3.60) can be rewritten as

$$\frac{\mathrm{d}\vartheta}{(1+\vartheta)} = \left(-\frac{\bar{k}(x)}{\lambda + x\bar{k}(x)}\right)\mathrm{d}x,$$

namely

$$\frac{\mathrm{d}\vartheta}{(1+\vartheta)} = \left(-\frac{\sin(x) - x}{x\sin(x)}\right)\mathrm{d}x.$$

By integration,

$$\ln\bigl(1+\vartheta(x)\bigr) = \ln\left(c\frac{\sin(x)}{x(1+\cos(x))}\right),$$

where c is a constant, whence one finds that $\vartheta(x) = \frac{c\sin(x)-x-x\cos(x)}{x(1+\cos(x))} = (\frac{1}{2}c - 1) + \frac{1}{24}cx^2 + O(x^3)$; imposing that $\vartheta(0) = 0$, one can fix the value of c, $c = 2$, thus obtaining the linearizing transformation $y = x + x\vartheta(x) = 2\frac{\sin(x)}{1+\cos(x)} = x + \frac{1}{12}x^3 + O(x^4)$, which satisfies $L_f\varphi = \varphi$, with $\varphi(x) = x + x\vartheta(x)$.

Corollary 3.3 *Assume that* $f(x) \in \mathbb{R}^n$ *and* $g(x) \in \mathbb{R}^n$ *are analytic at* $x = 0$, $f(0) = 0$ *and* $g(0) = 0$, *with linear parts* Ax *and* Bx, *respectively*; A *and* B *need not be semi-simple. If there exist two constants* $a, b \in \mathbb{R}$ *such that* $aA + bB = E$ *and* $[f, g] = 0$, *then there exists a near-identity diffeomorphism* $y = \varphi(x)$ *such that* $\varphi_* f(y) = Ay$.

Proof If $[f, g] = 0$, then $[f, \hat{g}] = 0$, with $\hat{g} = af + bg$; since the linear part of $\hat{g}$ is x, the proof of the theorem follows from Theorem 3.35. □

Corollary 3.4 *Let* $n = 2$. *Assume that* $f(x) \in \mathbb{R}^2$ *and* $g(x) \in \mathbb{R}^2$ *are analytic at* $x = 0$, $f(0) = 0$ *and* $g(0) = 0$, *with diagonal linear parts* Ax *and* Bx, *and such that* $[f, g] = 0$. *If* Ax *and* Bx *are not co-linear over* $\mathbb{R}$ *(i.e., if* $\det([Ax\ Bx]) \neq 0$), *then there exist two constants* $a, b \in \mathbb{R}$ *such that* $aA + bB = E$, *whence there exists a near-identity diffeomorphism* $y = \varphi(x)$ *such that* $\varphi_* f(y) = Ay$.

Proof Clearly, from $A = \mathrm{diag}\{A_{1,1}, A_{2,2}\}$ and $B = \mathrm{diag}\{B_{1,1}, B_{2,2}\}$, taking into account that $\det([Ax\ Bx]) = (A_{1,1}B_{2,2} - A_{2,2}B_{1,1})x_1x_2$, one concludes that condition $\det([Ax\ Bx]) \neq 0$ is equivalent to $A_{1,1}B_{2,2} - A_{2,2}B_{1,1} \neq 0$. Now, $aA + bB = \mathrm{diag}\{aA_{1,1} + bB_{1,1}, aA_{2,2} + bB_{2,2}\}$, whence from $aA + bB = E$, one obtains $a = \frac{B_{2,2}-B_{1,1}}{A_{1,1}B_{2,2}-A_{2,2}B_{1,1}}$ and $b = \frac{A_{1,1}-A_{2,2}}{A_{1,1}B_{2,2}-A_{2,2}B_{1,1}}$. □

Example 3.37 Let $f(x) = \begin{bmatrix} c_1 x_1 \\ c_2 x_2 + c_3 x_1^2 \end{bmatrix}$, $g(x) = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, which clearly satisfy $[f, g] = 0$ (by construction, f represents the set of all vector functions being polynomial and homogeneous of degree 0 with respect to g); the linear parts of f and g are Ax and Bx, respectively, where $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. By $aA + bB = a\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} ac_1+b & 0 \\ 0 & ac_2+2b \end{bmatrix}$, imposing that $aA + bB = E$, one obtains $a = \frac{1}{2c_1 - c_2}$, $b = \frac{c_1 - c_2}{2c_1 - c_2}$, under the assumption that $2c_1 - c_2 \neq 0$, which is equivalent to say that term $\begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}$ is not resonant. Then, $\hat{g}(x) = \frac{1}{2c_1 - c_2}\begin{bmatrix} c_1 x_1 \\ c_2 x_2 + c_3 x_1^2 \end{bmatrix} + \frac{c_1 - c_2}{2c_1 - c_2}\begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + \frac{c_3}{2c_1 - c_2} x_1^2 \end{bmatrix}$ is a symmetry of f, analytic at $x = 0$, $\hat{g}(0) = 0$, and with linear part x, which implies, by Corollary 3.3, that f can be linearized; in particular, the linearizing near-identity diffeomorphism is $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} x_1 \\ x_2 - \frac{c_3}{2c_1 - c_2} x_1^2 \end{bmatrix},$$

which satisfies $L_f \varphi = A\varphi$.

The following theorem shows how to compute a diffeomorphism $y = \varphi(x)$, when it exists. Some hints for its applicability are given in [92].

Theorem 3.37 *There exists a diffeomorphism $y = \varphi(x)$, $\varphi(x) : \mathscr{U}^* \to \mathbb{R}^n$, for some open and connected $\mathscr{U}^* \subseteq \mathscr{U}$, with $0 \in \mathscr{U}^*$, such that $g = (\frac{\partial \varphi}{\partial x})^{-1}\varphi$ if and only if there exist $h_1(x), \dots, h_n(x) \in \mathbb{R}^n$ (h_i being analytic on $\mathscr{U}^*$) such that*

$$h_i + [g, h_i] = 0, \quad i = 1, \dots, n, \tag{3.61a}$$

$$[h_i, h_j] = 0, \quad i, j \in \{1, \dots, n\}, \tag{3.61b}$$

$$\det(H) \neq 0, \tag{3.61c}$$

where $H = [h_1 \ \dots \ h_n]$. Furthermore, the rows of $H(0)H^{-1}(x)$ are exact one-forms and $\frac{\partial \varphi(x)}{\partial x} = H(0)H^{-1}(x)$, which means that $y = \varphi(x)$ can be computed by integrating the Jacobian matrix $H(0)H^{-1}(x)$. If (3.61a)–(3.61c) hold on the whole $\mathbb{R}^n$ and the vector functions h_i are complete (i.e., the flow associated with h_i is defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), then the diffeomorphism $y = \varphi(x)$ thus computed is global.

Proof Necessity of (3.61a)–(3.61c): let $\tilde{g}(y) = Ey$ and define $\tilde{h}_i(y) := e_i$, where e_i is the ith column of the $n \times n$ identity matrix E. Then,

$$\tilde{h}_i + [\tilde{g}, \tilde{h}_i] = \tilde{h}_i + L_{\tilde{g}}\tilde{h}_i - L_{\tilde{h}_i}\tilde{g} = e_i - e_i = 0,$$

$$[\tilde{h}_i, \tilde{h}_j] = [e_i, e_j] = 0.$$

Then, $h_i(x) = (\frac{\partial \varphi(x)}{\partial x})^{-1} e_i$, $i = 1, \dots, n$, satisfy (3.61a) and (3.61b), by the invariance of the Lie bracket to diffeomorphisms, and also (3.61c), because with this choice $H(x) = (\frac{\partial \varphi(x)}{\partial x})^{-1}$ and $H(0) = E$.

Sufficiency of (3.61a)–(3.61c): let $K(x) = H(x)H^{-1}(0)$, and let $k_i(x)$ denote the ith column of $K(x)$. It is easy to see that

$$k_i + [g, k_i] = 0, \quad i = 1, \ldots, n, \tag{3.62a}$$

$$[k_i, k_j] = 0, \quad i, j \in \{1, \ldots, n\}, \tag{3.62b}$$

$$K(0) = E. \tag{3.62c}$$

By the Frobenius Theorem 1.9 at p. 21, in view of (3.62b) and (3.62c) the rows of matrix $K^{-1}(x) = H(0)H^{-1}(x)$ are exact one-forms. Let $\frac{\partial\varphi}{\partial x} = K^{-1}(x)$, i.e., define each $\varphi_i(x)$, $i = 1, \ldots, n$, as the integral of the ith row of $K^{-1}(x)$ such that $\varphi_i(0) = 0$. In view of (3.62a)–(3.62c), $\varphi(x)$ is a near-identity diffeomorphism. Since $\tilde{k}_i(y) = (\frac{\partial\varphi}{\partial x}k_i) \circ \varphi^{-1}(y) = e_i$, $i = 1, \ldots, n$, (3.62a) implies that

$$e_i + [\tilde{g}, e_i] = 0, \quad i = 1, \ldots, n,$$

whence, taking into account that $g(0) = 0 \Rightarrow \tilde{g}(0) = 0$, one has that $\tilde{g}(y) = y$. If (3.61a) and (3.61b) hold in the whole $\mathbb{R}^n$, where all the functions involved are assumed to be analytic, $\det(H)(x) \neq 0$ in the whole $\mathbb{R}^n$, and the vector functions h_i are complete (i.e., the flow $\Phi_{h_i}(t, x)$ is defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), for all $i = 1, \ldots, n$, then $y = \varphi(x)$ is a global diffeomorphism, as well as its inverse $x = \varphi^{-1}(y)$. □

Example 3.38 Consider

$$f(x) = \begin{bmatrix} x_2 + x_1^2 \\ x_1 - 2x_1x_2 - 2x_1^3 \\ -4x_1x_2 + 4x_1x_2^2 + 4x_1^3x_2 - 2x_1^3 + x_3 + x_1^2 + x_2^2 \end{bmatrix}.$$

The linear part of f is given by

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$ as eigenvalues; matrix A is semi-simple, but with an infinite number of resonances: therefore, the linearization of f cannot be addressed by the Poincaré–Dulac Theorem 3.33. Nevertheless, f can be actually linearized because it admits the following symmetry:

$$g(x) = \begin{bmatrix} x_1 \\ x_2 - x_1^2 \\ x_3 - x_1^2 - x_2^2 + 2x_1^2x_2 \end{bmatrix}$$

having as linear part

$$B = \left.\frac{\partial g(x)}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 3.37 can be used to linearize $g(x)$. Define

$$h_1(x) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad h_2(x) := \begin{bmatrix} 0 \\ 1 \\ -2x_2 \end{bmatrix}, \qquad h_3(x) := \begin{bmatrix} 1 \\ -2x_1 \\ -2x_1 + 4x_1x_2 \end{bmatrix}.$$

Then, it is easy to check that (3.61a) and (3.61b) hold. Hence, defining

$$H(x) := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2x_1 \\ 1 & -2x_2 & -2x_1 + 4x_1x_2 \end{bmatrix},$$

since condition (3.61c) holds, one can compute the linearizing transformation by integrating the three rows of

$$H(0)H^{-1}(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2x_1 & 2x_2 & 1 \\ 2x_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2x_1 & 1 & 0 \\ 2x_1 & 2x_2 & 1 \end{bmatrix},$$

which define the global diffeomorphism $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + x_1^2 + x_2^2 \end{bmatrix};$$

in this case, it is easy to compute the flows of h_1, h_2 and h_3 and verify that h_1, h_2 and h_3 are complete.

Theorem 3.38 *Assume that $B \in \mathbb{R}^{n\times n}$ is semi-simple and that $h(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $h(0) = 0$. Then,*

$$\big[Bx, \big[Bx, h(x)\big]\big] = 0 \quad \Longleftrightarrow \quad \big[Bx, h(x)\big] = 0.$$

Proof The implication $[Bx, [Bx, h(x)]] = 0 \Longleftarrow [Bx, h(x)] = 0$ is trivial. Consider the implication $[Bx, [Bx, h(x)]] = 0 \Longrightarrow [Bx, h(x)] = 0$. Since B is semi-simple, both operators $[Bx, [Bx, \cdot]]$ and $[Bx, \cdot]$ are linear and semi-simple. By the invariance of the Lie bracket to diffeomorphisms, assume that B is diagonal. Hence, consider each monomial term $x_1^{n_1} \cdots x_n^{n_n} e_k$ constituting $h(x)$, in its formal Taylor series expansion, separately from the others. Now, from

$$\big[Bx, \big[Bx, x_1^{n_1} \cdots x_n^{n_n} e_k\big]\big] = (n_1\lambda_1 + \cdots + n_n\lambda_n - \lambda_k)^2 x_1^{n_1} \cdots x_n^{n_n} e_k,$$
$$\big[Bx, x_1^{n_1} \cdots x_n^{n_n} e_k\big] = (n_1\lambda_1 + \cdots + n_n\lambda_n - \lambda_k) x_1^{n_1} \cdots x_n^{n_n} e_k,$$

it is easy to see that $[Bx, [Bx, x_1^{n_1} \cdots x_n^{n_n} e_k]] = 0$ implies that $n_1\lambda_1 + \cdots + n_n\lambda_n - \lambda_k = 0$, which implies that $[Bx, x_1^{n_1} \cdots x_n^{n_n} e_k] = 0$. □

Next theorem states that if a symmetry g of f is in the Poincaré–Dulac normal form, then the linear part of g is also a symmetry of f. This is a very technical result that is useful in Chap. 7.

Theorem 3.39 *Assume that $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^n$ are analytic at $x = 0$, $f(0) = 0$ and $g(0) = 0$, with linear parts characterized by $A = \frac{\partial f(x)}{\partial x}|_{x=0}$ and $B = \frac{\partial g(x)}{\partial x}|_{x=0}$, $f(x) = Ax + h(x)$ and $g(x) = Bx + k(x)$. Assume that A and B are semi-simple. If $g(x)$ is in the Poincaré–Dulac normal form, $[Bx, k(x)] = 0$, then*

$$\big[f(x), g(x)\big] = 0 \quad \Longrightarrow \quad \big[f(x), Bx\big] = 0.$$

Proof Let $h(x) = \sum_{i=-\infty}^{-1} h^{[i]}(x)$ and $k(x) = \sum_{i=-\infty}^{-1} k^{[i]}(x)$, where $h^{[i]}(x)$ and $k^{[i]}(x)$ are homogeneous of degree i with respect to the standard dilation, i.e., $[h^{[i]}(x), x] = ih^{[i]}(x)$ and $[k^{[i]}(x), x] = ik^{[i]}(x)$. Then, relation $[f(x), g(x)] = 0$ can be rewritten as

$$\begin{aligned} 0 = \big[f(x), g(x)\big] &= \big[Ax + h^{[-1]}(x) + \ldots, Bx + k^{[-1]}(x) + \cdots\big] \\ &= \big([Ax, Bx]\big) + \big(\big[Ax, k^{[-1]}(x)\big] + \big[h^{[-1]}(x), Bx\big]\big) \\ &\quad + \big(\big[Ax, k^{[-2]}(x)\big] + \big[h^{[-1]}(x), k^{[-1]}(x)\big] + \big[h^{[-2]}(x), Bx\big]\big) + \cdots, \end{aligned}$$

where $[Ax, Bx]$ is homogeneous of degree 0 with respect to the standard dilation, $[Ax, k^{[-1]}(x)] + [h^{[-1]}(x), Bx]$ of degree -1, $[Ax, k^{[-2]}(x)] + [h^{[-1]}(x), k^{[-1]}(x)] + [h^{[-2]}(x), Bx]$ of degree -2 and so on. Therefore, condition $0 = [f(x), g(x)]$ is equivalent to conditions

$$[Ax, Bx] = 0, \tag{3.63a}$$

$$\big[Ax, k^{[-1]}(x)\big] = \big[Bx, h^{[-1]}(x)\big], \tag{3.63b}$$

$$\vdots$$

where the dots indicate the countable equations corresponding respectively to brackets of a given degree of homogeneity. From (3.63b), one obtains

$$\big[Bx, \big[Ax, k^{[-1]}(x)\big]\big] = \big[Bx, \big[Bx, h^{[-1]}(x)\big]\big]; \tag{3.64}$$

by the Jacobi identity,

$$\big[Bx, \big[Ax, k^{[-1]}(x)\big]\big] = \big[Ax, \big[Bx, k^{[-1]}(x)\big]\big] + \big[k^{[-1]}(x), [Ax, Bx]\big],$$

from which it can be deduced that $[Bx, [Ax, k^{-1}(x)]] = 0$, because $[Ax, Bx] = 0$ and condition $[Bx, k(x)] = 0$ implies $[Bx, k^{[-1]}(x)] = 0$. Hence, by (3.64),

$$\big[Bx, \big[Bx, h^{[-1]}(x)\big]\big] = 0,$$

which, taking into account that B is semi-simple, implies (by Theorem 3.38) that $[Bx, h^{[-1]}(x)] = 0$. Proceeding in this way, one can show that $[Bx, h^{[i]}(x)] = 0$ for all $i \in \mathbb{Z}^{<}$, whence that $[f(x), Bx] = 0$. □

3.13 Homogeneity and Resonance of Continuous-Time Nonlinear Systems

Aim of this section is to point out the relationship existing between homogeneity and resonance in the continuous-time case.

Let $f^{[m]}$ and g be analytic at $x = 0$, with $f^{[m]}(0)$ that need not be equal to zero. Assume that g is linear, $g(x) = Bx$, with B being real and diagonal, $B = \mathrm{diag}\{\gamma_1, \dots, \gamma_n\}$. Assume that $f^{[m]}$ is homogeneous of degree $m \in \mathbb{Z}$ with respect to g, $[f^{[m]}(x), Bx] = mf^{[m]}(x)$. Since B is semi-simple, operator $[\cdot, Bx]$ is linear and semi-simple too. Therefore, each term $(x_1^{n_1} \cdots x_n^{n_n})e_k$, $n_i \in \mathbb{Z}^{\geq}$, of $f^{[m]}$ satisfies

$$\left[\left(x_1^{n_1} \cdots x_n^{n_n}\right)e_k, Bx\right] = m\left(x_1^{n_1} \cdots x_n^{n_n}\right)e_k; \tag{3.65}$$

taking into account that, if $\hat{f}(x) = (x_1^{n_1} \cdots x_n^{n_n})e_k$, then

$$\begin{aligned} L_{\hat{f}} Bx &= \left(x_1^{n_1} \cdots x_n^{n_n}\right)Be_k = \gamma_k\left(x_1^{n_1} \cdots x_n^{n_n}\right)e_k, \\ L_{Bx}\hat{f}(x) &= (n_1\gamma_1 + \cdots + n_n\gamma_n)\left(x_1^{n_1} \cdots x_n^{n_n}\right)e_k, \end{aligned}$$

condition (3.65) leads to the following *continuous-time generalized resonance condition* (briefly, *generalized resonance condition*):

$$\gamma_k - m = n_1\gamma_1 + \cdots + n_n\gamma_n, \quad n_i \in \mathbb{Z}^{\geq}. \tag{3.66}$$

This means that if $f^{[m]}$ is analytic at $x = 0$ and homogeneous of degree m with respect to g, then it is polynomial and each its term $(x_1^{n_1} \cdots x_n^{n_n})e_k$ satisfies the generalized resonance condition (3.66).

Example 3.39 Consider $B = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, with $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = 2$. Let $m = 2$. Then, the generalized resonance conditions relative to γ_1 and γ_2 (i.e., for $k \in \{1, 2\}$ in (3.66)) are never satisfied; the generalized resonance condition relative to γ_3 (i.e., for $k = 3$) yields

$$\frac{\begin{array}{rll} \gamma_3 - 2 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow \quad x_1^{n_1}x_2^{n_2}x_3^{n_3}e_3 \end{array}}{\begin{array}{rll} 0 & = (0)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow \quad e_3 \end{array}},$$

which is satisfied if and only if $n_1 = n_2 = n_3 = 0$. This means that the term $x_1^0x_2^0x_3^0e_3$ is the only one that can be present is $f^{[2]}$, which is therefore given by

$$f^{[2]}(x) = a_1e_3 = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}.$$

Let $m = 1$. Then, the generalized resonance condition relative to γ_1 yields

$$\begin{array}{rlcr} \gamma_1 - 1 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_1 \\ \hline 0 & = (0)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & e_1 \end{array};$$

the generalized resonance condition relative to γ_2 yields

$$\begin{array}{rlcr} \gamma_2 - 1 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_2 \\ \hline 0 & = (0)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & e_2 \end{array};$$

the generalized resonance condition relative to γ_3 yields

$$\begin{array}{rlcr} \gamma_3 - 1 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_3 \\ \hline 1 & = (1)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_1e_3 \\ 1 & = (0)\gamma_1 + (1)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_2e_3 \end{array};$$

hence,

$$f^{[1]}(x) = a_1e_1 + a_2e_2 + a_3x_1e_3 + a_4x_2e_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3x_1 + a_4x_2 \end{bmatrix}.$$

Let $m = 0$. Then, the generalized resonance condition relative to γ_1 yields

$$\begin{array}{rlcr} \gamma_1 - 0 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_1 \\ \hline 1 & = (1)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_1e_1 \\ 1 & = (0)\gamma_1 + (1)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_2e_1 \end{array};$$

the generalized resonance condition relative to γ_2 yields

$$\begin{array}{rlcr} \gamma_2 - 0 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_2 \\ \hline 1 & = (1)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_1e_2 \\ 1 & = (0)\gamma_1 + (1)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_2e_2 \end{array};$$

the generalized resonance condition relative to γ_3 yields

$$\begin{array}{rlcr} \gamma_2 - 0 & = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 & \Rightarrow & x_1^{n_1}x_2^{n_2}x_3^{n_3}e_3 \\ \hline 2 & = (2)\gamma_1 + (0)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_1^2e_3 \\ 2 & = (1)\gamma_1 + (1)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_1x_2e_3 \\ 2 & = (0)\gamma_1 + (2)\gamma_2 + (0)\gamma_3 & \Rightarrow & x_2^2e_3 \\ 2 & = (0)\gamma_1 + (0)\gamma_2 + (1)\gamma_3 & \Rightarrow & x_3e_3 \end{array};$$

hence,

$$\begin{aligned} f^{[0]}(x) = {} & a_1x_1e_1 + a_2x_2e_1 + a_3x_1e_2 + a_4x_2e_2 + a_5x_1^2e_3 + a_6x_1x_2e_3 \\ & + a_7x_2^2e_3 + a_8x_3e_3 \end{aligned}$$

$$= \begin{bmatrix} a_1x_1 + a_2x_2 \\ a_3x_1 + a_4x_2 \\ a_5x_1^2 + a_6x_1x_2 + a_7x_2^2 + a_8x_3 \end{bmatrix}.$$

3.14 The Belitskii Normal Form of Continuous-Time Nonlinear Systems

The Belitskii normal form is a concept similar to the Poincaré–Dulac normal form that applies also when the linear part of f is not semi-simple [16, 45, 113].

Throughout this section, assume that $f(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $f(0) = 0$. The *linear part* of f is Ax, with $A = \frac{\partial f(x)}{\partial x}|_{x=0}$ that need not be semi-simple. Assume that matrix A can be expressed as $A = A_{s,n} + A_n$, where $A_{s,n} \in \mathbb{R}^{n\times n}$ is normal, $A_n \in \mathbb{R}^{n\times n}$ is nilpotent, and $[A_{s,n}, A_n] = [A_{s,n}, A_n^\top] = 0$ (by Lemma 2.5 at p. 39, this can be obtained for any $A \in \mathbb{R}^{n\times n}$ using a real linear transformation).

Example 3.40 If $A = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$, then $A_{s,n} = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$ and $A_n = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

If

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

then

$$A_{s,n} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A_n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 3.15 Vector function $f(x) = Ax + h(x)$, with $h(x)$ being analytic at $x = 0$, $h(0) = 0$, and having linear part equal to zero, is in the *Belitskii normal form* if

$$\left[h(x), A^\top x\right] = 0. \tag{3.67}$$

By Remark 3.5, under the above positions, f is in the Belitskii normal form if and only if

$$h\left(e^{A^\top t}x\right) = e^{A^\top t}h(x).$$

Given $A \in \mathbb{R}^{n\times n}$, let $\{M_0, \dots, M_{r-1}\}$ be a basis of $\mathscr{L}_c(A^\top)$. All $h \in \mathscr{C}_C(A^\top x)$ are parameterized by $h(x) = \mu_0 M_0 x + \cdots + \mu_{r-1} M_{r-1} x$, where $\mu_0, \dots, \mu_{r-1} \in \mathscr{I}_C(A^\top x)$. Hence, $f(x) = Ax + h(x)$ is in the Belitskii normal form if and only if $h \in \mathscr{C}_C(A^\top x)$, h is analytic at $x = 0$, $h(0) = 0$, with zero linear part.

Remark 3.39 If A is normal, a system in the Belitskii normal form is in the Poincaré–Dulac normal form, and vice versa.

Remark 3.40 By Lemma 2.4 at p. 37, taking $A^\top = A_{s,n}^\top + A_n^\top$, with $A_{s,n}^\top$ being normal, $A_n^\top$ being nilpotent and $[A_{s,n}^\top, A_n^\top] = [A_{s,n}^\top, A_n] = 0$, one concludes that $\mathscr{L}_c(A^\top) = \mathscr{L}_c(A_{s,n}^\top + A_n^\top) = \mathscr{L}_c(A_{s,n}^\top) \cap \mathscr{L}_c(A_n^\top)$; if $A_{s,n}$ is diagonal, then $\mathscr{L}_c(A^\top) = \mathscr{L}_c(A_{s,n}) \cap \mathscr{L}_c(A_n^\top)$. Under the above assumptions, this means that, in order to find all f in the Belitskii normal form and with linear part Ax, one can first find all $f_{s,n}$ being in the Poincaré–Dulac normal form with linear part $A_{s,n}x$, then $f(x) = Ax + f_{s,n}(x)$ is in the Belitskii normal form under the additional requirement that $[f(x), A^\top x] = 0$; such a further requirement generally restricts the set of admissible f.

Example 3.41 Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};$$

such an A is nilpotent. A basis of $\mathscr{L}_c(A^\top)$ is $\{E, A^\top, (A^\top)^2\}$; set $\mathscr{I}_C(A^\top x)$ is constituted by all arbitrary functions of $I_1(x) = x_1$ and $I_2(x) = 2x_1x_3 - x_2^2$. Then, the set of all f being in the Belitskii normal form is parameterized by

$$\begin{aligned} f(x) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left(\mu_0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 \\ x_1 \\ x_2 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ 0 \\ x_1 \end{bmatrix} \right) \\ &= \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_0 x_1 \\ \mu_0 x_2 + \mu_1 x_1 \\ \mu_0 x_3 + \mu_1 x_2 + \mu_2 x_1 \end{bmatrix}, \end{aligned}$$

where μ_0, μ_1, μ_2 are arbitrary functions of I_1, I_2, such that $h(x) = f(x) - Ax$ is analytic at $x = 0$, $h(0) = 0$, with zero linear part.

Example 3.42 Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

then, $A = A_{s,n} + A_n$, where

$$A_{s,n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is normal and

$$A_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent. A basis of $\mathscr{L}_c(A^\top)$ is $\{E, A^\top, (A^\top)^2\}$; set $\mathscr{I}_C(A^\top x)$ is constituted by all arbitrary functions of $I_1(x) = \frac{x_1^2}{x_3}$ and $I_2(x) = -\frac{x_2}{x_1} + \ln(|\frac{x_3}{x_1}|)$. Then, the set of all f having Ax as linear part, in the Belitskii normal form, is parameterized by

$$\begin{aligned} f(x) &= \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 2x_3 \end{bmatrix} + \left(\mu_0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \mu_1 \begin{bmatrix} x_1 \\ x_1 + x_2 \\ 2x_3 \end{bmatrix} + \mu_2 \begin{bmatrix} x_1 \\ 2x_1 + x_2 \\ 4x_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} x_1 + x_2 + \mu_0 x_1 + \mu_1 x_1 + \mu_2 x_1 \\ x_2 + \mu_0 x_2 + \mu_1(x_1 + x_2) + \mu_2(2x_1 + x_2) \\ 2x_3 + \mu_0 x_3 + 2\mu_1 x_3 + 4\mu_2 x_3 \end{bmatrix}, \end{aligned}$$

where μ_0, μ_1 and μ_2 are arbitrary functions of I_1, I_2 such that $h(x) = f(x) - Ax$ is analytic at $x = 0$, $h(0) = 0$, with zero linear part. In particular, $h(x) = f(x) - Ax$ is analytic at $x = 0$, $h(0) = 0$, with zero linear part, if and only if $\mu_0 = aI_1$, $\mu_1 = -2aI_1$ and $\mu_2 = aI_1$, for an arbitrary $a \in \mathbb{R}$,

$$f(x) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 2x_3 + ax_1^2 \end{bmatrix}. \tag{3.68}$$

For deducing the Belitskii normal form associated with the given matrix A, one can use the procedure described in Remark 3.40. All $f_{s,n}$ in the Poincaré–Dulac normal form, with linear part $A_{s,n}x$, are given by $f_{s,n}(x) = A_{s,n}x + h_{s,n}(x)$, where

$$h_{s,n}(x) = \begin{bmatrix} 0 \\ 0 \\ a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 \end{bmatrix}.$$

The three reals a_1, a_2, a_3 are now to be taken so that $[h_{s,n}(x), A^\top x] = 0$; since

$$\left[h_{s,n}(x), A^\top x\right] = \begin{bmatrix} 0 \\ 0 \\ -a_2 x_1^2 - 2a_3 x_1 x_2 \end{bmatrix},$$

it must be $a_2 = a_3 = 0$, with $a_1 \in \mathbb{R}$ being arbitrary; then the resulting $f(x) = Ax + h_{s,n}(x)$ coincides with the f given in (3.68), with $a = a_1$.

The following theorem is taken from [16].

Theorem 3.40 *Assume that $A = A_{s,n} + A_n$, where $A_{s,n} \in \mathbb{R}^{n\times n}$ is normal and $A_n \in \mathbb{R}^{n\times n}$ is nilpotent, and satisfy $[A_{s,n}, A_n] = [A_{s,n}, A_n^\top] = 0$. Given a vector function*

$f(x) = Ax + h(x)$, being C^∞ at $x = 0$ and with $h(x)$ having zero linear part, there exists a diffeomorphism $y = \varphi(x)$ being C^∞ in a neighborhood of the origin of $\mathbb{R}^n$, with $\varphi(0) = 0$ and $\frac{\partial \varphi(x)}{\partial x}|_{x=0} = E$, such that the push-forward of f takes the form $\varphi_* f = \tilde{f}_b + \alpha$, with $\tilde{f}_b$ in the Belitskii normal form and with the vector function α being C^∞ and flat at $y = 0$.

The meaning of the above theorem is that any C^∞-system can be transformed in the Belitskii normal form though a polynomial diffeomorphism, up to a certain order of approximation, which can be arbitrarily fixed.

Under further convergence conditions, the diffeomorphism $y = \varphi(x)$ of Theorem 3.40 is analytic at $x = 0$.

3.15 Nonlinear Transformations of Linear Systems

Let $f(x), g(x) \in \mathbb{R}$; let $x = \Phi_g(\tau, y)$ be the flow associated with g. Expanding in Taylor series with respect to τ, one obtains the following formula known as the *Hadamard Lemma*:

$$\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g = f + \tau[g, f] + \frac{\tau^2}{2!}[g, [g, f]] + \frac{\tau^3}{3!}[g, [g, [g, f]]] + \cdots . \tag{3.69}$$

Example 3.43 Take $f(x) = [x_1 + x_2 \ x_1^2]^\top$ and $g(x) = [x_1 \ -2x_2 + x_1^2]^\top$. The flow associated with g is $\Phi_g(\tau, y) = [\mathrm{e}^\tau y_1 \ \mathrm{e}^{-2\tau} y_2 + (\frac{1}{4}\mathrm{e}^{2\tau} - \frac{1}{4}\mathrm{e}^{-2\tau}) y_1^2]^\top$. Then,

$$\begin{aligned}
&\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g(\tau, y) \\
&= \begin{bmatrix} y_1 + y_2 \mathrm{e}^{-3\tau} + \frac{1}{4}\mathrm{e}^\tau y_1^2 - \frac{1}{4} y_1^2 \mathrm{e}^{-3\tau} \\ \frac{1}{2} y_1^2 \mathrm{e}^{4\tau} - \frac{1}{2} y_1 \mathrm{e}^\tau y_2 - \frac{1}{8} y_1^3 \mathrm{e}^{5\tau} + \frac{1}{4} y_1^3 \mathrm{e}^\tau + \frac{1}{2} y_1^2 + \frac{1}{2}\mathrm{e}^{-3\tau} y_1 y_2 - \frac{1}{8}\mathrm{e}^{-3\tau} y_1^3 \end{bmatrix} \\
&= \begin{bmatrix} y_1 + y_2 \\ y_1^2 \end{bmatrix} + \tau \begin{bmatrix} -3y_2 + y_1^2 \\ -2y_1 y_2 + 2y_1^2 \end{bmatrix} + \frac{\tau^2}{2!} \begin{bmatrix} 9y_2 - 2y_1^2 \\ 4y_1 y_2 + 8y_1^2 - 4y_1^3 \end{bmatrix} + O(\tau^3),
\end{aligned}$$

where

$$\begin{bmatrix} y_1 + y_2 \\ y_1^2 \end{bmatrix} = f(y), \qquad \begin{bmatrix} -3y_2 + y_1^2 \\ -2y_1 y_2 + 2y_1^2 \end{bmatrix} = [g(y), f(y)],$$

$$\begin{bmatrix} 9y_2 - 2y_1^2 \\ 4y_1 y_2 + 8y_1^2 - 4y_1^3 \end{bmatrix} = [g(y), [g(y), f(y)]].$$

Formula (3.69) is particularly useful to understand which nonlinear terms can be generated from a linear system by a near-identity diffeomorphism $x = \Phi_g(\tau, y)$, for some g; note that, given a vector function g analytic at $x = 0$, for $\Phi_g(\tau, y)$ to be near-identity it is necessary and sufficient that the linear part of g is zero: this is assumed hereafter. By Statement (a) of Proposition 6.1 of [57], for any formal near-identity diffeomorphism $y = \varphi(x)$ and for any arbitrary $\tau \in \mathbb{R}^{>}$, there exists a formal $g(x)$ such that $\varphi(x) = \Phi_g(-\tau, x)$; g can be called the *logarithm* of φ (see also [93]). In particular, note that $\Phi_g(\tau, y) \approx y + \tau g(y)$, for a small τ. Therefore, the following reasoning applies to arbitrary near-identity diffeomorphisms.

Assume that $f(x) = Ax$; hence, formula (3.69) can be rewritten as

$$\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} A\Phi_g(\tau, y) = Ay + \tau\big[g(y), Ay\big] + \frac{\tau^2}{2!}\big[g(y), \big[g(y), Ay\big]\big] + \frac{\tau^3}{3!}\big[g(y), \big[g(y), \big[g(y), Ay\big]\big]\big] + \cdots . \tag{3.70}$$

Now, if $g(x)$ is homogeneous of degree m with respect to the standard dilation, $[g(x), x] = mg(x)$, i.e., its entries are of degree $1 - m$, then the vector function appearing in (3.70) multiplied by τ^h is homogeneous of degree hm with respect to the standard dilation, for $h \in \mathbb{Z}^{>}$: $[g(y), Ay]$ has degree m, $[g(y), [g(y), Ay]]$ has degree $2m$, $[g(y), [g(y), [g(y), Ay]]]$ has degree $3m$ and so on.

Example 3.44 Take $A = \text{diag}\{1, 2\}$; then, the only resonant term is $x_1^2 e_2$. Take a vector function g being homogeneous of degree -1 with respect to x,

$$g(x) = \begin{bmatrix} a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 \\ a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 \end{bmatrix}. \tag{3.71}$$

Then, from

$$\big[g(x), Ax\big] = \begin{bmatrix} -a_1x_1^2 - 2a_2x_1x_2 - 3a_3x_2^2 \\ -a_5x_1x_2 - 2a_6x_2^2 \end{bmatrix},$$

which is homogeneous of degree -1 with respect to x, one can easily check that $x_1^2 e_2$ cannot be generated by any near-identity diffeomorphism. This, in particular, implies that the push-forward $\varphi_* f(y)$, with $f(x) = [x_1 \; 2x_2 + x_1^2]^\top$, cannot be equal to Ax, for any near-identity diffeomorphism $y = \varphi(x)$.

Example 3.45 Take $A = \text{diag}\{1, 3\}$; then, the only resonant term is $x_1^3 e_2$. Take the vector function g being homogeneous of degree -1 with respect to x, given in (3.71). Then, from

$$\big[g(x), [g(x), Ax]\big] = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix},$$

where

$$G_1(x) = -4a_2a_4x_1^3 + (2a_1a_2 - 2a_2a_5 - 12a_3a_4)x_1^2x_2 + (8a_1a_3 - 8a_3a_5)x_1x_2^2 + (2a_2a_3 - 4a_3a_6)x_2^3,$$

$$G_2(x) = (4a_4a_1 - 2a_5a_4)x_1^3 + (8a_2a_4 - 8a_6a_4)x_1^2x_2 + (12a_3a_4 + 2a_2a_5 \\ - 2a_5a_6)x_1x_2^2 + 4a_5a_3x_2^3,$$

one can easily see that the resonant term $x_1^3e_2$ appears in the push-forward of Ax if and only if $4a_4a_1 - 2a_5a_4 \neq 0$.

3.16 Invariant Distributions and Dual Semi-Invariants

In the above sections, it has been shown that the concept of semi-invariant associated with f, $L_f\omega = \lambda\omega$, generalizes the concept of left eigenvector of a square matrix $A \in \mathbb{R}^{n\times n}$, $u^\top A = \lambda u^\top$, in the sense that $\omega(x) = u^\top x$ is a semi-invariant of the linear system $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$. Assume that A is semi-simple. Then, there are n left eigenvectors $u_1, \ldots, u_n$ of A being linearly independent over $\mathbb{C}$, $u_i^\top A = \lambda_i u_i^\top$, and matrix $U = [u_1 \ \ldots \ u_n]^\top$ is invertible. Let v_i be the ith column of matrix $V = U^{-1}$; as well known, v_i is a right eigenvector of matrix A, $Av_i = \lambda_i v_i$. The left and right eigenvectors thus defined are *dual*, in the sense that

$$u_i^\top v_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Objective of this section is to give a dual concept of semi-invariant of a nonlinear system, thus generalizing the concept of right eigenvector.

Definition 3.16 A vector function $d(x) \in \mathbb{R}^n$, $d \neq 0$, is a *column semi-invariant* of system (1.1a) if it satisfies

$$[d, f] = \lambda d, \tag{3.72}$$

for some *characteristic function* $\lambda(x) \in \mathbb{R}$.

If $f(x) = Ax$ and $d = v$, with v being a right eigenvector of matrix A, then $[d, f] = \lambda d$, where λ is the eigenvalue associated with v.

A vector function d satisfying (3.72) is nothing else that a vector function having f as orbital symmetry. In particular, let I be a first integral associated with f (i.e., $L_f I = 0$) such that $L_d I \neq 0$; then, according to Theorem 3.6, f has $\hat{d} := \frac{1}{L_d I} d$ as symmetry. Vice versa, if g is any symmetry of f, then $[d, f] = \lambda d$ holds with $d = g$ and $\lambda = 0$.

Furthermore, let $\mathscr{D}$ be the distribution spanned by d, $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{d\}$; then, according to Definition 3.41 of [100] (see also [69]), $\mathscr{D}$ is *f-invariant*, i.e., $[\mathscr{D}, f] \subseteq \mathscr{D}$. By the assumption $d \neq 0$, let x^o be a regular point of $\mathscr{D}$. Then, there exists a diffeomorphism $y = \varphi(x)$, $\varphi(\cdot) : \mathscr{U}^* \to \mathbb{R}^n$, with $\mathscr{U}^*$ being some neighborhood of x^o, such that the push-forward of d takes the form $\varphi_* d = e_1$ and

$[e_1, \varphi_* f] = (\varphi_* \lambda) e_1$. Now, $[e_1, \varphi_* f] = (\varphi_* \lambda) e_1$ is equivalent to the condition that the last $n-1$ entries of $\tilde{f} = \varphi_* f$ do not depend on the first entry y_1 of y,

$$\left.\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tilde{f}_1(y_1, y_2, \ldots, y_n), \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= \tilde{f}_2(y_2, \ldots, y_n), \\ &\vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}t} &= \tilde{f}_n(y_2, \ldots, y_n). \end{aligned}\right\} \tag{3.73}$$

Now, under the further assumption that there exist $y_2^*, \ldots, y_n^*$ such that $\tilde{f}_i(y_2^*, \ldots, y_n^*) = 0$, $i = 2, \ldots, n$, then the set of points $\mathscr{I}_d = \{x \in \mathbb{R}^n : \varphi_i(x) = y_i^*, i = 2, \ldots, n\}$ is invariant. It is worth pointing out that the above assumption is certainly satisfied if x^o is a singular point of f.

Example 3.46 Let $f(x) = [x_1 \; -x_2 + x_1^2]^\top$; it is easy to see that $d_1(x) = [1 \; \frac{2}{3} x_1]^\top$ and $d_2(x) = [0 \; \frac{1}{3}]^\top$ satisfy $[d_1, f] = d_1$ and $[d_2, f] = -d_2$, namely they satisfy condition (3.72) with respective characteristic functions $\lambda_1 = 1$ and $\lambda_2 = -1$. Clearly, $x^o = 0$ is a singular point for f and a regular point for both d_1 and d_2. Consider the diffeomorphism $y = \varphi(x)$, with $\varphi(x) = [x_1 \; 3x_2 - x_1^2]^\top$, $\varphi(0) = 0$, in a neighborhood of $x^o = 0$; since $L_{d_i} \varphi = e_i$, $i = 1, 2$, then $\mathscr{I}_{d_1} = \{x \in \mathbb{R}^2 : \varphi_2(x) = 3x_2 - x_1^2 = 0\}$ and $\mathscr{I}_{d_2} = \{x \in \mathbb{R}^2 : \varphi_1(x) = x_1 = 0\}$ are invariant for the considered system.

Note that, if the first of (3.73) is neglected, the remaining ones constitute a reduction of the given system, according to the terminology introduced in Sect. 3.5. The approach of obtaining a reduction using f-invariant distributions is generalized in Sect. 3.17.

Definition 3.17 Assume that system (1.1a) has n functionally independent semi-invariants ω_i, $L_f \omega_i = \lambda_i \omega_i$, $i = 1, \ldots, n$. Moreover, assume that system (1.1a) has n column semi-invariants d_i, $[d_i, f] = \lambda_i d_i$, being linearly independent over $\mathscr{K}_n$ and having the same respective characteristic function λ_i. If

$$\left(\frac{\partial}{\partial x} \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \right)^{-1} = [d_1 \; \ldots \; d_n], \tag{3.74}$$

then ω_i and d_i, $i = 1, \ldots, n$, are called *dual semi-invariants* of system (1.1a).

If condition (3.74) holds, then, by Remark 1.8 at p. 22, the vector functions d_i are necessarily pairwise commuting, $[d_i, d_j] = 0$, $\forall i, j$, and matrix $[d_1 \; \ldots \; d_n]$ has rank n in some open and connected set $\mathscr{U}^*$.

Theorem 3.41 *Assume the existence of n column semi-invariants $d_i(x) \in \mathbb{R}^n$ such that*

(3.41.1) $[d_i, f] = \lambda_i d_i$, *for some characteristic function* λ_i, $i = 1, \dots, n$;
(3.41.2) $[d_i, d_j] = 0$, $\forall i, j$, *and matrix* $[d_1 \ \dots \ d_n]$ *has rank n in some open and connected set* $\mathscr{U}^*$.

Then, there exist n functionally independent functions ω_i satisfying (3.74) *such that $L_f \omega_i = \gamma_i(\omega_i)$, for some function γ_i of ω_i, $i = 1, \dots, n$. In particular, $\gamma_i(\omega_i) = \lambda_i \omega_i$ if and only if λ_i is constant*; *in such a case*, *ω_i is a semi-invariant of system* (1.1a) *with characteristic value λ_i.*

Proof By conditions (3.41.2) of the theorem, the rows of matrix $[d_1 \ \dots \ d_n]^{-1}$ are exact one-forms and the diffeomorphism $y = \varphi(x)$ given by $y_i = \omega_i$, $i = 1, \dots, n$, with the function ω_i obtained by integrating the ith row of $[d_1 \ \dots \ d_n]^{-1}$, jointly straightens $d_1, \dots, d_n$, which in the y-coordinates become $\varphi_* d_1 = e_1, \dots, \varphi_* d_n = e_n$, with e_i being the ith column of the identity matrix E. By the invariance of the Lie bracket to diffeomorphisms, $[e_i, \varphi_* f] = (\varphi_* \lambda_i) e_i$; hence, letting $\tilde{f} = \varphi_* f$ and $\tilde{\lambda}_i = \varphi_* \lambda_i$, one concludes that

$$\frac{\partial \tilde{f}_i(y)}{\partial y_j} = \begin{cases} \tilde{\lambda}_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

which shows that both $\tilde{\lambda}_i$ and $\tilde{f}_i$ are functions of $y_i = \omega_i$, $\tilde{f}_i(y_i) = \int_0^{y_i} \tilde{\lambda}_i(\theta)\, \mathrm{d}\theta_i + c_i$, for some constant c_i. Since $\frac{\mathrm{d}y_i}{\mathrm{d}t} = \tilde{f}_i$, then $L_f \omega_i = \gamma_i(\omega_i)$ with $\gamma_i(\omega_i) = \tilde{f}_i(\omega_i)$. Clearly, $\int_0^{y_i} \tilde{\lambda}_i(\theta)\, \mathrm{d}\theta_i + c_i$ is a linear function of y_i if and only if $\tilde{\lambda}_i$ is constant and $c_i = 0$. □

Corollary 3.5 *If the assumptions of Theorem* 3.41 *hold and the characteristic functions λ_i are constant*, *then system* (1.1a) *is linear in the local coordinates $y_i = \omega_i$, $i = 1, \dots, n$.*

Example 3.47 Continue Example 3.46. Clearly, d_1 and d_2 are commuting, $[d_1, d_2] = 0$, and therefore the rows of matrix $[d_1(x) \ d_2(x)]^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ -2x_1 & 3 \end{smallmatrix}\right]$ are exact one-forms; then, by integration, one can compute two semi-invariants of system (1.1a): $\omega_1(x) = x_1$ and $\omega_2(x) = -x_1^2 + 3x_2$, with respective characteristic functions $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus, d_1 and d_2 are dual of semi-invariants ω_1 and ω_2, respectively.

Example 3.48 Consider $f(x) = \left[\begin{smallmatrix} -ax_1^2 + 2bx_1x_2 + ax_2^2 \\ -bx_1^2 - 2ax_1x_2 + bx_2^2 \end{smallmatrix}\right]$; hence, $\omega_1(x) = x_1^2 + x_2^2$ and $\omega_2(x) = bx_1 + ax_2$ are two functionally independent semi-invariants associated with f, with respective characteristic functions $\lambda_1(x) = 2(-ax_1 + bx_2)$ and $\lambda_2(x) = 2(-ax_1 + bx_2)$. Hence, ω_1 and ω_2 are two functionally independent semi-invariants associated with the normalized $\bar{f}(x) = \frac{1}{-ax_1 + bx_2} f(x)$, with respective characteristic values $\lambda_1 = 2$ and $\lambda_2 = 2$. By computing d_1 and d_2 as:

$$\begin{bmatrix} d_1(x) \; d_2(x) \end{bmatrix} = \left(\frac{\partial}{\partial x} \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \end{bmatrix} \right)^{-1} = \begin{bmatrix} -\frac{1}{2}\frac{a}{-ax_1+bx_2} & \frac{x_2}{-ax_1+bx_2} \\ \frac{1}{2}\frac{b}{-ax_1+bx_2} & -\frac{x_1}{-ax_1+bx_2} \end{bmatrix},$$

and verifying that $[d_i, \tilde{f}] = 2d_i$, $i = 1, 2$, one concludes that $d_1(x) = \begin{bmatrix} -\frac{1}{2}\frac{a}{-ax_1+bx_2} \\ \frac{1}{2}\frac{b}{-ax_1+bx_2} \end{bmatrix}$ and $d_2(x) = \begin{bmatrix} \frac{x_2}{-ax_1+bx_2} \\ -\frac{x_1}{-ax_1+bx_2} \end{bmatrix}$ are dual of semi-invariants ω_1 and ω_2. In particular, by Corollary 3.5, the system characterized by $\bar{f}$ can be linearized by a change of coordinates, and therefore the system characterized by f can be linearized by a change of coordinates and a state-dependent change of time scale.

3.17 Decomposition of Continuous-Time Nonlinear Systems

The decomposition of nonlinear systems has been widely used in connection with structural properties of nonlinear control systems (see, e.g., [69, 100]).

The following theorem shows that a decomposition of a nonlinear system follows from the existence of invariant distributions.

Theorem 3.42 *Let $\mathscr{D}$ be an involutive distribution, having constant rank m in a neighborhood of a regular point $x = x^o$. Assume that such a distribution is f-invariant, $[\mathscr{D}, f] \subseteq \mathscr{D}$. Then, in a neighborhood of the regular point $x = x^o$, there exists a diffeomorphism $y = \varphi(x)$ such that the nonlinear system* (1.1a) *can be* decomposed, *in the local y-coordinates, as*

$$\frac{\mathrm{d}y_a}{\mathrm{d}t} = \tilde{f}_a(y_a, y_b), \tag{3.75a}$$

$$\frac{\mathrm{d}y_b}{\mathrm{d}t} = \tilde{f}_b(y_b), \tag{3.75b}$$

where $y_a = [y_1 \; \dots \; y_m]^\top$, $y_b = [y_{m+1} \; \dots \; y_n]^\top$ and $\tilde{f}^\top = [\tilde{f}_a^\top \; \tilde{f}_b^\top]$, $\tilde{f} = \varphi_* f$.

Proof By the Frobenius Theorem 1.9 at p. 21, there exist $h_i(x) \in \mathbb{R}^n$, $i = 1, \dots, m$, such that $[h_i, h_j] = 0$, $\mathrm{rank}_{\mathbb{R}}([h_1(x) \; \dots \; h_m(x)]) = m$, for all x in a neighborhood of x^o, and $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{h_1, \dots, h_m\}$. Let $y = \varphi(x)$ be a diffeomorphism that, in a neighborhood of the regular point x^o, jointly straightens $h_1, \dots, h_m$, which become $\varphi_* h_i = e_i$, $i = 1, \dots, m$. Let $\tilde{\mathscr{D}} = \mathrm{span}_{\mathscr{K}_n}\{e_1, \dots, e_m\}$ and $\tilde{f} = \varphi_* f$. Now, since all $\tilde{d} \in \tilde{\mathscr{D}}$ have the form $\tilde{d}(y) = [\tilde{d}_1(y) \; \dots \; \tilde{d}_m(y) \; 0 \; \dots \; 0]^\top$ and condition $[\tilde{\mathscr{D}}, \tilde{f}] \subseteq \tilde{\mathscr{D}}$ implies condition $[e_i, \tilde{f}] \subseteq \tilde{\mathscr{D}}$, $i = 1, \dots, m$, this allows to conclude that the last $n - m$ entries of $\tilde{f}$ do not depend on $y_1, \dots, y_m$, and therefore that the nonlinear system (1.1a) can be *decomposed*, in the local y-coordinates, as in (3.75a), (3.75b). □

A stronger structure of the nonlinear system can be recognized if the vector functions $h_1, \dots, h_m$, in addition to the above assumptions, are column semi-invariants associated with f; as a matter of fact, this implies that $[e_i, \tilde{f}] = \tilde{\lambda}_i e_i$, and therefore that the ith entry of $\tilde{f}$, with $i = 1, \dots, m$, depends only on y_i, y_b instead of y_a, y_b.

Example 3.49 Consider

$$f(x) = \begin{bmatrix} x_1 + x_3 + x_1^3 \\ -x_1^2 + 2x_1x_3 + 2x_1^4 + 3x_2 - x_3^2 - 2x_1^3x_3 - x_1^6 \\ -3x_1^3 - 3x_1^2x_3 - 3x_1^5 + x_3^2 + 2x_1^3x_3 + x_1^6 \end{bmatrix};$$

let $h_1(x) = [1\ 2x_1\ -3x_1^2]^\top$ and $h_2(x) = [0\ 1\ 0]^\top$. Clearly, h_1 and h_2 are two column semi-invariants associated with f, since $[h_1, f] = h_1$, $[h_2, f] = 3h_2$; moreover, since $[h_1, h_2] = 0$ and $\text{rank}_{\mathbb{R}}([h_1(x)\ h_2(x)]) = 2$ on the whole $\mathbb{R}^3$, the distribution $\mathscr{D}$ spanned by h_1, h_2 is regular on the whole $\mathbb{R}^3$, it is involutive and f-invariant. In particular, consider the global diffeomorphism $y = \varphi(x)$, with $\varphi(x) = [x_1\ x_2 - x_1^2\ x_3 + x_1^3]^\top$. Since $L_{h_i}\varphi = e_i$, then in the y-coordinates, the system is decomposed with respect to the given distribution,

$$\tilde{f}(y) = \begin{bmatrix} y_1 + y_3 \\ 3y_2 - y_3^2 \\ y_3^2 \end{bmatrix}. \tag{3.76}$$

The proof of the following theorem is classical (see, e.g., [69, 100]).

Theorem 3.43 *Assume $f(x,t) = f_0(x) + \sum_{i=1}^{p} f_i(x)u_i(t)$, for some functions $u_i(t)$ of t. Let $\mathscr{D}$ be an involutive distribution, having constant rank m in a neighborhood of a regular point $x = x^o$. Assume that such a distribution is f_0-invariant, $[\mathscr{D}, f_0] \subseteq \mathscr{D}$. If $f_i \in \mathscr{D}$, $i = 1, \dots, p$, then, in a neighborhood of x^o, there exists a diffeomorphism $y = \varphi(x)$ such that the nonlinear system* (1.1a) *can be* decomposed, *in the local y-coordinates, as*

$$\frac{\mathrm{d}y_a}{\mathrm{d}t} = \tilde{f}_{0,a}(y_a, y_b) + \sum_{i=1}^{p} \tilde{f}_{i,a}(y_a, y_b)u_i, \tag{3.77a}$$

$$\frac{\mathrm{d}y_b}{\mathrm{d}t} = \tilde{f}_{0,b}(y_b), \tag{3.77b}$$

where $y_a = [y_1\ \dots\ y_m]^\top$, $y_b = [y_{m+1}\ \dots\ y_n]^\top$ and $\tilde{f}_i^\top = [\tilde{f}_{i,a}^\top\ \tilde{f}_{i,b}^\top]$, $\tilde{f}_i = \varphi_* f_i$, $i = 0, 1, \dots, p$.

Example 3.50 Continue Example 3.49. Let f_0 be equal to the f given in Example 3.49. Let $p = 1$ and $f_1(x) = h_1(x) + x_3h_2(x) = [1\ 2x_1 + x_3\ -3x_1^2]^\top$, which belongs by construction to $\mathscr{D}$. Clearly, $\tilde{f}_0$ is equal to the $\tilde{f}$ given in (3.76), whereas

$$\tilde{f}_1(y) = \begin{bmatrix} 1 \\ y_3 - y_1^3 \\ 0 \end{bmatrix},$$

according to (3.77a), (3.77b).

The decomposition (3.77a), (3.77b) clarifies that the functions u_i do not influence in any way the state variables in y_b. Another important decomposition can be obtained when the nonlinear systems is endowed with output variables; for simplicity, just the case of a scalar output is studied here (see [69, 100] for the general case).

Consider now the nonlinear system (1.1a) endowed with an output function

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{3.78a}$$

$$y = h(x), \tag{3.78b}$$

where $h(x) \in \mathbb{R}$ is meromorphic. Consider the directional derivatives of h by f, $L_f^0 h = h$ and $L_f^{i+1} h = L_f(L_f^i h)$. Let index q be such that $L_f^0 h, \ldots, L_f^{q-1} h$ are functionally independent, but $L_f^0 h, \ldots, L_f^q h$ are functionally dependent. Then, there exists a meromorphic function $\Theta(z_1, \ldots, z_{q+1})$ such that $\Theta(L_f^0 h, \ldots, L_f^q h) = 0$ identically. Since $L_f^0 h, \ldots, L_f^{q-1} h$ are functionally independent, it is impossible that $\frac{\partial \Theta(z_1, \ldots, z_{q+1})}{\partial z_{q+1}}$ is identically equal to zero, whence $\Theta(L_f^0 h, \ldots, L_f^q h) = 0$ implies that the identity $L_f^q h = \Xi_1(L_f^0 h, \ldots, L_f^{q-1} h)$ holds locally, for some meromorphic function Ξ_1. This means that

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_q \end{bmatrix} = \begin{bmatrix} L_f^0 h(x) \\ \vdots \\ L_f^{q-1} h(x) \end{bmatrix}$$

qualifies as a partial diffeomorphism such that nonlinear system (3.78a), (3.78b) is transformed into:

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= \xi_2, \\ &\vdots \\ \frac{\mathrm{d}\xi_{q-1}}{\mathrm{d}t} &= \xi_q, \\ \frac{\mathrm{d}\xi_q}{\mathrm{d}t} &= \Xi_1(\xi_1, \ldots, \xi_q), \end{aligned}$$

$$\frac{d\eta}{dt} = \Xi_2(\xi_1, \dots, \xi_q, \eta),$$
$$y = \xi_1,$$

where $\eta \in \mathbb{R}^{n-q}$ are suitable additional state variables that complete the choice of the local variables ξ. Note that, letting $y_a = \eta$ and $y_b = \xi$, the structure (3.75a), (3.75b) is again obtained.

3.18 Symmetries of Algebraic Equations

After having considered symmetries of differential equations, one can deal with the conceptually simpler case of symmetries for a system of algebraic equations:

$$a_i(x) = 0, \quad i = 1, \dots, \nu, \tag{3.79}$$

for $\nu \in \mathbb{Z}^>$ and $a_i(x) \in \mathbb{R}$, $i = 1, \dots, \nu$, being analytic functions on $\mathscr{U}$. Here, the adjective *algebraic* is only used to distinguish this simpler case from the case of systems of differential equations previously considered. Consider the infinitesimal generator $\frac{dx}{d\tau} = g(x)$ of a one-parameter group of transformations $x = \Phi_g(\tau, y)$, where Φ_g is the flow associated with g.

Two possible definitions of symmetry of an algebraic system can be given [102]:

(1) a symmetry transforms any solution of the algebraic system (3.79) into a solution of the same system;
(2) a symmetry transforms the algebraic system (3.79) into the same system.

Such two different concepts are defined formally as follows.

Definition 3.18 A point $x = x^s$, $x^s \in \mathscr{U}$, is a *solution* of the algebraic system (3.79) if $a_i(x^s) = 0$ for $i = 1, \dots, \nu$. The one-parameter group of transformations $x = \Phi_g(\tau, y)$ (briefly, the infinitesimal generator g) is a *symmetry for the solutions of the algebraic system* (3.79) if $x = \Phi_g(\tau, x^s)$ is a solution of (3.79), $a_i(\Phi_g(\tau, x^s)) = 0$, $i = 1, \dots, \nu$, for any admissible $\tau \in \mathbb{R}$ and for any solution $x = x^s$, $x^s \in \mathscr{U}$, of (3.79), whereas it is a *symmetry for the algebraic system* (3.79) if $a_i(\Phi_g(\tau, y)) = a_i(y)$, $\forall y \in \mathscr{U}$, $i = 1, \dots, \nu$, for any admissible $\tau \in \mathbb{R}$.

Remark 3.41 Clearly, if $x = \Phi_g(\tau, y)$ is a symmetry for the algebraic system (3.79), it is necessarily a symmetry for its solutions, whereas the converse need not be true. As an example, consider $g(x) = [x_1 \; 3x_2]^\top$ and compute $\Phi_g(\tau, y) = [e^\tau y_1 \; e^{3\tau} y_2]^\top$. Clearly, $x = \Phi_g(\tau, y)$ is a symmetry for the solutions of the algebraic equation $3x_2 - 2x_1^3 = 0$, whose solutions are parameterized by $x_1 = c$, $x_2 = \frac{2}{3}c^3$, for $c \in \mathbb{R}$, but not for the equation; as a matter of fact, letting $[x_1 \; x_2]^\top = [e^\tau y_1 \; e^{3\tau} y_2]^\top_{y_1=c, y_2=\frac{2}{3}c^3} = [e^\tau c \; \frac{2}{3}e^{3\tau} c^3]^\top$, one has $(3x_2 - 2x_1^3)|_{x_1=e^\tau c, x_2=\frac{2}{3}e^{3\tau}c^3} = 0$, for any $c \in \mathbb{R}$, but it is worth pointing out that $(3x_2 - 2x_1^3)|_{x_1=e^\tau y_1, x_2=e^{3\tau} y_2} = e^{3\tau}(3y_2 - 2y_1^3) \neq 3y_2 - 2y_1^3$

if $\tau \neq 0$. In the same way, it is easy to see that the same $x = \Phi_g(\tau, y)$ is a symmetry for the algebraic equation (and hence for its solutions) $3 - 2\frac{x_1^3}{x_2} = 0$; as a matter of fact, $(3 - 2\frac{x_1^3}{x_2})|_{x_1=e^\tau y_1, x_2=e^{3\tau} y_2} = 3 - 2\frac{y_1^3}{y_2}$ for all $y \in \mathbb{R}^2$, $y_2 \neq 0$.

Theorem 3.44 *The one-parameter group of transformations $x = \Phi_g(\tau, y)$ (briefly, the infinitesimal generator g) is a symmetry for the algebraic system (3.79) if and only if a_i is a first integral associated with g, $a_i \in \mathscr{I}_C(g)$ (i.e., $L_g a_i = 0$), $i = 1, \ldots, \nu$; $x = \Phi_g(\tau, y)$ is a symmetry for the solutions of the algebraic system (3.79) if and only if*

$$L_g a_i|_{a_1=0,\ldots,a_\nu=0} = 0, \quad i = 1, \ldots, \nu; \tag{3.80}$$

in particular, if a_i is a semi-invariant associated with g (i.e., $L_g a_i = \lambda_i a_i$) for $i = 1, \ldots, \nu$, then $x = \Phi_g(\tau, y)$ a symmetry for the solutions of the algebraic system (3.79).

Proof Condition

$$a_i\big(\Phi_g(\tau, y)\big) = a_i(y), \quad \forall y \in \mathscr{U}, i = 1, \ldots, \nu, \tag{3.81}$$

is equivalent to the same condition computed at $\tau = 0$ (which certainly holds since $x = \Phi_g(\tau, y)$ is the identity for $\tau = 0$) and the condition obtained by taking the derivative of both sides of (3.81) by τ:

$$(L_g a_i) \circ \Phi_g(\tau, y) = 0, \quad \forall y \in \mathscr{U},\ i = 1, \ldots, \nu, \tag{3.82}$$

which is clearly satisfied if and only if $L_g a_i = 0$, $i = 1, \ldots, \nu$, i.e., if and only if a_i is a first integral associated with g. The proof of (3.80) is similar. Moreover, if a_i is a semi-invariant associated with g, $L_g a_i = \lambda_i a_i$, then (3.80) holds. □

Example 3.51 Consider again the vector function $g(x) = [x_1\ 3x_2]^\top$ introduced in Remark 3.41. Clearly, $\omega_1(x) = 3x_2 - 2x_1^3$ is a Darboux polynomial associated with g and $I(x) = 3 - 2\frac{x_1^3}{x_2}$ is a first integral associated with g.

3.19 Symmetries and Dimensional Analysis

Dimensional analysis is probably one of the concepts in engineering that have wider applicability, because it is used in many fields such as fluid-dynamics or heat transfer problems (see, e.g., [15, 24, 26] and the references therein), both to prove theorems or to have suggestions on how to describe efficiently some problems by the use of dimensionless quantities. Here, to relate the dimensional analysis to the topics in this book, its application to the very simple case of the oscillations of a pendulum is considered.

Consider a mechanical pendulum constituted by a pendulum blob of mass m, suspended from a frictionless joint by a link of length l, in a gravitational field of acceleration g, without other forces or torques acting on it. Consider a motion of the pendulum of constant period $\frac{2\pi}{\varpi}$, and let θ be the positive angular position of the pendulum when the angular velocity of the pendulum changes the sign. The physical dimensions of these five parameters, which are seen as the entries of a vector x, are

x	x_1	x_2	x_3	x_4	x_5
	m	l	g	ϖ	θ
$[x]$	M	L	LT^{-2}	T^{-1}	1

where M is the *mass* unit, L the *length* unit and T the *time* unit. It is known that the process under study is completely described by the parameters above and that there is some further relationship among them, in the sense that observations (or the knowledge of the problem) indicate that not all of them are functionally independent.

Assume that the three units are changed according to the rules $M \to e^{\tau_1} M$, $L \to e^{\tau_2} L$ and $T \to e^{\tau_3} T$; the physical dimensions of the five parameters are changed accordingly $m \to e^{\tau_1} m$, $l \to e^{\tau_2} l$, $g \to e^{\tau_2} e^{-2\tau_3} g$, $\varpi \to e^{-\tau_3} \varpi$, $\theta \to \theta$, which is a three-parameters group of transformations

$$\Phi(\tau_1, \tau_2, \tau_3, x) = \begin{bmatrix} e^{\tau_1} x_1 \\ e^{\tau_2} x_2 \\ e^{\tau_2} e^{-2\tau_3} x_3 \\ e^{-\tau_3} x_4 \\ x_5 \end{bmatrix}.$$

The three infinitesimal generators g_1, g_2 and g_3 of the group are obtained by the formula $g_i(x) = \frac{\partial \Phi(\tau_1, \tau_2, \tau_3, x)}{\partial \tau_i}\big|_{\tau_1=0, \tau_2=0, \tau_3=0}$,

$$g_1(x) = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix}, \qquad g_3(x) = \begin{bmatrix} 0 \\ 0 \\ -2x_3 \\ -x_4 \\ 0 \end{bmatrix}.$$

Now, by looking for the first integrals that g_1, g_2 and g_3 have in common (by the technique shown in Remark 1.9 at p. 27), one easily obtains $5 - 3 = 2$ dimensionless quantities; in particular, matrix B (which, in this framework, is called the *units matrix*) is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 \end{bmatrix}.$$

The kernel of B is spanned by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and each element of the basis corresponds to a dimensionless quantity

$$I_1(x) = \frac{x_2 x_4^2}{x_3} = \frac{l\varpi^2}{g}, \qquad I_2(x) = x_5 = \theta.$$

Now, by the *Buckingham Pi Theorem* [26], any dimensionless relationship among the five quantities can be expressed as a function of I_1 and I_2, only. Hence, if the given five parameters are related, this has to be through an equation of the form $H(\frac{l\varpi^2}{g}, \theta) = 0$. As an example, this means that

$$\varpi = h(\theta)\sqrt{\frac{g}{l}},$$

where $h(\theta)$ is some function to be determined. Note that, since m does not appear in I_1 and I_2, the mass m of the pendulum blob can be neglected in the above analysis.

3.20 Symmetries of Scalar Ordinary Differential Equations

Systems of ordinary differential equations of first order have been considered in the previous sections. Such an analysis is extended in this section by considering scalar ordinary differential equations of an arbitrary finite order n [20, 22, 67, 111],

$$h\big(t, y, y^{(1)}, \dots, y^{(n)}\big) = 0, \tag{3.83}$$

where $t \in \mathbb{R}$ is the independent variable (the time), $y(t) \in \mathbb{R}$ is dependent variable and $y^{(i)}(t) = \frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i}$, $i = 1, \dots, n$.

The meaning of the following technical assumption will be clarified later.

Assumption 3.1 The partial derivatives $\frac{\partial h}{\partial t}, \frac{\partial h}{\partial y}, \dots, \frac{\partial h}{\partial y^{(n)}}$ of h, considered as functions of $t, y, \dots, y^{(n)}$, are not all identically equal to zero when $h = 0$.

The analysis of the previous sections is widened by considering the above scalar equation in the sense that h may depend on time t, equation $h = 0$ may be an implicit function of $y^{(n)}$, and the order n of the equation may be greater than 1. Note that the case of a scalar ordinary differential equation can always be reduced to the case of a system of ordinary differential equations, $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, $x \in \mathbb{R}^n$, when the scalar equation $h = 0$ can be rendered explicit with respect to $y^{(n)}$ and h does not depend explicitly on time t, by taking $x = [y\ y^{(1)}\ \dots\ y^{(n-1)}]^\top$ as state vector. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(t, y) = \begin{bmatrix} \tau(t, y) \\ \theta(t, y) \end{bmatrix},$$

where $\tau(t, y), \theta(t, y) \in \mathbb{R}$. Clearly, the flow Φ_g associated with g,

$$\begin{bmatrix} t \\ y \end{bmatrix} = \Phi_g\left(\varepsilon, \begin{bmatrix} \tilde{t} \\ \tilde{y} \end{bmatrix}\right), \quad \varepsilon \in \mathbb{R},$$

qualifies as a one-parameter group of transformations, which can be rewritten as:

$$\begin{aligned} t &= t\big(\varepsilon, \tilde{t}, \tilde{y}\big) = \tilde{t} + \varepsilon\tau\big(\tilde{t}, \tilde{y}\big) + O\big(\varepsilon^2\big), \\ y &= y\big(\varepsilon, \tilde{t}, \tilde{y}\big) = \tilde{y} + \varepsilon\theta\big(\tilde{t}, \tilde{y}\big) + O\big(\varepsilon^2\big), \end{aligned}$$

where $O(\varepsilon^2)$ denotes second and higher order terms with respect to ε. Let $\widetilde{y^{(i)}}(\tilde{t}) = \frac{\mathrm{d}^i \tilde{y}(\tilde{t})}{\mathrm{d}\tilde{t}^i}$, $i = 1, \dots, n$. Clearly, such a transformation on t, y yields an induced transformation on the derivatives of y, $(t, y, \dots, y^{(i)}(t)) \to \widetilde{y^{(i)}}$, $i = 1, \dots, n$. It is possible to show [102] that the whole transformation,

$$\begin{bmatrix} t \\ y \\ y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix} \to \begin{bmatrix} \tilde{t} \\ \tilde{y} \\ \widetilde{y^{(1)}} \\ \vdots \\ \widetilde{y^{(n)}} \end{bmatrix}, \tag{3.84}$$

is again a one-parameter group of transformation, whose infinitesimal generator,

$$g_e\left(\begin{bmatrix} t \\ y \\ y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix}\right) = \begin{bmatrix} \tau \\ \theta \\ \theta^{[1]} \\ \vdots \\ \theta^{[n]} \end{bmatrix}, \tag{3.85}$$

with $\theta^{[i]}(t, y, y^{(1)}, \dots, y^{(i)}) \in \mathbb{R}$, $i = 1, \dots, n$, can be computed as follows. By definition,

$$\begin{aligned} \widetilde{y^{(1)}} &= \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}(\tilde{y} + \varepsilon\theta + O(\varepsilon^2))}{\mathrm{d}(\tilde{t} + \varepsilon\tau + O(\varepsilon^2))} = \frac{\mathrm{d}\tilde{y} + \varepsilon\,\mathrm{d}\theta + O(\varepsilon^2)}{\mathrm{d}\tilde{t} + \varepsilon\,\mathrm{d}\tau + O(\varepsilon^2)} \\ &= \frac{\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} + \varepsilon\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{t}} + O(\varepsilon^2)}{1 + \varepsilon\frac{\mathrm{d}\tau}{\mathrm{d}\tilde{t}} + O(\varepsilon^2)}; \end{aligned}$$

hence, taking into account that (3.84) becomes the identity transformation when $\varepsilon = 0$, the infinitesimal generator $\theta^{[1]}$ is given by:

$$\begin{aligned}\theta^{[1]} &= \left.\frac{\mathrm{d}\widetilde{y^{(1)}}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\ &= \left.\frac{(\frac{\mathrm{d}\theta}{\mathrm{d}t} + O(\varepsilon))(1 + \varepsilon\frac{\mathrm{d}\tau}{\mathrm{d}t} + O(\varepsilon^2)) - (\frac{\mathrm{d}y}{\mathrm{d}t} + \varepsilon\frac{\mathrm{d}\theta}{\mathrm{d}t} + O(\varepsilon^2))(\frac{\mathrm{d}\tau}{\mathrm{d}t} + O(\varepsilon^2))}{(1 + \varepsilon\frac{\mathrm{d}\tau}{\mathrm{d}t} + O(\varepsilon^2))^2}\right|_{\varepsilon=0},\end{aligned}$$

namely

$$\theta^{[1]} = \frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}\tau}{\mathrm{d}t} y^{(1)}, \tag{3.86a}$$

where it is stressed that $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the total derivative with respect to t,

$$\theta^{[1]} = \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial y} y^{(1)} - \left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial y} y^{(1)}\right) y^{(1)}.$$

Similarly, one has

$$\theta^{[i+1]} = \frac{\mathrm{d}\theta^{[i]}}{\mathrm{d}t} - \frac{\mathrm{d}\tau}{\mathrm{d}t} y^{(i+1)}, \quad i = 1, \dots, n-1. \tag{3.86b}$$

Example 3.52 Let $\tau = at$ and $\theta = by$. Hence,

$$\begin{aligned}\theta^{[1]} &= \frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}\tau}{\mathrm{d}t} y^{(1)} = (b-a) y^{(1)}, \\ \theta^{[2]} &= \frac{\mathrm{d}\theta^{[1]}}{\mathrm{d}t} - \frac{\mathrm{d}\tau}{\mathrm{d}t} y^{(2)} = (b-2a) y^{(2)},\end{aligned}$$

and, by induction, one has $\theta^{[i]} = (b - ia) y^{(i)}$, $i = 1, \dots, n$.

Substituting (3.84) into the left-hand side of (3.83), one obtains the following function of $\tilde{t}, \tilde{y}, \widetilde{y^{(1)}}, \dots, \widetilde{y^{(n)}}$:

$$h\Big(t\big(\varepsilon, \tilde{t}, \tilde{y}\big), y\big(\varepsilon, \tilde{t}, \tilde{y}\big), y^{(1)}\Big(\varepsilon, \tilde{t}, \tilde{y}, \widetilde{y^{(1)}}\Big), \dots, y^{(n)}\Big(\varepsilon, \tilde{t}, \tilde{y}, \widetilde{y^{(1)}}, \dots, \widetilde{y^{(n)}}\Big)\Big). \tag{3.87}$$

Definition 3.19 Vector function $g = [\tau \ \theta]^\top$ is a symmetry of the scalar ordinary differential equation (3.83) if

$$\begin{aligned}&h\Big(t\big(\varepsilon, \tilde{t}, \tilde{y}\big), y\big(\varepsilon, \tilde{t}, \tilde{y}\big), y^{(1)}\Big(\varepsilon, \tilde{t}, \tilde{y}, \widetilde{y^{(1)}}\Big), \dots, y^{(n)}\Big(\varepsilon, \tilde{t}, \tilde{y}, \widetilde{y^{(1)}}, \dots, \widetilde{y^{(n)}}\Big)\Big) \\ &\quad = h\Big(\tilde{t}, \tilde{y}, \widetilde{y^{(1)}}, \dots, \widetilde{y^{(n)}}\Big), \quad \forall \tilde{t}, \tilde{y}, \widetilde{y^{(1)}}, \dots, \widetilde{y^{(n)}} \in \mathbb{R},\end{aligned} \tag{3.88}$$

for any $\varepsilon \in \mathbb{R}$ for which both sides of the above equation are defined.

Clearly, (3.88) holds for $\varepsilon = 0$. Compute the derivative of (3.88) with respect to ε and then let $\varepsilon = 0$ in the result,

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial y^{(1)}}\theta^{[1]} + \cdots + \frac{\partial h}{\partial y^{(n)}}\theta^{[n]} = 0, \tag{3.89}$$

where $\theta^{[k]}$ is computed iteratively by (3.86a), (3.86b).

Hence, $g = [\tau\ \theta]^\top$ is a symmetry of (3.83) only if (3.89) holds. Actually, under Assumption 3.1, it is possible to show [111] that $g = [\tau\ \theta]^\top$ is a symmetry of (3.83) if and only if (3.89) holds, where it is worth pointing out that (3.89) must hold, modulo the equality $h = 0$, which constrains $t, y, y^{(1)}, \ldots, y^{(n)}$ all together. The sufficiency can be proven easily as in [111], by taking into account that equality (3.89) is invariant with respect to diffeomorphisms. By the flow box Theorem 3.3, apart from a diffeomorphism about any regular point of g, assume that $\tau = 1$ and $\theta = 0$, which implies $\theta^{[i]} = 0$ for any $i = 1, \ldots, n$. Hence, equality (3.89) becomes

$$\frac{\partial h}{\partial t} = 0, \tag{3.90}$$

which shows that h does not depend explicitly on time t. The one-parameter group of transformation is

$$t = \tilde{t} + \varepsilon, \qquad y = \tilde{y}, \qquad y^{(i)} = \widetilde{y^{(i)}}, \quad i = 1, \ldots, n,$$

which substituted into an equation $h = 0$ independent of t yields exactly the same equation; this shows that if (3.90) holds, then $g = [1\ 0]^\top$ is a symmetry of $h = 0$, which by the invariance to diffeomorphisms proves the sufficiency of (3.89).

By Assumption 3.1, equations like $(y^{(1)} - ty)^2 = 0$ are ruled out; for the above equation, $\frac{\partial h}{\partial t} = -2(y^{(1)} - ty)y$, $\frac{\partial h}{\partial y} = -2(y^{(1)} - ty)t$ and $\frac{\partial h}{\partial y^{(1)}} = 2(y^{(1)} - ty)$, which are all equal to zero when $(y^{(1)} - ty)^2 = 0$, although the scalar ordinary differential equation depends on time t. This shows that Assumption 3.1 means that condition $\frac{\partial h}{\partial t} = 0$ implies that h is independent of t.

Remark 3.42 Consider $h(t, y, \dot{y}) = \dot{y} - f(t, y)$, where $\dot{y} = y^{(1)}$. Let $\theta = \bar{g}(t, y)$ and $\theta^{[1]} = \frac{d\bar{g}}{dt} - \frac{d\tau}{dt}\dot{y}$. Hence,

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} = -\frac{\partial f}{\partial t}\tau - \frac{\partial f}{\partial y}\bar{g} + \frac{\partial \bar{g}}{\partial t} + \frac{\partial \bar{g}}{\partial y}\dot{y} - \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial y}\dot{y}\right)\dot{y}.$$

Substituting $\dot{y} = f(t, y)$, one obtains

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} = -\frac{\partial f}{\partial t}\tau - \frac{\partial f}{\partial y}\bar{g} + \frac{\partial \bar{g}}{\partial t} + \frac{\partial \bar{g}}{\partial y}f - \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial y}f\right)f.$$

For instance, if $\tau = 0$, then one has

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} = \frac{\partial \bar{g}}{\partial t} + [f, \bar{g}],$$

namely, in such a case, $g = [0\ \bar{g}]^\top$ is a symmetry of h if and only if

$$\frac{\partial \bar{g}}{\partial t} = -[f, \bar{g}].$$

Note that, when $\bar{g}$ does not depend on t, the concept of symmetry of the previous sections is recovered. Let $\tau = at$ and $\theta = by$, which yield $\theta^{[1]} = (b-a)\dot{y}$. Hence,

$$\begin{aligned}\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} &= -\frac{\partial f}{\partial t}at - \frac{\partial f}{\partial y}by + (b-a)\dot{y}\\ &= -\frac{\partial f}{\partial t}at - \frac{\partial f}{\partial y}by + (b-a)f.\end{aligned}$$

Letting $\bar{g} = [at\ by]^\top$, condition $\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} = 0$ is equivalent to $L_{\bar{g}} f = (b-a)f$, i.e., $\bar{g}$ is a symmetry of the equation $\dot{y} = f(t, y)$ if and only if f is a homogeneous function of t, y of degree $b-a$ with respect to $\bar{g}$. For instance, letting $b - a = k$ and $\bar{g}_e = [at\ by\ kf]^\top$, two functionally independent first integrals associated with $\bar{g}_e$ are $I_1 = y^a t^{-b}$ and $I_2 = f y^{-k/b}$, whence, by Theorem 3.16, all scalar ordinary differential equations $\dot{y} = f(t, y)$ having $\bar{g}$ as symmetry are characterized by $f(t, y) = y^{k/b} C(y^a t^{-b})$, where C is an arbitrary function of the argument.

Example 3.53 Consider the equation $h = 0$, with $h(t, y, \dot{y}, \ddot{y}) = \ddot{y} - t^h y^k$, $h, k \in \mathbb{Z}$, $h, k \geq 1$, where $\dot{y} = y^{(1)}$ and $\ddot{y} = y^{(2)}$. Take $\tau = at$ and $\theta = by$, which yields $\theta^{[1]} = (b-a)\dot{y}$ and $\theta^{[2]} = (b-2a)\ddot{y}$. Hence,

$$\begin{aligned}\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} + \frac{\partial h}{\partial \ddot{y}}\theta^{[2]} &= -ht^{h-1}y^k at - kt^h y^{k-1} by + (b-2a)\ddot{y}\\ &= -(ha+bk)t^h y^k + (b-2a)\ddot{y}.\end{aligned}$$

Substituting $\ddot{y} = t^h y^k$, one has

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} + \frac{\partial h}{\partial \ddot{y}}\theta^{[2]} = -\big((h+2)a + (k-1)b\big)t^h y^k,$$

which is identically equal to zero if and only if $a = -(k-1)c$ and $b = (h+2)c$, for an arbitrary $c \in \mathbb{R}$.

Example 3.54 Consider the equation $h = 0$, with $h(y, y^{(1)}, y^{(2)}, y^{(3)}) = 2y^{(1)}y^{(3)} - 3(y^{(2)})^2$. Take $\tau = at$ and $\theta = by$, which yields $\theta^{[i]} = (b - ia)y^{(i)}$, $i = 1, 2, 3$. Hence, one has the relation

$$2y^{(3)}(b-a)y^{(1)} + 2y^{(1)}(b-3a)y^{(3)} - 6y^{(2)}(b-2a)y^{(2)} = 0,$$

from which

$$2(b-2a)\Big(2y^{(1)}y^{(3)} - 3\big(y^{(2)}\big)^2\Big) = 0.$$

Condition $h = 0$ implies that the above equation holds for any $a, b \in \mathbb{R}$.

About any regular point of g, assume that $g = [0\ 1]^\top$, namely that $\tau = 0$ and $\theta = 1$, which implies that $\theta^{[i]} = 0$, $i \in \mathbb{Z}^>$. By condition

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \sum_{i=1}^{n} \frac{\partial h}{\partial y^{(i)}}\theta^{[i]} = \frac{\partial h}{\partial y} = 0,$$

one concludes that g is a symmetry of the equation $h = 0$ if and only if h does not depend on y, $h(t, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)})$; hence, by defining $z := y^{(1)}$, the equation of reduced order $h(t, z, z^{(1)}, \ldots, z^{(n-1)}) = 0$ is obtained.

