EXPLICIT DIFFERENTIAL CHARACTERIZATION OF THE NEWTONIAN FREE PARTICLE SYSTEM IN $m \geq 2$ DEPENDENT VARIABLES

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ABSTRACT. In 1883, as an early result, Sophus Lie established an *explicit* necessary and sufficient condition for an analytic second order ordinary differential equation $y_{xx} = F(x, y, y_x)$ to be equivalent, through a point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$, to the Newtonian free particle equation $Y_{XX} = 0$. This result, preliminary to the deep group-theoretic classification of second order analytic ordinary differential equations, was parachieved later in 1896 by Arthur Tresse, a French student of S. Lie. In the present paper, following closely the original strategy of proof of S. Lie, which we firstly expose and restitute in length, we generalize this explicit characterization to the case of several second order ordinary differential equations. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, or more generally any field of characteristic zero equipped with a valuation, so that $\mathbb{K}$-analytic functions make sense. Let $x \in \mathbb{K}$, let $m \geq 2$, let $y := (y^1, \dots, y^m) \in \mathbb{K}^m$ and let

$$y^1_{xx} = F^1 (x, y, y_x), \dots\dots, y^m_{xx} = F^m (x, y, y_x),$$

be a collection of m analytic second order ordinary differential equations, in general nonlinear. We provide an explicit necessary and sufficient condition in order that this system is equivalent, under a point transformation

$$(x, y^1, \dots, y^m) \mapsto \big(X(x, y), Y^1(x, y), \dots, Y^m(x, y)\big),$$

to the Newtonian free particle system $Y^1_{XX} = \cdots = Y^m_{XX} = 0$. Strikingly, the (complicated) differential system that we obtain is of first order in the case $m \geq 2$, whereas it is of second order in S. Lie's original case $m = 1$.

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§1. INTRODUCTION

1.1. Motivation. Since the deep mathematical investigations of S. Lie (*see* the first chapters of the historical monograph [H2001]), the symmetry group of a system of partial differential equations is now understood as its very core, the most fundamental object attached to it, invariant and coordinate-free, analogous in a rigorous sense to the Galois group of an algebraic equation and as well hidden behind the crude expression of the equation. In a specific class of differential systems, it is often of great interest to focus the mathematical attention on the most symmetric differential equations — namely on those having a group of maximal possible dimension — at least as a first step in classification.

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For instance, according to an early and often cited theorem due to S. Lie, the Newtonian free particle equation $Y_{XX} = 0$, with one degree Y of freedom and one independent variable X (we use the index notation to denote partial derivatives), is the unique (up to coordinate changes) second order ordinary differential equation which admits a point symmetry group of maximal dimension equal to eight ; then its symmetry group is unique and consists of the full group of two-dimensional projective transformations. Sometimes, this equation is called *flat*, such a terminology being inspired by the notion of zero curvature in the sense of C.F. Gauss and B. Riemann, after that É. Cartan attached in [Ca1924] a projective connection, together with some curvature, to any second order differential equation, the class of equations equivalent to $Y_{XX} = 0$ being precisely characterized by the *vanishing* of the curvature. More importantly, S. Lie also obtained in 1883 an explicit characterization of the local equivalence to the Newtonian free particle equation with one degree of freedom $Y_{XX} = 0$, which appears to be rather fundamental, with respect to practical purposes.

Theorem 1.2. (Sophus LIE, [Lie1883], pp. 362–365) *Let $\mathbb{K} = \mathbb{R}$ of $\mathbb{C}$. Let $x \in \mathbb{K}$ and $y \in \mathbb{K}$. A local second order ordinary differential equation $y_{xx} = F(x, y, y_x)$ is equivalent under an invertible point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ to the free particle equation $Y_{XX} = 0$ if and only if the following two conditions are satisfied*:

(i) *$F_{y_x y_x y_x y_x} = 0$, or equivalently F is a degree three polynomial in y_x, namely there exist four functions G, H, L and M of (x, y) such that F can be written as*

$$F(x, y, y_x) = G(x, y) + y_x \cdot H(x, y) + (y_x)^2 \cdot L(x, y) + (y_x)^3 \cdot M(x, y); \tag{1.3}$$

(ii) *the four functions G, H, L and M satisfy the following system of two second order quasi-linear partial differential equations*:

$$\begin{cases} 0 = -2\,G_{yy} + \frac{4}{3}\,H_{xy} - \frac{2}{3}\,L_{xx} + \\ \qquad\qquad + 2\,(G\,L)y - 2\,G_x\,M - 4\,G\,M_x + \frac{2}{3}\,H\,L_x - \frac{4}{3}\,H\,H_y, \\ 0 = -\frac{2}{3}\,H_{yy} + \frac{4}{3}\,L_{xy} - 2\,M_{xx} + \\ \qquad\qquad + 2\,G\,M_y + 4\,G_y\,M - 2\,(H\,M)_x - \frac{2}{3}\,H_y\,L + \frac{4}{3}\,L\,L_x. \end{cases} \tag{1.4}$$

Notice that the second equation in (1.4) is obtained formally from the first equation in (1.4), by replacing $(G, H, L, M) \longmapsto (-M, -L, -H, -G)$ and $(x, y) \mapsto (y, x)$.

Secction 2 of this paper is devoted to a detailed exposition of the original proof of Theorem 1.2 due to S. Lie himself; in fact, to the author's knowledge, there is no modern restitution of this proof in the contemporary literature, whereas the description of an alternative proof of Theorem 1.2 by means of É. Cartan's equivalence method appears in the references [HK1989], [GTW1989], [OL1995], [NS2003]; *see* [Ste1982], [G1989], [OL1995] for background about É. Cartan's theory. We note that in these references (except notably [HK1989]), the already substantial computations are stopped just after the reduction to an $\{e\}$-structure on a eight-dimensional (local) principal bundle over the three-dimensional first order jet space. The vanishing of two (among four) fundamental tensors in the structure equations of the obtained $\{e\}$-structure yields two partial differential equations satisfied by the right hand side $F(x, y, y_x)$, which are equivalent to **(i)** and **(ii)** of Theorem 1.2. We mention that with the help of Maple programming, the complete reduction to an $\{e\}$-structure *on the base* (not only on the principal bundle) is achieved in [HK1989], in the simpler case of so-called *fiber-preserving* transformations, namely point transformations leaving invariant the "vertical" foliation $\{x = \mathrm{ct.}\}$. In the classical reference [TR1896], applying S. Lie's theory of differential invariants (*cf.* [T1934], [OL1995]) and S. Lie's group classification of second order differential equations, A. Tresse produces the complete list of differential invariants for each class of differential equation with fixed group, uneder general point transformations. To the autor's knowledge, the complete confirmation of A. Tresse's results by means of É. Cartan's method has never been achieved, possibly because the computations are much harder than in S. Lie's theory; even a complete modern rewriting of A. Tresse's thesis would require a substantial amount of work.

In sum, we would like to point out that for just characterizing the flat equation $Y_{XX} = 0$, S. Lie's original proof in [Lie1883] is, *from the point of view of the size of computations*, much shorter than the reduction to an $\{e\}$-structure through É. Cartan's method of equivalence. In the references [GTW1989], [OL1995], [NS2003], most straightforward intermediate computations for the reduction to an $\{e\}$-structure are essentially left to the reader (as well as in most of É. Cartan's works), but they are consequent.