Example 3.55 Consider the equation $h = 0$, with $h(t, y, \dot{y}, \ddot{y}) = \ddot{y} + \alpha_1(t)\dot{y} + \alpha_0(t)y$, where $\dot{y} = y^{(1)}$ and $\ddot{y} = y^{(2)}$ and $\alpha_0(t), \alpha_1(t) \in \mathbb{R}$. Let $\tau = 0$ and $\theta = by$, which implies $\theta^{[1]} = b\dot{y}$ and $\theta^{[2]} = b\ddot{y}$. Hence,

$$\frac{\partial h}{\partial t}\tau + \frac{\partial h}{\partial y}\theta + \frac{\partial h}{\partial \dot{y}}\theta^{[1]} + \frac{\partial h}{\partial \ddot{y}}\theta^{[2]} = b(\alpha_0 y + \alpha_1 \dot{y} + \ddot{y}) = 0.$$

Hence, $g = [0\ by]^\top$ is a symmetry of the equation $h = 0$ for any $b \in \mathbb{R}$. For the sake of simplicity, let $b = 1$. Let $\tilde{t} = t$ and $\tilde{y} = \ln(|y|)$ be a diffeomorphism straightening g (i.e., $L_g\tilde{t} = 0$ and $L_g\tilde{y} = 1$). Hence,

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}, \qquad \frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{t}^2} = -\frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{1}{y}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2},$$

which can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}}, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = y\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{t}^2} + y\left(\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}}\right)^2;$$

by substituting the above expressions into the equation $h = 0$, one obtains

$$y\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{t}^2} + y\left(\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}}\right)^2 + \alpha_1 y\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} + \alpha_0 y = 0,$$

namely (if $y \neq 0$) the *Riccati differential equation*

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{t}} + \tilde{z}^2 + \alpha_1\tilde{z} + \alpha_0 = 0,$$

where $\tilde{z} = \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}}$.

Chapter 4
Analysis of Discrete-Time Nonlinear Systems

4.1 Semi-invariants and Darboux Polynomials of Discrete-Time Nonlinear Systems

In this section, the results of Sect. 3 are extended to the discrete-time case [93].

Definition 4.1 A *semi-invariant* of system (1.1b) is a meromorphic scalar function $\omega(x) \in \mathbb{R}$ such that

$$\omega \circ F = \lambda \omega,$$

with $\lambda(x) \in \mathbb{R}$ being meromorphic and such that there is no zero/pole cancelation between λ and ω; if ω and λ are polynomial, then ω is said to be a *Darboux polynomial*; λ is called the *characteristic function* (respectively, the *characteristic polynomial*) of the semi-invariant (respectively, of the Darboux polynomial). If λ is constant, then it is called the *characteristic value*.

A semi-invariant (respectively, a Darboux polynomial) of system (1.1b) is also called a DT-semi-invariant (respectively, a DT-Darboux polynomial) associated with F. If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *semi-invariant* is used instead of DT-semi-invariant.

Clearly, if not empty, set $\mathscr{I}_\omega = \{x \in \mathscr{U} : \omega(x) = 0\}$ is invariant, i.e., if $x(0) \in \mathscr{I}_\omega$, then $x(t) \in \mathscr{I}_\omega$ for all $t \in \mathbb{Z}$, $t \geq 0$, possibly close to 0; as a matter of fact, letting $\bar{\omega}(t) = \omega(x(t))$ and $\bar{\lambda}(t) = \lambda(x(t))$, if $\bar{\omega}(t) = 0$, then $\bar{\omega}(t+1) = \bar{\lambda}(t)\bar{\omega}(t) = 0$ (clearly, if $\bar{\lambda}(t) \neq 0$, then $\bar{\omega}(t+1) = 0$ implies $\bar{\omega}(t) = 0$). From Definition 4.1, a first integral associated with F is a semi-invariant associated with F, with $\lambda = 1$.

For simplicity, the following theorem considers the Darboux polynomials associated with F, although some of such properties hold for semi-invariants too, subject to some amendments.

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Theorem 4.1 *Assume that F is polynomial.*

(4.1.1) *If $I = \frac{\omega_1}{\omega_2}$ is a first integral of system* (1.1b), *with ω_1 and ω_2 being co-prime polynomials, then ω_1 and ω_2 are Darboux polynomials of system* (1.1b), *with the same characteristic polynomial $\lambda_1 = \lambda_2$.*

(4.1.2) *Let ω_1 and ω_2 be Darboux polynomials of system* (1.1b) *with respective characteristic polynomials λ_1 and λ_2; then, the product $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system* (1.1b) *for any pair $n_1, n_2 \in \mathbb{Z}^{\geq}$, with characteristic polynomial $\lambda_1^{n_1}\lambda_2^{n_2}$.*

Proof First, consider Statement (4.1.1) of the theorem. Since I is a first integral of system (1.1b), it follows that $I \circ F = \frac{\omega_1 \circ F}{\omega_2 \circ F} = \frac{\omega_1}{\omega_2}$, which implies $(\omega_1 \circ F)\omega_2 = (\omega_2 \circ F)\omega_1$; this last equality shows, taking into account that ω_1 and ω_2 are co-prime, that ω_1 is a factor of $\omega_1 \circ F$ and ω_2 is a factor of $\omega_2 \circ F$, with $\lambda_1 = \frac{\omega_1 \circ F}{\omega_1}$ and $\lambda_2 = \frac{\omega_2 \circ F}{\omega_2}$ being the respective characteristic polynomials; substituting these expressions in $(\omega_1 \circ F)\omega_2 = (\omega_2 \circ F)\omega_1$, one finds that $\omega_1\omega_2(\lambda_1 - \lambda_2) = 0$, which shows that $(\lambda_1 - \lambda_2) = 0$, because $\omega_1\omega_2$ is not identically equal to zero. As for statement (4.1.2) of the theorem, the computations

$$\left(\omega_1^{n_1}\omega_2^{n_2}\right) \circ F = (\omega_1 \circ F)^{n_1}(\omega_2 \circ F)^{n_2} = (\lambda_1\omega_1)^{n_1}(\lambda_2\omega_2)^{n_2} = \left(\lambda_1^{n_1}\lambda_2^{n_2}\right)\left(\omega_1^{n_1}\omega_2^{n_2}\right),$$

show that $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system (1.1b). □

Remark 4.1 To compare Theorem 4.1 with the similar Theorem 3.1 at p. 56 that holds in the continuous-time case, recall that if $\omega = \omega_1\omega_2$ is a Darboux polynomial of system (1.1a), with ω_1 and ω_2 being polynomials, one concludes that its factors ω_1 and ω_2 are certainly Darboux polynomials associated with f; the same need not hold in the discrete-time case. As an illustrative example, let $F(x) = [x_2 \; x_3 \; 0]^\top$; clearly, $\omega(x) = x_3 p(x)$ is a Darboux polynomial associated with F, with characteristic value $\lambda = 0$, for any polynomial $p(x)$, as well as its factor $\omega_1(x) = x_3$,

$$\omega \circ F = \left(F_3 p(F)\right)\big|_{F_1=x_2, F_2=x_3, F_3=0} = 0,$$

but the other factor $p(x)$, being an arbitrary polynomial, is not, in general, a Darboux polynomial associated with F.

4.2 A "Computational" Result for the Darboux Polynomials of Discrete-Time Nonlinear Systems

For the sake of simplicity, assume that F is polynomial, and consider its Darboux polynomials; note that the algorithm proposed in this section can be adapted to cover the computation of semi-invariants associated with F, when F is not polynomial, as shown in the subsequent Examples 4.2, 4.3 and 4.4.

Assume that ω is a Darboux polynomial associated with F, with characteristic polynomial λ, i.e., $\omega \circ F = \lambda\omega$. Assume, in addition, that ω is a linear combination with real and constant coefficients c_i of some functionally independent polynomials $p_1, p_2, \ldots, p_k$, for some $k > 0$, $\omega = \sum_{i=1}^{k} c_i p_i$. Consider the square $k \times k$ matrix

$$\Gamma = \begin{bmatrix} p_1 & p_2 & \ldots & p_k \\ \Delta p_1 & \Delta p_2 & \ldots & \Delta p_k \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{k-1} p_1 & \Delta^{k-1} p_2 & \ldots & \Delta^{k-1} p_k \end{bmatrix}, \tag{4.1}$$

where $\Delta p_j = p_j \circ F$, $\Delta^2 p_j = p_j \circ F \circ F$ and so on.

Theorem 4.2 [93] *Under the above positions, if* $\det(\Gamma) \neq 0$, *then* ω *is a factor of* $\det(\Gamma)$.

Proof Assume $\omega = \sum_{i=1}^{k} c_i p_i$, for $c_i \in \mathbb{R}$; with no loss of generality, apart from a reordering of polynomials p_i, assume that $c_k \neq 0$. First, note that if ω is a Darboux polynomial associated with F, with characteristic polynomial λ, i.e., $\Delta\omega = \lambda\omega$, then for any $i \in \mathbb{Z}^{>}$, $\Delta^i \omega = \lambda_i \omega$, for some polynomial λ_i, with $\lambda_1 = \lambda$. This fact can be proven as follows:

$$\begin{aligned} \Delta\omega &= \lambda\omega = \lambda_1\omega, & \lambda_1 &:= \lambda, \\ \Delta^2\omega &= (\Delta\lambda_1)(\Delta\omega) = (\Delta\lambda_1)\lambda_1\omega = \lambda_2\omega, & \lambda_2 &:= (\Delta\lambda_1)\lambda_1, \\ &\vdots & &\vdots \\ \Delta^{k-1}\omega &= \lambda_{k-1}\omega, & \lambda_{k-1} &:= (\Delta\lambda_{k-2})\lambda_{k-2}; \end{aligned}$$

note that if λ is constant, $\Delta\lambda = \lambda$, whence $\lambda_i = \lambda^i$; in particular, if $\lambda = 0$, then $\lambda_i = 0$, $i = 1, \ldots, k-1$. Since $\omega = \sum_{i=1}^{k} c_i p_i$, it follows that $\Delta^j \omega = \sum_{i=1}^{k} c_i \Delta^j p_i$, $j = 0, \ldots, k-1$. For this reason,

$$\Gamma \cdot \begin{bmatrix} 1 & 0 & \ldots & 0 & c_1 \\ 0 & 1 & \ldots & 0 & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & c_{k-1} \\ 0 & 0 & \ldots & 0 & c_k \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & p_2 & \ldots & \sum_{i=1}^{k} c_i p_i \\ \Delta p_1 & \Delta p_2 & \ldots & \sum_{i=1}^{k} c_i \Delta p_i \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{k-1} p_1 & \Delta^{k-1} p_2 & \ldots & \sum_{i=1}^{k} c_i \Delta^{k-1} p_i \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & p_2 & \dots & \omega \\ \Delta p_1 & \Delta p_2 & \dots & \Delta\omega \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{k-1} p_1 & \Delta^{k-1} p_2 & \dots & \Delta^{k-1}\omega \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & p_2 & \dots & \omega \\ \Delta p_1 & \Delta p_2 & \dots & \lambda_1\omega \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{k-1} p_1 & \Delta^{k-1} p_2 & \dots & \lambda_{k-1}\omega \end{bmatrix} = \hat{\Gamma},$$

whence $\det(\Gamma) = \frac{1}{c_k}\det(\hat{\Gamma})$, from which the theorem follows. □

Remark 4.2 When $\det(\Gamma) \neq 0$, Theorem 4.2 guarantees that if a Darboux polynomial ω, associated with F, is a linear combination with constant coefficients of $p_1, \dots, p_k$, then ω is a factor of $\det(\Gamma)$. But in the application of the theorem, all factors of $\det(\Gamma)$ or of the determinants of its minors, not only those that are linear combinations of $p_1, \dots, p_k$, are good candidates to be Darboux polynomials associated with F, because Γ could be a minor of another matrix $\check{\Gamma}$ found with an enlarged choice of the polynomials $p_1, \dots, p_{\check{k}}$.

Remark 4.3 When $\det(\Gamma) = 0$, Theorem 4.2 cannot be applied: in such a case, good candidates to be Darboux polynomials associated with F are the factors of the determinants of minors of Γ that are not zero. As a matter of fact, one typical reason for $\det(\Gamma)$ to be identically equal to zero is that two or more different linear combinations, with constant coefficients, of some polynomials $p_1, \dots, p_k$ are Darboux polynomials associated with F, with the same characteristic polynomial.

Example 4.1 Let $F(x) = [x_2 \; x_2 + x_2^2 - x_1^2]^\top$. Take as basis polynomials $p_1(x) = x_2$, $p_2(x) = x_1^2$. Then,

$$\Gamma(x) = \begin{bmatrix} x_2 & x_1^2 \\ x_2 + x_2^2 - x_1^2 & x_2^2 \end{bmatrix},$$

with $\det(\Gamma(x)) = (x_1^2 - x_2)(x_1^2 - x_2^2)$. Let $\omega(x) = x_1^2 - x_2$; since

$$\Delta\omega(x) = \left[F_1^2 - F_2\right]_{F_1 = x_2, F_2 = x_2 + x_2^2 - x_1^2} = x_1^2 - x_2 = \omega(x),$$

ω is a Darboux polynomial associated with F, with characteristic value equal to 1, i.e., ω is a first integral associated with F.

Example 4.2 Let $F(x) = \frac{x-3}{1+x}$. Take as basis polynomials $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$. Then,

$$\Gamma(x) = \begin{bmatrix} 1 & x & x^2 \\ 1 & \frac{x-3}{1+x} & \frac{(x-3)^2}{(1+x)^2} \\ 1 & -\frac{3+x}{x-1} & \frac{(3+x)^2}{(x-1)^2} \end{bmatrix},$$

with $\det(\Gamma(x)) = -2\frac{(3+x^2)^3}{(1+x)^2(x-1)^2}$. Let $\omega_1(x) = 3 + x^2$ and $\omega_2(x) = \frac{(3+x^2)^3}{(1+x)^2(x-1)^2}$; since

$$\Delta\omega_1(x) = \left(3 + F^2\right)\Big|_{F=\frac{x-3}{1+x}} = 4\frac{3+x^2}{(1+x)^2} = \lambda_1\omega_1(x),$$

with $\lambda_1(x) = \frac{4}{(1+x)^2}$, and

$$\Delta\omega_2(x) = \frac{(3+F^2)^3}{(1+F)^2(F-1)^2}\Bigg|_{F=\frac{x-3}{1+x}} = \frac{(3+x^2)^3}{(1+x)^2(x-1)^2} = \lambda_2\omega_2(x),$$

with $\lambda_2 = 1$, one concludes that ω_1 and ω_2 are semi-invariants associated with F; in particular, since $\lambda_2 = 1$, ω_2 is a first integral associated with F.

Example 4.3 Consider the algorithm for the computation of the square root of a positive real number a^2, with $a > 0$, as described by the discrete-time system (1.1b), with $F(x) = \frac{a^2-1}{a^2}x + \frac{1}{x}$. Take as basis polynomials $p_1(x) = 1$ and $p_2(x) = x$. Then,

$$\Gamma(x) = \begin{bmatrix} 1 & x \\ 1 & \frac{a^2-1}{a^2}x + \frac{1}{x} \end{bmatrix}$$

with $\det(\Gamma(x)) = -\frac{x^2-a^2}{a^2x}$. Clearly,

$$\Delta\left(x^2 - a^2\right) = \left(F^2 - a^2\right)\Big|_{F=\frac{a^2-1}{a^2}x+\frac{1}{x}} = \frac{(a^2x - x + a)(a^2x - x - a)}{a^4x^2}\left(x^2 - a^2\right),$$

which shows that $\omega(x) = x^2 - a^2$ is a semi-invariant associated with F, with characteristic function $\lambda(x) = \frac{(a^2x-x+a)(a^2x-x-a)}{a^4x^2}$.

Example 4.4 Consider the Lyness-type system characterized by $F(x) = [x_2 \ \frac{x_2}{x_1}]^\top$ (see, e.g., [77]). Take as basis polynomials $p_1(x) = x_1$, $p_2(x) = x_2$, $p_3(x) = x_1^2$, $p_4(x) = x_1x_2$, $p_5(x) = x_2^2$, $p_6(x) = x_1^3$, $p_7(x) = x_1^2x_2$, $p_8(x) = x_1x_2^2$, $p_9(x) = x_2^3$ (i.e., all monomials of degree less than 4, with respect to the standard dilation). Matrix Γ corresponding to such a choice has not full generic rank (its generic rank is 6). Taking the minor $\hat{\Gamma}$, found from Γ deleting the columns 4, 6 and 9 and the rows 7, 8 and 9 (actually, this corresponds to exclude monomials p_4, p_6 and p_9 from the chosen basis),

$$\hat{\Gamma}(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 \\ x_2 & x_2/x_1 & x_2^2 & x_2^2/x_1^2 & x_2^3/x_1 & x_2^3/x_1^2 \\ x_2/x_1 & 1/x_1 & x_2^2/x_1^2 & 1/x_1^2 & x_2^2/x_1^3 & x_2/x_1^3 \\ 1/x_1 & 1/x_2 & 1/x_1^2 & 1/x_2^2 & 1/(x_2 x_1^2) & 1/(x_1 x_2^2) \\ 1/x_2 & x_1/x_2 & 1/x_2^2 & x_1^2/x_2^2 & x_1/x_2^3 & x_1^2/x_2^3 \\ x_1/x_2 & x_1 & x_1^2/x_2^2 & x_1^2 & x_1^3/x_2^2 & x_1^3/x_2 \end{bmatrix},$$

and computing its determinant, one finds that $\det(\hat{\Gamma}) = q\omega_1\omega_2$, where $\omega_1(x) = x_1 + x_2 + x_1x_2^2 + x_1^2x_2 + x_1^2 + x_2^2$, $\omega_2(x) = \frac{x_1+x_2+x_1x_2^2+x_1^2x_2+x_1^2+x_2^2}{x_1x_2}$ and $q(x)$ is another rational function; in particular, ω_1 and ω_2 are semi-invariants associated with F, with respective characteristic functions $\lambda_1(x) = \frac{x_2}{x_1^2}$ and $\lambda_2(x) = 1$: actually, ω_2 is a first integral associated with F.

4.3 Symmetries of Discrete-Time Nonlinear Systems

For any *admissible* τ (to be considered as a constant parameter),

$$x = \Phi_g(\tau, y) \tag{4.2}$$

qualifies as a local analytic diffeomorphism (actually, it is a local one-parameter group of transformations), with inverse

$$y = \Phi_g(-\tau, x); \tag{4.3}$$

system (1.1b) is transformed, according to such a diffeomorphism, as follows:

$$\Delta y = \Phi_g(-\tau, \cdot) \circ F(\cdot) \circ \Phi_g(\tau, y). \tag{4.4}$$

Definition 4.2 [93] The diffeomorphism (4.2) is a *symmetry* of system (1.1b) and system (1.2) is its *infinitesimal generator* if

$$\Phi_g(-\tau, \cdot) \circ F(\cdot) \circ \Phi_g(\tau, y) = F(y), \quad \forall (\tau, y) \in \mathscr{V}, \tag{4.5}$$

where $\mathscr{V}$ is an open and connected set of $\mathbb{R} \times \mathbb{R}^n$ including $\{0\} \times \mathscr{U}$. If (4.5) holds, by abuse of notation, also the infinitesimal generator (1.2) is called a *symmetry* of the discrete-time system (1.1b); similarly, g is called a *DT-symmetry* of F.

If no confusion can arise between the continuous-time and discrete-time cases, the simpler nomenclature *symmetry* is used instead of *DT-symmetry*. It is worth pointing out that symmetries for higher order difference equations can be defined [66] similarly to what has been done in Sect. 3.20 for higher order differential equations.

Theorem 4.3 *Vector function g is a symmetry of F if and only if $\lfloor F, g \rfloor = 0$.*

Proof Clearly, condition (4.5) is equivalent to:

$$F(\cdot) \circ \Phi_g(\tau, x) = \Phi_g(\tau, \cdot) \circ F(x). \tag{4.6}$$

Condition (4.6) holds for $\tau = 0$. Taking the derivative with respect to τ of both sides of (4.6), one obtains

$$\left(\frac{\partial F}{\partial x} g\right) \circ \Phi_g = g \circ F \circ \Phi_g,$$

whence $\lfloor F, g \rfloor \circ \Phi_g = 0$. □

If $y = F(x)$ qualifies as a diffeomorphism in some open and connected subset of $\mathbb{R}^n$ and $\lfloor F, g \rfloor = 0$ therein, then each orbit of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = g(x)$ is mapped by $y = F(x)$ into the same orbit of the same system, preserving the time-parameterization along the orbit; to be more precise,

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\partial F(x)}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\partial F(x)}{\partial x} g(x) = g\big(F(x)\big) = g(y).$$

Remark 4.4 If symmetry g is linear, $g(x) = Bx$ for some $B \in \mathbb{R}^{n \times n}$, then (4.6) becomes $F(\mathrm{e}^{B\tau} x) = \mathrm{e}^{B\tau} F(x)$, according to the fact that $\lfloor F(x), Bx \rfloor = [F(x), Bx]$.

Remark 4.5 By a simple modification of the flow box Theorem 3.3 at p. 57, about any regular point of g, there exist local coordinates such that $g(x) = x_1 e_1$, where e_1 is the first column of the $n \times n$ identity matrix E. First, consider the case $n = 2$ and $g(x) = [x_1 \; 0]^\top$. Let F have g as symmetry; then, the equalities

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} - \begin{bmatrix} x_1 \frac{\partial F_1}{\partial x_1} \\ x_1 \frac{\partial F_2}{\partial x_1} \end{bmatrix},$$

imply

$$x_1 \frac{\partial F_1}{\partial x_1} = F_1, \qquad \frac{\partial F_2}{\partial x_1} = 0,$$

namely F has g as symmetry if and only if

$$F(x) = \begin{bmatrix} x_1 \beta_1 \\ \beta_2 \end{bmatrix},$$

where β_1 and β_2 are arbitrary functions of x_2. In the general case, assume that $g(x) = x_1 e_1$, with e_1 being the first column of the $n \times n$ identity matrix E; with a similar reasoning, it is easy to show that F has g as symmetry if and only if

$$F(x) = \begin{bmatrix} x_1\beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix},$$

with the β_i's being arbitrary functions of $x_2, \ldots, x_n$.

Theorem 4.4 *Let $y = \varphi(x)$ be a diffeomorphism analytic on $\mathscr{U}$. Let $\varphi_* F = \varphi \circ F \circ \varphi^{-1}$ and $\varphi_* g = (\frac{\partial \varphi}{\partial x} g) \circ \varphi^{-1}$. Then, $\varphi_* g$ is a symmetry of $\varphi_* F$ if and only if g is a symmetry of F,*

$$\lfloor \varphi_* F, \varphi_* g \rfloor = 0 \quad \Longleftrightarrow \quad \lfloor F, g \rfloor = 0.$$

Proof Equation (4.6) yields

$$\varphi \circ F \circ \varphi^{-1} \circ \varphi \circ \Phi_g \circ \varphi^{-1} = \varphi \circ \Phi_g \circ \varphi^{-1} \circ \varphi \circ F \circ \varphi^{-1}; \tag{4.7}$$

since $\varphi_* F = \varphi \circ F \circ \varphi^{-1}$ and $\Phi_{\varphi_* g} = \varphi \circ \Phi_g \circ \varphi^{-1}$, (4.7) becomes $(\varphi_* F) \circ \Phi_{\varphi_* g} = \Phi_{\varphi_* g} \circ (\varphi_* F)$, which holds (by Theorem 4.3) if and only if $\lfloor \varphi_* F, \varphi_* g \rfloor = 0$. □

Remark 4.5 and Theorem 4.4 yield the following theorem.

Theorem 4.5 *Let $g(x) \in \mathbb{R}^n$ be given. Let $y = \varphi(x)$ be a diffeomorphism such that the push-forward of g is $\varphi_* g(y) = y_1 e_1$, with e_1 being the first column of the $n \times n$ identity matrix. Then, F has g as symmetry if and only if*

$$F(x) = \varphi^{-1} \circ \begin{bmatrix} y_1\beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \circ \varphi(x),$$

with the β_i's being arbitrary functions of $y_2 = \varphi_2(x), \ldots, y_n = \varphi_n(x)$.

Example 4.5 Let $g(x) = [x_1 \ \ -x_2]^\top$. A diffeomorphism $y = \varphi(x)$, such that the push-forward of g is $\tilde{g}(y) = \varphi_* g(y) = [y_1 \ 0]^\top$, is given by $\varphi(x) = [x_1 \ x_1 x_2]^\top$, with inverse $\varphi^{-1}(y) = [y_1 \ \frac{y_2}{y_1}]^\top$. Then, the set of all $\tilde{F}$ having $\tilde{g}$ as symmetry is parameterized by

$$\tilde{F}(y) = \begin{bmatrix} y_1\beta_1 \\ \beta_2 \end{bmatrix},$$

where β_1, β_2 are arbitrary functions of y_2. By the pull-back of $\tilde{F}$, one concludes that the set of all F having g as symmetry is parameterized by

$$F(x) = \varphi^{-1} \circ \tilde{F} \circ \varphi(x) = \left[\left[\begin{bmatrix} \tilde{F}_1 \\ \frac{\tilde{F}_2}{\tilde{F}_1} \end{bmatrix}_{\tilde{F}_1 = y_1\beta_1, \tilde{F}_2 = \beta_2} \right]_{y_1 = x_1, y_2 = x_1 x_2} \right] = \begin{bmatrix} x_1\beta_1 \\ \frac{1}{x_1}\frac{\beta_2}{\beta_1} \end{bmatrix}, \quad (4.8)$$

where β_1, β_2 are arbitrary functions of $x_1 x_2$.

Note that the diffeomorphism $y = \varphi(x)$ in Example 4.5 is not invertible at $x = 0$. Nevertheless, it can be used to find vector functions $F(x) \in \mathbb{R}^2$ that are analytic at $x = 0$; as an example, letting $\beta_2 = x_1 x_2 \beta_1$ with β_1 analytic at $x = 0$, one has $F(x) = [x_1\beta_1 \; x_2]^\top$, which is analytic at $x = 0$.

Remark 4.6 By the definitions $[f, g] = L_f g - L_g f$ and $\lfloor F, g \rfloor = g \circ F - L_g F$, it is easy to see that $\lfloor F(x), Bx \rfloor = BF(x) - L_{Bx}F(x) = [F(x), Bx]$, whereas $\lfloor Ax, g(x) \rfloor = g(Ax) - L_g Ax$ need not coincide with $[Ax, g(x)]$. Since the g considered in Example 4.5 is linear, the set of all F having g as DT-symmetry, given in (4.8), is coincident with the set of all f having g as CT-symmetry, given in (3.15). As a matter of fact, the two sets coincide by letting $\beta_1 = \alpha + \frac{\beta}{2x_1x_2}$, $\beta_2 = -x_1x_2\alpha^2 + \frac{1}{4x_1x_2}\beta^2$, where α and β are arbitrary functions of x_1x_2. In particular, by Theorem 3.10 at p. 66, if $g(x) = Bx$ and $\{M_0, \dots, M_{r-1}\}$ is a basis of $\mathscr{L}_c(B)$, then g is both a CT-symmetry and a DT-symmetry of any element of $\mathscr{I}_C(Bx) \otimes \mathscr{L}_c(B) \equiv \mathscr{C}_C(Bx) \equiv \mathscr{C}_D(Bx)$ (see the notation introduced just before Theorem 3.11 at p. 66).

4.4 Symmetries of Scalar Discrete-Time Nonlinear Systems

The following theorem (which is inspired by [82]) gives a necessary and sufficient condition for a scalar discrete-time nonlinear system to be diffeomorphic to the special form

$$y(t+1) = y(t) + c, \quad (4.9)$$

with $c \in \mathbb{R}$.

Theorem 4.6 *Let $F(x) \in \mathbb{R}$. There exists a diffeomorphism $y = \varphi(x)$ such that $\varphi_* F(y) = y + c$, where $c \in \mathbb{R}$ is a constant, if and only if there exists a symmetry $g(x) \in \mathbb{R}$, $g \neq 0$, of $F(x)$, $\lfloor F, g \rfloor = 0$. In such a case, there exists a (non-trivial) first integral $I(x)$ associated with $F(x)$.*

Proof Assume that $\varphi_* F(y) = y + c$. Let $\tilde{g}(y) = 1$; clearly, $\lfloor \varphi_* F, \tilde{g} \rfloor = 0$, and therefore, by Theorem 4.4, one has $\lfloor F, g \rfloor = 0$, with $g = \varphi^* \tilde{g} \neq 0$. Conversely, assume that $\lfloor F, g \rfloor = 0$, with $g \neq 0$. Let $y = \varphi(x)$, with $\varphi(x) = \int_0^x \frac{1}{g(\xi)}\, \mathrm{d}\xi$, which is well defined in a neighborhood of any regular point of g. Hence, $\varphi_* g(y) = (\frac{1}{g(x)} g(x)) \circ \varphi^{-1}(y) = 1$. By Theorem 4.4, condition $\lfloor F, g \rfloor = 0$ implies $\lfloor \varphi_* F, \varphi_* g \rfloor = 0$. Now, since $\lfloor \tilde{F}(y), \varphi_* g(y) \rfloor = 1 - \frac{\partial \tilde{F}(y)}{\partial y}$ for any $\tilde{F}(y) \in \mathbb{R}$, condition $\lfloor \varphi_* F, \varphi_* g \rfloor = 0$ implies that $\frac{\partial \varphi_* F(y)}{\partial y} = 1$, i.e., $\varphi_* F(y) = y + c$. Note that if $c = 0$ in (4.9), then $\varphi(x)$ is a first integral associated with F; conversely, if $\varphi(x)$ is a non-constant first integral associated with F, then $y = \varphi(x)$ is a diffeomorphism such that $\varphi_* F(y) = y$. For $c \neq 0$, a first integral of (4.9) is $\tilde{I}(y) = \sin(\frac{2\pi}{c} y)$, whence $I = \varphi^* \tilde{I}$. As a matter of fact, letting $\tilde{F}(y) = y + c$, one has $\tilde{I} \circ \tilde{F}(y) = \sin(\frac{2\pi}{c}(y + c)) = \sin(\frac{2\pi}{c} y + 2\pi) = \sin(\frac{2\pi}{c} y) = \tilde{I}(y)$. □

Remark 4.7 Theorem 4.6 gives a complete picture of scalar discrete-time systems admitting a symmetry, which can be summarized by saying that the following statements are equivalent:

(4.7.1) the scalar discrete-time system admits a symmetry $g(x)$;
(4.7.2) the scalar discrete-time system is diffeomorphic by $y = \varphi(x)$ to form (4.9), for some $c \in \mathbb{R}$;
(4.7.3) the scalar discrete-time system admits a (non-trivial) first integral $I(x)$.

If g is a symmetry associated with F, then $\varphi(x) = \int_0^x \frac{1}{g(\xi)} \mathrm{d}\xi$; if $c = 0$, then $I(x) = \varphi(x)$, otherwise $I(x) = \sin(\frac{2\pi}{c} \varphi(x))$. If $y = \varphi(x)$ is a diffeomorphism such that $\varphi_* F(y) = y + c$, then $g = (\frac{\partial \varphi}{\partial x})^{-1}$ is a symmetry associated with F; as before, if $c = 0$, then $I(x) = \varphi(x)$, otherwise $I(x) = \sin(\frac{2\pi}{c} \varphi(x))$. If I is a first integral associated with F, then $g = (\frac{\partial I}{\partial x})^{-1}$ is a symmetry associated with F and the diffeomorphism $y = \varphi(x)$, with $\varphi = I$, is such that $\varphi_* F(y) = y$.

Example 4.6 Let $F(x) = \frac{ax+b}{cx+d}$ and look for a symmetry of F of the form $g(x) = \alpha x^2 + \beta x + \gamma$; from

$$g \circ F(x) = \alpha \frac{(ax+b)^2}{(cx+d)^2} + \beta \frac{ax+b}{cx+d} + \gamma,$$

$$\frac{\partial F(x)}{\partial x} g(x) = \frac{ad - cb}{(cx+d)^2} \big(\alpha x^2 + \beta x + \gamma \big),$$

one has that $\lfloor F, g \rfloor = 0$ if and only if the following algebraic system has a real solution in the unknowns α, β, γ:

$$\big(ad - cb - a^2\big)\alpha - ac\beta - c^2\gamma = 0, \tag{4.10a}$$

$$-2ab\alpha - 2bc\beta - 2cd\gamma = 0, \tag{4.10b}$$

$$-b^2\alpha - bd\beta + \big(-d^2 - cb + ad\big)\gamma = 0. \tag{4.10c}$$

In particular, one of the solutions of (4.10a)–(4.10c) is $\alpha = -c, \beta = a - d, \gamma = b$, which yields the symmetry $g(x) = -cx^2 + (a - d)x + b$. For the sake of simplicity, consider the case $a = 3$, $b = 1$, $c = -1$ and $d = 1$,

$$F(x) = \frac{3x+1}{1-x}, \qquad g(x) = x^2 + 2x + 1.$$

The resulting diffeomorphism, which is well defined in a neighborhood of $x = 0$, is $y = \varphi(x)$, with

$$\varphi(x) = \int_0^x \frac{1}{\xi^2 + 2\xi + 1}\,\mathrm{d}\xi = \frac{x}{x+1},$$

with inverse $\varphi^{-1}(y) = \frac{y}{1-y}$. It is easy to verify that $\varphi_* F(y) = \varphi \circ F \circ \varphi^{-1}(y) = ((\frac{F}{F+1})|_{F=\frac{3x+1}{1-x}})|_{x=\frac{y}{1-y}} = y + \frac{1}{2}$. Since a first integral associated with $\varphi_* F$ is $\sin(4\pi y)$, a first integral associated with F is $I(x) = \sin(4\pi \frac{x}{x+1})$; as a matter of fact, one can check

$$\begin{aligned} I \circ F(x) &= \sin\left(4\pi \frac{3x+1}{(1-x)(\frac{3x+1}{1-x}+1)}\right) = \sin\left(4\pi \frac{x}{x+1} + 2\pi\right) \\ &= \sin\left(4\pi \frac{x}{x+1}\right) \\ &= I(x). \end{aligned}$$

Now, consider the case $a = 1$, $b = -3$, $c = 1$ and $d = 1$,

$$F(x) = \frac{x-3}{x+1}, \qquad g(x) = -x^2 - 3.$$

The resulting diffeomorphism, which is well defined in a neighborhood of $x = 0$, is $y = \varphi(x)$, where

$$\varphi(x) = \int_0^x \frac{1}{-\xi^2 - 3}\,\mathrm{d}\xi = -\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right),$$

with inverse $\varphi^{-1}(y) = -\sqrt{3}\tan(\sqrt{3}y)$. It is easy to verify that $\varphi_* F(y) = y + \frac{\sqrt{3}\pi}{9}$. In Example 4.2, it has been shown that the diffeomorphism $y = \frac{(3+x^2)^3}{(1+x)^2(x-1)^2}$ transforms system $x(t+1) = \frac{x(t)-3}{x(t)+1}$ into the linear system $y(t+1) = y(t)$. Hence, a symmetry $g(x)$ of $F(x)$ can be computed as follows:

$$g(x) = \left(\frac{\partial \varphi(x)}{\partial x}\right)^{-1} = \frac{(1+x)^3(x-1)^3}{2(3+x^2)^2(x^2-9)x};$$

it is left to the reader to show that $\lfloor F, g \rfloor = 0$ for this choice.

Theorem 4.6 is extended to the case $n > 1$ by the following theorem.

Theorem 4.7 *Let $F(x) \in \mathbb{R}^n$. There exists a diffeomorphism $y = \varphi(x)$ such that $\varphi_* F(y) = y + c$, where $c \in \mathbb{R}^n$ is a constant, if and only if there exist n symmetries $g_i(x) \in \mathbb{R}^n$ of $F(x)$, $\lfloor F, g_i \rfloor = 0$, $i = 1, \dots, n$, such that $[g_i, g_j] = 0$, for all $i, j \in \{1, \dots, n\}$, and* $\det([g_1 \ \dots \ g_n]) \neq 0$. *In such a case, there exist n functionally independent first integrals $I_i(x)$, $i = 1, \dots, n$, associated with $F(x)$.*

Proof Assume that $\varphi_* F(y) = y + c$. Let $\tilde{g}_i(y) = e_i$, $i = 1, \dots, n$; clearly, $\lfloor \varphi_* F, \tilde{g}_i \rfloor = 0$, and therefore, by Theorem 4.4, one has $\lfloor F, g_i \rfloor = 0$, with $g_i = \varphi^* \tilde{g}_i$, $i = 1, \dots, n$; in particular, by construction, the vector functions g_i are pairwise commuting, $[g_i, g_j] = 0$, and satisfy $\det([g_1 \ \dots \ g_n]) \neq 0$. Conversely, assume that $\lfloor F, g_i \rfloor = 0$, with the vector functions g_i being pairwise commuting, $[g_i, g_j] = 0$, and satisfying $\det([g_1 \ \dots \ g_n]) \neq 0$. Hence, by Remark 1.8 at p. 22, all rows of $[g_1 \ \dots \ g_n]^{-1}$ are exact one-forms. Let $y = \varphi(x)$ be a diffeomorphism such that

$$\frac{\partial \varphi(x)}{\partial x} = \begin{bmatrix} g_1(x) & \dots & g_n(x) \end{bmatrix}^{-1},$$

which is well defined about any point x^o such that $\det([g_1(x^o) \ \dots \ g_n(x^o)]) \neq 0$. Hence, $\varphi_* g_i(y) = ([g_1(x) \ \dots \ g_n(x)]^{-1} g_i(x)) \circ \varphi^{-1}(y) = e_i$. By Theorem 4.4, condition $\lfloor F, g_i \rfloor = 0$ implies $\lfloor \varphi_* F, \varphi_* g_i \rfloor = 0$. Now, since $\lfloor \tilde{F}(y), \varphi_* g_i(y) \rfloor = e_i - \frac{\partial \tilde{F}(y)}{\partial y_i}$ for any $\tilde{F}(y) \in \mathbb{R}^n$, condition $\lfloor \varphi_* F, \varphi_* g_i \rfloor = 0$ implies that $\frac{\partial \varphi_* F(y)}{\partial y_i} = e_i$, i.e., $\varphi_* F(y) = y_i e_i + c_i(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$, for some function c_i being independent of y_i. Letting i vary in $\{1, \dots, n\}$, one concludes that $\varphi_* F(y) = y + c$, for a constant $c = [c_1 \ \dots \ c_n]^\top$. If $c_i = 0$, then $\tilde{I}_i(y) = y_i$ is a first integral associated with $\varphi_* F$, otherwise $\tilde{I}_i(y) = \sin(\frac{2\pi}{c} y_i)$ is a first integral associated with $\varphi_* F$. In particular, such first integrals are functionally independent. By the pull-back to the original coordinates, one obtains the functionally independent first integrals $I_i = \varphi^* \tilde{I}_i$, $i = 1, \dots, n$, associated with $F(x)$. □

Example 4.7 Consider the discrete-time system described by

$$F(x) = \begin{bmatrix} -4x_2^4 - 8x_1x_2^2 + 4x_2^3 - 4x_1^2 + 4x_1x_2 - 4x_2^2 - 3x_1 + 2x_2 \\ -2x_2^2 - 2x_1 + x_2 - 1 \end{bmatrix}.$$

Let

$$g_1(x) = \begin{bmatrix} 1 + 4x_1x_2 + 4x_2^3 \\ -2x_1 - 2x_2^2 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} -2x_2 \\ 1 \end{bmatrix}.$$

It is easy to check that $\lfloor F, g_i \rfloor = 0$, $i = 1, 2$, $[g_1, g_2] = 0$ and $\det([g_1 \ g_2]) \neq 0$. Hence the rows of

$$\begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2x_2 \\ 2x_1 + 2x_2^2 & 1 + 4x_1x_2 + 4x_2^3 \end{bmatrix}$$

are exact one-forms and their integrals yield the diffeomorphism $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} x_1 + x_2^2 \\ x_2 + x_1^2 + 2x_1x_2^2 + x_2^4 \end{bmatrix}, \qquad \varphi^{-1}(y) = \begin{bmatrix} y_1 - y_2^2 + 2y_2y_1^2 - y_1^4 \\ y_2 - y_1^2 \end{bmatrix}.$$

Compute the push-forward

$$\varphi_* F(y) = \begin{bmatrix} 1 + y_1 \\ y_2 \end{bmatrix};$$

the first integrals associated with $\varphi_* F(y)$ are $\tilde{I}_1(y) = \sin(2\pi y_1)$ and $\tilde{I}_2(y) = y_2$. Hence, two functionally independent first integrals associated with $F(x)$ can be computed by the pull-back to the original coordinates,

$$I_1(x) = \varphi^* \tilde{I}_1(x) = \sin\bigl(2\pi\bigl(x_1 + x_2^2\bigr)\bigr), \qquad I_2(x) = \varphi^* \tilde{I}_2(x) = x_2 + x_1^2 + 2x_1x_2^2 + x_2^4.$$

4.5 Reduction of Discrete-Time Nonlinear Systems

Let $F(x), g(x) \in \mathbb{R}^n$ be such that $\lfloor F, g \rfloor = 0$. Let $\mathscr{I}_C(g)$ be the set of the CT-first integrals associated with g, i.e., by Remark 3.12 at p. 77, the set of all functions being homogeneous of degree 0 with respect to g. Then, there exist $n-1$ functionally independent elements $J_1, \ldots, J_{n-1}$ of $\mathscr{I}_C(g)$ that generate the whole $\mathscr{I}_C(g)$, i.e., any $J \in \mathscr{I}_C(g)$ can be expressed as $C(J_1, \ldots, J_{n-1})$, where C is an arbitrary function of the arguments. Since $J_i \in \mathscr{I}_C(g)$, it follows that $J_i \circ F \in \mathscr{I}_C(g)$: as a matter of fact, taking into account that $\lfloor F, g \rfloor = 0$ implies $F \circ \Phi_g = \Phi_g \circ F$ and that $J_i \in \mathscr{I}_C(g)$ implies $J_i \circ \Phi_g = J_i$, one concludes that

$$J_i \circ F \circ \Phi_g = J_i \circ \Phi_g \circ F = J_i \circ F,$$

as to be shown. Since $J_i \circ F \in \mathscr{I}_C(g)$, there exists a function C_i such that $J_i \circ F = C_i(J_1, \ldots, J_{n-1})$. Therefore, by the *projection* $\mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\xi_i = J_i(x)$, $i = 1, \ldots, n-1$, a discrete-time nonlinear system, of reduced dimension $n-1$, is found.

As in the continuous-time case, the reduced system does not describe wholly the given system, but, being of lower dimension, it can be useful to study it. For instance, the meaning of an equilibrium point of the reduced system is the same as in the continuous-time case.

Example 4.8 Consider $F(x) = [x_1 + x_2 \; x_2 \; 3x_3 + a_1x_1^2 + a_2x_1x_2 + a_3x_2^2]^\top$, $g(x) = [x_1 \; x_2 \; 2x_3]^\top$; clearly, $\lfloor F, g \rfloor = 0$. Two functionally independent CT-first integrals associated with g are $J_1(x) = \frac{x_2}{x_1}$ and $J_2(x) = \frac{x_3}{x_1^2}$; then, by the projection $\xi_1 = \frac{x_2}{x_1}$, $\xi_2 = \frac{x_3}{x_1^2}$, taking into account that (with the substitution, $F_1(x) = x_1 + x_2$, $F_2(x) = x_2$ and $F_3(x) = 3x_3 + a_1x_1^2 + a_2x_1x_2 + a_3x_2^2$)

$$\xi_1 \circ F(x) = \frac{F_2(x)}{F_1(x)} = \frac{\frac{x_2}{x_1}}{1 + \frac{x_2}{x_1}}, \qquad \xi_2 \circ F(x) = \frac{F_3(x)}{F_1^2(x)} = \frac{a_1 + 3\frac{x_3}{x_1^2} + a_2\frac{x_2}{x_1} + a_3\frac{x_2^2}{x_1^2}}{1 + 2\frac{x_2}{x_1} + \frac{x_2^2}{x_1^2}},$$

one obtains $\Delta\xi = F_r(\xi)$, with $F_r(\xi) = [\frac{\xi_1}{1+\xi_1} \; \frac{a_1 + 3\xi_2 + a_2\xi_1 + a_3\xi_1^2}{(1+\xi_1)^2}]^\top$.

4.6 A Property of Discrete-Time Nonlinear Planar Systems

Throughout this section assume that $x \in \mathbb{R}^2$.

Definition 4.3 A scalar function $\omega \neq 0$ is an *inverse integrating factor* associated with F if

$$\omega \circ F = \det\left(\frac{\partial F}{\partial x}\right)\omega. \tag{4.11}$$

Actually, any function ω such that (4.11) holds is a semi-invariant associated with F, with characteristic function $\lambda = \det(\frac{\partial F}{\partial x})$, provided that there is no zero/pole cancelation between λ and ω.

The name "inverse integrating factor" given in Definition 4.3 is motivated by the following reasoning. Assume that there exist a $\delta_T \in \mathbb{R}$, $\delta_T > 0$, and a vector function $f(x) \in \mathbb{R}^n$ such that $F(x) = \Phi_f(\delta_T, x)$, i.e., the discrete-time system (1.1b) is the sampling of the continuous-time system (1.1a) with sampling time δ_T or, equivalently, f is the *logarithm* of F (see [93]). If ω is an inverse integrating factor associated with f, in the sense of Definition 3.9 at p. 85, then ω is an inverse integrating factor associated with F, in the sense of Definition 4.3.

Lemma 4.1

(4.1.1) *If ω_1 and ω_2 are two inverse integrating factors associated with F, then $I = \frac{\omega_1}{\omega_2}$ is a first integral associated with F.*

(4.1.2) *If ω and I are, respectively, an inverse integrating factor and a first integral associated with F, then $\hat{\omega} = \omega I$ is an inverse integrating factor associated with F.*

(4.1.3) *If ω_1 and ω_2 are two inverse integrating factors associated with F, then $\omega = a_1\omega_1 + a_2\omega_2$ in an inverse integrating factor associated with F, $\forall a_1, a_2 \in \mathbb{R}$.*

Proof Proof of (4.1.1). If $\omega_i \circ F = \det(\frac{\partial F}{\partial x})\omega_i$, $i = 1, 2$, then

$$I \circ F = \frac{\omega_1 \circ F}{\omega_2 \circ F} = \frac{\det(\frac{\partial F}{\partial x})\omega_1}{\det(\frac{\partial F}{\partial x})\omega_2} = \frac{\omega_1}{\omega_2} = I.$$

Proof of (4.1.2). If $\omega \circ F = \det(\frac{\partial F}{\partial x})\omega$ and $I \circ F = I$, then

$$\hat{\omega} \circ F = (\omega \circ F)(I \circ F) = \left(\det\left(\frac{\partial F}{\partial x}\right)\omega\right)(I) = \det\left(\frac{\partial F}{\partial x}\right)\hat{\omega}.$$

Proof of (4.1.3). If $\omega_i \circ F = \det(\frac{\partial F}{\partial x})\omega_i$, $i = 1, 2$, then

$$\begin{aligned}\omega \circ F &= a_1\omega_1 \circ F + a_2\omega_2 \circ F = a_1 \det\left(\frac{\partial F}{\partial x}\right)\omega_1 + a_2 \det\left(\frac{\partial F}{\partial x}\right)\omega_2 \\ &= \det\left(\frac{\partial F}{\partial x}\right)(a_1\omega_1 + a_2\omega_2) = \det\left(\frac{\partial F}{\partial x}\right)\omega.\end{aligned}$$

□

Example 4.9 Take $F(x) = [x_1\ 3x_2 + x_1^2]^\top$. Compute the semi-invariants associated with F that are linear combinations of $p_1(x) = x_1$ and $p_2(x) = x_2$, by the technique of Sect. 4.2; then, from

$$\Gamma(x) = \begin{bmatrix} x_1 & x_2 \\ x_1 & 3x_2 + x_1^2 \end{bmatrix},$$

one computes $\det(\Gamma(x)) = x_1(2x_2 + x_1^2)$, from which $\omega_1(x) = x_1$ and $\omega_2(x) = 2x_2 + x_1^2$ are found as candidates to be semi-invariants associated with F. In particular, $\omega_1 \circ F = \lambda_1\omega_1$ and $\omega_2 \circ F = \lambda_2\omega_2$, with $\lambda_1 = 1$ and $\lambda_2 = 3$; since, $\lambda_1 = 1$ and $\lambda_2 = \det(\frac{\partial F}{\partial x})$, ω_1 is a first integral and ω_2 is an inverse integrating factor associated with F.

Theorem 4.8 *If ω and I are, respectively, an inverse integrating factor and a first integral associated with F, then*

$$g = \omega S\left(\frac{\partial I}{\partial x}\right)^\top, \tag{4.12}$$

where $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, is a symmetry of F. Vice versa, if g is a symmetry of F and I is a first integral associated with both g and F ($L_g I = 0$ and $I \circ F = F$), then g can be rewritten as $g = \omega S(\frac{\partial I}{\partial x})^\top$ for some ω that is an inverse integrating factor associated with F.

Proof First, it is pointed out that $BSB^\top = \det(B)S$, for any matrix B, as shown by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & bc - ad \\ ad - bc & 0 \end{bmatrix} = (ad - bc)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};$$

furthermore, note that $I \circ F = I$ implies $\frac{\partial I}{\partial x}|_{x=F}\frac{\partial F}{\partial x} = \frac{\partial I}{\partial x}$. Since

$$g \circ F = (\omega \circ F)S\left(\frac{\partial I}{\partial x}\bigg|_{x=F}\right)^\top = \det\left(\frac{\partial F}{\partial x}\right)\omega S\left(\frac{\partial I}{\partial x}\bigg|_{x=F}\right)^\top,$$

and

$$\begin{aligned} \frac{\partial F}{\partial x}g = \omega\left(\frac{\partial F}{\partial x}\right)S\left(\frac{\partial I}{\partial x}\right)^\top &= \omega\left(\frac{\partial F}{\partial x}\right)S\left(\frac{\partial F}{\partial x}\right)^\top\left(\frac{\partial I}{\partial x}\bigg|_{x=F}\right)^\top \\ &= \omega\det\left(\frac{\partial F}{\partial x}\right)S\left(\frac{\partial I}{\partial x}\bigg|_{x=F}\right)^\top, \end{aligned}$$

one concludes that $g \circ F = \frac{\partial F}{\partial x}g$. Vice versa, if I is a CT-first integral associated with g, then g can be rewritten as in (4.12), for some ω; in addition, if I is a DT-first integral associated with F, then $\frac{\partial I}{\partial x}|_{x=F}\frac{\partial F}{\partial x} = \frac{\partial I}{\partial x}$. Therefore, since g is a symmetry of F, $g \circ F = \frac{\partial F}{\partial x} g$; therefore,

$$(\omega \circ F) S \left(\frac{\partial I}{\partial x}\bigg|_{x=F} \right)^{\top} = \det\left(\frac{\partial F}{\partial x}\right) \omega S \left(\frac{\partial I}{\partial x}\bigg|_{x=F} \right)^{\top},$$

which implies $\omega \circ F = \det(\frac{\partial F}{\partial x})\omega$. □

Example 4.10 Consider again the vector function F introduced in Example 4.9. A symmetry g of F is

$$g(x) = \omega_2 S \left(\frac{\partial \omega_1(x)}{\partial x} \right)^{\top} = \left(2x_2 + x_1^2\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1^2 + 2x_2 \end{bmatrix}.$$

4.7 Lax Pairs for Discrete-Time Nonlinear Systems

The concept of the Lax pair, which is classical in the continuous-time case, can be extended to the discrete-time case (see [94]). The notation in this section is somewhat different from the one in the rest of the book, e.g., matrices A and B are not constant here.

Let a vector function $F(x) \in \mathbb{R}^n$ be given. Given a matrix function $A(x) \in \mathbb{R}^{\nu \times \nu}$, with entries $A_{i,j}(x)$, the symbol $A \circ F$ clearly denotes the matrix function having $A_{i,j} \circ F$ as entries.

Definition 4.4 Given a vector function $F(x) \in \mathbb{R}^n$, a *DT-Lax pair* (briefly, a Lax pair if no confusion can arise) associated with $F(x)$ is an ordered pair of matrix functions (A, B), with $A(x), B(x) \in \mathbb{R}^{\nu \times \nu}$, $\nu^2 \geq n$, and B being invertible over the field of meromorphic functions, such that

$$A \circ F = B A B^{-1}. \tag{4.13}$$

Theorem 4.9 *Let (A, B) be a Lax pair associated with a given F. Then, for any $k \in \mathbb{Z}^{\geq}$, $I = \mathrm{trace}(A^k)$ is a first integral associated with F.*

Proof Taking into account that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, one concludes that

$$I \circ F = \mathrm{trace}\big((A \circ F)^k\big) = \mathrm{trace}\big(B A^k B^{-1}\big) = \mathrm{trace}\big(A^k\big) = I.$$ □

Using a reasoning similar to the one used in the continuous-time case, it is possible to show that the eigenvalues of A, as well as the coefficients of the characteristic polynomial of A, as well as $\det(A) = \prod_i \lambda_i$, are first integrals associated with F. This, in particular, shows that at most ν functionally independent first integrals associated with F can be computed from the knowledge of A.

Remark 4.8 For given $A(x), B(x) \in \mathbb{R}^{\nu\times\nu}$ and an unknown $F(x) \in \mathbb{R}^n$, (4.13) is a set of ν^2 algebraic equations in the n unknown entries of F. If such a system has a unique solution F, then (A, B) is called a *regular Lax pair* associated with the vector function F thus identified. For instance, take $\nu = 2$ and $n = 3$,

$$A(x) = \begin{bmatrix} x_1 & x_2 \\ 1 & x_3 \end{bmatrix}, \qquad B(x) = \begin{bmatrix} 2 + x_1 - x_2 - x_3 & 1 \\ 1 & 1 \end{bmatrix};$$

then,

$$A \circ F(x) = \begin{bmatrix} F_1 & F_2 \\ 1 & F_3 \end{bmatrix},$$

$$BAB^{-1} = \begin{bmatrix} x_1 - x_2 + 1 & -x_2x_3 + x_3 - x_2^2 + x_1x_2 + 3x_2 - x_1 - 1 \\ 1 & x_2 + x_3 - 1 \end{bmatrix},$$

from which (A, B) is a regular Lax pair associated with

$$F(x) = \begin{bmatrix} -x_2 + x_1 + 1 \\ -x_2x_3 + x_3 - x_2^2 + x_1x_2 + 3x_2 - x_1 - 1 \\ x_3 + x_2 - 1 \end{bmatrix}. \tag{4.14}$$

Hence, $I_1(x) = \text{trace}(A(x)) = x_1 + x_3$ and $I_2(x) = \text{trace}(A^2(x)) = x_1^2 + 2x_2 + x_3^2$ are two functionally independent first integrals associated with F.

Theorem 4.10 *Let (A, B) be a Lax pair associated with a given F. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a polynomial scalar function of the argument. Then, $(\alpha(A), B)$ is a Lax pair associated with F.*

Proof First, it is shown how (A^k, B) is a Lax pair associated with f, for any $k \in \mathbb{Z}^{\geq}$,

$$A^k \circ F = (A \circ F)^k = BA^kB^{-1}.$$

Clearly, if (A, B) is a Lax pair associated with F, then (aA, B) is a Lax pair associated with F, for any constant $a \in \mathbb{R}$. Finally, if (A_1, B) and (A_2, B) are two Lax pairs associated with F, then $(A_1 + A_2, B)$ is a Lax pair associated with F,

$$(A_1 + A_2) \circ F = A_1 \circ F + A_2 \circ F = BA_1B^{-1} + BA_2B^{-1} = B(A_1 + A_2)B^{-1}.$$

□

Theorem 4.11 *Let (A, B_1) be a Lax pair associated with a given F. Then, (A, B_2), with* $\det(B_2) \neq 0$, *is a Lax pair associated with F if and only if* $[A, B_2^{-1}B_1] = 0$.

Proof If (A, B_1) and (A, B_2) are two Lax pairs associated with F, then

$$A \circ F = B_1 A B_1^{-1}, \qquad A \circ F = B_2 A B_2^{-1}$$
$$\downarrow$$
$$B_2^{-1} B_1 A = A B_2^{-1} B_1.$$

Vice versa, if $A \circ F = B_1 A B_1^{-1}$ and $B_2^{-1} B_1 A = A B_2^{-1} B_1$, then $A \circ F = B_2 A B_2^{-1}$. □

Theorem 4.12 *Let (A, B) be a Lax pair associated with a given F. Then, for any matrix $M(x) \in \mathbb{R}^{\nu \times \nu}$ invertible over the field of meromorphic functions, pair $(\tilde{A}, \tilde{B})$, with*

$$\tilde{A} = M A M^{-1}, \qquad \tilde{B} = (M \circ F) B M^{-1}, \tag{4.15}$$

is a Lax pair associated with F.

Proof Taking into account that $A \circ F = BAB^{-1}$, one concludes that

$$\begin{aligned} \tilde{A} \circ F &= (M \circ F)(A \circ F)(M \circ F)^{-1} = (M \circ F) BAB^{-1} (M \circ F)^{-1} \\ &= (M \circ F) B M^{-1} \tilde{A} M B^{-1} (M \circ F)^{-1} = \tilde{B} \tilde{A} \tilde{B}^{-1}. \end{aligned}$$ □

Theorem 4.13 *Let $I_1, \dots, I_m$ be $m \leq n$ functionally independent first integrals associated with a given F. Let $M(x) \in \mathbb{R}^{n \times n}$ be invertible over the field of meromorphic functions. Then,*

$$A = M \Lambda M^{-1}, \qquad B = (M \circ F) M^{-1}$$

where $\Lambda = \mathrm{diag}\{I_1, \dots, I_m, c_{m+1}, \dots, c_n\}$ and the c_i's are arbitrary constants, is a Lax pair associated with f.

Proof The proof follows from Theorem 4.12, taking into account that (Λ, E) is a Lax pair associated with F. □

Example 4.11 Consider the vector function F given in (4.14); $I_1(x) = x_1 + x_3$ and $I_2(x) = x_1^2 + 2x_2 + x_3^2$ are two functionally independent first integrals associated with F. Take the simple polynomial matrix $M(x)$, with polynomial inverse

$$M(x) = \begin{bmatrix} 1 & 0 & x_2 \\ x_1 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M^{-1}(x) = \begin{bmatrix} 1 & 0 & -x_2 \\ -x_1 & 1 & -x_3 + x_1 x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

for which

$$M \circ F(x) = \begin{bmatrix} 1 & 0 & -x_2 x_3 + x_3 - x_2^2 + x_1 x_2 + 3x_2 - x_1 - 1 \\ x_1 - x_2 + 1 & 1 & x_2 + x_3 - 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let (any constant value is acceptable as the (3, 3)-entry of Λ)

$$\Lambda(x) = \begin{bmatrix} x_1 + x_3 & 0 & 0 \\ 0 & x_1^2 + 2x_2 + x_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

then, (A, B) with

$$\begin{aligned} A(x) &= M(x)\Lambda(x)M^{-1}(x) \\ &= \left[\begin{matrix} x_1 + x_3 & 0 \\ x_1^2 + x_1x_3 - x_1^3 - 2x_1x_2 - x_1x_3^2 & x_1^2 + 2x_2 + x_3^2 \\ 0 & 0 \end{matrix}\right. \\ &\qquad \left.\begin{matrix} -x_1x_2 - x_2x_3 + x_2 \\ -x_1^2x_2 - x_1x_2x_3 - x_1^2x_3 + x_1^3x_2 - 2x_2x_3 + 2x_1x_2^2 - x_3^3 + x_1x_2x_3^2 + x_3 \\ 1 \end{matrix}\right], \end{aligned}$$

and

$$\begin{aligned} B(x) &= \big(M \circ F(x)\big)M^{-1}(x) \\ &= \begin{bmatrix} 1 & 0 & 2x_2 - 1 - x_2x_3 + x_3 - x_2^2 + x_1x_2 - x_1 \\ -x_2 + 1 & 1 & -(x_1 - x_2 + 1)x_2 + x_1x_2 + x_2 - 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

is a Lax pair associated with F (different from the regular one given in Remark 4.8).

Remark 4.9 Once a Lax pair (A, B) of the vector function F has been identified, some of the first integrals associated with F can be computed, as well as (by possible factorization) some of the semi-invariants associated with F. The concept of the Lax pair can be generalized for the direct computation of semi-invariants. A *generalized DT-Lax pair* (briefly, a generalized Lax pair) associated with F is an ordered pair (A, B) of matrices $A, B \in \mathbb{R}^{n\times n}$ such that $A \circ F$ and BAB^{-1} are co-linear over the field of meromorphic functions, i.e., such that

$$A \circ F = \alpha BAB^{-1},$$

for some scalar function $\alpha(x) \in \mathbb{R}$. In such a case, if $\operatorname{trace}(A^k)$ and α have not zero/pole in common, then $\omega = \operatorname{trace}(A^k)$ is a semi-invariant associated with F, with characteristic function α^k. As a matter of fact,

$$\begin{aligned} \omega \circ F &= \operatorname{trace}\big((A \circ F)^k\big) = \operatorname{trace}\big(\alpha^k BA^kB^{-1}\big) = \alpha^k \operatorname{trace}\big(A^k\big) \\ &= \alpha^k \omega. \end{aligned}$$

If $M(x) \in \mathbb{R}^{\nu\times\nu}$ is invertible over the field of meromorphic functions and (A, B) is a generalized Lax pair associated with F, then the pair $(\tilde{A}, \tilde{B})$ given in (4.15) is a generalized Lax pair associated with F, for the same function α. Define the diagonal matrix $\Lambda := \operatorname{diag}\{\omega_1, \ldots, \omega_m, 0, \ldots, 0\}$, with the ω_i's being semi-invariants associated with F, with the same characteristic function $\lambda_i = \alpha$. Clearly, (Λ, E) is a generalized Lax pair associated with F, since $\Lambda \circ F = \alpha \Lambda$. Therefore, $A = M \Lambda M^{-1}$ and $B = (M \circ F) M^{-1}$ constitute a generalized Lax pair associated with F, for any matrix $M(x) \in \mathbb{R}^{n \times n}$ being invertible over the field of meromorphic functions.

Example 4.12 Consider the vector function

$$F(x) = \begin{bmatrix} x_2^2 x_3 + x_2^5 - x_1 - x_2^2 x_4 - x_1 x_2^2 \\ -x_2 \\ -x_3 - x_2^3 \\ -x_2^2 x_3 - x_2^5 + x_2^2 x_4 + x_1 x_2^2 + x_2^3 - x_4 \end{bmatrix}.$$

A generalized Lax pair associated with F is (A, B), with

$$A = \begin{bmatrix} x_3 & x_1 \\ x_2 & x_4 + x_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & x_2^2 \\ 0 & 1 \end{bmatrix},$$

which satisfy $A \circ F = -BAB^{-1}$. Then, $\omega_1 = \operatorname{trace}(A) = x_1 + x_3 + x_4$ and $\omega_2 = \operatorname{trace}(A^2) = x_3^2 + 2x_1x_2 + x_4^2 + 2x_1x_4 + x_1^2$ are two Darboux polynomials with characteristic values $\lambda_1 = -1$ and $\lambda_2 = 1$.

4.8 The Poincaré–Dulac Normal Form for Discrete-Time Nonlinear Systems

In this section, the Poincaré–Dulac normal form is introduced for discrete-time nonlinear systems [5, 29, 57].

Throughout this section, assume that $F(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $F(0) = 0$. The *linear part* of F is Ax, with $A = \frac{\partial F(x)}{\partial x}|_{x=0}$. If not otherwise specified, assume throughout this section that A is *semi-simple*, i.e., that can be diagonalized over $\mathbb{C}$.

Definition 4.5 Vector function $F(x) = Ax + H(x)$, with A being semi-simple, $H(x)$ being analytic at $x = 0$, $H(0) = 0$, and having linear part equal to zero, is in the *Poincaré–Dulac normal form* if

$$\lfloor Ax, H(x) \rfloor = 0. \tag{4.16}$$

Remark 4.10 The Poincaré–Dulac normal form is often introduced under the assumption that the linear part Ax of F is characterized by A being normal, instead of simply semi-simple. The two definitions coincide, apart from a linear transformation, because, by Lemma 2.5 at p. 39, any semi-simple matrix can be rendered normal by a linear transformation, and any normal matrix is certainly semi-simple. Let $F(x) = A_s x + H_s(x)$, with A_s being semi-simple; let $x = Qy$,

$\det(Q) \neq 0$, be a linear transformation such that $\tilde{A}_{s,n} = Q^{-1}AQ$ is normal, and let $\tilde{H}_{s,n}(y) = Q^{-1}H_s(Qy)$. By Theorem 4.4, the following relation holds:

$$\lfloor A_s x, H_s(x) \rfloor = 0 \quad \Longleftrightarrow \quad \lfloor \tilde{A}_{s,n} y, \tilde{H}_{s,n}(y) \rfloor = 0.$$

Case $A = E$ is trivial, because any H satisfies (4.16) in such a case,

$$\lfloor Ex, H(x) \rfloor = H(Ex) - EH(x) = 0,$$

whence the Poincaré–Dulac normal form of a system with a linear part Ex does not give any insight about its properties.

Theorem 4.14 $\lfloor Ax, H(x) \rfloor = 0 \iff H(A^t x) = A^t H(x)$.

Proof The proof that $\lfloor Ax, H(x) \rfloor = 0$ implies $H(A^t x) = A^t H(x)$ is done by induction on integer t. Such an implication is clearly satisfied for $t = 0$ and for $t = 1$, taking into account that $\lfloor Ax, H(x) \rfloor = H(Ax) - AH(x)$. Assume that $H(A^t x) = A^t H(x)$, then

$$H(A^{t+1}x) = H(Ay) = AH(y) = AH(A^t x) = A^{t+1}H(x),$$

where $y = A^t x$. The converse can be proven by letting $t = 1$ in $H(A^t x) = A^t H(x)$. □

Theorem 4.15 *Assume that A is semi-simple and that 0 is not eigenvalue of A. Let $\{M_0, M_1, \ldots, M_{r-1}\}$ be a basis of the linear centralizer $\mathscr{L}_c(A)$ of A and let $\mathscr{I}_D(Ax)$ be the set of all first integrals of the discrete-time system $\Delta x = Ax$. Assume that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part. Then*

$$\lfloor Ax, H(x) \rfloor = 0 \quad \Longleftrightarrow \quad H(x) = \sum_{i=0}^{r-1} \mu_i M_i x, \quad \mu_i \in \mathscr{I}_D(Ax).$$

Proof If $\mu \in \mathscr{I}_D(Ax)$, then $\mu(A^t x) = \mu(x)$. Then,

$$H\big(A^t x\big) = \sum_{i=0}^{r-1} \mu_i\big(A^t x\big) M_i A^t x = A^t \sum_{i=0}^{r-1} \mu_i(x) M_i x = A^t H(x)$$

implies, by Theorem 4.14, that $\lfloor Ax, H(x) \rfloor = 0$. Conversely, thanks to Theorem 4.4, assume that A is diagonal, $A = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$; H is the linear combination (possibly, infinite) of terms (with $n_i \in \mathbb{Z}^{\geq}$, $\sum_{i=1}^n n_i \geq 2$)

$$\big(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\big) e_k, \tag{4.17}$$

where e_k if the kth column of the $n \times n$ identity matrix E. First, note that, since A is semi-simple, the operator $\lfloor Ax, \cdot \rfloor$ is linear and semi-simple too, in the sense that each term $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n} e_k$ is mapped by the operator $\lfloor Ax, \cdot \rfloor$ into a term proportional to $(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k$:

$$\begin{aligned}\lfloor Ax, \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k \rfloor &= (\lambda_1 x_1)^{n_1} (\lambda_2 x_2)^{n_2} \cdots (\lambda_n x_n)^{n_n} e_k - \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) A e_k \\ &= \left(\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_n^{n_n} - \lambda_k\right)\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k.\end{aligned}$$

Then, condition $\lfloor Ax, H(x) \rfloor = 0$ is equivalent to $\lfloor Ax, (x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k \rfloor = 0$ for each $(n_1, \ldots, n_n, k)$, and condition $\lfloor Ax, (x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}) e_k \rfloor = 0$ holds if and only if the following *discrete-time resonance condition* (briefly, *resonance condition* if no confusion can arise between the continuous-time and discrete-time cases) among the eigenvalues of A holds:

$$\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_n^{n_n} = \lambda_k, \quad n_i \in \mathbb{Z}^{\geq}, \sum_{i=1}^{n} n_i \geq 2. \tag{4.18}$$

If (4.18) holds, then the term (4.17) is called *resonant*; note that such a resonant term need not appear into the linear combination constituting H (it depends on the values of its coefficient into the linear combination constituting H). A monomial $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ is *resonant* if (4.18) holds for some k. It is worth pointing out that the resonance condition (4.18) implies that $\frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k}$ is a first integral of the discrete-time system $\Delta x = Ax$, since $\lambda_k \neq 0$ implies:

$$\begin{aligned}\Delta \frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k} &= \frac{(\Delta x_1)^{n_1} (\Delta x_2)^{n_2} \cdots (\Delta x_n)^{n_n}}{\Delta x_k} = \frac{(\lambda_1 x_1)^{n_1} (\lambda_2 x_2)^{n_2} \cdots (\lambda_n x_n)^{n_n}}{\lambda_k x_k} \\ &= \frac{(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n})}{x_k}.\end{aligned}$$

Then,

$$\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k = \frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k} [0 \ \ldots \ e_k \ \ldots \ 0] \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix}.$$

Since the coefficient matrix $\bar{M}_k := [0 \ \ldots \ e_{k+1} \ \ldots \ 0]$ commutes with matrix A and the coefficient $\frac{x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}}{x_k}$ is a first integral of the discrete-time system $\Delta x = Ax$, one finds that $H(x) = \sum_{i=0}^{n-1} \mu_i \bar{M}_i x$, with $\bar{M}_0, \bar{M}_1, \ldots, \bar{M}_{n-1}$ belonging to the linear centralizer $\mathscr{L}_c(A)$ of A and the coefficients μ_i being DT-first integrals of $\Delta x = Ax$. □

By Proposition 2.1 of [57], any F, analytic at $x = 0$, with a semi-simple linear part can be *formally* transformed into its Poincaré–Dulac normal form through a *for-*

mal series; some convergence conditions guarantee, in some cases, that such a transformation is analytic. When the series does not converge, by the Borel Lemma [62], there exists a C^∞-diffeomorphism such that the push-forward of F differs from its normal form for a vector function being flat at $x = 0$; this means that, for any arbitrarily large integer $m > 0$, there exists a polynomial diffeomorphism such that the push-forward of F differs from its normal form for terms of order higher than m.

Remark 4.11 Consider the simplest case $n = 1$. Let $A = \lambda$, with λ being real. If $\lambda = 0$, since $\lfloor \lambda x, H(x) \rfloor = H(\lambda x) - \lambda H(x)$, condition $\lfloor \lambda x, H(x) \rfloor = 0$ is satisfied by any $H(x)$, analytic at $x = 0$, with $H(0) = 0$, $\frac{\partial H(x)}{\partial x}|_{x=0} = 0$. Assume that $\lambda \neq 0$. The linear centralizer of A is spanned by $E = 1$; if $|\lambda| \neq 1$, then the discrete-time system $\Delta x = \lambda x$ has no first integrals and, therefore, the Poincaré–Dulac normal form associated with A is $F(x) = \lambda x$. If $\lambda = 1$, then a first integral of the discrete-time system $\Delta x = x$ is x and, therefore, the Poincaré–Dulac normal form associated with A is $F(x) = x + x\mu(x)$, where μ is an arbitrary function of x (in such a case, no insight about the dynamics of the system can be found from the Poincaré–Dulac normal form). Finally, if $\lambda = -1$, then a first integral of the discrete-time system $\Delta x = -x$ is x^2 and, therefore, the Poincaré–Dulac normal form associated with A is $F(x) = x + x\mu(x^2)$, where μ is an arbitrary function of x^2.

Example 4.13 Let $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right]$. The linear centralizer of A is spanned by E and A, whereas the set of all first integrals of the discrete-time system $\Delta x = Ax$ is constituted by all arbitrary functions of $\frac{x_1^2}{x_2}$; then, the Poincaré–Dulac normal form $F(x) = Ax + H(x)$ associated with such an A is characterized by

$$H(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\mu_0 + 2\mu_1)x_1 \\ (\mu_0 + 4\mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I(x) = \frac{x_1^2}{x_2}$ such that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part. Then, necessarily $\mu_1(I) + 2\mu_2(I) = 0$ and $\mu_1(I) + 4\mu_2(I) = aI = a\frac{x_1^2}{x_2}$, for some $a \in \mathbb{R}$. Then, one concludes that $F(x) = [2x_1 \; 4x_2 + ax_1^2]^\top$, with the only resonant term $[0 \; ax_1^2]^\top$.

Example 4.14 Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. The linear centralizer of A is spanned by E and A, whereas the set of all first integrals of the discrete-time system $\Delta x = Ax$ is constituted by all arbitrary functions of x_1; then, the Poincaré–Dulac normal form $F(x) = Ax + H(x)$ associated with such an A is characterized by

$$H(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\mu_0 + \mu_1)x_1 \\ (\mu_0 + 2\mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I(x) = x_1$ such that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part, which yields

$$F(x) = \begin{bmatrix} x_1(1 + \mu_0 + \mu_1) \\ x_2(2 + \mu_0 + 2\mu_1) \end{bmatrix}.$$

Example 4.15 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The linear centralizer of A is spanned by E and A, whereas the set of all first integrals of the discrete-time system $\Delta x = Ax$ is constituted by all arbitrary functions of x_1 and x_2^2; then, the Poincaré–Dulac normal form $F(x) = Ax + H(x)$ associated with such an A is characterized by

$$H(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\mu_0 + \mu_1)x_1 \\ (\mu_0 - \mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I_1(x) = x_1$ and $I_2(x) = x_2^2$ such that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part, which yields

$$F(x) = \begin{bmatrix} x_1(1 + \mu_0 + \mu_1) \\ x_2(-1 + \mu_0 - \mu_1) \end{bmatrix}.$$

Example 4.16 Let $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$. The linear centralizer of A is spanned by E and A, whereas the set of all first integrals of the discrete-time system $\Delta x = Ax$ is constituted by all arbitrary functions of $x_1 x_2$; then, the Poincaré–Dulac normal form $F(x) = Ax + H(x)$ associated with such an A is characterized by

$$H(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\mu_0 + \frac{1}{2}\mu_1)x_1 \\ (\mu_0 + 2\mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I(x) = x_1 x_2$ such that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part, which yields

$$F(x) = \begin{bmatrix} (\frac{1}{2} + \mu_0 + \frac{1}{2}\mu_1)x_1 \\ (2 + \mu_0 + 2\mu_1)x_2 \end{bmatrix}.$$

Example 4.17 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The linear centralizer of A is spanned by E and A. In order to find all first integrals of the discrete-time system $\Delta x = Ax$, apply the procedure of Sect. 4.2. Taking as basis polynomials $p_1(x) = x_1$, $p_2(x) = x_2$, $p_3(x) = x_1^2$, $p_4(x) = x_1 x_2$ and $p_5(x) = x_2^2$, one has

$$\Gamma(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_2 & -x_1 & x_2^2 & -x_1x_2 & x_1^2 \\ -x_1 & -x_2 & x_1^2 & x_1x_2 & x_2^2 \\ -x_2 & x_1 & x_2^2 & -x_1x_2 & x_1^2 \\ x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}.$$

The generic rank of Γ is 4. Computing the minor $\hat{\Gamma}$ found by deleting the fifth column and fifth row of Γ, one finds that $\det(\hat{\Gamma}(x)) = 4x_1x_2(x_1^2 + x_2^2)^2$. Letting $\omega_1(x) = x_1 x_2$ and $\omega_2(x) = x_1^2 + x_2^2$, one concludes that $\Delta\omega_1 = -\omega_1$ and $\Delta\omega_2 = \omega_2$, which shows that $I_1(x) = \omega_1^2(x) = x_1^2 x_2^2$ and $I_2(x) = \omega_2(x) = x_1^2 + x_2^2$ are DT-first integrals of the discrete-time system $\Delta x = Ax$. Then, the Poincaré–Dulac normal form $F(x) = Ax + H(x)$ associated with such an A is characterized by

$$H(x) = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_0 x_1 + \mu_1 x_2 \\ \mu_0 x_2 - \mu_1 x_1 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I_1(x) = \omega_1^2(x) = x_1^2 x_2^2$ and $I_2(x) = \omega_2(x) = x_1^2 + x_2^2$ such that H is analytic at $x = 0$, $H(0) = 0$, with zero linear part, which yields

$$F(x) = \begin{bmatrix} \mu_0 x_1 + (1 + \mu_1) x_2 \\ -(1 + \mu_1) x_1 + \mu_0 x_2 \end{bmatrix}.$$

Remark 4.12 Assume that $A = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, for $\lambda_i \in \mathbb{R}^{>}$, and that there exists some $\varepsilon > 0$ such that $\gamma_i := \frac{\ln(\lambda_i)}{\ln(\varepsilon)}$, $i = 1, \ldots, n$, are positive integers; this means that $A = \mathrm{e}^{G \ln(\varepsilon)}$, where $G = \mathrm{diag}\{\gamma_1, \ldots, \gamma_n\}$. Assume that the number of DT-resonances among the eigenvalues of A is finite, which implies that the number of CT-resonances among the eigenvalues of G is finite. Assume that $F(x) = Ax + H(x)$ is in the Poincaré–Dulac normal form, i.e, $\lfloor Ax, H(x) \rfloor = 0$; hence,

$$\begin{aligned} \lfloor Ax, F(x) \rfloor &= \lfloor Ax, Ax + H(x) \rfloor = \big(Ax + H(x)\big) \circ Ax - A\big(Ax + H(x)\big) \\ &= A^2 x + H(Ax) - A^2 x - AH(x) = \lfloor Ax, H(x) \rfloor = 0. \end{aligned}$$

This yields $F \circ (Ax) = AF(x)$. Let $\omega(x) \in \mathbb{R}$ be homogeneous of degree m with respect to Gx; by Theorem 3.15 at p. 74, $\omega(Ax) = \omega(\mathrm{e}^{G \ln(\varepsilon)} x) = \mathrm{e}^{m \ln(\varepsilon)} \omega(x) = \varepsilon^m \omega(x)$. If $\omega(x) = x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ is a resonant monomial, $\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_n^{n_n} = \lambda_k$, then $\omega(Ax) = \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_n^{n_n} \omega(x) = \lambda_k \omega(x) = \varepsilon^{\gamma_k} \omega(x)$, which implies that the resonant monomial is homogeneous of degree γ_k with respect to Gx. Vice versa, if ω is analytic at $x = 0$ and homogeneous of degree γ_k with respect to Gx, then it is a linear combination with real coefficients of resonant monomials of degree λ_k. Under the assumption that $\omega(x) = x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ is a resonant monomial, by $\omega \circ F \circ (Ax) = \omega \circ (AF) = \varepsilon^{\gamma_k} (\omega \circ F)$, one concludes that $\omega \circ F$ is a linear combination with real coefficients of resonant monomials of degree λ_k. Since the number of resonant monomials has been assumed to be finite, one concludes that a discrete-time nonlinear system in the Poincaré–Dulac normal form can be linearized by taking as additional state variables the resonant monomials (see [95]).

Example 4.18 Let $A = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, with $\lambda_1 = 2$, $\lambda_2 = 4$ and $\lambda_3 = 8$; hence, $\lambda_i = \mathrm{e}^{\gamma_i \ln(\varepsilon)}$, $i = 1, 2, 3$, with $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 3$ and $\varepsilon = 2$. Since $\lambda_2 = \lambda_1^2 \lambda_2^0 \lambda_3^0$, $\lambda_3 = \lambda_1^3 \lambda_2^0 \lambda_3^0$ and $\lambda_3 = \lambda_1^1 \lambda_2^1 \lambda_3^0$ are the only resonances among the eigenvalues of A, all discrete-time systems, having Ax as linear part, in the Poincaré–Dulac normal form are parameterized by

$$F(x) = \begin{bmatrix} 2x_1 \\ 4x_2 + a_1 x_1^2 \\ 8x_3 + a_2 x_1^3 + a_3 x_1 x_2 \end{bmatrix},$$

where a_1, a_2 and a_3 are constant parameters. Such a system can be linearized by taking as additional state variables $x_4 = x_1^2$, $x_5 = x_1^3$ and $x_6 = x_1 x_2$. To be more precise, the dynamics of x_4 are described by $\Delta x_4 = F_1^2(x) = 4x_1^2 = 4x_4$, the dynamics of x_5 are described by $\Delta x_5 = F_1^3(x) = 8x_1^3 = 8x_5$, and the dynamics of x_6 are described by $\Delta x_6 = F_1(x)F_2(x) = 2a_1 x_1^3 + 8x_1 x_2 = 2a_1 x_5 + 8x_6$. Then, one has the extended linear system $\Delta x_e = A_e x_e$, with

$$A_e = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 8 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2a_1 & 8 \end{bmatrix}, \qquad x_e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Note that, under the assumption that the real numbers a_i are non-zero, the Jordan form of A_e is

$$J_e = \left[\begin{array}{c|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right],$$

namely, although the original A is semi-simple, the state immersion has generated in A_e Jordan blocks of dimension greater than 1 (A_e is not semi-simple), and this justifies the name *resonance* used to represent this phenomenon. It is worth pointing out that if some a_i is equal to zero, i.e., if some resonant term is missing in the Poincaré–Dulac normal form, then the Jordan form of A_e may differ from the above reported J_e. For instance, if $a_3 = 0$, the Jordan form of A_e is

$$\bar{J}_e = \left[\begin{array}{c|cc|cc|c} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 8 \end{array}\right].$$

4.9 Linearization of Discrete-Time Nonlinear Systems

The following theorem is the extension of Theorem 3.35 at p. 121 to the discrete-time case, giving a necessary and sufficient condition for the linearization of a discrete-time system.

Theorem 4.16 *Assume that $F(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $F(0) = 0$, with linear part Ax, where A need not be semi-simple. There exists a near-identity diffeomor-*

phism $y = \varphi(x)$ such that $\varphi_* F(y) = \varphi \circ F \circ \varphi^{-1}(y) = Ay$ if and only if there exists a $g(x) \in \mathbb{R}^n$, analytic at $x = 0$, $g(0) = 0$, such that $\lfloor F, g \rfloor = 0$ and the linear part of g is x.

Proof If $\tilde{F}(y) = Ay$, then $\tilde{g}(y) = y$ satisfies $\lfloor \tilde{F}, \tilde{g} \rfloor = 0$. By the pull-back of $\tilde{g}$ one obtains $g(x) = (\frac{\partial \varphi(x)}{\partial x})^{-1} \varphi(x)$ that is analytic at $x = 0$, satisfies $g(0) = 0$, and has x as linear part. Furthermore, by Theorem 4.4, $\lfloor F, g \rfloor = 0$. Conversely, for any g being analytic at $x = 0$, $g(0) = 0$, and with linear part x, the Poincaré–Dulac Theorem 3.33 at p. 118 implies the existence of a near-identity diffeomorphism $y = \varphi(x)$ such that $\varphi_* g(y) = y$, by virtue of the absence of resonances among the eigenvalues of the linear part of g. If $\lfloor F, g \rfloor = 0$, then $\lfloor \varphi_* F(y), \varphi_* g(y) \rfloor = \lfloor \varphi_* F(y), y \rfloor = 0$; since $\lfloor \varphi_* F(y), By \rfloor = [\varphi_* F(y), By]$, for any $B \in \mathbb{R}^{n \times n}$, condition $\lfloor \varphi_* F(y), y \rfloor = 0$ implies that $\varphi_* F$ is homogeneous of degree 0 with respect to the standard dilation; since $\varphi_* F$ is analytic at $y = 0$, $\varphi_* F(0) = 0$, it is necessarily linear. □

Example 4.19 Consider

$$F(x) = \begin{bmatrix} x_2 + x_1^2 \\ -x_1 - x_2^2 - 2x_1^2 x_2 - x_1^4 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} x_1 \\ x_2 - x_1^2 \end{bmatrix},$$

and call F_1 and F_2 the two entries of F. Since $\lfloor F(x), g(x) \rfloor = 0$, g is a symmetry of F. In particular, since g is analytic at $x = 0$, $g(0) = 0$, and the linear part of g is x, there exists a diffeomorphism $y = \varphi(x)$ such that the push-forward of g is $\varphi_*(y) = y$; therefore, such a diffeomorphism $y = \varphi(x)$ linearizes the discrete-time system. In particular, $\varphi(x) = [x_1 \; x_2 + x_1^2]^\top$ and

$$\varphi_* F(y) = \varphi \circ F \circ \varphi^{-1}(y) = \begin{bmatrix} F_1(x) \\ F_2(x) + F_1^2(x) \end{bmatrix}_{x_1 = y_1, x_2 = y_2 - y_1^2} = \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}.$$

4.10 Homogeneity and Resonance of Discrete-Time Nonlinear Systems

The definition of homogeneity of a vector function in the discrete-time case must be properly amended with respect to the continuous-time case, whereas it remains unchanged in case of scalar functions.

Definition 4.6 Let $g(x) = Bx$, with $B \in \mathbb{R}^{n \times n}$ being semi-simple and such that there exist a $G \in \mathbb{R}^{n \times n}$ and a positive scalar ε such that $B = e^{G \ln(\varepsilon)}$. Then, $\Delta x = F(x)$ is *homogeneous* of degree m with respect to g if $F(Bx) = B^m F(x)$, namely if $\lfloor Bx, F(x) \rfloor = -BF(x) + B^m F(x)$.

By [37], the equation $B = e^{G \ln(\varepsilon)}$ has real solutions $G \in \mathbb{R}^{n \times n}$, $\varepsilon \in \mathbb{R}^{\geq 0}$, if and only if $\det(B) \neq 0$ and each Jordan block of B corresponding to a negative eigenvalue appears an even number of times in the Jordan form of B.

According to Definition 4.6, which holds in the discrete-time case, if $F(x) \in \mathbb{R}^n$ is homogeneous of degree 1 with respect to Bx, then $\lfloor Bx, F(x) \rfloor = 0$; note that, according to Definition 3.8 at p. 73, which holds in the continuous-time case, if $[Bx, f(x)] = 0$, then $f(x) \in \mathbb{R}^n$ is homogeneous of degree 0 with respect to Bx.

If $\omega(x) \in \mathbb{R}$ is homogeneous of degree m with respect to Gx, then $\omega(Bx) = \omega(\mathrm{e}^{G\ln(\varepsilon)}x) = \mathrm{e}^{m\ln(\varepsilon)}\omega(x) = \varepsilon^m\omega(x)$.

Theorem 4.17 $\lfloor Bx, F(x) \rfloor = -BF(x) + B^m F(x) \iff F(B^t x) = B^{mt} F(x)$, $\forall t \in \mathbb{Z}^{\geq}$.

Proof Note that $\det(B) \neq 0$, by $B = \mathrm{e}^{G\ln(\varepsilon)}$. If $\lfloor Bx, F(x) \rfloor = -BF(x) + B^m F(x)$, namely if $F(Bx) = B^m F(x)$, then clearly $F(B^t x) = B^{mt} F(x), \forall t \in \mathbb{Z}$. Conversely, if $F(B^t x) = B^{mt} F(x), \forall t \in \mathbb{Z}$, letting $t = 1$, one concludes that $F(Bx) = B^m F(x)$. □

In the remainder of this section, assume that F is analytic at $x = 0$. For the sake of simplicity, assume that B is diagonal $B = \mathrm{diag}\{\gamma_1, \ldots, \gamma_n\}$. Under the assumption that $F(x)$ is analytic at $x = 0$, then it is the (possibly, infinite) sum of terms

$$\left(x_1^{n_1} x_2^{n_2} \ldots x_n^{n_n}\right) e_k,$$

where e_k is the kth column of the $n \times n$ identity matrix. Since B is semi-simple and diagonal, then $\lfloor Bx, \cdot \rfloor$ is semi-simple too. Then, condition $\lfloor Bx, F(x) \rfloor = -BF(x) + B^m F(x)$ is satisfied if and only if

$$\left\lfloor Bx, \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k \right\rfloor = -B\left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k + B^m \left(x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}\right) e_k;$$

such a condition holds if and only if the following *discrete-time generalized resonance condition* (briefly, *generalized resonance condition*) holds:

$$\gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_n^{n_n} = \gamma_k^m, \tag{4.19}$$

for some $n_1, n_2, \ldots, n_n \in \mathbb{Z}^{\geq}$ such that $n_1 + n_2 + \cdots + n_n \geq 1$.

Example 4.20 Let $B = \mathrm{diag}\{\gamma_1, \gamma_2\}$, with $\gamma_1 = 2$ and $\gamma_2 = 4$. Let $m = 1$. Then, the generalized resonance condition relative to γ_1 yields

$$\begin{array}{c} \gamma_1^1 = \gamma_1^{n_1} \gamma_2^{n_2} \Rightarrow x_1^{n_1} x_2^{n_2} e_1 \\ \hline 2 = \gamma_1^1 \gamma_2^0 \quad \Rightarrow \qquad x_1 e_1 \end{array};$$

the generalized resonance condition relative to γ_2 yields

$$\begin{array}{c} \gamma_2^1 = \gamma_1^{n_1} \gamma_2^{n_2} \Rightarrow x_1^{n_1} x_2^{n_2} e_2, \\ \hline 4 = \gamma_1^2 \gamma_2^0 \quad \Rightarrow \qquad x_1^2 e_2 \\ 4 = \gamma_1^0 \gamma_2^1 \quad \Rightarrow \qquad x_2 e_2 \end{array};$$

hence,

$$F^{[1]}(x) = a_1 x_1 e_1 + a_2 x_1^2 e_2 + a_3 x_2 e_2 = \begin{bmatrix} a_1 x_1 \\ a_2 x_1^2 + a_3 x_2 \end{bmatrix}.$$

Let $m = 2$. Then, the generalized resonance condition relative to γ_1 yields

$$\begin{array}{l} \gamma_1^2 = \gamma_1^{n_1}\gamma_2^{n_2} \Rightarrow x_1^{n_1}x_2^{n_2}e_1 \\ \hline 2^2 = \gamma_1^2\gamma_2^0 \quad \Rightarrow \qquad x_1^2 e_1 \\ 2^2 = \gamma_1^0\gamma_2^1 \quad \Rightarrow \qquad x_2 e_1 \end{array};$$

the generalized resonance condition relative to γ_2 yields

$$\begin{array}{l} \gamma_2^2 = \gamma_1^{n_1}\gamma_2^{n_2} \Rightarrow x_1^{n_1}x_2^{n_2}e_2 \\ \hline 4^2 = \gamma_1^4\gamma_2^0 \quad \Rightarrow \qquad x_1^4 e_2 \\ 4^2 = \gamma_1^2\gamma_2^1 \quad \Rightarrow \quad x_1^2 x_2 e_2 \\ 4^2 = \gamma_1^0\gamma_2^2 \quad \Rightarrow \qquad x_2^2 e_2 \end{array};$$

hence,

$$\begin{aligned} F^{[2]}(x) &= a_1 x_1^2 e_1 + a_2 x_2 e_1 + a_3 x_1^4 e_2 + a_4 x_1^2 x_2 e_2 + a_5 x_2^2 e_2 \\ &= \begin{bmatrix} a_1 x_1^2 + a_2 x_2 \\ a_3 x_1^4 + a_4 x_1^2 x_2 + a_5 x_2^2 \end{bmatrix}. \end{aligned}$$

Let $m = 3$. Then, the generalized resonance condition relative to γ_1 yields

$$\begin{array}{l} \gamma_1^3 = \gamma_1^{n_1}\gamma_2^{n_2} \Rightarrow x_1^{n_1}x_2^{n_2}e_1 \\ \hline 2^3 = \gamma_1^3\gamma_2^0 \quad \Rightarrow \qquad x_1^3 e_1 \\ 2^3 = \gamma_1^1\gamma_2^1 \quad \Rightarrow \quad x_1 x_2 e_1 \end{array};$$

the generalized resonance condition relative to γ_2 yields

$$\begin{array}{l} \gamma_2^3 = \gamma_1^{n_1}\gamma_2^{n_2} \Rightarrow x_1^{n_1}x_2^{n_2}e_2 \\ \hline 4^3 = \gamma_1^6\gamma_2^0 \quad \Rightarrow \qquad x_1^6 e_2 \\ 4^3 = \gamma_1^4\gamma_2^1 \quad \Rightarrow \quad x_1^4 x_2 e_2 \\ 4^3 = \gamma_1^2\gamma_2^2 \quad \Rightarrow \quad x_1^2 x_2^2 e_2 \\ 4^3 = \gamma_1^0\gamma_2^3 \quad \Rightarrow \qquad x_2^3 e_2 \end{array};$$

hence,

$$\begin{aligned} F^{[3]}(x) &= a_1 x_1^3 e_1 + a_2 x_1 x_2 e_1 + a_3 x_1^6 e_2 + a_4 x_1^4 x_2 e_2 + a_5 x_1^2 x_2^2 e_2 + a_6 x_2^3 e_2 \\ &= \begin{bmatrix} a_1 x_1^3 + a_2 x_1 x_2 \\ a_3 x_1^6 + a_4 x_1^4 x_2 + a_5 x_1^2 x_2^2 + a_6 x_2^3 \end{bmatrix}. \end{aligned}$$

4.11 The Belitskii Normal Form of Discrete-Time Nonlinear Systems

Throughout this section, assume that $F(x) \in \mathbb{R}^n$ is analytic at $x = 0$, $F(0) = 0$. The *linear part* of F is Ax, with $A = \frac{\partial F(x)}{\partial x}|_{x=0}$ that need not be semi-simple. Assume that matrix A can be expressed as $A = A_{s,n} + A_n$, where $A_{s,n} \in \mathbb{R}^{n\times n}$ is normal, $A_n \in \mathbb{R}^{n\times n}$ is nilpotent, and $[A_{s,n}, A_n] = [A_{s,n}, A_n^\top] = 0$ (by Lemma 2.5 at p. 39, this can be obtained for any $A \in \mathbb{R}^{n\times n}$ using a real linear transformation).

Definition 4.7 Vector function $F(x) = Ax + H(x)$, with $H(x)$ being analytic at $x = 0$, $H(0) = 0$, and having linear part equal to zero, is in the *Belitskii normal form* if

$$\lfloor A^\top x, H(x)\rfloor = 0. \tag{4.20}$$

Remark 4.13 If A is normal, then the Belitskii normal form of a discrete-time nonlinear system coincides with its Poincaré–Dulac normal form.

By the definition of the DT-Lie bracket, under the above positions, F is in the Belitskii normal form if and only if

$$H\big(A^\top x\big) = A^\top H(x),$$

which implies

$$H\big(\big(A^\top\big)^t x\big) = \big(A^\top\big)^t H(x), \quad \forall t \in \mathbb{Z}\ \big(t \geq 0 \text{ if } \det(A) = 0\big).$$

Given $A \in \mathbb{R}^{n\times n}$, with 0 that is not eigenvalue of A, let $\{M_0, \dots, M_{r-1}\}$ be a basis of $\mathscr{L}_c(A^\top)$. All $H \in \mathscr{C}_D(A^\top x)$ are parameterized by $H(x) = \mu_0 M_0 x + \cdots + \mu_{r-1} M_{r-1} x$, with $\mu_0, \dots, \mu_{r-1} \in \mathscr{I}_D(A^\top x)$. Hence, $F(x) = Ax + H(x)$ is in the Belitskii normal form if and only if $H \in \mathscr{C}_D(A^\top x)$, H is analytic at $x = 0$, $H(0) = 0$, with zero linear part.

By Proposition 2.1 of [57], any F, analytic at $x = 0$, can be *formally* transformed into its Belitskii normal form through a *formal* series; some convergence conditions guarantee, in some cases, that such a transformation is analytic. When the series does not converge, by the Borel Lemma [62], there exists a C^∞-diffeomorphism such that the push-forward of F differs from its normal form for a vector function being flat at $x = 0$; this also means that, for any arbitrarily large integer $m > 0$, there exists a polynomial diffeomorphism such that the push-forward of F differs from its normal form for terms of order higher than m.

Remark 4.14 By Lemma 2.4 at p. 37, since $A^\top = A_{s,n}^\top + A_n^\top$, with $A_{s,n}^\top$ being normal, $A_n^\top$ being nilpotent and $[A_{s,n}^\top, A_n^\top] = [A_{s,n}^\top, A_n] = 0$, one concludes that $\mathscr{L}_c(A^\top) = \mathscr{L}_c(A_{s,n}^\top + A_n^\top) = \mathscr{L}_c(A_{s,n}^\top) \cap \mathscr{L}_c(A_n^\top)$; if $A_{s,n}$ is diagonal, then $\mathscr{L}_c(A^\top) = \mathscr{L}_c(A_{s,n}) \cap \mathscr{L}_c(A_n^\top)$. This means that in order to find all F in the Belitskii normal form and with linear part Ax, one can first find all $F_{s,n}$ being in the

Poincaré–Dulac normal form with linear part $A_{s,n}x$, then $F(x) = A_n x + F_{s,n}(x)$ is in the Belitskii normal form if it satisfies the further requirement $\lfloor A^\top x, F(x)\rfloor = 0$.

Example 4.21 Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$; $\mathscr{L}_c(A^\top) = \mathrm{span}_{\mathbb{R}}\{E, A^\top\}$, and the set of all first integrals of the discrete-time system $\Delta x = A^\top x$ is given by all arbitrary functions of x_1; then, the Belitskii normal form associated with such an A is

$$F(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (1+\mu_0+\mu_1)x_1 + x_2 \\ \mu_1 x_1 + (1+\mu_0+\mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I(x) = x_1$, such that F is analytic at $x = 0$, $F(0) = 0$, with linear part Ax.

Example 4.22 Let $A = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$. The linear centralizer of $A^\top$ is spanned by E and $A^\top$, and the set of all first integrals of the discrete-time system $\Delta x = A^\top x$ is constituted by all arbitrary functions of x_1^2; then the Belitskii normal form associated with such an A is

$$F(x) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (-1+\mu_0-\mu_1)x_1 + x_2 \\ \mu_1 x_1 + (-1+\mu_0-\mu_1)x_2 \end{bmatrix},$$

where μ_0 and μ_1 are arbitrary functions of $I(x) = x_1^2$, such that F is analytic at $x = 0$, $F(0) = 0$, with linear part Ax.

Example 4.23 Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

which can be decomposed as $A = A_{s,n} + A_n$,

$$A_{s,n} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad A_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $A_{s,n}$ being normal and A_n being nilpotent. There are three resonances among the eigenvalues of $A_{s,n}$,

$$2^2 2^0 4^0 = 4 \quad \Rightarrow \quad x_1^2 e_3, \qquad 2^1 2^1 4^0 = 4 \quad \Rightarrow \quad x_1 x_2 e_3,$$
$$2^0 2^2 4^0 = 4 \quad \Rightarrow \quad x_2^2 e_3,$$

thus obtaining $H_{s,n}$,

$$H_{s,n}(x) = \begin{bmatrix} 0 \\ 0 \\ a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 \end{bmatrix}.$$

Now, since

$$H_{s,n}(A^\top x) = \begin{bmatrix} 0 \\ 0 \\ (a_3 + 4a_1 + 2a_2)x_1^2 + (4a_3 + 4a_2)x_1 x_2 + 4a_3 x_2^2 \end{bmatrix}$$

$$A^\top H_{s,n}(x) = \begin{bmatrix} 0 \\ 0 \\ 4a_1 x_1^2 + 4a_2 x_1 x_2 + 4a_3 x_2^2 \end{bmatrix},$$

the condition $H_{s,n}(A^\top x) = A^\top H_{s,n}(x)$ leads to the equations

$$(a_3 + 4a_1 + 2a_2) = 4a_1, \qquad (4a_3 + 4a_2) = 4a_2, \qquad 4a_3 = 4a_3,$$

which have solution $a_2 = 0$, $a_3 = 0$ and a_1 arbitrary, which yields the following F in the Belitskii normal form:

$$F(x) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_1 x_1^2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 2x_2 \\ 4x_3 + a_1 x_1^2 \end{bmatrix}.$$

4.12 Decomposition of Discrete-Time Nonlinear Systems

This section extends some of the results of Sect. 3.17 to discrete-time systems (some related results can be found in [100]).

Theorem 4.18 *Let $g_1(x), \dots, g_m(x) \in \mathbb{R}^n$ be m linearly independent (over $\mathcal{K}_n$) and pairwise commuting symmetries of F, i.e.,*

$$\lfloor F, g_i \rfloor = 0, \quad i = 1, \dots, m, \tag{4.21a}$$

$$\operatorname{rank}_{\mathcal{K}_n}\big([g_1 \; \dots \; g_m]\big) = m, \tag{4.21b}$$

$$[g_i, g_j] = 0, \quad i, j \in \{1, \dots, m\}. \tag{4.21c}$$

Then, there exist local coordinates $y = \varphi(x)$ such that the nonlinear system (1.1*b*) *can be* decomposed *in the y-coordinates as*

$$y_a(t+1) = \tilde{F}_a\big(y_a(t), y_b(t)\big),$$

$$y_b(t+1) = \tilde{F}_b\big(y_b(t)\big),$$

where $y_a = [y_1 \; \dots \; y_m]^\top$, $y_b = [y_{m+1} \; \dots \; y_n]^\top$ and $\tilde{F}^\top = [\tilde{F}_a^\top \; \tilde{F}_b^\top]$.

Proof By (4.21b), (4.21c), there exists a diffeomorphism $y = \varphi(x)$ such that the push-forward of g_i is straightened $\varphi_* g_i = e_i$, $i = 1, \dots, m$. Then, the condition $\lfloor \varphi_* F, \varphi_* g_i \rfloor = 0$ can be rewritten as follows, with $\tilde{F} = \varphi_* F$:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\partial \tilde{F}_1}{\partial y_i} \\ \vdots \\ \frac{\partial \tilde{F}_{i-1}}{\partial y_i} \\ \frac{\partial \tilde{F}_i}{\partial y_i} \\ \frac{\partial \tilde{F}_{i+1}}{\partial y_i} \\ \vdots \\ \frac{\partial \tilde{F}_n}{\partial y_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, \dots, m,$$

which shows that the last $n - m$ entries of $\tilde{F}$ do not depend on y_i, $i = 1, \dots, m$. □

Consider now the nonlinear system (1.1b) endowed with an output function, which, for simplicity, is assumed to be scalar,

$$\Delta x = F(x), \tag{4.22a}$$

$$y = h(x), \tag{4.22b}$$

where $h(x) \in \mathbb{R}$ is meromorphic. Consider the functions $\Delta^0 h = h$, $\Delta^{i+1} h = \Delta(\Delta^i h) = h \circ \underbrace{F \circ \cdots \circ F}_{(i+1)\text{-times}}$. Let index q be such that $\Delta^0 h, \dots, \Delta^{q-1} h$ are functionally independent, but $\Delta^0 h, \dots, \Delta^q h$ are functionally dependent. Then, there exists a meromorphic function $\Theta(z_1, \dots, z_{q+1})$ such that $\Theta(\Delta^0 h, \dots, \Delta^q h) = 0$ identically. Since $\Delta^0 h, \dots, \Delta^{q-1} h$ are functionally independent, it is impossible that $\frac{\partial \Theta(z_1, \dots, z_{q+1})}{\partial z_{q+1}}$ is identically equal to zero, whence $\Theta(\Delta^0 h, \dots, \Delta^q h) = 0$ implies that $\Delta^q h = \Xi_1(\Delta^0 h, \dots, \Delta^{q-1} h)$ holds locally, for some meromorphic function Ξ_1. This means that

$$\xi = \begin{bmatrix} \Delta^0 h(x) \\ \vdots \\ \Delta^{q-1} h(x) \end{bmatrix}$$

qualifies as a partial diffeomorphism such that the nonlinear system (4.22a), (4.22b) is transformed into

$$\begin{aligned} \Delta \xi_1 &= \xi_2, \\ &\vdots \\ \Delta \xi_{q-1} &= \xi_q, \end{aligned}$$

$$\begin{aligned}
\Delta\xi_q &= \Xi_1(\xi_1, \ldots, \xi_q),\\
\Delta\eta &= \Xi_2(\xi_1, \ldots, \xi_q, \eta),\\
y &= \xi_1,
\end{aligned}$$

where $\eta \in \mathbb{R}^{n-q}$ are suitable additional state variables that complete the choice of the local variables ξ.

Chapter 5
Analysis of Hamiltonian Systems

5.1 Euler–Lagrange Equations

Consider a mechanical system whose configuration is described by a vector $q \in \mathbb{R}^\nu$ of *generalized coordinates*: similarly, $\dot{q} := \frac{\mathrm{d}q}{\mathrm{d}t}$ and $\ddot{q} := \frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$ are the vectors of *generalized velocities* and *generalized accelerations*, respectively. The kinetic energy of the mechanical system is $T(q,\dot{q}) = \frac{1}{2}\dot{q}^\top B(q)\dot{q}$, where $B(q) \in \mathbb{R}^{\nu \times \nu}$ is the *generalized inertia matrix*, with $\det(B(q)) \neq 0$ and $B(q)$ being symmetric and positive definite. Let $U(q)$ be the potential energy of the system and assume that it is possible to neglect the non-conservative forces (such as friction). Then, letting $L(q,\dot{q}) = T(q,\dot{q}) - U(q) = \frac{1}{2}\dot{q}^\top B(q)\dot{q} - U(q)$ be the *Lagrangian function*, from the *Hamilton least action principle* (see [54]), one concludes that the motion of the mechanical system is described by the *Euler–Lagrange equations*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)^\top = 0, \tag{5.1}$$

which can be rewritten as

$$B(q)\ddot{q} + \frac{\mathrm{d}B(q)}{\mathrm{d}t}\dot{q} + \left(\frac{\partial U(q)}{\partial q}\right)^\top = 0. \tag{5.2}$$

Consider a nonlinear system $\frac{\mathrm{d}q}{\mathrm{d}\tau} = g(q)$, $g(q) \in \mathbb{R}^\nu$; let $\Phi_g(\tau, q)$ be the flow associated with g. Consider the one-parameter group of transformations $q = \Phi_g(\tau, \tilde{q})$ and compute, accordingly, $\dot{q} = \frac{\partial \Phi_g(\tau,\tilde{q})}{\partial \tilde{q}}\dot{\tilde{q}}$. Then, the Lagrangian function can be rewritten as a function of $\tilde{q}$ and $\dot{\tilde{q}}$ as follows:

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}) = L\left(\Phi_g(\tau, \tilde{q}), \frac{\partial \Phi_g(\tau, \tilde{q})}{\partial \tilde{q}}\dot{\tilde{q}}\right). \tag{5.3}$$

Definition 5.1 [6] The one-parameter group of transformations $q = \Phi_g(\tau, \tilde{q})$ (also, briefly, its infinitesimal generator g) is a *symmetry* of the Lagrangian function

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$L(q,\dot{q})$ if

$$\tilde{L}(q,\dot{q}) = L(q,\dot{q}), \quad \text{for any admissible pair } (q,\dot{q}) \in \mathbb{R}^\nu \times \mathbb{R}^\nu, \tag{5.4}$$

namely if the Lagrangian function is invariant under the one-parameter group of transformations (see, also, Sect. 3.18).

Remark 5.1 The Euler–Lagrange equation (5.2) can be rewritten as a first-order system by defining as state vector $x^\top = [x_1^\top \; x_2^\top] := [q^\top \; \dot{q}^\top]$,

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = x_2, \tag{5.5a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}\tau} = B^{-1}(x_1)\left(-\frac{\mathrm{d}B(x_1)}{\mathrm{d}t}x_2 - \left(\frac{\partial U(x_1)}{\partial x_1}\right)^\top\right). \tag{5.5b}$$

Given $g(x_1) \in \mathbb{R}^\nu$, according to Sect. 3.20 , let $g^{[1]}(x_1,x_2) = \frac{\partial g(x_1)}{\partial x}x_2$. Clearly, if $g(q)$ is a symmetry of the Lagrangian function $L(q,\dot{q})$, then

$$g_e(x_1,x_2) = \begin{bmatrix} g(x_1) \\ g^{[1]}(x_1,x_2) \end{bmatrix} \tag{5.6}$$

is a symmetry of system (5.5a), (5.5b). It is worth pointing out that there exist symmetries $g_e(x)$ of system (5.5a), (5.5b) that have not form (5.6).

Theorem 5.1 *If the one-parameter group of transformations $q = \Phi_g(\tau,\tilde{q})$ is a symmetry of the Lagrangian function $L(q,\dot{q})$, then $I = \frac{\partial L}{\partial \dot{q}}g$ is a first integral of the Euler–Lagrange* (5.1).

Proof By (5.4), $\tilde{L}$ is independent of τ, i.e., $\frac{\partial \tilde{L}}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau} + \frac{\partial \tilde{L}}{\partial \dot{q}}\frac{\mathrm{d}\dot{q}}{\mathrm{d}\tau} = 0$. By (5.4), $\frac{\partial \tilde{L}}{\partial q} = \frac{\partial L}{\partial q}$ and $\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}}$, which yields

$$\frac{\partial L}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\dot{q}}{\mathrm{d}\tau} = 0.$$

By the Euler–Lagrange equation (5.1), $\frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}$, whence

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)\frac{\mathrm{d}q}{\mathrm{d}\tau} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\frac{\mathrm{d}q}{\mathrm{d}t}}{\mathrm{d}\tau} = 0.$$

This implies $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}q}{\mathrm{d}\tau}) = 0$, namely $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}}g) = 0$. □

Example 5.1 Let $q = \left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right]$, $q_i \in \mathbb{R}$, $T = \frac{1}{2}[\dot{q}_1 \; \dot{q}_2]B(q_2)\left[\begin{smallmatrix} \dot{q}_1 \\ \dot{q}_2 \end{smallmatrix}\right]$ and $U = U(q_2)$ (i.e., the Lagrangian function is independent of q_1). Let $g(q) = e_1$, with e_1 being the first column of the 2×2 identity matrix; $\Phi_g(\tau,q) = [\tau + q_1 \; q_2]^\top$ is the flow associated with g. Then, for a constant τ, letting $q_1 = \tau + \tilde{q}_1$, $q_2 = \tilde{q}_2$ yields $\dot{q}_1 = \dot{\tilde{q}}_1$, $\dot{q}_2 = \dot{\tilde{q}}_2$; thus,

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\begin{bmatrix}\dot{\tilde{q}}_1 & \dot{\tilde{q}}_2\end{bmatrix} B(\tilde{q}_2) \begin{bmatrix}\dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2\end{bmatrix} - U(\tilde{q}_2) = L(\tilde{q}, \dot{\tilde{q}}),$$

whence g is a symmetry of L. Therefore, $I = \frac{\partial L}{\partial \dot{q}} g = \frac{\partial L}{\partial \dot{q}} e_1 = \frac{\partial L}{\partial \dot{q}_1}$ is a first integral of the Euler–Lagrange equations, according to the fact that if L does not depend on q_1, $\frac{\partial L}{\partial q_1} = 0$, then by the Euler–Lagrange equations (5.1), one concludes that $\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} = 0$.

Example 5.2 Consider $g(q) = Sq$, where $S \in \mathbb{R}^{\nu \times \nu}$ is constant and skew-symmetric; the flow associated with g is $\Phi_g(\tau, q) = \mathrm{e}^{S\tau} q$. Defining the transformation $q = \mathrm{e}^{S\tau} \tilde{q}$, one concludes that $\dot{q} = \mathrm{e}^{S\tau} \dot{\tilde{q}}$. Let the Lagrangian function L be given by $L = \frac{1}{2} \dot{q}^\top B \dot{q} - U(q)$, for a constant generalized inertia matrix B, with the corresponding Euler–Lagrange equations $\ddot{q}^\top B + \frac{\partial U}{\partial q} = 0$. In particular, assume that $B \in \mathscr{L}_c(S)$, which implies that $B \in \mathscr{L}_c(\mathrm{e}^{S\tau})$, and assume that $U \in \mathscr{I}_C(Sx)$, which implies that $U(\mathrm{e}^{S\tau} q) = U(q)$ and $L_{Sq} U = 0$. Therefore, $q = \mathrm{e}^{S\tau} \tilde{q}$ is a symmetry of L,

$$\begin{aligned}\tilde{L}(\tilde{q}, \dot{\tilde{q}}) &= \frac{1}{2} \dot{\tilde{q}}^\top \mathrm{e}^{S^\top \tau} B \mathrm{e}^{S\tau} \dot{\tilde{q}} - U(\mathrm{e}^{S\tau} \tilde{q}) = \frac{1}{2} \dot{\tilde{q}}^\top \mathrm{e}^{-S\tau} \mathrm{e}^{S\tau} B \dot{\tilde{q}} - U(\tilde{q}) \\ &= \frac{1}{2} \dot{\tilde{q}}^\top B \dot{\tilde{q}} - U(\tilde{q}) = L(\tilde{q}, \dot{\tilde{q}}).\end{aligned}$$

Hence, $I = \frac{\partial L}{\partial \dot{q}} g = \dot{q}^\top B S q$ is a first integral of the Euler–Lagrange equations. As a matter of fact,

$$\dot{I} = \ddot{q}^\top B S q + \dot{q}^\top B S \dot{q} = -\frac{\partial U}{\partial q} S q + \dot{q}^\top B S \dot{q}$$

and both terms $\frac{\partial U}{\partial q} S q$ and $\dot{q}^\top B S \dot{q}$ are equal to zero; the first one is equal to zero since $\frac{\partial U}{\partial q} S q = L_{Sq} U$ and the second one is equal to zero because matrix BS is skew-symmetric, $(BS)^\top = S^\top B = -SB = -BS$. As a simple example, take $L(q, \dot{q}) = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} k (q_1^2 + q_2^2)$, for constant $m, k \in \mathbb{R}$; $g(q) = [q_2 \; -q_1]^\top$ is a symmetry of L, whence $I = \frac{\partial L}{\partial \dot{q}} g = m(\dot{q}_1 q_2 - \dot{q}_2 q_1)$ is a first integral of the corresponding Euler–Lagrange equations.

5.2 Hamiltonian Systems

The special class of nonlinear systems termed as Hamiltonian is considered in this section, in view of its importance for modeling many physical systems; the reader interested in a more extensive treatment is referred to [6, 54, 86, 100, 102], where most of the topics analyzed in this section are reported.

Let $\mathscr{H}$ be the set of all functions $H(x) \in \mathbb{R}$ being analytic in some open and connected domain $\mathscr{U}$ of $\mathbb{R}^n$, hereafter called the *Hamiltonian functions*. More in general, H could be meromorphic, because in that case there exists an open and connected domain $\mathscr{U}$ where it is analytic. For any $u \in \mathscr{H}$, $\nabla u = (\frac{\partial u}{\partial x})^\top$ is the column gradient of u.

Definition 5.2 Assume that an operation $\{\cdot,\cdot\} : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is defined so to satisfy the following properties, with functions $u, v, z \in \mathscr{H}$ and constants $a, b \in \mathbb{R}$ being arbitrary:

(5.2.1) $\{u, v\} = -\{v, u\}$ (*skew-symmetry*);
(5.2.2) $\{au + bv, z\} = a\{u, z\} + b\{v, z\}$ and $\{u, av + bz\} = a\{u, v\} + b\{u, z\}$ (*bi-linearity*);
(5.2.3) $\{u, \{v, z\}\} + \{v, \{z, u\}\} + \{z, \{u, v\}\} = 0$ (*the Jacobi identity*);
(5.2.4) $\{u, vz\} = \{u, v\}z + \{u, z\}v$ (*the Leibniz rule*);

such an operation $\{u, v\}$ is called the *Poisson bracket* [6, 54, 86, 100] of u and v.

By the bi-linearity (5.2.2) and the Leibniz rule (5.2.4), given an analytic function $v \in \mathscr{H}$, map $u \to \{u, v\}$ defines a *derivation* on $\mathscr{H}$, and hence, by Theorem 1.3 at p. 6, there exists a locally unique vector function $f_v(x) \in \mathbb{R}^n$ such that $L_{f_v} u = \{u, v\}$, for any $u \in \mathscr{H}$ (see also [102, p. 392]).

Definition 5.3 Let a Poisson bracket $\{\cdot,\cdot\}$ be given. The vector function $f_v(x) \in \mathbb{R}^n$ such that $L_{f_v} u = \{u, v\}$, for any $u \in \mathscr{H}$, is the *Hamiltonian vector function* associated with the *Hamiltonian function* $v \in \mathscr{H}$. Let $\mathscr{F}_{\mathscr{H}}$ be the set of all Hamiltonian vector functions f_v, $v \in \mathscr{H}$, associated with the given Poisson bracket $\{\cdot,\cdot\}$.

The proof of Theorem 1.3 at p. 6 yields the following formula for the Hamiltonian vector function $f_v(x)$ associated with the Hamiltonian function $v \in \mathscr{H}$:

$$f_v(x) = \begin{bmatrix} \{x_1, v\} \\ \{x_2, v\} \\ \vdots \\ \{x_n, v\} \end{bmatrix}, \tag{5.7}$$

which can be used as an alternative definition of the Hamiltonian vector function.

The following theorem is very important in the subsequent Sect. 5.5, where the generality of the Hamiltonian approach is explored.

Theorem 5.2 *Let an operation* $\{\cdot,\cdot\} : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ *be given, satisfying the skew-symmetry* (5.2.1), *the bi-linearity* (5.2.2) *and the Leibniz rule* (5.2.3). *For an arbitrary* $z \in \mathscr{H}$, *let* $f_z(x)$ *be the vector function such that* $L_{f_z} u = \{u, z\}$, *for any* $u \in \mathscr{H}$. *Then,*

$$L_{f_z}\{u, v\} = \{L_{f_z} u, v\} + \{u, L_{f_z} v\}, \quad \forall u, z \in \mathscr{H}, \tag{5.8}$$

is equivalent to the Jacobi identity.

Proof Clearly, $\{L_{f_z}u, v\} = \{\{u, z\}, v\}$ and $\{u, L_{f_z}v\} = \{u, \{v, z\}\}$, and therefore, equation (5.8) becomes

$$\big\{\{u, v\}, z\big\} = \big\{\{u, z\}, v\big\} + \big\{u, \{v, z\}\big\},$$

which, by skew-symmetry and bi-linearity, gives

$$-\big\{z, \{u, v\}\big\} = \big\{v, \{z, u\}\big\} + \big\{u, \{v, z\}\big\},$$

i.e., the Jacobi identity. On the other hand, if the Jacobi identity holds, then

$$\begin{aligned}\{L_{f_z}u, v\} + \{u, L_{f_z}v\} &= \big\{\{u, z\}, v\big\} + \big\{u, \{v, z\}\big\} = \big\{v, \{z, u\}\big\} + \big\{u, \{v, z\}\big\} \\ &= -\big\{z, \{u, v\}\big\} = \big\{\{u, v\}, z\big\} \\ &= L_{f_z}\{u, v\},\end{aligned}$$

thus obtaining (5.8). □

By (5.7),

$$\begin{aligned}L_{f_v}u &= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} f_{v,i} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\{x_i, v\} = -\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\{v, x_i\} \\ &= -\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}\{x_j, x_i\} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\{x_i, x_j\} = \frac{\partial u}{\partial x}\ S\ \nabla v,\end{aligned}$$

where the (i, j)th entry of matrix function $S(x) \in \mathbb{R}^{n\times n}$ is equal to $\{x_i, x_j\}$, $S_{i,j}(x) = \{x_i, x_j\}$. Since $L_{f_v}u = \frac{\partial u}{\partial x} f_v$, the arbitrariness of u implies that

$$f_v(x) = S(x)\nabla v(x). \tag{5.9}$$

Note that (5.9) is often used an another alternative definition of the Hamiltonian vector function.

Since $L_{f_H}I = \{I, H\}$ for any $I(x) \in \mathbb{R}$, I is a first integral associated with f_H if and only if $\{I, H\} = 0$; since $\{H, H\} = 0$ (by the skew-symmetry), then H is a first integral associated with f_H, for any $H \in \mathscr{H}$. If K_1 and K_2 are two first integrals associated with f_H, $\{K_i, H\} = 0$, then $\{K_1, K_2\}$ is a (possibly, trivial) first integral associated with f_H. As a matter of fact, by the Jacobi identity,

$$\big\{H, \{K_1, K_2\}\big\} + \big\{K_1, \{K_2, H\}\big\} + \big\{K_2, \{H, K_1\}\big\} = 0,$$

one has

$$\big\{\{K_1, K_2\}, H\big\} = -\big\{H, \{K_1, K_2\}\big\} = 0.$$

Theorem 5.3 *Let a Poisson bracket* $\{\cdot,\cdot\}$ *be given. If* $f_H, f_K \in \mathscr{F}_{\mathscr{H}}$, *then* $[f_H, f_K] \in \mathscr{F}_{\mathscr{H}}$ *and, precisely,* $[f_H, f_K] = -f_{\{H,K\}} = f_{\{K,H\}}$ *for all* $H, K \in \mathscr{H}$.

Proof Equality $[f_H, f_K] = -f_{\{H,K\}}$ is equivalent to $L_{[f_H,f_K]}a = -L_{f_{\{H,K\}}}a$ for any $a \in \mathscr{H}$ (actually, since $L_f x_i$ is the ith entry of f, for any vector function f, it is necessary and sufficient to take $a = x_i$, $i = 1, \ldots, n$); therefore,

$$\begin{aligned} L_{[f_H,f_K]}a &= L_{f_H}L_{f_K}a - L_{f_K}L_{f_H}a = -L_{f_H}\{K,a\} - L_{f_K}\{a,H\} \\ &= \big\{H,\{K,a\}\big\} + \big\{K,\{a,H\}\big\} = -\big\{a,\{H,K\}\big\} = -L_{f_{\{H,K\}}}a. \end{aligned}$$ □

Note the inversion of position of H and K in equality $[f_H, f_K] = f_{\{K,H\}}$.

Definition 5.4 Let a Poisson bracket $\{\cdot,\cdot\}$ be given. Vector function g is a symmetry of $f_H \in \mathscr{F}_{\mathscr{H}}$ if $[f_H, g] = 0$; g is a *Noether symmetry* [6, 100] of f_H if, in addition to $[f_H, g] = 0$, one has $g \in \mathscr{F}_{\mathscr{H}}$, namely if there exists $K \in \mathscr{H}$ such that $g = f_K$.

Theorem 5.4 *Let a Poisson bracket* $\{\cdot,\cdot\}$ *be given;* $f_K \in \mathscr{F}_{\mathscr{H}}$ *is a Noether symmetry of* $f_H \in \mathscr{F}_{\mathscr{H}}$ *if and only if* $\{K, H\} = c$, *for some constant* $c \in \mathbb{R}$.

Proof Clearly, $f_K \in \mathscr{F}_{\mathscr{H}}$ is a Noether symmetry of $f_H \in \mathscr{F}_{\mathscr{H}}$ if and only if $[f_H, f_K] = 0$; since $[f_H, f_K] = f_{\{K,H\}}$, one has $f_{\{K,H\}} = 0$ if and only $\{K, H\} = c$, for some constant $c \in \mathbb{R}$. □

By Theorem 5.4, if K is a first integral associated with f_H, i.e., $\{K, H\} = 0$, then f_K is a symmetry of f_H. Conversely, if f_K is a symmetry of f_H, then K need not be a first integral associated with f_H; actually, if $\{K, H\} = c$, then $I = K - ct$ is a *time-varying first integral* associated with f_H in the sense that $\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + L_{f_K} I = 0$.

Theorem 5.5 *Let a Poisson bracket* $\{\cdot,\cdot\}$ *be given. Then there exists a matrix function* $S(x) \in \mathbb{R}^{n\times n}$ *such that* $\{u, v\} = \frac{\partial u}{\partial x}\, S\, \nabla v$, *for any* $u, v \in \mathscr{H}$.

Proof By (5.9), letting $S_{i,j}(x) = \{x_i, x_j\}$, $i, j \in \{1, \ldots, n\}$, one has

$$\{u, v\} = L_{f_v} u = \frac{\partial u}{\partial x}\, S\, \nabla v.$$ □

Theorem 5.6 *Given a matrix function* $S(x) \in \mathbb{R}^{n\times n}$, $\{u, v\} = \frac{\partial u}{\partial x}\, S\, \nabla v$ *is a Poisson bracket if and only if*

(5.6.1) *S is skew-symmetric,* $S^\top = -S$;
(5.6.2) *the entries $S_{i,j}$ of S satisfy*

$$\sum_{\ell=1}^{n}\left(S_{i,\ell}\frac{\partial S_{j,k}}{\partial x_\ell} + S_{j,\ell}\frac{\partial S_{k,i}}{\partial x_\ell} + S_{k,\ell}\frac{\partial S_{i,j}}{\partial x_\ell}\right) = 0, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{5.10}$$

Proof By definition, the operation $\{u, v\} = \frac{\partial u}{\partial x} \; S \; \nabla v$ is automatically bi-linear and satisfies the Leibniz rule. The skew-symmetry of matrix S is clearly equivalent to the skew-symmetry of the Poisson bracket. Thus, one only needs to verify the equivalence of (5.10) with the Jacobi identity. Here, it is shown that (5.10) is equivalent to

$$\big\{x_i, \{x_j, x_k\}\big\} + \big\{x_j, \{x_k, x_i\}\big\} + \big\{x_k, \{x_i, x_j\}\big\} = 0, \quad \forall i, j, k \in \{1, \dots, n\}, \tag{5.11}$$

leaving the equivalence of (5.11) with the Jacobi identity to the proof of Proposition 6.8 of [102]. Clearly,

$$\begin{aligned}\big\{x_i, \{x_j, x_k\}\big\} &= -\big\{\{x_j, x_k\}, x_i\big\} = -\{S_{j,k}, x_i\} = -\sum_{\ell=1}^{n} \frac{\partial S_{j,k}}{\partial x_\ell}\{x_\ell, x_i\} \\ &= \sum_{\ell=1}^{n} \frac{\partial S_{j,k}}{\partial x_\ell}\{x_i, x_\ell\} = \sum_{\ell=1}^{n} S_{i,\ell}\frac{\partial S_{j,k}}{\partial x_\ell},\end{aligned}$$

and similarly

$$\big\{x_j, \{x_k, x_i\}\big\} = \sum_{\ell=1}^{n} S_{j,\ell}\frac{\partial S_{k,i}}{\partial x_\ell}, \qquad \big\{x_k, \{x_i, x_j\}\big\} = \sum_{\ell=1}^{n} S_{k,\ell}\frac{\partial S_{i,j}}{\partial x_\ell}.$$

Then, (5.10) is equivalent to (5.11). □

Definition 5.5 Any matrix $S(x) \in \mathbb{R}^{n\times n}$ satisfying conditions (5.6.1) and (5.6.2) of Theorem 5.6 is called a *structure matrix*.

Remark 5.2 Any constant and skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$ satisfies conditions (5.6.1) and (5.6.2) of Theorem 5.6, whence it is a structure matrix.

The results above indicate that, in given local coordinates (e.g., x), a practical way to describe a Poisson bracket is to specify its structure matrix $S(x)$, satisfying conditions (5.6.1) and (5.6.1) of Theorem 5.6. For later use, the symbol $\{\cdot, \cdot\}_{S(x)}$ indicates the Poisson bracket characterized by the structure matrix $S(x)$, in the x-coordinates. The indication of $S(x)$ is omitted when this causes no confusion.

Remark 5.3 Let $n = 3$; for any $w \in \mathscr{H}$, let

$$S = \begin{bmatrix} 0 & -\frac{\partial w}{\partial x_3} & \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_3} & 0 & -\frac{\partial w}{\partial x_1} \\ -\frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_1} & 0 \end{bmatrix}. \tag{5.12}$$

By direct substitution it is easy to check such a matrix S satisfies conditions (5.6.1) and (5.6.2) of Theorem 5.6, whence it is a structure matrix (see also Sect. 5.5).

Remark 5.4 The "mechanical" systems considered in Sect. 5.1 can be very naturally recast in the Hamiltonian formalism, so that Hamiltonian systems can be seen as a generalization of Euler–Lagrange systems. To describe the systems considered in Sect. 5.1 as Hamiltonian systems, define the *generalized momenta* $p := (\frac{\partial L}{\partial \dot{q}})^\top = B(q)\dot{q}$; hence, define the Hamiltonian function as the total energy of the mechanical system:

$$H := T + U = \left(\frac{1}{2}\dot{q}^\top B(q)\dot{q} + U(q)\right)\Big|_{\dot{q}=B^{-1}(q)p} = \frac{1}{2}p^\top B^{-1}(q)p + U(q).$$

Then, by definition,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = B^{-1}(q)p = \left(\frac{\partial H}{\partial p}\right)^\top,$$

and (taking into account (5.1))

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)^\top = \left(\frac{\partial L}{\partial q}\right)^\top = -\left(\frac{\partial H}{\partial q}\right)^\top,$$

where the last equality can be proven taking into account that $L = T - U$ and, by (3.13), that $\frac{\partial B}{\partial q_i} = -B\frac{\partial B^{-1}}{\partial q_i}B$:

$$\begin{aligned}\frac{\partial}{\partial q_i}\big(\dot{q}^\top B(q)\dot{q}\big) &= \dot{q}^\top \frac{\partial B(q)}{\partial q_i}\dot{q} = -\dot{q}^\top B\frac{\partial B^{-1}}{\partial q_i}B\dot{q} = -p^\top \frac{\partial B^{-1}}{\partial q_i}p \\ &= -\frac{\partial}{\partial q_i}\big(p^\top B^{-1}(q)p\big).\end{aligned}$$

Letting $x = [q^\top \ p^\top]^\top$ and $S = \left[\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right]$, one concludes that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = S\left(\frac{\partial H}{\partial x}\right)^\top =: f_H(x),$$

where f_H is the *Hamiltonian vector function* associated with the Hamiltonian function H, through the Poisson bracket defined by such a matrix S, which is certainly a structure matrix being constant and skew-symmetric.

Using Theorem 5.5, it is possible to understand how a diffeomorphism $y = \varphi(x)$ (with inverse $x = \phi(y)$) transforms the structure matrix $S(x)$ of a given Poisson bracket (given in the x-coordinates) into the structure matrix $\tilde{S}(y)$ of the same Poisson bracket (expressed in the y-coordinates). By Theorem 5.5, there exists a matrix function $\tilde{S}(y)$ such that

$$\left(\frac{\partial u(x)}{\partial x}S(x)\nabla v(x)\right) \circ \phi(y) = \frac{\partial u(\phi(y))}{\partial y}\tilde{S}(y)\nabla v\big(\phi(y)\big),$$

where the symbol ∇ in the right-hand side is referred to the y -coordinates; hence, it follows that

$$S \circ \phi(y) = \frac{\partial \phi(y)}{\partial y} \, \tilde{S}(y) \left(\frac{\partial \phi(y)}{\partial y} \right)^{\top}, \tag{5.13}$$

or, equivalently,

$$\begin{aligned} \tilde{S}(y) &= \left(\frac{\partial \varphi(x)}{\partial x} S(x) \left(\frac{\partial \varphi(x)}{\partial x} \right)^{\top} \right) \circ \phi(y) \\ &= \left(\frac{\partial \phi(y)}{\partial y} \right)^{-1} S(x) \circ \phi(y) \left(\frac{\partial \phi(y)}{\partial y} \right)^{-\top}. \end{aligned} \tag{5.14}$$

Definition 5.6 Let a Poisson bracket $\{\cdot, \cdot\}$ be given. A diffeomorphism $y = \varphi(x)$ (with inverse $x = \phi(y)$) is a *Poisson map* if it preserves the given Poisson bracket, i.e., if

$$\tilde{S}(y) = S(y), \quad \forall y \in \varphi(\mathscr{U}), \tag{5.15}$$

where $\tilde{S}(y)$ is given in (5.14).

Example 5.3 Let

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

which clearly satisfies conditions (5.6.1) and (5.6.2) of Theorem 5.6. Let $x = \phi(y)$, with

$$\phi(y) = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Since (5.15) holds, $x = \phi(y)$ is a Poisson map.

The following theorem shows that a Poisson map $x = \phi(y)$ transforms the Hamiltonian vector function f_H associated with the Hamiltonian function H into the Hamiltonian vector function $f_{H \circ \phi}$ associated with the Hamiltonian function $H \circ \phi$.

Theorem 5.7 *Let a Poisson bracket* $\{\cdot, \cdot\}$ *be given. If* $x = \phi(y)$ *is a Poisson map, then*

$$\left(\frac{\partial \phi}{\partial y} \right)^{-1} f_H \circ \phi = f_{H \circ \phi}. \tag{5.16}$$

Proof Consider the ith entry $f_{H,i}$ of f_H, for which one has

$$f_{H,i} \circ \phi = \{x_i, H\} \circ \phi = \{x_i \circ \phi, H \circ \phi\} = \{\phi_i, H \circ \phi\} = \frac{\partial \phi_i}{\partial y} f_{H \circ \phi};$$

hence, $f_H \circ \phi = \frac{\partial \phi}{\partial y} f_{H \circ \phi}$. □

If $\varphi(x) = \phi^{-1}(x)$, then (5.16) can be rewritten as

$$\varphi_* f_H = f_{\varphi_* H}.$$

Example 5.4 Consider the Poisson bracket defined by the structure matrix $S(x)$ given in Example 5.3. Let $H(x) = \frac{1}{2}(\frac{x_1^2}{\mathbb{I}_1} + \frac{x_2^2}{\mathbb{I}_2} + \frac{x_3^2}{\mathbb{I}_3})$ for some constant $\mathbb{I}_i \neq 0$, $i = 1, 2, 3$. Then, the Hamiltonian vector function f_H associated with H is

$$f_H(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{x_1}{\mathbb{I}_1} \\ \frac{x_2}{\mathbb{I}_2} \\ \frac{x_3}{\mathbb{I}_3} \end{bmatrix} = \begin{bmatrix} \frac{\mathbb{I}_2 - \mathbb{I}_3}{\mathbb{I}_2 \mathbb{I}_3} x_2 x_3 \\ \frac{\mathbb{I}_3 - \mathbb{I}_1}{\mathbb{I}_1 \mathbb{I}_3} x_1 x_3 \\ \frac{\mathbb{I}_1 - \mathbb{I}_2}{\mathbb{I}_1 \mathbb{I}_2} x_1 x_2 \end{bmatrix}.$$

Hence, $\frac{\mathrm{d}x}{\mathrm{d}t} = f_H(x)$ is the system of the equations of motion of a rigid body that rotates about its center of mass, with inertia matrix $\mathbb{I} = \mathrm{diag}\{\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3\}$. Let

$$\phi(y) = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

which is a Poisson map by Example 5.3. The Hamiltonian function is transformed into

$$\tilde{H}(y) = H \circ \phi(y) = \frac{1}{2}\bigg(\frac{(-\frac{1}{3}y_1 + \frac{2}{3}y_2 + \frac{2}{3}y_3)^2}{\mathbb{I}_1} + \frac{(\frac{2}{3}y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_3)^2}{\mathbb{I}_2} + \frac{(\frac{2}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3)^2}{\mathbb{I}_3}\bigg).$$

The Hamiltonian vector function associated with $\tilde{H}$ is

$$\begin{aligned} f_{\tilde{H}}(y) &= \tilde{S}(y)\nabla\tilde{H}(y) = S(y)\nabla\tilde{H}(y) \\ &= \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{-\frac{1}{3}y_1 + \frac{2}{3}y_2 + \frac{2}{3}y_3}{\mathbb{I}_1} \\ \frac{\frac{2}{3}y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_3}{\mathbb{I}_2} \\ \frac{\frac{2}{3}y_1 + \frac{2}{3}y_2 - \frac{1}{3}y_3}{\mathbb{I}_3} \end{bmatrix}, \end{aligned}$$

which coincides with vector function f_H transformed by $x = \phi(y)$, i.e., $f_{\tilde{H}}(y) = \varphi_* f_H(x)$, if $\varphi = \phi^{-1}$.

Theorem 5.8 *Let a Poisson bracket* $\{\cdot,\cdot\}$ *be given. For any* $K \in \mathscr{H}$*, let* Φ_{f_K} *be the flow associated with the Hamiltonian vector function* f_K*. Then,* $x = \Phi_{f_K}(\tau, y)$ *is a Poisson map for any admissible* $\tau \in \mathbb{R}$.

Proof Diffeomorphism $x = \Phi_{f_K}(\tau, y)$ is a Poisson map if

$$\left\{u \circ \Phi_{f_K}(\tau, y), v \circ \Phi_{f_K}(\tau, y)\right\} = \{u, v\} \circ \Phi_{f_K}(\tau, y), \quad \forall u, v \in \mathscr{H}. \tag{5.17}$$

Such an equation holds for all admissible $\tau \in \mathbb{R}$ if and only if it holds for $\tau = 0$ and the relation obtained by taking the derivative of both sides of (5.17) with respect to τ is satisfied for all admissible $\tau \in \mathbb{R}$. Clearly, (5.17) holds for $\tau = 0$, because $x = \Phi_{f_K}(0, y)$ is the identity transformation. Taking into account that $\frac{\mathrm{d}\Phi_{f_K}(\tau,y)}{\mathrm{d}\tau} = f_K \circ \Phi_{f_K}$, the derivative of (5.17) with respect to τ yields the following equation computed at $x = \Phi_{f_K}(\tau, y)$:

$$\{L_{f_K} u, v\} + \{u, L_{f_K} v\} = L_{f_K}\{u, v\}. \tag{5.18}$$

Now, since $L_{f_K} u = \{u, K\}$ and $L_{f_K} v = \{v, K\}$ and $L_{f_K}\{u, v\} = \{\{u, v\}, K\}$, (5.18) becomes the Jacobi identity, which is satisfied since $\{\cdot,\cdot\}$ is a Poisson bracket. □

Remark 5.5 Any constant and skew-symmetric matrix S satisfies conditions (5.6.1) and (5.6.2) of Theorem 5.6, whence it is a structure matrix. Since all eigenvalues of any skew-symmetric matrix S have zero real part (see Statement 4.7.20 of [83]), then $\det(S) = 0$ if n is odd, because in such a case S has necessarily an eigenvalue equal to zero; in particular, the rank of S is even, $\mathrm{rank}_{\mathbb{R}}(S) = 2\nu$, where the number of eigenvalues equal to zero is $n - 2\nu \geq 0$. For any constant and skew-symmetric matrix S, there exists a constant $Q \in \mathbb{R}^{n\times n}$, with $Q^\top = Q^{-1}$ (it is a consequence of Statement 4.10.3 of [83]), such that $Q^{-1}SQ = Q^\top SQ$ is in the real Jordan form, which, taking into account that S is semi-simple, takes the form:

$$Q^\top SQ = Q^{-1}SQ = \left[\begin{array}{cc|c|cc|ccc} 0 & \omega_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -\omega_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \dots & 0 & \omega_\nu & 0 & \dots & 0 \\ 0 & 0 & \dots & -\omega_\nu & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right],$$

for constant $\omega_i \in \mathbb{R}^>$, $i = 1, \dots, \nu$. This implies that there exists a constant $\hat{Q}$ (in such a case $\hat{Q}^\top$ need not be the inverse of $\hat{Q}$) such that

$$\hat{Q}^\top S \hat{Q} = \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where E is the $\nu \times \nu$ identity matrix.

Remark 5.5 can be extended to the case of non-constant structure matrices by the subsequent Theorem 5.9, which is called the *Darboux Theorem* [102].

Definition 5.7 A point $x^o \in \mathbb{R}^n$ is a *regular point* of the Poisson bracket characterized by the structure matrix $S(x)$ if $S(x)$ has constant rank $2\nu \le n$ in a neighborhood $\mathscr{U}^*$ of x^o; 2ν is called the *rank* of the Poisson bracket at x^o.

Theorem 5.9 *Let $x^o \in \mathbb{R}^n$ be a regular point of the Poisson bracket; in particular, let 2ν be the rank of the Poisson bracket at x^o and $m = n - 2\nu \ge 0$. Then, there exists a diffeomorphism $y = \varphi(x)$, with $\varphi(\cdot) : \mathscr{U}^* \to \mathbb{R}^n$, with inverse $x = \phi(y)$, where $\mathscr{U}^*$ is a neighborhood of x^o, such that the transformed structure matrix $\tilde{S}(y)$ given by* (5.14) *takes the* canonical form:

$$\tilde{S}(y) = \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.19}$$

with E being the $\nu \times \nu$ identity matrix.

Proof If the rank of the Poisson bracket at x^o is 0, then S is identically equal to zero in $\mathscr{U}^*$, and therefore it is of form (5.19), with $\nu = 0$. Then, assume $\nu \ge 1$. By this assumption, there exists a $z \in \mathscr{H}$ such that x^o is a regular point of f_z, $f_z(x^o) \ne 0$. Let $y = \varphi(x)$ be the diffeomorphism straightening f_z, $L_{f_z}\varphi = e_1$. Take u equal to the first entry of φ, which implies $\{u, z\} = L_{f_z}u = 1$. Since $[f_z, f_u] = f_{\{u,z\}}$ and $\{u, z\} = 1$, one concludes that $[f_z, f_u] = 0$, namely that f_u and f_z are commuting and both have x^o as regular point. By the Frobenius Theorem 1.9 at p. 21, there exist $n - 2$ functions $\psi_1, \dots, \psi_{n-2}$ that are joint first integrals associated with both f_u and f_z, and such that $\{u, z, \psi_1, \dots, \psi_{n-2}\}$ is a set of n functions being functionally independent at $x = x^o$. Hence, letting $q_1 = u$, $p_1 = z$ and $y_i = \psi_i$, $i = 1, \dots, n-2$, one concludes that

$$\begin{aligned} \{q_1, p_1\} &= L_{f_z}u = 1, \\ \{y_i, q_1\} &= L_{f_z}y_i = 0, \quad i = 1, \dots, n-2, \\ \{y_i, p_1\} &= L_{f_u}y_i = 0, \quad i = 1, \dots, n-2. \end{aligned}$$

This means that, in such coordinates, the structure matrix has the form

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \hat{S} \end{bmatrix},$$

where $\hat{S} \in \mathbb{R}^{(n-2)\times(n-2)}$ need not be constant. By the Jacobi identity,

$$\big\{q_1, \{y_i, y_j\}\big\} + \big\{y_i, \{y_j, q_1\}\big\} + \big\{y_j, \{q_1, y_i\}\big\} = 0,$$

one shows that

$$\big\{\{y_i, y_j\}, q_1\big\} = 0;$$

in addition,

$$\begin{aligned} \big\{\{y_i, y_j\}, q_1\big\} = \big\{\hat{S}_{i,j}, q_1\big\} &= \begin{bmatrix} \frac{\partial \hat{S}_{i,j}}{\partial q_1} & \frac{\partial \hat{S}_{i,j}}{\partial p_1} & \frac{\partial \hat{S}_{i,j}}{\partial y} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \hat{S} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= -\frac{\partial \hat{S}_{i,j}}{\partial p_1}, \end{aligned}$$

which shows that $\frac{\partial \hat{S}_{i,j}}{\partial p_1} = 0$. Similarly, it can be shown that $\frac{\partial \hat{S}_{i,j}}{\partial q_1} = 0$, namely that matrix $\hat{S}$ is independent of (q_1, p_1) and hence is the structure matrix of a Poisson bracket in the y-variables of rank two less than that of S, from which the induction step is proven, up to the final step at which the remaining $\hat{S}$ is equal to zero or vanishes. □

Example 5.5 Consider the structure matrix

$$S(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2x_1 \\ -1 & 2x_1 & 0 \end{bmatrix},$$

which has rank 2 on the whole $\mathbb{R}^3$. Consider the Hamiltonian function $v(x) = x_3 + x_1^2 + x_2^3$ and the associated Hamiltonian vector function

$$f_v(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2x_1 \\ -1 & 2x_1 & 0 \end{bmatrix} \begin{bmatrix} 2x_1 \\ 3x_2^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2x_1 \\ -2x_1 + 6x_1x_2^2 \end{bmatrix},$$

which has no singular points. Such a Hamiltonian vector function f_v is straightened by the diffeomorphism $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} x_1 \\ x_3 + x_1^2 + x_2^3 \\ x_2 + x_1^2 \end{bmatrix}.$$

Define $u(x) := x_1$, which yields

$$f_u(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2x_1 \\ -1 & 2x_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Clearly, $z(x) = x_2 + x_1^2$ is a first integral associated with both f_u and f_v, and therefore the diffeomorphism $y = \varphi(x)$ brings the structure matrix $S(x)$ in canonical form. As a matter of fact,

$$\begin{aligned} \frac{\partial \varphi(x)}{\partial x} S(x) \left(\frac{\partial \varphi(x)}{\partial x} \right)^{\top} &= \begin{bmatrix} 1 & 0 & 0 \\ 2x_1 & 3x_2^2 & 1 \\ 2x_1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2x_1 \\ -1 & 2x_1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2x_1 & 2x_1 \\ 0 & 3x_2^2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Remark 5.6 Apart from a diffeomorphism $y = \varphi(x)$, assume that

$$\tilde{S}(y) = \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, letting

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_\nu \end{bmatrix}, \qquad p = \begin{bmatrix} p_1 \\ \vdots \\ p_\nu \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}, \qquad y = \begin{bmatrix} q \\ p \\ z \end{bmatrix},$$

the Poisson bracket takes the form

$$\{u, v\}_{\tilde{S}} = \sum_{i=1}^{\nu} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right);$$

such local coordinates are called *canonical* and satisfy

$$\begin{aligned} &\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \quad \forall i, j, \\ &\{q_i, p_j\} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \forall i, j, \\ &\{q_i, z_j\} = \{p_i, z_j\} = \{z_i, z_j\} = 0, \quad \forall i, j. \end{aligned}$$

The Hamiltonian vector function $f_{\tilde{H}}$ associated with the Hamiltonian function $\tilde{H}$ is therefore

$$f_{\tilde{H}}(q,p,z) = \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (\frac{\partial \tilde{H}}{\partial q})^\top \\ (\frac{\partial \tilde{H}}{\partial p})^\top \\ (\frac{\partial \tilde{H}}{\partial z})^\top \end{bmatrix} = \begin{bmatrix} (\frac{\partial \tilde{H}}{\partial p})^\top \\ -(\frac{\partial \tilde{H}}{\partial q})^\top \\ 0 \end{bmatrix}.$$

This means that there exist local canonical coordinates (q,p,z) such that any Hamiltonian system can be written as

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= \left(\frac{\partial \tilde{H}}{\partial p}\right)^\top, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\left(\frac{\partial \tilde{H}}{\partial q}\right)^\top, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 0, \end{aligned}$$

from which it is easy to see that the functions z_i, $i = 1,\dots,m$, are first integrals associated with H, for any H, namely $\{z_i, H\} = 0$ is equal to zero for any H. Such functions $z_i(x)$ are called either *Casimir's functions*, when one is referred to the Poisson bracket, or *distinguished first integrals*, when one is referred to any Hamiltonian system associated with the Poisson bracket. Note that the $m = n - 2\nu$ functionally independent Casimir functions do not depend on the specific Hamiltonian function H, but only on the Poisson bracket. Clearly, $\{c, H\}$ is equal to zero for any H and for any constant c; such trivial quantities are not referred to as Casimir's functions.

Example 5.6 Let $S(x)$ be given as in (5.12), for a given non-constant $w \in \mathscr{H}$. Since

$$\{w,v\}_S = \begin{bmatrix} \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 & -\frac{\partial w}{\partial x_3} & \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_3} & 0 & -\frac{\partial w}{\partial x_1} \\ -\frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \\ \frac{\partial v}{\partial x_3} \end{bmatrix} = 0$$

and

$$\begin{bmatrix} \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 & -\frac{\partial w}{\partial x_3} & \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_3} & 0 & -\frac{\partial w}{\partial x_1} \\ -\frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_1} & 0 \end{bmatrix} = [0\ 0\ 0],$$

one concludes that $\{w, v\}_S = 0$ for any v, whence that w is a Casimir function associated with the Poisson bracket $\{\cdot, \cdot\}_S$. Any Hamiltonian system described by

$$f_H = \begin{bmatrix} 0 & -\frac{\partial w}{\partial x_3} & \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial x_3} & 0 & -\frac{\partial w}{\partial x_1} \\ -\frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x_2}\frac{\partial H}{\partial x_3} - \frac{\partial w}{\partial x_3}\frac{\partial H}{\partial x_2} \\ \frac{\partial w}{\partial x_3}\frac{\partial H}{\partial x_1} - \frac{\partial w}{\partial x_1}\frac{\partial H}{\partial x_3} \\ \frac{\partial w}{\partial x_1}\frac{\partial H}{\partial x_2} - \frac{\partial w}{\partial x_2}\frac{\partial H}{\partial x_1} \end{bmatrix}$$

has w as distinguished first integral, in addition to the first integral H, for any Hamiltonian function H.

Let $S \in \mathbb{R}^{n\times n}$ be a constant and skew-symmetric matrix; then, $\{u, v\}_S = \frac{\partial u}{\partial x} S \nabla v$ is a Poisson bracket. If $H(x) = \frac{1}{2}x^\top Px$, with $P \in \mathbb{R}^{n\times n}$ being constant and symmetric, then the Hamiltonian vector function f_H associated with H is linear, $f_H(x) = SPx$. Vice versa, under the assumption that $\det(S) \neq 0$ (which implies that n is even), $f(x) = Ax$ is Hamiltonian if and only if A is S^{-1}-*symmetric*, namely if and only if $S^{-1}A$ is symmetric; in particular, the corresponding Hamiltonian function is $H(x) = \frac{1}{2}x^\top S^{-1}Ax$. Since $\det(S) \neq 0$, assume $S = \left[\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right]$. Then $S^{-1} = \left[\begin{smallmatrix} 0 & -E \\ E & 0 \end{smallmatrix}\right] = -S$, and therefore $S^{-1}A$ is symmetric (A is S^{-1}-symmetric) if and only if SA is symmetric (A is S-symmetric). Thanks to the structure of S, by the equalities $SA = (SA)^\top = A^\top S^\top = -A^\top S$, A is S-*symmetric* if and only if

$$SA + A^\top S = 0. \tag{5.20}$$

Letting $A = \left[\begin{smallmatrix} A_{q,q} & A_{q,p} \\ A_{p,q} & A_{p,p} \end{smallmatrix}\right]$, (5.20) becomes

$$\begin{bmatrix} A_{p,q} - A_{p,q}^\top & A_{p,p} + A_{q,q}^\top \\ -A_{q,q} - A_{p,p}^\top & -A_{q,p} + A_{q,p}^\top \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, A is S-symmetric if and only if $A_{q,p}$ and $A_{p,q}$ are symmetric, and

$$A_{p,p} + A_{q,q}^\top = 0.$$

If $n = 2$, this reduces to the fact that A is S-symmetric if and only if it has zero trace.

5.3 Normal Forms of Hamiltonian Systems

In this section, apart from a diffeomorphism, assume that

$$S = \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where ν is the dimension of E, and let $m = n - 2\nu \geq 0$.

Remark 5.7 Let f_K be a Noether symmetry of f_H, i.e., $\{K, H\} = c$, for some constant $c \in \mathbb{R}$. Assume that H and K are analytic at $x = 0$ and $H(0) = K(0) = 0$ and $\frac{\partial H(x)}{\partial x}|_{x=0} = \frac{\partial K(x)}{\partial x}|_{x=0} = 0$. Hence, $\{K(x), H(x)\}|_{x=0} = \frac{\partial K(x)}{\partial x}|_{x=0}\ S\ \nabla H(x)|_{x=0} = 0$, which implies that if $\{K, H\} = c$, then $c = 0$.

Theorem 5.10 [98] *Let $H = H_2 + H_{\geq 3}$ be analytic at $x = 0$, with $H_2(x) = \frac{1}{2}x^\top Px$, for some constant and symmetric $P \in \mathbb{R}^{n\times n}$, and $H_{\geq 3}$ denoting third and higher order terms with respect to $x = 0$. Let $A = SP$ and assume that A is semi-simple. Then, f_H is in the Poincaré–Dulac normal form if and only if $H_{\geq 3} \in \mathscr{I}_C(Ax)$.*

Proof Clearly, $f_H = f_{H_2} + f_{H_{\geq 3}}$, where $f_{H_2}(x) = SPx$ and $f_{H_{\geq 3}} = S\nabla H_{\geq 3}$. Under the assumptions of the theorem, f_H is in the Poincaré–Dulac normal form if and only if $[f_{H_{\geq 3}}, f_{H_2}] = 0$. Since $[f_{H_{\geq 3}}, f_{H_2}] = -f_{\{H_{\geq 3}, H_2\}}$, f_H is in the Poincaré–Dulac normal form if and only if $\{H_{\geq 3}, H_2\} = c$. Now, $c = 0$ by Remark 5.7, and $\{H_{\geq 3}, H_2\} = 0$ is equivalent to the fact that $H_{\geq 3}$ is a first integral associated with $f_{H_2}(x) = Ax$. □

The Poincaré–Dulac normal form for Hamiltonian systems takes also a special name as in the following definition.

Definition 5.8 Let $H = H_2 + H_{\geq 3}$ be analytic at $x = 0$, with $H_2(x) = \frac{1}{2}x^\top Px$ and $H_{\geq 3}$ denoting third and higher order terms with respect to $x = 0$. Let $A = SP$ and assume that A is semi-simple. If $H_{\geq 3} \in \mathscr{I}_C(Ax)$, then H is in the *Birkhoff–Gustavson normal form* [19, 59].

The proof of the following corollary follows from the proof of Theorem 5.4.

Corollary 5.1 *Let $H = H_2 + \hat{H}$, where $H_2(x) = \frac{1}{2}x^\top Px$, for some constant and symmetric $P \in \mathbb{R}^{n\times n}$; let $A = SP$. Then, the Hamiltonian vector function $f_H(x) = Ax + S\nabla\hat{H}(x)$ has the Hamiltonian vector function $f_{H_2}(x) = Ax$ as symmetry if $\hat{H} \in \mathscr{I}_C(Ax)$, namely if $\{\hat{H}, H_2\} = 0$. Vice versa, if the Hamiltonian vector function f_H has the Hamiltonian vector function $f_{H_2} = Ax$ as symmetry, then $\{\hat{H}, H_2\} = c$, for some constant c.*

Example 5.7 Let $H(x) = \frac{1}{2}x^\top Px + \hat{H}$, with $P = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ and $\hat{H}$ being analytic at $x = 0$. Then, $A = \left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ is semi-simple and $\mathscr{I}_C(Ax)$ is constituted by all arbitrary functions of $I(x) = q^2 + p^2$. Then, f_H is in the Poincaré–Dulac normal form if and only if $\hat{H}$ is an arbitrary function of $q^2 + p^2$, such that $\hat{H}(0) = 0$, $\frac{\partial \hat{H}(x)}{\partial x}|_{x=0} = 0$ and $\frac{\partial^2 \hat{H}(x)}{\partial x^2}|_{x=0} = 0$. For instance, if $\hat{H}(x) = I^2(x) = (q^2 + p^2)^2$, then the following f_H is in the Poincaré–Dulac normal form:

$$f_H(x) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}q\\ p\end{bmatrix} + 4\big(q^2 + p^2\big)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}q\\ p\end{bmatrix} = \begin{bmatrix}p + 4p(q^2 + p^2)\\ -q - 4q(q^2 + p^2)\end{bmatrix}.$$

According to Corollary 5.1, if $\hat{H}$ does not satisfy $\hat{H}(0) = 0$, $\frac{\partial \hat{H}(x)}{\partial x}|_{x=0} = 0$ and $\frac{\partial^2 \hat{H}(x)}{\partial x^2}|_{x=0} = 0$, but still satisfies $\hat{H} \in \mathscr{I}(Ax)$, although the resulting f_H is not in the Poincaré–Dulac normal form, f_H has $f_{\frac{1}{2}x^\top Px}(x) = Ax$ as symmetry. For instance, taking $\hat{H}(x) = \sqrt{I(x)} = \sqrt{q^2 + p^2}$, the resulting

$$f_H(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{q}{\sqrt{q^2+p^2}} \\ \frac{p}{\sqrt{q^2+p^2}} \end{bmatrix} = \left(1 + \frac{1}{\sqrt{q^2+p^2}}\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

is not in the Poincaré–Dulac normal form, but has Ax as symmetry, because it is co-linear with Ax, with a multiplicative coefficient $1 + \frac{1}{\sqrt{q^2+p^2}} \in \mathscr{I}(Ax)$.

Theorem 5.11 *Assume that $n = 2\nu$. Let $H = H_2 + H_{\geq 3}$ be analytic at $x = 0$, with $H_2(x) = \frac{1}{2}x^\top Px$, for some constant and symmetric $P \in \mathbb{R}^{n\times n}$, and $H_{\geq 3}$ denoting third and higher order terms with respect to $x = 0$. Let $A = SP$ and assume that $A = A_{s,n} + A_n$, with $A_{s,n}$ being normal, A_n being nilpotent and $[A_{s,n}, A_n] = [A_{s,n}, A_n^\top] = 0$. Then, f_H is in the Belitskii normal form if and only if $H_{\geq 3} \in \mathscr{I}_C(A^\top x)$.*

Proof Clearly, $f_H = f_{H_2} + f_{H_{\geq 3}}$, where $f_{H_2}(x) = SPx$ and $f_{H_{\geq 3}} = S\nabla H_{\geq 3}$. Under the assumptions of the theorem, f_H is in the Belitskii normal form if and only if $[f_{H_{\geq 3}}(x), A^\top x] = 0$. The assumption $n = 2\nu$ implies that S is invertible with inverse $S^{-1} = -S$. Since $A = SP$, one concludes that $\bar{P} = S^{-1}A^\top = -SPS^\top$ is symmetric. Since $[f_{H_{\geq 3}}(x), A^\top x] = -f_{\{H_{\geq 3}(x), \frac{1}{2}x^\top \bar{P}x\}}(x)$, f_H is in the Poincaré–Dulac normal form if and only if $\{H_{\geq 3}(x), \frac{1}{2}x^\top \bar{P}x\} = c$. Now, $c = 0$ by Remark 5.7, and $\{H_{\geq 3}(x), \frac{1}{2}x^\top \bar{P}x\} = 0$ implies that $H_{\geq 3}$ is a first integral associated with $f_{H_2}(x) = A^\top x$. □

Example 5.8 Let $H(x) = \frac{1}{2}x^\top Px + H_{\geq 3}(x)$, with $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent and $\mathscr{I}_C(A^\top x)$ is constituted by all arbitrary functions of $I(x) = q$. Then, f_H is in the Poincaré–Dulac normal form if and only if $H_{\geq 3}$ is an arbitrary function of q, such that $H_{\geq 3}(0) = 0$, $\frac{\partial H_{\geq 3}(x)}{\partial x}|_{x=0} = 0$ and $\frac{\partial^2 H_{\geq 3}(x)}{\partial x^2}|_{x=0} = 0$. For instance, if $H_{\geq 3}(x) = \frac{1}{3}q^3$, then the following f_H is in the Belitskii normal form:

$$f_H(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} - q \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} p \\ -q^2 \end{bmatrix}.$$

Let H be a Hamiltonian function and f_H be the associated Hamiltonian vector function. Let K be a Hamiltonian function, f_K be the associated Hamiltonian vector function and $\Phi_{f_K}(\tau, \tilde{x})$ be the flow associated with f_K; by Theorem 5.8, $x = \Phi_{f_K}(1, \tilde{x})$ is a Poisson map (assume that $\tau = 1$ is admissible). Hence, letting

$\tilde{H}(\tilde{x}) = H(x) \circ \Phi_{f_K}(1, \tilde{x})$ and denoting by $f_{\tilde{H}}(\tilde{x})$ the associated Hamiltonian vector function, by (3.69), one concludes that

$$f_{\tilde{H}}(\tilde{x}) = f_H(\tilde{x}) + \left[f_K(\tilde{x}), f_H(\tilde{x})\right] + \frac{1}{2!}\left[f_K(\tilde{x}), \left[f_K(\tilde{x}), f_H(\tilde{x})\right]\right] + \cdots.$$

Therefore,

$$\tilde{H}(\tilde{x}) = H(\tilde{x}) - \left\{K(\tilde{x}), H(\tilde{x})\right\} + \frac{1}{2!}\left\{K(\tilde{x}), \left\{K(\tilde{x}), H(\tilde{x})\right\}\right\} - \cdots.$$

Given $H(x)$, the objective is to choose $K(\tilde{x})$ so that $\tilde{H}(\tilde{x})$ is simpler than $H(x)$, for instance so that $\tilde{H}(\tilde{x})$ is in the Birkhoff–Gustavson normal form, and therefore quadratic in absence of resonances.

Now, note that if K_i (respectively, H_j) is homogeneous of degree i (respectively, j) with respect to the standard dilation, then $\{K_i, H_j\}$ is homogeneous of degree $i + j - 2$ with respect to the standard dilation. Therefore, if $H = \sum_{j=2}^{+\infty} H_j$, then

$$\begin{aligned}\tilde{H} &= \sum_{j_1=2}^{+\infty} H_{j_1} - \left\{K, \sum_{j_2=2}^{+\infty} H_{j_2}\right\} + \frac{1}{2!}\left\{K, \left\{K, \sum_{j_3=2}^{+\infty} H_{j_3}\right\}\right\} - \cdots \\ &= \sum_{j_1=2}^{+\infty} H_{j_1} - \sum_{j_2=2}^{+\infty}\{K, H_{j_2}\} + \frac{1}{2!}\sum_{j_3=2}^{+\infty}\{K, \{K, H_{j_3}\}\} - \cdots.\end{aligned}$$

In particular, if $K = K_i$ is homogeneous of degree $i \geq 3$, then $\{K, H_{j_2}\}$ has degree $i + j_2 - 2$ (its degree is equal to i when $j_2 = 2$), $\{K, \{K, H_{j_3}\}\}$ has degree $2i + j_3 - 3$ (its degree is equal to $2i$ when $j_3 = 3$) and so on. Therefore, letting $\tilde{H} = \sum_{j=2}^{+\infty} \tilde{H}_j$, one finds that $\tilde{H}_j = H_j$ for all $j \in \{2, \ldots, i - 1\}$, $\tilde{H}_i = H_i - \{K_i, H_2\}$. This shows how the canonical transformation $x = \Phi_{f_K}(1, \tilde{x})$ with $K = K_i$, $i \geq 3$, does not alter the homogeneous terms of H having degree less than i, modifies the homogeneous term H_i of H having degree i with a change given by $\tilde{H}_i = H_i - \{K_i, H_2\}$, whereas the terms of order higher than i are modified in a more cumbersome way, but irrelevant, as shown in the following example; for instance, the Poisson map linearizing a Hamiltonian system, when it exists, can be obtained by a sequence of such diffeomorphisms, by taking first $i = 3$, then $i = 4$ and so on. It is worth pointing out that if a Hamiltonian system is not linearizable by a Poisson map, then this need not imply that such a Hamiltonian system is not linearizable by a diffeomorphism.

Example 5.9 Let $n = 2$ and $x = [q\ \ p]^\top$. Let $H = \sum_{j=2}^{12} H_j$, where $H_2(x) = q^2 + 2p^2$, $H_3(x) = 4pq^2$, $H_4(x) = 2q^4 + 2qp^3$, $H_5(x) = 6p^2q^3$, $H_6(x) = 6pq^5 + p^6$, $H_7(x) = 6p^5q^2 + 2q^7$, $H_8(x) = 15p^4q^4$, $H_9(x) = 20p^3q^6$, $H_{10}(x) = 15p^2q^8$, $H_{11}(x) = 6pq^{10}$ and $H_{12}(x) = q^{12}$. Consider first the Poisson map $x = \Phi_{f_{K_3}}(1, \tilde{x})$, with K_3 being homogeneous of degree 3 with respect to the standard dilation:

$$K_3(\tilde{x}) = a_1\tilde{q}^3 + a_2\tilde{q}^2\tilde{p} + a_3\tilde{q}\tilde{p}^2 + a_4\tilde{p}^3,$$

where the coefficients a_i are real numbers to be fixed so that

$$\begin{aligned}\tilde{H}_3(\tilde{x}) &= H_3(\tilde{x}) - \left\{K_3(\tilde{x}), H_2(\tilde{x})\right\} \\ &= 2a_2\tilde{q}^3 + (4 - 12a_1 + 4a_3)\tilde{q}^2\tilde{p} + (-8a_2 + 6a_4)\tilde{q}\tilde{p}^2 - 4a_3\tilde{p}^3\end{aligned}$$

is as simple as possible. In particular, by imposing that $\tilde{H}_3 = 0$ identically, one obtains the following algebraic system:

$$\begin{aligned} 2a_2 &= 0, \\ 4 - 12a_1 + 4a_3 &= 0, \\ -8a_2 + 6a_4 &= 0, \\ -4a_3 &= 0, \end{aligned}$$

with the unique solutions $a_1 = \frac{1}{3}, a_2 = 0, a_3 = 0, a_4 = 0$, which yields $K_3(\tilde{x}) = \frac{1}{3}\tilde{q}^3$. The Hamiltonian system with the Hamiltonian function K_3 is described by the Hamiltonian vector function $f_{K_3}(x) = [0\ -\tilde{q}^2]^\top$; the flow of such a system is $\Phi_{f_{K_3}}(\tau, \tilde{x}) = [\tilde{q}\ -\tilde{q}^2\tau + \tilde{p}]^\top$. Then, letting $\tau = 1$ and $\tilde{H}(\tilde{q}, \tilde{p}) = H(\tilde{q}, -\tilde{q}^2 + \tilde{p})$, one finds that $\tilde{H} = \tilde{H}_2 + \tilde{H}_4 + \tilde{H}_6$, where $\tilde{H}_2(\tilde{x}) = \tilde{q}^2 + 2\tilde{p}^2$, $\tilde{H}_4(\tilde{x}) = 2\tilde{q}\tilde{p}^3$ and $\tilde{H}_6(\tilde{x}) = \tilde{p}^6$. Consider now the Poisson map $\tilde{x} = \Phi_{f_{K_4}}(1, \hat{x})$, with K_4 being homogeneous of degree 4 with respect to the standard dilation:

$$K_4(\hat{x}) = b_1\hat{q}^4 + b_2\hat{q}^3\hat{p} + b_3\hat{q}^2\hat{p}^2 + b_4\hat{q}\hat{p}^3 + b_5\hat{p}^4,$$

where the coefficients b_i are real numbers to be fixed so that

$$\begin{aligned}\hat{H}_4(\hat{x}) = \tilde{H}_4(\hat{x}) - \left\{K_4(\hat{x}), \tilde{H}_2(\hat{x})\right\} &= 2b_2\hat{q}^4 + (4b_3 - 16b_1)\hat{q}^3\hat{p} \\ &+ (6b_4 - 12b_2)\hat{q}^2\hat{p}^2 + (2 + 8b_5 - 8b_3)\hat{q}\hat{p}^3 - 4b_4\hat{p}^4\end{aligned}$$

is as simple as possible. In particular, by imposing that $\hat{H}_4 = 0$ identically, one obtains the following algebraic system:

$$\begin{aligned} 2b_2 &= 0, \\ -16b_1 + 4b_3 &= 0, \\ -12b_2 + 6b_4 &= 0, \\ -8b_3 + 2 + 8b_5 &= 0, \\ -4b_4 &= 0, \end{aligned}$$

with the following set of solutions:

$$b_1 = \frac{1}{16} + \frac{1}{4}c, \qquad b_2 = 0, \qquad b_3 = \frac{1}{4} + c, \qquad b_4 = 0, \qquad b_5 = c,$$

where $c \in \mathbb{R}$ is an arbitrary constant. Then, $K_4(\hat{x}) = (\frac{1}{16} + \frac{1}{4}c)\hat{q}^4 + (\frac{1}{4} + c)\hat{q}^2\hat{p}^2 + c\hat{p}^4$; for instance, choosing $c = -\frac{1}{4}$, one has $K_4(\hat{x}) = -\frac{1}{4}\hat{p}^4$. The Hamiltonian system with the Hamiltonian function K_4 is described by the Hamiltonian vector function $f_{K_4}(\hat{x}) = [-\hat{p}^3 \;\; 0]^\top$; the flow of such a system is $\Phi_{f_{K_4}}(\tau, \hat{x}) = [-\hat{p}^3\tau + \hat{q} \;\; \hat{p}]^\top$. Then, letting $\tau = 1$ and $\hat{H}(\hat{q}, \hat{p}) = \tilde{H}(-\hat{p}^3 + \hat{q}, \hat{p})$, one finds that $\hat{H} = \hat{H}_2$, with $\hat{H}_2(\hat{x}) = \hat{q}^2 + 2\hat{p}^2$. In conclusion, by defining the Poisson map $\Phi_{f_{K_3}}(1, \tilde{x}) \circ \Phi_{f_{K_4}}(1, \hat{x})$,

$$
\begin{aligned}
q &= -\hat{p}^3 + \hat{q}, \\
p &= -\big(-\hat{p}^3 + \hat{q}\big)^2 + \hat{p},
\end{aligned}
$$

with inverse

$$\hat{q} = q + \big(p + q^2\big)^3, \tag{5.21a}$$

$$\hat{p} = p + q^2, \tag{5.21b}$$

one concludes that $H = (q + (p + q^2)^3)^2 + 2(p + q^2)^2 = \hat{q}^2 + 2\hat{p}^2$, namely that the Hamiltonian system can be linearized by the Poisson map (5.21a), (5.21b).