On the contrary, in this paper, since we want to provide *two* generalizations of S. Lie's Theorem 1.2, we shall *carefully detail each intermediate computational step*, seeking first the combinatorics of the formal calculations in the case $m = 1$ and devising then the underlying combinatorics for the case $m \geq 2$. Actually, the size of differential expressions is relatively impressive, as will become clear soon.

1.5. Systems of second-order ordinary differential equations. First of all, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ or more generally any field of characteristic zero equipped with a valuation, so that the local $\mathbb{K}$-analyticity of formal power series with coefficients in $\mathbb{K}$ possesses a mathematical signification. Let $x \in \mathbb{K}$, let $m \geq 2$, let $y := (y^1, \dots, y^m) \in \mathbb{K}^m$ and let

$$y^1_{xx}(x) = F^1\left(x, y(x), y_x(x)\right), \dots\dots, y^m_{xx}(x) = F^m\left(x, y(x), y_x(x)\right) \tag{1.6}$$

be a collection of $\mathbb{K}$-analytic second order ordinary differential equations, possibly nonlinear, of the most general form. In [GG1983], it was shown that the Lie symmetry group of this system is at most of dimension $m^2 + 4m + 3$, with the upper bound attained for the flat system $Y^j_{XX} = 0$, $j = 1, \dots, m$. The infinitesimal Lie symmetry algebra of this system is isomorphic to $\mathfrak{sl}(m+2, \mathbb{K})$. In [Ch1939] (for fiber-preserving transformations) and in [Fe1995] (for arbitrary point transformations), the É. Cartan method of equivalence is conducted trough absorptions of torsion, normalizations and prolongations up to the reduction to an $\{e\}$-structure. Because of their real complexity, the computations are achieved in a non-parametric way in these references. As a byproduct of the uniqueness of the obtained $\{e\}$-structure for which all invariants vanish, it is deduced in [Fe1995] that the flat system $Y^j_{XX} = 0$, $j = 1, \dots, m$, is, up to equivalence, the only system of second order possessing a symmetry group of maximal dimension. Similar results appeared previously in [G1988], where an explicit necessary and sufficient condition for the local flatness of linear second order systems is given.

Our first main theorem provides the generalization of S. Lie's theorem to the case of $m \geq 2$ dependent variables. Before stating it, we would like to mention that, throughout this article, we shall *not* adopt the summation convention. Indeed, from our point of view, it is more convenient to see explicitely the classical summation symbol $\sum$ when dealing with rather massive expressions, because it helps to see clear differences between apparently similar terms. In fact, although in the statement of Theorems 1.7 and 1.23 below the summation convention applies implicitely so that one can drop the sums and just look at repetitions of indices, it will happen in the sequel that it is totally impossible to maintain the summation convention coherently, without being forced to mention the numerous repeated indices that should not be summed; *see* for instance the majority of the mathematical equations of Section 4, in which we *never* sum over the repeated indices l_1 or l_2. This is why *we abandon this convention*, definitely. Also, we always put commas between the indices, to prevent ambiguities like: "does 11 mean "one-one" or "eleven"?"; finally, a partial differentiation of an indexed quantity is always appended in index notation, also separated from the indices by a comma: for instance $L^j_{l_1,l_3,y^{l_2}}$ denotes $\partial L^j_{l_1,l_3}/\partial y^{l_2}$ shortly.

Theorem 1.7. ($m = 2$: [N2003]) Suppose $m \geq 2$. *A local system of m second order ordinary differential equations $y^j_{xx} = F^j(x, y, y_x)$, $j = 1, \ldots, m$, is equivalent under an invertible point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ to the free particle system $Y^j_{XX} = 0$, $j = 1, \ldots, m$, if and only if the following two conditions are satisfied*:

(i) *There exist local $\mathbb{K}$-analytic functions G^j, $H^j_{l_1}$, $L^j_{l_1,l_2}$ and M_{l_1,l_2}, where $j, l_1, l_2 = 1, \ldots, m$, enjoying the symmetries $L^j_{l_1,l_2} = L^j_{l_2,l_1}$ and $M_{l_1,l_2} = M_{l_2,l_1}$ and depending only on (x, y) such that the right hand side $F^j(x, y, y_x)$ may be written as the following specific cubic polynomial with respect to y_x*:

$$y^j_{xx} = G^j + \sum_{l_1=1}^m y^{l_1}_x \, H^j_{l_1} + \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x \, y^{l_2}_x \, L^j_{l_1,l_2} + y^j_x \cdot \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x \, y^{l_2}_x \, M_{l_1,l_2}. \tag{1.8}$$

(ii) *The functions G^j, $H^j_{l_1}$, $L^j_{l_1,l_2}$ and M_{l_1,l_2} satisfy the following system of* four families *of* first order *partial differential equations*:

$$\text{(I)} \qquad \left\{ \begin{aligned} 0 = \; & -2\, G^j_{y^{l_1}} + 2\, \delta^j_{l_1} \, G^{l_2}_{y^{l_2}} + H^j_{l_1,x} - \delta^j_{l_1} \, H^{l_2}_{l_2,x} + \\ & + 2 \sum_{k=1}^m G^k \, L^j_{l_1,k} - 2\, \delta^j_{l_1} \sum_{k=1}^m G^k \, L^{l_2}_{l_2,k} + \\ & + \frac{1}{2}\, \delta^j_{l_1} \sum_{k=1}^m H^k_{l_2} \, H^{l_2}_k - \frac{1}{2} \sum_{k=1}^m H^k_{l_1} \, H^j_k, \end{aligned} \right.$$

where the indices j, l_1 *vary in* $\{1, 2, \ldots, m\}$;

$$
\text{(II)}\qquad
\left\{
\begin{aligned}
0 = & -\frac{1}{2}\, H^j_{l_1, y^{l_2}} + \frac{1}{6}\, \delta^j_{l_1}\, H^{l_2}_{l_2, y^{l_2}} + \frac{1}{3}\, \delta^j_{l_2}\, H^{l_1}_{l_1, y^{l_1}} + \\
& + L^j_{l_1, l_2, x} - \frac{1}{3}\, \delta^j_{l_1}\, L^{l_2}_{l_2, l_2, x} - \frac{2}{3}\, \delta^j_{l_2}\, L^{l_1}_{l_1, l_1, x} + \\
& + G^j\, M_{l_1, l_2} - \frac{1}{3}\, \delta^j_{l_1}\, G^{l_2}\, M_{l_2, l_2} - \frac{2}{3}\, \delta^j_{l_2}\, G^{l_1}\, M_{l_1, l_1} + \\
& + \frac{1}{3}\, \delta^j_{l_1} \sum_{k=1}^m G^k\, M_{l_2, k} - \frac{1}{3}\, \delta^j_{l_2} \sum_{k=1}^m G^k\, M_{l_1, k} - \\
& - \frac{1}{2} \sum_{k=1}^m H^j_k\, L^k_{l_1, l_2} + \frac{1}{2} \sum_{k=1}^m H^k_{l_1}\, L^j_{l_2, k} + \\
& + \delta^j_{l_1} \left(\frac{1}{6} \sum_{k=1}^m H^{l_2}_k L^k_{l_2, l_2} - \frac{1}{6} \sum_{k=1}^m H^k_{l_2}\, L^{l_2}_{l_2, k} \right) + \\
& + \delta^j_{l_2} \left(\frac{1}{3} \sum_{k=1}^m H^{l_1}_k\, L^k_{l_1, l_1} - \frac{1}{3} \sum_{k=1}^m H^k_{l_1}\, L^{l_1}_{l_1, k} \right),
\end{aligned}
\right.
$$