5.4 Hamiltonian Planar Systems

When $n = 2$, any skew-symmetric matrix $S(x)$ can be rewritten as $S = \left[\begin{smallmatrix} 0 & \omega \\ -\omega & 0 \end{smallmatrix}\right]$, where $\omega(x) \in \mathbb{R}$; if S is the structure matrix of a Poisson bracket, then $\omega(x) = \{x_1, x_2\}$. When considering the Jacobi condition

$$\big\{x_i, \{x_j, x_k\}\big\} + \big\{x_j, \{x_k, x_i\}\big\} + \big\{x_k, \{x_i, x_j\}\big\} = 0, \tag{5.22}$$

for all $i, j, k \in \{1, 2\}$, there are only two possible cases: either all indices are equal (in this case (5.22) trivially holds) or only two of them are equal. In this last case, for instance, assume $i = j = 1$ and $k = 2$; then,

$$\big\{x_1, \{x_1, x_2\}\big\} + \big\{x_1, \{x_2, x_1\}\big\} + \big\{x_2, \{x_1, x_1\}\big\} = \big\{x_1, \omega(x)\big\} + \big\{x_1, -\omega(x)\big\},$$

which shows that (5.22) holds, whence that S is a structure matrix, for all $\omega(x) \in \mathbb{R}$. Therefore, apart from a diffeomorphism, assume that coordinates $x \in \mathbb{R}^2$, $x = [x_1 \;\; x_2]^\top = [q \;\; p]^\top$, are canonical. This imply that either $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ or $S = 0$; the last case, which corresponds to $f_H = 0$ for any $H \in \mathscr{H}$, is trivial. Thus, in the rest of this section, assume that $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. In this simple case, the Hamiltonian vector function $f_H(x) \in \mathbb{R}^2$ associated with H is

$$f_H = S\left(\frac{\partial H}{\partial x}\right)^\top = \begin{bmatrix} \frac{\partial H}{\partial x_2} \\ -\frac{\partial H}{\partial x_1} \end{bmatrix}. \tag{5.23}$$

Note that the class $\mathscr{F}_{\mathscr{H}}$ given by (5.23) coincides with the class of all $f(x) \in \mathbb{R}^2$ having an inverse integrating factor ω equal to 1; as a matter of fact, if $\omega = 1$ is an inverse integrating factor associated with f, then $[f_2 - f_1]$ is exact, i.e., there exists a first integral I such that $\frac{\partial I}{\partial x_1} = f_2$ and $\frac{\partial I}{\partial x_2} = -f_1$, which is locally unique apart from the sum of an arbitrary constant. Hence, $f = f_H$, with $H = -I$. Conversely, an inverse integrating factor associated with (5.23) is $\omega = 1$. Since $L_f\omega = \mathrm{div}(f)\omega$, if $\omega = 1$, then $\mathrm{div}(f) = 0$; vice versa, if $\mathrm{div}(f) = 0$, then any constant (whence, also $\omega = 1$) is an inverse integrating factor associated with f. Therefore, condition $\omega = 1$ is equivalent to condition $\mathrm{div}(f) = 0$.

Consider a diffeomorphism $x = \phi(\tilde{x})$, with $\tilde{x} = [\tilde{q}\ \tilde{p}]^\top$, where $\tilde{x} = [\tilde{x}_1\ \tilde{x}_2]^\top = \phi^{-1}(x) = \varphi(x)$ is the inverse. Let $\tilde{H}(\tilde{x}) = H \circ \phi(\tilde{x})$ and $f_{\tilde{H}}(\tilde{x})$ be the associated Hamiltonian vector function. Is it true that $f_{\tilde{H}} = (\frac{\partial \phi}{\partial \tilde{x}})^{-1} f_H \circ \phi$, namely that $\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = f_H \circ \phi$? A partial answer is already known from the general case (see Theorem 5.7): if ϕ is a Poisson map, then $\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = f_H \circ \phi$ holds. The complete answer in the planar case can be obtained by the following relations:

$$\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = \frac{\partial \phi}{\partial \tilde{x}} S \left(\frac{\partial \tilde{H}}{\partial \tilde{x}} \right)^\top,$$

$$f_H \circ \phi = S \left(\frac{\partial H}{\partial x} \right)^\top \circ \phi;$$

taking into account that $\frac{\partial \tilde{H}}{\partial \tilde{x}} = (\frac{\partial H}{\partial x} \circ \phi) \frac{\partial \phi}{\partial \tilde{x}}$, it follows that

$$\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = \frac{\partial \phi}{\partial \tilde{x}} S \left(\frac{\partial \phi}{\partial \tilde{x}} \right)^\top \left(\frac{\partial H}{\partial x} \right)^\top \circ \phi.$$

Since $BSB^\top = \det(B)S$, for any matrix $B \in \mathbb{R}^{2\times 2}$, one concludes that

$$\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = \det\left(\frac{\partial \phi}{\partial \tilde{x}} \right) S \left(\frac{\partial H}{\partial x} \right)^\top \circ \phi.$$

Therefore, $\frac{\partial \phi}{\partial \tilde{x}} f_{\tilde{H}} = f_H \circ \phi$ if and only if $\det(\frac{\partial \phi}{\partial \tilde{x}}) = \det(\frac{\partial \varphi}{\partial x}) = 1$. Let $\varphi = [u\ v]^\top$; then:

$$\det\left(\frac{\partial \varphi}{\partial x} \right) = \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ \frac{\partial v}{\partial q} & \frac{\partial v}{\partial p} \end{bmatrix} \right) = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q}$$
$$= \{u, v\},$$

namely the Poisson bracket $\{u, v\}$ of u and v coincides with the determinant of the Jacobian matrix of $\varphi = [u\ v]^\top$. Local coordinates $\tilde{q} = u(q, p)$ and $\tilde{p} = v(q, p)$ are *canonical coordinates* if $\{u, v\} = 1$; similarly, diffeomorphism ϕ (respectively, φ) is called *canonical*. Clearly, $\{q, p\} = 1$. By the above analysis, $(\frac{\partial \phi}{\partial \tilde{x}})^{-1} f_H \circ \phi$ is Hamiltonian with the Hamiltonian function $\tilde{H} = H \circ \phi$ if and only if $\tilde{q}, \tilde{p}$ are canonical, namely the following diagram is commutative if and only if $\tilde{q}, \tilde{p}$ are canonical:

$$\begin{array}{ccc} H(x) & \xrightarrow{\tilde{x}=\varphi(x)} & \tilde{H}(\tilde{x}) \\ \downarrow & & \downarrow \\ f_H(x) & \xrightarrow[\tilde{x}=\varphi(x)]{} & f_{\tilde{H}}(\tilde{x}) \end{array}$$

Note that $\tilde{q} = p$, $\tilde{p} = -q$ is a canonical diffeomorphism, since $\{p, -q\} = \{q, p\} = 1$. This shows that the role of generalized coordinate and generalized momentum can be interchanged.

Example 5.10 Pair $(\tilde{q}, \tilde{p}) = (\ln(\frac{1}{q}\sin(p)), q\cot(p))$ is canonical (see, also, [54]),

$$\{\tilde{q}, \tilde{p}\} = \det\left(\begin{bmatrix} -\frac{1}{q} & \frac{\cos(p)}{\sin(p)} \\ \cot(p) & -(1+\cot^2(p))q \end{bmatrix}\right) = 1.$$

Pair $(\tilde{q}, \tilde{p}) = (\ln(1+\sqrt{q}\cos(p)), 2(1+\sqrt{q}\cos(p))\sqrt{q}\sin(p))$ is canonical (see, also, [54]),

$$\{\tilde{q}, \tilde{p}\} = \det\left(\begin{bmatrix} \frac{1}{2\sqrt{q}}\frac{\cos(p)}{1+\sqrt{q}\cos(p)} & -\sqrt{q}\frac{\sin(p)}{1+\sqrt{q}\cos(p)} \\ (\frac{1}{\sqrt{q}}+2\cos(p))\sin(p) & 2(\sqrt{q}\cos(p)-q\sin^2(p)+q\cos^2(p)) \end{bmatrix}\right)$$
$$= 1.$$

Remark 5.8 Let $\tilde{x} = \varphi(x)$ be a canonical diffeomorphism, $\varphi = [u \ v]^\top$ and $\{u, v\} = \det(\frac{\partial\varphi}{\partial x}) = 1$. Let $[f \ \ g] = (\frac{\partial\varphi}{\partial x})^{-1}$; by construction f and g are commuting, $[f, g] = 0$, and $\omega = 1$ is an inverse integrating factor associated with both f and g, whence both f and g are Hamiltonian. Since $\frac{\partial\varphi}{\partial x}[f \ g] = E$, u is a first integral associated with g and v is a first integral associated with f. In particular, denoting by f_i and g_i the ith entries of f and g, respectively, since $\left[\begin{smallmatrix} f_1 & g_1 \\ f_2 & g_2 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} g_2 & -g_1 \\ -f_2 & f_1 \end{smallmatrix}\right]$, $K = -u$ is the Hamiltonian function associated with g and $H = v$ is the Hamiltonian function associated with f. For instance, as in Example 5.10, choose $u = \ln(\frac{1}{q}\sin(p))$ and $v = q\cot(p)$. Then, by

$$\left(\frac{\partial\varphi}{\partial x}\right)^{-1} = \begin{bmatrix} -\frac{1}{q} & \frac{\cos(p)}{\sin(p)} \\ \cot(p) & (-1-\cot^2(p))q \end{bmatrix}^{-1} = \begin{bmatrix} q(-1-\cot^2(p)) & -\frac{\cos(p)}{\sin(p)} \\ -\cot(p) & -\frac{1}{q} \end{bmatrix},$$

one shows that $f_H(x) = \left[\begin{smallmatrix} q(-1-\cot^2(p)) \\ -\cot p \end{smallmatrix}\right]$ is Hamiltonian with the Hamiltonian function $H(x) = v(x) = q\cot(p)$, $f_K(x) = \left[\begin{smallmatrix} -\frac{\cos(p)}{\sin(p)} \\ -\frac{1}{q} \end{smallmatrix}\right]$ is Hamiltonian with the Hamiltonian function $K(x) = -u(x) = -\ln(\frac{1}{q}\sin(p))$, and $[f_K, f_H] = f_{\{H,K\}} = 0$.

Remark 5.9 By Remark 5.8 and the analysis of Sect. 1.6, the flow of a Hamiltonian f_K can be rewritten as $\Phi_{f_K}(\tau, x) = \varphi^{-1}(\tau e_1 + \varphi(x))$, for some canonical diffeomorphism $\tilde{x} = \varphi(x)$. Since $\det(\frac{\partial \Phi_{f_K}(\tau,x)}{\partial x}) = (\frac{\partial \varphi^{-1}(x)}{\partial x} \circ (\tau e_1 + \varphi(x)))\frac{\partial \varphi(x)}{\partial x}$ and $\det(\frac{\partial \varphi(x)}{\partial x}) = \det(\frac{\partial \varphi^{-1}(\tilde{x})}{\partial \tilde{x}}) = 1$, one concludes that $\Phi_{f_K}(\tau, x)$ is a canonical diffeomorphism for any Hamiltonian f_K (in agreement with Theorem 5.8). As an example, consider the Hamiltonian vector function $f_K(x) = [1 \ \sin(q)]^\top$ associated with the Hamiltonian function $H(x) = p + \cos(q)$; then, $\Phi_{f_K}(\tau, x) = [\tau + q \ \ p + \cos(q) - \cos(\tau + q)]^\top$. Since

$$\det\left(\frac{\partial \Phi_{f_K}(\tau, x)}{\partial x}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ -\sin(q) + \sin(\tau + q) & 1 \end{bmatrix}\right) = 1,$$

$\Phi_{f_K}(\tau, x)$ is a canonical diffeomorphism for any $\tau \in \mathbb{R}$.

Remark 5.10 A linear transformation $x = Q\tilde{x}$, $Q \in \mathbb{R}^{2\times 2}$, is canonical if and only if $\det(Q) = 1$.

By Theorem 2.10 at p. 45, for a square matrix B, one finds that $\varpi(\tau) = \mathrm{e}^{\mathrm{trace}(B)\tau}\varpi(0)$, where $\varpi(\tau) = \det(\mathrm{e}^{B\tau})$. Hence, taking into account that $\varpi(0) = 1$,

$$\mathrm{trace}(B) = 0 \quad \Longleftrightarrow \quad \varpi(\tau) = 1, \quad \forall \tau \in \mathbb{R}.$$

This means that if $g(x) = Bx$ is Hamiltonian, then $Q = \mathrm{e}^{B\tau}$ is a canonical linear transformation for any $\forall \tau \in \mathbb{R}$ and, vice versa, if $Q = \mathrm{e}^{B\tau}$ is a canonical linear transformation for any $\forall \tau \in \mathbb{R}$, then $g(x) = Bx$ is Hamiltonian. However, note that $Q = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ is a canonical linear transformation since $\det(Q) = 1$, but there exists no $B \in \mathbb{R}^{2\times 2}$ such that $Q = \mathrm{e}^{B\tau}$, for some $\tau \in \mathbb{R}$, because B has an odd number of Jordan blocks with negative eigenvalues. Actually, there exists no Hamiltonian vector function $f_H(x) \in \mathbb{R}^2$ analytic at $x = 0$, $f_H(0) = 0$, such that $\Phi_{f_H}(\tau, x) = Qx$ for some $\tau \in \mathbb{R}$, because if $f_H(x) = Bx + \cdots$ and $\Phi_{f_H}(\tau, x) = Q_1(\tau)x + \cdots$, then $\mathrm{e}^{B\tau} = Q_1(\tau)$.

Remark 5.11 When $n = 2$, the proof that the Poisson bracket described by $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ satisfies properties (5.2.1)–(5.2.4) of Definition 5.2, with $u(q, p)$, $v(q, p)$, $z(q, p) \in \mathbb{R}$, can be carried out by substitution.

Proof of (5.2.1):

$$\{u, v\} = \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ \frac{\partial v}{\partial q} & \frac{\partial v}{\partial p} \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \frac{\partial v}{\partial q} & \frac{\partial v}{\partial p} \\ \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \end{bmatrix}\right) = -\{v, u\}.$$

Proof of (5.2.2):

$$\{au + bv, z\} = \det\left(\begin{bmatrix} a\frac{\partial u}{\partial q} + b\frac{\partial v}{\partial q} & a\frac{\partial u}{\partial p} + b\frac{\partial v}{\partial p} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial p} \end{bmatrix}\right)$$

$$= a \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial p} \end{bmatrix}\right) + b \det\left(\begin{bmatrix} \frac{\partial v}{\partial q} & \frac{\partial v}{\partial p} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial p} \end{bmatrix}\right) = a\{u, z\} + b\{v, z\}.$$

Proof of (5.2.3): the statement is easily proven by summing the following three equalities

$$\begin{aligned}
\big\{u, \{v, z\}\big\} &= \frac{\partial u}{\partial q}\frac{\partial}{\partial p}\{v, z\} - \frac{\partial u}{\partial p}\frac{\partial}{\partial q}\{v, z\} \\
&= \frac{\partial u}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial q}\frac{\partial z}{\partial p} - \frac{\partial v}{\partial p}\frac{\partial z}{\partial q}\right) - \frac{\partial u}{\partial p}\frac{\partial}{\partial q}\left(\frac{\partial v}{\partial q}\frac{\partial z}{\partial p} - \frac{\partial v}{\partial p}\frac{\partial z}{\partial q}\right) \\
&= \frac{\partial u}{\partial q}\left(\frac{\partial^2 v}{\partial q \partial p}\frac{\partial z}{\partial p} + \frac{\partial v}{\partial q}\frac{\partial^2 z}{\partial p^2} - \frac{\partial^2 v}{\partial p^2}\frac{\partial z}{\partial q} - \frac{\partial v}{\partial p}\frac{\partial^2 z}{\partial q \partial p}\right) \\
&\quad - \frac{\partial u}{\partial p}\left(\frac{\partial^2 v}{\partial q^2}\frac{\partial z}{\partial p} + \frac{\partial v}{\partial q}\frac{\partial^2 z}{\partial q \partial p} - \frac{\partial^2 v}{\partial q \partial p}\frac{\partial z}{\partial q} - \frac{\partial v}{\partial p}\frac{\partial^2 z}{\partial q^2}\right), \\
\big\{v, \{z, u\}\big\} &= \frac{\partial v}{\partial q}\left(\frac{\partial^2 z}{\partial q \partial p}\frac{\partial u}{\partial p} + \frac{\partial z}{\partial q}\frac{\partial^2 u}{\partial p^2} - \frac{\partial^2 z}{\partial p^2}\frac{\partial u}{\partial q} - \frac{\partial z}{\partial p}\frac{\partial^2 u}{\partial q \partial p}\right) \\
&\quad - \frac{\partial v}{\partial p}\left(\frac{\partial^2 z}{\partial q^2}\frac{\partial u}{\partial p} + \frac{\partial z}{\partial q}\frac{\partial^2 u}{\partial q \partial p} - \frac{\partial^2 z}{\partial q \partial p}\frac{\partial u}{\partial q} - \frac{\partial z}{\partial p}\frac{\partial^2 u}{\partial q^2}\right), \\
\big\{z, \{u, v\}\big\} &= \frac{\partial z}{\partial q}\left(\frac{\partial^2 u}{\partial q \partial p}\frac{\partial v}{\partial p} + \frac{\partial u}{\partial q}\frac{\partial^2 v}{\partial p^2} - \frac{\partial^2 u}{\partial p^2}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial p}\frac{\partial^2 v}{\partial q \partial p}\right) \\
&\quad - \frac{\partial z}{\partial p}\left(\frac{\partial^2 u}{\partial q^2}\frac{\partial v}{\partial p} + \frac{\partial u}{\partial q}\frac{\partial^2 v}{\partial q \partial p} - \frac{\partial^2 u}{\partial q \partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial p}\frac{\partial^2 v}{\partial q^2}\right).
\end{aligned}$$

Proof of (5.2.4):

$$\begin{aligned}
\{u, vz\} &= \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ \frac{\partial vz}{\partial q} & \frac{\partial vz}{\partial p} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ z\frac{\partial v}{\partial q} & z\frac{\partial v}{\partial p} \end{bmatrix}\right) + \det\left(\begin{bmatrix} \frac{\partial u}{\partial q} & \frac{\partial u}{\partial p} \\ v\frac{\partial z}{\partial q} & v\frac{\partial z}{\partial p} \end{bmatrix}\right) \\
&= \{u, v\}z + \{u, z\}v.
\end{aligned}$$

The equations of motion of a planar Hamiltonian system can be rewritten, using the Poisson bracket, as

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} &= \{q, H\}, \\
\frac{\mathrm{d}p}{\mathrm{d}t} &= \{p, H\}.
\end{aligned}$$

Consider now the Noether symmetries defined in Definition 5.4 and recall Theorem 5.4. It is easy to see that the Noether symmetry f_K of f_H, $\{K, H\} = c$, is trivial if and only if $c = 0$; if $\{K, H\} = 0$, then K is a first integral associated with f_H, whence $K = C(H)$ for some function C, because the Hamiltonian system is planar and, therefore, cannot have more than one functionally independent first integral.

Now, excluding the trivial Noether symmetries corresponding to case $c = 0$ and, apart from the division for a constant $c \neq 0$, all the Noether symmetries are given by f_K, with $\{K, H\} = 1$. Let $\{K, H\} = 1$ (if $\{K, H\} = c$, for $c \neq 0$, then take $\frac{1}{c}K$ instead of K). By the above discussion, $\tilde{q} = K(q, p)$, $\tilde{p} = H(q, p)$ is a canonical diffeomorphism and $\tilde{q}, \tilde{p}$ qualify as canonical coordinates. In the local coordinates $\tilde{q}, \tilde{p}$, both f_H and f_K are straightened; as a matter of fact, the dynamics of the system are described, in the local coordinates $\tilde{q}, \tilde{p}$, by

$$\frac{\mathrm{d}\tilde{q}}{\mathrm{d}t} = \{K, H\} = 1,$$
$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} = \{H, H\} = 0,$$

whereas the dynamics of the Noether symmetry are described, in the local coordinates $\tilde{q}, \tilde{p}$, by

$$\frac{\mathrm{d}\tilde{q}}{\mathrm{d}t} = \{K, K\} = 0,$$
$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} = \{H, K\} = -1.$$

The following theorem parametrizes all the Noether symmetries of a Hamiltonian vector function f_H, whereas the parameterization of all symmetries (not necessarily of the Noether type) is given in Theorem 3.9 at p. 64.

Theorem 5.12 *Let $\bar{K} \neq 0$ be a particular solution of $\{K, H\} = 1$ (whose existence is ensured about any regular point of f_H); then, all solutions of $\{K, H\} = 1$ are given by $K = \bar{K} + C(H)$, where C is an arbitrary function of H.*

Proof By the bi-linearity of the Poisson bracket, the set of all solutions in K of $\{K, H\} = 1$, is generated by finding a particular solution $\bar{K}$ of $\{K, H\} = 1$ and by adding to $\bar{K}$ an arbitrary solution of the homogeneous equation $\{K, H\} = 0$. Any solution of $\{K, H\} = 0$ is a first integral associated with f_H; in the planar case, all first integrals associated with f_H are functions of H. □

Example 5.11 Consider the Hamiltonian function $H(x) = p - q^2$ and the associated vector function

$$f_H(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2q \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2q \end{bmatrix}.$$

A solution of $\{K, H\} = 1$ is $\bar{K}(x) = q$, since $\{q, p - q^2\} = \det\left(\left[\begin{smallmatrix} 1 & 0 \\ -2q & 1 \end{smallmatrix}\right]\right) = 1$. Then, all solutions of $\{K, H\} = 1$ are $K(x) = q + C(p - q^2)$, where C is an arbitrary function of the argument. Then, apart from the division for some non-zero constant, all the non-trivial Noether symmetries of f_H are given by f_K,

$$f_K(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 - 2qG \\ G \end{bmatrix} = \begin{bmatrix} G \\ -1 + 2qG \end{bmatrix},$$

where $G(\xi) = \frac{\mathrm{d}C(\xi)}{\mathrm{d}\xi}$ is an arbitrary function of $p - q^2$. For instance, taking $G = (p - q^2)^2$ (i.e., $C = \frac{1}{3}(p - q^2)^3$), one concludes that a Noether symmetry of f_H is given by $f_K(x) = [(p - q^2)^2 \;\; -1 + 2q(p - q^2)^2]^\top$. As a matter of fact, the Hamiltonian function associated with f_K is $K(x) = q + \frac{1}{3}(p - q^2)^3$ and

$$\begin{aligned}\left[f_H(x), f_K(x)\right] &= \begin{bmatrix} 4q(-p + q^2) & 2p - 2q^2 \\ 2(-p + q^2)(-p + 5q^2) & -4q(-p + q^2) \end{bmatrix} \begin{bmatrix} 1 \\ 2q \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} (p - q^2)^2 \\ -1 + 2q(p - q^2)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned}$$

Both f_H and f_K are straightened by the canonical diffeomorphism $\tilde{q} = K(x) = q + \frac{1}{3}(p - q^2)^3$, $\tilde{p} = H(x) = p - q^2$, as can be easily verified by taking into account that $\{H, H\} = 0$, $\{K, K\} = 0$ and $\{K, H\} = 1$.

Remark 5.12 Let $f(x) = Ax$. By relation (5.20), $f = f_H$ for some $H \in \mathscr{H}$ (namely, Ax is Hamiltonian) if and only if A has zero trace. Let $f_H(x) = Ax$ and $f_K(x) = Bx$ be Hamiltonian, i.e., let A and B have zero trace, $A = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right]$. Then, f_K is a Noether symmetry of f_H if and only if A and B are commuting. Since

$$[A, B] = \begin{bmatrix} \beta c - \gamma b & 2\alpha b - 2\beta a \\ 2\gamma a - 2\alpha c & \gamma b - \beta c \end{bmatrix},$$

A and B are commuting if and only if A and B are co-linear over $\mathbb{R}$, i.e., $B = \kappa A$, for some constant $\kappa \in \mathbb{R}$. Therefore, f_K is a trivial Noether symmetry of f_H.

Theorem 5.13 *Let H be a Hamiltonian function and f_H be the associated vector function. Then, there are local canonical coordinates $(\tilde{q}, \tilde{p}) = (u(q, p), v(q, p))$, $\{u, v\} = 1$, such that $f_{\tilde{H}}$ is linear with $\tilde{H}(u(x), v(x)) = H(x)$ if and only if $\tilde{H}(u, v) = \frac{1}{2}[u \;\, v]P\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, for some constant and symmetric $P \in \mathbb{R}^{2\times 2}$.*

Proof If $\tilde{H}(u, v) = \frac{1}{2}[u \;\, v]P\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ for some u, v such that $\{u, v\} = 1$, then letting $(\tilde{q}, \tilde{p}) = (u, v)$, one has that $\tilde{H}$ is a quadratic function of $\tilde{q}, \tilde{p}$, whence the associated $f_{\tilde{H}}$ is linear, $f_{\tilde{H}}(x) = SPx$. Conversely, if $f_{\tilde{H}}$ is linear, then the associated Hamiltonian function (choosing zero integration constant) is a quadratic function of $\tilde{q}$, $\tilde{p}$,

$\tilde{H}(\tilde{q}, \tilde{p}) = \frac{1}{2}[\tilde{q}\ \tilde{p}]P\left[\begin{smallmatrix}\tilde{q}\\ \tilde{p}\end{smallmatrix}\right]$. If (u, v) is a canonical diffeomorphism, $\{u, v\} = 1$, then the Hamiltonian function of the transformed Hamiltonian system is $H = \tilde{H}(u, v)$. □

Corollary 5.2 *Let H be a Hamiltonian function and f_H be the associated vector function. Then, there are local canonical coordinates $(\tilde{q}, \tilde{p}) = (u(q, p), v(q, p))$, $\{u, v\} = 1$, such that $f_{\tilde{H}}$ is linear, $f_{\tilde{H}}(x) = SPx$, with SP being diagonal, if and only if $\tilde{H}(u, v) = -\lambda uv$, with $\lambda \in \mathbb{R}$ being constant. In particular, u and v are two semi-invariants associated with f_H.*

Proof Let $\tilde{H}(u, v) = \frac{1}{2}[u\ v]P\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]$; then, $A = SP$ and, therefore, $P = -SA$, with A being diagonal with zero trace:

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & -\lambda \\ -\lambda & 0 \end{bmatrix}.$$

Therefore, $\tilde{H}(u, v) = \frac{1}{2}[u\ v]\left[\begin{smallmatrix}0 & -\lambda\\ -\lambda & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] = -\lambda uv$. □

Corollary 5.2 is particularly helpful when the Hamiltonian function is polynomial, as shown in the following example.

Example 5.12 Let $H(x) = qp + ap^4$, which can be clearly factorized as $H(x) = (q + ap^3)p$. Since $\{q + ap^3, p\} = \det\left(\left[\begin{smallmatrix}1 & 3ap^2\\ 0 & 1\end{smallmatrix}\right]\right) = 1$, then defining the canonical coordinates $\tilde{q} := q + ap^3$ and $\tilde{p} := p$, one obtains a linear system

$$\frac{\mathrm{d}\tilde{q}}{\mathrm{d}t} = \left\{q + ap^3, qp + ap^4\right\} = \det\left(\begin{bmatrix} 1 & 3ap^2 \\ p & q + 4ap^3 \end{bmatrix}\right) = q + ap^3 = \tilde{q},$$
$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} = \left\{p, qp + ap^4\right\} = \det\left(\begin{bmatrix} 0 & 1 \\ p & q + 4ap^3 \end{bmatrix}\right) = -p = -\tilde{p}.$$

5.5 Systems Having an Inverse Jacobi Last Multiplier Equal to 1

Goal of this section is to show that any vector function f having an inverse Jacobi last multiplier (as defined in Sect. 3.8) equal to one can be written as a Hamiltonian vector function. In particular, by using the concept of the Nambu bracket [99], it is possible to define a Poisson bracket having as the Casimir functions some functionally independent first integrals associated with f, so that the considered system can be rewritten as Hamiltonian with respect to such a Poisson bracket.

In the first part of this section, assume that $x \in \mathbb{R}^3$, $x = [x_1\ x_2\ x_3]^\top$: such an assumption is removed in the final part of the section. Let $x^o \in \mathbb{R}^3$ be a regular point of $f(x) \in \mathbb{R}^3$, $f(x^o) \neq 0$. By the flow box Theorem 3.3 at p. 57, there exists an analytic diffeomorphism $y = \varphi(x) : \mathscr{U} \to \mathbb{R}^3$ such that $L_f\varphi = e_1$, with $\mathscr{U}$ being a neighborhood of x^o. In particular, f is just the first column of $(\frac{\partial\varphi}{\partial x})^{-1}$, namely f

is the solution of $\frac{\partial\varphi}{\partial x} f = e_1$, and the other two columns g_1 and g_2 of $(\frac{\partial\varphi}{\partial x})^{-1}$ are two commuting symmetries of f. Letting $\varphi = [I_0\ I_1\ I_2]^\top$, one obtains $L_f I_0 = 1$, $L_f I_1 = 0$ and $L_f I_2 = 0$; hence, $I_0 - t$ is a time-varying first integral associated with f, whereas I_1 and I_2 are first integrals. Letting $f = [f_1\ f_2\ f_3]^\top$, by the *Cramer rules* applied to $\frac{\partial\varphi}{\partial x} f = e_1$, one finds that $f_i = \sigma_0\sigma_i$, $i = 1, 2, 3$, where

$$\sigma_0^{-1} = \det\left(\frac{\partial\varphi}{\partial x}\right) = \det\left(\begin{bmatrix} \frac{\partial I_0}{\partial x_1} & \frac{\partial I_0}{\partial x_2} & \frac{\partial I_0}{\partial x_3} \\ \frac{\partial I_1}{\partial x_1} & \frac{\partial I_1}{\partial x_2} & \frac{\partial I_1}{\partial x_3} \\ \frac{\partial I_2}{\partial x_1} & \frac{\partial I_2}{\partial x_2} & \frac{\partial I_2}{\partial x_3} \end{bmatrix}\right)$$

and

$$\sigma_1 = \det\left(\begin{bmatrix} 1 & \frac{\partial I_0}{\partial x_2} & \frac{\partial I_0}{\partial x_3} \\ 0 & \frac{\partial I_1}{\partial x_2} & \frac{\partial I_1}{\partial x_3} \\ 0 & \frac{\partial I_2}{\partial x_2} & \frac{\partial I_2}{\partial x_3} \end{bmatrix}\right), \qquad \sigma_2 = \det\left(\begin{bmatrix} \frac{\partial I_0}{\partial x_1} & 1 & \frac{\partial I_0}{\partial x_3} \\ \frac{\partial I_1}{\partial x_1} & 0 & \frac{\partial I_1}{\partial x_3} \\ \frac{\partial I_2}{\partial x_1} & 0 & \frac{\partial I_2}{\partial x_3} \end{bmatrix}\right),$$

$$\sigma_3 = \det\left(\begin{bmatrix} \frac{\partial I_0}{\partial x_1} & \frac{\partial I_0}{\partial x_2} & 1 \\ \frac{\partial I_1}{\partial x_1} & \frac{\partial I_1}{\partial x_2} & 0 \\ \frac{\partial I_2}{\partial x_1} & \frac{\partial I_2}{\partial x_2} & 0 \end{bmatrix}\right).$$

Since the cross product $a \times b$ of vectors $a = [a_1\ a_2\ a_3]^\top$ and $b = [b_1\ b_2\ b_3]^\top$ is formally given by

$$a \times b = \det\left(\begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right),$$

where e_i is the ith column of the 3×3 identity matrix E, one concludes that

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \nabla I_1 \times \nabla I_2.$$

Finally, since $\sigma_0 = \det(\frac{\partial\varphi}{\partial x})^{-1} = \det([f\ g_1\ g_2])$, then $\omega = \sigma_0$ is an inverse Jacobi last multiplier associated with f; in particular, it is called the inverse Jacobi last multiplier *corresponding* to I_1 and I_2, because a different choice of I_1 and I_2 would yield a different ω. Hence, any f can be locally rewritten as

$$f = \omega \nabla I_1 \times \nabla I_2, \tag{5.24}$$

where I_1 and I_2 are two functionally independent first integrals associated with f and ω is the corresponding inverse Jacobi last multiplier.

Let $a = [a_1\ a_2\ a_3]^\top$ and define the skew-symmetric matrix

$$S(a) := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}; \tag{5.25}$$

clearly, $a \times b = S(a)b$, for any $a, b \in \mathbb{R}^3$. Then, (5.24) can be rewritten as

$$f = \omega S(\nabla I_1)\nabla I_2. \tag{5.26}$$

Definition 5.9 Given the functions $u, v, z \in \mathscr{H}$, the *Nambu bracket* $\langle u, v, z\rangle$ of the ordered triplet (u, v, z) is [86, 99]

$$\langle u, v, z\rangle = \det\left(\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ z \end{bmatrix}\right).$$

Given a scalar function $G(x) \in \mathbb{R}$ and writing $S(\nabla G)$ as in (5.25), define the candidate Poisson bracket $\{\cdot,\cdot\}_{S(\nabla G)}$ associated with G as follows:

$$\{K, H\}_{S(\nabla G)} := \frac{\partial K}{\partial x}(\nabla G \times \nabla H) = \frac{\partial K}{\partial x} S(\nabla G)\nabla H. \tag{5.27}$$

Note that such a candidate Poisson bracket can be rewritten as a Nambu bracket, since

$$\{K, H\}_{S(\nabla G)} = \langle K, G, H\rangle.$$

It is easy to verify that $\{\cdot,\cdot\}_{S(\nabla G)}$ satisfies the skew-symmetry property, the bi-linearity and the Leibniz rule. Let $f_H = S(\nabla G)\nabla H$; clearly, $L_{f_H}K = \{K, H\}_{S(\nabla G)} = \langle K, G, H\rangle$, for any $K \in \mathscr{H}$. Then, taking into account the properties of the matrix determinant,

$$\begin{aligned} L_{f_H}\{F, K\}_{S(\nabla G)} &= L_{f_H}\langle F, G, K\rangle \\ &= \langle L_{f_H}F, G, K\rangle + \langle F, L_{f_H}G, K\rangle + \langle F, G, L_{f_H}K\rangle \\ &= \big\langle\langle F, G, H\rangle, G, K\big\rangle + \big\langle F, \langle G, G, H\rangle, K\big\rangle + \big\langle F, G, \langle K, G, H\rangle\big\rangle \\ &= \big\langle\langle F, G, H\rangle, G, K\big\rangle + \big\langle F, G, \langle K, G, H\rangle\big\rangle \\ &= \{L_{f_H}F, K\}_{S(\nabla G)} + \{F, L_{f_H}K\}_{S(\nabla G)}. \end{aligned}$$

Thus, the Jacobi identity follows from Theorem 5.2, and it is proven that $\{\cdot,\cdot\}_{S(\nabla G)}$ is actually a Poisson bracket.

Lemma 5.1 *The following equalities hold*:

$$\{K, H\}_{S(\nabla G)} = \{G, K\}_{S(\nabla H)} = \{H, G\}_{S(\nabla K)}.$$

Proof Since $\{K, H\}_{S(\nabla G)} = \langle K, G, H\rangle$, the lemma follows from the properties of the determinant. For instance, $\{K, H\}_{S(\nabla G)} = \langle K, G, H\rangle$ and $\{G, K\}_{S(\nabla H)} = \langle G, H, K\rangle$, but

$$\langle K, G, H\rangle = \det\left(\frac{\partial}{\partial x}\begin{bmatrix} K \\ G \\ H \end{bmatrix}\right) = \det\left(\frac{\partial}{\partial x}\begin{bmatrix} G \\ H \\ K \end{bmatrix}\right) = \langle G, H, K\rangle.$$

□

Since $e_i^\top S(\nabla I_1)\nabla I_2 = \{x_i, I_2\}_{S(\nabla I_1)}$, by (5.26) one has

$$f = \omega S(\nabla I_1)\nabla I_2 = \omega E S(\nabla I_1)\nabla I_2 = \omega \begin{bmatrix} \{x_1, I_2\}_{S(\nabla I_1)} \\ \{x_2, I_2\}_{S(\nabla I_1)} \\ \{x_3, I_2\}_{S(\nabla I_1)} \end{bmatrix},$$

namely, any system in $\mathbb{R}^3$ can be locally written as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \omega\{x_i, I_2\}_{S(\nabla I_1)}, \quad i = 1, 2, 3,$$

where ω is an inverse Jacobi last multiplier and I_1, I_2 are two functionally independent first integrals associated with f.

The next remark is the crucial result of this section written for the case $n = 3$.

Remark 5.13 In $\mathbb{R}^3$, any system admitting an inverse Jacobi last multiplier ω equal to 1, namely any system that can be written as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, H\}_{S(\nabla G)}, \quad i = 1, 2, 3,$$

is Hamiltonian, where $G(x) \in \mathbb{R}$ defines the Poisson bracket and $H(x) \in \mathbb{R}$ is the Hamiltonian function; all first integrals of the Hamiltonian systems are arbitrary functions of the distinguished first integral G and of the first integral H.

Example 5.13 Let $G(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ and $H(x) = \frac{1}{2}(\frac{x_1^2}{\mathbb{I}_1} + \frac{x_2^2}{\mathbb{I}_2} + \frac{x_3^2}{\mathbb{I}_3})$ for some constant $\mathbb{I}_i \neq 0$, $i = 1, 2, 3$. Then, the corresponding Hamiltonian system is characterized by

$$f_H(x) = S\big(\nabla G(x)\big)\nabla H(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}\begin{bmatrix} \frac{x_1}{\mathbb{I}_1} \\ \frac{x_2}{\mathbb{I}_2} \\ \frac{x_3}{\mathbb{I}_3} \end{bmatrix} = \begin{bmatrix} \frac{\mathbb{I}_2-\mathbb{I}_3}{\mathbb{I}_2\mathbb{I}_3}x_3x_2 \\ \frac{\mathbb{I}_3-\mathbb{I}_1}{\mathbb{I}_1\mathbb{I}_3}x_1x_3 \\ \frac{\mathbb{I}_1-\mathbb{I}_2}{\mathbb{I}_2\mathbb{I}_1}x_1x_2 \end{bmatrix},$$

thus obtaining the *Euler equations*, which describe the motion of a rigid body rotating around its center of mass [54, 56, 69].

Example 5.14 Let $G(x) = \frac{1}{2}(x_1^2 - x_2^2)$ and $H(x) = \frac{1}{2}(x_2^2 - x_3^2)$. Then, the corresponding Hamiltonian system is characterized by

$$f_H(x) = S\big(\nabla G(x)\big)\nabla H(x) = \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix}.$$

The Nambu bracket for $n = 3$ has been a useful tool in order to prove that the operation $\{\cdot, \cdot\}_{S(\nabla G)}$ is indeed a Poisson bracket. The next remark specifies that, in a neighborhood of a regular point, any non-trivial Poisson bracket can be written in such a form.

Remark 5.14 For a given Poisson bracket in $\mathbb{R}^3$, let x_o be one of its regular points and let $\operatorname{rank}(S(x)) = 2$, with $S(x)$ being its structure matrix. By the Darboux Theorem 5.9, there exists a diffeomorphism $y = \varphi(x)$ such that in the y-coordinates the structure matrix of the Poisson bracket is

$$\tilde{S}(y) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the y-coordinates, $\tilde{S}(y) = S(\nabla\tilde{G}(y))$, where $\tilde{G}(y) = -y_3$. Going back to the original x-coordinates, one has that $S(x) = S(\nabla G(x))$, with $G(x) = \tilde{G} \circ \varphi^{-1}(x)$.

The results that have been described for $n = 3$ can be replicated for the general case $n \in \mathbb{Z}^>$, by defining the *Nambu bracket* of n functions $(u_1, u_2, \ldots, u_{n-1}, u_n)$ as

$$\langle u_1, u_2, \ldots, u_{n-1}, u_n\rangle := \det\left(\frac{\partial u}{\partial x}\right),$$

where $u(x) = [u_1(x) \ \ldots \ u_n(x)]^\top$. Let $S_{u_2,\ldots,u_{n-1}}(x)$ be the skew-symmetric matrix such that (its existence is ensured by the multi-linearity of the determinant)

$$\langle u_1, u_2, \ldots, u_{n-1}, u_n\rangle = \frac{\partial u_1}{\partial x} S_{u_2,\ldots,u_{n-1}} \nabla u_n.$$

It can be shown that $S_{u_2,\ldots,u_{n-1}}$ is a structure matrix for all $u_2, \ldots, u_{n-1} \in \mathscr{H}$ and that it defines the Poisson bracket

$$\{u_1, u_n\}_{S_{u_2,\ldots,u_{n-1}}} = \langle u_1, u_2, \ldots, u_{n-1}, u_n\rangle.$$

By a reasoning wholly similar to the one used to obtain (5.26), any $f(x) \in \mathbb{R}^n$ can be written as $f = \omega S_{I_1,\ldots,I_{n-2}} \nabla I_{n-1}$, where ω is the inverse Jacobi last multiplier corresponding to the functionally independent first integrals $I_1, \ldots, I_{n-2}, I_{n-1}$ associated with f; if $\omega = 1$, then f is the Hamiltonian vector function corresponding to the Hamiltonian function I_{n-1} and to the Poisson bracket defined by the structure matrix $S_{I_1,\ldots,I_{n-2}}$. Moreover, the functions $u_2, \ldots, u_{n-1}$ are the Casimir functions associated with the Poisson bracket $\{u_1, u_n\}_{S_{u_2,\ldots,u_{n-1}}}$.

Example 5.15 Assume $n = 4$ and consider $a(x), b(x), c(x), d(x) \in \mathbb{R}$, being non-constant; let $b_i = \frac{\partial b}{\partial x_i}$ and $c_i = \frac{\partial c}{\partial x_i}$, $i = 1, \ldots, 4$. Then,

$$\langle a, b, c, d\rangle = \frac{\partial a}{\partial x} S_{b,c} \nabla d,$$

where

$$S_{b,c} = \begin{bmatrix} 0 & b_3c_4 - b_4c_3 & -b_2c_4 + c_2b_4 & b_2c_3 - c_2b_3 \\ -b_3c_4 + b_4c_3 & 0 & b_1c_4 - c_1b_4 & -b_1c_3 + c_1b_3 \\ b_2c_4 - c_2b_4 & -b_1c_4 + c_1b_4 & 0 & b_1c_2 - c_1b_2 \\ -b_2c_3 + c_2b_3 & b_1c_3 - c_1b_3 & -b_1c_2 + c_1b_2 & 0 \end{bmatrix}$$

is a structure matrix and, therefore, defines a Poisson bracket; b and c are the Casimir functions of such a Poisson bracket.

In order to appreciate the generality of the theory developed in this section, two considerations can be made. If, after a first choice of the straightening diffeomorphism, it turns out that $\omega(x) \neq 1$, then it might be that with a different choice of the first integrals $I_1, \ldots, I_{n-1}$ one can obtain $\omega(x) = 1$. In particular, note that the role of I_{n-1} (the Hamiltonian function) is not to be considered here different from the role of any of the Casimir functions $I_1, \ldots, I_{n-2}$. On the other hand, if $\omega(x) \neq 1$, one can consider a state-dependent time scaling such that the new time variable τ satisfies $\mathrm{d}\tau = \omega(x)\mathrm{d}t$, so that in the new time scale the system is described by

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = S_{I_1,\ldots,I_{n-2}} \nabla I_{n-1},$$

and it is therefore Hamiltonian. Apart from some possible equilibrium points, a state-dependent time scaling does not alter the orbits of the system, but only their time parameterization, as already discussed for orbital symmetries.

Chapter 6
Lie Algebras

6.1 Abstract Lie Algebras

Lie algebras, as well as some operations defined on them, can be defined in an abstract way. In this section, basic definitions and properties are recalled; the reader interested in a more deep description is referred, i.e., to [44, 65, 68, 70, 107, 109, 114].

Given a subset $\mathfrak{X}$ of a vector space $\mathfrak{Z}$ over a field $\mathscr{F}$ and an operation $[\cdot,\cdot]:\mathfrak{Z}\times\mathfrak{Z}\to\mathfrak{Z}$, which is *bilinear* (i.e., $[a_1 f_1 + a_2 f_2, g] = a_1[f_1, g] + a_2[f_2, g]$ and $[g, a_1 f_1 + a_2 f_2] = a_1[g, f_1] + a_2[g, f_2]$, $\forall a_1, a_2 \in \mathscr{F},\ f_1, f_2, g \in \mathfrak{Z}$), *skew-symmetric* (i.e., $[f, g] = -[g, f]$ and $[f, f] = 0$, $\forall f, g \in \mathfrak{Z}$) and satisfies the *Jacobi identity* (i.e., $[f,[g,h]] + [g,[h, f]] + [h,[f, g]] = 0$, $\forall f, g, h \in \mathfrak{Z}$), $\mathfrak{X}$ is a *Lie algebra* over $\mathscr{F}$ if $\mathfrak{X}$ is a vector space over $\mathscr{F}$ and $\mathfrak{X}$ is closed under $[\cdot,\cdot]$, $[\mathfrak{X},\mathfrak{X}] \subseteq \mathfrak{X}$, where symbol $[\mathfrak{X},\mathfrak{X}] \subseteq \mathfrak{X}$ means $[f, g] \in \mathfrak{X}$, $\forall f, g \in \mathfrak{X}$. Such an operation $[\cdot,\cdot]:\mathfrak{Z}\times\mathfrak{Z}\to\mathfrak{Z}$ is called the *Lie bracket*.

A *basis* (respectively, the *dimension*) of $\mathfrak{X}$ as a Lie algebra is a basis (respectively, the dimension) of $\mathfrak{X}$ as a vector space over $\mathscr{F}$: if $\{f_1,\ldots, f_r\}$ is a basis of a finite dimensional Lie algebra $\mathfrak{X}$, then one can write $\mathfrak{X} = \mathrm{span}_{\mathscr{F}}\{f_1,\ldots, f_r\}$, saying that $\mathfrak{X}$ is *spanned* by $\{f_1,\ldots, f_r\}$.

Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a vector subspace of $\mathfrak{X}$. Hence, $\mathfrak{Y}$ is a Lie algebra if $[\mathfrak{Y},\mathfrak{Y}] \subseteq \mathfrak{Y}$: $\mathfrak{Y}$ is called a *Lie sub-algebra* of $\mathfrak{X}$; $\mathfrak{Y}$ is called a *Lie ideal* of $\mathfrak{X}$ if $[\mathfrak{X},\mathfrak{Y}] \subseteq \mathfrak{Y}$. Clearly, a Lie ideal is also a Lie sub-algebra. Given $f_1,\ldots, f_p \in \mathfrak{X}$, denote by $\{f_1,\ldots, f_p\}_{\mathscr{F}}$ the smallest Lie sub-algebra of $\mathfrak{X}$ containing $f_1,\ldots, f_p$; $\{f_1,\ldots, f_p\}_{\mathscr{F}}$ is called the Lie algebra *generated* by $f_1,\ldots, f_p$ over $\mathscr{F}$. The Lie algebra $\mathfrak{X}$ generated by $f_1,\ldots, f_p$ over $\mathscr{F}$ can be computed by induction on integer i as follows: let $\mathfrak{X}_0 = \mathrm{span}_{\mathscr{F}}\{f_1,\ldots, f_p\}$; let $\{f_1,\ldots, f_{p_i}\}$ be a basis of $\mathfrak{X}_i$ as vector space over $\mathscr{F}$ and define

$$\begin{aligned}\mathfrak{X}_{i+1} := \mathrm{span}_{\mathscr{F}}\big\{&f_1,\ldots, f_{p_i}, [f_1, f_2],\ldots,[f_1, f_{p_i}],[f_2, f_3],\ldots,\\ &[f_2, f_{p_i}],\ldots,[f_{p_i-1}, f_{p_i}]\big\};\end{aligned}$$

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hence, $\mathfrak{X} = \lim_{i \to +\infty} \mathfrak{X}_i$. If $\mathfrak{Z}$ as vector space over $\mathscr{F}$ is finite dimensional, then $\mathfrak{X}$ can be computed in a finite number of steps, because there exists an integer i^* such that $\mathfrak{X}_{i^*+1} = \mathfrak{X}_{i^*}$, which implies $\mathfrak{X} = \mathfrak{X}_{i^*}$.

Since the Lie algebra $\mathfrak{X}$ is closed under the Lie bracket, $[\mathfrak{X}, \mathfrak{X}] \subseteq \mathfrak{X}$, if $\mathfrak{X}$ has finite dimension r and $\{f_1, \dots, f_r\}$ is one of its bases, then there exist a finite number of scalars $c_{i,j;\ell} \in \mathscr{F}$ such that $[f_i, f_j] = \sum_{k=\ell}^{r} c_{i,j;\ell} f_\ell$; the scalars $c_{i,j;\ell}$ are called the *structure scalars* and the rules $[f_i, f_j] = \sum_{k=\ell}^{r} c_{i,j;\ell} f_\ell$ are called *commutation relations*. Conversely, if $\mathfrak{X}$ is a vector space over $\mathscr{F}$ with a basis $\{f_1, \dots, f_r\}$ satisfying the commutation relations $[f_i, f_j] = \sum_{k=\ell}^{r} c_{i,j;\ell} f_\ell$ for some scalars $c_{i,j;\ell} \in \mathscr{F}$, then $\mathfrak{X}$ is a Lie algebra.

The following definition of isomorphism is one of the most important concepts related with Lie algebras, because it allows a classification of Lie algebras that is very useful.

Definition 6.1 Two Lie algebras $\mathfrak{X} = \mathrm{span}_{\mathscr{F}}\{f_1, \dots, f_r\}$ and $\mathfrak{Y} = \mathrm{span}_{\mathscr{F}}\{g_1, \dots, g_r\}$, having the same dimension r, are *isomorphic* if there exists a linear transformation $g_i = \sum_{j=1}^{r} Q_{i,j} \bar{g}_j$, where matrix Q with entries $Q_{i,j} \in \mathscr{F}$ satisfies $\det(Q) \neq 0$, such that

$$[f_i, f_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} f_\ell \quad \Longleftrightarrow \quad [\bar{g}_i, \bar{g}_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} \bar{g}_\ell,$$

for some structure scalars $c_{i,j;\ell} \in \mathscr{F}$.

Remark 6.1 Under the assumption that $\mathscr{F} = \mathbb{R}$, it is known (see [70]) that any Lie algebra of dimension $r \in \{1, 2, 3\}$ is isomorphic to one of the following Lie algebras.

(Case $r = 1$) The only Lie algebra of dimension one, $\mathfrak{X} = \mathrm{span}_{\mathscr{F}}\{f\}$, is described by

(6.1.1) $[f, f] = 0$.

(Case $r = 2$) There are only two non-isomorphic Lie algebras of dimension two, $\mathfrak{X} = \mathrm{span}_{\mathscr{F}}\{f_1, f_2\}$:

(6.1.2) $[f_1, f_2] = 0$;
(6.1.3) $[f_1, f_2] = f_1$.

(Case $r = 3$) There are five classes of non-isomorphic Lie algebras of dimension three, $\mathfrak{X} = \mathrm{span}_{\mathscr{F}}\{f_1, f_2, f_3\}$:

(6.1.4) $[f_i, f_j] = 0, \forall i, j \in \{1, 2, 3\}$;
(6.1.5) $[f_1, f_2] = f_3$, $[f_1, f_3] = 0$, $[f_2, f_3] = 0$ (the *Heisenberg* Lie algebra);
(6.1.6) $[f_1, f_2] = f_1$, $[f_1, f_3] = 0$, $[f_2, f_3] = 0$;
(6.1.7) $[f_1, f_2] = 0$, $[f_1, f_3] = A_{1,1} f_1 + A_{1,2} f_2$, $[f_2, f_3] = A_{2,1} f_1 + A_{2,2} f_2$, where matrix A having entries $A_{i,j}$ satisfies $\det(A) \neq 0$;

(6.1.8) $[f_1, f_2] = f_3$, $[f_1, f_3] = af_2$, $[f_2, f_3] = bf_1$, where $a, b \in \mathbb{R}$ are arbitrary constants, $ab \neq 0$ (when $a = 2$ and $b = -2$, one has the *split three-dimensional simple* Lie algebra, whereas when $a = -1$ and $b = 1$, one has the Lie algebra of *rotations* in $\mathbb{R}^3$).

For the proof of the above statements the reader is referred to [70]. Just as an example, the proof of Statement (6.1.3) is reported. Let $\{g_1, g_2\}$ be a basis of a two-dimensional Lie algebra, with the commutation relation $[g_1, g_2] = c_{1,2;1}g_1 + c_{1,2;2}g_2$, with one of the constants $c_{1,2;\ell} \neq 0$. Consider the transformation $f_1 = Q_{1,1}g_1 + Q_{1,2}g_2$, $f_2 = Q_{2,1}g_1 + Q_{2,2}g_2$, with inverse $g_1 = \frac{1}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}(Q_{2,2}f_1 - Q_{1,2}f_2)$, $g_2 = \frac{1}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}(-Q_{2,1}f_2 + Q_{1,1}f_2)$. Hence,

$$\begin{aligned}
[f_1, f_2] &= (Q_{1,1}Q_{2,2} - Q_{1,2}Q_{2,1})[g_1, g_2] \\
&= c_{1,2;1}(Q_{2,2}f_1 - Q_{1,2}f_2) + c_{1,2;2}(-Q_{2,1}f_2 + Q_{1,1}f_2) \\
&= (c_{1,2;1}Q_{2,2} - c_{1,2;2}Q_{2,1})f_1 + (-c_{1,2;1}Q_{1,2} + c_{1,2;2}Q_{1,1})f_2 \\
&= \tilde{c}_{1,2;1}f_1 + \tilde{c}_{1,2;2}f_2,
\end{aligned}$$

which yields the transformation law for the structure constants

$$\begin{bmatrix} \tilde{c}_{1,2;1} \\ \tilde{c}_{1,2;2} \end{bmatrix} = \begin{bmatrix} Q_{2,2} & -Q_{2,1} \\ -Q_{1,2} & Q_{1,1} \end{bmatrix} \begin{bmatrix} c_{1,2;1} \\ c_{1,2;2} \end{bmatrix}. \tag{6.1}$$

To obtain the relation $[f_1, f_2] = f_1$, let $\tilde{c}_{1,2;1} = 1$ and $\tilde{c}_{1,2;2} = 0$; then, solve the resulting equation (6.1) in the unknowns $Q_{i,j}$. If $c_{1,2;1} \neq 0$, one can choose $Q_{1,2} = Q_{1,1}\frac{c_{1,2;2}}{c_{1,2;1}}$, $Q_{2,2} = \frac{Q_{2,1}c_{1,2;2}+1}{c_{1,2;1}}$, for arbitrary $Q_{2,1}, Q_{1,1} \in \mathscr{F}$, whereas if $c_{1,2;2} \neq 0$, one can choose $Q_{2,1} = \frac{Q_{2,2}c_{1,2;1}-1}{c_{1,2;2}}$, $Q_{1,1} = Q_{1,2}\frac{c_{1,2;1}}{c_{1,2;2}}$, for arbitrary $Q_{1,2}, Q_{2,2} \in \mathscr{F}$.

Assume that $\mathfrak{X}$ is a finite dimensional Lie algebra. A sequence of Lie algebras $\mathfrak{X}_0, \mathfrak{X}_1, \ldots$ can be recursively defined by $\mathfrak{X}_0 = \mathfrak{X}$ and $\mathfrak{X}_{i+1} = [\mathfrak{X}, \mathfrak{X}_i]$ ($\mathfrak{X}_i$ is a Lie ideal of $\mathfrak{X}$); since $\mathfrak{X}_{i+1} \subseteq \mathfrak{X}_i$ and $\mathfrak{X}_0$ is finite dimensional, such a sequence terminates, i.e., there exists an integer i^* such that $\mathfrak{X}_{i^*+1} = \mathfrak{X}_{i^*}$; if $\mathfrak{X}_{i^*} = \emptyset$, then $\mathfrak{X}$ is said to be *nilpotent* (if $i^* = 1$, then $\mathfrak{X}$ is said to be *Abelian*). Another sequence of Lie algebras $\mathfrak{X}^0, \mathfrak{X}^1, \ldots$ can be similarly defined by $\mathfrak{X}^0 = \mathfrak{X}$ and $\mathfrak{X}^{i+1} = [\mathfrak{X}^i, \mathfrak{X}^i]$; since $\mathfrak{X}^{i+1} \subseteq \mathfrak{X}^i$ and $\mathfrak{X}^0$ is finite dimensional, such a sequence terminates, i.e., there exists an integer i^* such that $\mathfrak{X}^{i^*+1} = \mathfrak{X}^{i^*}$; if $\mathfrak{X}^{i^*} = \emptyset$, then $\mathfrak{X}$ is said to be *solvable*. If $\mathfrak{X}$ is nilpotent, then $\mathfrak{X}$ is solvable. Clearly, $\mathfrak{X}_1 = \mathfrak{X}^1 =: \mathfrak{X}'$; $\mathfrak{X}'$ is called the *derived Lie algebra*; $\mathfrak{X}$ is solvable if and only if $\mathfrak{X}'$ is nilpotent [70].

Remark 6.2 Lie algebras (6.1.1), (6.1.2) and (6.1.4) are Abelian, whence both nilpotent and solvable; the Lie algebra (6.1.5) is nilpotent, whence solvable, but not Abelian; Lie algebras (6.1.3), (6.1.6) and (6.1.7) are solvable, but not nilpotent neither Abelian; Lie algebras (6.1.8) are not solvable, whence neither Abelian nor nilpotent.

6.2 Lie Algebras of Matrices

The notion of linear symmetry given in Definition 2.1 at p. 31 has been generalized to the notion of linear orbital symmetry given in Definition 2.5 at p. 41, which can be further generalized to the notion of Lie algebra of matrices [44] over $\mathbb{R}$.

Definition 6.2 Let $M_1, \ldots, M_r \in \mathbb{R}^{n\times n}$ be r matrices being linearly independent over $\mathbb{R}$. If there exist some *structure constants* $c_{i,j;\ell} \in \mathbb{R}$ such that

$$[M_i, M_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} M_\ell,$$

then $\mathfrak{M} = \operatorname{span}_{\mathbb{R}}\{M_1, \ldots, M_r\}$ is a *matrix Lie algebra* over $\mathbb{R}$ of dimension r.

Clearly, for any vector subspace $\mathfrak{M}$ of $\mathbb{R}^{n\times n}$, $\mathfrak{M}$ is a Lie algebra of matrices if and only if $[A, B] \in \mathfrak{M}$ for all $A, B \in \mathfrak{M}$.

Remark 6.3 Since $[B_1, B_2] \in \mathbb{R}^{n\times n}$ for all $B_1, B_2 \in \mathbb{R}^{n\times n}$, one concludes that $\mathbb{R}^{n\times n}$ endowed with the Lie bracket $[\cdot,\cdot]$ is a Lie algebra over $\mathbb{R}$; a basis of $\mathbb{R}^{n\times n}$ is $\{e_1 e_1^\top, \ldots, e_1 e_n^\top, \ldots, e_n e_1^\top, \ldots, e_n e_n^\top\}$, where e_i is the ith column of the $n \times n$ identity matrix E. For any matrix $A \in \mathbb{R}^{n\times n}$, since $[B_1, B_2] \in \mathscr{L}_c(A)$ for all $B_1, B_2 \in \mathscr{L}_n(A)$ (whence, also for all $B_1, B_2 \in \mathscr{L}_c(A)$) and $\mathscr{L}_c(A) \subseteq \mathscr{L}_n(A)$, then $\mathscr{L}_n(A)$ is a Lie sub-algebra of $\mathbb{R}^{n\times n}$ and $\mathscr{L}_c(A)$ is a Lie ideal of $\mathscr{L}_n(A)$.

Remark 6.4 Some Lie algebras of matrices $A \in \mathbb{R}^{n\times n}$, with entries $A_{i,j}$, are listed in the following:

(6.4.1) the set $\mathfrak{M}$ of all diagonal A, i.e., $A_{i,j} = 0$ if $i \neq j$: a basis of $\mathfrak{M}$ is given by $e_i e_i^\top$, for $i \in \{1, \ldots, n\}$;
(6.4.2) the set $\mathfrak{M}$ of all skew-symmetric A, i.e., $A^\top + A = 0$: a basis of $\mathfrak{M}$ is given by $e_i e_j^\top - e_j e_i^\top$, for $i, j \in \{1, \ldots, n\}$, $i < j$;
(6.4.3) the set $\mathfrak{M}$ of all upper (respectively, lower) triangular A, i.e., $A_{i,j} = 0$ if $i < j$ (respectively, if $i > j$): a basis of $\mathfrak{M}$ is given by $e_i e_j^\top$, for $i, j \in \{1, \ldots, n\}$, $i \geq j$ (respectively, $i \leq j$);
(6.4.4) the set $\mathfrak{M}$ of all strictly upper (respectively, lower) triangular A, i.e., $A_{i,j} = 0$ if $i \leq j$ (respectively, if $i \geq j$): a basis of $\mathfrak{M}$ is given by $e_i e_j^\top$, for $i, j \in \{1, \ldots, n\}$, $i > j$ (respectively, $i < j$);
(6.4.5) the set of all A having zero trace, i.e., $\operatorname{trace}(A) = \sum_{i=1}^{n} A_{i,i} = 0$;
(6.4.6) given $B \in \mathbb{R}^{m\times n}$ (respectively, $B \in \mathbb{R}^{n\times m}$), the set of all A such that $BA = 0$ (respectively, $AB = 0$).

It is worth pointing out that the matrix Lie algebra (6.4.4) is nilpotent, whence solvable, whereas the matrix Lie algebra (6.4.3) is solvable, but not necessarily nilpotent. Since any skew-symmetric matrix has zero trace, the matrix Lie algebra (6.4.2) is a Lie sub-algebra of (6.4.5).

Theorem 6.1 *Let $\mathfrak{M}$ be a matrix Lie algebra (possibly coincident with $\mathbb{R}^{n\times n}$) and let $t \in \mathbb{R}$. Then, $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt} \in \mathfrak{M}$ for all $A, B \in \mathfrak{M}$ and $t \in \mathbb{R}$. In particular, $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt} = A$ for all $B \in \mathfrak{M}$ if and only if $[\mathfrak{M}, \{A\}] = \{0\}$.*

Proof Taking into account that $\frac{\mathrm{d}\mathrm{e}^{Bt}}{\mathrm{d}t} = B\mathrm{e}^{Bt} = \mathrm{e}^{Bt}B$, one can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-Bt}A\mathrm{e}^{Bt}\right) &= -\mathrm{e}^{-Bt}BA\mathrm{e}^{Bt} + \mathrm{e}^{-Bt}AB\mathrm{e}^{Bt}\\ &= \mathrm{e}^{-Bt}[B, A]\mathrm{e}^{Bt},\end{aligned}$$

which, for any $A \in \mathfrak{M}$, shows that taking the derivative of $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt}$ with respect to t is equivalent to substituting matrix A with the Lie bracket $[B, A]$, and therefore by induction on integer $h \geq 1$ that

$$\frac{\mathrm{d}^h}{\mathrm{d}t^h}\left(\mathrm{e}^{-Bt}A\mathrm{e}^{Bt}\right) = \mathrm{e}^{-Bt}\underbrace{\big[B, \ldots \big[B, [B}_{h \text{ times}}, A]\big]\ldots\big]\mathrm{e}^{Bt}.$$

Hence, one obtains the following formula known as the *Hadamard Lemma*:

$$\mathrm{e}^{-Bt}A\mathrm{e}^{Bt} = A + t[B, A] + \frac{t^2}{2!}\big[B, [B, A]\big] + \frac{t^3}{3!}\big[B, \big[B, [B, A]\big]\big] + \cdots. \tag{6.2}$$

Now, since the fact that $\mathfrak{M}$ is a Lie algebra implies $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}$, if $A, B \in \mathfrak{M}$, then A, $[B, A]$, $[B, [B, A]]$, $[B, [B, [B, A]]] \in \mathfrak{M}$, and so on; hence, the Hadamard Lemma implies that $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt} \in \mathfrak{M}$. The last statement is trivial since (6.2) implies that $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt} = A$, $\forall t \in \mathbb{R}$, if and only if $[B, A] = 0$. □

Example 6.1 Consider the set $\mathfrak{M}$ of matrices $A \in \mathbb{R}^{2\times 2}$, with zero trace, i.e., $A = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$, with $a, b, c \in \mathbb{R}$ being arbitrary; since the trace of a matrix is a linear operation, such a set is a vector space over $\mathbb{R}$; to be more precise, if A_1, A_2 have zero trace, then $\alpha_1 A_1 + \alpha_2 A_2$ has zero trace for all $\alpha_1, \alpha_2 \in \mathbb{R}$. Clearly, a basis of $\mathfrak{M}$ is

$$\left\{M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

Since

$$[M_1, M_2] = -2M_2, \qquad [M_1, M_3] = 2M_3, \qquad [M_2, M_3] = -M_1,$$

$\mathfrak{M}$ is a Lie algebra over $\mathbb{R}$. As an example, for

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \mathrm{e}^{Bt} = \begin{bmatrix} \mathrm{e}^{2t} & \frac{1}{4}\mathrm{e}^{2t} - \frac{1}{4}\mathrm{e}^{-2t} \\ 0 & \mathrm{e}^{-2t} \end{bmatrix},$$

it is easy to see that

$$\mathrm{e}^{-Bt} A \mathrm{e}^{Bt} = \left(a + \frac{1}{4}c - \frac{1}{4}c\mathrm{e}^{4t}\right) M_1 \\ + \left(\frac{1}{2}a + \frac{1}{8}c - \frac{1}{16}c\mathrm{e}^{4t} + \left(b - \frac{1}{2}a - \frac{1}{16}c\right)\mathrm{e}^{-4t}\right) M_2 + \left(c\mathrm{e}^{4t}\right) M_3,$$

namely that $\mathrm{e}^{-Bt} A \mathrm{e}^{Bt} \in \mathfrak{M}$, for all $a, b, c \in \mathbb{R}$.

6.3 Lie Algebras of Vector Functions

Let $\mathfrak{Z}$ be the set of all $f(x) \in \mathbb{R}^n$ with entries in $\mathscr{K}_n$. Let $f_1, f_2 \in \mathfrak{Z}$. Since $\alpha_1 f_1 + \alpha_2 f_2 \in \mathfrak{Z}$ for all real constants $\alpha_1, \alpha_2 \in \mathbb{R}$, $\mathfrak{Z}$ has the structure of vector space over $\mathbb{R}$, which is infinite dimensional. Since $\alpha_1 f_1 + \alpha_2 f_2 \in \mathfrak{Z}$ for all $\alpha_1, \alpha_2 \in \mathscr{K}_n$, $\mathfrak{Z}$ has also the structure of vector space over $\mathscr{K}_n$, which has finite dimension n. To be more precise, let $f_1, \ldots, f_n \in \mathfrak{Z}$ be n vector functions such that $\det([f_1 \ \ldots \ f_n]) \neq 0$; then, any $f \in \mathfrak{Z}$ can be expressed as $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$, where functions α_i are meromorphic. If a vector space $\mathfrak{X} \subseteq \mathfrak{Z}$ over $\mathscr{K}_n$ (respectively, $\mathbb{R}$), possibly coincident with $\mathfrak{Z}$, is closed under the Lie bracket, $[\mathfrak{X}, \mathfrak{X}] \subseteq \mathfrak{X}$, then there exist *structure functions* $c_{i,j;\ell} \in \mathscr{K}_n$ (respectively, *structure constants* $c_{i,j;\ell} \in \mathbb{R}$), such that $[f_i, f_j] = \sum_{\ell=1}^{n} c_{i,j;\ell} f_k$; under the above assumption, $\mathfrak{X}$ endowed with $[\cdot, \cdot]$ is a *Lie algebra* of meromorphic vector functions over $\mathscr{K}_n$ (respectively, $\mathbb{R}$). Clearly, $\mathfrak{Z}$ is a Lie algebra over $\mathscr{K}_n$ of dimension n and it is an infinite dimensional Lie algebra over $\mathbb{R}$.

Definition 6.3 A point $x^o \in \mathbb{R}^n$ is *regular* for a Lie algebra $\mathfrak{X} \subseteq \mathfrak{Z}$ of vector functions over either $\mathbb{R}$ or $\mathscr{K}_n$ if there exists a basis $\{f_1, \ldots, f_r\}$ of $\mathfrak{X}$ such that $[f_1(x) \ \ldots \ f_r(x)]$ has constant rank over $\mathbb{R}$ for all $x \in \mathscr{B}_{x^o}$, where $\mathscr{B}_{x^o}$ is a neighborhood of x^o.

A distribution $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{f_1, \ldots, f_r\}$, with $[f_1 \ \ldots \ f_r]$ having full generic rank r, is a Lie sub-algebra of $\mathfrak{Z}$ over $\mathscr{K}_n$ if and only if it is involutive.

By the Hadamard Lemma (3.69), it is possible to show that

$$\left(\frac{\partial \Phi_g}{\partial y}\right)^{-1} f \circ \Phi_g \in \mathfrak{X}, \quad \forall f, g \in \mathfrak{X},$$

if and only if $\mathfrak{X} \subseteq \mathfrak{Z}$ is closed under the Lie bracket, $[\mathfrak{X}, \mathfrak{X}] \subseteq \mathfrak{X}$. In particular, if $f, g \in \mathfrak{X}$, then $f, [g, f], [g, [g, f]] \in \mathfrak{X}$ and so on, which yields $(\frac{\partial \Phi_g}{\partial y})^{-1} f \circ \Phi_g \in \mathfrak{X}$.

Remark 6.5 Given a vector function $g(x) \in \mathbb{R}^n$, let $J_0, J_1, \ldots, J_{n-1}$ be functionally independent functions such that $L_g J_0 = 1$ and $L_g J_i = 0$, $i = 1, \ldots, n-1$; let $J = [J_0 \ J_1 \ \ldots \ J_{n-1}]^\top$. By statement (3.9.1) of Theorem 3.9 at p. 64, the centralizer

$\mathscr{C}_C(g)$ is spanned by the columns $f_1, \dots, f_n$ of $(\frac{\partial J}{\partial x})^{-1}$ (which are pairwise commuting, $[f_i, f_j] = 0$), with coefficients being arbitrary meromorphic functions of $J_1, \dots, J_{n-1}$, $\mathscr{C}_C(g) = \text{span}_{\mathscr{I}_c(g)}\{f_1, \dots, f_n\}$, which is a Lie algebra over the field $\mathscr{I}_c(g)$ of the meromorphic functions of $J_1, \dots, J_{n-1}$.

Remark 6.6 If the symmetry g mentioned in Remark 6.5 is linear with positive integer eigenvalues, $g(x) = Bx$, $B = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i \in \mathbb{Z}$, the subset $\bar{\mathscr{C}}_C(g)$ of $\mathscr{C}_C(g)$, which is constituted by all the vector functions that are analytic at $x = 0$, is a finite dimensional Lie algebra over $\mathbb{R}$. For instance, if $g(x) = [x_1 \; mx_2]^\top$, $m \in \mathbb{Z}$, $m \geq 2$, then a basis of $\bar{\mathscr{C}}_C(g)$ is

$$f_1(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \qquad f_3(x) = \begin{bmatrix} 0 \\ x_1^m \end{bmatrix},$$

with the commutation relations $[f_1, f_2] = 0$, $[f_1, f_3] = mf_3$, $[f_2, f_3] = -f_3$.

Example 6.2 Consider again the Lie algebra $\mathfrak{M}$ of matrices with zero trace, considered in Example 6.1. The vector functions associated with the considered basis of $\mathfrak{M}$ are

$$f_1(x) = M_1 x = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}, \qquad f_2(x) = M_2 x = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \qquad f_3(x) = M_3 x = \begin{bmatrix} 0 \\ x_1 \end{bmatrix},$$

and satisfy the commutation relations

$$[f_1, f_2] = -2f_2, \qquad [f_1, f_3] = 2f_3, \qquad [f_2, f_3] = -f_1.$$

The set $\mathfrak{X}$ of all linear combinations of f_1, f_2 and f_3 over $\mathbb{R}$ is a Lie algebra of dimension three. Let $\mathscr{F}$ be the field of all rational functions of $I(x) = \frac{x_1}{x_2}$ and note that

$$L_{f_1} I(x) = 2\frac{x_1}{x_2} = 2I(x), \qquad L_{f_2} I(x) = 1, \qquad L_{f_3} I(x) = -\frac{x_1^2}{x_2^2} = -I^2(x),$$

namely that $L_{f_i} I \in \mathscr{F}$ for any $I \in \mathscr{F}$. Consider the set $\mathfrak{Y}$ generated by taking linear combinations of f_1 and f_2 (but a similar result can be obtained by replacing f_1 or f_2 with f_3) over $\mathscr{F}$. It is easy to check that

$$f_3(x) = -\frac{x_1}{x_2} f_1(x) + \frac{x_1^2}{x_2^2} f_2(x) \in \mathfrak{Y}.$$

Let $g = \sum_{i=1}^{2} \alpha_i f_i$, with $\alpha_i \in \mathscr{F}$; then, taking into account that $[\sum_{i=1}^{2} \alpha_i f_i, f_j] = \sum_{i=0}^{1} (\alpha_i [f_i, f_j] + (L_{f_j} \alpha_i) f_i) \in \mathfrak{Y}$, one concludes that $\mathfrak{Y}$ is a Lie algebra over $\mathscr{F}$ of dimension two. In particular, one has that $\mathfrak{X} \subset \mathfrak{Y}$ as a set, but $\mathfrak{X}$ is not a sub-algebra of $\mathfrak{Y}$ because $\mathfrak{X}$ and $\mathfrak{Y}$ are not algebras over the same field.

Remark 6.7 Apart from a diffeomorphism about a regular point, any two-dimensional Lie algebra of vector functions $f(x) \in \mathbb{R}^2$ over $\mathbb{R}$ is isomorphic to one of the following Lie algebras spanned over $\mathbb{R}$ by the following pairs (see [68]):

(6.7.1) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $[f_1, f_2] = 0$, $\det([f_1\ f_2]) \neq 0$;

(6.7.2) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $[f_1, f_2] = 0$, $\det([f_1\ f_2]) = 0$;

(6.7.3) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = f_1$, $\det([f_1\ f_2]) \neq 0$;

(6.7.4) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = f_1$, $\det([f_1\ f_2]) = 0$.

Remark 6.8 Apart from a diffeomorphism about a regular point, any three-dimensional Lie algebra of vector functions $f(x) \in \mathbb{R}^2$ over $\mathbb{R}$ is isomorphic to one of the following Lie algebras spanned over $\mathbb{R}$ by the following triplets (see [68]):

(6.8.1) $f_1(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$, $[f_1, f_2] = f_1$, $[f_1, f_3] = 2f_2$, $[f_2, f_3] = f_3$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.2) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1^2 \\ x_1x_2 \end{bmatrix}$, $[f_1, f_2] = 2f_1$, $[f_1, f_3] = f_2$, $[f_2, f_3] = 2f_3$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.3) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$, $[f_1, f_2] = f_1$, $[f_1, f_3] = 2f_2$, $[f_2, f_3] = f_3$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 1$;

(6.8.4) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1 \\ ax_2 \end{bmatrix}$, $a \notin \{0, 1\}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = af_2$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.5) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} (1-a)x_1 \\ x_2 \end{bmatrix}$, $a \notin \{0,\ 1\}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = af_2$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.6) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = f_2$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.7) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = f_2$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 1$;

(6.8.8) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = f_1 + f_2$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.9) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = f_2 - f_1$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.10) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = 0$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.11) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_1$, $[f_2, f_3] = 0$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.12) $f_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $[f_1, f_2] = 0$, $[f_1, f_3] = f_2$, $[f_2, f_3] = 0$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 2$;

(6.8.13) $f_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$, $f_3(x) = \begin{bmatrix} 0 \\ p(x_1) \end{bmatrix}$, $p(x_1) \neq 0$, $[f_1, f_2] = 0$, $[f_1, f_3] = 0$, $[f_2, f_3] = 0$, $\text{rank}_{\mathscr{K}_n}([f_1\ f_2\ f_3]) = 1$,

where a is any real number and $p(x_1)$ is any meromorphic function.

6.4 Representation of Lie Algebras by Vector Functions

By the Ado Theorem [114, Sect. 3.17], it is known that any abstract finite dimensional Lie algebra $\mathfrak{X}$ over $\mathbb{R}$ can be represented by a matrix Lie algebra. Any finite dimensional Lie algebra admits various matrix representations: one of these is the *adjoint matrix representation*. For each $f \in \mathfrak{X}$, denote by $\mathrm{ad}_f(\cdot) : \mathfrak{X} \to \mathfrak{X}$ the linear mapping $h \to [f, h]$, $\forall h \in \mathfrak{X}$, which is called the *adjoint representation* of f. The mapping $\mathrm{ad}_f(\cdot)$ is usually represented [70] by a matrix M, which has as entries of the ith row the coordinates of $\mathrm{ad}_f(f_i)$ with respect to the basis $\{f_1, \dots, f_r\}$. Let $\{f_1, \dots, f_r\}$ be a basis of $\mathfrak{X}$, such that $[f_i, f_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} f_\ell$; let M_i be the matrix representing the linear mapping $\mathrm{ad}_{f_i}(\cdot)$. Then, by [70], it is known that $\{M_1, \dots, M_r\}$ spans a matrix Lie algebra such that $[M_i, M_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} M_\ell$, for the same structure constants $c_{i,j;\ell}$: it is worth pointing out that the dimension of $\mathrm{span}_{\mathbb{R}}\{M_1, \dots, M_r\}$ may be less than r, because matrices $M_1, \dots, M_r$ can be linearly dependent. For proving the statement above, it is sufficient to show that

$$\mathrm{ad}_{[f,g]}(h) = \big[\mathrm{ad}_f(h), \mathrm{ad}_g(h)\big], \quad \forall f, g, h \in \mathfrak{X}, \tag{6.3}$$

as in the following relations (see (1.3)):

$$\begin{aligned}\mathrm{ad}_{[f,g]}(h) &= \big[[f, g], h\big] = \big[f, [g, h]\big] - \big[g, [f, h]\big] \\ &= \mathrm{ad}_f\big([g, h]\big) - \mathrm{ad}_g\big([f, h]\big) = \mathrm{ad}_f\big(\mathrm{ad}_g(h)\big) - \mathrm{ad}_g\big(\mathrm{ad}_f(h)\big) \\ &= \big[\mathrm{ad}_f(h), \mathrm{ad}_g(h)\big].\end{aligned}$$

As a matter of fact, using relation (6.3), one has

$$\begin{aligned}\big[\mathrm{ad}_{f_i}(h), \mathrm{ad}_{f_j}(h)\big] &= \mathrm{ad}_{[f_i, f_j]}(h) = \mathrm{ad}_{\sum_{\ell=1}^{r} c_{i,j;\ell} f_\ell}(h) \\ &= \sum_{\ell=1}^{r} c_{i,j;\ell}\, \mathrm{ad}_{f_\ell}(h),\end{aligned}$$

which proves the assertion thanks to the arbitrariness of $h \in \mathfrak{X}$. Therefore, $\{\hat{f}_1(x) = M_1 x, \dots, \hat{f}_r(x) = M_r x\}$ is a representation of the given Lie algebra with linear vector functions. Since x is a linear symmetry of any of the above $f_i(x)$, $\mathfrak{X}$ can be represented by nonlinear vector functions of dimension $n - 1$, which are obtained by the projection $y_1 = \frac{x_1}{x_n}, \dots, y_{n-1} = \frac{x_{n-1}}{x_n}$ (according to Sect. 3.5, the y_i's are first integrals of the symmetry x).