where the indices j, l_1, l_2 *vary in* $\{1, 2, \ldots, m\}$;

$$
\text{(III)}\qquad
\left\{
\begin{aligned}
0 = & \, L^j_{l_1, l_2, y^{l_3}} - L^j_{l_1, l_3, y^{l_2}} + \delta^j_{l_3}\, M_{l_1, l_2, x} - \delta^j_{l_2}\, M_{l_1, l_3, x} + \\
& + \frac{1}{2}\, H^j_{l_3}\, M_{l_1, l_2} - \frac{1}{2}\, H^j_{l_2}\, M_{l_1, l_3} + \\
& + \frac{1}{2}\, \delta^j_{l_1} \sum_{k=1}^m H^k_{l_3}\, M_{l_2, k} - \frac{1}{2}\, \delta^j_{l_1} \sum_{k=1}^m H^k_{l_2}\, M_{l_3, k} + \\
& + \frac{1}{2}\, \delta^j_{l_3} \sum_{k=1}^m H^k_{l_1}\, M_{l_2, k} - \frac{1}{2}\, \delta^j_{l_2} \sum_{k=1}^m H^k_{l_1}\, M_{l_3, k} + \\
& + \sum_{k=1}^m L^k_{l_1, l_3}\, L^j_{l_2, k} - \sum_{k=1}^m L^k_{l_1, l_2}\, L^j_{l_3, k},
\end{aligned}
\right.
$$

where the indices j, l_1, l_2, l_3 *vary in* $\{1, \ldots m\}$; *and*

$$
\text{(IV)}\qquad
\left\{
0 = M_{l_1, l_2, y^{l_3}} - M_{l_1, l_3, y^{l_2}} - \sum_{k=1}^m L^k_{l_1, l_2}\, M_{l_3, k} + \sum_{k=1}^m L^k_{l_1, l_3}\, M_{l_2, k},
\right.
$$

where the indices l_1, l_2, l_3 *vary in* $\{1, \ldots, m\}$.

Of course, the form of the right hand side in (1.8) is the analog of the form of the right hand side in (1.3) of S. Lie's theorem (however, we notice that the right hand side of (1.8) is not the most general degree three polynomial in the variables y^j_x, $j = 1, \ldots, m$: some coefficients of the cubic terms vanish). On the contrary, whereas the system satisfied by the functions G, H, L and M was of second order in S. Lie's Theorem 1.2, for $m \geq 2$, the system satisfied by the functions G^j, $H^j_{l_1}$, $L^j_{l_1, l_2}$ and M_{l_1, l_2} is of first order.

We mention that we recover the main theorem of [G1988] in the linear case where $F^j := G^j_0(x) + \sum_{l=1}^m y^l\, G^j_{1,l}(x) + \sum_{l_1=1}^m y^{l_1}_x\, H^j_{l_1}(x)$: putting this expression in the system (I) (the three others vanish identically), one recovers the necessary and sufficient condition discovered in [G1988] for local flatness of linear systems. We also mention that in [Fe1995], the parametric computations are achieved after restriction to the identity of the (prolonged) (prolonged) G-structure, which lightens substantially the computations. The vanishing of the two tensors $\widetilde{P}_i^j$ and $\widetilde{S}_{ikl}^j$ *at the identity* of the structure group, namely, as computed in Lemma 4.1 of [Fe1995], yields (translating into our notation)

(1.9)
$$\left\{\begin{aligned} 0 = & \ (\widetilde{S}^j_{ikl})\Big|_{\mathrm{Id}} = F^j_{y^i_x y^k_x y^l_x} - \frac{1}{n+2} \sum_{l_1=1}^m \sum_{\sigma \in \mathfrak{S}_3} \delta^j_{\sigma(l)} \, F^{l_1}_{y^{l_1}_x y^{\sigma(i)}_x y^{\sigma(k)}_x}, \\ 0 = & \ (\widetilde{P}^j_i)\Big|_{\mathrm{Id}} = \frac{1}{2}\, D\left(F^j_{y^i_x}\right) - F^j_{y^i} - \frac{1}{4} \sum_{k=1}^m F^j_{y^k_x} \, F^k_{y^i_x} - \\ & \qquad - \frac{1}{m}\, \delta^j_i \left[\frac{1}{2}\, D\left(\sum_{k=1}^m F^k_{y^k_x} \right) - \sum_{k=1}^m F^k_{y^k} - \frac{1}{4} \sum_{k=1}^m \sum_{l=1}^m F^l_{y^k_x} \, F^k_{y^l_x} \right], \end{aligned}\right.$$

where we set $D := \frac{\partial}{\partial x} + \sum_{l=1}^m y_x^l \, \frac{\partial}{\partial y^l} + \sum_{l=1}^m F^l \, \frac{\partial}{\partial y_x^l}$, and where $i, \, j, \, k, \, l = 1, \dots, m$. Strikingly, one may check that the first equation is equivalent to the form (1.8) and then that the second equation yields the (complicated) four families of first order partial differential equations (I), (II), (III) and (IV) of Theorem 1.7. Hence the necessary conditions found in [Fe1995] (whose sufficiency was open) were in fact also sufficient! This phenomenon may be explained as follows: as soon as the tensors $\widetilde{S}_{ikl}^j$ vanish, the system enjoys a projective connection (appendix of [Fe1995]); with such a connection, the tensors $\widetilde{P}_i^j$ then transform according to a specific rule via tensorial rotation formulas and their general expression may be deduced from their expression at the identity (*cf.* [M2003]; a similar phenomenon has been observed in [Bi2003]).

Finally, even if the expressions (1.9) are more compact than the (equivalent) conditions in Theorem 1.7, we prefer the expressions of Theorem 1.7, since they are more explicit. If the reader prefers compact expressions and "short" theorems, (s)he may replace the conditions of Theorem 1.7 by (1.9).

1.10. Complete systems of second order partial differential equations in $n \geq 2$ independent variables. We shall also study a second generalization of S. Lie's theorem to the case of one dependent variable $y \in \mathbb{K}$ and $n \geq 2$ independent variables $x = (x^1, \dots, x^n) \in \mathbb{K}^n$. Consider a complete system of local $\mathbb{K}$-analytic second order partial differential equations of the form

$$y_{x^{j_1} x^{j_2}}(x) = F^{j_1, j_2}\left(x, y(x), y_{x^1}(x), \dots, y_{x^n}(x)\right), \qquad j_1, j_2 = 1, \dots n, \tag{1.11}$$

where $F^{j_1,j_2} = F^{j_2,j_1}$. Of course, we assume that this system is completely integrable, namely that the vector field system associated to (1.11) in the second order jet space enjoys the Frobenius involutivity condition (*cf.* for instance [Stk2000], Ch. 1). Concretely, this integrability condition just amounts to say that

$$D_{x^{j_3}}\left(F^{j_1,j_2}\right) = D_{x^{j_2}}\left(F^{j_1,j_3}\right), \tag{1.12}$$

for all $j_1, j_2, j_3 = 1, \dots, n$, where, for $j = 1, \dots, n$, the D_{x^j} are the *total differentiation operators* defined by

$$D_{x^j} := \frac{\partial}{\partial x^j} + y_{x^j} \, \frac{\partial}{\partial y} + \sum_{l=1}^n F^{j,l} \, \frac{\partial}{\partial y_{x^l}}. \tag{1.13}$$