Example 6.3 Consider the split three-dimensional simple Lie algebra $\mathfrak{X}$ defined by [70]

$$[f_1, f_2] = 2 f_1, \qquad [f_1, f_3] = f_2, \qquad [f_2, f_3] = 2 f_3.$$

The adjoint map $\mathrm{ad}_{f_1}(\cdot)$ is defined by $\mathrm{ad}_{f_1}(f_1) = [f_1, f_1] = 0$, $\mathrm{ad}_{f_1}(f_2) = [f_1, f_2] = 2 f_1$ and $\mathrm{ad}_{f_1}(f_3) = [f_1, f_3] = f_2$; the matrix M_1 representing $\mathrm{ad}_{f_1}(\cdot)$

is

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The adjoint map $\mathrm{ad}_{f_2}(\cdot)$ is defined by $\mathrm{ad}_{f_2}(f_1) = [f_2, f_1] = -2f_1$, $\mathrm{ad}_{f_2}(f_2) = [f_2, f_2] = 0$ and $\mathrm{ad}_{f_2}(f_3) = [f_2, f_3] = 2f_3$; the matrix M_2 representing $\mathrm{ad}_{f_2}(\cdot)$ is

$$M_2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Finally, the adjoint map $\mathrm{ad}_{f_3}(\cdot)$ is defined by $\mathrm{ad}_{f_3}(f_1) = [f_3, f_1] = -f_2$, $\mathrm{ad}_{f_3}(f_2) = [f_3, f_2] = -2f_3$ and $\mathrm{ad}_{f_3}(f_3) = [f_3, f_3] = 0$; the matrix M_3 representing $\mathrm{ad}_{f_3}(\cdot)$ is

$$M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $[M_1, M_2] = 2M_1$, $[M_1, M_3] = M_2$ and $[M_2, M_3] = 2M_3$, as expected. Therefore, $\mathfrak{X}$ is represented by the Lie algebra of vector functions spanned over $\mathbb{R}$ by the three-dimensional linear vector functions

$$\hat{f}_1(x) = \begin{bmatrix} 0 \\ 2x_1 \\ x_2 \end{bmatrix}, \qquad \hat{f}_2(x) = \begin{bmatrix} -2x_1 \\ 0 \\ 2x_3 \end{bmatrix}, \qquad \hat{f}_3(x) = \begin{bmatrix} -x_2 \\ -2x_3 \\ 0 \end{bmatrix}.$$

By the projection $y_1 = \frac{x_1}{x_3}$, $y_2 = \frac{x_2}{x_3}$, from the triplet $\hat{f}_1, \hat{f}_2, \hat{f}_3$, one obtains a representation of $\mathfrak{X}$ by the Lie algebra of vector functions over $\mathbb{R}$ spanned by the two-dimensional nonlinear vector functions:

$$\tilde{f}_1(y) = \begin{bmatrix} -y_1 y_2 \\ 2y_1 - y_2^2 \end{bmatrix}, \qquad \tilde{f}_2(y) = \begin{bmatrix} -4y_1 \\ -2y_2 \end{bmatrix}, \qquad \tilde{f}_3 = \begin{bmatrix} -y_2 \\ -2 \end{bmatrix}.$$

It is easy to check that $[\tilde{f}_1, \tilde{f}_2] = 2\tilde{f}_1$, $[\tilde{f}_1, \tilde{f}_3] = \tilde{f}_2$ and $[\tilde{f}_2, \tilde{f}_3] = 2\tilde{f}_3$, as expected.

Example 6.4 Consider the Lie algebra $\mathfrak{X}$ given in Statement (6.1.7) of Remark 6.1: $[f_1, f_2] = 0$, $[f_1, f_3] = A_{1,1}f_1 + A_{1,2}f_2$, $[f_2, f_3] = A_{2,1}f_1 + A_{2,2}f_2$, where matrix A having entries $A_{i,j}$ satisfies $\det(A) \neq 0$. A matrix representation of $\mathfrak{X}$ is given by

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{1,1} & A_{1,2} & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} -A_{1,1} & -A_{1,2} & 0 \\ -A_{2,1} & -A_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A representation of $\mathfrak{X}$ by nonlinear vector functions is

$$f_1(x) = \begin{bmatrix} -A_{1,1}x_1^2 - A_{1,2}x_1x_2 \\ -A_{1,1}x_1x_2 - A_{1,2}x_2^2 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} -A_{2,1}x_1^2 - A_{2,2}x_1x_2 \\ -A_{2,1}x_1x_2 - A_{2,2}x_2^2 \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} -A_{1,1}x_1 - A_{1,2}x_2 \\ -A_{2,1}x_1 - A_{2,2}x_2 \end{bmatrix}.$$

6.5 Nonlinear Superposition

After the works of S. Lie [81], the concept of nonlinear superposition principle indicates a pair of formulas (the explicit and implicit nonlinear superposition formulas) that allow to express the general solution of a system of ordinary differential equations in terms of a finite number of particular solutions and of a certain number of arbitrary constants. Systems of linear time-invariant differential equations are a remarkable case, in which the explicit superposition formula allows to express the general solution as a linear combination of n particular solutions, with n arbitrary constants, where n is the dimension of the state. Other important classes of systems admitting a nonlinear superposition principle are the bilinear ones [44] and the linear switched systems [84]. The knowledge of an explicit nonlinear superposition formula is important not only for the possibility of computing any solution of the considered system, but also for the possibility of deducing some properties of the general solution (such as stability and attractivity), on the basis of the properties of some particular solutions. For some classes of systems, the computation of the nonlinear superposition formulas has been achieved in closed form [2, 110].

Consider the class of time-varying nonlinear systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\big(t, x(t)\big), \tag{6.4}$$

where $x(t), f(t,x) \in \mathbb{R}^n$; it is assumed that a unique solution of (6.4) exists for an open set of initial conditions and small times t. A special subclass of systems belonging to class (6.4) is constituted by the linear ones:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t), \tag{6.5}$$

where $A(t) \in \mathbb{R}^{n\times n}$. Class (6.5) is very important because of the *linear superposition principle*; given n solutions $\xi^i(t) \in \mathbb{R}^n$, $i = 1, \dots, n$, of (6.5),

$$\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t} = A(t)\xi^i(t), \quad i = 1, \dots, n, \tag{6.6}$$

such that $\det([\xi^1(t_0) \ \dots \ \xi^n(t_0)]) \neq 0$, for some initial time t_0, the linear superposition principle allows to express any solution $x(t) \in \mathbb{R}^n$ of (6.5) as a linear combination of $\xi^1(t), \dots, \xi^n(t)$,

$$x(t) = k_1\xi^1(t) + \cdots + k_n\xi^n(t), \tag{6.7}$$

where the constant vector $k = [k_1 \ \dots \ k_n]^\top \in \mathbb{R}^n$ is given by

$$k = \left[\xi^1(t) \ \dots \ \xi^n(t)\right]^{-1} x(t), \tag{6.8}$$

and the inverse $[\xi^1(t) \ \dots \ \xi^n(t)]^{-1}$ exists for all t in a sufficiently small open interval $\mathscr{T}_{t_0}$ containing the initial time t_0. Equation (6.7) is the *explicit linear superposition formula* and (6.8) is the *implicit linear superposition formula* for systems (6.5). It is worth pointing out that each entry of the vector on the right-hand side of (6.8) is a first integral of the *extended system* constituted by system (6.5) and its replicas (6.6), i.e.,

$$\frac{\partial k}{\partial x} A(t)x + \sum_{i=1}^{n} \frac{\partial k}{\partial \xi^i} A(t)\xi^i = 0, \quad \forall t \in \mathscr{T}_{t_0}.$$

Example 6.5 Consider a linear oscillator with time-varying frequency,

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = -\omega(t)x_1, \end{cases} \tag{6.9}$$

where $\omega(t)$ is the time-varying oscillation frequency. Consider two replicas of the oscillator,

$$\begin{cases} \frac{\mathrm{d}\xi_1^1}{\mathrm{d}t} = \xi_2^1, \\ \frac{\mathrm{d}\xi_2^1}{\mathrm{d}t} = -\omega(t)\xi_1^1, \end{cases} \qquad \begin{cases} \frac{\mathrm{d}\xi_1^2}{\mathrm{d}t} = \xi_2^2, \\ \frac{\mathrm{d}\xi_2^2}{\mathrm{d}t} = -\omega(t)\xi_1^2. \end{cases} \tag{6.10}$$

The explicit and implicit superposition formulas are, respectively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \end{bmatrix} + k_2 \begin{bmatrix} \xi_1^2 \\ \xi_2^2 \end{bmatrix}, \qquad \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \frac{\xi_2^2 x_1 - \xi_1^2 x_2}{\xi_1^1 \xi_2^2 - \xi_1^2 \xi_2^1} \\ \frac{\xi_1^1 x_2 - \xi_2^1 x_1}{\xi_1^1 \xi_2^2 - \xi_1^2 \xi_2^1} \end{bmatrix}.$$

In this section, only time-varying nonlinear systems (6.4) that are sufficiently "close" to class (6.5) are considered: in particular, only those nonlinear systems that admit superposition formulas similar to (6.7) and (6.8).

Consider m particular solutions of (6.4), i.e., m functions $\xi^i(t) \in \mathbb{R}^n$, $i = 1, \dots, m$, such that

$$\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t} = f\left(t, \xi^i(t)\right), \quad i = 1, \dots, m; \tag{6.11}$$

conditions on integer m and functions $\xi^i(t)$ are given in the following.

Following S. Lie (see [111]), equation (6.4) admits a *nonlinear superposition principle* if there exists a function $\Psi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that any solution $x(t)$ of (6.4) can be written for all t in a sufficiently small open interval $\mathscr{T}_{t_0}$ containing the initial time t_0 as

$$x(t) = \Psi\left(\xi^1(t), \dots, \xi^m(t), k\right), \tag{6.12}$$

where $k \in \mathbb{R}^n$ is constant; in particular, it is required that (6.12) computed at $t = t_0$ is locally invertible with respect to k, so that k can be expressed as a function of $x(t_0), \xi^1(t_0), \dots, \xi^m(t_0)$. It is worth pointing out that function Ψ does not depend explicitly on time t. Equation (6.12) is called an *explicit nonlinear superposition formula*. By the Implicit Function Theorem (see [48]), the explicit superposition formula (6.12) can be locally inverted with respect to k, i.e., there exists a function $\Theta : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that the following equation holds for all $t \in \mathscr{T}_{t_0}$:

$$k = \Theta\bigl(x(t), \xi^1(t), \dots, \xi^m(t)\bigr); \tag{6.13}$$

equation (6.13) is called an *implicit nonlinear superposition formula*. In general, the implicit nonlinear superposition formula (6.13) holds on an open dense subset of $\mathbb{R}^{n(m+1)}$ rather than on the whole $\mathbb{R}^{n(m+1)}$. It is worth pointing out that formula (6.13) is invariant with respect to any permutation of the $m + 1$ vector arguments of Θ; for example, in case $m = 1$, if $\Theta(x(t), \xi^1(t))$ is a first integral of the extended system, then $\Theta(\xi^1(t), x(t))$ is a first integral too.

Example 6.6 Consider the single-input linear control system $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t)$, $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $u(t) \in \mathbb{R}$, $B \in \mathbb{R}^n$; let $t_0 = 0$. Consider $m = n + 1$ particular solutions $\xi^i(t) \in \mathbb{R}^n$, $i = 0, \dots, n$, of such a control system, i.e., such that $\frac{\mathrm{d}\xi^i(t)}{\mathrm{d}t} = A\xi^i(t) + Bu(t)$, $i = 0, \dots, n$. Clearly, letting $\gamma^i(t) = \xi^i(t) - \xi^0(t)$, one has $\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t} = A\gamma^i(t)$, $i = 1, \dots, n$, and therefore letting $\Gamma = [\gamma^1 \ \dots \ \gamma^n]$, one has $\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} = A\Gamma(t)$, which yields $\Gamma(t) = \mathrm{e}^{At}\Gamma(0)$; this implies $\mathrm{e}^{At} = \Gamma(t)\Gamma^{-1}(0)$, under the assumption that $\det(\Gamma(0)) \neq 0$ (this is the condition to be satisfied in order that the particular solutions $\xi^0(t), \dots, \xi^n(t)$ can be used in the superposition formula). Finally, since $x(t) = \mathrm{e}^{At}c + \xi^0(t)$, for some constant $c \in \mathbb{R}^n$, the explicit and implicit nonlinear superposition formulas are, respectively, obtained:

$$x = \xi^0 + \bigl[\xi^1 - \xi^0 \ \dots \ \xi^n - \xi^0\bigr]k = \xi^0 + \sum_{i=1}^{n}\bigl(\xi^i - \xi^0\bigr)k_i,$$

$$k = \bigl[\xi^1 - \xi^0 \ \dots \ \xi^n - \xi^0\bigr]^{-1}\bigl(x - \xi^0\bigr).$$

The following theorem dates back to S. Lie [111].

Theorem 6.2 *The time-varying nonlinear system* (6.4) *admits the superposition formulas* (6.12), (6.13) *if and only if*

$$f(t, x) = \sum_{i=1}^{p} u_i(t) f_i(x), \tag{6.14}$$

where $u_i(t) \in \mathbb{R}$, $i = 1, \dots, p$, are some functions of time and $f_1(x), \dots, f_p(x) \in \mathbb{R}^n$ are time-invariant vector functions such that the smallest Lie algebra over $\mathbb{R}$ that contains $f_1(x), \dots, f_p(x)$ is finite dimensional.

Proof Although the proof of the theorem is outside the scope of the book, a sketch of it is given for the simplest case $n = 1$, i.e., $x(t), f(t,x) \in \mathbb{R}$.

(Necessity) Assume that $k = \Theta(x, \xi^1, \ldots, \xi^m)$, with $\Theta(\cdot, \cdot, \ldots, \cdot) : \mathbb{R}^{m+1} \to \mathbb{R}$, is an implicit nonlinear superposition formula for system (6.4). Define the extended vector function

$$\bar{f}(t, x, \xi^1, \ldots, \xi^m) := \begin{bmatrix} f(t,x) \\ f(t,\xi^1) \\ \vdots \\ f(t,\xi^m) \end{bmatrix}.$$

Clearly, $L_{\bar{f}(t,x,\xi^1,\ldots,\xi^m)}\Theta(x, \xi^1, \ldots, \xi^m) = 0$ for all $t \in \mathscr{T}_{t_0}$. Fix $t_1, \ldots, t_m$ such that $\bar{f}_1(x, \xi^1, \ldots, \xi^m) := \bar{f}(t_1, x, \xi^1, \ldots, \xi^m), \ldots, \bar{f}_m(x, \xi^1, \ldots, \xi^m) := \bar{f}(t_m, x, \xi^1, \ldots, \xi^m)$ are linearly independent over $\mathbb{R}$. The m time-invariant vector functions $\bar{f}_1, \ldots, \bar{f}_m \in \mathbb{R}^{m+1}$ share the same first integral $k = \Theta(x, \xi^1, \ldots, \xi^m)$ and, by the Frobenius Theorem 1.9 at p. 21, they span an involutive distribution, whence they span an m-dimensional Lie algebra over the field of meromorphic functions; therefore, there exist structure functions $c_{i,j;\ell}(x, \xi^1, \ldots, \xi^m)$ such that $[\bar{f}_i, \bar{f}_j] = \sum_{\ell=1}^m c_{i,j;\ell}\bar{f}_\ell$. Consider the Lie bracket

$$[\bar{f}_1, \bar{f}_2] = \begin{bmatrix} \frac{\partial f(t_2,x)}{\partial x} & 0 & \ldots & 0 \\ 0 & \frac{\partial f(t_2,\xi^1)}{\partial \xi^1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\partial f(t_2,\xi^m)}{\partial \xi^m} \end{bmatrix} \begin{bmatrix} f(t_1,x) \\ f(t_1,\xi^1) \\ \vdots \\ f(t_1,\xi^m) \end{bmatrix} - \begin{bmatrix} \frac{\partial f(t_1,x)}{\partial x} & 0 & \ldots & 0 \\ 0 & \frac{\partial f(t_1,\xi^1)}{\partial \xi^1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\partial f(t_1,\xi^m)}{\partial \xi^m} \end{bmatrix} \begin{bmatrix} f(t_2,x) \\ f(t_2,\xi^1) \\ \vdots \\ f(t_2,\xi^m) \end{bmatrix}.$$

Looking at the first entry of $[\bar{f}_1, \bar{f}_2] = \sum_{\ell=1}^m c_{1,2;\ell}\bar{f}_\ell$, one has

$$\frac{\partial f(t_2,x)}{\partial x} f(t_1,x) - \frac{\partial f(t_1,x)}{\partial x} f(t_2,x) = \sum_{\ell=1}^m c_{1,2;\ell}\big(x, \xi^1, \ldots, \xi^m\big) f(t_\ell, x),$$

which implies that the structure functions $c_{1,2;\ell}$, $\ell = 1, \ldots, m$, are independent of $\xi^1, \ldots, \xi^m$; this and the similar relations obtained by considering the other entries of $[\bar{f}_1, \bar{f}_2] = \sum_{\ell=1}^m c_{1,2;\ell}\bar{f}_\ell$ show that the structure functions $c_{1,2;\ell}$, $\ell = 1, \ldots, m$, are constant. Repeating this reasoning for all the Lie brackets $[\bar{f}_i, \bar{f}_j]$ shows that $\bar{f}_1, \ldots, \bar{f}_m$ span an m-dimensional Lie algebra over $\mathbb{R}$, as well as the scalar functions $f_1(x) := f(t_1, x), \ldots, f_m(x) := f(t_m, x)$. The arbitrariness of times $t_1, \ldots, t_m$ imply that $f(t,x)$ belongs to the Lie algebra over $\mathbb{R}$ spanned by $f_1(x), \ldots, f_m(x)$ for all $t \in \mathscr{T}_{t_0}$, i.e., there exist functions $u_i(t)$, $i = 1, \ldots, m$, such that (6.14) holds with $p = m$.

(Sufficiency) Let r be the dimension of the Lie algebra $\mathfrak{X}$ generated by the vector functions $f_1(x), \dots, f_p(x) \in \mathbb{R}^n$ over $\mathbb{R}$ and let $\{g_1, \dots, g_r\}$ be a basis of $\mathfrak{X}$; then, $f_i(x) = \sum_{j=1}^r Q_{i,j} g_j(x)$, for $Q_{i,j} \in \mathbb{R}$; this, in particular, implies that

$$f(t,x) = \sum_{i=1}^p u_i(t) \sum_{j=1}^r Q_{i,j} g_j(x) = \sum_{j=1}^r \sum_{i=1}^p Q_{i,j} u_i(t) g_j(x)$$
$$= \sum_{j=1}^r v_j(t) g_j(x),$$

where $v_j(t) = \sum_{i=1}^p Q_{i,j} u_i(t)$. Therefore, there is no loss of generality in assuming that $p = r$, that $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_r\}$ is an r-dimensional Lie algebra over $\mathbb{R}$ and that $\{f_1, \dots, f_r\}$ is an arbitrary basis of $\mathfrak{X}$. Define the extended vector functions

$$f_{i,e}\big(x, \xi^1, \dots, \xi^r\big) := \begin{bmatrix} f_i(x) \\ f_i(\xi^1) \\ \vdots \\ f_i(\xi^r) \end{bmatrix}, \quad i = 1, \dots, r.$$

Clearly, $f_{1,e}(x), \dots, f_{r,e}(x)$ span a Lie algebra over $\mathbb{R}$, $\mathfrak{X}_e = \mathrm{span}_{\mathbb{R}}\{f_{1,e}, \dots, f_{r,e}\}$. Since $\mathfrak{X}_e$ is r-dimensional and $[\mathfrak{X}_e, \mathfrak{X}_e] \subseteq \mathfrak{X}_e$, the vector functions $f_{1,e}(x), \dots,$ $f_{r,e}(x)$ admit a joint first integral $\Theta(x, \xi^1, \dots, \xi^r)$. It can be seen that, since $f_i \neq 0$, Θ must depend on $\mathfrak{X}$, whence the implicit nonlinear superposition formula $k = \Theta(x, \xi^1, \dots, \xi^r)$ is obtained; the explicit nonlinear superposition formula is obtained by the Implicit Function Theorem (see [48]). □

As explained in the proof above, there is no loss of generality in assuming that $p = r$, that $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_r\}$ is an r-dimensional Lie algebra over $\mathbb{R}$ and that $\{f_1, \dots, f_r\}$ is an arbitrary basis of $\mathfrak{X}$.

Remark 6.9 According to Theorem 6.2, for any time-varying linear system (6.5), one can write

$$A(t)x(t) = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}(t) M_{i,j} x(t),$$

where the n^2 matrices $M_{i,j} := e_i e_j^\top$ constitute a basis of the matrix Lie algebra $\mathbb{R}^{n \times n}$.

Since functions Ψ and Θ appearing in the nonlinear superposition formulas (6.12) and (6.13) are independent of time, the expressions of Ψ and Θ do not depend on scalar functions $u_i(t) \in \mathbb{R}$, $i = 1, \dots, r$, but only on vector functions $f_1(x), \dots, f_r(x) \in \mathbb{R}^n$; two systems $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^r u_i(t) f_i(x)$ and $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^r v_i(t) f_i(x)$ are described by the same superposition formulas, by the arbitrariness of functions u_i and v_i.

By (6.13), it is easy to see that the entries Θ_i, $i = 1, \ldots, n$, of Θ are functionally independent first integrals of the extended system constituted by equations (6.4), (6.11), whence, by the arbitrariness of the scalar functions $u_i(t)$, they are functionally independent joint first integrals associated with the extended vector functions

$$f_{1,e}\big(x, \xi^1, \ldots, \xi^m\big) = \begin{bmatrix} f_1(x) \\ f_1(\xi^1) \\ \vdots \\ f_1(\xi^m) \end{bmatrix}, \ldots, f_{r,e}\big(x, \xi^1, \ldots, \xi^m\big) = \begin{bmatrix} f_r(x) \\ f_r(\xi^1) \\ \vdots \\ f_r(\xi^m) \end{bmatrix},$$

which certainly exist when m is taken sufficiently high, because $f_{1,e}, \ldots, f_{r,e}$ generate a finite dimensional Lie algebra over $\mathbb{R}$. In particular, taking into account that $f_{i,e}(x_o, \xi_o^1, \ldots, \xi_o^m) \in \mathbb{R}^{n(m+1)}$, if there exists a point $(x_o, \xi_o^1, \ldots, \xi_o^m)$ such that $f_{1,e}(x_o, \xi_o^1, \ldots, \xi_o^m), \ldots, f_{r,e}(x_o, \xi_o^1, \ldots, \xi_o^m)$ are linearly independent, then the number of first integrals associated with $f_{1,e}, \ldots, f_{r,e}$, being functionally independent about point $(x_o, \xi_o^1, \ldots, \xi_o^m)$, is $n(m+1) - r$, which must be greater than or equal to n, thus yielding the inequality $nm \geq r$.

Remark 6.10 An important class of systems that can be written as (6.4) is that of bilinear systems [44],

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_0 x + \sum_{i=1}^{\nu} A_i x u_i(t),$$

for which nonlinear superposition formulas always exist, because $A_0 x, \ldots, A_\nu x$ generate a finite dimensional Lie algebra for any $A_0, \ldots, A_\nu \in \mathbb{R}^{n \times n}$.

Remark 6.11 Two operations preserve the structure of the Lie algebra, whence the existence of nonlinear superposition formulas, although their expressions in closed form may change: a nonlinear transformation on the state x, and a linear transformation on the time functions u_i.

(6.11.1) Given a finite dimensional Lie algebra over $\mathbb{R}$ $\mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_r\}$ and a diffeomorphism $y = \varphi(x)$, one has that $\mathrm{span}_{\mathbb{R}}\{\varphi_* f_1, \ldots, \varphi_* f_r\}$ is a finite dimensional Lie algebra over $\mathbb{R}$, characterized by the same characteristic constants as $\mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_r\}$.

(6.11.2) Given an invertible matrix $Q \in \mathbb{R}^{r \times r}$, $u = Qv$, where $u = [u_1 \; \ldots \; u_r]^\top$ and $v = [v_1 \; \ldots \; v_r]^\top$, (6.14) can be recast as follows:

$$\begin{aligned} f(t,x) &= \sum_{i=1}^{r} u_i(t) f_i(x) = \sum_{i=1}^{r} \sum_{j=1}^{r} Q_{i,j} v_j(t) f_i(x) \\ &= \sum_{j=1}^{r} \sum_{i=1}^{r} Q_{i,j} f_i(x) v_j(t) = \sum_{j=1}^{r} v_j(t) g_j(x), \end{aligned}$$

where $g_j(x) = \sum_{i=1}^{r} Q_{i,j} f_i(x)$, $j = 1, \dots, r$. The two Lie algebras $\mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_r\}$ and $\mathrm{span}_{\mathbb{R}}\{g_1, \dots, g_r\}$ over $\mathbb{R}$ are isomorphic, but in general they are described by different structure constants (see Remark 6.1).

Example 6.7 Consider the case $n = 1$. By [111], it is known that any Lie algebra over $\mathbb{R}$ spanned by scalar functions is at most three-dimensional. Hence, assume that $\mathfrak{X}$ is three-dimensional, i.e., $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$, with $\{f_1, f_2, f_3\}$ being a basis of $\mathfrak{X}$, and that $\mathrm{rank}_{\mathscr{K}_1}\{f_1, f_2, f_3\} = 1$. About a regular point of f_i, apart from a diffeomorphism, it can be assumed that $f_i = 1$; by $[1, g] = \frac{\partial g}{\partial x}$, one concludes that any g commuting with f_i satisfies $g = cf_i$, for some constant c. Therefore, it can be assumed that $[f_i, f_j]$ is not identically equal to zero, because otherwise $\{f_1, f_2, f_3\}$ is not a basis of $\mathfrak{X}$. By Remark 6.1 (see [70]), it is known that the only three-dimensional Lie algebras over $\mathbb{R}$ satisfying the conditions $[f_i, f_j] \neq 0$, $i, j \in \{1, 2, 3\}$, $i \neq j$, are, apart from a proper choice of the Lie algebra basis, the Lie algebras listed in Statement (6.1.1) of Remark 6.1; in particular, if f_1, f_2, f_3 are scalar functions of $x \in \mathbb{R}$, then it can be shown [111] that, apart from a proper choice of the Lie algebra basis, the only Lie algebra satisfying the conditions $[f_i, f_j] \neq 0$, $i, j \in \{1, 2, 3\}$, $i \neq j$, is the split three-dimensional simple Lie algebra, described by the commutation relations

$$[f_1, f_2] = 2f_1, \qquad [f_1, f_3] = f_2, \qquad [f_2, f_3] = 2f_3.$$

Assume, apart from a diffeomorphism about any regular point, that $f_1(x) = 1$. Hence,

$$\begin{aligned}
[f_1, f_2] = 2f_1 \quad &\Longrightarrow \quad \frac{\partial f_2(x)}{\partial x} = 2 \quad \Longrightarrow \quad f_2(x) = 2x + c_2, \\
[f_1, f_3] = f_2 \quad &\Longrightarrow \quad \frac{\partial f_3(x)}{\partial x} = 2x + c_2 \quad \Longrightarrow \quad f_3(x) = x^2 + c_2 x + c_3, \\
[f_2, f_3] = 2f_3 \quad &\Longrightarrow \quad c_2^2 - 4c_3 = 0 \quad \Longrightarrow \quad c_3 = \frac{1}{4}c_2^2,
\end{aligned}$$

which shows that $\{1, 2x + c_2, x^2 + c_2 x + \frac{1}{4}c_2^2\}$ is a basis of $\mathfrak{X}$; another basis of $\mathfrak{X}$ is $\{1, x, x^2\}$, which shows that any scalar differential equation, which admits nonlinear superposition formulas, is diffeomorphic to the scalar *Riccati differential equation*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u_1(t) + u_2(t)x + u_3(t)x^2. \tag{6.15}$$

Let $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ and define the extended vector functions

$$f_{1,e}\big(\xi^0, \xi^1, \xi^2, \xi^3\big) := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad f_{2,e}\big(\xi^0, \xi^1, \xi^2, \xi^3\big) := \begin{bmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \\ \xi^3 \end{bmatrix},$$

$$f_{3,e}\big(\xi^0,\xi^1,\xi^2,\xi^3\big) := \begin{bmatrix}(\xi^0)^2\\(\xi^1)^2\\(\xi^2)^2\\(\xi^3)^2\end{bmatrix},$$

where $\xi^0 = x$. All first integrals associated with $f_{1,e}$ are given by arbitrary functions of $\xi^i - \xi^j$, for $i, j \in \{0,1,2,3\}$; all first integrals associated with $f_{2,e}$ are given by arbitrary functions of $\frac{\xi^i-\xi^j}{\xi^h-\xi^k}$ for $i, j, h, k \in \{0,1,2,3\}$, $(i, j) \neq (h, k)$; all first integrals associated with $f_{3,e}$ are given by arbitrary functions of $\frac{1}{\xi^i} - \frac{1}{\xi^j} = \frac{\xi^j-\xi^i}{\xi^i\xi^j}$, for $i, j \in \{0,1,2,3\}$. Hence, these three vector functions admit as joint first integrals the arbitrary functions of the following quantity, which is often referred to as the *cross ratio*:

$$\Theta = \frac{(\xi^0-\xi^1)(\xi^2-\xi^3)}{(\xi^0-\xi^2)(\xi^1-\xi^3)}.$$

This gives the implicit nonlinear superposition formula $k = \Theta$; by solving such an equation by $x = \xi^0$, one obtains the explicit nonlinear superposition formula $x = \Psi$, with

$$\Psi = \frac{k\xi^2(\xi^1-\xi^3) - \xi^1(\xi^2-\xi^3)}{k(\xi^1-\xi^3) - (\xi^2-\xi^3)}.$$

Similar explicit nonlinear superposition formulas can be easily determined by a permutation of the solutions ξ^1, ξ^2, ξ^3; for instance, if the triplet (ξ^1, ξ^2, ξ^3) is replaced with the triplet (ξ^2, ξ^3, ξ^1), one obtains the explicit nonlinear superposition formula

$$x = \frac{k\xi^3(\xi^2-\xi^1) - \xi^2(\xi^3-\xi^1)}{k(\xi^2-\xi^1) - (\xi^3-\xi^1)}.$$

Now, consider a planar linear system $\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y$, where $y \in \mathbb{R}^2$ and

$$A(t) = \begin{bmatrix}A_{1,1}(t) & A_{1,2}(t)\\A_{2,1}(t) & A_{2,2}(t)\end{bmatrix}.$$

Since $[A(t), E] = 0$ for any $t \in \mathbb{R}$, according to Sect. 3.5, consider the projection $x = \frac{y_1}{y_2}$, which transform $\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y$ into

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \begin{bmatrix}\frac{1}{y_2} & -\frac{y_1}{y_2^2}\end{bmatrix}\begin{bmatrix}A_{1,1}(t) & A_{1,2}(t)\\A_{2,1}(t) & A_{2,2}(t)\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\\&= A_{1,2}(t) + \big(A_{1,1}(t) - A_{2,2}(t)\big)\frac{y_1}{y_2} - A_{2,1}(t)\frac{y_1^2}{y_2^2}\\&= A_{1,2}(t) + \big(A_{1,1}(t) - A_{2,2}(t)\big)x - A_{2,1}(t)x^2,\end{aligned}$$

i.e., the Riccati differential equation (6.15) with $u_1(t) = A_{1,2}(t)$, $u_2(t) = A_{1,1}(t) - A_{2,2}(t)$ and $u_3(t) = -A_{2,1}(t)$. Therefore, this shows that any scalar differential equation that admits nonlinear superposition formulas can be immersed into a planar linear system (see [111]), thus justifying the assertion that the scalar nonlinear systems that admit nonlinear superposition formulas are “close” to the linear ones.

If one of the scalar functions $u_i(t)$ appearing in (6.14) is identically equal to zero, the computation of the explicit and implicit nonlinear superposition formulas can be simplified as shown in the following example, which shows also that the explicit and implicit nonlinear superposition formulas, also modulo permutation of the particular solutions, are not unique.

Example 6.8 Consider again the linear oscillator with time-varying frequency (6.9). Define the vector functions $f_1(x) := [x_2\ \ 0]^\top$ and $f_2(x) := [0\ \ x_1]^\top$. Compute the Lie bracket $[f_1(x), f_2(x)] = [-x_1\ \ x_2]^\top$ and let $f_3(x) := [-x_1\ \ x_2]^\top$. Since $[f_1, f_2] = f_3$, $[f_1, f_3] = -2f_1$ and $[f_2, f_3] = 2f_2$, $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra. Compute the extended vector functions $f_{1,e}(x, \xi^1, \xi^2) = [x_2\ 0\ \xi_2^1\ 0\ \xi_2^2\ 0]^\top$, $f_{2,e}(x, \xi^1, \xi^2) = [0\ x_1\ 0\ \xi_1^1\ 0\ \xi_1^2]^\top$ (there is no need to compute $f_{3,e}$). Two joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are given by $x_1\xi_2^1 - \xi_1^1 x_2$ and $x_1\xi_2^2 - \xi_1^2 x_2$. The implicit superposition formula is

$$k_1 = x_1\xi_2^1 - \xi_1^1 x_2,$$
$$k_2 = x_1\xi_2^2 - \xi_1^2 x_2;$$

by the inverse with respect to x, the explicit superposition formula,

$$x_1 = \frac{k_1\xi_1^2 - k_2\xi_1^1}{\xi_2^1\xi_1^2 - \xi_1^1\xi_2^2},$$
$$x_2 = \frac{k_1\xi_2^2 - k_2\xi_2^1}{\xi_2^1\xi_1^2 - \xi_1^1\xi_2^2},$$

is obtained under the assumption that $\det([\xi^1\ \ \xi^2]) = \xi_1^1\xi_2^2 - \xi_2^1\xi_1^2$ is not identically zero.

Example 6.9 (A knife edge [8]) Consider the kinematic equations of motion of a *knife edge*

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \cos(x_3)u_1(t),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \sin(x_3)u_1(t),$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = u_2(t).$$

Define $f_1(x) := [\cos(x_3)\ \ \sin(x_3)\ 0]^\top$ and $f_2(x) := [0\ 0\ 1]^\top$. Since $[f_1(x), f_2(x)] = [\sin(x_3)\ \ -\cos(x_3)\ 0]^\top$, define $f_3(x) := [\sin(x_3)\ \ -\cos(x_3)\ 0]^\top$. Since $[f_1, f_2] = f_3$,

$[f_1, f_3] = 0$ and $[f_2, f_3] = -f_1$, one concludes that $\mathfrak{X} = \text{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra over $\mathbb{R}$. Compute the extended vector functions $f_{1,e}(x,\xi^1) = [\cos(x_3)\ \sin(x_3)\ 0\ \cos(\xi_3^1)\ \sin(\xi_3^1)\ 0]^\top$, $f_{2,e}(x,\xi^1) = [0\ 0\ 1\ 0\ 0\ 1]^\top$ (there is no need to compute $f_{3,e}$). Three joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are $x_1 - \xi_1^1 \cos(\xi_3^1 - x_3) - \xi_2^1 \sin(\xi_3^1 - x_3)$, $x_2 + \xi_1^1 \sin(\xi_3^1 - x_3) - \xi_2^1 \cos(\xi_3^1 - x_3)$ and $(x_3 - \xi_3^1)$, thus obtaining the implicit nonlinear superposition formula:

$$
\begin{aligned}
k_1 &= x_1 - \xi_1^1 \cos\bigl(\xi_3^1 - x_3\bigr) - \xi_2^1 \sin\bigl(\xi_3^1 - x_3\bigr),\\
k_2 &= x_2 + \xi_1^1 \sin\bigl(\xi_3^1 - x_3\bigr) - \xi_2^1 \cos\bigl(\xi_3^1 - x_3\bigr),\\
k_3 &= x_3 - \xi_3^1;
\end{aligned}
$$

by the inverse, the explicit nonlinear superposition formula is obtained,

$$
\begin{aligned}
x_1 &= \xi_1^1 \cos(k_3) - \xi_2^1 \sin(k_3) + k_1,\\
x_2 &= \xi_1^1 \sin(k_3) + \xi_2^1 \cos(k_3) + k_2,\\
x_3 &= \xi_3^1 + k_3.
\end{aligned}
$$

Example 6.10 (Chained system [8]) Consider a three-dimensional *chained system*:

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= u_1(t),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= u_2(t),\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_2 u_1(t).
\end{aligned}
$$

Define $f_1(x) := [1\ 0\ x_2]^\top$ and $f_2(x) := [0\ 1\ 0]^\top$. Letting $f_3(x) := [0\ 0\ 1]^\top$, it is easy to see that $[f_1, f_2] = -f_3$, $[f_1, f_3] = 0$ and $[f_2, f_3] = 0$, whence that $\mathfrak{X} = \text{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$ is a three-dimensional Lie algebra over $\mathbb{R}$. Compute the extended vector functions $f_{1,e}(x,\xi^1) = [1\ 0\ x_2\ 1\ 0\ \xi_2^1]^\top$, $f_{2,e}(x,\xi^1) = [0\ 1\ 0\ 0\ 1\ 0]^\top$ (there is no need to compute $f_{3,e}$). Three joint functionally independent first integrals associated with $f_{1,e}$ and $f_{2,e}$ are given by $x_1 - \xi_1^1$, $x_2 - \xi_2^1$ and $x_3 - \xi_3^1 - \xi_1^1 x_2 + \xi_1^1 \xi_2^1$, thus obtaining the implicit nonlinear superposition formula:

$$
\begin{aligned}
k_1 &= x_1 - \xi_1^1,\\
k_2 &= x_2 - \xi_2^1,\\
k_3 &= x_3 - \xi_3^1 - \xi_1^1 x_2 + \xi_1^1 \xi_2^1;
\end{aligned}
$$

by the inverse, the explicit nonlinear superposition formula is obtained,

$$
x_1 = \xi_1^1 + k_1,
$$

$$x_2 = \xi_2^1 + k_2,$$
$$x_3 = \xi_3^1 + \xi_1^1 k_2 + k_3.$$

Example 6.11 (DC-to-DC electric power conversion systems [44]) Consider a DC-to-DC electric power conversion systems described by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{u(t)-1}{L}x_2 + \frac{E}{L},$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{u(t)-1}{L}x_1 - \frac{1}{RC}x_2,$$

where the DC supply is E and the load resistance is R. The state variables are the current x_1 through the inductor L and the output voltage x_2 on the capacitor C; $u(t)$ is a piecewise constant function of time, $u(t) \in \{0, 1\}$. Since the system parameters E, L, R and C my be subject to time-varying uncertainties, it would be nice to obtain a superposition formula independent of them. Define

$$f_1(x) := \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad f_2(x) := \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \qquad f_3(x) := \begin{bmatrix} 0 \\ x_1 \end{bmatrix},$$
$$f_4(x) := \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \qquad f_5(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad f_6(x) := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which span a six-dimensional Lie algebra over $\mathbb{R}$, described by the commutation relations $[f_1, f_2] = -f_2$, $[f_1, f_3] = -f_3$, $[f_1, f_4] = 0$, $[f_1, f_5] = -f_5$, $[f_1, f_6] = 0$, $[f_2, f_3] = f_4 - f_1$, $[f_2, f_4] = -f_2$, $[f_2, f_5] = 0$, $[f_2, f_6] = -f_5$, $[f_3, f_4] = f_3$, $[f_3, f_5] = -f_6$, $[f_3, f_6] = 0$, $[f_4, f_5] = 0$, $[f_4, f_6] = -f_6$, $[f_5, f_6] = 0$. Proceeding as in the previous examples, explicit and implicit nonlinear superposition formulas are, respectively, obtained:

$$x_1 = \xi_1^1 + \left(\xi_1^2 - \xi_1^1\right)k_1 + \left(\xi_1^3 - \xi_1^1\right)k_2,$$
$$x_2 = \xi_2^1 + \left(\xi_2^2 - \xi_2^1\right)k_1 + \left(\xi_2^3 - \xi_2^1\right)k_2,$$

and

$$k_1 = \frac{-\xi_2^3 x_1 + \xi_1^1\xi_2^3 + \xi_2^1 x_1 + \xi_1^3 x_2 - \xi_1^3\xi_2^1 - \xi_1^1 x_2}{-\xi_1^2\xi_2^3 + \xi_1^2\xi_2^1 + \xi_1^1\xi_2^3 + \xi_1^3\xi_2^2 - \xi_1^3\xi_2^1 - \xi_1^1\xi_2^2},$$
$$k_2 = \frac{\xi_2^2 x_1 - \xi_1^1\xi_2^2 - \xi_2^1 x_1 - \xi_1^2 x_2 + \xi_1^2\xi_2^1 + \xi_1^1 x_2}{-\xi_1^2\xi_2^3 + \xi_1^2\xi_2^1 + \xi_1^1\xi_2^3 + \xi_1^3\xi_2^2 - \xi_1^3\xi_2^1 - \xi_1^1\xi_2^2}.$$

6.6 Nonlinear Superposition Formulas for Solvable Lie Algebras

Let $\mathfrak{X}$ be a finite dimensional Lie algebra (of vector functions $f(x) \in \mathbb{R}^n$) over $\mathbb{R}$; let r be its dimension and $\{f_1, \ldots, f_r\}$ be one of its bases. There exist structure constants $c_{i,j;\ell} \in \mathbb{R}$ such that $[f_i, f_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} f_\ell$; the Lie algebra is uniquely described by the basis $\{f_1, \ldots, f_r\}$ and by the structure constants $c_{i,j;\ell}$. According to [114], as is well known, if $\mathfrak{X}$ is solvable, then there exists a basis $\{f_1, \ldots, f_r\}$ such that for $i = 1, \ldots, r$:

$$[f_1, f_i] = c_{1,i;1} f_1, \tag{6.16a}$$

$$[f_2, f_i] = c_{2,i;1} f_1 + c_{2,i;2} f_2, \tag{6.16b}$$

$$\vdots$$

$$[f_r, f_i] = c_{r,i;1} f_1 + c_{r,i;2} f_2 + \cdots + c_{r,i;r} f_r. \tag{6.16c}$$

For the sake of simplicity, assume the existence of a regular point $x^o \in \mathbb{R}^n$ of the Lie algebra such that $\mathrm{rank}_{\mathbb{R}}\{f_1(x^o), \ldots, f_r(x^o)\} = r$, although the procedure outlined in the following can be easily extended in the case of a regular point $x^o \in \mathbb{R}^n$ of the Lie algebra such that $\mathrm{rank}_{\mathbb{R}}\{f_1(x), \ldots, f_r(x)\}$ is constant about x^o, but less than r. Assume that $n \geq r$. The computation of nonlinear superposition formulas can be carried out by a repeated application of the flow box Theorem 3.3 at p. 57. Since $f_1(x^o) \neq 0$, there exists about x^o a diffeomorphism $y = \varphi(x)$ such that $\varphi_* f_1(y) = e_1$, where e_1 is the first column of the $n \times n$ identity matrix E. From (6.16a) rewritten in the y-coordinates, $[e_1, \varphi_* f_i] = c_{1,i;1} e_1$, $i = 2, \ldots, r$, it follows that

$$\varphi_* f_i = \begin{bmatrix} c_{1,i;1} y_1 + \alpha_i(y_b) \\ \tilde{f}_i(y_b) \end{bmatrix},$$

where $y_b = [y_2 \ \ldots \ y_n]^\top$, $\alpha_i(y_b) \in \mathbb{R}$ and $\tilde{f}_i(y_b) \in \mathbb{R}^{n-1}$. Now, consider the $r-1$ vector functions $\tilde{f}_i(y_b) \in \mathbb{R}^{n-1}$, $i = 2, \ldots, r$, which satisfy relations (6.16b), (6.16c), $i = 2, \ldots, r$, with $c_{1,i;1} = \cdots = c_{r,i;1} = 0$; this can be easily seen by noticing that vector functions $\varphi_* f_i$ satisfy relations (6.16a)–(6.16c) with f_i substituted by $\varphi_* f_i$ for all (y_1, y_b), whence also for all $(0, y_b)$. Since $\tilde{f}_2(y_b^o) \neq 0$, where $y^o = \varphi(x^o) = [y_1^o \ (y_b^o)^\top]^\top$, there exists about y_b^o a diffeomorphism $z_b = \chi(y_b)$ such that $\chi_* \tilde{f}_2(z) = e_1$, where e_1 is now the first column of the $(n-1) \times (n-1)$ identity matrix. From (6.16b), with $c_{2,i;1} = 0$, rewritten in the z_b-coordinates, $[e_1, \chi_* \tilde{f}_i] = c_{2,i;2} e_1$, $i = 3, \ldots, r$, it follows that

$$\chi_* \tilde{f}_i = \begin{bmatrix} c_{2,i;2} z_2 + \beta_i(z_c) \\ \hat{f}_i(z_c) \end{bmatrix},$$

where $z_b = [z_3 \ \dots \ z_n]^\top$, $\beta_i(z_c) \in \mathbb{R}$ and $\hat{f}_i(z_c) \in \mathbb{R}^{n-2}$. Proceeding in this way, one concludes, apart from a diffeomorphism about the regular point x^o, that vector functions $f_1, \dots, f_r$ can be rewritten as

$$f_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} c_{1,2;1}x_1 + \alpha_2(x_2, \dots, x_n) \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} c_{1,3;1}x_1 + \alpha_3(x_2, \dots, x_n) \\ c_{2,3;2}x_2 + \beta_3(x_3, \dots, x_n) \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots,$$

$$f_r(x) = \begin{bmatrix} c_{1,r;1}x_1 + \alpha_r(x_2, \dots, x_n) \\ c_{2,r;2}x_2 + \beta_r(x_3, \dots, x_n) \\ c_{3,r;3}x_3 + \gamma_r(x_4, \dots, x_n) \\ c_{4,r;4}x_4 + \delta_r(x_5, \dots, x_n) \\ \vdots \\ c_{r-1,r;r-1}x_{r-1} + \omega_r(x_r, \dots, x_n) \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As shown in the following with reference to Lie algebras of dimension two, since in these local coordinates the vector functions f_i have such a special triangular structure, the computation of first integrals, whence of the nonlinear superposition formulas, is highly simplified. Therefore, by a pull-back, nonlinear superposition formulas can be computed in the original coordinates.

6.6.1 Two-Dimensional Lie Algebras

Consider $f(t,x) = u_1(t)f_1(x) + u_2(t)f_2(x)$, where $f_1(x), f_2(x) \in \mathbb{R}^n$ span a two-dimensional Lie algebra over $\mathbb{R}$. Assume $n \geq 2$. By Remark 6.1 (see [70]), it is known that, apart from a change of basis of the Lie algebra (which corresponds to a linear transformation in the control inputs), there are only two possible cases: $[f_1, f_2] = 0$ and $[f_1, f_2] = f_1$, which are analyzed in the following sections.

6.6.1.1 The Lie Algebra $[f_1, f_2] = 0$

This case can be split into two different cases: f_1 and f_2 are not (respectively, are) co-linear over $\mathscr{K}_n$.

(1) Assume that f_1 and f_2 are not co-linear over $\mathscr{K}_n$. Let $x^o \in \mathbb{R}^n$ be a regular point of the Lie algebra such that $\mathrm{rank}_{\mathbb{R}}([f_1(x^o)\ f_2(x^o)]) = 2$. Since f_1 and f_2 are commuting, about x^o there exists a diffeomorphism $y = \varphi(x)$, with inverse $x = \phi(y)$, such that $\tilde{f}_1(y) = \varphi_* f_1(y) = e_1$ and $\tilde{f}_2(y) = \varphi_* f_2(y) = e_2$, where e_i is ith column of the identity matrix E. Define the extended vector functions $\tilde{f}_{1,e}(y,\eta) := [\tilde{f}_1^\top(y)\ \tilde{f}_1^\top(\eta)]^\top$ and $\tilde{f}_{2,e}(y,\eta) := [\tilde{f}_2^\top(y)\ \tilde{f}_2^\top(\eta)]^\top$, where $\eta = \varphi(\xi)$; in this case, superposition formulas with $m = 1$ can be found. It is easy to see that $\tilde{f}_{1,e}$ and $\tilde{f}_{2,e}$ admit $2n-1$ functionally independent joint first integrals:

$$I_1 = y_1 - \eta_1, \qquad I_2 = y_2 - \eta_2, \qquad I_3 = y_3, \dots, I_n = y_n,$$
$$I_{n+1} = \eta_3, \dots, I_{2n-1} = \eta_n,$$

from which one of the possible implicit nonlinear superposition formulas is obtained by noticing that $I_i - I_{n+i-1}$, $i = 2, \dots, n$, are functionally independent joint first integrals associated with $\tilde{f}_{1,e}$ and $\tilde{f}_{2,e}$:

$$k = y - \eta.$$

By the pull-back to the original coordinates, an implicit nonlinear superposition formula is obtained in the original coordinates:

$$k = \varphi(x) - \varphi(\xi).$$

An explicit nonlinear superposition formula is obtained by inversion:

$$x = \phi\big(k + \varphi(\xi)\big).$$

(2) Now, assume that f_1 and f_2 are co-linear over $\mathscr{K}_n$; exclude the trivial case $f_2 = af_1$, for a constant $a \in \mathbb{R}$. Let $x^o \in \mathbb{R}^n$ be a regular point of the Lie algebra, i.e., a point such that $\mathrm{rank}_{\mathbb{R}}([f_1(x^o)\ f_2(x^o)]) = 1$. About x^o, there exists a diffeomorphism $y = \varphi(x)$, with inverse $x = \phi(y)$, such that $\tilde{f}_1(y) = \varphi_* f_1(y) = e_1$. Since f_2 is co-linear with f_1 over $\mathscr{K}_n$, one concludes that $\tilde{f}_2(y) = \varphi_* f_2(y) = \alpha(y)e_1$, for some scalar function $\alpha(y) \in \mathbb{R}$. Condition $[\tilde{f}_1, \tilde{f}_2] = 0$, which is equivalent to

$\frac{\partial \tilde{f}_2}{\partial y} e_1 = 0$, implies that α does not depend on y_1. Since α is not constant by assumption, apart from a reordering of the variables y_i, assume that $\frac{\partial \alpha}{\partial y_2} \neq 0$; therefore, $z = \chi(y) = [y_1 \; \alpha(y_2, \ldots, y_n) \; y_3 \; \ldots \; y_n]^\top$ qualifies as a diffeomorphism such that $\hat{f}_1(z) = \chi_* \tilde{f}_1(z) = e_1$ and $\hat{f}_2(z) = \chi_* \tilde{f}_2(z) = z_2 e_1$. Define the extended vector functions $\hat{f}_{i,e}^\top(z, \zeta^1, \zeta^2) := [\hat{f}_i^\top(z) \; \hat{f}_i^\top(\zeta^1) \; \hat{f}_i^\top(\zeta^2)]$, $i = 1, 2$, where $\zeta^i = \chi \circ \varphi(\xi^i)$, $i = 1, 2$; in this case, superposition formulas with $m = 2$ can be found. Proceeding as in the previous case, an implicit nonlinear superposition formula is easily obtained on the basis of two particular solutions $\zeta^1(t), \zeta^2(t) \in \mathbb{R}^n$:

$$k_1 = \frac{z_1 - \zeta_1^1}{z_2 - \zeta_2^1} - \frac{z_1 - \zeta_1^2}{z_2 - \zeta_2^2},$$
$$k_i = z_i, \quad i = 2, \ldots, n.$$

Let $J = \varphi \circ \chi$, so that $z = J(x)$; by the pull-back to the original coordinates, an implicit nonlinear superposition formula is obtained in the original coordinates on the basis of two particular solutions $\xi^1 = J^{-1}(\zeta^1)$ and $\xi^2 = J^{-1}(\zeta^2)$:

$$k_1 = \frac{J_1(x) - J_1(\xi^1)}{J_2(x) - J_2(\xi^1)} - \frac{J_1(x) - J_1(\xi^2)}{J_2(x) - J_2(\xi^2)},$$
$$k_i = J_i(x), \quad i = 2, \ldots, n,$$

where $J_i(x)$ denotes the ith entry of $J(x)$.

6.6.1.2 The Lie Algebra $[f_1, f_2] = f_1$

Also this case can be split into two different cases: f_1 and f_2 are not (respectively, are) co-linear over $\mathcal{K}_n$.

(3) Assume that f_1 and f_2 are not co-linear over $\mathcal{K}_n$. Let $x^o \in \mathbb{R}^n$ be a regular point of the Lie algebra such that $\mathrm{rank}_{\mathbb{R}}([f_1(x^o) \; f_2(x^o)]) = 2$. About x^o, there exists a diffeomorphism $y = \varphi(x)$, with inverse $x = \phi(y)$, such that $\tilde{f}_1(y) = \varphi_* f_1(y) = e_1$ and $\tilde{f}_2(y) = \varphi_* f_2(y) = (y_1 + \alpha(y_2, y_3, \ldots, y_n))e_1 + e_2$, where e_i is ith column of the identity matrix E and $\alpha(y_2, y_3, \ldots, y_n) \in \mathbb{R}$. Consider the additional diffeomorphism

$$z = \chi(y) = \left[y_1 - \int_0^{y_2} \mathrm{e}^{(y_2 - \theta)} \alpha(\theta, y_3, \ldots, y_n) \, \mathrm{d}\theta \; y_2 \; \ldots \; y_n \right]^\top;$$

in the z-coordinates, one has $\chi_* \tilde{f}_1(z) = e_1$ and $\chi_* \tilde{f}_2(z) = z_1 e_1 + e_2$. Let $J = \varphi \circ \chi$, so that $z = J(x)$. Proceeding as in the previous case, an implicit nonlinear superposition formula is easily obtained on the basis of one particular solution $\zeta(t) \in \mathbb{R}^n$, $\zeta = J(\xi)$ (therefore, $m = 1$ in this case):

$$k_1 = (z_1 - \zeta_1)\mathrm{e}^{-\zeta_2},$$

$$k_2 = z_2 - \zeta_2,$$
$$k_i = z_i, \quad i = 3, \dots, n.$$

By the pull-back to the original coordinates, an implicit nonlinear superposition formula is obtained in the original coordinates on the basis of one particular solution $\xi = J^{-1}(\zeta)$:

$$k_1 = \big(J_1(x) - J_1(\xi)\big)\mathrm{e}^{-J_2(\xi)},$$
$$k_2 = J_2(x) - J_2(\xi),$$
$$k_i = J_i(x), \quad i = 3, \dots, n,$$

where $J_i(x)$ denotes the ith entry of $J(x)$.

(4) Now, assume that f_1 and f_2 are co-linear over $\mathcal{K}_n$; exclude the trivial case $f_2 = af_1$, for a constant $a \in \mathbb{R}$. Let $x^o \in \mathbb{R}^n$ be a regular point of the Lie algebra, i.e., a point such that $\mathrm{rank}_{\mathbb{R}}([f_1(x^o)\ f_2(x^o)]) = 1$. About x^o, there exists a diffeomorphism $y = \varphi(x)$, with inverse $x = \phi(y)$, such that $\tilde{f}_1(y) = \varphi_* f_1(y) = e_1$. Since f_2 is co-linear with f_1 over $\mathcal{K}_n$, one concludes that $\tilde{f}_2(y) = \varphi_* f_2(y) = \alpha(y)e_1$, for some scalar function $\alpha(y) \in \mathbb{R}$. Condition $[\tilde{f}_1, \tilde{f}_2] = \tilde{f}_1$, which is equivalent to $\frac{\partial \tilde{f}_2}{\partial y} e_1 = e_1$, yields $\alpha(y) = y_1 + \beta(y_2, \dots, y_n)$. As in case (3), consider the additional diffeomorphism

$$z = \chi(y) = \left[y_1 - \int_0^{y_2} \mathrm{e}^{(y_2-\theta)}\alpha(\theta, y_3, \dots, y_n)\,\mathrm{d}\theta\ \ y_2\ \dots\ y_n \right]^\top;$$

in the z-coordinates, one has $\chi_* \tilde{f}_1(z) = e_1$ and $\chi_* \tilde{f}_2(z) = z_1 e_1$. Let $J = \varphi \circ \chi$. Proceeding as in the previous case, an implicit nonlinear superposition formula is easily obtained on the basis of one particular solution $\zeta(t) \in \mathbb{R}^n$, $\zeta = J(\xi)$ (therefore, $m = 1$ in this case):

$$k_1 = \frac{z_1}{\zeta_1},$$
$$k_i = z_i, \quad i = 2, \dots, n.$$

By the pull-back to the original coordinates, an implicit nonlinear superposition formula is obtained in the original coordinates on the basis of one particular solution $\xi = J^{-1}(\zeta)$:

$$k_1 = \frac{J_1(x)}{J_1(\xi)},$$
$$k_i = J_i(x), \quad i = 2, \dots, n,$$

where $J_i(x)$ denotes the ith entry of $J(x)$.

6.7 Darboux Polynomials of a Lie Algebra

The following definition extends Definition 3.1 at p. 55 to the case of the time-varying system (6.4), (6.14), where, for the sake of simplicity, it is assumed that $f_1, \ldots, f_p$ are polynomial vector functions; the extension to semi-invariants in case of meromorphic vector functions is easy.

Definition 6.4 A *Darboux polynomial* of system (6.4), (6.14) is a scalar polynomial $\omega(x) \in \mathbb{R}$ such that its derivative $\frac{\mathrm{d}\omega(x)}{\mathrm{d}t}$ along the solutions of (6.4), (6.14) satisfies, for any p-plet of functions $u_1(t), \ldots, u_p(t)$, the following equation:

$$\frac{\mathrm{d}\omega(x)}{\mathrm{d}t} = \lambda(t, x)\omega(x), \tag{6.17}$$

where $\lambda(t, x) \in \mathbb{R}$ is a polynomial in x with time-varying coefficients being functions of $u_1, \ldots, u_p$; $\lambda(t, x)$ is called the *characteristic polynomial*. If $\lambda(t, x)$ is identically equal to zero, then $\omega(x)$ is called a *polynomial first integral* of system (6.4), (6.14).

If $f_1, \ldots, f_p$ generate a finite dimensional Lie algebra $\{f_1, \ldots, f_p\}_{\mathbb{R}}$ over $\mathbb{R}$, taking into account that $f(t, x)$ in (6.14) is an element of $\{f_1, \ldots, f_p\}_{\mathbb{R}}$ for arbitrary constant functions $u_1, \ldots, u_p$, the arbitrariness of the functions $u_1, \ldots, u_p$ shows that any Darboux polynomial of system (6.4), (6.14) is a joint Darboux polynomial associated with all $f \in \{f_1, \ldots, f_p\}_{\mathbb{R}}$, whence, in particular, it is a joint Darboux polynomial associated with all $f_1, \ldots, f_p$. Conversely, any joint Darboux polynomial $\omega(x) \in \mathbb{R}$ associated with all $f_1, \ldots, f_p$, $L_{f_i}\omega(x) = \lambda_i(x)\omega(x)$, $i = 1, \ldots, p$, is a Darboux polynomial of system (6.4), (6.14),

$$\frac{\mathrm{d}\omega(x)}{\mathrm{d}t} = \sum_{i=1}^{p} u_i(t) L_{f_i}\omega(x) = \left(\sum_{i=1}^{p} u_i(t)\lambda_i(x)\right)\omega(x).$$

If $f_1, \ldots, f_p$ generate a finite dimensional Lie algebra $\{f_1, \ldots, f_p\}_{\mathbb{R}}$, a Darboux polynomial of system (6.4), (6.14) can be termed as a *Darboux polynomial of the Lie algebra* $\{f_1, \ldots, f_p\}_{\mathbb{R}}$.

If not empty, the set $\mathscr{I}_\omega = \{x \in \mathscr{U} : \omega(x) = 0\}$ is invariant along the solutions of system (6.4), (6.14), i.e., if $x(0) \in \mathscr{I}_\omega$, then $x(t) \in \mathscr{I}_\omega$ for all $t \in \mathbb{R}$, possibly close to 0.

The following theorem describes some characteristics of the Darboux polynomials of system (6.4), (6.14).

Theorem 6.3 (6.3.1) *If $I(x) = \frac{\omega_1(x)}{\omega_2(x)}$, with ω_1 and ω_2 being co-prime polynomials, satisfies $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$ for any choice of $u_1, \ldots, u_p$ (it is a* rational first integral *of system (6.4), (6.14)), then $\omega_1(x)$ and $\omega_2(x)$ are Darboux polynomials of system (6.4), (6.14), with characteristic polynomials $\lambda_1(t, x)$ and $\lambda_2(t, x)$ such that $\lambda_1(t, x) - \lambda_2(t, x) = 0$.*

(6.3.2) *Let $\omega(x)$, $\omega_1(x)$ and $\omega_2(x)$ be Darboux polynomials of system* (6.4), (6.14) *with respective characteristic polynomials $\lambda(t,x)$, $\lambda_1(t,x)$ and $\lambda_2(t,x)$; then, all irreducible factors of $\omega(x)$ are Darboux polynomials of system* (6.4), (6.14), *and the product $\omega_1^{n_1}(x)\omega_2^{n_2}(x)$ is a Darboux polynomial of system* (6.4), (6.14) *for arbitrary constants $n_1, n_2 \in \mathbb{Z}^{\geq}$, with characteristic polynomial $n_1\lambda_1(t,x)+n_2\lambda_2(t,x)$.*

Proof As for Statement (6.3.1) of the theorem, one finds that

$$0 = \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\omega_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} - \omega_1 \frac{\mathrm{d}\omega_2}{\mathrm{d}t}}{\omega_2^2}.$$

Taking into account that ω_1 and ω_2 are co-prime and that $\omega_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \omega_1 \frac{\mathrm{d}\omega_2}{\mathrm{d}t}$, one concludes that ω_1 is a factor of $\frac{\mathrm{d}\omega_1}{\mathrm{d}t}$ and ω_2 is a factor of $\frac{\mathrm{d}\omega_2}{\mathrm{d}t}$, with $\lambda_1 = \frac{1}{\omega_1}\frac{\mathrm{d}\omega_1}{\mathrm{d}t}$ and $\lambda_2 = \frac{1}{\omega_2}\frac{\mathrm{d}\omega_2}{\mathrm{d}t}$ being the respective characteristic polynomials; substituting these expressions in $\omega_2 \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \omega_1 \frac{\mathrm{d}\omega_2}{\mathrm{d}t}$, one concludes that $\omega_1\omega_2(\lambda_1 - \lambda_2) = 0$, which shows that $\lambda_1 - \lambda_2 = 0$, because $\omega_1\omega_2$ is not the zero polynomial. As for Statement (6.3.2) of the theorem, in order to show that $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system (6.4), (6.14), compute

$$\begin{aligned}\frac{\mathrm{d}\omega_1^{n_1}\omega_2^{n_2}}{\mathrm{d}t} &= \omega_2^{n_2}\frac{\mathrm{d}\omega_1^{n_1}}{\mathrm{d}t} + \omega_1^{n_1}\frac{\mathrm{d}\omega_2^{n_2}}{\mathrm{d}t} = n_1\omega_1^{n_1-1}\omega_2^{n_2}\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + n_2\omega_1^{n_1}\omega_2^{n_2-1}\frac{\mathrm{d}\omega_2}{\mathrm{d}t}\\ &= (n_1\lambda_1 + n_2\lambda_2)\omega_1^{n_1}\omega_2^{n_2}.\end{aligned}$$

In order to show that all irreducible factors of ω are Darboux polynomials of system (6.4), (6.14), let $\omega = \omega_1^{n_1}\omega_2$, with ω_1 being irreducible and pair ω_1, ω_2 being co-prime. Then,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega_1^{n_1}\omega_2}{\mathrm{d}t} = n_1\omega_1^{n_1-1}\omega_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \omega_1^{n_1}\frac{\mathrm{d}\omega_2}{\mathrm{d}t},$$

which implies (because $\frac{\mathrm{d}\omega}{\mathrm{d}t} = \lambda\omega$)

$$n_1\omega_1^{n_1-1}\omega_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \omega_1^{n_1}\frac{\mathrm{d}\omega_2}{\mathrm{d}t} = \lambda\omega_1^{n_1}\omega_2.$$

From this equality, $\omega_1^{n_1}$ divides $n_1\omega_1^{n_1-1}\omega_2\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \omega_1^{n_1}\frac{\mathrm{d}\omega_2}{\mathrm{d}t}$; now, since ω_1 and ω_2 are co-prime, ω_1 must divide $\frac{\mathrm{d}\omega_1}{\mathrm{d}t}$, with the ratio $\frac{1}{\omega_1}\frac{\mathrm{d}\omega_1}{\mathrm{d}t}$ being the characteristic polynomial of ω_1. □

Remark 6.12 A *greatest common divisor* of polynomials $p_i(x), p_j(x) \in \mathbb{R}$ is a polynomial $h(x) \in \mathbb{R}$ such that:

(6.12.1) h divides p_1 and p_2,
(6.12.2) if $k(x) \in \mathbb{R}$ is another polynomial that divides p_1 and p_2, then k divides h.

A polynomial $h(x) \in \mathbb{R}$ is a *least common multiple* of $p_i(x), p_j(x) \in \mathbb{R}$ if:

(6.12.3) p_i divides h and p_j divides h,
(6.12.4) h divides any polynomial that both p_i and p_j divide.

Similar definitions can be given in case of multiple polynomials $p_1(x), \ldots, p_k(x) \in \mathbb{R}$. A greatest common divisor of some polynomials $p_1(x), \ldots, p_k(x) \in \mathbb{R}$ is denoted by $\mathrm{GCD}(p_1, \ldots, p_k)$; note that $\mathrm{GCD}(p_1, \ldots, p_k) = \frac{p_1 \cdots p_k}{\mathrm{LCM}(p_1, \ldots, p_k)}$, where $\mathrm{LCM}(p_1, \ldots, p_k)$ is a least common multiple of $p_1, \ldots, p_k$. Both $\mathrm{GCD}(p_1, \ldots, p_k)$ and $\mathrm{LCM}(p_1, \ldots, p_k)$ are unique up to multiplication by a constant.

Assume that $\{f_1, \ldots, f_r\}$ is a basis of a Lie algebra of vector functions $f(x) \in \mathbb{R}^n$ over $\mathbb{R}$ and that $r \geq n$. Let

$$\Omega(x) = \big[f_1(x) \; \ldots \; f_r(x)\big],$$

and assume that the generic rank of Ω is n; let $\{p_1, \ldots, p_k\}$ be the set of the determinants of all $n \times n$ minors of Ω.

Theorem 6.4 *Under the above assumptions and positions, polynomial* $\omega(x) = \mathrm{GCD}(p_1(x), \ldots, p_k(x))$ *is a Darboux polynomial of system* (6.4), (6.14).

Proof Assume first that $r = n$. Compute the directional derivative of ω along any one of the vector functions f_i, say f_1, where $\omega = \det(\Omega)$. Taking into account that $L_{f_1} f_j - L_{f_j} f_1 = [f_1, f_j] = \sum_{k=1}^r c_{1,j;k} f_k$, it is found that

$$\begin{aligned}
L_{f_1}\omega &= \det\big([L_{f_1} f_1 \; f_2 \; \ldots \; f_r]\big) + \det\big([f_1 \; L_{f_1} f_2 \; \ldots \; f_r]\big) \\
&\quad + \cdots + \det\big([f_1 \; f_2 \; \ldots \; L_{f_1} f_r]\big) \\
&= \det\big([L_{f_1} f_1 \; f_2 \; \ldots \; f_r]\big) + \det\left(\left[f_1 \; L_{f_2} f_1 + \sum_{k=1}^r c_{1,2;k} f_k \; \ldots \; f_r\right]\right) \\
&\quad + \cdots + \det\left(\left[f_1 \; f_2 \; \ldots \; L_{f_r} f_1 + \sum_{k=1}^r c_{1,r;k} f_k\right]\right) \\
&= \det\left(\left[\frac{\partial f_1}{\partial x} f_1 \; f_2 \; \ldots \; f_r\right]\right) + \det\left(\left[f_1 \; \frac{\partial f_1}{\partial x} f_2 + c_{1,2;2} f_2 \; \ldots \; f_r\right]\right) \\
&\quad + \cdots + \det\left(\left[f_1 \; f_2 \; \ldots \; \frac{\partial f_1}{\partial x} f_r + c_{1,r;r} f_r\right]\right);
\end{aligned}$$

therefore, by the multi-linearity of the determinant, one has

$$L_{f_1}\omega(x) = \left(\mathrm{div}\big(f_1(x)\big) + \sum_{k=2}^r c_{1,k;k}\right)\omega(x).$$

Similarly,

$$L_{f_i}\omega(x) = \left(\operatorname{div}\big(f_i(x)\big) + \sum_{k=1,k\neq i}^{r} c_{i,k;k}\right)\omega(x), \quad i = 1, \dots, r,$$

which implies

$$\frac{d\omega(x)}{dt} = \left(\sum_{i=1}^{r} u_i(t)\left(\operatorname{div}\big(f_i(x)\big) + \sum_{k=1,k\neq i}^{r} c_{i,k;k}\right)\right)\omega(x).$$

For the general case $r > n$, it is sufficient to repeat the same arguments for each $n \times n$ minor of Ω, and then taking the greatest common divisor. □

Remark 6.13 Assume that $r < n$. In this case, a good candidate to be a Darboux polynomial is a greatest common divisor of the determinants of all $r \times r$ minors of $\Omega = [f_1 \ \dots \ f_r]$. As another possibility, one can augment the set $\{f_1, \dots, f_r\}$ with other vector functions $f_{r+1}(x), \dots, f_{\bar{r}}(x) \in \mathbb{R}^n$ such that $\bar{r} \geq n$ and $\{f_1, \dots, f_{\bar{r}}\}$ is a basis of a Lie algebra having $\operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_r\}$ as sub-algebra (this is always possible) and such that $\Omega = [f_1 \ \dots \ f_{\bar{r}}]$ has full generic rank equal to n. It is worth pointing out that such a choice should be judicious, because a generic choice of $f_{r+1}, \dots, f_{\bar{r}}$ would yield a Darboux polynomial equal to 1.

Now, assume $r = n$ and consider any polynomial diffeomorphism $y = \varphi(x)$ with inverse $x = \varphi^{-1}(y)$. Let

$$\tilde{\Omega}(y) = \big[\varphi_* f_1(y) \ \dots \ \varphi_* f_n(y)\big];$$

clearly,

$$\det\big(\tilde{\Omega}(y)\big) = \left(\det\left(\frac{\partial \varphi}{\partial x}\right)\det(\Omega)\right) \circ \varphi^{-1}(y),$$

which shows how the Darboux polynomial computed with this technique is changed by a polynomial diffeomorphism.

Example 6.12 Consider the Lie algebra of vector functions over $\mathbb{R}$ spanned by

$$f_1(x) = \begin{bmatrix} 2 \\ x_1 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} x_1^2 - 2x_2 \\ x_2 x_1 \end{bmatrix}, \qquad f_3(x) = \begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix},$$

satisfying the commutation relations $[f_1, f_2] = f_3$, $[f_1, f_3] = 2f_1$, $[f_2, f_3] = -2f_2$. The determinants of the minors of dimension two of

$$\Omega(x) = \begin{bmatrix} 2 & x_1^2 - 2x_2 & 2x_1 \\ x_1 & x_1 x_2 & 4x_2 \end{bmatrix}$$

are

$$p_1(x) = 2x_2\big(x_1^2 - 4x_2\big), \qquad p_2(x) = -2\big(x_1^2 - 4x_2\big), \qquad p_3(x) = -x_1\big(x_1^2 - 4x_2\big),$$

which yield the Darboux polynomial $\omega(x) = \mathrm{GCD}(p_1(x), p_2(x), p_3(x)) = x_1^2 - 4x_2$. It is worth pointing out that $L_{f_1}\omega(x) = 0$, $L_{f_2}\omega(x) = 2x_1\omega(x)$ and $L_{f_3}\omega(x) = 4\omega(x)$ and, consequently, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = (2x_1u_2 + 4u_3)\omega$.

Example 6.13 Consider the Lie algebra of vector functions over $\mathbb{R}$ spanned by

$$f_1(x) = \begin{bmatrix} x_1^2+1 \\ x_1x_2 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} x_1x_2 \\ x_2^2+1 \end{bmatrix}, \qquad f_3(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix},$$

satisfying the commutation relations $[f_1, f_2] = -f_3$, $[f_1, f_3] = f_2$, $[f_2, f_3] = -f_1$. The determinants of the minors of dimension two of

$$\Omega(x) = \begin{bmatrix} x_1^2+1 & x_1x_2 & -x_2 \\ x_1x_2 & 1+x_2^2 & x_1 \end{bmatrix}$$

are

$$p_1(x) = x_1^2 + x_2^2 + 1, \qquad p_2(x) = x_1\big(x_1^2 + x_2^2 + 1\big), \qquad p_3(x) = x_2\big(x_1^2 + x_2^2 + 1\big),$$

which yield the Darboux polynomial $\omega(x) = \mathrm{GCD}(p_1(x), p_2(x), p_3(x)) = x_1^2 + x_2^2 + 1$. It is worth pointing out that $L_{f_1}\omega(x) = 2x_1\omega(x)$, $L_{f_2}\omega(x) = 2x_2\omega(x)$ and $L_{f_3}\omega(x) = 0$ and, consequently, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = (2x_1u_1 + 2x_2u_2)\omega$.

Remark 6.14 Theorem 6.4 can be applied to all the non-isomorphic Lie algebras listed in Remark 6.8 for which matrix $\Omega(x)$ has full generic rank, thus obtaining:

(6.14.1) $\Omega(x) = \left[\begin{smallmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}\big(x_2 - x_1, (x_2 - x_1)(x_1 + x_2), x_1x_2(x_2 - x_1)\big) = x_2 - x_1$;

(6.14.2) $\Omega(x) = \left[\begin{smallmatrix} 1 & 2x_1 & x_1^2 \\ 0 & x_2 & x_1x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(x_2, x_1x_2, x_1^2x_2) = x_2$;

(6.14.3) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & x_2 & x_2^2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 1$;

(6.14.4) $\Omega(x) = \left[\begin{smallmatrix} 1 & 0 & x_1 \\ 0 & 1 & ax_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(1, ax_2, -x_1) = 1$;

(6.14.5) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & (1-a)x_1 \\ 1 & x_1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(0, -(1-a)x_1, -(1-a)x_1^2) = x_1$;

(6.14.6) $\Omega(x) = \left[\begin{smallmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(1, x_2, -x_1) = 1$;

(6.14.7) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & x_1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 1$;

(6.14.8) $\Omega(x) = \left[\begin{smallmatrix} 1 & 0 & x_1+x_2 \\ 0 & 1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(1, x_2, -x_1 - x_2) = 1$;

(6.14.9) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & x_1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(0, -1, -x_1) = 1$;

(6.14.10) $\Omega(x) = \left[\begin{smallmatrix} 1 & 0 & x_1 \\ 0 & 1 & 0 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(1, 0, -x_1) = 1$;

(6.14.11) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & x_1 \\ 1 & x_1 & x_2 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(0, -x_1, -x_1^2) = x_1$;

(6.14.12) $\Omega(x) = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1 \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 2$, $\mathrm{GCD}(1, x_1, 0) = 1$;

(6.14.13) $\Omega(x) = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & x_1 & p(x_1) \end{smallmatrix}\right]$, $\mathrm{rank}_{\mathscr{K}_2}(\Omega(x)) = 1$.

Example 6.14 Consider the linear system $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + bu$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Such a system can be rewritten as $\frac{\mathrm{d}x}{\mathrm{d}t} = u_1(t) f_1(x) + u_2(t) f_2(x)$, where $f_1(x) = Ax$, $f_2(x) = b$, $u_1(t) = 1$ and $u_2(t) = u(t)$. Consider the Lie algebra over $\mathbb{R}$ spanned by $Ax, b, Ab, \ldots, A^{n-1}b$: the proof that such vector functions span a Lie algebra is very simple, taking into account the Cayley–Hamilton Theorem (see Theorem 3.28.2 of [83]), by $[A^i b, Ab^j] = 0$ and $[A^i b, Ax] = A^{i+1}b$. Let

$$\Omega(x) = [Ax \; b \; Ab \; \ldots \; A^{n-1}b].$$

Since $\det([b \; Ab \; \ldots \; A^{n-1}b])$ of Ω is constant, the only possibility for ω to be non-constant is that pair (A, b) is not controllable. As an example, take $A = \left[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, which yield

$$\Omega(x) = \begin{bmatrix} a_1 x_1 & b_1 & a_1 b_1 \\ a_2 x_2 & b_2 & a_2 b_2 \end{bmatrix}.$$

Condition $0 = \det\left(\left[\begin{smallmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{smallmatrix}\right]\right) = b_1 b_2 (a_2 - a_1)$ yields two possible cases: one of the two entries b_i of b, say b_1, is equal to zero, and the two eigenvalues of A coincide, $a_1 = a_2$. In the first case, one has the Darboux polynomial $\omega(x) = \mathrm{GCD}(0, a_1 a_2 b_2 x_1, a_1 b_2 x_1) = x_1$, i.e., if $x_1(0) = 0$, then $x_1(t) = 0$ for all $t \geq 0$, for any input function $u(t)$; in the second case, one has the Darboux polynomial $\omega(x) = \mathrm{GCD}(0, a_2^2(b_2 x_1 - b_1 x_2), a_2(b_2 x_1 - b_1 x_2)) = b_2 x_1 - b_1 x_2$, i.e., if $b_2 x_1(0) = b_1 x_2(0)$, then $b_2 x_1(t) = b_1 x_2(t)$ for all $t \geq 0$, for any input function $u(t)$.

As in the example above, Darboux polynomials can be used to study the controllability also for nonlinear systems written in the form (6.4), (6.14). If the Lie algebra generated by $\{f_1, \ldots, f_p\}$ has a non-constant Darboux polynomial, then this allow to identify the invariant set $\mathscr{I}_\omega$, if not empty; the inputs u_i do not influence the solution of (6.4), (6.14) along $\mathscr{I}_\omega$.

6.8 The Joint Poincaré–Dulac Normal Form

The problem of finding a diffeomorphism that jointly linearizes a set of nonlinear systems is relatively old one [108]. Such a problem can be relaxed to finding a diffeomorphism such that the transformed systems are in the joint Poincaré–Dulac normal form [34].

Definition 6.5 Let vector functions $f_i(x) \in \mathbb{R}^n$, $i = 1, \ldots, r$, be analytic at $x = 0$, $f_i(0) = 0$, with linear part $A_i x$, where $A_i = \frac{\partial f_i(x)}{\partial x}|_{x=0}$ is semi-simple. Then, $f_i(x) = A_i x + h_i(x)$, $i = 1, \ldots, m$, are in the *joint Poincaré–Dulac normal form* if the following relation holds for each $i \in \{1, \ldots, r\}$:

$$\left[h_i(x), A_j x\right] = 0, \quad j = 1, \ldots, r.$$

For the proof of the following theorem see [34].

Theorem 6.5 *Let vector functions $f_i(x) \in \mathbb{R}^n$, $i = 1, \ldots, r$, be analytic at $x = 0$, $f_i(0) = 0$, with linear part $A_i x$, where $A_i = \frac{\partial f_i(x)}{\partial x}|_{x=0}$ is normal. Assume that $\{f_1, \ldots, f_r\}$ is a basis of a nilpotent Lie algebra $\mathfrak{X}$ of vector functions over $\mathbb{R}$. Then, there exists a formal diffeomorphism $y = \varphi(x)$ such that the push-forwards $\varphi_* f_i$ are in the joint Poincaré–Dulac normal form. Under further convergence conditions, $y = \varphi(x)$ is analytic at $x = 0$.*

Since $\{f_1, \ldots, f_r\}$ is a basis of a finite dimensional nilpotent Lie algebra $\mathfrak{X}$ of vector functions over $\mathbb{R}$, it satisfies the commutation relations $[f_i, f_j] = \sum_{\ell=1}^{r} c_{i,j;\ell} f_\ell$, for some structure constants $c_{i,j;\ell}$; therefore, the linear vector functions $A_1 x, \ldots, A_r x$ satisfy the commutation relations $[A_i x, A_j x] = \sum_{\ell=1}^{r} c_{i,j;\ell} A_\ell x$, for the same structure constants, whence the distribution $\mathscr{D} = \mathrm{span}_{\mathscr{K}_n}\{A_1 x, \ldots, A_r x\}$ is involutive. Now, assume that, about a regular point of such a distribution, its rank is $\hat{r}$. By the Frobenius Theorem 1.9 at p. 21, there exists $n - \hat{r}$ functionally independent functions $I_1(x), \ldots, I_{n-\hat{r}}(x) \in \mathscr{I}_C(A_1 x) \cap \cdots \cap \mathscr{I}_C(A_r x)$, namely joint first integrals of the linear systems $\frac{dx}{dt} = A_i x$, $i = 1, \ldots, r$. Note that the linear parts $A_1 x, \ldots, A_r x$ could be linearly dependent. Let $\{M_0, \ldots, M_{\bar{r}-1}\}$ be a basis of the linear centralizer $\mathscr{L}_c(A_1, \ldots, A_r)$, i.e., of the set of all matrices B that commute under the matrix product with all A_i, $BA_i - A_i B = 0$, $i = 1, \ldots, r$. Then, $f_i(x) = A_i x + h_i(x)$, $i = 1, \ldots, r$, are in the joint Poincaré–Dulac normal form if and only if

$$h_i(x) = \mu_{i,0} M_0 x + \cdots + \mu_{i,\bar{r}-1} M_{\bar{r}-1} x,$$

where $\mu_{i,j} \in \mathscr{I}_C(A_1 x) \cap \cdots \cap \mathscr{I}_C(A_r x)$, and $h_i(x)$ is analytic at $x = 0$, $h_i(0) = 0$, with zero linear part.

Example 6.15 Consider the matrices $A_1 = \mathrm{diag}\{0, 0, 0, 1\}$, $A_2 = \mathrm{diag}\{1, -1, 0, 0\}$ and $A_2 = \mathrm{diag}\{1, 0, 1, 1\}$. Compute the kernel of

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

which is spanned by $[-1\ -1\ 1\ 0]^\top$. Then, set $\mathscr{I}_C(A_1x) \cap \mathscr{I}_C(A_2x) \cap \mathscr{I}_C(A_3x)$ is constituted by the arbitrary functions of $I = \frac{x_3}{x_1x_2}$. A basis of $\mathscr{L}_c(A_1, A_2, A_3)$ is

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, for each $i \in \{1, 2, 3\}$, $f_i(x) = A_i x + h_i(x)$ is in the joint Poincaré–Dulac normal form if $h_i(x) = [x_1\mu_{i,0}\ x_2\mu_{i,1}\ x_3\mu_{i,2}\ x_4\mu_{i,3}]^\top$, $h_i(x)$ is analytic at $x = 0$, $h_i(0) = 0$, with zero linear part, where $\mu_{i,j}$ are arbitrary functions of $\frac{x_3}{x_1x_2}$. Clearly, the only possible h_i, $i = 1, 2, 3$, are obtained by taking $\mu_{i,0} = 0$, $\mu_{i,1} = 0$, $\mu_{i,2} = a_i\frac{x_1x_2}{x_3}$, $\mu_{i,3} = 0$, thus obtaining $h_i(x) = [0\ 0\ a_i x_1 x_2\ 0]^\top$, where a_i is an arbitrary real, $i = 1, 2, 3$. It is worth pointing out that the three vector functions

$$f_1(x) = \begin{bmatrix} 0 \\ 0 \\ a_1x_1x_2 \\ x_4 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} x_1 \\ -x_2 \\ a_2x_1x_2 \\ 0 \end{bmatrix}, \qquad f_3(x) = \begin{bmatrix} x_1 \\ 0 \\ x_3 + a_3x_1x_2 \\ x_4 \end{bmatrix}$$

span an Abelian Lie algebra over $\mathbb{R}$.

6.9 The Exponential Notation

The formalism introduced in this section (similar to the formalisms used in [1]) has the advantage of simplifying the computations related with the solution of nonlinear differential equations, and involving flows associated with vector functions and the Lie brackets.