1.14. Motivation from Cauchy-Riemann geometry. The interest to such systems was raised in the early beginnings of the domain of research nowadays called *Cauchy-Riemann geometry* (CR geometry for short). One of the main purposes of this field is to study the invariants of real analytic hypersurfaces M in $\mathbb{C}^{n+1}$ under the pseudo-group of biholomorphic transformations $(z, w) \mapsto (Z(z, w), W(z, w))$, where $Z^1, \dots, Z^n, W$ are holomorphic functions of the $(n + 1)$ complex variables (z, w). Three classical founder memoirs of the subject are [P1907], [Se1931] and [Ca1932]. Let us recall how systems of the form (1.11) are naturally associated to real analytic hypersurfaces in $\mathbb{C}^{n+1}$, an observation firstly made by B. Segre in [Se1931] and then exploited further in [Ca1932], [Ch1975]. We refer the reader to [BER1999], Ch. 1, for elementary background about the local geometry of CR manifolds (however, in this reference, nothing can be found about differential symmetries and invariants of differential equations) and to [Su2001], [NS2003], [GM2003], [GM2004] for more about the canonical correspondence between CR manifolds and certain systems of partial differential equations.

In suitable local holomorphic coordinates $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^{n+1}$, such a real analytic hypersurface passing through the origin may be represented locally as the set of $(z, w) \in \mathbb{C}^{n+1}$ satisfying a holomorphic graphed equation of the form

$$w = \overline{\Theta}(z, \bar{z}, \bar{w}), \tag{1.15}$$

where $\bar{z}$, $\bar{w}$ are the complex conjugates of z, w, where $\overline{\Theta}(z, \bar{z}, \bar{w}) = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \sum_{k \in \mathbb{N}} \Theta_{\alpha,\beta,k}\, z^\alpha\, \bar{z}^\beta\, \bar{w}^k$ is a power series converging in some neighborhood of the origin in $\mathbb{C}^{2n+1}$ which, together with its conjugate $\Theta(\bar{z}, z, w)$, satisfies the functional equation

$$w \equiv \overline{\Theta}(z, \bar{z}, \Theta(\bar{z}, z, w)), \tag{1.16}$$

stemming from the fact that M is a *real* hypersurface in $\mathbb{C}^{n+1}$. Replacing now $\bar{z}$ and $\bar{w}$ in (1.12) by new independent variables $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ and $w \in \mathbb{C}$, we may view the sets

$$\{(z, w) \in \mathbb{C}^{n+1} : \; w - \overline{\Theta}(z, \zeta, \xi) = 0\} \tag{1.17}$$

as a family of *complex hypersurfaces* (complex submanifolds of codimension one in $\mathbb{C}^{n+1}$) graphed over the z space and parametrized by $(\zeta, \xi) \in \mathbb{C}^{n+1}$. Differentiating w with respect to z_k for $k = 1, \dots, n$, we get

$$w_{z_k} = \overline{\Theta}_{z_k}(z, \zeta, \xi). \tag{1.18}$$

In generic cases, the rank at $(\zeta, \xi) = (0, 0)$ of the mapping

$$(\zeta, \xi) \longmapsto \big(\overline{\Theta}(0, \zeta, \xi),\; \overline{\Theta}_{z_1}(0, \zeta, \xi), \dots\dots, \overline{\Theta}_{z_k}(0, \zeta, \xi)\big) \in \mathbb{C}^{n+1} \tag{1.19}$$

is maximal equal to $n + 1$; technically, this rank property holds if and only if the real analytic hypersurface is *Levi-nondegenerate* at the origin, *cf.* [BER1999], Ch. 1 for the definition. In this circumstance, we can solve, by means of the complex analytic implicit function theorem, the parameters (ζ, ξ) with respect to the variables (z, w, w_z), which yields

$$(\zeta, \xi) = \Psi(z, w, w_z), \tag{1.20}$$

for some local $\mathbb{C}^{n+1}$-valued holomorphic mapping Ψ. Finally, differentiating twice w with respect to $z_{j_1}z_{j_2}$ for $j_1, j_2 = 1, \ldots, n$, and replacing (ζ, ξ), we obtain

$$\text{(1.21)} \qquad \begin{cases} w_{z_{j_1}z_{j_2}} & = \overline{\Theta}_{z_{j_1}z_{j_2}}(z, \zeta, \xi) \\ & = \overline{\Theta}_{z_{j_1}z_{j_2}}\left(z, \Psi(z, w, w_z)\right) \\ & =: F^{j_1,j_2}(z, w, w_z). \end{cases}$$

In conclusion we obtain a system of the form (1.11), with different notations. Of course, the system (1.21) is completely integrable, just because $w(z) := \overline{\Theta}(z, \zeta, \xi)$ was the general solution from which it was constructed! More information about the correspondence between a system of partial differential equations of the form (1.9) and an associated *submanifold of solutions* of equation $\{(z, w, \zeta, \xi) \in \mathbb{C}^4 : \ w - \overline{\Theta}(z, \zeta, \xi) = 0\}$ can be found in [GM2003], [M2004a]. In particular, this reference contains a new theory of Lie symmetries of partial differential equations, valuable in the $\mathbb{K}$-analytic category, by concentrating on the submanifold of solutions instead of looking at the skeleton associated to the differential system in the suitable jet space.

The equivalence method for Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$ has been considered by É. Cartan in part II of [Ca1932]. In fact, immediately after he discovered the observation of B. Segre that a second order ordinary differential equation may be associated to a real analytic hypersurface in $\mathbb{C}^2$, É. Cartan, who knew perfectly S. Lie's and A. Tresse's works on differential equations, started his memoir on pseudo-conformal invariants of hypersurfaces in $\mathbb{C}^2$. Later, in 1974, S.-S. Chern (a student of É. Cartan) jointly with J.K. Moser studied in [CM1974] the equivalence method for Levi nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ for $n \geq 2$. In 1975, with slight modifications, S.-S. Chern applied in [Ch1975] the equivalence method for complete, completely integrable systems of partial differential equations of the form (1.21), coming from a hypersurface. As observed in [Fa1980], not all systems of the form (1.11) come from a real analytic hypersurface, but the reduction to an $\{e\}$-structure achieved by S.-S. Chern in [Ch1975] is valid, without almost any modification, for all systems of the general form (1.11), *cf.* also [BN2002] and [Bi2003]. It is important to emphasize that S.-S. Chern's computations have never been achieved in a parametric way, though there has been a vivid continuation of S.-S. Chern and J.K. Moser's work on real analytic hypersurfaces (*cf.* for instance [We1977], [Be1979] [Lo1981], [Kr1987], [Vi1990], [Is1996], [EIS1999], [Lo2001], [Eb2001], [Su2002]. and the references therein).