Given a vector function $f(x) \in \mathbb{R}^n$, $f = [f_1\ \dots\ f_n]^\top$, the *vector field* X_f associated with f is defined as $X_f := f_1\frac{\partial}{\partial x_1} + \dots + f_n\frac{\partial}{\partial x_n}$. Given a scalar function $h(x) \in \mathbb{R}$, the Lie derivative of h by f, $L_f h$, can also be denoted by $X_f h = f_1\frac{\partial h}{\partial x_1} + \dots + f_n\frac{\partial h}{\partial x_n}$; similarly, given a vector function $h(x) \in \mathbb{R}^n$, the vector field associated with the vector function $L_f h$ is denoted by $X_f h = (L_f h_1)\frac{\partial}{\partial x_1} + \dots + (L_f h_n)\frac{\partial}{\partial x_n}$. Since $[f, g] = L_f g - L_g f$, $g(x) \in \mathbb{R}^n$, then it is natural to define the Lie bracket of the vector fields X_f and X_g associated with f and g, respectively, as $[X_f, X_g] := X_f X_g - X_g X_f$, which is again a vector field. Given a vector field X_f associated with a vector function f, the flow $\Phi_f(t, x)$ associated with f is represented by $e^{tX_f}x$: since, by Theorem 3.4 at p. 61, relation $[f, f] = 0$

implies $L_f\Phi_f(t,x) = \frac{\partial\Phi_f(t,x)}{\partial x}f(x) = f\circ\Phi_f(t,x)$, the notation $X_f\mathrm{e}^{tX_f}x$ denotes $\frac{\partial\Phi_f(t,x)}{\partial x}f(x)$ (whence, $X_f\mathrm{e}^{-tX_f}x$ denotes $(\frac{\partial\Phi_f(t,x)}{\partial x})^{-1}f(x)$) and the notation $\mathrm{e}^{tX_f}X_fx$ denotes $f\circ\Phi_f(t,x)$, from which the formal property that X_f and e^{tX_f} commute in the product, $X_f\mathrm{e}^{tX_f}x = \mathrm{e}^{tX_f}X_fx$, i.e., $\mathrm{e}^{-tX_f}X_f\mathrm{e}^{tX_f}x = \mathrm{e}^{tX_f}X_f\mathrm{e}^{-tX_f}x = X_fx$. Similarly, $\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}x$ represents $\Phi_g(t,\cdot)\circ\Phi_f(t,x)$; for instance, if $f(x) = Ax$ and $g(x) = Bx$, then $\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}x$ represents $\mathrm{e}^{Bt}\mathrm{e}^{At}x$ (note the inversion of ordering). Similarly, if $f(x) = Ax$ and $g(x) = Bx$, then $\mathrm{e}^{tX_g}X_fx$ is the vector field that represents the vector function $A\mathrm{e}^{Bt}x$ and $X_f\mathrm{e}^{tX_g}x$ is the vector field that represents the vector function $\mathrm{e}^{Bt}Ax$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_f(t,x) = f\circ\Phi_f(t,x) = \frac{\partial\Phi_f(t,x)}{\partial x}f(x)$, one derives the formal property $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tX_f}x = X_f\mathrm{e}^{tX_f}x = \mathrm{e}^{tX_f}X_fx$, which justify the use of the exponential notation e^{tX_f} for the formal series $\mathrm{e}^{tX_f} = X_f^0 + \sum_{i=1}^{+\infty}\frac{t^i}{i!}X_f^i$, where $X_f^i = \underbrace{X_fX_f\cdots X_f}_{i\ \text{times}}$, and X_f^0 is the identity vector field defined by $X_f^0x = x$. In this way, $\mathrm{e}^{tX_f}|_{t=0} = X_f^0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tX_f} = X_f\mathrm{e}^{tX_f} = \mathrm{e}^{tX_f}X_f$. Taking the derivative with respect to t of $\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x$, it is found that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x\big) &= \mathrm{e}^{tX_g}X_gX_f\mathrm{e}^{-tX_g}x - \mathrm{e}^{-tX_g}X_fX_g\mathrm{e}^{tX_g}x\\ &= -\mathrm{e}^{tX_g}(X_fX_g - X_gX_f)\mathrm{e}^{-tX_g}x\\ &= -\mathrm{e}^{tX_g}[X_f, X_g]\mathrm{e}^{-tX_g}x.\end{aligned}$$

It is worth pointing out that if $f(x) = Ax$ and $g(x) = Bx$, then $\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x$ represents $\mathrm{e}^{-Bt}A\mathrm{e}^{Bt}x$, whence $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-Bt}A\mathrm{e}^{Bt}x) = -\mathrm{e}^{-Bt}[A,B]\mathrm{e}^{Bt}x$, as expected.

Taking into account that $\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x|_{t=0} = X_fx$, this yields the formal property

$$\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x = X_fx \quad\Longleftrightarrow\quad [X_f, X_g] = 0,$$

which is equivalent to the property stated in Theorem 3.4:

$$\left(\frac{\partial\Phi_g(t,x)}{\partial x}\right)^{-1} f\circ\Phi_g(t,x) = f \quad\Longleftrightarrow\quad [f,g] = 0.$$

Similarly, one has

$$\begin{aligned}&\big(\mathrm{e}^{tX_f}\mathrm{e}^{tX_g} - \mathrm{e}^{tX_g}\mathrm{e}^{tX_f}\big)\big|_{t=0} = 0,\\ &\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{tX_f}\mathrm{e}^{tX_g} - \mathrm{e}^{tX_g}\mathrm{e}^{tX_f}\big)\right)\bigg|_{t=0}\\ &\quad= \big(X_f\mathrm{e}^{tX_f}\mathrm{e}^{tX_g} + \mathrm{e}^{tX_f}X_g\mathrm{e}^{tX_g} - X_g\mathrm{e}^{tX_g}\mathrm{e}^{tX_f} - \mathrm{e}^{tX_g}X_f\mathrm{e}^{tX_f}\big)\big|_{t=0}\\ &\quad= X_f + X_g - X_g - X_f = 0,\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}-\mathrm{e}^{tX_g}\mathrm{e}^{tX_f}\right)\right)\Big|_{t=0}\\
&=\left(X_f^2\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}+X_f\mathrm{e}^{tX_f}X_g\mathrm{e}^{tX_g}+X_f\mathrm{e}^{tX_f}X_g\mathrm{e}^{tX_g}+\mathrm{e}^{tX_f}X_g^2\mathrm{e}^{tX_g}\right.\\
&\quad\left.-X_g^2\mathrm{e}^{tX_g}\mathrm{e}^{tX_f}-X_g\mathrm{e}^{tX_g}X_f\mathrm{e}^{tX_f}-X_g\mathrm{e}^{tX_g}X_f\mathrm{e}^{tX_f}-\mathrm{e}^{tX_g}X_f^2\mathrm{e}^{tX_f}\right)\big|_{t=0}\\
&=X_f^2+X_fX_g+X_fX_g+X_g^2-X_g^2-X_gX_f-X_gX_f-X_f^2\\
&=2[X_f,X_g],
\end{aligned}$$

which shows the formal property

$$\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}x-\mathrm{e}^{tX_g}\mathrm{e}^{tX_f}x=t^2[X_f,X_g]x+O\left(t^3\right),$$

where $O(t^3)$ denotes terms of order higher than or equal to 3; this is equivalent to the formula:

$$\Phi_g(t,\cdot)\circ\Phi_f(t,x)-\Phi_f(t,\cdot)\circ\Phi_g(t,x)=t^2[f,g]+O\left(t^3\right),$$

which shows that $[f,g]$ is a "measure" of how much $\Phi_f(t,x)$ and $\Phi_g(t,x)$ fail to commute.

As above, it is easy to compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}\right)=\mathrm{e}^{tX_g}[X_g,X_f]\mathrm{e}^{-tX_g}\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{tX_g}[X_g,X_f]\mathrm{e}^{-tX_g}\right)=\mathrm{e}^{tX_g}\left[X_g,[X_g,X_f]\right]\mathrm{e}^{-tX_g}\\
&\frac{\mathrm{d}^h}{\mathrm{d}t^h}\left(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}\right)=\mathrm{e}^{tX_g}\underbrace{\left[X_g,\ldots\left[X_g,[X_g,X_f]\right]\ldots\right]}_{h\text{ times}}\mathrm{e}^{-tX_g},\quad h\geq 2.
\end{aligned}$$

Hence, since

$$\begin{aligned}
&\left(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}\right)\big|_{t=0}=X_f,\\
&\left(\frac{\mathrm{d}^h}{\mathrm{d}t^h}\left(\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}\right)\right)\Big|_{t=0}=\underbrace{\left[X_g,\ldots\left[X_g,[X_g,X_f]\right]\ldots\right]}_{h\text{ times}},\quad h\geq 1,
\end{aligned}$$

taking the Taylor series expansion of $\mathrm{e}^{tX_g}X_f\mathrm{e}^{-tX_g}x$ with respect to t, one obtains the following formula known as the *Hadamard Lemma*:

$$\mathrm{e}^{tX_g} X_f \mathrm{e}^{-tX_g} = X_f + t[X_g, X_f] + \frac{t^2}{2!}\big[X_g, [X_g, X_f]\big] + \frac{t^3}{3!}\big[X_g, \big[X_g, [X_g, X_f]\big]\big] + \cdots, \tag{6.18}$$

which is equivalent to the formula (3.69) with τ replaced by t. Let $X_{\mathfrak{X}}$ be the finite dimensional Lie algebra of the vector fields X_f associated with the vector functions $f \in \mathfrak{X}$, where $\mathfrak{X}$ is a Lie algebra of meromorphic vector functions over $\mathbb{R}$. By (6.18), if $X_f, X_g \in X_{\mathfrak{X}}$, then $\mathrm{e}^{tX_g} X_f \mathrm{e}^{-tX_g} \in X_{\mathfrak{X}}$. In particular, if $\{f_1, \ldots, f_r\}$ is a basis of $\mathfrak{X}$, and $\{X_{f_1}, \ldots, X_{f_r}\}$ is the corresponding basis of $X_{\mathfrak{X}}$, then $\mathfrak{X}$ and $X_{\mathfrak{X}}$ are isomorphic and, in particular, they are described by the same structure constants. Hence, if $X_f, X_g \in X_{\mathfrak{X}}$, then $\mathrm{e}^{tX_g} X_f \mathrm{e}^{-tX_g} = a_1(t) X_{f_1} + \cdots + a_r(t) X_{f_r}$, where the scalar functions $a_1(t), \ldots, a_r(t) \in \mathbb{R}$ only depend on the structure constants representing the Lie algebra: they do not depend on the particular vector fields used for the representation of the Lie algebra, and particularly they do not depend on the local coordinates chosen to represent the vector fields.

The following examples show how the scalar functions $a_1(t), \ldots, a_r(t) \in \mathbb{R}$ can be computed in practice.

Example 6.16 Assume that $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2\}$, where $[f_1, f_2] = f_1$ and f_1, f_2 are linearly independent over $\mathbb{R}$. First, compute $\mathrm{e}^{tX_{f_2}} X_{f_1} \mathrm{e}^{-tX_{f_2}}$. By a repeated substitution of $[X_{f_2}, X_{f_1}] = -X_{f_1}$ in (6.18), one obtains

$$\begin{aligned}
\mathrm{e}^{tX_{f_2}} X_{f_1} \mathrm{e}^{-tX_{f_2}} &= X_{f_1} + t[X_{f_2}, X_{f_1}] + \frac{t^2}{2!}\big[X_{f_2}, [X_{f_2}, X_{f_1}]\big] \\
&\quad + \frac{t^3}{3!}\big[X_{f_2}, \big[X_{f_2}, [X_{f_2}, X_{f_1}]\big]\big] + \cdots \\
&= X_{f_1} - tX_{f_1} + \frac{t^2}{2!} X_{f_1} - \frac{t^3}{3!} X_{f_1} + \cdots \\
&= \mathrm{e}^{-t} X_{f_1}.
\end{aligned}$$

A second approach consists in taking the derivative with respect to t of the equality $\mathrm{e}^{tX_{f_2}} X_{f_1} \mathrm{e}^{-tX_{f_2}} = a_1 X_{f_1} + a_2 X_{f_2}$, where $a_1(t), a_2(t) \in \mathbb{R}$:

$$\mathrm{e}^{tX_{f_2}} [X_{f_2}, X_{f_1}] \mathrm{e}^{-tX_{f_2}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2},$$

which yields

$$-\mathrm{e}^{tX_{f_2}} X_{f_1} \mathrm{e}^{-tX_{f_2}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2},$$

namely

$$-a_1 X_{f_1} - a_2 X_{f_2} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2}.$$

This equation, by the linear independence of X_{f_1}, X_{f_2} over $\mathbb{R}$, yields the differential equations

$$\frac{da_1}{dt} = -a_1, \qquad \frac{da_2}{dt} = -a_2;$$

from $X_{f_1} = (e^{tX_{f_2}} X_{f_1} e^{-tX_{f_2}})|_{t=0} = a_1(0)X_{f_1} + a_2(0)X_{f_2}$, one has $a_1(0) = 1$ and $a_2(0) = 0$. Therefore, $a_1(t) = e^{-t}$ and $a_2(t) = 0$. A slight modification of the second approach is based on the fact that the solution a_1, a_2 of equation $e^{tX_{f_2}} X_{f_1} e^{-tX_{f_2}} = a_1 X_{f_1} + a_2 X_{f_2}$ is independent of the particular representation of the Lie algebra. Hence, one can choose a matrix representation $\mathrm{span}_{\mathbb{R}}\{M_1, M_2\}$ of $\mathfrak{X}$, with the requirement that M_1 and M_2 are linearly independent. For instance, in this case, one can choose the adjoint matrix representation:

$$M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The term $e^{-tM_2} M_1 e^{tM_2}$, which is a representation of $e^{tX_{f_2}} X_{f_1} e^{-tX_{f_2}}$, can be computed explicitly,

$$e^{-tM_2} M_1 e^{tM_2} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e^{-t} & 0 \end{bmatrix},$$

and therefore the scalar functions a_1, a_2 can be computed by solving the linear equation $e^{-tM_2} M_1 e^{tM_2} = a_1 M_1 + a_2 M_2$,

$$\begin{bmatrix} 0 & 0 \\ e^{-t} & 0 \end{bmatrix} = a_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_2 & 0 \\ a_1 & 0 \end{bmatrix},$$

which has the unique solution $a_1 = e^{-t}$ and $a_2 = 0$. Now, compute $e^{tX_{f_1}} X_{f_2} e^{-tX_{f_1}}$. By a repeated substitution of $[X_{f_1}, X_{f_2}] = X_{f_1}$ in (6.18), one obtains

$$\begin{aligned} e^{tX_{f_1}} X_{f_2} e^{-tX_{f_1}} &= X_{f_2} + t[X_{f_1}, X_{f_2}] + \frac{t^2}{2!}[X_{f_1}, [X_{f_1}, X_{f_2}]] \\ &\quad + \frac{t^3}{3!}[X_{f_1}, [X_{f_1}, [X_{f_1}, X_{f_2}]]] + \cdots \\ &= X_{f_2} + tX_{f_1}. \end{aligned}$$

The same result can be computed by taking the derivative with respect to t of $e^{tX_{f_1}} X_{f_2} e^{-tX_{f_1}} = a_1 X_{f_1} + a_2 X_{f_2}$, where $a_1(t), a_2(t) \in \mathbb{R}$:

$$e^{tX_{f_1}} [X_{f_1}, X_{f_2}] e^{-tX_{f_1}} = \frac{da_1}{dt} X_{f_1} + \frac{da_2}{dt} X_{f_2},$$

which yields

$$e^{tX_{f_1}} X_{f_1} e^{-tX_{f_1}} = \frac{da_1}{dt} X_{f_1} + \frac{da_2}{dt} X_{f_2},$$

namely

$$X_{f_1} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2}.$$

This equation, by the linear independence of X_{f_1}, X_{f_2} over $\mathbb{R}$, yields the differential equations

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}a_2}{\mathrm{d}t} = 0;$$

similarly, from $X_{f_2} = (\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}})|_{t=0} = a_1(0)X_{f_1} + a_2(0)X_{f_2}$, one has $a_1(0) = 0$ and $a_2(0) = 1$. Hence, $a_1(t) = t$ and $a_2(t) = 1$. Finally, the term $\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1}$, which is a representation of $\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}}$, with M_1, M_2 as above, can be computed explicitly,

$$\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1} = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix},$$

and therefore the scalar functions a_1, a_2 can be computed by solving the linear equation $\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1} = a_1 M_1 + a_2 M_2$,

$$\begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix} = a_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_2 & 0 \\ a_1 & 0 \end{bmatrix},$$

which has the unique solution $a_1 = t$ and $a_2 = 1$.

Example 6.17 Consider again the split three-dimensional simple Lie algebra introduced in Example 6.3, $\mathfrak{X} = \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$, where $[f_1, f_2] = 2f_1$, $[f_1, f_3] = f_2$ and $[f_2, f_3] = 2f_3$, where f_1, f_2, f_3 are linearly independent. A matrix representation of $\mathfrak{X}$ is given by the adjoint matrix representation:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.19}$$

Clearly, $\mathrm{e}^{tX_{f_i}} X_{f_i} \mathrm{e}^{-tX_{f_i}} = X_{f_i}$, $i = 1, 2, 3$. The three procedures outlined in Example 6.16 can be applied to compute $\mathrm{e}^{tX_{f_i}} X_{f_j} \mathrm{e}^{-tX_{f_i}}$, $i \neq j$.

(1) Computation of $\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}}$. By a repeated substitution of $[X_{f_1}, X_{f_2}] = 2X_{f_1}$ in (6.18), one obtains:

$$\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}} = X_{f_2} + t[X_{f_1}, X_{f_2}] + \frac{t^2}{2!}\big[X_{f_1}, [X_{f_1}, X_{f_2}]\big]$$
$$+ \frac{t^3}{3!}\big[X_{f_1}, \big[X_{f_1}, [X_{f_1}, X_{f_2}]\big]\big] + \cdots$$

$$= X_{f_2} + 2t X_{f_1}.$$

The same result can be computed by taking the derivative with respect to t of $\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}} = a_1 X_{f_1} + a_2 X_{f_2} + a_3 X_{f_3}$, where $a_1(t), a_2(t), a_3(t) \in \mathbb{R}$:

$$\mathrm{e}^{tX_{f_1}} [X_{f_1}, X_{f_2}] \mathrm{e}^{-tX_{f_1}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3},$$

which yields

$$2\mathrm{e}^{tX_{f_1}} X_{f_1} \mathrm{e}^{-tX_{f_1}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3},$$

namely

$$2X_{f_1} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3}.$$

This equation, by the linear independence of $X_{f_1}, X_{f_2}, X_{f_3}$ over $\mathbb{R}$, yields the differential equations

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = 2, \qquad \frac{\mathrm{d}a_2}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}a_3}{\mathrm{d}t} = 0;$$

similarly, from $X_{f_2} = (\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}})|_{t=0} = a_1(0) X_{f_1} + a_2(0) X_{f_2} + a_3(0) X_{f_3}$, one has $a_1(0) = 0$, $a_2(0) = 1$ and $a_3(0) = 0$. Therefore, $a_1(t) = 2t$, $a_2(t) = 1$ and $a_3(t) = 0$. Finally, the term $\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1}$, which is a representation of $\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}}$, with M_1, M_2 in (6.19), can be computed explicitly,

$$\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1} = \begin{bmatrix} 1 & 0 & 0 \\ -2t & 1 & 0 \\ t^2 & -t & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2 & t & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 4t & 0 & 0 \\ 0 & 2t & 2 \end{bmatrix},$$

and therefore the scalar functions a_1, a_2, a_3 can be computed by solving the linear equation $\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1} = a_1 M_1 + a_2 M_2 + a_3 M_3$,

$$\begin{bmatrix} -2 & 0 & 0 \\ 4t & 0 & 0 \\ 0 & 2t & 2 \end{bmatrix} = \begin{bmatrix} -2a_2 & -a_3 & 0 \\ 2a_1 & 0 & -2a_3 \\ 0 & a_1 & 2a_2 \end{bmatrix},$$

which has the unique solution $a_1 = 2t$, $a_2 = 1$ and $a_3 = 0$.

(2) Computation of $\mathrm{e}^{tX_{f_1}} X_{f_3} \mathrm{e}^{-tX_{f_1}}$. By a repeated substitution of $[X_{f_1}, X_{f_3}] = X_{f_2}$ and $[X_{f_1}, X_{f_2}] = 2X_{f_1}$ in (6.18), one obtains:

$$\begin{aligned}\mathrm{e}^{tX_{f_1}} X_{f_3} \mathrm{e}^{-tX_{f_1}} &= X_{f_3} + t[X_{f_1}, X_{f_3}] + \frac{t^2}{2!}\big[X_{f_1}, [X_{f_1}, X_{f_3}]\big] \\ &\quad + \frac{t^3}{3!}\big[X_{f_1}, \big[X_{f_1}, [X_{f_1}, X_{f_3}]\big]\big] + \cdots\end{aligned}$$

$$= X_{f_3} + tX_{f_2} + t^2 X_{f_1}.$$

The same result can be computed by taking the derivative with respect to t of $\mathrm{e}^{tX_{f_1}} X_{f_3} \mathrm{e}^{-tX_{f_1}} = a_1 X_{f_1} + a_2 X_{f_2} + a_3 X_{f_3}$, where $a_1(t), a_2(t), a_3(t) \in \mathbb{R}$:

$$\mathrm{e}^{tX_{f_1}} [X_{f_1}, X_{f_3}] \mathrm{e}^{-tX_{f_1}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3},$$

which yields

$$\mathrm{e}^{tX_{f_1}} X_{f_2} \mathrm{e}^{-tX_{f_1}} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3},$$

namely

$$X_{f_2} + 2tX_{f_1} = \frac{\mathrm{d}a_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}a_2}{\mathrm{d}t} X_{f_2} + \frac{\mathrm{d}a_3}{\mathrm{d}t} X_{f_3}.$$

This equation, by the linear independence of $X_{f_1}, X_{f_2}, X_{f_3}$ over $\mathbb{R}$, yields the differential equations

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = 2t, \qquad \frac{\mathrm{d}a_2}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}a_3}{\mathrm{d}t} = 0;$$

similarly, from $X_{f_3} = (\mathrm{e}^{tX_{f_1}} X_{f_3} \mathrm{e}^{-tX_{f_1}})|_{t=0} = a_1(0)X_{f_1} + a_2(0)X_{f_2} + a_3(0)X_{f_3}$, one has $a_1(0) = 0$, $a_2(0) = 0$ and $a_3(0) = 1$. Therefore, $a_1(t) = t^2$, $a_2(t) = t$ and $a_3(t) = 1$. Finally, the term $\mathrm{e}^{-tM_1} M_3 \mathrm{e}^{tM_1}$, which is a representation of $\mathrm{e}^{tX_{f_1}} X_{f_3} \mathrm{e}^{-tX_{f_1}}$, with M_1, M_3 in (6.19), can be computed explicitly,

$$\begin{aligned} \mathrm{e}^{-tM_1} M_3 \mathrm{e}^{tM_1} &= \begin{bmatrix} 1 & 0 & 0 \\ -2t & 1 & 0 \\ t^2 & -t & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2 & t & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2t & -1 & 0 \\ 2t^2 & 0 & -2 \\ 0 & t^2 & 2t \end{bmatrix}, \end{aligned}$$

and therefore the scalar functions a_1, a_2, a_3 can be computed by solving the linear equation $\mathrm{e}^{-tM_1} M_2 \mathrm{e}^{tM_1} = a_1 M_1 + a_2 M_2 + a_3 M_3$,

$$\begin{bmatrix} -2t & -1 & 0 \\ 2t^2 & 0 & -2 \\ 0 & t^2 & 2t \end{bmatrix} = \begin{bmatrix} -2a_2 & -a_3 & 0 \\ 2a_1 & 0 & -2a_3 \\ 0 & a_1 & 2a_2 \end{bmatrix},$$

which has the unique solution $a_1 = t^2$, $a_2 = t$ and $a_3 = 1$. By applying the same methods, it is possible to compute:

$$\begin{aligned} &\mathrm{e}^{tX_{f_2}} X_{f_1} \mathrm{e}^{-tX_{f_2}} = \mathrm{e}^{-2t} X_{f_1}, \qquad \mathrm{e}^{tX_{f_2}} X_{f_3} \mathrm{e}^{-tX_{f_2}} = \mathrm{e}^{2t} X_{f_3}, \\ &\mathrm{e}^{tX_{f_3}} X_{f_1} \mathrm{e}^{-tX_{f_3}} = X_{f_1} - tX_{f_2} + t^2 X_{f_3}, \qquad \mathrm{e}^{tX_{f_3}} X_{f_2} \mathrm{e}^{-tX_{f_3}} = X_{f_2} - 2tX_{f_3}. \end{aligned}$$

6.10 The Wei–Norman Equations

Consider the time-varying system (6.4), (6.14), where $p = r$ and $\{f_1, \ldots, f_r\}$ is a basis of a Lie algebra $\mathfrak{X}$ over $\mathbb{R}$. Let $X_f = u_1 X_{f_1} + \cdots + u_r X_{f_r}$ be the vector field associated with the vector function $f(t,x) = u_1(t) f_1(x) + \cdots + u_r(t) f_r(x)$. The goal here is to express the solution of system (6.4), (6.14) from the initial condition $x(0) = x_0$ in the form $x(t) = \Phi_{f_r}(\gamma_r(t), \cdot) \circ \cdots \circ \Phi_{f_2}(\gamma_2(t), \cdot) \circ \Phi_{f_1}(\gamma_1(t), x_0)$, where $\gamma_1(t), \ldots, \gamma_r(t) \in \mathbb{R}$ are functions of time to be computed, satisfying $\gamma_i(0) = 0$. Clearly, using the exponential notation, such an expression can be found if and only if

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}}\right)\\ &= \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}} (u_1 X_{f_1} + u_2 X_{f_2} + \cdots + u_r X_{f_r}).\end{aligned} \tag{6.20}$$

From (6.20), it follows that

$$\begin{aligned}&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} X_{f_1} \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}} + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \mathrm{e}^{\gamma_1 X_{f_1}} X_{f_2} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}} + \cdots\\ &\qquad + \frac{\mathrm{d}\gamma_r}{\mathrm{d}t} \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots X_{f_r} \mathrm{e}^{\gamma_r X_{f_r}}\\ &= \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}} (u_1 X_{f_1} + u_2 X_{f_2} + \cdots + u_r X_{f_r}),\end{aligned}$$

from which by left multiplying for $\mathrm{e}^{-\gamma_r X_{f_r}} \cdots \mathrm{e}^{-\gamma_2 X_{f_2}} \mathrm{e}^{-\gamma_1 X_{f_1}}$, one concludes that

$$\begin{aligned}&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \mathrm{e}^{-\gamma_r X_{f_r}} \cdots \mathrm{e}^{-\gamma_2 X_{f_2}} \mathrm{e}^{-\gamma_1 X_{f_1}} X_{f_1} \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}}\\ &\qquad + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \mathrm{e}^{-\gamma_r X_{f_r}} \cdots \mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_2} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}} + \cdots + \frac{\mathrm{d}\gamma_r}{\mathrm{d}t} \mathrm{e}^{-\gamma_r X_{f_r}} X_{f_r} \mathrm{e}^{\gamma_r X_{f_r}}\\ &= u_1 X_{f_1} + u_2 X_{f_2} + \cdots + u_r X_{f_r}.\end{aligned}$$

Taking into account that

$$\begin{aligned}&\mathrm{e}^{-\gamma_r X_{f_r}} \cdots \mathrm{e}^{-\gamma_2 X_{f_2}} \mathrm{e}^{-\gamma_1 X_{f_1}} X_{f_1} \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}}\\ &\quad = a_{1,1} X_{f_1} + a_{1,2} X_{f_2} + \cdots + a_{1,r} X_{f_r},\\ &\mathrm{e}^{-\gamma_r X_{f_r}} \cdots \mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_2} \mathrm{e}^{\gamma_2 X_{f_2}} \cdots \mathrm{e}^{\gamma_r X_{f_r}}\\ &\quad = a_{2,1} X_{f_1} + a_{2,2} X_{f_2} + \cdots + a_{2,r} X_{f_r},\\ &\quad \vdots\\ &\mathrm{e}^{-\gamma_r X_{f_r}} X_{f_r} \mathrm{e}^{\gamma_r X_{f_r}}\\ &\quad = a_{r,1} X_{f_1} + a_{r,2} X_{f_2} + \cdots + a_{r,r} X_{f_r},\end{aligned}$$

where the functions $a_{i,j}(\gamma_1, \ldots, \gamma_r)$, which can be computed as in Sect. 6.9, only depend on the structure constants of the Lie algebra $\mathfrak{X}$, and not on its particular representation given by $\{X_{f_1}, \ldots, X_{f_r}\}$, one has

$$\sum_{i=1}^{r} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \sum_{j=1}^{r} a_{i,j} X_{f_j} = \sum_{j=1}^{r} u_j X_{f_j};$$

hence,

$$\sum_{j=1}^{r} \left(\sum_{i=1}^{r} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} a_{i,j} \right) X_{f_j} = \sum_{j=1}^{r} u_j X_{f_j},$$

which, taking into account the linear independence of $X_{f_1}, \ldots, X_{f_r}$ over $\mathbb{R}$, yields the differential equations:

$$\sum_{i=1}^{r} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} a_{i,j} = u_j, \quad j = 1, \ldots, r. \tag{6.21}$$

System (6.21) can be rewritten in compact form as

$$A^{\top}(\gamma) \frac{\mathrm{d}\gamma}{\mathrm{d}t} = u,$$

where $a_{i,j}$ is the (i, j)th entry of $A(\gamma)$, $\gamma = [\gamma_1 \ \ldots \ \gamma_r]^{\top}$ and $u = [u_1 \ \ldots \ u_r]^{\top}$. In [120, 121], it is proven that matrix $A(\gamma)$ is invertible for small $t \geq 0$, whence for small γ; the *Wei–Norman equations* are given in vector form by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = A^{-\top}(\gamma) u. \tag{6.22}$$

This means that for small $t \geq 0$ the solution of (6.4), (6.14), from the initial condition $x(0) = x_0$, is given by

$$x(t) = \varPhi_{f_r}(\gamma_r, \cdot) \circ \varPhi_{f_{r-1}}(\gamma_{r-1}, \cdot) \circ \cdots \circ \varPhi_{f_1}(\gamma_1, x_0),$$

where $\gamma = [\gamma_1 \ \ldots \ \gamma_r]^{\top}$ is the solution of (6.22) from the initial condition $\gamma(0) = 0$. In [121], it is proven that, when $\mathfrak{X}$ is solvable, there exists a choice of the basis of its representation such that $A(\gamma)$ is invertible for all $t \geq 0$.

Remark 6.15 If all vector functions $f_1, \ldots, f_r$ are pairwise commuting, $[f_i, f_j] = 0$, then the Wei–Norman equation (6.22) becomes $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = u$.

Example 6.18 Consider the Lie algebra having $\{f_1, f_2\}$ as basis, with $[f_1, f_2] = f_1$. From

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \right) = \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} (u_1 X_{f_1} + u_2 X_{f_2}),$$

it can be found

$$\frac{d\gamma_1}{dt} X_{f_1} e^{\gamma_1 X_{f_1}} e^{\gamma_2 X_{f_2}} + \frac{d\gamma_2}{dt} e^{\gamma_1 X_{f_1}} X_{f_2} e^{\gamma_2 X_{f_2}}$$
$$= e^{\gamma_1 X_{f_1}} e^{\gamma_2 X_{f_2}} (u_1 X_{f_1} + u_2 X_{f_2}). \tag{6.23}$$

From (6.23), by left multiplication for $e^{-\gamma_2 X_{f_2}} e^{-\gamma_1 X_{f_1}}$, one has

$$\frac{d\gamma_1}{dt} e^{-\gamma_2 X_{f_2}} e^{-\gamma_1 X_{f_1}} X_{f_1} e^{\gamma_1 X_{f_1}} e^{\gamma_2 X_{f_2}} + \frac{d\gamma_2}{dt} e^{-\gamma_2 X_{f_2}} X_{f_2} e^{\gamma_2 X_{f_2}} = (u_1 X_{f_1} + u_2 X_{f_2}).$$

Since $e^{-\gamma_1 X_{f_1}} X_{f_1} e^{\gamma_1 X_{f_1}} = X_{f_1}$, $e^{-\gamma_2 X_{f_2}} X_{f_1} e^{\gamma_2 X_{f_2}} = e^{\gamma_2} X_{f_1}$ and $e^{-\gamma_2 X_{f_2}} X_{f_2} e^{\gamma_2 X_{f_2}} = X_{f_2}$, it can be found that

$$\frac{d\gamma_1}{dt} e^{\gamma_2} X_{f_1} + \frac{d\gamma_2}{dt} X_{f_2} = u_1 X_{f_1} + u_2 X_{f_2},$$

from which one concludes that

$$\frac{d\gamma_1}{dt} e^{\gamma_2} = u_1, \qquad \frac{d\gamma_2}{dt} = u_2,$$

namely the Wei–Norman equations are obtained

$$\frac{d\gamma_1}{dt} = e^{-\gamma_2} u_1,$$
$$\frac{d\gamma_2}{dt} = u_2.$$

By integration, it is found that

$$\gamma_1(t) = \int_0^t \exp\left(-\int_0^\tau u_2(\theta)\, d\theta\right) u_1(\tau)\, d\tau,$$
$$\gamma_2(t) = \int_0^t u_2(\tau)\, d\tau.$$

For instance, letting $u_1(t) = t$ and $u_2(t) = 1$, one computes $\gamma_1(t) = -e^{-t} t - e^{-t} + 1$ and $\gamma_2(t) = t$. As a particular example, the solution of (6.4), (6.14), with

$$f_1(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}, \qquad u_1(t) = t, \qquad u_2(t) = 1,$$

for which $[f_1, f_2] = f_1$, is given by $x(t) = \Phi_{f_2}(\gamma_2, \cdot) \circ \Phi_{f_1}(\gamma_1, x_0)$, where $\Phi_{f_1}(\gamma_1, x) = \left[\begin{smallmatrix} x_1 \\ \gamma_1 x_1 + x_2 \end{smallmatrix}\right]$ and $\Phi_{f_2}(\gamma_2, x) = \left[\begin{smallmatrix} e^{-\gamma_2} x_1 \\ x_2 \end{smallmatrix}\right]$, namely

$$x(t) = \begin{bmatrix} e^{-\gamma_2} x_{0,1} \\ \gamma_1 x_{0,1} + x_{0,2} \end{bmatrix}\Bigg|_{\gamma_1 = -e^{-t} t - e^{-t} + 1, \gamma_2 = t} = \begin{bmatrix} e^{-t} x_{0,1} \\ (-e^{-t} t - e^{-t} + 1) x_{0,1} + x_{0,2} \end{bmatrix}.$$

Example 6.19 Let $g(x) = [x_1\ 2x_2]^\top$. Consider the set $\mathfrak{X}$ of all vector functions f being homogeneous of degree 0 with respect to g, $[f, g] = 0$. Clearly, $\mathfrak{X}$ is a Lie algebra over $\mathbb{R}$ of dimension three and one of its basis is $\{f_1, f_2, f_3\}$, where $f_1(x) = [x_1\ 0]^\top$, $f_2(x) = [0\ x_2]^\top$ and $f_3(x) = [0\ x_1^2]^\top$. The flows associated with f_1, f_2 and f_3 are, respectively, $\Phi_{f_1}(t, x) = [\mathrm{e}^t x_1\ x_2]^\top$, $\Phi_{f_2}(t, x) = [x_1\ \mathrm{e}^t x_2]^\top$ and $\Phi_{f_3}(t, x) = [x_1\ x_1^2 t + x_2]^\top$. The objective is the computation of the solution of (6.4), (6.14), $r = 3$, from an arbitrary initial condition $x(0) = x_0$, for $u_1 = 1$, $u_2 = 1$ and $u_3 = t$. Clearly, $[f_1, f_2] = 0$, $[f_1, f_3] = 2f_3$ and $[f_2, f_3] = -f_3$. From

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}}\right) = \mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}}(u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3}),$$

it can be found

$$\begin{aligned}
&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} X_{f_1}\mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}} + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}\mathrm{e}^{\gamma_1 X_{f_1}} X_{f_2}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}} \\
&\qquad + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}} X_{f_3}\mathrm{e}^{\gamma_3 X_{f_3}} \\
&\quad = \mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}}(u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3}).
\end{aligned}$$

From the above equation, by left multiplication for $\mathrm{e}^{-\gamma_3 X_{f_3}}\mathrm{e}^{-\gamma_2 X_{f_2}}\mathrm{e}^{-\gamma_1 X_{f_1}}$, one has

$$\begin{aligned}
&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}\mathrm{e}^{-\gamma_3 X_{f_3}}\mathrm{e}^{-\gamma_2 X_{f_2}}\mathrm{e}^{-\gamma_1 X_{f_1}} X_{f_1}\mathrm{e}^{\gamma_1 X_{f_1}}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}} \\
&\qquad + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}\mathrm{e}^{-\gamma_3 X_{f_3}}\mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_2}\mathrm{e}^{\gamma_2 X_{f_2}}\mathrm{e}^{\gamma_3 X_{f_3}} + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_3}\mathrm{e}^{\gamma_3 X_{f_3}} \\
&\quad = (u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3}).
\end{aligned}$$

Since $\mathrm{e}^{-\gamma_i X_{f_i}} X_{f_i}\mathrm{e}^{\gamma_i X_{f_i}} = X_{f_i}$, $i = 1, 2, 3$, $\mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_1}\mathrm{e}^{\gamma_2 X_{f_2}} = X_{f_1}$, $\mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_1}\mathrm{e}^{\gamma_3 X_{f_3}} = X_{f_1} + 2\gamma_3 X_{f_3}$ and $\mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_2}\mathrm{e}^{\gamma_3 X_{f_3}} = X_{f_2} - \gamma_3 X_{f_3}$, it is found that

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} X_{f_1} + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} X_{f_2} + \left(2\gamma_3\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} - \gamma_3\frac{\mathrm{d}\gamma_2}{\mathrm{d}t} + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\right) X_{f_3} = u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3},$$

from which one concludes that

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = u_1, \qquad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} = u_2, \qquad 2\gamma_3\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} - \gamma_3\frac{\mathrm{d}\gamma_2}{\mathrm{d}t} + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t} = u_3,$$

namely the Wei–Norman equations are obtained,

$$\begin{aligned}
\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} &= u_1, \\
\frac{\mathrm{d}\gamma_2}{\mathrm{d}t} &= u_2, \\
\frac{\mathrm{d}\gamma_3}{\mathrm{d}t} &= -2\gamma_3 u_1 + \gamma_3 u_2 + u_3.
\end{aligned}$$

Letting $u_1 = 1$, $u_2 = 1$ and $u_3 = t$, one computes

$$\gamma_1(t) = t, \qquad \gamma_2(t) = t, \qquad \gamma_3(t) = \mathrm{e}^{-t} + t - 1.$$

From

$$\Phi_{f_3}(\gamma_3, \cdot) \circ \Phi_{f_2}(\gamma_2, \cdot) \circ \Phi_{f_1}(\gamma_1, x_0) = \begin{bmatrix} \mathrm{e}^{\gamma_1} x_{0,1} \\ \mathrm{e}^{2\gamma_1} \gamma_3 x_{0,1}^2 + \mathrm{e}^{\gamma_2} x_{0,2} \end{bmatrix},$$

the solution of the considered system is obtained

$$x(t) = \begin{bmatrix} \mathrm{e}^{t} x_{0,1} \\ \mathrm{e}^{2t}(\mathrm{e}^{-t} + t - 1) x_{0,1}^2 + \mathrm{e}^{t} x_{0,2} \end{bmatrix}.$$

Example 6.20 Consider the linear oscillator with time-varying frequency (6.9), which can be rewritten as

$$\frac{dx}{dt} = u_1(t) f_1(x) + u_2(t) f_2(x) + u_3(t) f_3(x),$$

where $u_1(t) = 1$, $u_2(t) = -\omega(t)$ and $u_3(t) = 0$, and the three vector functions $f_1(x) = [x_2 \; 0]^\top$, $f_2(x) = [0 \; x_1]^\top$ and $f_3(x) = [-x_1 \; x_2]^\top$ satisfy the commutation relations $[f_1, f_2] = f_3$, $[f_1, f_3] = -2f_1$ and $[f_2, f_3] = 2f_2$. Proceeding as in Example 6.19, one has

$$\begin{aligned}
&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \mathrm{e}^{-\gamma_3 X_{f_3}} \mathrm{e}^{-\gamma_2 X_{f_2}} \mathrm{e}^{-\gamma_1 X_{f_1}} X_{f_1} \mathrm{e}^{\gamma_1 X_{f_1}} \mathrm{e}^{\gamma_2 X_{f_2}} \mathrm{e}^{\gamma_3 X_{f_3}} \\
&\qquad + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \mathrm{e}^{-\gamma_3 X_{f_3}} \mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_2} \mathrm{e}^{\gamma_2 X_{f_2}} \mathrm{e}^{\gamma_3 X_{f_3}} + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t} \mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_3} \mathrm{e}^{\gamma_3 X_{f_3}} \\
&\quad = (u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3}).
\end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{e}^{-\gamma_i X_{f_i}} X_{f_i} \mathrm{e}^{\gamma_i X_{f_i}} = X_{f_i}, \quad i = 1, 2, 3, \\
&\mathrm{e}^{-\gamma_2 X_{f_2}} X_{f_1} \mathrm{e}^{\gamma_2 X_{f_2}} = X_{f_1} - \gamma_2^2 X_{f_2} + \gamma_2 X_{f_3}, \\
&\mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_1} \mathrm{e}^{\gamma_3 X_{f_3}} = \mathrm{e}^{-2\gamma_3} X_{f_1}, \qquad \mathrm{e}^{-\gamma_3 X_{f_3}} X_{f_2} \mathrm{e}^{\gamma_3 X_{f_3}} = \mathrm{e}^{2\gamma_3} X_{f_2},
\end{aligned}$$

it is found that

$$\begin{aligned}
&\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \left(\mathrm{e}^{-2\gamma_3} X_{f_1} - \gamma_2^2 \mathrm{e}^{2\gamma_3} X_{f_2} + \gamma_2 X_{f_3} \right) + \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \mathrm{e}^{2\gamma_3} X_{f_2} + \frac{\mathrm{d}\gamma_3}{\mathrm{d}t} X_{f_3} \\
&\quad = (u_1 X_{f_1} + u_2 X_{f_2} + u_3 X_{f_3}),
\end{aligned}$$

from which one concludes that

$$\mathrm{e}^{-2\gamma_3} \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = u_1, \qquad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \mathrm{e}^{2\gamma_3} - \mathrm{e}^{2\gamma_3} \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \gamma_2^2 = u_2, \qquad \frac{\mathrm{d}\gamma_3}{\mathrm{d}t} + \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \gamma_2 = u_3,$$

namely (substituting $u_1(t) = 1$, $u_2(t) = -\omega(t)$ and $u_3(t) = 0$) the Wei–Norman equations are obtained

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = \mathrm{e}^{2\gamma_3},$$
$$\frac{\mathrm{d}\gamma_2}{\mathrm{d}t} = -\mathrm{e}^{-2\gamma_3}\omega + \mathrm{e}^{2\gamma_3}\gamma_2^2,$$
$$\frac{\mathrm{d}\gamma_3}{\mathrm{d}t} = -\mathrm{e}^{2\gamma_3}\gamma_2.$$

6.11 Commutation Rules

Theorem 6.6 *Assume that $\{f_1, \dots, f_r\}$ is a basis of a finite dimensional Lie algebra $\mathfrak{X}$ of vector functions over $\mathbb{R}$. For any $f \in \mathfrak{X}$, there exist r functions $\gamma_1(t), \dots, \gamma_r(t) \in \mathbb{R}$ such that*

$$\mathrm{e}^{tX_f}x = \mathrm{e}^{\gamma_1(t)X_{f_1}} \cdots \mathrm{e}^{\gamma_{r-1}(t)X_{f_{r-1}}}\mathrm{e}^{\gamma_r(t)X_{f_r}}x, \quad \forall t \in \mathscr{T}_0,$$

namely such that

$$\Phi_f(t,x) = \Phi_{f_r}\big(\gamma_r(t), \cdot\big) \circ \Phi_{f_{r-1}}\big(\gamma_{r-1}(t), \cdot\big) \circ \cdots \circ \Phi_{f_1}\big(\gamma_1(t), x\big), \quad \forall t \in \mathscr{T}_0,$$

where $\mathscr{T}_0$ is a sufficiently small interval containing $t = 0$.

Proof Choosing functions u_i being constant, $u_i(t) = b_i$, $i = 1, \dots, r$, by the Wei–Norman formula (6.22), it can be concluded, for arbitrary b_i, $i = 1, \dots, r$, the existence (at least for small $|t|$) of functions $\gamma_1(t), \dots, \gamma_r(t) \in \mathbb{R}$ such that

$$\mathrm{e}^{t(b_1X_{f_1}+\cdots+b_rX_{f_r})}x = \mathrm{e}^{\gamma_1(t)X_{f_1}} \cdots \mathrm{e}^{\gamma_{r-1}(t)X_{f_{r-1}}}\mathrm{e}^{\gamma_r(t)X_{f_r}}x,$$

namely such that

$$\Phi_{b_1f_1+\cdots+b_rf_r}(t,x) = \Phi_{f_r}\big(\gamma_r(t), \cdot\big) \circ \Phi_{f_{r-1}}\big(\gamma_{r-1}(t), \cdot\big) \circ \cdots \circ \Phi_{f_1}\big(\gamma_1(t), x\big);$$

since $f \in \mathfrak{X}$ implies the existence of constants $b_1, \dots, b_r \in \mathbb{R}$ such that $f = b_1f_1 + \cdots + b_rf_r$, there exist (at least for small $|t|$) functions $\gamma_1(t), \dots, \gamma_r(t) \in \mathbb{R}$ such that

$$\mathrm{e}^{tX_f}x = \mathrm{e}^{\gamma_1(t)X_{f_1}} \cdots \mathrm{e}^{\gamma_{r-1}(t)X_{f_{r-1}}}\mathrm{e}^{\gamma_r(t)X_{f_r}}x. \qquad \square$$

Note that the functions $\gamma_1(t), \dots, \gamma_r(t)$ only depend on the structure constants of the Lie algebra, and not on its particular representation; in particular, they can be computed by integration of the Wei–Norman formula (6.22) from the initial condition $\gamma(0) = 0$. More easily, such functions can be computed through a matrix representation of the Lie algebra, as detailed in the following example.

Example 6.21 Consider a Lie algebra of vector functions over $\mathbb{R}$ with basis $\{f_1, f_2\}$, such that $[f_1, f_2] = f_1$. A matrix representation of the Lie algebra is $M_1 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. An arbitrary element $f = c_1 f_1 + c_2 f_2$, $c_1, c_2 \in \mathbb{R}$, is represented by $M = c_1 M_1 + c_2 M_2 = \left[\begin{smallmatrix} -c_2 & 0 \\ c_1 & 0 \end{smallmatrix}\right]$; for the sake of simplicity, assume that $c_i \neq 0$, $i = 1, 2$. Compute

$$\mathrm{e}^{\gamma_1 M_1} = \begin{bmatrix} 1 & 0 \\ \gamma_1 & 1 \end{bmatrix}, \qquad \mathrm{e}^{\gamma_2 M_2} = \begin{bmatrix} \mathrm{e}^{-\gamma_2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathrm{e}^{tM} = \begin{bmatrix} \mathrm{e}^{-tc_2} & 0 \\ -c_1 \frac{\mathrm{e}^{-tc_2}-1}{c_2} & 1 \end{bmatrix}.$$

From the equality $\mathrm{e}^{tM} = \mathrm{e}^{\gamma_2 M_2} \mathrm{e}^{\gamma_1 M_1}$,

$$\begin{bmatrix} \mathrm{e}^{-tc_2} & 0 \\ -c_1 \frac{\mathrm{e}^{-tc_2}-1}{c_2} & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\gamma_2} & 0 \\ \gamma_1 & 1 \end{bmatrix},$$

one can determine in a unique manner $\gamma_1(t) = \frac{c_1}{c_2}(1 - \mathrm{e}^{-tc_2})$, $\gamma_2(t) = tc_2$, thus obtaining the relation

$$\mathrm{e}^{tX_f} x = \mathrm{e}^{\frac{c_1}{c_2}(1-\mathrm{e}^{-tc_2}) X_{f_1}} \mathrm{e}^{tc_2 X_{f_2}} x, \quad \forall t \in \mathbb{R},$$

namely

$$\Phi_f(t, x) = \Phi_{f_2}(tc_2, \cdot) \circ \Phi_{f_1}\left(\frac{c_1}{c_2}\left(1 - \mathrm{e}^{-tc_2}\right), x\right), \quad \forall t \in \mathbb{R}.$$

The following theorem generalizes Theorem 6.6 (in case of linear vector fields, see [44]).

Theorem 6.7 *Assume that $\mathfrak{X} = \{f_1, \dots, f_p\}_{\mathbb{R}}$, the Lie algebra generated by the vector functions $f_1(x), \dots, f_p(x) \in \mathbb{R}^n$ over $\mathbb{R}$, is finite dimensional. For any $f \in \mathfrak{X}$, there exist an integer q, q functions $\gamma_1(t), \dots, \gamma_q(t) \in \mathbb{R}$, and q integers $i_1, \dots, i_q \in \{1, \dots, p\}$ such that*

$$\mathrm{e}^{tX_f} x = \mathrm{e}^{\gamma_1(t) X_{f_{i_1}}} \cdots \mathrm{e}^{\gamma_{q-1}(t) X_{f_{i_{q-1}}}} \mathrm{e}^{\gamma_q(t) X_{f_{i_q}}} x, \quad \forall t \in \mathscr{T}_0,$$

namely such that

$$\Phi_f(t, x) = \Phi_{f_{i_q}}\left(\gamma_q(t), \cdot\right) \circ \Phi_{f_{i_{q-1}}}\left(\gamma_{q-1}(t), \cdot\right) \circ \cdots \circ \Phi_{f_{i_1}}\left(\gamma_1(t), x\right), \quad \forall t \in \mathscr{T}_0,$$

where $\mathscr{T}_0$ is a sufficiently small interval containing $t = 0$.

Proof If $\mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_p\} = \{f_1, \dots, f_p\}_{\mathbb{R}}$, then the theorem is proven by Theorem 6.6. Otherwise, there exist two f_i, f_j, $i, j \in \{1, \dots, p\}$, such that $[f_i, f_j] \in \{f_1, \dots, f_p\}_{\mathbb{R}}$, but $[f_i, f_j] \notin \mathrm{span}_{\mathbb{R}}\{f_1, \dots, f_p\}$. Hence, by the Hadamard Lemma 6.18, $\mathrm{e}^{\tau X_{f_i}} X_{f_j} \mathrm{e}^{-\tau X_{f_i}} = X_{f_j} + \tau[X_{f_i}, X_{f_j}] + \cdots$, there exists a sufficiently small $|\tau|$ such that $e^{\tau X_{f_i}} X_{f_j} e^{-\tau X_{f_i}} \in \{X_{f_1}, \ldots, X_{f_p}\}_{\mathbb{R}}$, but such that $e^{\tau X_{f_i}} X_{f_j} e^{-\tau X_{f_i}} \notin \mathrm{span}_{\mathbb{R}}\{X_{f_1}, \ldots, X_{f_p}\}$. Let $f_{p+1}(x) \in \mathbb{R}^n$ be the vector function such that $X_{f_{p+1}} = e^{\tau X_{f_i}} X_{f_j} e^{-\tau X_{f_i}}$; clearly, $e^{tX_{f_{p+1}}} = e^{\tau X_{f_i}} e^{tX_{f_j}} e^{-\tau X_{f_i}}$. Now, if $\mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_p, f_{p+1}\} = \{f_1, \ldots, f_p\}_{\mathbb{R}}$, then the theorem is proven by Theorem 6.6, taking into account that $e^{tX_{f_{p+1}}} = e^{\tau X_{f_i}} e^{tX_{f_j}} e^{-\tau X_{f_i}}$. Otherwise, there exist two f_h, f_k, for $h, k \in \{1, \ldots, p, p+1\}$, such that $[f_h, f_k] \in \{f_1, \ldots, f_p\}_{\mathbb{R}}$, but $[f_h, f_k] \notin \mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_p, f_{p+1}\}$. By the Hadamard Lemma,

$$e^{\theta X_{f_h}} X_{f_k} e^{-\theta X_{f_h}} = X_{f_k} + \theta [X_{f_h}, X_{f_k}] + \cdots,$$

there exists a sufficiently small $|\theta|$ such that $e^{\theta X_{f_h}} X_{f_k} e^{-\theta X_{f_h}} \in \{X_{f_1}, \ldots, X_{f_p}\}_{\mathbb{R}}$, but $e^{\theta X_{f_h}} X_{f_k} e^{-\theta X_{f_h}} \notin \mathrm{span}_{\mathbb{R}}\{X_{f_1}, \ldots, X_{f_p}, X_{f_{p+1}}\}$. Let $f_{p+2}(x) \in \mathbb{R}^n$ be the vector function such that $X_{f_{p+2}} = e^{\theta X_{f_h}} X_{f_k} e^{-\theta X_{f_h}}$; clearly, one has $e^{tX_{f_{p+2}}} = e^{\theta X_{f_h}} e^{tX_{f_k}} e^{-\theta X_{f_h}}$. Note that if $f_k = f_{p+1}$, then $e^{tX_{f_{p+2}}} = e^{\theta X_{f_h}} e^{\tau X_{f_i}} e^{tX_{f_j}} e^{-\tau X_{f_i}} e^{-\theta X_{f_h}}$. Continuing in this way, one can compute $f_{p+1}(x), \ldots, f_r(x)$, where r is the dimension of $\{f_1, \ldots, f_p\}_{\mathbb{R}}$, such that $\mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_p, \ldots, f_r\} = \{f_1, \ldots, f_p\}_{\mathbb{R}}$ and such that, for each $j \in \{p+1, \ldots, r\}$, one can write

$$e^{tX_{f_j}} x = e^{\gamma_1(t) X_{f_{i_1}}} \cdots e^{\gamma_{m-1}(t) X_{f_{i_{m-1}}}} e^{\gamma_m(t) X_{f_{i_m}}} x,$$

where $m \in \mathbb{Z}^{>}$ and $i_1, \ldots, i_m \in \{1, \ldots, p\}$. The proof is then completed by Theorem 6.6. □

Remark 6.16 Theorem 6.7 can be easily understood in case of rotations of a rigid body, which is represented by the matrix Lie algebra spanned by

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad M_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad M_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assume that two reference frames are defined: an inertial reference frame and a moving reference frame rigidly connected with the body. At the initial time $t = 0$, the two frames coincide, and their origins coincide with the center of mass of the body. Assume that the body can rotate about its x- and y-axes, but not about its z-axis (this can be due to underactuation). The objective is to rotate the body of a certain angle α about the z-axis of the inertial frame. Although the rigid body cannot rotate about its z-axis, the objective can be achieved, because M_z belongs to the Lie algebra $\{M_x, M_y\}_{\mathbb{R}}$ generated by M_x, M_y. As an example, the objective can be easily obtained by (1) a rotation about the x-axis of the moving frame of $\frac{\pi}{2}$ radians so that the y-axis of the moving frame is aligned and orientated as the z-axis of the inertial frame, (2) a rotation about the y-axis of the moving frame of the angle α, (3) a rotation about the x-axis of the moving frame of $-\frac{\pi}{2}$ radians. This is equivalent to the formula $e^{M_z \alpha} = e^{M_x \pi/2} e^{M_y \alpha} e^{-M_z \pi/2}$, which can be easily checked as follows:

$$\begin{bmatrix}\cos(\alpha) & -\sin(\alpha) & 0\\ \sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\cos(\alpha) & 0 & \sin(\alpha)\\ 0 & 1 & 0\\ -\sin(\alpha) & 0 & \cos(\alpha)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}.$$

In particular, this means that for any three-dimensional Lie algebra $\mathrm{span}_{\mathbb{R}}\{f_x, f_y, f_z\}$ of vector functions over $\mathbb{R}$ satisfying the commutation relations

$$[f_x, f_y] = -f_z, \qquad [f_x, f_z] = f_y, \qquad [f_y, f_z] = -f_x,$$

one has

$$\Phi_{f_z}(\alpha, x) = \Phi_{f_x}(\pi/2, \cdot) \circ \Phi_{f_y}(\alpha, \cdot) \circ \Phi_{f_x}(-\pi/2, x).$$

Example 6.22 Consider the Lie algebra $\mathfrak{X}$ generated by $f_1(x), f_2(x) \in \mathbb{R}^2$ over $\mathbb{R}$, where $f_1(x) = [1\ 1]^\top$ and $f_2(x) = [x_1^2\ x_2^2]^\top$. It is easy to see that $\{f_1, f_2\}_{\mathbb{R}}$ is three-dimensional and has $\{f_1, f_2, f_3\}$ as basis, where $f_3(x) = [x_1\ x_2]^\top$. The commutation relations of $\{f_1, f_2\}_{\mathbb{R}}$ are

$$[f_1, f_2] = 2f_3, \qquad [f_1, f_3] = f_1, \qquad [f_2, f_3] = -f_2.$$

Consider the matrix representation of $\{f_1, f_2\}_{\mathbb{R}}$ given by

$$M_1 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 1 & 0 & 0\end{bmatrix}, \qquad M_2 = \begin{bmatrix}0 & 0 & -2\\ 0 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}, \qquad M_3 = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and the respective exponential matrices:

$$\mathrm{e}^{M_1 t} = \begin{bmatrix}1 & 0 & 0\\ t^2 & 1 & 2t\\ t & 0 & 1\end{bmatrix}, \qquad \mathrm{e}^{M_2 t} = \begin{bmatrix}1 & t^2 & -2t\\ 0 & 1 & 0\\ 0 & -t & 1\end{bmatrix}, \qquad \mathrm{e}^{M_3 t} = \begin{bmatrix}\mathrm{e}^{-t} & 0 & 0\\ 0 & \mathrm{e}^{t} & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Compute

$$\mathrm{e}^{-M_1 t} M_2 \mathrm{e}^{M_1 t} = \begin{bmatrix}-2t & 0 & -2\\ 0 & 2t & 2t^2\\ t^2 & -1 & 0\end{bmatrix};$$

clearly, $\mathrm{e}^{-M_1 t} M_2 \mathrm{e}^{M_1 t} \in \mathrm{span}_{\mathbb{R}}\{M_1, M_2, M_3\}$ (in particular, $\mathrm{e}^{-M_1 t} M_2 \mathrm{e}^{M_1 t} = t^2 M_1 + M_2 + 2t M_3$) for all $t \in \mathbb{R}$, but $\mathrm{e}^{-M_1 t} M_2 \mathrm{e}^{M_1 t} \notin \mathrm{span}_{\mathbb{R}}\{M_1, M_2\}$. For the sake of simplicity, let $t = 1$ and

$$N_3 = \mathrm{e}^{-M_1} M_2 \mathrm{e}^{M_1} = \begin{bmatrix}-2 & 0 & -2\\ 0 & 2 & 2\\ 1 & -1 & 0\end{bmatrix}.$$

By construction, $\{M_1, M_2, N_3\}$ is another basis of $\{M_1, M_2\}_{\mathbb{R}}$. This means that there exists a solution to the equation $e^{M_3 t} = e^{M_1\gamma_1}e^{M_2\gamma_2}e^{N_3\gamma_3}$, where

$$e^{N_3\gamma_3} = \begin{bmatrix} 1-2\gamma_3+\gamma_3^2 & \gamma_3^2 & 2\gamma_3^2-2\gamma_3 \\ \gamma_3^2 & 1+2\gamma_3+\gamma_3^2 & 2\gamma_3^2+2\gamma_3 \\ \gamma_3-\gamma_3^2 & -\gamma_3-\gamma_3^2 & 1-2\gamma_3^2 \end{bmatrix};$$

in particular, one computes two solutions

$$\gamma_1(t) = e^{\frac{1}{2}t} - e^t, \qquad \gamma_2(t) = e^{-\frac{1}{2}t} - 1, \qquad \gamma_3(t) = -1 + e^{\frac{1}{2}t}, \tag{6.24a}$$

$$\gamma_1(t) = -e^{\frac{1}{2}t} - e^t, \qquad \gamma_2(t) = -e^{-\frac{1}{2}t} - 1, \qquad \gamma_3(t) = -1 - e^{\frac{1}{2}t}. \tag{6.24b}$$

Taking into account that $e^{N_3\gamma_3} = e^{-M_1}e^{M_2\gamma_3}e^{M_1}$, one obtains

$$e^{M_3 t} = e^{M_1\gamma_1}e^{M_2\gamma_2}e^{-M_1}e^{M_2\gamma_3}e^{M_1},$$

which implies

$$e^{tX_{f_3}} = e^{X_{f_1}}e^{\gamma_3 X_{f_2}}e^{-X_{f_1}}e^{\gamma_2 X_{f_2}}e^{\gamma_1 X_{f_1}},$$

where γ_i, $i = 1, 2, 3$, are given in (6.24a), (6.24b). Taking into account that

$$\Phi_{f_1}(t, x) = \begin{bmatrix} t + x_1 \\ t + x_2 \end{bmatrix}, \qquad \Phi_{f_2}(t, x) = \begin{bmatrix} \frac{x_1}{1-tx_1} \\ \frac{x_2}{1-tx_2} \end{bmatrix}, \qquad \Phi_{f_3}(t, x) = \begin{bmatrix} e^t x_1 \\ e^t x_2 \end{bmatrix},$$

it is easy to verify that

$$\Phi_{f_3}(t, x) = \Phi_{f_1}(\gamma_1, \cdot) \circ \Phi_{f_2}(\gamma_2, \cdot) \circ \Phi_{f_1}(-1, \cdot) \circ \Phi_{f_2}(\gamma_3, \cdot) \circ \Phi_{f_1}(1, x).$$

For sufficiently small $t, \tau \geq 0$, the *Campbell–Baker–Hausdorff formula* is

$$e^{tX_f}e^{\tau X_g}x = e^{X_h}x, \tag{6.25}$$

where (see [114])

$$\begin{aligned} h = {} & (tf + \tau g) + \frac{1}{2}[tf, \tau g] + \frac{1}{12}\big(\big[[tf, \tau g], \tau g\big] - \big[[tf, \tau g], tf\big]\big) \\ & - \frac{1}{48}\big(\big[\tau g, \big[tf, [tf, \tau g]\big]\big] + \big[tf, \big[\tau g, [tf, \tau g]\big]\big]\big) + \cdots, \end{aligned} \tag{6.26}$$

and the dots denote repeated Lie brackets involving only f and g; this yields that if $f, g \in \mathfrak{X}$, then there exists $h \in \mathfrak{X}$ such that (6.25) holds. Therefore, as before, there exists (at least for small $t, \tau \geq 0$) functions $\gamma_1(t, \tau), \ldots, \gamma_r(t, \tau) \in \mathbb{R}$ such that

$$e^{tX_f}e^{\tau X_g}x = e^{\gamma_1(t,\tau)X_{f_1}} \cdots e^{\gamma_{r-1}(t,\tau)X_{f_{r-1}}}e^{\gamma_r(t,\tau)X_{f_r}}x,$$

namely that

$$\begin{aligned}&\Phi_g(\tau,\cdot)\circ\Phi_f(t,x)\\&\quad=\Phi_{f_r}\big(\gamma_r(t,\tau),\cdot\big)\circ\Phi_{f_{r-1}}\big(\gamma_{r-1}(t,\tau),\cdot\big)\circ\cdots\circ\Phi_{f_1}\big(\gamma_1(t,\tau),x\big).\end{aligned}$$

For the sake of compactness, introduce the *iterated Lie bracket*

$$[X_1,X_2,\dots,X_{p-1},X_p]=\big[X_1,\big[X_2,\dots,[X_{p-1},X_p]\dots\big]\big],$$

and the compact notation

$$\big[X_1^{h_1},X_2^{h_2},\dots,X_{p-1}^{h_{p-1}},X_p^{h_p}\big]=[\underbrace{X_1,\dots,X_1}_{h_1\text{ times}},\underbrace{X_2,\dots,X_2}_{h_2\text{ times}},\dots,\underbrace{X_p,\dots,X_p}_{h_p\text{ times}}].$$

The following combinatorial expression for the Campbell–Baker–Hausdorff expansion (with $\tau=t$) is due to Dynkin [43]:

$$\begin{aligned}\mathrm{e}^{tX_f}\mathrm{e}^{tX_g}=\sum_{p\ge 1}&\frac{(-1)^{p+1}}{p}\frac{1}{(i_1+j_1)+\cdots+(i_p+j_p)}\frac{1}{i_1!j_1!\cdots i_p!j_p!}\\&\times\big[tX_f^{i_1},tX_g^{j_1},\dots,tX_f^{i_p},tX_g^{j_p}\big],\end{aligned}$$

where the sum is taken over all non-negative $2p$-tuples $(i_1,j_1,\dots,i_p,j_p)$ satisfying $i_h+j_h\ge 1$.

Example 6.23 If $\{f_1,f_2\}$ is a basis of a two-dimensional Lie algebra $\mathfrak{X}$ of vector functions over $\mathbb{R}$ characterized by $[f_1,f_2]=c_1f_1+c_2f_2$, the above reasoning shows that the equation

$$\mathrm{e}^{\gamma_1X_{f_1}}\mathrm{e}^{\gamma_2X_{f_2}}=\mathrm{e}^{\eta_2X_{f_2}}\mathrm{e}^{\eta_1X_{f_1}},$$

in the unknowns η_1,η_2, has solution for small $|\gamma_1|,|\gamma_2|$. In particular, since such a solution does not depend on the particular representation of the Lie algebra, the expressions of η_1 and η_2 can be found by considering a matrix representation of $\mathfrak{X}$. For instance, a matrix representation of $\mathfrak{X}$ is given by $M_1=\left[\begin{smallmatrix}0&0\\c_1&c_2\end{smallmatrix}\right]$ and $M_2=\left[\begin{smallmatrix}-c_1&-c_2\\0&0\end{smallmatrix}\right]$. For the sake of simplicity, assume $c_i\ne 0$, $i=1,2$. In particular, the equation $\mathrm{e}^{\gamma_2M_2}\mathrm{e}^{\gamma_1M_1}=\mathrm{e}^{\eta_1M_1}\mathrm{e}^{\eta_2M_2}$,

$$\begin{aligned}&\begin{bmatrix}\mathrm{e}^{-\gamma_2c_1}&c_2\frac{\mathrm{e}^{-\gamma_2c_1}-1}{c_1}\\0&1\end{bmatrix}\begin{bmatrix}1&0\\c_1\frac{\mathrm{e}^{\gamma_1c_2}-1}{c_2}&\mathrm{e}^{\gamma_1c_2}\end{bmatrix}\\&\quad=\begin{bmatrix}1&0\\c_1\frac{\mathrm{e}^{\eta_1c_2}-1}{c_2}&\mathrm{e}^{\eta_1c_2}\end{bmatrix}\begin{bmatrix}\mathrm{e}^{-\eta_2c_1}&c_2\frac{\mathrm{e}^{-\eta_2c_1}-1}{c_1}\\0&1\end{bmatrix},\end{aligned}$$

in the unknowns η_1, η_2 has the unique solution

$$\eta_1 = -\frac{\ln(e^{-\gamma_2 c_1 + \gamma_1 c_2} - e^{\gamma_1 c_2} + 1) - \gamma_1 c_2 + \gamma_2 c_1}{c_2},$$

$$\eta_2 = -\frac{\ln(e^{-\gamma_2 c_1 + \gamma_1 c_2} - e^{\gamma_1 c_2} + 1)}{c_1}.$$

Note that $\lim_{c_1 \to 0} \eta_1 = \gamma_1$ and $\lim_{c_1 \to 0} \eta_2 = e^{\gamma_1 c_2} \gamma_2$.

Chapter 7
Linearization by State Immersion

7.1 Sufficient Conditions for the Existence of a Linearizing State Immersion

The aim of this section is to study the following problem (see [89, 95]).

Problem 7.1 Find a state immersion $x_e = \varphi_e(x)$, with $\varphi_e(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n_e}$ and $n_e \geq n$, such that systems (1.1a), (1.1b) expressed in the new x_e-coordinates are linear and the rank of $\frac{\partial \varphi_e}{\partial x}$ is full in some open and connected subset $\mathscr{U}^*$ of $\mathscr{U}$.

The concept of immersion is used in the literature (see [101] and references therein) for systems having inputs and outputs, to indicate a mapping transforming the state (and possibly increasing its dimension) but preserving the input-output map for an open set of initial conditions. Here the concept is similar (the definition given in [101] applies), but with the simplification that there are no inputs and the output is the original state vector. The use of the word "immersion" made in this section is coherent with the one given at p. 35 of [100], referred to a map between smooth manifolds.

Remark 7.1 The use of monomials $x_1^{h_1} \cdots x_n^{h_n}$ as additional state variables is a step of the classical Carleman linearization (see [112], where such a procedure is used for obtaining a bilinear approximation of a nonlinear control system); the drawback of the Carleman linearization is that the resulting linear system is, in general, infinite dimensional, and only finite dimensional approximations of the given nonlinear system can be obtained.

The following two sections, which extend the analysis carried out in Remark 3.33 at p. 115 and Remark 4.12 at p. 177 through the Poincaré–Dulac normal forms, give an answer to Problem 7.1, without any assumption on the position of the eigenvalues of the linear part of f, such as the belonging to the Poincaré domain or that the number of the possible resonant terms associated with f is finite.

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7.1.1 Linearization of Continuous-Time Systems by State Immersion

In this section, the continuous-time case is considered only.

Theorem 7.1 [89] *Let g be a symmetry of f and let there exists a diffeomorphism $y=\varphi(x)$ such that $\varphi_* g = (\frac{\partial\varphi}{\partial x} g)\circ\varphi^{-1}$ is in the Poincaré–Dulac normal form, $\varphi_* g(y) = By + k(y)$, with B diagonal, $k(y)\in\mathbb{R}^n$ and $[By, k(y)] = 0$. If the following conditions hold*:

(7.1.1) $\varphi_* f = (\frac{\partial\varphi}{\partial x} f)\circ\varphi^{-1}$ *is analytic at* $y=0$,
(7.1.2) *all eigenvalues of B are rational and have the same sign, then system* (1.1a) *can be immersed into a finite dimensional extended linear system.*

Proof If g is a symmetry of f, then $\varphi_* g$ is a symmetry of $\varphi_* f$. If $\varphi_* g(y) = By + k(y)$ is in the Poincaré–Dulac normal form and the vector function $\varphi_* f$ is analytic at $y=0$, then $\hat{g}(y) = By$ is a linear symmetry of $\varphi_* f$ [34]. Let $\hat{g}(y) = [w_1 y_1 \ \dots \ w_n y_n]^\top$, with the eigenvalues w_i being rational, different from 0, and having the same sign: if $\hat{g}(y) = By$ is a symmetry of $\varphi_* f$, then $\check{g}(y) = kBy$ is a symmetry of $\varphi_* f$ for any non-zero integer k; hence, with no loss of generality, it is assumed that the w_i's are all positive integers, possibly repeated and ordered so that $0 < w_1 \le w_2 \le \dots \le w_n$. Let $J_0(y) = \frac{1}{w_1}\ln(|y_1|)$, $J_1(y) = \frac{y_1^{w_2}}{y_2^{w_1}}, \dots, J_{n-1}(y) = \frac{y_1^{w_n}}{y_n^{w_1}}$; clearly, $L_g J_0 = 1$ and $L_g J_i = 0$, $i = 1, 2, \dots, n-1$ and $\det(\frac{\partial J}{\partial y}) \neq 0$. Therefore, one concludes that

$$\left(\frac{\partial J(y)}{\partial y}\right)^{-1} = \begin{bmatrix} w_1 y_1 & 0 & \dots & 0 \\ w_2 y_2 & -\frac{1}{w_1}\frac{y_2^{w_1+1}}{y_1^{w_2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_n y_n & 0 & \dots & -\frac{1}{w_1}\frac{y_n^{w_1+1}}{y_1^{w_n}} \end{bmatrix}.$$

Then, by Statement (3.9.1) of Theorem 3.9, all vector functions $\tilde{f}$ having $\hat{g}(y) = [w_1 y_1 \ \dots \ w_n y_n]^\top$ as a symmetry are given by

$$\tilde{f}(y) = \begin{bmatrix} w_1 y_1 C_0 \\ w_2 y_2 C_0 - \frac{1}{w_1}\frac{y_2^{w_1+1}}{y_1^{w_2}} C_1 \\ \vdots \\ w_n y_n C_0 - \frac{1}{w_1}\frac{y_n^{w_1+1}}{y_1^{w_n}} C_{n-1} \end{bmatrix},$$

with the C_i's being arbitrary functions of $\frac{y_1^{w_2}}{y_2^{w_1}}, \dots, \frac{y_1^{w_n}}{y_n^{w_1}}$. For the sake of simplicity assume that $w_i \neq w_j$ (the case of repeated eigenvalues is similar); moreover, for the sake of brevity, consider most of the following equalities valid locally in a neighborhood $\mathscr{U}^*$ of the origin. If $\tilde{f}$ is analytic at $y=0$, then C_0 is necessarily constant, $C_0=a_0$, whence the first entry of $\tilde{f}$ is $\tilde{f}_1(y)=a_0 w_1 y_1$; if w_2 is an integer multiple of w_1 (i.e., if $w_2=h_{2,1}w_1$, for some positive integer $h_{2,1}$), then necessarily $C_1=a_1\frac{y_1^{\frac{w_2}{w_1}}}{y_2}+a_2\left(\frac{y_1^{\frac{w_2}{w_1}}}{y_2}\right)^{1+\frac{h_{2,1}}{w_2}}$, whence the second entry of $\tilde{f}$ is $\tilde{f}_2(y)=(w_2a_0-\frac{a_1}{w_1})y_2-\frac{1}{w_1}a_2y_1^{h_{2,1}}$; if w_2 is not an integer multiple of w_1, then necessarily $C_1=a_1\frac{y_1^{\frac{w_2}{w_1}}}{y_2}$, whence the second entry of $\tilde{f}$ is $\tilde{f}_2(y)=(w_2a_0-\frac{a_1}{w_1})y_2$; and so on. In this way, it is easy to see that $\tilde{f}=\varphi_* f$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^w x$, with $w=[w_1\ \dots\ w_n]^\top$. Once f has been transformed by φ into a block triangular form $\tilde{f}$, corresponding to the fact that $\tilde{f}$ is polynomial and homogeneous of degree 0 with respect to a positive integer dilation, then it can be easily immersed into a larger state space so that the nonlinear system thus immersed is finite dimensional and linear; as a matter of fact, let $\mathscr{M}^{w_n}$ be the set of all monomials having degree less than or equal to w_n, with respect to the given dilation: such a set is clearly finite. Let $y_1^{h_1}\cdots y_n^{h_n}\in\mathscr{M}^{w_n}$ be of degree m, for arbitrary $h_1,\dots,h_n\in\mathbb{Z}^{\geq}$; since $\tilde{f}$ has degree 0, then $L_{\tilde{f}}(y_1^{h_1}\cdots y_n^{h_n})$ has degree m and therefore it is an element of $\mathscr{M}^{w_n}$. □

Note that if g satisfies the conditions of the Poincaré–Dulac Theorem 3.33 at p. 118, then there exists a near-identity diffeomorphism, analytic at $x=0$, such that the symmetry g in the new coordinates is in the Poincaré–Dulac normal form; a further linear transformation can be used to render its linear part diagonal.

Remark 7.2 Condition (7.1.2) of Theorem 7.1 can seen a strong one, but it is actually necessary for the solvability of Problem 7.1 at least for two notable classes of systems. First, the existence of a symmetry of f having the identity as linear part is a necessary and sufficient condition for the linearization of a nonlinear system through a change of coordinates (see Theorem 3.35 at p. 121). Secondly, it can be proven that any f with a semi-simple linear part whose eigenvalues are real and belong to the Poincaré domain admits a symmetry g with rational eigenvalues having the same sign: this class includes the systems considered in Remark 3.33 at p. 115. To prove this statement, with no loss of generality, assume that in the original x-coordinates the linear part of f is diagonal, and consider the near-identity diffeomorphism $y=\varphi(x)$ such that $\frac{\mathrm{d}y}{\mathrm{d}t}=\tilde{f}(y)$ is in the Poincaré–Dulac normal form, which is indeed characterized by a finite number of resonances; in particular, assuming that all eigenvalues have been ordered so that $\lambda_i\leq\lambda_j$ if $i\leq j$, the kth resonance is characterized by the following equation:

$$\lambda_{i_k}=\ell_{1,k}\lambda_1+\ell_{2,k}\lambda_2+\cdots+\ell_{i_k-1,k}\lambda_{i_k-1}, \tag{7.1}$$

where $\ell_{j,k}\in\mathbb{Z}$, $\ell_{j,k}\geq 0$, $\ell_{1,k}+\ell_{2,k}+\cdots+\ell_{i_k,k}\geq 2$. Consider the following algebraic linear system:

$$\xi_{i_k}=\ell_{1,k}\xi_1+\ell_{2,k}\xi_2+\cdots+\ell_{i_k-1,k}\xi_{i_k-1},\quad k=1,\dots,N, \tag{7.2}$$

in the n unknowns ξ_i, $i = 1, \dots, n$. It can be easily seen that, since its coefficients are integer numbers, and it admits at least a non-zero solution (the set of the eigenvalues λ_i), then the vector space of its solutions has a basis of m vectors $\{v_1, \dots, v_m\}$ having integer elements. By construction, given any $\xi_1, \dots, \xi_n$ solution of system (7.2) the vector function

$$\tilde{g}(y) = [\xi_1 y_1 \ \dots \ \xi_n y_n]^\top \tag{7.3}$$

is a symmetry of $\tilde{f}$. Now, note that the vector $[\lambda_1 \ \dots \ \lambda_n]^\top$ (whose elements are either all positive or all negative) can be written as a linear combination of the vectors $v_1, \dots, v_m$ with real coefficients c_i; then, by a sufficiently accurate rational approximation of such a linear combination (obtained substituting each c_i with a rational approximation $\hat{c}_i$), an approximation $\hat{\xi}_1, \dots, \hat{\xi}_n$ of the eigenvalues $\lambda_1, \dots, \lambda_n$ can be obtained, such that all $\hat{\xi}_i$ have the same sign. Considering the vector $\tilde{g}$ obtained by replacing ξ_i in (7.3) by $\hat{\xi}_i$, one concludes that $g = (\frac{\partial \varphi}{\partial x})^{-1} \tilde{g} \circ \varphi$ is a symmetry of f satisfying condition (7.1.2) of Theorem 7.1.

Remark 7.3 The dimension n_e of the state space of the extended system is at most the number of elements of $\mathscr{M}^{w_n}$. In some cases, an extended system of lower dimension can be obtained, if some of the monomials in $\mathscr{M}^{w_n}$ do not appear in f nor in any of the directional derivatives of the elements of $\mathscr{M}^{w_n}$ by f.

Remark 7.4 The assumption that the eigenvalues of B have the same sign (besides being rational) is crucial. Let $a \neq 0$; all vector functions f described by $f(x) = [ax_1 C_0 \ C_1]^\top$, with C_0 and C_1 being analytic functions of $x_2 \in \mathbb{R}$, are analytic on the whole $\mathbb{R}^2$ and have $g(x) = [ax_1 \ 0]^\top$ as symmetry. Clearly, the systems described by such an f cannot be, in general, immersed into extended linear systems. On the other hand, all vector functions f described by $f(x) = [ax_1 C_0 \ \ -ax_2 C_0 + x_1 x_2^2 C_1]^\top$, with C_0 and C_1 being analytic functions of $x_1 x_2$, are analytic on the whole $\mathbb{R}^2$ and have $g(x) = [ax_1 \ \ -ax_2]^\top$ as a symmetry. Such systems too cannot be, in general, immersed into extended linear systems.

Next, some examples are proposed to illustrate the applicability of Theorem 7.1. In the first example, the vector w of weights has repeated entries.

Example 7.1 Let $g(x) = [x_1 + x_2^2 \ x_2 \ \ -x_1 + x_2^2 + 2x_3]^\top$ and $J_0(x) = \ln(|x_1 - x_2^2|)$, $J_1(x) = \frac{x_1 - x_2^2}{x_2}$, $J_2(x) = \frac{(x_1 - x_2^2)^2}{x_3 - x_1 + x_2^2}$, satisfying $L_g J_0 = 1$, $L_g J_1 = 0$ and $L_g J_2 = 0$; then, all vector functions f having g as a symmetry are given by $f = (\frac{\partial J}{\partial x})^{-1}[C_0 \ C_1 \ C_2]^\top$, with $J = [J_0 \ J_1 \ J_2]^\top$ and the C_i's being arbitrary functions of J_1, J_2. By a simple analysis, it is easy to see that f is analytic at $x = 0$ if and only if $C_0 = a_1 + a_2 \frac{1}{J_1}$, $C_1 = a_2 + a_3 J_1 + a_4 J_1^2$ and $C_2 = a_5 J_2 + a_6 J_2^2 + 2a_2 \frac{J_2}{J_1} + a_7 \frac{J_2^2}{J_1} + a_8 \frac{J_2^2}{J_1^2}$, where the a_i's are arbitrary reals. Consider the transformation $y_1 = x_1 - x_2^2$, $y_2 = x_2$, $y_3 = x_3 - x_1 + x_2^2$; in the new y-coordinates, one finds that $\tilde{g}(y) = [y_1 \; y_2 \; 2y_3]^\top$ and

$$\tilde{f}(y) = \begin{bmatrix} a_1 y_1 + a_2 y_2 \\ -a_4 y_1 + (-a_3 + a_1) y_2 \\ -a_6 y_1^2 - a_7 y_1 y_2 - a_8 y_2^2 + (-a_5 + 2a_1) y_3 \end{bmatrix}.$$

Such a system can be immersed into a larger state space with the positions $y_4 = y_1^2$, $y_5 = y_1 y_2$, $y_6 = y_2^2$.

Note that the classical approaches for linearization through state immersion [12, 72, 112] can be applied to the system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ considered in Example 7.1, because there exist an integer m and matrices M_k such that

$$L_f^m x = \sum_{k=0}^{m-1} M_k L_f^k x; \tag{7.4}$$

actually, in such a case, the integer m can be either 2 or 3, depending on the values of the parameters a_i. However, Theorem 7.1 can be applied to systems for which such existing approaches do not work, as shown in the next example.

Example 7.2 Consider the system (1.1a) with

$$f(x) = \begin{bmatrix} x_1(x_1 + 1) \\ 2x_2 + \frac{x_1^2}{(x_1+1)^2} \end{bmatrix}.$$

It is easy to show that

$$L_f^k x = \begin{bmatrix} (k+1)!x_1^{k+1} + p_k(x_1) \\ 2^k x_2 + b_k \frac{x_1^2}{(x_1+1)^2} \end{bmatrix},$$

where, for each $k \geq 1$, $p_k(x_1)$ is a polynomial of order smaller than $k + 1$ and b_k is a real constant. Hence, it is clear that condition (7.4) cannot be satisfied by any integer m. On the other hand, consider $g(x) = [x_1(x_1 + 1) \; 2x_2]^\top$, which is a symmetry of f. Since the ith component of g is a function of x_i only, and the eigenvalues of its linear part are not zero, then the diffeomorphism $y = \varphi(x) = [\frac{x_1}{x_1+1} \; x_2]^\top$ that brings g into its Poincaré–Dulac normal form $\tilde{g}(y) = [y_1 \; 2y_2]^\top$ can be computed by integration. Such a g satisfies conditions (7.1.1) and (7.1.2) of Theorem 7.1, since $\tilde{f}(y) = [y_1 \; 2y_2 + y_1^2]^\top$ and $B = \mathrm{diag}\{1, 2\}$, whence the given system can be immersed into a linear system by the state immersion

$$x_e = \varphi_e(x) = \begin{bmatrix} \frac{x_1}{x_1+1} \\ x_2 \\ \frac{x_1^2}{(x_1+1)^2} \end{bmatrix}.$$

Note that the system (1.1a) considered in Example 7.2 (and the one considered in Example 7.1 as well, if a_6, a_7 and a_8 are not all zero) cannot be linearized by a diffeomorphism because of the non-zero resonant term $x_1^2 e_2$. Hence, such a system cannot be immersed into a linear system by means of the classical techniques.

7.1.2 Linearization of Discrete-Time Systems by State Immersion

In this section, the discrete-time case is considered only.

Theorem 7.2 [95] *Let g be a symmetry of F and let there exists a diffeomorphism $y = \varphi(x)$ such that $\varphi_* g = (\frac{\partial \varphi}{\partial x} g) \circ \varphi^{-1}$ is linear, $\varphi_* g(y) = By$, with $B \in \mathbb{R}^{n\times n}$ diagonal. If the following conditions hold*:

(7.2.1) $\varphi_* F(y) = \varphi \circ F \circ \varphi^{-1}(y)$ *is analytic at* $y = 0$,

(7.2.2) *all eigenvalues of B are rational and have the same sign*, *then*, *system* (1.1b) *can be immersed into a finite dimensional extended linear system.*

Proof By Theorem 4.4 at p. 160, consider all vector functions expressed in the y-coordinates. Since $\lfloor \varphi_* F(y), By \rfloor = 0$ implies $[\varphi_* F(y), By] = 0$ and $\varphi_* F(y)$ is analytic at $y = 0$, then $\varphi_* F(y)$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^w y$, with $w = [w_1 \ \dots \ w_n]^\top$ and $0 \le w_1 \le \cdots \le w_n$ being the eigenvalues of B, $B = \mathrm{diag}\{w_1, \dots, w_n\}$. This means that $\varphi_* F(\mathrm{e}^{B\tau} y) = \mathrm{e}^{B\tau} \varphi_* F(y)$. Let $\mathscr{M}^{w_n}$ be the set of all monomials having degree less than or equal to w_n, with respect to $\delta_\varepsilon^w y$: such a set is clearly finite, since $w_i > 0$, $\forall i$. Let $k(y) = y_1^{h_1} \cdots y_n^{h_n} \in \mathscr{M}^{w_n}$; in particular, m is its degree if and only if $k(\mathrm{e}^{B\tau} y) = \mathrm{e}^{m\tau} k(y)$. Clearly, $k \circ \varphi_* F \in \mathscr{M}^{w_n}$ and its degree is still m, since

$$k \circ \varphi_* F\big(\mathrm{e}^{B\tau} y\big) = k\big(\varphi_* F\big(\mathrm{e}^{B\tau} y\big)\big) = k\big(\mathrm{e}^{B\tau} \varphi_* F(y)\big) = \mathrm{e}^{m\tau} k\big(\varphi_* F(y)\big). \qquad \square$$

Note that if g satisfies the conditions of the Poincaré–Dulac Theorem 3.33, then there exists a near-identity diffeomorphism, analytic at $x = 0$, such that the symmetry g in the new coordinates is in the Poincaré–Dulac normal form. For g to be useful for Theorem 7.2, its Poincaré–Dulac normal form has to be linear; a further linear transformation can be used to render its linear part diagonal.

Remark 7.5 Theorem 7.2 is somewhat weaker than the corresponding Theorem 7.1 valid in continuous-time. In fact, for continuous-time systems, one needs to know a symmetry g of f and the change of coordinates that brings g in the Poincaré–Dulac normal form (which need not be linear), whereas for discrete-time systems, the Poincaré–Dulac normal form of the symmetry g must be linear.

Remark 7.6 The dimension n_e of the state space of the extended system is at most the number of elements of $\mathscr{M}^{w_n}$. In some cases, an extended system of lower dimension could be obtained, if some of the monomials in $\mathscr{M}^{w_n}$ do not appear in the vector function.

Example 7.3 Let

$$F(x) = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 + a_3 x_1^2 \\ a_4 x_3 + a_5 x_1^3 + a_6 x_1 x_2 \end{bmatrix},$$

for arbitrary constants $a_1, \dots, a_6 \in \mathbb{R}$; F is homogeneous of degree 0 with respect to the integer dilation $\delta_\varepsilon^w x$, with $w = [1\ 2\ 3]^\top$, according to Definition 3.7 at p. 71. Set $\mathscr{M}^3$ is given by $\mathscr{M}^3 = \{x_1, x_2, x_1^2, x_3, x_1^3, x_1 x_2\}$. Define the variables $x_4 := x_1^2$, $x_5 := x_1^3$ and $x_6 := x_1 x_2$ and compute their dynamics, $\Delta x_4 = F_1^2 = (a_1 x_1)^2 = a_1^2 x_1^2 = a_1^2 x_4$, $\Delta x_5 = F_1^3 = (a_1 x_1)^3 = a_1^3 x_1^3 = a_1^3 x_5$ and $\Delta x_6 = F_1 F_2 = a_1 a_3 x_1^3 + a_1 a_2 x_1 x_2 = a_1 a_3 x_5 + a_1 a_2 x_6$. Hence, the extended linear system $\Delta x_e = A_e x_e$ is obtained, with

$$A_e = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 & a_5 & a_6 \\ 0 & 0 & 0 & a_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1^3 & 0 \\ 0 & 0 & 0 & 0 & a_1 a_3 & a_1 a_2 \end{bmatrix}, \qquad x_e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

7.2 Computation of the Flow by State Immersion

Assume the existence of a state immersion $x_e = \varphi_e(x)$, with $\varphi_e(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n_e}$ and $n_e \geq n$, such that system (1.1a) (respectively, (1.1b)) expressed in the new coordinates x_e is linear, $\Delta x_e = A_e x_e$. Apart from a preliminary diffeomorphism, assume that the first n entries of $\varphi_e(x)$ coincide with x. Then, the flow associated with f (respectively, F) is given by

$$\Phi_f(t, x) = [E\ 0] \mathrm{e}^{A_e t} \varphi_e(x), \quad \big(\text{respectively, } \Psi_F(t, x) = [E\ 0] A_e^t \varphi_e(x)\big),$$

where E is the $n \times n$ identity matrix and matrix $[E\ 0]$ is used to select the first n entries of the vector on the right.

Example 7.4 Let $f(x) = F(x) = [c_1 x_1\ \ c_2 x_2 + c_3 x_1^2\ \ c_4 x_3 + c_5 x_1^3 + c_6 x_1 x_2]^\top$. Both in the continuous-time and discrete-time cases, the system can be linearized by taking as additional state variables $x_4 = x_1^2$, $x_5 = x_1^3$ and $x_6 = x_1 x_2$, obtaining the extended linear system characterized by $A_e = A_{C,e}$ in the continuous-time case and by $A_e = A_{D,e}$ in the discrete-time case, where

$$A_{C,e} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & 0 & c_5 & c_6 \\ 0 & 0 & 0 & 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3c_1 & 0 \\ 0 & 0 & 0 & 0 & c_3 & c_1 + c_2 \end{bmatrix},$$

and

$$A_{D,e} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & 0 & c_5 & c_6 \\ 0 & 0 & 0 & c_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^3 & 0 \\ 0 & 0 & 0 & 0 & c_1c_3 & c_1c_2 \end{bmatrix}.$$

As an example assume that the three monomials x_1^2, x_1^3 and x_1x_2 are resonant, $c_4 = 3c_1, c_2 = 2c_1$, in the continuous-time case, and $c_4 = c_1^3, c_2 = c_1^2$, in the discrete-time case; under such an assumption, one has

$$e^{A_{C,e}t} = \begin{bmatrix} e^{tc_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2tc_1} & 0 & c_3te^{2tc_1} & 0 & 0 \\ 0 & 0 & e^{3tc_1} & 0 & \frac{1}{2}te^{3tc_1}(2c_5 + c_6c_3t) & c_6te^{3tc_1} \\ 0 & 0 & 0 & e^{2tc_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{3tc_1} & 0 \\ 0 & 0 & 0 & 0 & c_3te^{3tc_1} & e^{3tc_1} \end{bmatrix},$$

which yields

$$\begin{aligned} \Phi_f(t,x) &= \begin{bmatrix} e^{tc_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2tc_1} & 0 & c_3te^{2tc_1} & 0 & 0 \\ 0 & 0 & e^{3tc_1} & 0 & \frac{1}{2}te^{3tc_1}(2c_5 + c_6c_3t) & c_6te^{3tc_1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1^2 \\ x_1^3 \\ x_1x_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{tc_1}x_1 \\ e^{2tc_1}x_2 + tc_3e^{2tc_1}x_1^2 \\ e^{3tc_1}x_3 + \frac{1}{2}te^{3tc_1}(2c_5 + tc_6c_3)x_1^3 + tc_6e^{3tc_1}x_1x_2 \end{bmatrix}, \end{aligned}$$

and

$$A_{D,e}^t = \begin{bmatrix} c_1^t & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1^{2t} & 0 & c_3tc_1^{2t-2} & 0 & 0 \\ 0 & 0 & c_1^{3t} & 0 & \frac{1}{2}c_6c_3t(t-1)c_1^{-5+3t} + tc_5c_1^{3t-3} & c_6tc_1^{3t-3} \\ 0 & 0 & 0 & c_1^{2t} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^{3t} & 0 \\ 0 & 0 & 0 & 0 & c_3tc_1^{3t-2} & c_1^{3t} \end{bmatrix},$$

which yields

$$\Psi_F(t,x) = \begin{bmatrix} c_1^t & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1^{2t} & 0 & c_3 t c_1^{2t-2} & 0 & 0 \\ 0 & 0 & c_1^{3t} & 0 & \frac{1}{2}c_6 c_3 t(t-1)c_1^{-5+3t} + tc_5 c_1^{3t-3} & c_6 t c_1^{3t-3} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1^2 \\ x_1^3 \\ x_1 x_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1^t x_1 \\ c_1^{2t} x_2 + c_3 t c_1^{2t-2} x_1^2 \\ c_1^{3t} x_3 + (\frac{1}{2}c_6 c_3 t(t-1)c_1^{-5+3t} + c_5 t c_1^{3t-3})x_1^3 + c_1^{3t-3} c_6 t x_1 x_2 \end{bmatrix}.$$

7.3 Computation of a Linearizing Diffeomorphism by Using Semi-invariants

Under the assumptions of Theorems 7.1 and 7.2, in order to simplify the exposition, assume that g is linear and diagonal, with integer and positive eigenvalues, and that f, F are polynomial and homogeneous of degree 0 with respect to g, using Definition 3.8 of homogeneity even when the discrete-time system is considered; this corresponds to being already in the y-coordinates mentioned in Theorems 7.1 and 7.2. Denote by Ax the linear part of f or F, $A = \frac{\partial f(x)}{\partial x}|_{x=0} = \frac{\partial F(x)}{\partial x}|_{x=0}$, and by A_e the dynamic matrix of the extended linear system obtained by the state immersion: let x_e be the state of the extended linear system thus obtained and n_e be its dimension.

Let u be a real (respectively, complex) left eigenvector of matrix A_e with a real (respectively, complex) eigenvalue λ, $u^\top A_e = \lambda u^\top$; then, $\omega(x_e) = u^\top x_e$ (respectively, $\omega(x_e) = (u^{*\top} x_e)(u^\top x_e)$, where $*$ means complex conjugate) is a semi-invariant of the extended linear system. Then, by the pull-back to the original coordinates, if $\hat{\omega}(x) = \omega(x_e) = \omega \circ \varphi_e(x)$, then $\hat{\omega}(x)$ is a semi-invariant of the original nonlinear systems (1.1a), (1.1b). Hence, the set of points $x \in \mathbb{R}^n$ such that $\hat{\omega}(x) = 0$ is invariant for the nonlinear systems (1.1a), (1.1b).

Example 7.5 Let $g(x) = [x_1 \;\; 3x_2]^\top$; any f, F polynomial and homogeneous of degree 0 with respect to g are given by $f(x) = F(x) = [a_1 x_1 \;\; a_2 x_2 + a_3 x_1^3]^\top$, with a_1, a_2, a_3 being arbitrary reals. Such a system can be linearized with the position

$x_3 = x_1^3$, thus obtaining

$$A_{C,e} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & 0 & 3a_1 \end{bmatrix}$$

in the continuous-time case and

$$A_{D,e} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & 0 & a_1^3 \end{bmatrix}$$

in the discrete-time case. Under the assumption of absence of resonances (i.e., $a_2 \neq 3a_1$ in the continuous-time case and $a_2 \neq a_1^3$ in the discrete-time case), matrix A_e has three left eigenvectors being linearly independent over $\mathbb{R}$: $u_1^\top = [1\ 0\ 0]$, with eigenvalue $\lambda_1 = a_1$, $u_2^\top = [0\ 0\ 1]$, with eigenvalue $\lambda_2 = 3a_1$ if $\mathbb{T} = \mathbb{R}$ and $\lambda_2 = a_1^3$ if $\mathbb{T} = \mathbb{Z}$, and $u_3^\top = [0\ a_2 - 3a_1\ a_3]$ if $\mathbb{T} = \mathbb{R}$, and $u_3^\top = [0\ a_2 - a_1^3\ a_3]$ if $\mathbb{T} = \mathbb{Z}$, with eigenvalue $\lambda_3 = a_2$. The semi-invariants of the extended linear system $\Delta x_e = A_e x_e$ are $\omega_i(x_e) = u_i^\top x_e$, $i = 1, 2, 3$. Consequently, the semi-invariants associated with f are $\hat{\omega}_1(x) = x_1$, $\hat{\omega}_2(x) = x_1^3$, $\hat{\omega}_3(x) = (a_2 - 3a_1)x_2 + a_3 x_1^3$ and the semi-invariants associated with F are $\hat{\omega}_1(x) = x_1$, $\hat{\omega}_2(x) = x_1^3$, $\hat{\omega}_3(x) = (a_2 - a_1^3)x_2 + a_3 x_1^3$.

Using the concept of semi-invariant and a simple extension of it, it is possible to prove the following theorem, similar to Theorem 3.35 at p. 121 and Theorem 4.16 at p. 178, whose constructive proof gives an expression in closed form of the transformation $y = \varphi(x)$, assuming a diagonal linear symmetry g, having generic integer positive eigenvalues.

Theorem 7.3 *Let f, F be polynomial and homogeneous of degree* 0 *with respect to $g(x) = Bx$, with B diagonal and having integer and positive eigenvalues. Let $A = \frac{\partial f(x)}{\partial x}|_{x=0} = \frac{\partial F(x)}{\partial x}|_{x=0}$ and let A_e be the dynamic matrix of the extended linear system obtained by state immersion. If, for each left Jordan chain $\{u_1^\top, \ldots, u_h^\top\}$ of A relative to the eigenvalue λ (such as $u_i^\top A = \lambda u_i^\top + u_{i+1}$, $i = 1, \ldots, h-1$, and $u_h^\top A = \lambda u_h^\top$), there exist h vectors $\{\bar{u}_1, \ldots, \bar{u}_h\}$ such that $\{[u_1^\top\ \bar{u}_1^\top], \ldots, [u_h^\top\ \bar{u}_h^\top]\}$ is a left Jordan chain of A_e, relative to the eigenvalue λ, then there exist n functionally independent scalar functions $v_i(x)$ such that $y_i = v_i(x)$, $i = 1, \ldots, n$, qualifies as a diffeomorphism at $x = 0$ and the systems* (1.1a), (1.1b) *expressed in the new coordinates are linear.*

Remark 7.7 If A is semi-simple, the condition on its Jordan chains is simply that for each left eigenvector $u^\top$ of A, relative to the eigenvalue λ, there exists $\bar{u}^\top$ such that $[u^\top\ \bar{u}^\top]$ is a left eigenvector of A_e, relative to the eigenvalue λ. Note that if the eigenvalues of g are all different, then necessarily A is diagonal (and therefore its left eigenvectors $u^\top$ are trivial, although it is not always true that, for each of them, there exists $\bar{u}^\top$ such that $[u^\top\ \bar{u}^\top]$ is a left eigenvector of A_e). Another special case is when the eigenvalues of A have the same algebraic multiplicity as eigenvalues of

A_e: in such a case, from the triangular form of A_e, it follows that the hypothesis of Theorem 7.3 on the left Jordan chains of A is satisfied.

Proof The proof of the theorem is completely detailed in the case of real eigenvalues of A, leaving some details about the case of complex eigenvalues to the reader. For each eigenvalue λ of A, with algebraic multiplicity μ and geometric multiplicity m, $1 \leq m \leq \mu$, there exist μ generalized left eigenvectors of A, relative to λ, linearly independent over $\mathbb{C}$ and organized in m Jordan chains as follows: $\{u_{1,1}^\top, u_{1,2}^\top, \ldots, u_{1,h_1}^\top\}, \ldots, \{u_{k,1}^\top, u_{k,2}^\top, \ldots, u_{k,h_k}^\top\}, \ldots, \{u_{m,1}^\top, u_{m,2}^\top, \ldots, u_{m,h_m}^\top\}$, with $\sum_{k=1}^m h_k = \mu$, and, for each generalized left eigenvector $u_{k,j}^\top$ of A, there exists $\bar{u}_{k,j}$ such that $w_{k,j}^\top = [u_{k,j}^\top \; \bar{u}_{k,j}^\top]$ is a generalized left eigenvector of A_e, relative to λ. Then, for each real eigenvalue λ, consider the corresponding μ functions $\omega_{k,j}(x_e) = w_{k,j}^\top x_e$, for which $\Delta\omega_{k,j}(x_e) = \lambda\omega_{k,j}(x_e) + \omega_{k,j+1}(x_e)$ for $j = 1, \ldots, h_k - 1$, or $\Delta\omega_{k,j}(x_e) = \lambda\omega_{k,j}(x_e)$ for $j = h_k$; writing them in the original coordinates, for the corresponding μ functions $v_{k,j}(x) = \omega_{k,j}(\varphi(x))$, one has $\Delta v_{k,j}(x) = \lambda v_{k,j}(x) + v_{k,j+1}(x)$ for $j = 1, \ldots, h_k - 1$ or $\Delta v_{k,j}(x) = \lambda v_{k,j}(x)$ for $j = h_k$. Note that the m functions $\omega_{k,h_k}(x_e) = w_{k,h_k}^\top x_e$ are semi-invariants of the extended linear system, such that $\Delta\omega_{k,h_k}(x_e) = \lambda\omega_{k,h_k}(x_e)$; writing them in the original coordinates, the corresponding m functions $v_{k,h_k}(x) = \omega_{k,h_k}(\varphi(x))$, are semi-invariants of the original system such that $\Delta v_{k,h_k}(x) = \lambda v_{k,h_k}(x)$. Then, the set of n functions $v_i(x)$, $i = 1, \ldots, n$, can be taken collecting μ_r functions for each real eigenvalue λ_r of A with algebraic multiplicity μ_r and $2\mu_r$ functions (found in a similar way) for each pair (λ_r, λ_r^*) of complex conjugate eigenvalues of A having algebraic multiplicity μ_r. It is easy to see that such functions are functionally independent, and that, assuming a proper ordering of them, letting $y = [y_1 \; \ldots \; y_n]^\top$, one has $\Delta y = A_{q,d}\, y$, with matrix $A_{q,d}$ being block diagonal in the real Jordan form. □

Example 7.6 Continue Example 7.5. As for the continuous-time case, if $a_2 \neq 3a_1$ (in which case the eigenvalues of A have the same algebraic multiplicity as eigenvalues of A_e), then the two left eigenvectors of A, namely $[1\; 0]$ and $[0\; a_2 - 3a_1]$, can be "extended" to the corresponding left eigenvectors $u_1^\top$ and $u_3^\top$ (using the notation in Example 7.5) of A_e; therefore $y_1 = x_1$, $y_2 = (a_2 - 3a_1)x_2 + a_3 x_1^3$ qualifies as a polynomial diffeomorphism. In the new y-coordinates, one finds that $\frac{\mathrm{d}y_1}{\mathrm{d}t} = a_1 y_1$, $\frac{\mathrm{d}y_2}{\mathrm{d}t} = a_2 y_2$. As for the discrete-time case, if $a_2 \neq a_1^3$ (in which case the eigenvalues of A have the same algebraic multiplicity as eigenvalues of A_e), then the two left eigenvectors of A, namely $[1\; 0]$ and $[0\; a_2 - a_1^3]$, can be "extended" to the corresponding left eigenvectors $u_1^\top$ and $u_3^\top$ (using the notation in Example 7.5) of A_e; therefore $y_1 = x_1$, $y_2 = (a_2 - a_1^3)x_2 + a_3 x_1^3$ qualifies as a polynomial diffeomorphism. In the new y-coordinates, one computes $\Delta y_1 = a_1 y_1$, $\Delta y_2 = a_2 y_2$.

7.4 Linearization of Hamiltonian Planar Systems

In this section, it is assumed that $x = [x_1 \; x_2]^\top \in \mathbb{R}^2$ and that $\{u, v\} = \frac{\partial u}{\partial x}\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\nabla v$. For all notations and basic concepts about Hamiltonian systems see Chap. 5.

Theorem 7.4 [87] *Consider the Hamiltonian function* $H(x) = K \circ h(x)$, *where* $h(x) = [h_1(x)\ h_2(x)]^\top \in \mathbb{R}^2$ *is analytic at* $x = 0$, $h(0) = 0$ *and such that* $\{h_1, h_2\} = 1$. *Assume that* $K(y)$ *is polynomial and homogeneous of degree* $k = w_1 + w_2$ *with respect to* $\delta_\varepsilon^w y$, *with* $w = [w_1\ w_2]^\top$, $w_1, w_2 \in \mathbb{Z}$, $w_1, w_2 > 0$. *Let* f_H *be the Hamiltonian vector function associated with* H. *Then,*

(7.4.1) $g(x) = (\frac{\partial h(x)}{\partial x})^{-1}[w_1 h_1(x)\ w_2 h_2(x)]^\top$ *is a (not necessarily Hamiltonian) symmetry of* f_H;
(7.4.2) g *can be linearized by* $y = h(x)$, *thus finding in the new coordinates* $\tilde{g}(y) = [w_1 y_1\ w_2 y_2]^\top$;
(7.4.3) *since the Hamiltonian vector function* $\tilde{f}_K(y) = (\frac{\partial h}{\partial x} f_H) \circ h^{-1}(y)$ *is analytic at* $y = 0$ *and homogeneous of degree* 0 *with respect to* $\tilde{g}(y) = [w_1 y_1\ w_2 y_2]^\top$, $\tilde{f}_K$ *can be rendered linear by a finite dimensional state immersion*;
(7.4.4) *if* $w_1 = w_2 = 1$, *then* $\tilde{f}_K$ *is linear.*

Proof First, note that $y = h(x)$ qualifies as a canonical diffeomorphism. In the y-coordinates, one finds that $\tilde{g}(y) = (\frac{\partial h}{\partial x} g) \circ h^{-1}(y) = [w_1 h_1\ w_2 h_2]^\top \circ h^{-1}(y) = [w_1 y_1\ w_2 y_2]^\top$, thus proving Statement (7.4.2) of the theorem. In these coordinates, the Hamiltonian function $H(x)$ takes the form $K(y)$, and the Hamiltonian system takes the form $\tilde{f}_K(y) = [\frac{\partial K(y)}{\partial y_2}\ \ -\frac{\partial K(y)}{\partial y_1}]^\top$. Then, clearly, $\frac{\partial K}{\partial y_2}$ is polynomial and homogeneous of degree $k - w_2 = w_1$ and $-\frac{\partial K}{\partial y_1}$ is polynomial and homogeneous of degree $k - w_1 = w_2$, whence $\tilde{f}_K$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^w y$, whence $\tilde{g}(y) = [w_1 y_1\ w_2 y_2]^\top$ is a symmetry of $\tilde{f}_K$. Thanks to the invariance of the Lie bracket to diffeomorphisms, $g = (\frac{\partial h}{\partial x})^{-1}[w_1 h_1\ w_2 h_2]^\top$ is a symmetry of f_H, thus proving Statement (7.4.1) of the theorem. Since if $\tilde{f}_K$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^w y$, with $r = [1\ 1]^\top$, one concludes that $\tilde{f}_K$ is linear, thus proving Statement (7.4.4) of the theorem. If $\tilde{f}_K$ is polynomial and homogeneous of degree 0 with respect to $\delta_\varepsilon^w y$, then $\tilde{f}_{K,i}$ is the sum of some monomials $m_j^{w_i} = y_1^{j_1} y_2^{j_2}$ homogeneous with respect to $\delta_\varepsilon^w y$ of degree w_i. If $u(y)$ is any function homogeneous with respect to $\delta_\varepsilon^w y$ of a certain degree, then $L_{\tilde{f}_K} u = \{u, K\}$ is homogeneous with respect to $\delta_\varepsilon^w y$ of the same degree, which shows how $\tilde{f}_K$ can be linearized by taking as state variables all monomials $m_j^{w_i}$, $i = 1, 2$, thus proving Statement (7.4.3) of the theorem. □

Example 7.7 Take $h(x) = [x_1 + x_2^2\ x_2]^\top$, $K(h) = \frac{1}{2} h^\top B h$ and $B = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ (respectively, $B = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$), which yields the Hamiltonian system described by

$$f_H(x) = \begin{bmatrix} 2x_2^3 - 3x_2^2 + (2x_1 + 2)x_2 - x_1 \\ -x_1 - x_2^2 + x_2 \end{bmatrix}$$

$$\left(\text{respectively,}\ \ f_H(x) = \begin{bmatrix} 2x_2^3 + 6x_2^2 + (2x_1 + 3)x_2 + 2x_1 \\ -x_1 - x_2^2 - 2x_2 \end{bmatrix}\right);$$

a symmetry g of f_H is $g(x) = [x_1 - x_2^2 \;\; x_2]^\top$ (note that such a symmetry is not Hamiltonian, because $\mathrm{div}(g) = 2 \neq 0$). It is easy to check that g can be linearized by $y = [x_1 + x_2^2 \;\; x_2]^\top$, thus finding $\tilde{g}(y) = [y_1 \;\; y_2]^\top$; by the same diffeomorphism, one has $\tilde{f}_K(y) = [-y_1 + 2y_2 \;\; -y_1 + y_2]^\top$ (respectively, $\tilde{f}_K(y) = [2y_1 + 3y_2 \;\; -y_1 - 2y_2]^\top$). Clearly, $H(x) = \frac{1}{2}h^\top(x)Bh(x)$ is a first integral associated with f, $L_f H = 0$. In the second case, matrix $\left[\begin{smallmatrix} 2 & 3 \\ -1 & -2 \end{smallmatrix}\right]$ has $u_1^\top = [1 \; 1]$, $\lambda_1 = 1$ and $u_2^\top = [1 \; 3]$, $\lambda_2 = -1$ as real (left eigenvector, eigenvalue) pairs; this yields two Darboux polynomials of the original system, $\omega_1(x) = u_1^\top h(x) = x_1 + x_2^2 + x_2$ and $\omega_2(x) = u_2^\top h(x) = x_1 + x_2^2 + 3x_2$. As a matter of fact, $L_f\omega_1 = \omega_1$ and $L_f\omega_2 = -\omega_2$; actually, note that $H = \frac{1}{2}\omega_1\omega_2$, according to the fact that $L_f H = \frac{1}{2}(\omega_2 L_f\omega_1 + \omega_1 L_f\omega_2) = \frac{1}{2}(\omega_1\omega_2 - \omega_1\omega_2) = 0$.

Example 7.8 Take $h(x) = [x_1 + \frac{1}{2}x_2^2 \;\; x_1 + x_2 + \frac{1}{2}x_2^2]^\top$ and $K(h) = ah_1h_2 + \frac{b}{3}h_1^3$; it is easy to see that K is homogeneous of degree 3 with respect to $\delta_\varepsilon^w h$, with $w = [1 \; 2]^\top$, and that $\{h_1, h_2\} = \det\left(\left[\begin{smallmatrix} 1 & x_2 \\ 1 & 1+x_2 \end{smallmatrix}\right]\right) = 1$. The corresponding Hamiltonian system is described by $f_H = [f_{H,1} \;\; f_{H,2}]^\top$, with $f_{H,1}(x) = \frac{1}{4}bx_2^5 + (a + bx_1)x_2^3 + \frac{3}{2}ax_2^2 + (2ax_1 + bx_1^2)x_2 + ax_1$ and $f_{H,2}(x) = -\frac{1}{4}bx_2^4 + (-a - bx_1)x_2^2 - ax_2 - 2ax_1 - bx_1^2$; a symmetry g of f_H is then (note that such a symmetry is not Hamiltonian, because $\mathrm{div}(g) = 3 \neq 0$)

$$g(x) = \begin{bmatrix} x_1 - \frac{3}{2}x_2^2 - x_1x_2 - \frac{1}{2}x_2^3 \\ x_1 + \frac{1}{2}x_2^2 + 2x_2 \end{bmatrix}.$$

With the diffeomorphism $y_1 = x_1 + \frac{1}{2}x_2^2$, $y_2 = x_1 + x_2 + \frac{1}{2}x_2^2$, one has

$$\tilde{g}(y) = \begin{bmatrix} y_1 \\ 2y_2 \end{bmatrix}, \qquad \tilde{f}_K(y) = \begin{bmatrix} ay_1 \\ -ay_2 - by_1^2 \end{bmatrix}.$$

Clearly, $\tilde{f}_K$ can be immersed into a linear system with the position $y_3 = y_1^2$, thus finding the extended linear system

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = ay_1, \qquad \frac{\mathrm{d}y_2}{\mathrm{d}t} = -ay_2 - by_3, \qquad \frac{\mathrm{d}y_3}{\mathrm{d}t} = 2ay_3.$$

The flow of the above extended linear system is

$$\Phi_{A_e y_e}(t, y_e) = \mathrm{e}^{A_e t}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{at}y_1 \\ \mathrm{e}^{-at}y_2 - \frac{1}{3}b\frac{\mathrm{e}^{2ta} - \mathrm{e}^{-at}}{a}y_3 \\ \mathrm{e}^{2ta}y_3 \end{bmatrix},$$

which, taking into account that $y_3 = y_1^2$, yields the following flow of the system $\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}_K(y)$:

$$\Phi_{\tilde{f}_K}(t, y) = \begin{bmatrix} \mathrm{e}^{at}y_1 \\ \mathrm{e}^{-at}y_2 - \frac{1}{3}b\frac{\mathrm{e}^{2ta} - \mathrm{e}^{-at}}{a}y_1^2 \end{bmatrix}.$$

From $\Phi_{\tilde{f}_K}(t, y)$, taking into account that $x_1 = y_1 - \frac{1}{2}y_2^2 + y_2 y_1 - \frac{1}{2}y_1^2$, $x_2 = y_2 - y_1$, one can compute the flow of the original Hamiltonian system.

Such results can be easily extended to the case when some *dissipation* is present in the Hamiltonian system, as explained in the following. Consider a Hamiltonian function $H = \frac{1}{2}h^\top Bh$, with the entries h_1 and h_2 of h satisfying $\{h_1, h_2\} = 1$, and the corresponding Hamiltonian system described by $f_H = (\frac{\partial h}{\partial x})^{-1}SBh$, with $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$; a symmetry of f_H is $g = (\frac{\partial h}{\partial x})^{-1}h$. Some dissipative effects, which maintain some of the structure of the system, can be taken into account by substituting matrix S with matrix $S_d = \left[\begin{smallmatrix} 0 & 1 \\ -1 & -d \end{smallmatrix}\right]$, with d being a real constant: $f_{H,d} = (\frac{\partial h}{\partial x})^{-1}S_d Bh$. Since the entries of $y = h(x)$ qualify as canonical coordinates, both $f_{H,d}$ and g can be linearized by $y = h(x)$, $\tilde{f}_{K,d} = (\frac{\partial h}{\partial x} f_{H,d}) \circ h^{-1}(y) = S_d By$ and $\tilde{g}(y) = (\frac{\partial h}{\partial x} g) \circ h^{-1}(y) = y$; g is a symmetry of $f_{K,d}$, since $\tilde{g}$ is a symmetry of $\tilde{f}_{K,d}$.

Example 7.9 Take $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$; since $H = \frac{1}{2}h^\top Bh = \frac{1}{2}(h_2^2 - h_1^2)$ is not positive definite, the classical approach of using H as a Lyapunov function is not effective in this case. The semi-invariants of the Hamiltonian system, with the dissipation described by the vector function $f_{H,d}$, are $\omega_1 = h_1(\frac{1}{2}d - \frac{1}{2}\sqrt{d^2+4}) + h_2$ and $\omega_2 = h_1(\frac{1}{2}d + \frac{1}{2}\sqrt{d^2+4}) + h_2$ with respective (constant) characteristic functions $\lambda_1 = \frac{1}{2}d + \frac{1}{2}\sqrt{d^2+4}$ and $\lambda_2 = \frac{1}{2}d - \frac{1}{2}\sqrt{d^2+4}$: the origin of the Hamiltonian system is clearly unstable for all values of d (λ_1 is a positive function of d, and λ_2 is a negative function of d).

The philosophy behind Theorem 7.4 is simple: given a Hamiltonian system, find one of its symmetries such that there exists a diffeomorphism linearizing the symmetry and jointly transforming the Hamiltonian system into a polynomial form, homogeneous of degree 0 with respect to a certain integer dilation. Then, it is of interest to compute all symmetries of a Hamiltonian system.

Consider the Hamiltonian function H and the corresponding Hamiltonian vector function f_H. Let $K(x)$ be a function such that $\{K, H\} = 1$: since

$$\{K, H\} = \det\left(\begin{bmatrix} \frac{\partial K}{\partial x} \\ \frac{\partial H}{\partial x} \end{bmatrix}\right),$$

condition $\{K, H\} = 1$ can hold only about a regular point x^o of f_H, $f_H(x^o) \neq 0$ (namely, such that $\frac{\partial H(x)}{\partial x}|_{x=x^o} \neq 0$), according to the Frobenius Theorem 1.9 at p. 21. In the canonical coordinates $y_1 = K(x)$ and $y_2 = H(x)$, the Hamiltonian function takes the form $\tilde{H}(y) = y_2$, and the Hamiltonian system expressed in these coordinates is straightened, $\tilde{f}_{\tilde{H}}(y) = [\frac{\partial \tilde{H}(y)}{\partial y_2} \; -\frac{\partial \tilde{H}(y)}{\partial y_1}]^\top = [1 \; 0]^\top$. All symmetries of $\tilde{f}_{\tilde{H}}$ are parameterized by $\tilde{g}(y) = [C_0(y_2) \; C_1(y_2)]^\top$, where $C_i(y_2)$ is an arbitrary function of y_2, whence (by Statement (1.4.1) of Theorem 1.4 at p. 9) all symmetries g of f_H are parameterized by

$$g = \left(\frac{\partial}{\partial x}\begin{bmatrix} K \\ H \end{bmatrix}\right)^{-1} \begin{bmatrix} C_0(H) \\ C_1(H) \end{bmatrix}^\top.$$

Example 7.10 Consider the Hamiltonian function $H(x) = ax_1x_2 + \frac{b}{3}x_1^3$, with $a \neq 0$, and the corresponding Hamiltonian system given by $f_H(x) = [ax_1 \;\; -ax_2 - bx_1^2]^\top$. Define $K(x) := \frac{1}{a}\ln(x_1)$, for which $\{K(x), H(x)\} = \det\left(\begin{bmatrix} \frac{1}{ax_1} & 0 \\ ax_2+bx_1^2 & ax_1 \end{bmatrix}\right) = 1$. Then, all symmetries g of f_H are parameterized by

$$g(x) = \begin{bmatrix} ax_1C_0 \\ (-ax_2 - bx_1^2)C_0 + \frac{1}{ax_1}C_1 \end{bmatrix},$$

with C_0 and C_1 being arbitrary functions of H. In particular, taking $C_0 = \frac{1}{a}$ and $C_1 = 3H = 3(ax_1x_2 + \frac{b}{3}x_1^3)$, one obtains the symmetry $g(x) = [x_1 \; 2x_2]^\top$, according to the fact that f_H is homogeneous of degree 0 with respect to δ_ε^w, $w = [1 \; 2]^\top$.

It is also of interest, given a vector function g, to compute all the Hamiltonian systems having g as symmetry. Assume the existence of two functions J_0 and J_1 such that $L_gJ_0 = 1$, $L_gJ_1 = 0$ and $\{J_0, J_1\} = 1$. This means that $y_1 = J_0(x)$, $y_2 = J_1(x)$ qualify as canonical coordinates with respect to the given Poisson bracket and, in particular, in these coordinates g is straightened: $\tilde{g}(y) = [1 \; 0]^\top$. If f_H is Hamiltonian and has g as a symmetry, then system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_H(x)$ transformed into the y-coordinates $\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}_{\tilde{H}}(y)$ is still Hamiltonian and has $\tilde{g}(y)$ as a symmetry, because y_1 and y_2 are canonical. All $\tilde{f}_{\tilde{H}}$ having $\tilde{g}$ as a symmetry are parameterized by $\tilde{f}_{\tilde{H}} = [C_0 \; C_1]^\top$, with C_0 and C_1 being arbitrary functions of y_2; if $\tilde{f}_{\tilde{H}}$ is Hamiltonian, then it must be area preserving (namely, $\operatorname{div}(\tilde{f}_{\tilde{H}}) = 0$): then, all the Hamiltonian vector functions $\tilde{f}_{\tilde{H}}$ having $\tilde{g}$ as a symmetry are parameterized by $\tilde{f}_{\tilde{H}}(y) = [C_0(y_2) \; C_1]^\top$, with C_0 being an arbitrary function of y_2 and C_1 being constant, with the respective Hamiltonian function $K(y_1, y_2) = \int C_0(y_2)\,\mathrm{d}y_2 - C_1y_1$ (clearly, $\int C_0(y_2)\,\mathrm{d}y_2$ is an arbitrary function of y_2). By the pull-back to the original x-coordinates, one concludes that all the Hamiltonian vector functions f_H having g as a symmetry are parameterized by

$$f_H = \left(\frac{\partial}{\partial x}\begin{bmatrix} J_0 \\ J_1 \end{bmatrix}\right)^{-1} \begin{bmatrix} C_0(J_1) \\ C_1 \end{bmatrix},$$

with the Hamiltonian function $H(x) = K(J_0, J_1) = \int C_0(y_2)\,\mathrm{d}y_2|_{y_2=J_1} - C_1J_0$.

Example 7.11 Consider $g(x) = [1 + x_2 \;\; -1]^\top$. Clearly, $J_0(x) = x_1 + \frac{1}{2}x_2^2$ and $J_1(x) = x_1 + x_2 + \frac{1}{2}x_2^2$ satisfy $L_gJ_0 = 1$, $L_gJ_1 = 0$ and $\{J_0, J_1\} = 1$. Then, all the Hamiltonian f_H having g as a symmetry are parameterized by the Hamiltonian function $H(x) = C_2(x_1 + x_2 + \frac{1}{2}x_2^2) - (x_1 + \frac{1}{2}x_2^2)C_1$, where $C_2(y_2) = \int C_0(y_2)\,\mathrm{d}y_2$ is an arbitrary function of y_2 and C_1 is a constant. For instance, taking $C_2(y_2) = \frac{1}{2}y_2^2$ and $C_1 = 1$, one obtains the Hamiltonian function $H(x) = \frac{1}{2}(x_1 + x_2 + \frac{1}{2}x_2^2)^2 -$

$(x_1 + \frac{1}{2}x_2^2)$, with the respective Hamiltonian system described by

$$f_H(x) = \begin{bmatrix} x_1 + x_1x_2 + \frac{3}{2}x_2^2 + \frac{1}{2}x_2^3 \\ -(x_1 + x_2 + \frac{1}{2}x_2^2 - 1) \end{bmatrix}.$$

7.5 Linearization of Higher Order Hamiltonian Systems

In this section assume that $S = \left[\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right]$ and $x = [q^\top \; p^\top]^\top$. The proof of the following theorem is omitted; it is similar to the one of Theorem 7.4.

Theorem 7.5 *Let $H(x) = K \circ h(x)$, where $h(x) \in \mathbb{R}^{2n}$ is analytic at $x = 0$, $h(0) = 0$. Assume that $K(h)$ is polynomial and homogeneous of degree $k = w_i + w_{i+n}$, $i = 1, \dots, n$, with respect to $\delta_\varepsilon^w h$, with $w = [w_1 \; \dots \; w_n \; w_{n+1} \; \dots \; w_{2n}]^\top$, $w_i \in \mathbb{Z}$, $w_i > 0$. Consider the Hamiltonian vector function f_H associated with H; assume that*

$$\{h_i, h_j\} = 0, \qquad \{h_{i+n}, h_{j+n}\} = 0,$$

$$\{h_i, h_{j+n}\} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \qquad \forall i, j \in \{1, \dots, n\}.$$

Then,

(7.5.1) *$g = (\frac{\partial h}{\partial x})^{-1}[w_1h_1 \; \dots \; w_{2n}h_{2n}]^\top$ is a (not necessarily Hamiltonian) symmetry of f;*

(7.5.2) *g can be linearized by $y = h(x)$, thus finding in the new coordinates $\tilde{g}(y) = [w_1y_1 \; \dots \; w_{2n}y_{2n}]^\top$;*

(7.5.3) *since $\tilde{f}_K(y) = (\frac{\partial h}{\partial x} f_H) \circ h^{-1}(y)$ is analytic at $y = 0$ and homogeneous of degree 0 with respect to $\tilde{g}(y) = [w_1y_1 \; \dots \; w_{2n}y_{2n}]^\top$, $\tilde{f}_K$ can be rendered linear by a finite dimensional state immersion;*

(7.5.4) *if $w_i = w_{i+n} = 1$, $i = 1, \dots, n$, then $\tilde{f}_K$ is linear.*

Example 7.12 Assume that $K(y)$ is polynomial and homogeneous of degree 4 with respect to a dilation, with the vector of weights $w = [2 \; 1 \; 2 \; 3]^\top$, $K(y) = a_1y_1y_2^2 + a_2y_3y_2^2 + a_3y_1y_3 + a_4y_2^4 + a_5y_2y_4$. Let $y_i = h_i(x)$, $i = 1, \dots, 4$, and assume, in addition, that functions h_i's satisfy the following conditions: $\{h_1, h_2\} = 0$, $\{h_1, h_3\} = 1$, $\{h_1, h_4\} = 0$, $\{h_2, h_3\} = 0$, $\{h_2, h_4\} = 1$, and $\{h_3, h_4\} = 0$. These conditions ensure that $y_i = h_i(x)$, $i = 1, \dots, 4$, qualify as canonical coordinates, such that the transformed Hamiltonian system is described by the following vector function:

$$\tilde{f}_K(y) = \begin{bmatrix} \frac{\partial K}{\partial y_3} \\ \frac{\partial K}{\partial y_4} \\ -\frac{\partial K}{\partial y_1} \\ -\frac{\partial K}{\partial y_2} \end{bmatrix} = \begin{bmatrix} a_2 y_2^2 + a_3 y_1 \\ a_5 y_2 \\ -a_1 y_2^2 - a_3 y_3 \\ -2a_1 y_1 y_2 - 2a_2 y_2 y_3 - 4a_4 y_2^3 - a_5 y_4 \end{bmatrix}.$$

This system can be linearized by the state immersion $y_5 = y_2^2$, $y_6 = y_2^3$, $y_7 = y_1 y_2$, and $y_8 = y_2 y_3$.

Chapter 8
Stability Analysis

8.1 Background

This brief section summarizes some classical results that are explained in more detail in many textbooks such as, e.g., [9, 11, 60, 76, 115].

For $x \in \mathbb{C}^n$, $\|x\| = \sqrt{x^\top x}$ denotes the *Euclidean norm* of x. Such a choice is not restrictive, since all norms on $x \in \mathbb{C}^n$ are equivalent (see [41, Sect. 2]).

Definition 8.1 An equilibrium point $x_e \in \mathbb{R}^n$ of systems (1.1a), (1.1b), $f(x_e) = 0$ and $F(x_e) = x_e$, is *stable* if for any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$, such that, for every initial condition $x(0) \in \mathbb{R}^n$ for which $\|x(0) - x_e\| < \delta_\varepsilon$, the solution $\Phi_f(t, x(0))$ (respectively, $\Psi_F(t, x(0))$) of systems (1.1a), (1.1b) through $x(0)$ at $t = 0$ satisfies the inequality $\|\Phi_f(t, x(0)) - x_e\| < \varepsilon$ (respectively, $\|\Psi_F(t, x(0)) - x_e\| < \varepsilon$) for all $t \geq 0$. The equilibrium point x_e is said to be *unstable* if it is not stable.

Definition 8.2 An equilibrium point $x_e \in \mathbb{R}^n$ of systems (1.1a), (1.1b) is *attractive* if there exists a $\delta > 0$ such that $\lim_{t \to +\infty} \|\Phi_f(t, x(0)) - x_e\| = 0$ (respectively, $\lim_{t \to +\infty} \|\Psi_F(t, x(0)) - x_e\| = 0$), for all $x(0) \in \mathbb{R}^n$ for which $\|x(0) - x_e\| < \delta$. If the above limits hold for all $x(0) \in \mathbb{R}^n$, then the equilibrium point x_e of (1.1a), (1.1b) is *globally attractive*.

Definition 8.3 An equilibrium point $x_e \in \mathbb{R}^n$ of systems (1.1a), (1.1b) is *asymptotically stable* (respectively, *globally asymptotically stable*) if it is stable and attractive (respectively, globally attractive).

In the following, it will be assumed that the equilibrium is the origin, $x_e = 0$, with no loss of generality apart from a translation $y = x - x_e$.

Definition 8.4 A function $V(x) \in \mathbb{R}$, continuous at $x = 0$, is *positive definite* (respectively, *positive semi-definite*) about $x = 0$ if $V(0) = 0$ and $V(x) > 0$ (respectively, $V(x) \geq 0$) for all $x \in \mathscr{U}^*$, $x \neq 0$, with $\mathscr{U}^*$ being a neighborhood of $x = 0$;

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V is *globally positive definite* if $\mathscr{U}^* = \mathbb{R}^n$. Function V is *negative definite* (respectively, *negative semi-definite*) if $-V$ is positive definite (respectively, positive semi-definite); V is *globally negative definite* if $-V$ is globally positive definite. Function V is *radially unbounded* if

$$\lim_{\|x\| \to +\infty} V(x) = +\infty.$$

Note that, in the following, when V is used for a continuous-time system, it will be implicitly required that V is C^1, so that $L_f V$ is well defined.

Theorem 8.1 *Let $V(x)$ be analytic at $x = 0$. Let $\delta_\varepsilon^w x$ be a positive dilation, with $w = [w_1 \ \dots \ w_n]^\top$, with constants $w_i > 0$. Consider the Taylor expansion of $V(\delta_\varepsilon^w x)$ with respect to $\varepsilon = 0$, for x in a sufficiently small neighborhood $\mathscr{U}^*$ of $x = 0$,*

$$V\left(\delta_\varepsilon^w x\right) = \varepsilon^m V^{[m]}(x) + O\left(\varepsilon^{m+1}\right).$$

If $V^{[m]}(x)$ is positive definite about $x = 0$, then $V(x)$ is positive definite about $x = 0$.

Proof The proof follows from the fact that there exists a positive ε^* such that

$$V^{[m]}(x) > O(\varepsilon), \quad \forall \varepsilon \in \left(0, \varepsilon^*\right),$$

for each $x \in \mathscr{U}^*$ for which $V^{[m]}(x) > 0$, since $y = \delta_\varepsilon^w x$ comprises a neighborhood of $y = 0$ (apart from $y = 0$), when x and ε vary in $\mathscr{U}^*$ and $(0, \varepsilon^*)$, respectively. □

Example 8.1 Consider $V(x) = x_1^6 - 2x_1^3 x_3^3 - 2x_1^2 x_2^2 + x_2^2 + x_3^4$. Using the standard dilation $w = [1\ 1\ 1]^\top$, one has

$$\begin{aligned} V\left(\delta_\varepsilon^w x\right) &= \left[x_1^6 - 2x_1^3 x_3^3 - 2x_1^2 x_2^2 + x_2^2 + x_3^4\right]_{x_1 = \varepsilon x_1, x_2 = \varepsilon x_2, x_3 = \varepsilon x_3} \\ &= \varepsilon^2 x_2^2 + \varepsilon^4\left(x_3^4 - 2x_1^2 x_2^2\right) + O\left(\varepsilon^6\right) \end{aligned}$$

and no conclusion about the positive definiteness of $V(x)$ can be inferred. Consider the dilation with the vector of weights $w = [2\ 6\ 3]^\top$ and compute

$$\begin{aligned} V\left(\delta_\varepsilon^w x\right) &= \left[x_1^6 - 2x_1^3 x_3^3 - 2x_1^2 x_2^2 + x_2^2 + x_3^4\right]_{x_1 = \varepsilon^2 x_1, x_2 = \varepsilon^6 x_2, x_3 = \varepsilon^3 x_3} \\ &= \varepsilon^{12}\left(x_1^6 + x_2^2 + x_3^4\right) + O\left(\varepsilon^{13}\right). \end{aligned}$$

Since $V^{[12]}(x) = x_1^6 + x_2^2 + x_3^4$ is positive definite about $x = 0$, then $V(x)$ is positive definite too.

The following theorems are classical (see, e.g., [60, 115]).

Theorem 8.2 (The first Lyapunov theorem) *Assume that $f(0) = F(0) = 0$ and that f and F are C^1 at $x = 0$. If there exists a function $V(x)$ being positive definite and* *such that* $L_f V(x)$ *if* $\mathbb{T} = \mathbb{R}$ (*respectively*, $V \circ F(x) - V(x)$ *if* $\mathbb{T} = \mathbb{Z}$) *is negative semi-definite*, *then the origin of systems* (1.1a), (1.1b) *is a stable equilibrium point*.

A function satisfying the conditions of the first Lyapunov Theorem 8.2 is said to be a (*weak*) *Lyapunov function*.

Theorem 8.3 (The second Lyapunov theorem) *Assume that* $f(0) = F(x) = 0$ *and that* f *and* F *are* C^1 *at* $x = 0$. *If there exists a function* $V(x)$ *being positive definite and such that* $L_f V(x)$ *if* $\mathbb{T} = \mathbb{R}$ (*respectively*, $V \circ F(x) - V(x)$ *if* $\mathbb{T} = \mathbb{Z}$) *is negative definite*, *then the origin of systems* (1.1a), (1.1b) *is an asymptotically stable equilibrium point*. *If*, *in addition*, V *is globally positive definite and radially unbounded and* $L_f V(x)$ *if* $\mathbb{T} = \mathbb{R}$ (*respectively*, $V \circ F(x) - V(x)$ *if* $\mathbb{T} = \mathbb{Z}$) *is globally negative definite*, *then the origin is a globally asymptotically stable equilibrium point*.

A function satisfying the conditions of the second Lyapunov Theorem 8.3 is said to be a (*strict*) *Lyapunov function*. Note that if there exists an interval (t_i, t_f) such that $L_f V(x(t)) < 0$ or $V \circ F(x(t)) - V(x(t)) < 0$ for all $t \in (t_i, t_f)$, then $V(x(t))$ is strictly decreasing on (t_i, t_f).

The following theorem, due to Kurzweil [78] (see also Theorem 2.4 of [11]) in the continuous-time case and due to other several authors in the discrete-time case (see Remark 5 at p. 429 of [10]), gives the converse statement of the second Lyapunov Theorem 8.3.

Theorem 8.4 *Let* $f(x)$ *and* $F(x)$ *be continuous at* $x = 0$. *If the origin of systems* (1.1a), (1.1b) *is an asymptotically stable equilibrium point*, *then there exists a strict Lyapunov function* $V(x)$, C^∞ *at* $x = 0$ *if* $\mathbb{T} = \mathbb{R}$ (*respectively*, C^0 *at* $x = 0$ *if* $\mathbb{T} = \mathbb{Z}$).

Remark 8.1 Assume $\mathbb{T} = \mathbb{R}$. If the origin is an asymptotically stable equilibrium point, and the convergence to 0 is exponential (in such a case the origin is *exponentially stable*), i.e., if there exist $k, \lambda, \delta > 0$ such that

$$\left\| \Phi_f(t, x) \right\| \le k e^{-\lambda t} \|x\|, \quad \forall t \in \mathbb{R}^{\ge}, \ \forall \|x\| < \delta,$$

where $\|a\| = \sqrt{a^\top a}$ for $a \in \mathbb{R}^n$, then the construction of a Lyapunov function is very simple. Define

$$V(x) := \lim_{T \to +\infty} \int_0^T \Phi_f^\top(\tau, x) \Phi_f(\tau, x) \, d\tau. \tag{8.1}$$

Clearly,

$$V(x) = \lim_{T \to +\infty} \int_0^T \left\| \Phi_f(\tau, x) \right\|^2 d\tau \le \lim_{T \to +\infty} \int_0^T k^2 e^{-2\lambda \tau} \|x\|^2 \, d\tau \le \frac{k^2}{2\lambda} \|x\|^2,$$

which shows that $V(x)$ in (8.1) is well defined (and non-negative) for all $\|x\| < \delta$. By the uniqueness of the solution, $\Phi_f(\tau, x) = 0$ if and only if $x = 0$, whence such a $V(x)$ is positive definite. Let $x(t) = \Phi_f(t, \xi)$ and consider now $W(t) = V(x(t))$ and compute the time derivative of $W(t)$ (which coincides with $L_f V(x(t))$),

$$L_f V\big(x(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t} \lim_{T\to+\infty} \int_0^T \Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big)\Phi_f\big(\tau, \Phi_f(t,\xi)\big)\,\mathrm{d}\tau$$
$$= \lim_{T\to+\infty} \int_0^T 2\Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big)\left(\frac{\partial \Phi_f(\tau,x)}{\partial x}\right) \circ \Phi_f(t,\xi)\frac{\partial \Phi_f(t,\xi)}{\partial t}\,\mathrm{d}\tau.$$

Now, by definition of flow, $\frac{\partial \Phi_f(t,\xi)}{\partial t} = f \circ \Phi_f(t,\xi)$, whence

$$L_f V\big(x(t)\big) = \lim_{T\to+\infty} \int_0^T 2\Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big)\left(\frac{\partial \Phi_f(\tau,x)}{\partial x} f(x)\right) \circ \Phi_f(t,\xi)\,\mathrm{d}\tau.$$

Since $[f, f] = 0$, by Theorem 3.4, one has $\frac{\partial \Phi_f(\tau,x)}{\partial x} f(x) = f(x) \circ \Phi_f(\tau, x)$, which yields the negative definite function

$$\begin{aligned} L_f V\big(x(t)\big) &= \lim_{T\to+\infty} \int_0^T 2\Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big) f(x) \circ \Phi_f\big(\tau, \Phi_f(t,\xi)\big)\,\mathrm{d}\tau \\ &= \lim_{T\to+\infty} \int_0^T 2\Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big)\frac{\partial \Phi_f(\tau, \Phi_f(t,\xi))}{\partial \tau}\,\mathrm{d}\tau \\ &= \lim_{T\to+\infty} \big[\Phi_f^\top\big(\tau, \Phi_f(t,\xi)\big)\Phi_f\big(\tau, \Phi_f(t,\xi)\big)\big]_0^T \\ &= -\Phi_f^\top\big(0, \Phi_f(t,\xi)\big)\Phi_f\big(0, \Phi_f(t,\xi)\big) = -\Phi_f^\top(t,\xi)\Phi_f(t,\xi) \\ &= -x^\top(t)x(t). \end{aligned}$$

Example 8.2 Consider $f(x) = [-x_1 \;\; -3x_2 + ax_1^2]^\top$, where a is an arbitrary real constant. The flow of f can be easily computed for any a,

$$\Phi_f(t,x) = \begin{bmatrix} \mathrm{e}^{-t}x_1 \\ \mathrm{e}^{-3t}x_2 + a(\mathrm{e}^{-2t} - \mathrm{e}^{-3t})x_1^2 \end{bmatrix}.$$

By (8.1), one computes $V(x) = \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{a}{15}x_2x_1^2 + \frac{a^2}{60}x_1^4$, which is clearly positive definite, with the negative definite derivative $L_f V(x) = -x_1^2 - x_2^2$.

The following theorem is due to Krasowskii and LaSalle in the continuous-time case (see, [60]) and it can be found in [79] for the discrete-time case.

Theorem 8.5 (Krasowskii-LaSalle Theorem) *Assume that $f(0) = 0$, $F(0) = 0$ and that f and F are C^1 at $x = 0$. Let V be a weak Lyapunov function. If the greatest invariant set contained in $\{x \in \mathscr{U}^* : L_f V(x) = 0\}$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\{x \in \mathscr{U}^* : V \circ F(x) - V(x) = 0\}$ if $\mathbb{T} = \mathbb{Z}$), then the origin of systems* (1.1a), (1.1b) *is an asymptotically stable equilibrium point.*

The following theorem is well known [60, 79].

Theorem 8.6 (Stability analysis by linearization) *Assume that* $f(0) = 0$, $F(0) = 0$ *and that* f *and* F *are* C^1 *at* $x = 0$. *Let* $A_C = \frac{\partial f(x)}{\partial x}|_{x=0}$ *and* $A_D = \frac{\partial F(x)}{\partial x}|_{x=0}$.

(8.6.1) *If all eigenvalues of* A_C *have negative real part in the continuous-time case* (*respectively, all eigenvalues of* A_D *have modulus less than* 1 *in the discrete-time case*), *then the origin of systems* (1.1a), (1.1b) *is asymptotically stable.*
(8.6.2) *If matrix* A_C *has one eigenvalue with positive real part in the continuous-time case* (*respectively, matrix* A_D *has one eigenvalue with modulus greater than* 1 *in the discrete-time case*), *then the origin of systems* (1.1a), (1.1b) *is unstable.*

Remark 8.2 Although the proof of Statement (8.6.1) is omitted, Statement (8.6.1) can be easily understood in the continuous-time case, when:

(8.2.1) A_C (not necessarily semi-simple) has negative eigenvalues that do not present resonances, and
(8.2.2) A_C is semi-simple with negative integer eigenvalues.

In case (8.2.1), taking into account the Poincaré–Dulac Theorem 3.33 at p. 118 and Remark 3.40 at p. 133, system (1.1a) is diffeomorphic to its linear part. In case (8.2.2), since all eigenvalues of A_C are negative, by the Poincaré–Dulac Theorem 3.33 at p. 118, there exists a near-identity diffeomorphism $y = \varphi(x)$, analytic at $x = 0$, such that the push-forward of the nonlinear system is in the Poincaré–Dulac normal form. Hence, apart from such a diffeomorphism, assume that $f(x) = A_C x + h(x)$, $[h(x), A_C x] = 0$, $h(x)$ analytic at $x = 0$, $h(0) = 0$, with zero linear part. By Remark 3.33 at p. 115, taking as additional state variables the resonant monomials in $h(x)$, the nonlinear system can be immersed into an extended linear system having all eigenvalues of its dynamic matrix with negative real part.

8.2 Scalar Nonlinear Systems

Consider first the continuous-time case.

Assume that $n = 1$ and that $f(x)$ is C^1 at $x = 0$. As discussed in [61], in the case of a scalar differential equation, if every solution with initial value close to 0 approaches 0 as $t \to +\infty$, then it follows that 0 is stable, namely in case of scalar systems the attractivity implies the stability. However, this is not true when $n > 1$ and the concepts of stability and attractivity are, in general, independent.

Theorem 8.7 *The equilibrium point* $x = 0$ *of* (1.1a) *with* $f(0) = 0$ *and* $n = 1$ *is*:

(8.7.1) stable *if there is a* $\delta > 0$ *such that* $xf(x) \leq 0$ *for all* $x \in \mathbb{R}$, $|x| < \delta$;
(8.7.2) asymptotically stable *if there is a* $\delta > 0$ *such that* $xf(x) < 0$ *for all* $x \in \mathbb{R}$, $|x| < \delta$, $x \neq 0$;

(8.7.3) unstable *if there is a* $\delta > 0$ *such that* $xf(x) > 0$ *for all* $x \in \mathbb{R}$, *with either* $0 < x < \delta$ *or* $-\delta < x < 0$.

Proof The proof of Statements (8.7.1) and (8.7.2) of the theorem can be easily done with the Lyapunov function $V = \frac{1}{2}x^2$. The proof of Statement (8.7.3) of the theorem follows from the fact that, for any $\varepsilon < \delta$, solutions starting in any arbitrarily small neighborhood of x, on the side where $xf(x) > 0$, satisfy $\|x(t)\| > \varepsilon$ for t sufficiently large. In this case, it can be said that the origin is repulsive. □

Theorem 8.8 *If* f *is analytic at* $x = 0$, *then the sufficient conditions of Statements* (8.7.1) *and* (8.7.3) *of Theorem* 8.7 *are also necessary.*

Proof If $f = 0$, then the condition of Statement (8.7.1) of Theorem 8.7 must hold. Let $f(x) = x^h\psi(x)$, with $\psi(0) \neq 0$ and $h \geq 1$ being an arbitrary integer. By Theorem 2.1 of [16], properly amended to include the case of f analytic at $x = 0$, there exists an analytic diffeomorphism $y = \varphi(x)$, $\varphi(x) : \mathscr{U}_0 \to \mathbb{R}^n$, with $\mathscr{U}_0$ being a neighborhood of the origin, $\varphi(0) = 0$ and $\frac{\partial\varphi(x)}{\partial x}|_{x=0} = 1$, such that

$$\varphi_* f(y) = a_h y^h + a_{2h-1} y^{2h-1}, \tag{8.2}$$

with $a_h \neq 0$ (note that if $h = 1$, $\varphi_* f(y) = by$, with $b = 2a_1$). Therefore, there are only four possible cases: (i) h even and $a_h < 0$, (ii) h even and $a_h > 0$, (iii) h odd and $a_h < 0$ and (iv) h odd and $a_h > 0$. If, for any sufficiently small $\varepsilon > 0$, there exist initial conditions arbitrarily close to $y = 0$, such that the corresponding solutions become larger that ε, one of the three cases (i), (ii) and (iv) happens, and therefore the condition of Statement (8.7.3) of Theorem 8.7 must necessarily hold. □

Example 8.3 If $f(x) = x$, $f(x) = \pm x^2 + a_3x^3$ or $f(x) = x^3 + a_5x^5$, then the equilibrium point at $x = 0$ is unstable. If $f(x) = -x$ or $f(x) = -x^3 + a_5x^5$, then the equilibrium point at $x = 0$ is asymptotically stable. If $f(x)$ is analytic at $x = 0$, $f(0) = 0$, then the equilibrium point at $x = 0$ is stable but not asymptotically stable if and only if $f = 0$.

Example 8.4 If $f(x)$ is not analytic at $x = 0$, the analysis is more cumbersome than the one depicted by Theorem 8.8. Let [61]

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ -x^3 \sin(\frac{1}{x}), & \text{if } x \neq 0. \end{cases}$$

The equilibrium points of $f(x)$ are given by $x_{e,k} = \frac{1}{k\pi}$, $k \in \mathbb{Z}$, $k \neq 0$, and by $x_e = 0$. Clearly, $x_e = 0$ is an equilibrium point that is an accumulation point of the other equilibrium points $x_{e,k}$ for $|k| \to +\infty$. By applying Statement (8.7.3) of Theorem 8.7, it is easy to see that the equilibrium point $x_{e,k}$ is unstable if (i) k is even and positive, and (ii) k is odd and negative, whereas it is asymptotically stable if (i) k is even and negative, and (ii) k is odd and positive. Hence, $x_e = 0$ is stable, but not asymptotically stable. Although $x_e = 0$ is stable, condition of Statement (8.7.1) of Theorem 8.7 does not hold (there exists no $\delta > 0$ such that $xf(x) \le 0$ for all $x \in \mathbb{R}$ such that $|x| < \delta$), and Theorem 8.8 cannot be applied since f is not analytic at $x = 0$.

Consider now the discrete-time case.

Combining together the analysis carried out in Sect. 4.8 and Proposition 5.1 of [57], the stability analysis of scalar discrete-time systems $\Delta x = F(x)$, with F being analytic at $x = 0$, can be carried out as follows. Note that, despite F is analytic at $x = 0$, some of the involved diffeomorphisms may be only formal, but this does not invalidate the proposed results, because, in view of the comments made right after Theorem 3.33, this just implies that arbitrarily high order terms (irrelevant for the results discussed here) are neglected.

Let λx, with $\lambda \in \mathbb{R}$, be the linear part of $F(x)$. Assume that $F(x)$ is already in the Poincaré–Dulac normal form, $F(x) = \lambda x + H(x)$, with $H(x)$ being analytic at $x = 0$, $H(0) = 0$, $\frac{\partial H(x)}{\partial x}|_{x=0} = 0$ and $\lfloor \lambda x, H(x) \rfloor = 0$. The linear centralizer of λ is spanned by 1, for any λ. Therefore, $H(x) = \mu(x)x$, where $\mu \in \mathscr{I}_D(Ax)$.

(i) If $|\lambda| \neq 1$ and $\lambda \neq 0$, set $\mathscr{I}_D(Ax)$ is constituted by constants and, therefore, $H(x) = 0$, which implies that

$$F(x) = \lambda x;$$

if $|\lambda| < 1$ (respectively, $|\lambda| > 1$), the origin of the discrete-time system is asymptotically stable (respectively, unstable).

(ii) If $\lambda = 0$, since $\lfloor \lambda x, H(x) \rfloor = H(\lambda x) - \lambda H(x)$, condition $\lfloor \lambda x, H(x) \rfloor = 0$ is satisfied by any $H(x)$, analytic at $x = 0$ with $H(0) = 0$, $\frac{\partial H(x)}{\partial x}|_{x=0} = 0$, namely

$$F(x) = H(x) = x\tilde{H}(x),$$

for some $\tilde{H}(x)$ analytic at $x = 0$, $\tilde{H}(0) = 0$. Taking as a Lyapunov function $V(x) = x^2$, one computes $\Delta V(x) - V(x) = x^2(\tilde{H}^2(x) - 1)$, which is negative definite. Hence, the origin of the discrete-time system is asymptotically stable.

(iii.a) If $\lambda = 1$, set $\mathscr{I}_D(Ax)$ is constituted by arbitrary functions of x. Therefore, one has that $H(x) = C(x)x$, where $C(x)$ is an arbitrary function of x, such that $C(0) = 0$. Let $h \ge 2$ be such that $H(x) = x^h \hat{H}(x)$, with $\hat{H}(0) \neq 0$. By Proposition 5.1 of [57], there exists a formal diffeomorphism, $y = \varphi(x)$ such that

$$\varphi_* F(y) = y + a_h y^h + a_{2h-1} y^{2h-1},$$

with $a_h \neq 0$. Consider the Lyapunov function $V(y) = y^2$, for which

$$\begin{aligned}\Delta V(y) - V(y) &= \left(y + a_h y^h + a_{2h-1} y^{2h-1}\right)^2 - y^2 \\ &= 2a_h y^{h+1} + \left(a_h^2 + 2a_{2h-1}\right) y^{2h} + 2y^{3h-1} a_h a_{2h-1} + a_{2h-1}^2 y^{4h-2},\end{aligned}$$

which is negative definite (respectively, positive definite) if h is odd and $a_h < 0$ (respectively, $a_h > 0$), and therefore the origin is asymptotically stable (respectively, unstable) in such a case.

(iii.b) If $\lambda = -1$, set $\mathscr{I}_D(Ax)$ is constituted by arbitrary functions of x^2. Therefore, $H(x) = C(x^2)x$, where C is an arbitrary function of the argument, such that $C(0) = 0$. Let $h \geq 3$ be that odd number such that $H(x) = x^h \hat{H}(x)$, with $\hat{H}(0) \neq 0$. By Proposition 5.1 of [57], there exists a formal diffeomorphism $y = \varphi(x)$ such that

$$\varphi_* F(y) = -y + a_h y^h + a_{2h-1} y^{2h-1},$$

with $a_h \neq 0$. Consider the Lyapunov function $V(y) = y^2$, for which

$$\begin{aligned}
&\Delta V(y) - V(y) \\
&\quad = \left(-y + a_h y^h + a_{2h-1} y^{2h-1}\right)^2 - y^2 \\
&\quad = -2a_h y^{h+1} + \left(a_h^2 - 2a_{2h-1}\right) y^{2h} + 2a_h a_{2h-1} y^{3h-1} + a_{2h-1}^2 y^{4h-2},
\end{aligned}$$

which is negative definite (respectively, positive definite) if $a_h > 0$ (respectively, $a_h < 0$), and therefore the origin is asymptotically stable (respectively, unstable) in such a case.

8.3 Semi-invariants and Center Manifold for Planar Systems

In this section, the connection between semi-invariants and center manifold is pointed out, by using the Poincaré–Dulac normal form. For simplicity, the analysis is restricted to the planar case.

Consider first the continuous-time case.

When the linear part of a planar system has one simple eigenvalue equal to zero, the *center manifold* theory (see [27, 58] and [69]) is one of the most powerful tools for studying the stability of the origin.

As already mentioned, if ω is a semi-invariant, then the manifold described by $\omega = 0$, if not empty, is invariant. Assume that the matrix A of the linear part of f is diagonal and has two real eigenvalues $\lambda_1 = 0$ and $\lambda_2 = b \neq 0$, $A = \mathrm{diag}\{0, b\}$; assume also that $b < 0$, otherwise the origin is unstable by Theorem 8.6. Call the center the subspace of $\mathbb{R}^2$ spanned by the eigenvector with eigenvalue λ_1. If $\omega = 0$ is tangent with the center at $x = 0$, then $\omega = 0$ is a center manifold. Such planar systems can be studied easily either by the *Shoshitaishvili Theorem* (see [33, 34]), or by using the Poincaré–Dulac normal form. Let $y = \varphi(x)$ be the (possibly, formal) change of coordinates transforming $f(x)$ into its Poincaré–Dulac normal form $\tilde{f}(y)$. The linear centralizer of A is spanned by $\{E, A\}$, with E being the identity matrix; all first integrals of $\frac{\mathrm{d}y}{\mathrm{d}t} = Ay$ are of the form $I = G(y_1)$, with G being an arbitrary function of y_1. Then,

$$\tilde{f}(y) = Ay + \mu_0 Ey + \mu_1 Ay = \begin{bmatrix} \mu_0 y_1 \\ by_2 + y_2(\mu_0 + b\mu_1) \end{bmatrix}, \tag{8.3}$$

with μ_0 and μ_1 being arbitrary functions of y_1, being analytic at $y_1 = 0$ and satisfying

$$\mu_0(0) = \mu_1(0) = 0 \quad \text{and} \quad \left.\frac{\partial \mu_0(y_1)}{\partial y_1}\right|_{y_1=0} = \left.\frac{\partial \mu_1(y_1)}{\partial y_1}\right|_{y_1=0} = 0.$$

Note that, since $\mu_1 + \frac{1}{b}\mu_0 + 1 > 0$ in a neighborhood of the origin, one can define the normalized system $\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}_N(y)$, where

$$\tilde{f}_N(y) = \frac{1}{\mu_1 + \frac{1}{b}\mu_0 + 1}\tilde{f}(y) = \begin{bmatrix} \frac{b\mu_0}{b+\mu_0+b\mu_1} y_1 \\ b y_2 \end{bmatrix};$$

the phase portrait of the normalized system is topologically equivalent to the phase portrait of $\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}(y)$ and, in particular, the stability properties of the origin are the same for both systems. The reasoning above coincides with the Shoshitaishvili Theorem, restricted to planar systems.

A symmetry $\tilde{g}$ of $\tilde{f}$ is given by

$$\tilde{g}(y) = Ay = \begin{bmatrix} 0 \\ b y_2 \end{bmatrix}.$$

The corresponding inverse integrating factor is given by $\omega(y) = b\mu_0 y_1 y_2$ (i.e., $\omega = \det([\tilde{f}\ \tilde{g}])$ as in Sect. 3.6). This gives giving two semi-invariants $\omega_1(y) = y_1$ and $\omega_2(y) = y_2$. The center manifold is described by $\omega_2 = 0$, and $\frac{\mathrm{d}y_1}{\mathrm{d}t} = y_1\mu_0(y_1)$ is the corresponding reduced system. For $b < 0$, the origin is asymptotically stable for the given system if and only if it is such for the reduced system, i.e., if and only if $\mu_0(y_1) < 0$ for all $y_1 \neq 0$ belonging to a neighborhood of $y_1 = 0$; this can be verified, in the original coordinates, using as a Lyapunov function $V(x) = \frac{1}{2}\omega_1^2(x) + \frac{1}{2}\omega_2^2(x)$. Under the above assumption, if the transformation $\varphi(x)$ is convergent (respectively, formal), the system has at least two (respectively, formal) semi-invariants that coincide with the entries of $\varphi(x)$.

Example 8.5 Consider

$$f(x) = \begin{bmatrix} x_2^2(3x_1^2 + 2x_1 - 2) - x_1^3 + x_2^6 - x_2^4(3x_1 + 2) \\ x_1x_2 - x_2^3 - x_2 \end{bmatrix};$$

it can be checked that $g(x) = [2x_2^2\ x_2]^\top$ is a symmetry of f. The corresponding inverse integrating factor is $\omega(x) = -(x_1 - x_2^2)^3 x_2$, which yields two Darboux polynomials $\omega_1(x) = x_1 - x_2^2$ and $\omega_2(x) = x_2$, with corresponding characteristic functions $\lambda_1(x) = -\omega_1^2$ and $\lambda_2(x) = -1 + \omega_1$. The center manifold is characterized by $\omega_2 = 0$, which implies $x_2 = 0$; the corresponding reduced system is obtained from $\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2)$ by letting $x_2 = 0$, thus obtaining $\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1^3$ (this already clarifies that the origin is asymptotically stable by the center manifold theory). Note that $y_1 =$

$\omega_1(x)$, $y_2 = \omega_2(x)$ qualifies as a polynomial diffeomorphism, such that the push-forward of f is in the Poincaré–Dulac normal form, $\varphi_* f(y) = [-y_1^3 - y_2 + y_1 y_2]^\top$. Clearly, the origin is an asymptotically stable equilibrium point, as can be shown with the Lyapunov function $V(x) = \frac{1}{2}\omega_1^2(x) + \frac{1}{2}\omega_2^2(x) = \frac{1}{2}(x_1 - x_2^2)^2 + \frac{1}{2}x_2^2$, having directional derivative along f: $L_f V(x) = -\omega_1^4(x) + (-1 + \omega_1(x))\omega_2^2(x) = -(x_1 - x_2^2)^4 + (-1 + x_1 - x_2^2)x_2^2$.

Consider now the discrete-time case.

If ω is a semi-invariant, then the manifold described by $\omega = 0$, if not empty, is invariant. Assume that the matrix A of the linear part of F is diagonal and has two real eigenvalues $\lambda_1 = \pm 1$ and $\lambda_2 = b$, $|b| < 1$, $A = \mathrm{diag}\{\lambda_1, b\}$: for simplicity, assume also $b \neq 0$. Call center the subspace of $\mathbb{R}^2$ spanned by the eigenvector with eigenvalue λ_1. If $\omega = 0$ is tangent with the center at $x = 0$, then $\omega = 0$ is a center manifold.

Similarly to the continuous-time case, the analysis of the stability properties of the origin can be done using the Poincaré–Dulac normal form. Let $y = \varphi(x)$ be the (possibly, formal) change of coordinates transforming $F(x)$ into its Poincaré–Dulac normal form $\tilde{F}(y)$. The linear centralizer of A is spanned by $\{E, A\}$, being E the identity matrix; all first integrals of $\frac{\mathrm{d}y}{\mathrm{d}t} = Ay$ are of the form $I = G(y_1)$ if $\lambda_1 = 1$ and of the form $I = G(y_1^2)$ if $\lambda_1 = -1$, with G being an arbitrary function. Then,

$$\tilde{F}(y) = Ay + \mu_0 Ey + \mu_1 Ay = \begin{bmatrix} (\mu_0 \pm (\mu_1 + 1))y_1 \\ (\mu_0 + b(\mu_1 + 1))y_2 \end{bmatrix}, \tag{8.4}$$

with μ_0 and μ_1 being arbitrary functions of I, such that $H(y) = \mu_0 Ey + \mu_1 Ay$ is analytic at $x = 0$, $H(0) = 0$ and $\frac{\partial H(y)}{\partial y}|_{y=0} = 0$.

Clearly, $\omega_1(y) = y_1$ and $\omega_2(y) = y_2$ are two semi-invariants with characteristic functions $\lambda_1(y) = \lambda_1(y) = \mu_0 \pm \mu_1 \pm 1$ and $\lambda_2(y) = \mu_0 + b\mu_1 + b$. The center manifold is described by $\omega_2 = 0$, and $\Delta y_1 = (\mu_0 \pm (\mu_1 + 1))y_1$ is the corresponding reduced system. For $|b| < 1$, the origin is asymptotically stable for the given system if and only if it is such for the reduced system. The fact that the reduced system must be asymptotically stable, if the whole system is such, is evident because a solution of the reduced system can be rewritten in the original coordinates as a solution of the whole system. To prove that asymptotic stability of the reduced system implies the asymptotic stability of the whole system, consider a strict Lyapunov function $V_1(y_1)$ for the reduced system (which exists by Theorem 8.4), and use it to write the Lyapunov function $V(x) = V_1(\omega_1(x)) + \omega_2^2(x)$ for the whole system. By computing ΔV in the y-coordinates, one has

$$\Delta V(y) = \Delta V_1(y_1) + \big(\lambda_2^2(y_1) - 1\big)y_2^2.$$

Since V_1 is a strict Lyapunov function for the reduced system and $\lambda_2(0) = b$, $|b| < 1$, then there exists a neighborhood of the origin of $\mathbb{R}^2$ in which ΔV is negative definite, thus proving asymptotic stability. Note that a sufficient condition for asymptotic stability of the reduced system is $|\mu_0 \pm (\mu_1 + 1)| < 1$ for all $y_1 \neq 0$

belonging to a neighborhood of $y_1 = 0$. Under the above assumption, if the transformation $\varphi(x)$ is convergent (respectively, formal), the system has at least two (respectively, formal) semi-invariants that coincide with the entries of $\varphi(x)$.

Example 8.6 Consider

$$F(x) = \begin{bmatrix} -x_1^4x_2^2 - 4x_1^3x_2^4 + x_1^3 - 6x_1^2x_2^6 + 2x_1^2x_2^2 - 4x_1x_2^8 + x_1x_2^4 - x_1 - x_2^{10} - \frac{5}{4}x_2^2 \\ x_1^2x_2 + 2x_1x_2^3 + x_2^5 + \frac{1}{2}x_2 \end{bmatrix};$$

it can be checked that $\omega_1(x) = x_1 + x_2^2$ and $\omega_2(x) = x_2$ are two Darboux polynomials, with characteristic functions $\lambda_1(x) = (x_1 + x_2^2 - 1)(x_1 + x_2^2 + 1)$ and $\lambda_2(x) = x_1^2 + 2x_1x_2^2 + x_2^4 + \frac{1}{2}$, respectively. The center manifold is characterized by $\omega_2 = 0$, which implies $x_2 = 0$; the corresponding reduced system is obtained from $\Delta x_1 = F_1(x_1, x_2)$ by letting $x_2 = 0$, thus obtaining $\Delta x_1 = x_1^3 - x_1$, which has $x_1 = 0$ as an asymptotically stable equilibrium point, as one can see with the Lyapunov function $W(x_1) = x_1^2$, for which

$$\Delta W(x_1) - W(x_1) = -\big(2 - x_1^2\big)x_1^4.$$

Note that $y_1 = \omega_1(x)$, $y_2 = \omega_2(x)$ qualifies as a polynomial diffeomorphism, such that the push-forward of F is in the Poincaré–Dulac normal form,

$$\varphi_* F(y) = \begin{bmatrix} y_1^3 - y_1 \\ y_2y_1^2 + \frac{1}{2}y_2 \end{bmatrix}.$$

Clearly, the origin is an asymptotically stable equilibrium point of the original discrete-time system, as can be shown with the Lyapunov function $V(x) = \omega_1^2(x) + \omega_2^2(x) = (x_1 + x_2^2)^2 + x_2^2$; to see that $\Delta V - V$ is negative definite, it is sufficient to expand $\Delta V - V$ in series of homogeneous terms with respect to the dilation $\delta_\varepsilon^w x$, with $w = [2\ 1]^\top$,

$$\Delta V\big(\delta_\varepsilon^w x\big) - V\big(\delta_\varepsilon^w x\big) = -\varepsilon^4\left(2x_1^4 + \frac{3}{4}x_2^2\right) + O\big(\varepsilon^6\big).$$

8.4 Stability of Continuous-Time Critical Planar Systems

In this section, the analysis is restricted to the continuous-time case, with $n = 2$. Assume that f is analytic at $x = 0$, $f(0) = 0$; let $A = \frac{\partial f(x)}{\partial x}|_{x=0}$. By Theorem 8.6, the only cases in which the stability analysis cannot be done from the linear approximation are those in which one eigenvalue of A has zero linear part, and the second eigenvalue has non-positive real part: such cases are called *critical*. Apart from the case $A = 0$ (which is still very challenging) and apart from a linear transformation, there are only three critical cases studied in the following sections:

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \quad a \in \mathbb{R}^{>}, \tag{8.5a}$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{R}^{<}, \tag{8.5b}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{8.5c}$$

8.4.1 Linear Part with Imaginary Eigenvalues

Let f be in the Poincaré–Dulac normal form, with a linear part described by the dynamic matrix given in (8.5a); with no loss of generality, since a linear time scaling does not change the stability properties, assume that $a = 1$. Since the linear centralizer of A (i.e., the set of all matrices B commuting with A) is spanned by the identity matrix E and by A and since all first integrals of $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax$ are arbitrary functions of $x_1^2 + x_2^2$, the vector function f is given by $f = Ax + \mu_0 x + \mu_1 Ax$, with μ_0 and μ_1 being arbitrary functions of $x_1^2 + x_2^2$ such that $\mu_i(0) = 0$, $i = 1, 2$. Then, $g = Ax = [x_2 \;\; -x_1]^\top$ is a symmetry of f. The corresponding inverse integrating factor is

$$\omega = \det\left(\begin{bmatrix} x_2 + \mu_0 x_1 + \mu_1 x_2 & x_2 \\ -x_1 + \mu_0 x_2 - \mu_1 x_1 & -x_1 \end{bmatrix}\right) = -\left(x_1^2 + x_2^2\right)\mu_0.$$

Hence, one has the semi-invariant $\omega = x_1^2 + x_2^2$, with characteristic function $\lambda = 2\mu_0$:

$$L_f \omega = 2\mu_0(\omega)\omega. \tag{8.6}$$

Clearly, the origin $x = 0$ of system (1.1a) is asymptotically stable if and only if the origin $\omega = 0$ of the scalar system (8.6) is asymptotically stable. Since $\mu_0(\omega)$ is assumed analytic at $\omega = 0$, if $\mu_0 \neq 0$, then in a neighborhood of the origin one has $\mu_0(\omega) \approx b_m \omega^m$, for some integer m; if $b_m < 0$, independently of the fact that m is even or odd, then, with the Lyapunov function $V = \frac{1}{2}\omega^2 = \frac{1}{2}(x_1^2 + x_2^2)^2$, having derivative $L_f V = 2\mu_0\omega^2 \approx 2b_m(x_1^2 + x_2^2)^{m+2}$, it is easy to see that the origin of system (1.1a) is an asymptotically stable equilibrium point (exponentially stable if $m = 0$), independently of the expression of function μ_1 (see also [42, 58]).

8.4.2 A Simple Proof of a Bendixson Result for Planar Continuous-Time Systems

The aim of this section is to give a new and simple proof of the subsequent Theorem 8.9, which resumes some results ascribed to Bendixson [17], giving a necessary and sufficient condition for asymptotic stability of the origin for a class of planar systems having the linear part with eigenvalues $\lambda_1 = 0$ and $\lambda_2 < 0$, without using the center manifold.