Recently, S. Neut implemented in [N2003] a general Maple program for the É. Cartan equivalence algorithm. The program takes a differential system as input, the appropriate G-structure (which depends on the chosen class of transformations: fiber preserving, point, contact, Bäcklund, *etc.*) and it provides an associated $\{e\}$-structure together with all relations between the tensors appearing in the final structure equations. The main fruit of this program is the characterization of differential systems for which all invariant tensors on the $\{e\}$-structure are constant, hence having in most cases a symmetry group of maximal possible dimension. For the time being, the program does not incorporate the discussion of the relations between invariants in the case of lower dimensional symmetry groups, as is done for instance in [HK1989]. In the case $n = 2$, this program has been applied to the system (1.11). After three hours of Maple computations, the $\{e\}$-structure obtained in a non-parametric way by S.-S. Chern in [Ch1975] is obtained parametrically by the computer machine (with slightly different normalizations) together will all covariant differential relations between the tensors, whose storage costs 1.1 Mo of memory; one gets that the differential algebra generated by the tensors is generated by a single tensor, over a family of forty eight. The vanishing of this tensor yields the following linear system of second order partial differential equations satisfied by the right members $F^{j_1,j_2}(x, y, y_x)$, extracted from [BN2002] and [N2003]:

$$
(1.22) \qquad \begin{cases}
0 = \dfrac{\partial^2 F^{1,1}}{\partial y_{x^2} \partial y_{x^2}}, \\
0 = \dfrac{\partial^2 F^{2,2}}{\partial y_{x^1} \partial y_{x^1}}, \\
0 = \dfrac{\partial^2 F^{1,2}}{\partial y_{x^2} \partial y_{x^2}} - \dfrac{\partial^2 F^{1,1}}{\partial y_{x^1} \partial y_{x^2}}, \\
0 = \dfrac{\partial^2 F^{1,2}}{\partial y_{x^1} \partial y_{x^1}} - \dfrac{\partial^2 F^{2,2}}{\partial y_{x^1} \partial y_{x^2}}, \\
0 = \dfrac{\partial^2 F^{1,1}}{\partial y_{x^1} \partial y_{x^1}} - 4 \dfrac{\partial^2 F^{1,2}}{\partial y_{x^1} \partial y_{x^2}} + \dfrac{\partial^2 F^{2,2}}{\partial y_{x^2} \partial y_{x^2}}.
\end{cases}
$$

In fact, this system is easily seen to be equivalent to the formulation of Theorem 1.23 just below, in the case $n = 2$. In the case $n = 3$, the computer data are so huge that the complete explicit computation of the $\{e\}$-structure does not succed.

Our second main theorem treats the general case $n \geq 2$ by means of hand computations and following the strategy of S. Lie, *cf.* [M2003]. Since the proof resembles a lot to the proof of Theorem 1.7, we shall not write down the details. We would like to mention that C. Bièche also obtained recently in [Bi2003] a complete proof of this theorem by computing explicitely some (and sufficiently many) of the tensors of S.-S. Chern. But of course, she does not compute all the tensors explicitely, which does the computer program in the case $n = 2$. Finally, we mention that for general $n \geq 2$, the *only if* part of Theorem 1.23 just below was established in [BN2002], [N2003] (in a slightly different form) and that the "if" part was stated there as an open problem.

Theorem 1.23. ($n = 2$: [BN2002], [N2003]; general $n \geq 2$: [Bi2003], [M2003]) Suppose $n \geq 2$. *The above system* (1.11) *is equivalent to the system* $Y_{X^{j_1} X^{j_2}} = 0$, $j_1, j_2 = 1, \dots, n$ *if and only if there exist* arbitrary *functions* G_{j_1,j_2}, $H^{k_1}_{j_1,j_2}$, $L^{k_1}_{j_1}$ *and* M^{k_1} *of the variables* $(x^1, \dots, x^n, y)$ *for* $1 \leq j_1, j_2, k_1 \leq n$ *satisfying* (*of course*!) *the two symmetry conditions* $G_{j_1,j_2} = G_{j_2,j_1}$ *and* $H^{k_1}_{j_1,j_2} = H^{k_1}_{j_2,j_1}$, *such that the equation* (1.11) *is of the specific cubic polynomial form*

$$
(1.24) \qquad y_{x^{j_1} x^{j_2}} = G_{j_1,j_2} + \sum_{k_1=1}^{n} y_{x^{k_1}} \left(H^{k_1}_{j_1,j_2} + \frac{1}{2} y_{x^{j_1}} L^{k_1}_{j_2} + \frac{1}{2} y_{x^{j_2}} L^{k_1}_{j_1} + y_{x^{j_1}} y_{x^{j_2}} M^{k_1} \right),
$$

for $j_1, j_2 = 1, \dots, n$.

Again, the explicit form of the right hand side of (1.24) is the analog of the form of the right hand side of (1.3) in S. Lie's Theorem 1.2; however, we again notice that the right hand side of (1.24) is not the most general degree three polynomial in the variables y_{x^j}. Apparently, the statement seems to be much simpler (and perhaps mysterious or maybe false!) than the statement of Theorem 1.7 above, and even simpler than S. Lie's Theorem 1.2 in the case $n = 1$, because the analog of conditions **(ii)** there do not appear in Theorem 1.23. However, by generalizing S. Lie's computations for the proof of Theorem 1.2, we shall see that there exists in fact a system of *second order partial* differential equations satisfied by the functions G_{j_1,j_2}, $H^{k_1}_{j_1,j_2}$, $L^{k_1}_{j_2}$ and M^{k_1} which is the combinatorial counterpart of the second order system appearing in **(ii)** of Theorem 1.2. By a strange phenomenon, this system of second order partial differential equations is in fact a consequence of the compatibility conditions (1.12). A similar phenomenon also holds for Theorem 1.7: there also exists a system of *second order partial* differential equations satisfied by the functions G^j, $H_{l_1}^j$, L_{l_1,l_2}^j and M_{l_1,l_2} which is the combinatorial counterpart of the second order system appearing in **(ii)** of Theorem 1.2 and which is a consequence of the four families of first order partial differential equations (I), (II), (III) and (IV).

Hence to see explicitely what is the analog of (I), (II), (III) and (IV) in Theorem 1.23, we must develope the compatibility conditions (1.12), in the case where the right hand sides F^{j_1,j_2} are given by the cubic polynomial (1.24). After some nontrivial manual work, we obtain the equation (1.12) in length:

$$
\begin{aligned}
(1.25)\qquad 0 = {} & G_{j_1,j_2,x^{j_3}} - G_{j_1,j_3,x^{j_2}} + \sum_{k_1=1}^n G_{k_1,j_3}\, H_{j_1,j_2}^{k_1} - \sum_{k_1=1}^n G_{k_1,j_2}\, H_{j_1,j_3}^{k_1} + \\
& + \sum_{k_1=1}^n y_{x^{k_1}} \left[
\begin{aligned}
& \delta_{j_3}^{k_1} G_{j_1,j_2,y} - \delta_{j_2}^{k_1} G_{j_1,j_3,y} + H_{j_1,j_2,x^{j_3}}^{k_1} - H_{j_1,j_3,x^{j_2}}^{k_1} + \\
& + \frac{1}{2} G_{j_1,j_3}\, L_{j_2}^{k_1} - \frac{1}{2} G_{j_1,j_2}\, L_{j_3}^{k_1} + \\
& + \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_2=1}^n G_{k_2,j_3}\, L_{j_2}^{k_2} - \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_2=1}^n G_{k_2,j_2}\, L_{j_3}^{k_2} + \\
& + \frac{1}{2} \delta_{j_2}^{k_1} \sum_{k_2=1}^n G_{k_2,j_3}\, L_{j_1}^{k_2} - \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_2=1}^n G_{k_2,j_2}\, L_{j_1}^{k_2} + \\
& + \sum_{k_2=1}^n H_{k_2,j_3}^{k_1}\, H_{j_1,j_2}^{k_2} - \sum_{k_2=1}^n H_{k_2,j_2}^{k_1}\, H_{j_1,j_3}^{k_2}
\end{aligned}
\right] + \\
& + \sum_{k_1=1}^n \sum_{k_2=1}^n y_{x^{k_1}}\, y_{x^{k_2}} \left[
\begin{aligned}
& \delta_{j_3}^{k_2} H_{j_1,j_2,y}^{k_1} - \delta_{j_2}^{k_2} H_{j_1,j_3,y}^{k_1} + \frac{1}{2} \delta_{j_2}^{k_2} L_{j_1,x^{j_3}}^{k_1} - \frac{1}{2} \delta_{j_3}^{k_2} L_{j_1,x^{j_2}}^{k_1} + \\
& + \frac{1}{2} \delta_{j_1}^{k_2} L_{j_2,x^{j_3}}^{k_1} - \frac{1}{2} \delta_{j_1}^{k_2} L_{j_3,x^{j_2}}^{k_1} + \\
& + \delta_{j_2}^{k_2} G_{j_1,j_3}\, M^{k_1} - \delta_{j_3}^{k_2} G_{j_1,j_2}\, M^{k_1} + \\
& + \delta_{j_1,j_2}^{k_1,k_2} \sum_{k_3=1}^n G_{k_3,j_3}\, M^{k_3} - \delta_{j_1,j_3}^{k_1,k_2} \sum_{k_3=1}^n G_{k_3,j_2}\, M^{k_3} + \\
& + \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_3=1}^n H_{k_3,j_3}^{k_2}\, L_{j_2}^{k_3} - \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_3=1}^n H_{k_3,j_2}^{k_2}\, L_{j_3}^{k_3} + \\
& + \frac{1}{2} \delta_{j_2}^{k_1} \sum_{k_3=1}^n H_{k_3,j_3}^{k_2}\, L_{j_1}^{k_3} - \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_3=1}^n H_{k_3,j_2}^{k_2}\, L_{j_1}^{k_3} + \\
& + \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_3=1}^n H_{j_1,j_2}^{k_3}\, L_{k_3}^{k_2} - \frac{1}{2} \delta_{j_2}^{k_1} \sum_{k_3=1}^n H_{j_1,j_3}^{k_3}\, L_{k_3}^{k_2}
\end{aligned}
\right] +
\end{aligned}
$$

$$+\sum_{k_1=1}^n\sum_{k_2=1}^n\sum_{k_3=1}^n y_{x^{k_1}}\,y_{x^{k_2}}\,y_{x^{k_3}}\left[\begin{array}{l}\frac{1}{2}\,\delta_{j_3,j_1}^{k_3,k_2}\,L_{j_2,y}^{k_1}-\frac{1}{2}\,\delta_{j_2,j_1}^{k_3,k_2}\,L_{j_3,y}^{k_1}+\\ +\,\delta_{j_2,j_1}^{k_3,k_2}\,M_{x^{j_3}}^{k_1}-\delta_{j_3,j_1}^{k_3,k_2}\,M_{x^{j_2}}^{k_1}+\\ +\,\delta_{j_2,j_1}^{k_3,k_1}\sum_{k_4=1}^n H_{k_4,j_3}^{k_2}\,M^{k_4}-\\ -\,\delta_{j_3,j_1}^{k_3,k_1}\sum_{k_4=1}^n H_{k_4,j_2}^{k_2}\,M^{k_4}+\\ +\,\frac{1}{4}\,\delta_{j_1,j_3}^{k_1,k_3}\sum_{k_4=1}^n L_{k_4}^{k_2}\,L_{j_2}^{k_4}-\\ -\,\frac{1}{4}\,\delta_{j_1,j_2}^{k_1,k_3}\sum_{k_4=1}^n L_{k_4}^{k_2}\,L_{j_3}^{k_4}\end{array}\right].$$

By identifying to zero all the coefficients of this cubic polynomial, we obtain a system of four families, (I'), (II'), (III') and (IV') (*see* the equations after (1.5) in [M2004b]) of first order partial differential equations satisfied by the functions G_{j_1,j_2}, $H_{j_1,j_2}^{k_1}$, $L_{j_1}^{k_1}$ and M^{k_1}. Strikingly, this system is analogous to the system obtained in Theorem 1.7, by the notational correspondence

$$(G_{j_1,j_2}, H_{j_1,j_2}^{k_1}, L_{j_1}^{k_1}, M^{k_1}) \longmapsto (-M^{l_1,l_2}, -L_{l_1,l_2}^{j}, -H_{l_1}^{j}, -G^{j}) \tag{1.26}$$

and by the exchange of coordinates $(x^1,\dots,x^n,y)\mapsto(y^1,\dots,y^m,x)$, in the case $n=m$ of course. Based on this intuitive observation, one may then deduce formally that Theorem 1.23 is a corollary of Theorem 1.7, and conversely (*cf.* [M2003], [M2004b]).

1.27. Acknowledgment. All the formulas obtained in Sections 2, 3 and 4 were first treated completely by hand and then, some of them were confirmed afterwards with the help of MAPLE release 6 in the cases $m=2$ and $m=3$. The author is deeply indebted to Sylvain Neut and to Michel Petitot, from the University of Lille 1, for their precious help in computer checking. Also, it is a pleasure to thank Camille Bièche, Sylvain Neut and Michel Petitot for fruitful exchanges about the method of equivalence.

1.28. Organization of the paper. Section 2 is devoted to a thorough restitution of S. Lie's original proof of Theorem 1.3. Section 3 is devoted to the formulation of combinatorial formulas yielding the general form of a system equivalent to $Y_{XX}^j=0$, $j=1,\dots,m$, under a point transformation, for general $m\geq 2$; the proof of the main technical Lemma 3.32 is exposed in Section 5. Section 4 is devoted to the final proof of Theorem 1.3. In [M2004b], the totally similar proof of Theorem 1.23 is resumed.

1.29. Closing remark. According to the recent literature, Theorem 1.7 was considered as an open problem. However, during the preparation of this work, we discovered that in his thesis [Ha1937], M. Hachtroudi (a brilliant Iranian student of É. Cartan) obtained a proof of both Theorems 1.7 and 1.23. His techniques rely on the so-called *method of equivalence* and on some tricky shortcuts of formal computations, inspired from the last section of É. Cartan's work [Ca1924] on projective connections. In fact, in [Ha1937], all the invariant tensors building a projective connection in the case $n\geq 2$, $m=1$ are computed explicitely only in for $n=2$. Thus, the generalization of S. Lie's original proof (totally different from É. Cartan's) that we provide in this paper and especially the combinatorial formulas (3.14), (3.15), (3.33) and (3.31) below, seem to be new.