Note that there are systems, with the origin being an asymptotically stable equilibrium point, for which the origin is not asymptotically stable for the first approximation, for all admissible dilations. In the following Example 8.7 (which is a well known case study, see, e.g., [11]), it is shown that the Darboux polynomials may be actually used for the construction of Lyapunov functions also in this case. Later on, the connection with the mentioned Bendixson result is pointed out.

Example 8.7 Consider $f = f^{\langle 3\rangle} + f^{\langle 4\rangle}$, where

$$f^{\langle 3\rangle}(x) = \begin{bmatrix} -x_1^3 \\ -x_2 \end{bmatrix}, \qquad f^{\langle 4\rangle}(x) = \begin{bmatrix} a_1 x_1 x_2 + a_2 x_1^4 \\ a_3 x_1 x_2 + a_4 x_1^4 \end{bmatrix}.$$

It is clear that $f^{\langle 3\rangle}$ cannot be the first approximation with respect to any integer dilation; in fact, if $f^{\langle 3\rangle}$ was homogeneous (of order m) with respect to some dilation with positive weights w_1 and w_2, then it would be $w_1 - m = 3w_1$ and $w_2 - m = w_2$, thus implying $m = 0$ and $w_1 = 0$. A symmetry of $f^{\langle 3\rangle}$ is $g(x) = [k_1 x_1^3 \; k_2 x_2]^\top$, with $k_1 \neq k_2$; the resulting inverse integrating factor is

$$\omega^{\langle 3\rangle}(x) = \det\left(\begin{bmatrix} -x_1^3 & k_1 x_1^3 \\ -x_2 & k_2 x_2 \end{bmatrix}\right) = (k_1 - k_2) x_1^3 x_2,$$

which yields two Darboux polynomials $\omega_1(x) = x_1$ and $\omega_2(x) = x_2$. Since all monomials appearing in $f^{\langle 3\rangle}$ are homogeneous of degree 3 with respect to $\delta_\varepsilon^w x$, with $w = [1\;3]^\top$, and all monomials appearing in $f^{\langle 4\rangle}$ are homogeneous of degree 4 with respect to the same $\delta_\varepsilon^w x$, instead of constructing a Lyapunov function homogeneous with respect to $\delta_\varepsilon^w x$, it is required that $\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$ are homogeneous with respect to $\delta_\varepsilon^w x$ with the same degree, so that $L_{f^{\langle 3\rangle}} V$ and $L_{f^{\langle 4\rangle}} V$ are homogeneous with respect to $\delta_\varepsilon^w x$ with the degree of $L_{f^{\langle 4\rangle}} V$ being equal to the degree of $L_{f^{\langle 3\rangle}} V$ plus 1. In particular, such a Lyapunov function is $V(x) = \frac{1}{4}\omega_1^4(x) + \frac{1}{2}\omega_2^2(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ ($\frac{\partial V(x)}{\partial x_1} = x_1^3$ and $\frac{\partial V(x)}{\partial x_2} = x_2$ are both homogeneous with respect to $\delta_\varepsilon^w x$ of degree 3); then,

$$L_f V\big(\delta_\varepsilon^w x\big) = L_{f^{\langle 3\rangle}} V\big(\delta_\varepsilon^w x\big) + L_{f^{\langle 4\rangle}} V\big(\delta_\varepsilon^w x\big) = \varepsilon^6 L_{f^{\langle 3\rangle}} V(x) + \varepsilon^7 L_{f^{\langle 4\rangle}} V(x),$$

with

$$L_{f^{\langle 3\rangle}} V(x) = -x_1^6 - x_2^2 \quad \text{and} \quad L_{f^{\langle 4\rangle}} V(x) = a_1 x_1^4 x_2 + a_2 x_1^7 + a_3 x_1 x_2^2 + a_4 x_1^4 x_2.$$

Since $L_{f^{\langle 3\rangle}} V$ is negative definite, there exists ε^* such that $L_f V(x)$ is negative definite for $x = \delta_\varepsilon^w x$, $\|x\| = 1$ and $\varepsilon \in (0, \varepsilon^*)$, which implies that the origin of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = f^{\langle 3\rangle}(x) + f^{\langle 4\rangle}(x)$ is asymptotically stable for all possible values of the parameters a_i's. Not surprisingly, the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, which seems to be natural for the stability analysis of the origin of $\frac{\mathrm{d}x}{\mathrm{d}t} = f^{\langle 3\rangle}(x)$, is not useful for the stability analysis of the origin of $\frac{\mathrm{d}x}{\mathrm{d}t} = f^{\langle 3\rangle}(x) + f^{\langle 4\rangle}(x)$.

In the remainder of this section, the rationale that has been used in Example 8.7, i.e., the search for a Lyapunov function that is not homogeneous itself but having directional derivatives with respect to different parts of f that are (in the scalar sense) homogeneous with respect to a suitable dilation, is used for a constructive proof of the subsequent Theorem 8.9. Such a result is concerned with the systems considered in Sect. 8.3. Rather than finding the center manifold and studying the reduced system, or finding the Poincaré–Dulac normal form, the stability of the origin can be studied simply (and directly in the original coordinates), using the theory due to Bendixson, recalled hereafter.

Consider the system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ written component-wise:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = h_1(x_1, x_2),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = bx_2 + h_2(x_1, x_2),$$

where $b \in \mathbb{R}^{<}$ and functions h_1 and h_2 are zero at the origin together with their first order derivatives. By the Implicit Function Theorem (see [48]), there exists a unique solution $x_2 = k(x_1)$, $k(0) = 0$, of the equation $0 = bx_2 + h_2(x_1, x_2)$ in a neighborhood of $x = 0$. For subsequent developments, it is important to stress that $\frac{\partial k}{\partial x_1}|_{x_1=0} = 0$. Define the function $G(x_1) := h_1(x_1, k(x_1))$ and assume that there exists a finite integer $p \geq 2$ such that $G(x_1) = a_p x_1^p + \cdots$, with $a_p \neq 0$ (function G is assumed to be neither identically equal to zero in a neighborhood of $x_1 = 0$ nor flat at $x_1 = 0$). The following theorem collects some results ascribed to [17] (see also [42] and [71]).

Theorem 8.9 *Assume $b < 1$. If p is odd, the origin of $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ is asymptotically stable (respectively, unstable) if and only if $a_p < 0$ (respectively, $a_p > 0$).*

It is to be noted that the Bendixson analysis deals also with the case of p even, concluding that, in such a case, since the origin is a saddle-node, it is unstable.

Before giving the new proof of Theorem 8.9, the simplicity of its application is illustrated by means of the following classical example, taken from [27].

Example 8.8 Consider the system:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1x_2 + ax_1^3 + bx_1x_2^2,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2 + cx_1^2 + dx_1^2x_2.$$

The equation $0 = -x_2 + cx_1^2 + dx_1^2 x_2$ has the solution $x_2 = c\frac{x_1^2}{1-dx_1^2}$, which yields

$$G(x_1) = (c+a)x_1^3 + \big(cd + bc^2\big)x_1^5 + \big(cd^2 + 2bc^2 d\big)x_1^7 + \cdots,$$

where the dots stand for higher order terms. The origin is asymptotically stable (respectively, unstable) if

(8.8.1) $c + a < 0$ (respectively, $c + a > 0$),
(8.8.2) $c + a = 0$ and $cd + bc^2 < 0$ (respectively, $cd + bc^2 > 0$),
(8.8.3) $c + a = 0$, $cd + bc^2 = 0$ and $cd^2 + 2bc^2 d < 0$ (respectively, $cd^2 + 2bc^2 d > 0$).

Note that if $c + a = 0$, $c(d + bc) = 0$ and $cd(d + 2bc) = 0$, then either $a = c = 0$ (with b and c arbitrary) or $b = d = 0$ and $a = -c$. In both cases the proposed method does not apply, because $G(x_1) = 0$; however, $G(x_1) = 0$ implies that the origin is not an isolated equilibrium, therefore it is not asymptotically stable.

The proof of Theorem 8.9 has been given, in a rather complicated way, by the Bendixson method or by the Frommer method (see for instance [4]). Aim of this section is to show that the proof can be done in a much simpler way, by the selection of a Lyapunov function V, according to what has been done in Example 8.7. The advantage of the presented proof is its simplicity and the construction in closed form of a Lyapunov function; in addition, it seems that such an analysis can easily be extended to the case of a non-planar system of the form $\frac{\mathrm{d}x_1}{\mathrm{d}t} = h_1(x_1, x_2)$, $\frac{\mathrm{d}x_2}{\mathrm{d}t} = Ax_2 + h_2(x_1, x_2)$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{n-1}$, with h_i containing second and higher order terms and with the spectrum of A in the open left half-plane [69].

Proof Consider the change of coordinates $y_1 = x_1$, $y_2 = x_2 - k(x_1)$ and the corresponding transformed system. Taking the time derivative of y_1, one has

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = h_1\big(y_1, y_2 + k(y_1)\big);$$

defining $\chi_1(\theta) := h_1(y_1, \theta y_2 + k(y_1))$, from the equality

$$\chi_1(1) - \chi_1(0) = \int_0^1 \frac{\partial \chi_1}{\partial \theta}\,\mathrm{d}\theta$$

it follows that

$$h_1\big(y_1, y_2 + k(y_1)\big) - h_1\big(y_1, k(y_1)\big) = F_1(y_1, y_2)y_2,$$

where

$$F_1(y_1, y_2) = \int_0^1 \left.\frac{\partial h_1(x_1, x_2)}{\partial x_2}\right|_{x_1=y_1, x_2=\theta y_2 + k(y_1)} \mathrm{d}\theta$$

is analytic in a neighborhood of $y_1 = 0$ and satisfies $F_1(0,0) = 0$. With such a definition, in the new coordinates

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = G(y_1) + F_1(y_1, y_2)y_2.$$

By similarly expanding $h_2(y_1, y_2 + k(y_1))$, and taking into account the definition of $k(x_1)$ given above, one has

$$\begin{aligned}\frac{\mathrm{d}y_2}{\mathrm{d}t} &= by_2 + bk(y_1) + h_2\big(y_1, y_2 + k(y_1)\big) - \frac{\partial k(y_1)}{\partial y_1}\big(G(y_1) + F_1(y_1, y_2)y_2\big)\\ &= by_2 + bk(y_1) + h_2\big(y_1, k(y_1)\big) + \int_0^1 \frac{\partial h_2(x_1, x_2)}{\partial x_2}\bigg|_{x_1 = y_1, x_2 = \theta y_2 + k(y_1)} \mathrm{d}\theta y_2\\ &\quad - \frac{\partial k(y_1)}{\partial y_1}\big(G(y_1) + F_1(y_1, y_2)y_2\big) = by_2 - \frac{\partial k(y_1)}{\partial y_1}G(y_1) + F_2(y_1, y_2)y_2,\end{aligned}$$

where

$$F_2(y_1, y_2) = \int_0^1 \frac{\partial h_2(x_1, x_2)}{\partial x_2}\bigg|_{x_1 = y_1, x_2 = \theta y_2 + k(y_1)} \mathrm{d}\theta - \frac{\partial k(y_1)}{\partial y_1} F_1(y_1, y_2)$$

is analytic in a neighborhood of $y_1 = 0$ and satisfies $F_2(0,0) = 0$. Hence, in the new coordinates the system under study is given by

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t} &= G(y_1) + F_1(y_1, y_2)y_2,\\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= by_2 - \frac{\partial k(y_1)}{\partial y_1}G(y_1) + F_2(y_1, y_2)y_2.\end{aligned}$$

In the case $a_p < 0$, the Lyapunov function $V(y) = \frac{1}{p+1}y_1^{p+1} + \frac{1}{2}y_2^2$ has the following time derivative:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = y_1^p\big(G(y_1) + F_1(y_1, y_2)y_2\big) + y_2\left(by_2 - \frac{\partial k(y_1)}{\partial y_1}G(y_1) + F_2(y_1, y_2)y_2\right);$$

since $F_1(0,0) = F_2(0,0) = \frac{\partial k}{\partial y_1}|_{y_1=0} = 0$, the first term of the homogeneous expansion of $\frac{\mathrm{d}V}{\mathrm{d}t}$ with respect to the dilation characterized by the vector of weights $w = [1 \;\; p]^\top$ is $a_p y_1^{2p} + by_2^2$, which, being negative definite, shows that $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative definite in a sufficiently small neighborhood of the origin, thus implying asymptotic stability of the origin; an estimate of the basin of attraction is given by $\mathscr{U}_\varepsilon = \{y \in \mathbb{R}^2 : \frac{1}{p+1}y_1^{p+1} + \frac{1}{2}y_2^2 \le \varepsilon\}$, for a sufficiently small $\varepsilon > 0$. If, on the other hand, $a_p > 0$, consider the function $V(y) = \frac{1}{p+1}y_1^{p+1} - \frac{1}{2}y_2^2$, which is zero at the origin and is such that the origin is an accumulation point of the set in which $V > 0$. The first term of the homogeneous expansion of $\frac{\mathrm{d}V}{\mathrm{d}t}$ with respect to the same dilation considered above is $a_p y_1^{2p} - by_2^2$, which, being positive definite, shows that $\frac{\mathrm{d}V}{\mathrm{d}t}$

is a positive definite in a sufficiently small neighborhood of the origin, thus implying instability of the origin in view of the Lyapunov first instability theorem (see Exercise 4.11 of [76]). □

The following example illustrates how the new proof of Theorem 8.9 allows to find Lyapunov functions for studying the stability of the origin for the system considered in Example 8.8.

Example 8.9 (Example 8.8 continued) Since $k(x_1) = c\frac{x_1^2}{1-dx_1^2}$, consider the change of coordinates $y_1 = x_1$, $y_2 = x_2 - c\frac{x_1^2}{1-dx_1^2}$. In case (8.8.1) ($c + a \neq 0$), one has $p = 3$, whence if $c + a < 0$, the Lyapunov function that shows the asymptotic stability of the origin is $V(y) = \frac{1}{4}y_1^4 + \frac{1}{2}y_2^2$, whereas if $c + a > 0$, the function that shows the instability of the origin is $V(y) = \frac{1}{4}y_1^4 - \frac{1}{2}y_2^2$. In case (8.8.2) ($c + a = 0$ and $c(d + bc) \neq 0$), one has $p = 5$, whence if $cd + bc^2 < 0$, the Lyapunov function that shows the asymptotic stability of the origin is $V(y) = \frac{1}{6}y_1^6 + \frac{1}{2}y_2^2$, whereas if $cd + bc^2 > 0$, the function that shows the instability of the origin is $V(y) = \frac{1}{6}y_1^6 - \frac{1}{2}y_2^2$. Finally, in case (8.8.3) ($c + a = 0$, $c(d + bc) = 0$ and $cd(d + 2bc) \neq 0$), one has $p = 7$, whence if $cd^2 + 2bc^2d < 0$, the Lyapunov function that shows the asymptotic stability of the origin is $V(y) = \frac{1}{8}y_1^8 + \frac{1}{2}y_2^2$, whereas if $cd^2 + 2bc^2d > 0$, the function that shows the instability of the origin is $V(y) = \frac{1}{8}y_1^8 - \frac{1}{2}y_2^2$.

8.4.3 Stability Analysis for Planar Systems in the Belitskii Normal Form

In this section, a necessary and sufficient condition for asymptotic stability of the origin for a large class of planar systems having as linear part $\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2$, $\frac{\mathrm{d}x_2}{\mathrm{d}t} = 0$ (up to a linear change of coordinates) is provided. An example showing that such a condition can be used for the stabilization of the origin is given in [97]. As a preliminary step for the discussion to follow, a result due to Andreev [3] is resumed here.

Consider a nonlinear system of the following form:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + f_1(x_1, x_2),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2),$$

where $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ (which need not be polynomial) contain second and higher order terms. Assume that $x_2 = \phi(x_1)$, $\phi(0) = 0$, is such that $\phi(x_1) + f_1(x_1, \phi(x_1)) = 0$, $\forall x_1$ in a neighborhood of $x_1 = 0$ (the existence of such a function ϕ is ensured by the Implicit Function Theorem (see [48])). Let

$$G_1(x_1) = f_2\big(x_1, \phi(x_1)\big) = \gamma x_1^h + \cdots,$$

$$G_2(x_1) = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \bigg|_{x_2=\phi(x_1)} = \eta x_1^k + \cdots ,$$

for some non-zero real numbers γ, η and positive integers h, k; for the sake of simplicity, the cases when G_1 or G_2 are identically equal to zero (or flat) are excluded. Note that such a requirement on $G_1(x_1)$ implies that the origin is an isolated equilibrium.

Following Andreev [3], it is known that:

(Q.1) if h is odd, k is even, $\gamma < 0$ and $h > 2k+1$, then $x = 0$ is a (either attractive or repulsive) node;
(Q.2) if h is odd, k is even, $\gamma < 0$, $h = 2k+1$ and $\eta^2 + 4\gamma(k+1) \geq 0$, then $x = 0$ is a (either attractive or repulsive) node;
(Q.3) if h is odd, $\gamma < 0$, $h = 2k+1$ and $\eta^2 + 4\gamma(k+1) < 0$, then $x = 0$ is a center or a (either attractive or repulsive) focus;
(Q.4) if h is odd, $\gamma < 0$ and $h < 2k+1$, then $x = 0$ is a center or a (either attractive or repulsive) focus;

in all other cases, the origin is neither a node nor a center/focus (it is a cusp or a saddle or a saddle-node, or the phase portrait presents an elliptic sector), and therefore it is not asymptotically stable nor even stable.

The organization of the remainder of the section is as follows: first, for systems in the Belitskii normal form, a necessary and sufficient condition for asymptotic stability of the origin is stated in Theorem 8.10, which covers three possible cases, and in Lemma 8.1, which partially covers a fourth, more complex, situation. After such two results, Theorem 8.11 and the discussion leading to it illustrate how to deal with systems that are not given in the Belitskii normal form.

Consider system (1.1a) with $f(x)$ in the Belitskii normal form, in the case its linear part Ax is described by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{8.7}$$

Since the linear centralizer of $A^\top$ is spanned by the identity matrix E and by $A^\top$, and since all first integrals of $\frac{\mathrm{d}x}{\mathrm{d}t} = A^\top x$ are of the form $I = C(x_1)$, with $C(x_1)$ being an arbitrary function of x_1, then the vector function f in the Belitskii normal form is given by

$$f_b(x) = Ax + \alpha(x_1)x + \beta(x_1)A^\top x = \begin{bmatrix} x_2 + \alpha(x_1)x_1 \\ \alpha(x_1)x_2 + \beta(x_1)x_1 \end{bmatrix}, \tag{8.8}$$

where α and β are arbitrary functions of x_1 (that need not be polynomial), analytic at $x_1 = 0$ and satisfying $\alpha(0) = \beta(0) = 0$. As for the application of Andreev result, clearly,

$$\phi(x_1) = -\alpha(x_1)x_1, \tag{8.9a}$$

$$G_1(x_1) = x_1\big(-\alpha^2(x_1) + \beta(x_1)\big), \tag{8.9b}$$

$$G_2(x_1) = \alpha(x_1) + x_1 \frac{\partial \alpha(x_1)}{\partial x_1} + \alpha(x_1). \tag{8.9c}$$

Now, let $\alpha(x_1) = a_m x_1^m + \cdots$ and $\beta(x_1) = b_n x_1^n + \cdots$ for some non-zero real numbers a_m and b_n and positive integers m and n, if function $\beta(x_1)$ is not identically equal to zero, otherwise use the same notation for $\alpha(x_1)$ and let $n = +\infty$ ($G_2(x_1)$ not identically equal to zero implies that $\alpha(x_1)$ is not identically equal to zero). Hence, $G_2(x_1)$ can be expanded as follows:

$$G_2(x_1) = a_m(2+m)x_1^m + \cdots ,$$

from which one has $k = m$ and $\eta = a_m(2+m)$, whereas, excluding for simplicity the special case when $2m = n$ and $b_n = a_m^2$, one has

$$G_1(x_1) = \begin{cases} -a_m^2 x_1^{2m+1} + \cdots , & \text{if } 2m < n, \\ (b_n - a_m^2)x_1^{2m+1} + \cdots , & \text{if } 2m = n \text{ and } b_n - a_m^2 \neq 0, \\ b_n x_1^{n+1} + \cdots , & \text{if } 2m > n. \end{cases}$$

Consequently, since h is given by

$$h = \begin{cases} 2m+1, & \text{if } 2m \leq n, \\ n+1, & \text{if } 2m > n, \end{cases}$$

it can be seen that case (Q.1) of Andreev's study cannot occur; note that this is a consequence of the exclusion of the case when $2m = n$ and $b_n = a_m^2$. Therefore, taking also into account that

$$\gamma = \begin{cases} -a_m^2, & \text{if } 2m < n, \\ b_n - a_m^2, & \text{if } 2m = n \text{ and } b_n - a_m^2 \neq 0, \\ b_n, & \text{if } 2m > n, \end{cases}$$

the following four cases of interest, can be identified:

(A) if $2m < n$ and m is even, then all conditions of case (Q.2) are satisfied and $x = 0$ is a (either attractive or repulsive) node;

(B) if $2m = n$, m is even, and both

$$b_n - a_m^2 < 0, \tag{8.10a}$$

$$m^2 a_m^2 + 4(m+1)b_n \geq 0, \tag{8.10b}$$

then all conditions of case (Q.2) are satisfied and $x = 0$ is a (either attractive or repulsive) node;

(C) if $2m = n$, m is even, and both

$$b_n - a_m^2 < 0, \tag{8.11a}$$

$$m^2 a_m^2 + 4(m+1)b_n < 0, \tag{8.11b}$$

then all conditions of case (Q.3) are satisfied and $x = 0$ is either a center or a (either attractive or repulsive) focus;

(D) if $2m > n$, n and m are even and $b_n < 0$, then all conditions of case (Q.4) are satisfied and $x = 0$ is either a center or a (either attractive or repulsive) focus.

It is to be stressed that, apart from the special case of $2m = n$ and $b_n = a_m^2$, for all systems that do not belong to any of the classes (A)–(D), the origin is either a cusp or a saddle or a saddle-node or the phase portrait presents an elliptic sector (in all such cases the origin is unstable) or the origin is a center-focus. The problem of distinguishing a center from a focus is one of the classical unsolved problems in mathematics [53], and is partially dealt with in cases (C) and (D).

Note that the difference between cases (B) and (C) is the value of the parameter ξ defined as

$$\xi := \eta^2 + 4\gamma(k+1) = m^2 a_m^2 + 4(m+1)b_n;$$

in case (B) the two inequalities (8.10a), (8.10b) are equivalent to $-\frac{m^2 a_m^2}{4(m+1)} \leq b_n < a_m^2$ (i.e., b_n can be positive or negative but with "small" absolute value), whereas in case (C) the two inequalities (8.11a)–(8.11b) are equivalent to

$$b_n < -\frac{m^2 a_m^2}{4(m+1)}. \tag{8.12}$$

The following Theorem 8.10, which is reported from [97], gives a simple necessary and sufficient condition for asymptotic stability of the origin in the three cases (A), (B) and (C); the proof is based on the use of Lyapunov functions derived on the basis of Darboux polynomials for the first approximation of the given system with respect to a given dilation. The more complicate case (D) is dealt with partially in the subsequent Lemma 8.1; note that the necessary and sufficient condition in Theorem 8.10 and in Lemma 8.1 is the same.

Theorem 8.10 *Assume that the functions $G_1(x_1)$ and $G_2(x_1)$ defined in* (8.9a)–(8.9c) *are not identically equal to zero. Assume that the hypotheses of either one of the cases* (*A*), (*B*) *and* (*C*) *are satisfied. Then, the origin $x = 0$ is asymptotically stable for system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_b(x)$, with $f_b(x)$ analytic at $x = 0$ and of the form* (8.8), *if and only if*

$$a_m < 0. \tag{8.13}$$

Proof The proof uses the first approximation of $f_b(x)$ in (8.8) with respect to the vector function $g = [x_1 \; (m+1)x_2]^\top$. By the assumptions made, the first approximation of $f_b(x)$ with respect to g has degree $j^* = -m$ in all three cases (A), (B) and (C): it is denoted by $f^{[-m]}(x)$ and is given by $f^{[-m]}(x) = [x_2 + a_m x_1^{m+1} \; a_m x_1^m x_2]^\top$ in case (A), and by $f^{[-m]}(x) = [x_2 + a_m x_1^{m+1} \; a_m x_1^m x_2 + b_n x_1^{2m+1}]^\top$ in cases (B) and (C). The inverse integrating factor associated with the pair $(f^{[-m]}, g)$ is $\omega = \det([f^{[-m]} \; g])$ (see Sect. 3.6). In case (A), ω can be factorized as $\omega = \omega_1\omega_2$, thus yielding the two Darboux polynomials (with the respective characteristic polynomials):

$$\omega_1(x) = x_2, \qquad \lambda_1 = a_m x_1^m,$$
$$\omega_2(x) = (m+1)x_2 + m a_m x_1^{m+1}, \qquad \lambda_2 = (m+1)a_m x_1^m.$$

Following [88, 96], the positive definite Lyapunov function is

$$V(x) = \frac{1}{2}x_2^2 + \frac{1}{2}\big((m+1)x_2 + m a_m x_1^{m+1}\big)^2,$$

for which

$$L_{f^{[-m]}} V(x) = a_m x_1^m x_2^2 + (m+1)a_m x_1^m \big((m+1)x_2 + m a_m x_1^{m+1}\big)^2.$$

By Andreev's result applied to the first approximation, the origin is a node and therefore it can be either asymptotically stable or unstable (in the second case completely repulsive). If condition (8.13) holds, then $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative semi-definite. Hence, the first approximation is asymptotically stable, thus proving asymptotic stability of the origin for the system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_b(x)$, with $f_b(x)$ of the form (8.8), in view of [11, 106]. If, conversely, $a_m > 0$, then, with the same Lyapunov function, asymptotic stability of the origin can be proven for the system $\frac{\mathrm{d}x}{\mathrm{d}t} = -f_b(x)$, thus showing that the origin is unstable for system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_b(x)$. In case (B), the inverse integrating factor ω can be factorized as $\omega = (m+1)\omega_1\omega_2$, thus yielding the two Darboux polynomials (with the respective characteristic polynomials):

$$\omega_1(x) = x_2 - \frac{(-a_m m + \sqrt{\xi})}{2(m+1)} x_1^{m+1}, \qquad \lambda_1(x) = \frac{1}{2}\big(a_m(m+2) - \sqrt{\xi}\big)x_1^m,$$
$$\omega_2(x) = x_2 - \frac{(-a_m m - \sqrt{\xi})}{2(m+1)} x_1^{m+1}, \qquad \lambda_2(x) = \frac{1}{2}\big(a_m(m+2) + \sqrt{\xi}\big)x_1^m.$$

If $a_m < 0$, then $\frac{1}{2}(a_m(m+2) - \sqrt{\xi})$ is negative; hence, $\frac{\lambda_2}{x_1^m}$ is negative if and only if $\frac{\lambda_1\lambda_2}{x_1^{2m}}$ is positive; in particular

$$\frac{\lambda_1(x)\lambda_2(x)}{x_1^{2m}} = \frac{1}{4}\big(a_m^2(m+2)^2 - \xi^2\big) = \big(a_m^2 - b_n\big)(m+1), \tag{8.14}$$

which is positive by condition (8.10a). Then, a Lyapunov function is $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2$, whose time derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$, under condition (8.13), is negative semi-definite; since also in case (B) the origin is a node for the first approximation, the same reasoning made above for case (A) proves asymptotic stability of the origin for system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_b(x)$, with $f_b(x)$ of the form (8.8). If, conversely, $a_m > 0$, then $\frac{1}{2}(a_m(m+2) + \sqrt{\xi})$ is positive; since $\frac{\lambda_1\lambda_2}{x_1^{2m}}$ is positive (because (8.14) still holds), then also $\frac{1}{2}(a_m(m+2) - \sqrt{\xi})$ is positive; hence, $\frac{\mathrm{d}V}{\mathrm{d}t}$ is positive semi-definite and the instability of the origin follows, as in case (A), considering the system $\frac{dx}{dt} = -f_b(x)$. In case (C), the inverse integrating factor ω is

$$\omega(x) = \begin{bmatrix} x_1^{m+1} & x_2 \end{bmatrix} \begin{bmatrix} -b_n & \frac{a_m m}{2} \\ \frac{a_m m}{2} & (m+1) \end{bmatrix} \begin{bmatrix} x_1^{m+1} \\ x_2 \end{bmatrix}.$$

Since the inequality (8.12) implies that b_n is negative, then, using also condition (8.11a)–(8.11b), it can be seen that ω is a positive definite function of x, to be used as a Lyapunov function:

$$V(x) = x_2^2(m+1) + a_m m x_2 x_1^{m+1} - b_n x_1^{2+2m},$$

with time derivative

$$\frac{dV}{dt} = (2+m) a_m x_1^m \big(x_2^2(m+1) + a_m m x_2 x_1^{m+1} - b_n x_1^{2+2m}\big).$$

If condition (8.13) holds, $\frac{dV}{dt}$ is negative semi-definite, and, using the Krasowskii–LaSalle Theorem 8.5, it can be shown that the origin is asymptotically stable for the first approximation and, as a consequence [11, 106], for system $\frac{dx}{dt} = f_b(x)$, with $f_b(x)$ of the form (8.8). If, conversely, $a_m > 0$, then $\frac{dV}{dt}$ is positive semi-definite; in this case, the Krasowskii–LaSalle Theorem 8.5 can be used to prove that the origin is asymptotically stable for system $\frac{dx}{dt} = -f_b(x)$ and is therefore unstable for system $\frac{dx}{dt} = f_b(x)$. □

The following lemma partially deals with case (D); it is somewhat weaker than Theorem 8.10 because in its proof the asymptotic stability of the origin for system $\frac{dx}{dt} = f_b(x)$ is not proven by means of its first approximation.

Lemma 8.1 *Assume that the functions $G_1(x_1)$ and $G_2(x_1)$ defined in (8.9a)–(8.9c) are not identically equal to zero. Assume that the hypotheses of case (D) are satisfied and, moreover, $\beta(x_1) = b_n x_1^n + \sum_{s=\bar{s}}^{+\infty} \bar{b}_s x_1^{n+s}$, for some $\bar{s}$ such that $2\bar{s} > 2m - n$. Then, the equilibrium point $x = 0$ is asymptotically stable for system $\frac{dx}{dt} = f_b(x)$ with $f_b(x)$ of the form (8.8) if and only if*

$$a_m < 0. \tag{8.15}$$

Proof In this case, using the first approximation of $f_b(x)$ with respect to the vector function $g = [2x_1 \ (n+2)x_2]^\top$, which is given by

$$\frac{dx_1}{dt} = x_2,$$
$$\frac{dx_2}{dt} = b_n x_1^{n+1},$$

the corresponding inverse integrating factor,

$$\omega(x) = \det\left(\begin{bmatrix} x_2 & 2x_1 \\ b_n x_1^{n+1} & (n+2)x_2 \end{bmatrix}\right) = (n+2)x_2^2 - 2b_n x_1^{2+n},$$

is a positive definite function of x, by condition $b_n < 0$, and can be used as a Lyapunov function $V = \omega$. However, its time derivative is identically equal to zero for the first approximation, whence higher order terms are to be taken into account in order to distinguish between a center and a focus. In particular, letting $\alpha(x_1) = a_m x_1^m + \sum_{h=1}^{+\infty} \bar{a}_h x_1^{m+h}$, the time derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$ of $V = \omega$ for the whole system $\frac{\mathrm{d}x}{\mathrm{d}t} = f_b(x)$ can be written as

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 2(n+2)x_1^m\big(a_m W_1(x) + W_2(x)\big),\\
W_1(x) &= x_2^2 - b_n x_1^{2+n},\\
W_2(x) &= \sum_{h=1}^{+\infty} \bar{a}_h x_1^h\big(x_2^2 - b_n x_1^{2+n}\big) + x_1 x_2 \sum_{s=\bar{s}}^{+\infty} \bar{b}_s x_1^{n+s-m}.
\end{aligned}$$

Since W_1 is a positive definite homogeneous function of order $2(n+2)$, with respect to the mentioned dilation, and, under the hypothesis on $\bar{s}$, W_2 only contains terms of order higher than $2(n+2)$, then $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative semi-definite if condition (8.15) holds, whereas it is positive semi-definite if $a_m > 0$. The proof can be completed by the same reasoning made in case (C) of Theorem 8.10. □

It is stressed that the approach considered in this section is quite powerful, because the knowledge of the "exact" Belitskii normal form of the system (and of the change of coordinates that leads to it, which needs not be convergent) is not needed to study the asymptotic stability of the origin with the help of Theorem 8.10. Consider a given vector function f, being $\mathscr{C}^\nu$ at $x = 0$ for a (sufficiently high) integer ν. If $f(0) = 0$ and (8.7) holds, then there exists a near-identity polynomial diffeomorphism $y = T_\nu(x)$ (it can be found with simple computations, [16]) such that in the new coordinates

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{f}(y) = f_{b,\nu}(y) + r_{b,\nu}(y), \tag{8.16}$$

where $f_{b,\nu}(y)$ is polynomial and in the Belitskii normal form and $r_{b,\nu}(y)$ is of order higher than ν (with respect to the standard dilation). If f is analytic or $\mathscr{C}^\infty$ at $x = 0$, then $f_{b,\nu}(y)$ represents the "ν-order approximation" of the exact Belitskii normal form (which is, in general, hard to compute). In the cases when Theorem 8.10 proves asymptotic stability of the origin for the system $\frac{\mathrm{d}y}{\mathrm{d}t} = f_{b,\nu}(y)$, the proof of Theorem 8.10 uses asymptotic stability of the first approximation of $f_{b,\nu}(y)$ with respect to $[y_1\ (m+1)y_2]^\top$ and uses the Darboux polynomials corresponding to such a first approximation to find a Lyapunov function whose time derivative is negative semi-definite; in view of the results in [11] (Theorem 5.8, see also [106]), it is known that another Lyapunov function $\overline{V}$ exists for the first approximation with the property of being homogeneous and with time derivative with respect to the first approximation that is homogeneous and negative definite, this means that $\overline{V}$ can be used to infer asymptotic stability of the origin for the whole system. Such a reasoning allows to prove the following theorem.

Theorem 8.11 (8.11.1) *Consider system* (1.1a) *where* f *is* $\mathscr{C}^{\nu}$ *at* $x = 0$ *and* (8.7) *holds. Let* $f_{b,\nu}(y)$ *be defined as in* (8.16). *If the assumptions and hypotheses of Theorem* 8.10 *hold for* $f_{b,\nu}(y)$, *condition* (8.13) *holds, and* $\nu \geq 2m + 1$, *then the origin is an asymptotically stable equilibrium for system* (1.1a).

(8.11.2) *Consider system* (1.1a) *and assume that* f *is analytic at* $x = 0$ *and there exists an analytic change of coordinates* $z = \varphi(x)$ *that brings the system in the exact Belitskii normal form* $f_b(z)$. *If the assumptions and hypotheses of Theorem* 8.10 *hold for* $f_b(z)$, *then there exists an integer* ν *such that such assumptions and hypotheses hold for its* ν*-order approximation* $f_{b,\nu}(y)$, *and* $\nu \geq 2m + 1$.

Remark 8.3 From a practical point of view Statement (8.11.1) above is stronger than Statement (8.11.2); as a matter of fact it implies that one can infer the asymptotic stability of the origin just computing $f_{b,\nu}(y)$ without even worrying about the existence of an analytic transformation yielding the exact Belitskii normal form. On the other hand, Statement (8.11.2) clarifies that Statement (8.11.1) can be actually applied to a large class of systems. The condition $\nu \geq 2m + 1$ is necessary to ensure that the chosen order of approximation is sufficiently high.

8.5 Construction of Lyapunov Functions Through Darboux Polynomials for Linear Systems

The following theorem gives a way to construct Lyapunov functions, for studying the stability of a linear system, on the basis of Darboux polynomials. It is important to remark that the same Lyapunov function is found both in the continuous-time and discrete-time cases.

Theorem 8.12 *Let* $\omega_i(x)$ *be co-prime Darboux polynomials of systems* (2.1a), (2.1b), $\Delta\omega_i = \lambda_i \omega_i$.

(8.12.1) *If all real numbers* λ_i *are negative in case* $\mathbb{T} = \mathbb{R}$ (*respectively, have absolute value less than* 1 *in case* $\mathbb{T} = \mathbb{Z}$), *then the set described by* $\omega_1 = 0, \omega_2 = 0, \ldots$, *if not empty, is asymptotically stable.*

(8.12.2) *If* $V = \frac{1}{2}\sum_i \omega_i^2$ *is a positive definite function of* $x \in \mathbb{R}^n$ *and all real numbers* λ_i *are negative in case* $\mathbb{T} = \mathbb{R}$ (*respectively, have absolute value less than* 1 *in case* $\mathbb{T} = \mathbb{Z}$), *then the origin of systems* (2.1a), (2.1b) *is asymptotically stable.*

(8.12.3) *If one of the real numbers* λ_i *is equal to zero in case* $\mathbb{T} = \mathbb{R}$ (*respectively, equal to either* 1 *or* -1 *in case* $\mathbb{T} = \mathbb{Z}$), *then the origin of systems* (2.1a), (2.1b) *is not attractive, whence it is not asymptotically stable.*

(8.12.4) *If one of the real numbers* λ_i *is greater than* 0 *in case* $\mathbb{T} = \mathbb{R}$ (*respectively, has absolute value greater than* 1 *in case* $\mathbb{T} = \mathbb{Z}$), *then the origin of systems* (2.1a), (2.1b) *is unstable.*

Proof First, note that $\omega_i(0) = 0, \forall i$. Under the assumptions of Statements (8.12.1) and (8.12.2) of the theorem, one can observe that V is a positive definite function of $\omega_1, \omega_2, \ldots$ and $\frac{\mathrm{d}V}{\mathrm{d}t}$ (respectively, $\Delta V - V$) is a negative definite function of $\omega_1, \omega_2, \ldots$; hence, Statement (8.12.1) of the theorem follows directly, whereas Statement (8.12.2) of the theorem follows by remarking that since $V = 0$ has unique solution $x = 0$, system $\omega_1 = 0, \omega_2 = 0, \ldots$ has unique solution $x = 0$, which again implies that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ (respectively, $\Delta V - V = 0$) has unique solution $x = 0$ because all real numbers λ_i (respectively, $\lambda_i - 1$) are negative; this, noticing that $\frac{\mathrm{d}V}{\mathrm{d}t}$ (respectively, $\Delta V - V$) is non-positive, shows that $\frac{\mathrm{d}V}{\mathrm{d}t}$ (respectively, $\Delta V - V$) is negative definite, thus proving (8.12.1). Under the assumptions of Statement (8.12.3) of the theorem, if $\lambda_i = 0$ when $\mathbb{T} = \mathbb{R}$ (respectively, $|\lambda_i| = 1$ when $\mathbb{T} = \mathbb{Z}$), then $|\omega_i(x(t))| = |\omega_i(x(0))| \neq 0$ for all times $t \in \mathbb{T}$ and for all initial conditions $x(0)$ arbitrarily close to the origin of $\mathbb{R}^n$ such that $\omega_i(x(0)) \neq 0$, which prevents the attractivity to hold. Under the assumptions of Statement (8.12.4) of the theorem, if $\lambda_i > 0$ (respectively, $|\lambda_i| - 1 > 0$), then $\omega_i(x(t))$ tends to infinity for all initial conditions $x(0)$ arbitrarily close to the origin of $\mathbb{R}^n$ such that $\omega_i(x(0)) \neq 0$; since ω_i is a (non-constant) polynomial of x, at least one of the entry $x_i(t)$ of $x(t)$ tends to infinity for all initial conditions $x(0)$ arbitrarily close to the origin of $\mathbb{R}^n$ such that $\omega_i(x(0)) \neq 0$. □

Example 8.10 Consider again the matrix A introduced in Example 2.9 at p. 44. In this case, one computes

$$\begin{aligned}\omega(x) &= \det\big(\Omega(x)\big) = \det\big([Ax\ x]\big) = \det\left(\begin{bmatrix} x_2 & x_1 \\ \alpha x_1 + \beta x_2 & x_2 \end{bmatrix}\right) \\ &= x_2^2 - \beta x_1 x_2 - \alpha x_1^2. \end{aligned} \tag{8.17}$$

(1) If $\beta = 0$ and $\alpha = -\gamma^2$, $\gamma \neq 0$, then matrix A has a pair of imaginary eigenvalues; the resulting $\omega(x) = x_2^2 + \gamma^2 x_1^2$ yields the Lyapunov function candidate $V = \frac{1}{2}(x_2^2 + \gamma^2 x_1^2)^2$, for which

$$L_{Ax} V = \big(x_2^2 + \gamma^2 x_1^2\big)\big[2\gamma^2 x_1\ 2x_2\big]\begin{bmatrix} x_2 \\ -\gamma^2 x_1 \end{bmatrix} = 0, \quad \text{if } \mathbb{T} = \mathbb{R},$$

which shows that the origin is stable in the continuous-time case, and

$$\begin{aligned} V \circ Ax - V &= \frac{1}{2}\big(F_2^2 + \gamma^2 F_1^2\big)^2\big|_{F_1 = x_2, F_2 = -\gamma^2 x_1} - \frac{1}{2}\big(x_2^2 + \gamma^2 x_1^2\big)^2 \\ &= \frac{1}{2}\big(\gamma^4 - 1\big)\big(x_2^2 + \gamma^2 x_1^2\big)^2, \quad \text{if } \mathbb{T} = \mathbb{Z}, \end{aligned}$$

which shows in the discrete-time case that the origin is asymptotically stable if $0 < |\gamma| < 1$, stable if $|\gamma| = 1$ and unstable if $|\gamma| > 1$.

(2) If $\alpha = -\lambda_1\lambda_2$ and $\beta = \lambda_1 + \lambda_2$, with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then matrix A has a pair of negative eigenvalues (λ_1, λ_2); the resulting $\omega(x)$ can be factorized as

$\omega(x) = \omega_1(x)\omega_2(x)$, with $\omega_1(x) = (\lambda_2 x_1 - x_2)$ and $\omega_2(x) = (\lambda_1 x_1 - x_2)$; its factors can be used to construct the Lyapunov function candidate $V = \frac{1}{2}(\lambda_2 x_1 - x_2)^2 + \frac{1}{2}(\lambda_1 x_1 - x_2)^2$, for which

$$L_{Ax} V = \lambda_1(\lambda_2 x_1 - x_2)^2 + \lambda_2(\lambda_1 x_1 - x_2)^2, \quad \text{if } \mathbb{T} = \mathbb{R},$$

which shows that the origin is asymptotically stable in the continuous-time case if $\lambda_1, \lambda_2 < 0$, and

$$V \circ Ax - V = \frac{1}{2}\big(\lambda_1^2 - 1\big)(\lambda_2 x_1 - x_2)^2 + \frac{1}{2}\big(\lambda_2^2 - 1\big)(\lambda_1 x_1 - x_2)^2, \quad \text{if } \mathbb{T} = \mathbb{Z},$$

which shows that the origin is asymptotically stable in the discrete-time case if $|\lambda_1|, |\lambda_2| < 1$.

Under the assumptions of Theorem 2.15 at p. 52, the use of Darboux polynomials of systems (2.1a)–(2.1b) for the construction of a Lyapunov function associated with A, yields that the same Lyapunov function can be used for all systems having dynamic matrix B (and this is useful especially in case of hybrid systems), as shown in the following example.

Example 8.11 Let $A = \left[\begin{smallmatrix} 0 & 1 \\ -2 & -3 \end{smallmatrix}\right]$: such a matrix is semi-simple with distinct eigenvalues, whence $\mathscr{L}_c(A) = \operatorname{span}_{\mathbb{R}}\{E, A\}$. Hence, any $B \in \mathscr{L}_c(A)$ can be expressed as

$$B = \mu_0 \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_0 \\ -2\mu_0 & -3\mu_0 + \mu_1 \end{bmatrix}.$$

By letting $\Omega(x) = [Ax\ x]$ and $\omega(x) = \det(\Omega(x)) = (x_1 + x_2)(2x_1 + x_2)$, one finds that a Lyapunov function for all systems having the dynamic matrix belonging to $\mathscr{L}_c(A)$ is $V = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(2x_1 + x_2)^2$, both in the continuous-time and discrete-time cases. If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} L_{Bx} V &= [5x_1 + 3x_2\ \ 3x_1 + 2x_2] \begin{bmatrix} \mu_1 x_1 + \mu_0 x_2 \\ -2\mu_0 x_1 + (-3\mu_0 + \mu_1) x_2 \end{bmatrix} \\ &= (-2\mu_0 + \mu_1)(x_1 + x_2)^2 + (\mu_1 - \mu_0)(2x_1 + x_2)^2, \end{aligned}$$

which implies that the origin of the continuous-time system $\frac{\mathrm{d}x}{\mathrm{d}t} = Bx$ is asymptotically stable if $-2\mu_0 + \mu_1 < 0$ and $\mu_1 - \mu_0 < 0$; it can be seen, in this simple case, that the matrix B is Hurwitz if and only if such two conditions hold. If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &V \circ Bx - V \\ &\quad = \left(\frac{1}{2}(F_1 + F_2)^2 + \frac{1}{2}(2F_1 + F_2)^2\right)\Bigg|_{F_1 = \mu_1 x_1 + \mu_0 x_2,\, F_2 = -2\mu_0 x_1 + (-3\mu_0 + \mu_1) x_2} \\ &\qquad - \frac{1}{2}(x_1 + x_2)^2 - \frac{1}{2}(2x_1 + x_2)^2 \end{aligned}$$

$$= \frac{1}{2}\big((-2\mu_0 + \mu_1)^2 - 1\big)(x_1 + x_2)^2 + \frac{1}{2}\big((\mu_1 - \mu_0)^2 - 1\big)(2x_1 + x_2)^2,$$

which implies that the origin of the discrete-time system $x(t+1) = Bx(t)$ is asymptotically stable if $(-2\mu_0 + \mu_1)^2 < 1$ and $(\mu_1 - \mu_0)^2 < 1$; also in the discrete-time case, it can be verified that the eigenvalues of B have both modulus smaller than one if and only if such two conditions hold.

8.6 Construction of Lyapunov Functions Through Darboux Polynomials for Nonlinear Systems

What has been done in Sect. 8.5 can be extended to the nonlinear case, as shown in the following example, used to motivate the subsequent Theorem 8.13.

Example 8.12 Let $\pi = \mathbb{R}$ and assume that f is polynomial and homogeneous of degree -2 with respect to the dilation $\delta_\varepsilon^w x$, with the vector of weights $w = [1\ 3]^\top$, i.e., let

$$f(x) = \begin{bmatrix} a_1 x_2 + a_2 x_1^3 \\ a_3 x_1^5 + a_4 x_1^2 x_2 \end{bmatrix};$$

note that the linear part of f is nilpotent and non-zero if $a_1 \neq 0$ (its linear approximation cannot be directly used for stability analysis). Letting $g(x) = [x_1\ 3x_2]^\top$, a Darboux polynomial is given by the inverse integrating factor

$$\omega(x) = \det\big(\big[f(x)\ g(x)\big]\big) = 3a_1 x_2^2 + (3a_2 - a_4)x_1^3 x_2 - a_3 x_1^6.$$

Since $\omega = 0$, if not empty, is an invariant set, then the possible curves obtained by letting $\omega = 0$ divide the plane into open sectors such that if the initial state is in one of these sectors then the state remains there for all times. Darboux polynomials are given by the possible irreducible factors of ω, depending on the values of the parameters a_i's. Consider the systems S^A, S^B and S^C, described, respectively, by:

$$f^A(x) = \begin{bmatrix} x_2 - x_1^3 \\ -x_2 x_1^2 \end{bmatrix}, \qquad f^B(x) = \begin{bmatrix} x_2 - x_1^3 \\ -x_2 x_1^2 + x_1^5 \end{bmatrix},$$

$$f^C(x) = \begin{bmatrix} x_2 - x_1^3 \\ -x_2 x_1^2 + \frac{8}{3} x_1^5 \end{bmatrix};$$

the respective inverse integrating factors are

$$\omega^A(x) = x_2\big(3x_2 - 2x_1^3\big),$$

$$\omega^B(x) = \big(3x_2 + x_1^3\big)\big(x_2 - x_1^3\big),$$

$$\omega^C(x) = \frac{1}{3}\big(3x_2 + 2x_1^3\big)\big(3x_2 - 4x_1^3\big).$$

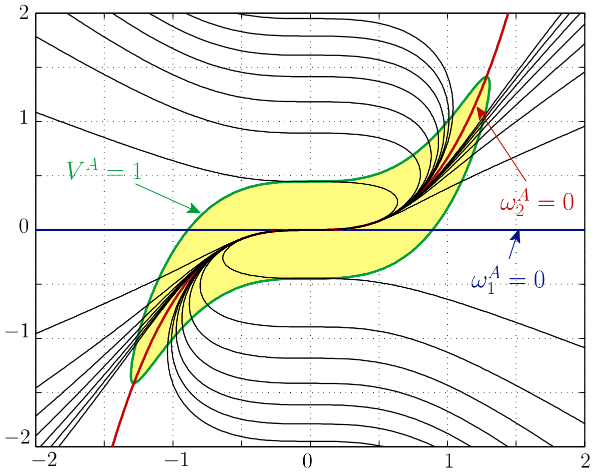


Fig. 8.1 In *black* the state trajectories of system S^A. In *blue* the invariant set $\mathscr{I}_{\omega_1}$, in *red* the invariant set $\mathscr{I}_{\omega_2}$. In *green* the level set $\{x : V^A(x) = 1\}$

By computing the irreducible factors of the inverse integrating factor, one has two Darboux polynomials in each case:

$$\omega_1^A(x) = x_2, \qquad \omega_2^A(x) = 3x_2 - 2x_1^3,$$
$$\omega_1^B(x) = 3x_2 + x_1^3, \qquad \omega_2^B(x) = x_2 - x_1^3,$$
$$\omega_1^C(x) = 3x_2 + 2x_1^3, \qquad \omega_2^C(x) = 3x_2 - 4x_1^3,$$

with respective characteristic polynomials:

$$\lambda_1^A(x) = -x_1^2, \qquad \lambda_2^A(x) = -3x_1^2,$$
$$\lambda_1^B(x) = 0, \qquad \lambda_2^B(x) = -4x_1^2,$$
$$\lambda_1^C(x) = x_1^2, \qquad \lambda_2^C(x) = -5x_1^2.$$

In case A, by choosing the Lyapunov function

$$V^A(x) = \frac{1}{2}\left(\omega_1^A(x)\right)^2 + \frac{1}{2}\left(\omega_2^A(x)\right)^2 = \frac{1}{2}x_2^2 + \frac{1}{2}\left(3x_2 - 2x_1^3\right)^2,$$

one has

$$\frac{\mathrm{d}V^A}{\mathrm{d}t} = -x_1^2\left(\omega_1^A(x)\right)^2 - 3x_1^2\left(\omega_2^A(x)\right)^2 = -x_1^2x_2^2 - 3x_1^2\left(3x_2 - 2x_1^3\right)^2,$$

which is negative semi-definite and, therefore, shows that the origin is stable; the further remark that the origin is the largest invariant set contained in $\frac{\mathrm{d}V^A}{\mathrm{d}t} = 0$ shows, by the Krasowskii–LaSalle Theorem 8.5, that the origin is asymptotically stable (since V^A is radially unbounded, then the origin is globally asymptotically stable). State trajectories and invariant sets of system S^A are depicted in Fig. 8.1. In case B, by choosing the Lyapunov function

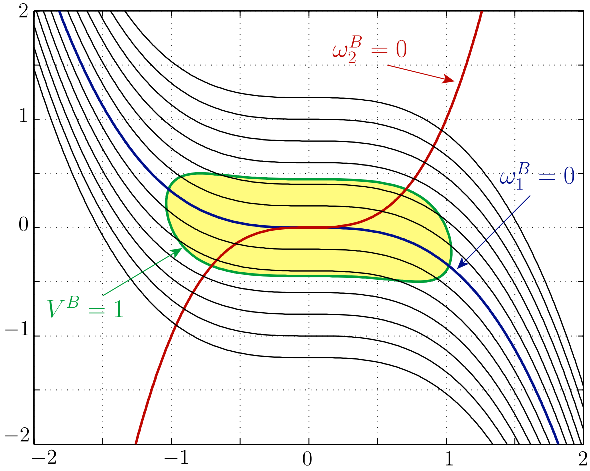


Fig. 8.2 In *black* the state trajectories of system S^B. In *blue* the invariant set $\mathscr{I}_{\omega_1}$, in *red* the invariant set $\mathscr{I}_{\omega_2}$. In *green* the level set $\{x : V^B(x) = 1\}$

$$V^B(x) = \frac{1}{2}\big(\omega_1^B(x)\big)^2 + \frac{1}{2}\big(\omega_2^B(x)\big)^2 = \frac{1}{2}\big(3x_2 + x_1^3\big)^2 + \frac{1}{2}\big(x_2 - x_1^3\big)^2,$$

one has

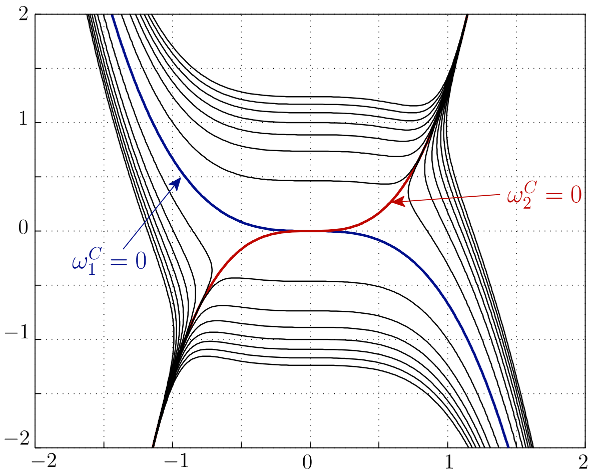
$$\frac{\mathrm{d}V^B}{\mathrm{d}t} = -4x_1^2\big(\omega_2^B(x)\big)^2 = -4x_1^2\big(x_2 - x_1^3\big)^2,$$

which is negative semi-definite and, therefore, shows that the origin is stable; since the curve described by $\omega_1^B = c$ (namely, $x_2 = -\frac{1}{3}x_1^3 + \frac{c}{3}$) is invariant for any real c (because $\frac{\mathrm{d}\omega_1^B}{\mathrm{d}t} = 0$), it does not pass through the origin for $c \neq 0$, and for $c \neq 0$ arbitrarily small it passes through points arbitrarily close to $x = 0$, then the origin is not attractive (this could have been deduced easily before proving stability since $x_2 = x_1^3$ is a set of equilibrium points). State trajectories and invariant sets of system S^B are depicted in Fig. 8.2. In case C, instability of the origin can be proven by means of the Chetaev Theorem (see [115]) using

$$V^C(x) = \frac{1}{2}\big(\omega_1^C(x)\big)^2 - \frac{1}{2}\big(\omega_2^C(x)\big)^2,$$

because, for all x in the set $\mathscr{A} = \{x_1 > 0 \text{ and } x_2 > \frac{1}{3}x_1^3\}$, one has both $V^C(x) > 0$ and $\frac{\mathrm{d}V^C(x)}{\mathrm{d}t} > 0$, and $V^C(x) = 0$ for $x \in \partial\mathscr{A}$. State trajectories and invariant sets of system S^C are depicted in Fig. 8.3. Note that f^A contains monomials of degree less than or equal to 3 with respect to the standard dilation, whereas f^B and f^C are obtained from f^A by adding a term of higher degree with respect to the standard dilation; in particular, the origin of S^A, which is asymptotically stable, is rendered simply stable by adding to f^A the term $h^B(x) = [0\ x_1^5]^\top$ ($f^B = f^A + h^B$) and unstable by adding to f^A the term $h^C(x) = [0\ \frac{8}{3}x_1^5]^\top$ ($f^C = f^A + h^C$). Actually, this has been simply done because f^A and the additional terms h^B, h^C have the same degree with respect to the chosen dilation, with weights $w_1 = 1$, $w_2 = 3$.

Fig. 8.3 In *black* the state trajectories of system S^C. In *blue* the invariant set $\mathscr{I}_{\omega_1}$, in *red* the invariant set $\mathscr{I}_{\omega_2}$



Now, the following theorem can be proven, which gives conditions for the stability analysis of the origin for systems (1.1a), (1.1b); in the cases when it can be applied, its proof gives also a Lyapunov function in closed form.

Theorem 8.13 *Assume that f and F are polynomial, with $f(0) = F(0) = 0$.*

(8.13.1) *Let $\omega_i, i = 1, 2, \dots, m$, be Darboux polynomials of systems* (1.1a), (1.1b), *with λ_i being the respective characteristic polynomials. Let $\lambda = [\lambda_1\ \lambda_2\ \dots\ \lambda_m]]^\top$; if there exist $k \geq 1$ row vectors of positive integers $h_i = [h_{i,1}\ h_{i,2}\ \dots\ h_{i,m}]$, $i = 1, 2, \dots, k$, such that $\tilde{\lambda}_i := h_i\lambda \leq 0$ if $\mathbb{T} = \mathbb{R}$ ($|\tilde{\lambda}_i| \leq 1$, with $\tilde{\lambda}_i := \prod_{j=1}^m \lambda_j^{h_{i,j}}$, if $\mathbb{T} = \mathbb{Z}$) in a neighborhood of the origin, and if $x = 0$ is the only solution of $\sum_{i=1}^k \tilde{\omega}_i^2 = 0$, with $\tilde{\omega}_i = \prod_{\ell=1}^m \omega_\ell^{h_{i,\ell}}$, then the origin is stable for systems* (1.1a), (1.1b). *If the greatest invariant set contained in $\sum_{i=1}^k \tilde{\lambda}_i \tilde{\omega}_i^2 = 0$ is $x = 0$, then the origin is asymptotically stable.*

(8.13.2) *Assume $\mathbb{T} = \mathbb{R}$. Let $\omega_i, i = 1, 2, \dots, m$, be Darboux polynomials of system* (1.1a), *with λ_i being the corresponding characteristic polynomials. Let $\lambda = [\lambda_1\ \lambda_2\ \dots\ \lambda_m]^\top$; if there exists a row vector of positive integers $h = [h_1\ h_2\ \dots\ h_m]$ such that $\tilde{\lambda} := h\lambda \geq 0$ in a neighborhood of the origin, and $\frac{\partial \tilde{\omega}(x)}{\partial x}|_{x=0} \neq 0$, with $\tilde{\omega} = \prod_{\ell=1}^m \omega_\ell^{h_\ell}$, then the origin is not attractive.*

Proof Consider Statement (8.13.1). Since $x = 0$ is the only solution in a neighborhood of the origin of $\sum_{i=1}^k \tilde{\omega}_i^2 = 0$, then $V = \frac{1}{2}\sum_{i=1}^k \tilde{\omega}_i^2$ is a positive definite function of x in a neighborhood of $x = 0$; then, the statement follows from the observation that $L_f V$ and $V \circ F - V$ are negative semi-definite.

Consider Statement (8.13.2). By construction, $\tilde{\omega}$ is a Darboux polynomial with characteristic polynomial $\tilde{\lambda}$. Since, by hypothesis, $\frac{\partial \tilde{\omega}(x)}{\partial x}|_{x=0} \neq 0$, then $y = \tilde{\omega}(x)$ qualifies as a partial diffeomorphism (it can be completed with other coordinates so to obtain a diffeomorphism in a neighborhood of the origin), such that $\frac{\mathrm{d}y}{\mathrm{d}t} =$

$\tilde{\lambda}y$. Since $\tilde{\lambda} \geq 0$ in a neighborhood of the origin, if $y(0) > 0$, then $\frac{\mathrm{d}y}{\mathrm{d}t} \geq 0$ for all admissible t, whence the origin is not attractive. □

The following example illustrates the applicability of Theorem 8.13 based on the computation of orbital symmetries.

Example 8.13 Let $\mathbb{T} = \mathbb{R}$ and $f(x) = [-x_1 x_2^2 \; x_1^3 - \frac{1}{10}x_2^3]^\top$; f is homogeneous of degree -2 with respect to $g = [x_1 \; x_2]^\top$, $\omega = \det([f \; g]) = \omega_1\omega_2\omega_3$, with

$$\omega_1 = x_1, \qquad \omega_2 = x_1 + \sqrt[3]{\frac{9}{10}}x_2, \qquad \omega_3 = x_1^2 - \sqrt[3]{\frac{9}{10}}x_1x_2 + \sqrt[3]{\frac{81}{100}}x_2^2,$$

and the respective characteristic functions are

$$\lambda_1 = -x_2^2, \qquad \lambda_2 = \sqrt[3]{\frac{9}{10}}x_1^2 - \sqrt[3]{\frac{81}{100}}x_1x_2 - \frac{1}{10}x_2^2,$$
$$\lambda_3 = -\sqrt[3]{\frac{9}{10}}x_1^2 + \sqrt[3]{\frac{81}{100}}x_1x_2 - \frac{1}{5}x_2^2;$$

λ_2 and λ_3 are not definite nor semi-definite, whereas both λ_1 and $\tilde{\lambda}_2(x) := \lambda_2(x) + \lambda_3(x) = -\frac{3}{10}x_2^2$ are negative semi-definite. Hence, the positive definite function

$$V(x) = \frac{1}{2}\omega_1^2(x) + \frac{1}{2}\big(\omega_2(x)\omega_3(x)\big)^2 = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_1^3 + \frac{9}{10}x_2^3\right)^2$$

(obtained as in the proof of Theorem 8.13, with $h_1 = [1\;0\;0]$ and $h_2 = [0\;1\;1]$) is such that $\frac{\mathrm{d}V}{\mathrm{d}t} = -x_2^2x_1^2 - \frac{3}{10}x_2^2(x_1^3 + \frac{9}{10}x_2^3)^2$. By the Krasowskii–LaSalle Theorem 8.5, the asymptotic stability of the origin follows.

In the following example, it is shown that the center manifold theory can be generalized through the concept of semi-invariant.

Example 8.14 Consider again system S^C of Example 8.12. The equation $\omega_2^C = 3x_2 - 4x_1^3 = 0$ can be locally rendered explicit with respect to x_2, obtaining $x_2 = \varphi_2(x_1) = \frac{4}{3}x_1^3$; the corresponding reduced system along $\omega_2^C = 0$ is $\frac{\mathrm{d}x_1}{\mathrm{d}t} = h_2(x_1) = \frac{1}{3}x_1^3$. Since $x_1h_2(x_1) = \frac{1}{3}x_1^4$ is positive for any $x_1 \neq 0$, then the origin of S^C is unstable. Consider again system S^B of Example 8.12. The equation $\omega_2^B = x_2 - x_1^3 = 0$ can be locally rendered explicit with respect to x_2, obtaining $x_2 = \varphi_2(x_1) = x_1^3$; the corresponding reduced system is $\frac{\mathrm{d}x_1}{\mathrm{d}t} = h_2(x_1) = 0$. Since $h_2 = 0$ for all x_1, then the origin of S^B is not attractive. Note that, for systems S^B and S^C, the center manifold cannot be defined because the linear approximation has two eigenvalues at the origin. Nevertheless, thanks to semi-invariants, it is still possible to study the stability on a reduced system. This concept can be extended to systems with $n > 2$.

To conclude this section, the following example shows that it is not necessary that f is analytic at $x = 0$ for the semi-invariants to be used for the stability analysis of the origin.

Example 8.15 Consider again the system studied in Example 3.17 at p. 89. Then, there are two semi-invariants $\omega_1(x) = x_1^2 + x_2^2$ and $\omega_2(x) = x_1^2 + x_2^2 - 1$, functionally dependent because $\omega_2 = \omega_1 - 1$. Since $\frac{d\omega_1}{dt} = 2\omega_1(\omega_1 - 1)$, then the equilibrium point $\omega_1 = 0$ and the invariant set $\omega_1 = 1$ are, respectively, asymptotically stable and unstable (use, respectively, the Lyapunov functions $V = \frac{1}{2}\omega_1^2$ and $V = \frac{1}{2}\omega_2^2$).

8.7 Examples of Construction of Lyapunov Functions

In this last section, some examples are proposed of derivation of Lyapunov functions using semi-invariants. Some of the considered systems are classical ones. Despite the fact that all such examples are continuous-time systems, many of the concepts involved can be used, with minor modifications, also for discrete-time systems.

Example 8.16 Let $f(x) = [-x_1 \quad -x_2 + x_1^2 + x_1^3]^\top$. Clearly the origin of this system is at least locally exponentially stable, because the linear part of f has two eigenvalues equal to -1. Since $L_f y_2 = 0$, then $y_2 = \frac{1}{2}\frac{2x_2 + 2x_1^2 + x_1^3 - 3x_1}{x_1}$ is a first integral associated with f, and $\omega_1 = 2x_2 + 2x_1^2 + x_1^3 - 3x_1$ and $\omega_2 = x_1$ are two Darboux polynomials with respective characteristic polynomials $\lambda_1 = -1$ and $\lambda_2 = -1$. The origin is globally asymptotically stable, as can be seen with the (positive definite in the whole and radially unbounded) Lyapunov function $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = \frac{1}{2}(2x_2 + 2x_1^2 + x_1^3 - 3x_1)^2 + \frac{1}{2}x_1^2$, with (negative definite in the whole) derivative $\frac{dV}{dt} = -\omega_1^2 - \omega_2^2 = -(2x_2 + 2x_1^2 + x_1^3 - 3x_1)^2 - x_1^2$.

Example 8.17 Let $f(x) = [x_1\psi(x_1x_2) \; x_2\varphi(x_1x_2)]^\top$, where ψ and φ are arbitrary functions of the argument. Clearly, such a vector function is homogeneous of degree 0 with respect to the dilation $\delta_\varepsilon^w x$, with $w = [-1 \; 1]^\top$. Then, a symmetry of f is $g = [-x_1 \; x_2]^\top$; the corresponding inverse integrating factor is

$$\omega = \det\left(\begin{bmatrix} x_1\psi & -x_1 \\ x_2\varphi & x_2 \end{bmatrix}\right) = x_1x_2(\psi + \varphi).$$

Two semi-invariants are given by $\omega_1 = x_1$ and $\omega_2 = x_2$, with corresponding characteristic functions $\lambda_1 = \psi$ and $\lambda_2 = \varphi$. Since $\psi(\xi)$ and $\varphi(\xi)$ are assumed analytic at $\xi = 0$, then in a neighborhood of the origin one has $\psi(\xi) \approx a\xi^n$ and $\varphi(\xi) \approx b\xi^m$; if $a, b < 0$ and integers n, m are even, then, with the Lyapunov function $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ having derivative $\frac{dV}{dt} = \psi\omega_1^2 + \varphi\omega_2^2 \approx a(x_1x_2)^n x_1^2 + b(x_1x_2)^m x_2^2$, it is easy to see that the origin is a stable equilibrium point (exponentially stable if $n = 0$ and $m = 0$).

Example 8.18 Let

$$f(x) = \begin{bmatrix} (-2 + 2\cos(x_1 + 2x_2) - x_1 \sin(x_1 x_2))^\top \\ (1 - \cos(x_1 + 2x_2) + x_2 \sin(x_1 x_2))^\top \end{bmatrix};$$

since $\mathrm{div}(f) = 0$, the system is area preserving, and, by Statement (3.18.3) of Remark 3.18, an orbital symmetry g of f is given by:

$$g(x) = \begin{bmatrix} \frac{-2+2\cos(x_1+2x_2)-x_1\sin(x_1x_2)}{(-2+2\cos(x_1+2x_2)-x_1\sin(x_1x_2))^2+(1-\cos(x_1+2x_2)+x_2\sin(x_1x_2))^2} \\ \frac{1-\cos(x_1+2x_2)+x_2\sin(x_1x_2)}{(-2+2\cos(x_1+2x_2)-x_1\sin(x_1x_2))^2+(1-\cos(x_1+2x_2)+x_2\sin(x_1x_2))^2} \end{bmatrix}.$$

The corresponding inverse integrating factor is $\omega = 1$. The one-form $[f_2 \ \ -f_1]$ is exact and its integration yields the first integral $I = x_1 + 2x_2 - \sin(x_1 + 2x_2) - \cos(x_1 x_2)$ (i.e., $L_f I = 0$). Since the curve $x_1 + 2x_2 - \sin(x_1 + 2x_2) - \cos(x_1 x_2) = C$, with C being an arbitrary constant, is invariant and does not pass through the origin if $C \neq -1$, whereas for $C \neq -1$ and $|C + 1|$ arbitrarily small such a curve pass through points arbitrarily close to the origin, then the origin is not attractive (this is strictly correlated with the fact that the system is area preserving). Consider the standard dilation with the vector of weights $w = [1 \ 1]^\top$; then $f(\delta_\varepsilon^w x) = \varepsilon^2 f^{[-1]} + O(\varepsilon^3)$, where $f^{[-1]} = [-(x_1 + 2x_2)^2 \ \frac{1}{2}(x_1 + 2x_2)^2]^\top$ is the vector function describing the dynamics of the system in the first approximation. An orbital symmetry $g^{[-1]}$ of $f^{[-1]}$ is $g^{[-1]} = [x_1 \ x_2]^\top$, for which $[f^{[-1]}, g^{[-1]}] = -f^{[-1]}$; the corresponding inverse integrating factor is $\omega^{[-1]} = \det\left(\begin{bmatrix} -(x_1+2x_2)^2 & x_1 \\ \frac{1}{2}(x_1+2x_2)^2 & x_2 \end{bmatrix}\right) = -\frac{1}{2}(x_1 + 2x_2)^3$. In this case, one has the Darboux polynomial $\omega_1^{[-1]} = x_1 + 2x_2$, with characteristic polynomial $\lambda_1^{[-1]} = 0$. Along the invariant curve $x_1 + 2x_2 = C$, with C being an arbitrary constant, one has $x_1 = C - 2x_2$, along which the dynamics of the first approximation are described by $\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{2}C^2$, which shows the instability of the origin of the first approximation. Since $\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{2}C^2$ is the first approximation of the dynamics of the original system along the invariant curve $x_1 + 2x_2 - \sin(x_1 + 2x_2) - \cos(x_1 x_2) = C$, thus proving the instability of the origin.

Example 8.19 Let

$$f(x) = \begin{bmatrix} -ax_1^2 + 2bx_1x_2 + ax_2^2 \\ -bx_1^2 - 2ax_1x_2 + bx_2^2 \end{bmatrix},$$

with a, b being arbitrary real numbers. Since $\frac{\partial f_1(x)}{\partial x_1} = \frac{\partial f_2(x)}{\partial x_2} = -2ax_1 + 2bx_2$ and $\frac{\partial f_1(x)}{\partial x_2} = -\frac{\partial f_2(x)}{\partial x_1} = -2ax_1 + 2bx_2$, by Statement (3.18.4) of Remark 3.18, a symmetry g of f is

$$g(x) = \begin{bmatrix} bx_1^2 + 2ax_1x_2 - bx_2^2 \\ -ax_1^2 + 2bx_1x_2 + ax_2^2 \end{bmatrix}.$$

The corresponding inverse integrating factor is

$$\omega(x) = \det\left(\begin{bmatrix} -ax_1^2 + 2bx_1x_2 + ax_2^2 & bx_1^2 + 2ax_1x_2 - bx_2^2 \\ -bx_1^2 - 2ax_1x_2 + bx_2^2 & -ax_1^2 + 2bx_1x_2 + ax_2^2 \end{bmatrix}\right)$$
$$= (a^2 + b^2)(x_1^2 + x_2^2)^2.$$

Since such an f is also homogeneous with respect to the standard dilation, then another inverse integrating factor is

$$\hat{\omega}(x) = \det\left(\begin{bmatrix} -ax_1^2 + 2bx_1x_2 + ax_2^2 & x_1 \\ -bx_1^2 - 2ax_1x_2 + bx_2^2 & x_2 \end{bmatrix}\right) = (x_1^2 + x_2^2)(bx_1 + ax_2).$$

Hence, $I(x) = \frac{\omega(x)}{\hat{\omega}(x)} = (a^2 + b^2)\frac{x_1^2+x_2^2}{bx_1+ax_2}$ is a first integral associated with f. The first integral $I(x) = \frac{x_1^2+x_2^2}{bx_1+ax_2}$ implies that the curve $x_1^2 + x_2^2 - C(bx_1 + ax_2) = 0$ is invariant for any arbitrary constant C. Assume $b \neq 0$. For any initial condition such that $bx_1(0) + ax_2(0) \neq 0$, the corresponding orbit is a circle, with center $(x_{1,c}, x_{2,c}) = (\frac{Cb}{2}, \frac{Ca}{2})$ and radius equal to $\frac{C}{2}\sqrt{a^2 + b^2}$, passing through $x = 0$, which shows that there are initial conditions arbitrarily close to $x = 0$ for which the corresponding solution, after going arbitrarily far from the origin, tends to zero as time goes to infinity (for such initial conditions, although the origin is attractive, one has an unstable behavior). If $bx_1(0) + ax_2(0) = 0$, since the set $bx_1 + ax_2 = 0$ is invariant (it corresponds to $\omega_2 = 0$), one has $x_1 = -\frac{a}{b}x_2$; the corresponding reduced dynamics is $\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{a^2+b^2}{b}x_2^2$, which shows that the origin is unstable (actually, one has a finite escape time for all initial conditions $(x_1, x_2) = (-\frac{a}{b}x_2, x_2)$, with $\frac{a^2+b^2}{b}x_2^2 > 0$). Assuming that the two points at infinity of the straight line $bx_1 + ax_2 = 0$ are the same point, then this is (for $a = 0$ and $b = 1$, it is the same example used in [60] and [107]) an example of a system having an unstable, but attractive equilibrium point.

Example 8.20 Let $f(x) = [\sin(x_2) \;\; -x_1^3]^\top$. Since f_1 is a function of x_2 and f_2 is a function of x_1, then $g(x) = [0 \;\; \frac{1}{\sin(x_2)}]^\top$ is an orbital symmetry of f. The corresponding inverse integrating factor is $\omega = 1$. Also this system is area preserving, and therefore its origin can be at most stable. A first integral associated with f is $I = \frac{1}{4}x_1^4 - \cos(x_2)$, i.e., $L_f I = 0$. Since $-\cos(x_2) = -1 + \frac{1}{2}x_2^2 + O(x_2^4)$, then a Lyapunov function is $V = \frac{1}{4}x_1^4 - \cos(x_2) + 1 = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + O(x_2^4)$, which shows the stability of the origin because $L_f V = 0$. Note that this is a critical case that cannot be studied with the linearized system.

Example 8.21 Consider a generalized Lotka–Volterra planar system described by

$$f(x) = \begin{bmatrix} a_1x_1 + b_1x_1x_2 & a_2x_2 + b_2x_1x_2 \end{bmatrix}^\top,$$

with a_1, a_2, b_1, b_2 being arbitrary real numbers. It is well known (see Example 4.1.8 of [56]) that a first integral of this system is $I = -b_2x_1 + b_1x_2 - a_2 \ln|x_1| + a_1 \ln|x_2|$. Since $\frac{\partial I}{\partial x_1} = -b_2 - \frac{\lambda_2}{x_1}$, then an orbital symmetry of f is given by $g = [-\frac{x_1}{b_2x_1+a_2}\ 0]^\top$. The corresponding inverse integrating factor is

$$\omega = \det\left(\begin{bmatrix} a_1x_1 + b_1x_1x_2 & -\frac{x_1}{b_2x_1+a_2} \\ a_2x_2 + b_2x_1x_2 & 0 \end{bmatrix}\right) = x_1x_2.$$

One has two Darboux polynomials $\omega_1 = x_1$ and $\omega_2 = x_2$, with respective characteristic polynomials $\lambda_1 = (a_1 + b_1x_2)$ and $\lambda_2 = (a_2 + b_2x_1)$; $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is a Lyapunov function, with derivative $\frac{\mathrm{d}V}{\mathrm{d}t} = \lambda_1\omega_1^2 + \lambda_2\omega_2^2 = (a_1 + b_1x_2)x_1^2 + (a_2 + b_2x_1)x_2^2$, and the origin is asymptotically stable (actually, exponentially stable) if $a_1, a_2 < 0$ (as could easily be seen from the linearized system).

Example 8.22 Let $f(x) = [x_2\ \ -2x_1(1+3x_1^2)(1+x_1^2) - 3(1+3x_1^2)x_2]^\top$; note that $A = \frac{\partial f}{\partial x}|_{x=0} = \left[\begin{smallmatrix} 0 & 1 \\ -2 & -3 \end{smallmatrix}\right]$ is a Hurwitz matrix. Hence, the origin is locally asymptotically stable. An orbital symmetry is $g(x) = [\frac{x_1+x_1^3}{1+3x_1^2}\ x_2]^\top$; the corresponding inverse integrating factor is

$$\begin{aligned}\omega(x) &= \det\left(\begin{bmatrix} x_2 & \frac{x_1+x_1^3}{1+3x_1^2} \\ -2x_1(1+3x_1^2)(1+x_1^2) - 3(1+3x_1^2)x_2 & x_2 \end{bmatrix}\right) \\ &= \left(x_2 + x_1 + x_1^3\right)\left(x_2 + 2x_1 + 2x_1^3\right).\end{aligned}$$

One has two Darboux polynomials $\omega_1 = x_2 + x_1 + x_1^3$ and $\omega_2 = x_2 + 2x_1 + 2x_1^3$, with respective characteristic polynomials $\lambda_1 = -2(3x_1^2 + 1)$ and $\lambda_2 = -(3x_1^2 + 1)$. One can construct the Lyapunov function $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = \frac{1}{2}(x_2 + x_1 + x_1^3)^2 + \frac{1}{2}(x_2 + 2x_1^3 + 2x_1)^2$, with derivative

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -2\left(3x_1^2 + 1\right)\omega_1^2 - \left(3x_1^2 + 1\right)\omega_2^2 \\ &= -2\left(3x_1^2 + 1\right)\left(x_2 + x_1 + x_1^3\right)^2 - \left(3x_1^2 + 1\right)\left(x_2 + 2x_1^3 + 2x_1\right)^2.\end{aligned}$$

Since V is positive definite in the whole and radially unbounded and $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative definite in the whole, then the origin is a globally asymptotically stable equilibrium point.

Example 8.23 Let $f(x) = [x_2 - x_1^3 + x_1^4\ \ x_1^2(x_1 - 1)(x_2 - x_1^3 + x_1^4)]^\top$. This is of the form $f(x) = [x_2 + ax_1\ \ ax_2 + a^2x_1]^\top$, with $a = -x_1^2 + x_1^3$; hence, it is in the Belitskii normal form. An orbital symmetry is $g = [0\ 1]^\top$; the corresponding inverse integrating factor is

$$\omega(x) = \det\left(\begin{bmatrix} x_2 - x_1^3 + x_1^4 & 0 \\ x_1^2(x_1-1)(x_2 - x_1^3 + x_1^4) & 1 \end{bmatrix}\right) = x_2 - x_1^3 + x_1^4.$$

One Darboux polynomial is given by the inverse integrating factor $\omega_1 = x_2 - x_1^3 + x_1^4$, with characteristic polynomial $\lambda_1 = (-4 + 5x_1)x_1^2$ and a second one by the first integral $\omega_2 = \int a(x_1)\,\mathrm{d}x_1 - x_2 = -\frac{1}{3}x_1^3 + \frac{1}{4}x_1^4 - x_2$, with characteristic polynomial $\lambda_2 = 0$. Then, one can construct a positive definite Lyapunov function $V = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 = \frac{1}{2}(x_2 - x_1^3 + x_1^4)^2 + \frac{1}{2}(-\frac{1}{3}x_1^3 + \frac{1}{4}x_1^4 - x_2)^2$, with derivative $\frac{\mathrm{d}V}{\mathrm{d}t} = \lambda_1\omega_1^2 + \lambda_2\omega_2^2 = (-4+5x_1)x_1^2(x_2 - x_1^3 + x_1^4)^2$ being negative semi-definite, which proves the stability of the origin. Since $\omega_1 = 0$ is an invariant set, one can consider the reduced system along it. The dynamics of the reduced system are described by $\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0$, thus showing that the origin is not attractive.

Example 8.24 Consider $f = f^{[-2]} + f^{[-3]}$, where $f^{[-2]}$ and $f^{[-3]}$ are polynomial and homogeneous of respective degrees -2 and -3 with respect to $g = [x_1 \;\; 3x_2]^\top$, i.e., $[f^{[-2]}, g] = -2f$ and $[f^{[-3]}, g] = -3f$. Take $f^{[-2]} = [x_2 - x_1^3 \;\; -x_1^2x_2]^\top$ and $f^{[-3]} = [a_1x_1^4 + a_2x_1x_2 \;\; a_3x_1^3x_2 + a_4x_1^6]^\top$, where the a_i' are arbitrary reals. The inverse integrating factor associated with $f^{[-2]}$ is $\omega^{[-2]} = \det(\left[\begin{smallmatrix} x_2 - x_1^3 & x_1 \\ -x_1^2x_2 & 3x_2 \end{smallmatrix}\right]) = x_2(3x_2 - 2x_1^3)$; the corresponding Darboux polynomials are $\omega_1^{[-2]} = x_2$ and $\omega_2^{[-2]} = 3x_2 - 2x_1^3$, with corresponding characteristic polynomials $\lambda_1 = -x_1^2$ and $\lambda_2 = -3x_1^2$. A Lyapunov function for the first approximation, with the characteristic of being homogeneous of degree 6 with respect to the given dilation, is $V = \frac{1}{2}(\omega_1^{[-2]})^2 + \frac{1}{2}(\omega_2^{[-2]})^2 = \frac{1}{2}x_2^2 + \frac{1}{2}(3x_2 - 2x_1^3)^2$; in particular, $L_fV = L_{f^{[-2]}}V + L_{f^{[-3]}}V$, with $L_{f^{[-2]}}V = -x_1^2x_2^2 - 3x_1^2(3x_2 - 2x_1^3)^2$ being negative semi-definite and homogeneous of degree 8 and $L_{f^{[-3]}}V = x_1^2b(x)$, where it is remarked that $b(x) = 12a_1x_1^7 - 6a_4x_1^7 - 6a_3x_1^4x_2 + 12a_2x_1^4x_2 - 18a_1x_1^4x_2 + 10x_1^4a_4x_2 + 10x_1a_3x_2^2 - 18a_2x_1x_2^2$ is homogeneous of degree 7. Therefore, from $L_fV(\delta_\varepsilon^r x) = \varepsilon^8 L_{f^{[-2]}}V + \varepsilon^9 L_{f^{[-3]}}V = -\varepsilon^8 x_1^2((x_2^2 + 3(3x_2 - 2x_1^3)^2) - \varepsilon b(x))$, since $x_2^2 + 3(3x_2 - 2x_1^3)^2$ is positive definite, there exists ε^* such that $L_fV(x)$ is semi-definite negative for $x = \delta_\varepsilon^r x$, $\|x\| = 1$ and $\varepsilon \in (0, \varepsilon^*)$, which implies that the origin is stable; then, since the largest invariant subspace contained in $L_fV(x) = 0$ is the origin, by the Krasowskii–LaSalle theorem, the origin is asymptotically stable. Note that, by the method of stability in the first approximation [11, 106], the analysis could have been carried out just on the first approximation $f^{[-2]}$, thus obtaining the same result.

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