§2. PROOF OF S. LIE'S THEOREM

2.1. Argument. This preliminary section contains a detailed exposition of S. Lie's original proof of Theorem 1.2. Since our goal is to guess the combinatorics of computations in several variables, it will be a crucial point for us to explain thoroughly and patiently each step of S. Lie's computation. Without such an intuitive control, it would be hopeless to conduct any generalization to several variables. Hence we shall respect a fundamental principle: always explain clearly and completely what sort of computation is achieved at each step. Also, we shall many times introduce some appropriate new notation.

2.2. Combinatorics of the second order prolongation of a point transformation. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(x, y) \mapsto (X(x, y), Y(x, y))$ be a local $\mathbb{K}$-analytic invertible transformation, defined in a neighborhood of the origin in $\mathbb{K}^2$, which maps the second order differential equation $y_{xx} = F(x, y, y_x)$ to the flat equation $Y_{XX} = 0$. By assumption, the Jacobian determinant

$$\Delta(x|y) := \begin{vmatrix} X_x & X_y \\ Y_x & Y_y \end{vmatrix} \tag{2.3}$$

is nowhere vanishing. Since the equation $Y_{XX} = 0$ is left unchanged by any affine transformation in the (X, Y) space, we can (and we shall) assume that the transformation is tangent to the identity at the origin, namely the above Jacobian matrix equals the identity matrix at $(x, y) = (0, 0)$.

The computation how the differential equation in the (X, Y) coordinates is related to the differential equation in the (x, y)-coordinates is classical, *cf.* [Lie1883], [TR1896], [BK1989], [IB1992]: let us remind it. A local graph $\{y = y(x)\}$ being transformed to a local graph $\{Y = Y(X)\}$, we have a direct formula for the first derivative Y_X:

$$Y_X := \frac{dY}{dX} = \frac{dx \cdot \partial Y(x, y(x))/\partial x}{dx \cdot \partial X(x, y(x))/\partial x} = \frac{Y_x + y_x Y_y}{X_x + y_x X_y}. \tag{2.4}$$

This yields the prolongation of the transformation to the first order jet space. For the second order prolongation, introducing the second order total differentiation operator (which geometrically corresponds to differentiation along graphs $\{(x, y(x))\}$) defined by

$$D := \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x}, \tag{2.5}$$

we may compute, simplify and reorder the expression of the second order derivative in the (X, Y)-coordinates:

(2.6)

$$\left\{ \begin{aligned} Y_{XX} := \frac{d^2Y}{dX^2} \equiv \frac{DY_X}{DX} &= \frac{D\left[(Y_x + y_x Y_y)(X_x + y_x X_x)^{-1}\right]}{X_x + y_x X_y} = \\ &= \frac{1}{[X_x + y_x X_y]^3} \left\{ y_{xx} \left[X_x Y_y - Y_x X_y \right] + X_x Y_{xx} - Y_x X_{xx} + \right. \\ &+ \ y_x \left[2(X_x Y_{xy} - Y_x X_{xy}) - (X_{xx} Y_y - Y_{xx} X_y) \right] + \\ &+ \ y_x y_x \left[X_x Y_{yy} - Y_x X_{yy} - 2(X_{xy} Y_y - Y_{xy} X_y) \right] + \\ &+ \left. \ y_x y_x y_x \left[-(X_{yy} Y_y - Y_{yy} X_y) \right] \right\}. \end{aligned} \right.$$

Even if not too complicated, the internal combinatorics of this expression has to be analyzed and expressed thoroughly. First of all, as $Y_{XX} = 0$ by assumption, we may erase the cubic factor $[X_x + y_x X_y]^{-3}$. Next, as the factor of y_{xx} in the right hand side of (2.6), we just recognize the Jacobian $\Delta(x|y)$ expressed in (2.3) above. Also, all the other factors are modifications of the Jacobian $\Delta(x|y)$, whose combinatorics may be understood as follows.

There exist exactly three possible distinct second order derivatives: xx, xy and yy. There are also exactly two columns in (2.3). By replacing each of the two columns of first order derivative in $\Delta(x|y)$ by any column of second order derivative (leaving X and Y unchanged), we may build exactly six new determinants

$$\left\{\begin{array}{lll}\Delta(xx|y) & \Delta(xy|y) & \Delta(yy|y) \\ \Delta(x|xx) & \Delta(x|xy) & \Delta(x|yy)\end{array}\right. \tag{2.7}$$

where for instance

$$\left\{\Delta(\underline{xx}|y) := \left|\begin{array}{cc} X_{\underline{xx}} & X_y \\ Y_{\underline{xx}} & Y_y \end{array}\right| \quad \text{and} \quad \Delta(x|\underline{xy}) := \left|\begin{array}{cc} X_x & X_{\underline{xy}} \\ Y_x & Y_{\underline{xy}} \end{array}\right|.\right. \tag{2.8}$$

Hence, by rewriting (2.6), we see that the equation $y_{xx} = F(x, y, y_x)$ equivalent to $Y_{XX} = 0$ may be written under the general explicit form, involving determinants

$$\left\{\begin{aligned} 0 = y_{xx} \cdot \left|\begin{array}{cc} X_x & X_y \\ Y_x & Y_y \end{array}\right| + \left|\begin{array}{cc} X_x & X_{xx} \\ Y_x & Y_{xx} \end{array}\right| + y_x \cdot \left\{2\left|\begin{array}{cc} X_x & X_{xy} \\ Y_x & Y_{xy} \end{array}\right| - \left|\begin{array}{cc} X_{xx} & X_y \\ Y_{xx} & Y_y \end{array}\right|\right\} + \\ + y_x y_x \cdot \left\{\left|\begin{array}{cc} X_x & X_{yy} \\ Y_x & Y_{yy} \end{array}\right| - 2\left|\begin{array}{cc} X_{xy} & X_y \\ Y_{xy} & Y_y \end{array}\right|\right\} + y_x y_x y_x \cdot \left\{-\left|\begin{array}{cc} X_{yy} & X_y \\ Y_{yy} & Y_y \end{array}\right|\right\} \end{aligned}\right. \tag{2.9}$$

or equivalently, after solving in y_{xx}, *i.e.* after dividing by the Jacobian $\Delta(x|y)$:

$$\left\{\begin{aligned} y_{xx} = & -\frac{\Delta(x|xx)}{\Delta(x|y)} + y_x \cdot \left\{-2\frac{\Delta(x|xy)}{\Delta(x|y)} + \frac{\Delta(xx|y)}{\Delta(x|y)}\right\} + \\ & + (y_x)^2 \cdot \left\{-\frac{\Delta(x|yy)}{\Delta(x|y)} + 2\frac{\Delta(xy|y)}{\Delta(x|y)}\right\} + (y_x)^3 \cdot \left\{\frac{\Delta(yy|y)}{\Delta(x|y)}\right\}. \end{aligned}\right. \tag{2.10}$$

At this point, it will be convenient to slightly contract the notation by introducing a new family of *square functions* as follows. We first index the coordinates (x, y) as (y^0, y^1), namely we introduce the two notational equivalences

$$\boxed{y^0 \equiv x, \quad y^1 \equiv y}, \tag{2.11}$$

which will be very convenient in the sequel, especially to write down general combinatorial formulas anticipating our treatment of the case of $m \geq 2$ dependent variables $(y^1, \dots, y^m)$, to be achieved in Sections 3, 4 and 5 below. With this convention at hand, our six square functions $\square^{k_1}_{y^{j_1} y^{j_2}}$, symmetric with respect to the lower indices, where $0 \leq j_1, j_2, k_1 \leq 1$, are defined by

$$\left\{\begin{array}{lll} \square^0_{xx} := \dfrac{\Delta(xx|y)}{\Delta(x|y)}, & \square^0_{xy} := \dfrac{\Delta(xy|y)}{\Delta(x|y)}, & \square^0_{yy} := \dfrac{\Delta(yy|y)}{\Delta(x|y)}, \\ \square^1_{xx} := \dfrac{\Delta(x|xx)}{\Delta(x|y)}, & \square^1_{xy} := \dfrac{\Delta(x|xy)}{\Delta(x|y)}, & \square^1_{yy} := \dfrac{\Delta(x|yy)}{\Delta(x|y)}. \end{array}\right. \tag{2.12}$$

Here of course, the upper index designates the column upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that the square functions are explicit rational expressions in terms of the second order jet of the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$. However, we shall be aware of not confusing the index in the square functions with a second order partial derivative of some function "$\square^j$", denoted by the square symbol: indeed, the partial derivatives are hidden in some determinant.

At this point, we may summarize what we have established so far.

Lemma 2.13. *The equation $y_{xx} = F(x, y, y_x)$ is equivalent to the flat equation $Y_{XX} = 0$ if and only if there exist two local $\mathbb{K}$-analytic functions $X(x, y)$ and $Y(x, y)$ such that it may be written under the form*

$$y_{xx} = -\square^1_{xx} + y_x \cdot \left(-2\,\square^1_{xy} + \square^0_{xx}\right) + (y_x)^2 \cdot \left(-\square^1_{yy} + 2\,\square^0_{xy}\right) + (y_x)^3 \cdot \square^0_{yy}. \tag{2.14}$$

At this point, for heuristic reasons, it may be useful to compare the right hand side of (2.14) with the classical expression of the prolongation to the second order jet space of a general vector field of the form $L := X(x, y)\frac{\partial}{\partial x} + Y(x, y)\frac{\partial}{\partial y}$, which is given, according to [Lie1883], [OL1986], [BK1989], by

(2.15)

$$\begin{cases} L^{(2)} = X\,\dfrac{\partial}{\partial x} + Y\,\dfrac{\partial}{\partial y} + \left[Y_x + y_x \cdot (Y_y - X_x) + (y_x)^2 \cdot (-X_y)\right]\,\dfrac{\partial}{\partial y_x} + \\ \qquad + \left[Y_{xx} + y_x \cdot (2Y_{xy} - X_{xx}) + (y_x)^2 \cdot (Y_{yy} - 2X_{xy}) + (y_x)^3 \cdot (-X_{yy}) + \right. \\ \qquad \left. + \; y_{xx} \cdot (Y_y - 2\,X_x) + y_x y_{xx} \cdot (-3\,X_y)\right]\,\dfrac{\partial}{\partial y_{xx}}. \end{cases}$$

We immediately see that (up to an overall minus sign) the right hand side of (2.14) is formally analogous to the second line of (2.15) : the letter X corresponds to the symbol $\square^0$ and the letter Y corresponds to the symbol $\square^1$. This analogy is no mystery, just because the formula for $L^{(2)}$ is classically obtained by differentiating at $\varepsilon = 0$ the second order prolongation $[\exp(\varepsilon L)(\cdot)]^{(2)}$ of the flow of L !

In fact, as we assumed that the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ is tangent to the identity at the origin, we may think that $X_x \cong 1$, $X_y \cong 0$, $Y_x \cong 0$ and $Y_y \cong 1$, whence the Jacobian $\Delta(x|y) \cong 1$ and moreover

$$\begin{cases} \square^0_{xx} \cong X_{xx}, & \square^0_{xy} \cong X_{xy}, & \square^0_{yy} \cong X_{yy}, \\ \square^1_{xx} \cong Y_{xx}, & \square^1_{xy} \cong Y_{xy}, & \square^1_{yy} \cong Y_{yy}. \end{cases} \tag{2.16}$$

By means of this (abusive) notational correspondence, we see that, up to an overall minus sign, the right hand side of (2.14) transforms precisely to the second line of (2.15). This analogy will be useful in devisingcombinatorial formulas for the generalization of Lemma 2.13 to the case of $m \geq 2$ variables $(y^1, \dots, y^m)$, *see* Lemmas 3.22 and 3.32 below.

2.17. Continuation. Clearly, since the right hand side of (2.14) is a polynomial of degree three in y_x, condition **(i)** of Theorem 1.1 immediately holds. We are therefore led to establish that condition **(ii)** is necessary and sufficient in order that there exist two local $\mathbb{K}$-analytic functions $X(x, y)$ and $Y(x, y)$ which solve the following system of nonlinear second order partial differential equations (remind that the second order jet of (X, Y) is hidden in the square functions):

$$\begin{cases} G = -\,\square^1_{xx}, \\ H = -\,2\,\square^1_{xy} + \square^0_{xx}, \\ L = -\,\square^1_{yy} + 2\,\square^0_{xy}, \\ M = \square^0_{yy}. \end{cases} \tag{2.18}$$

In the remainder of this section, following [Lie1883], p. 364, we shall study this second order system by introducing two auxiliary systems of partial differential equations which are *complete*, and we shall see in §2.38 below that the compatibility conditions (insuring involutivity, hence complete integrability) of the second auxiliary system exactly provide the two partial differential equations appearing as condition **(ii)** in S. Lie's Theorem 1.2.

2.19. First auxiliary system. We notice that in (2.18), there are two more square functions $\square_{xx}^0$, $\square_{xy}^0$, $\square_{yy}^0$, $\square_{xx}^1$, $\square_{xy}^1$, $\square_{yy}^1$, than functions G, H, L and M. Hence, as a trick, let us introduce six new independent functions $\Pi_{j_1,j_2}^{k_1}$ of (x,y), symmetric with respect to the lower indices, for $0 \leq j_1, j_2, k_1 \leq 1$ and let us seek necessary and sufficient conditions in order that there exist solutions (X,Y) to the *first auxiliary system*:

$$\text{(2.20)} \qquad \begin{cases} \square_{xx}^0 = \Pi_{0,0}^0, & \square_{xy}^0 = \Pi_{0,1}^0, \qquad \square_{yy}^0 = \Pi_{1,1}^0, \\ \square_{xx}^1 = \Pi_{0,0}^1, & \square_{xy}^1 = \Pi_{0,1}^1, \qquad \square_{yy}^1 = \Pi_{1,1}^1. \end{cases}$$

According to the (aprooximate) identities (2.16), this system looks like a complete second order system of partial differential equations in two variables (x,y) and in two unknowns (X,Y). More rigorously, by means of elementary algebraic operations, taking account of the fact that $X_x \cong 1$, $X_y \cong 0$, $Y_x \cong 0$ and $Y_y \cong 1$, one may transform this sytem in a true second order *complete* system, solved with respect to the top order derivatives, namely of the form

$$\text{(2.21)} \qquad \begin{cases} X_{xx} = \Lambda_{0,0}^0, & X_{xy} = \Lambda_{0,1}^0, \qquad X_{yy} = \Lambda_{1,1}^0, \\ Y_{xx} = \Lambda_{0,0}^1, & Y_{xy} = \Lambda_{0,1}^1, \qquad Y_{yy} = \Lambda_{1,1}^1, \end{cases}$$

where the $\Lambda_{j_1,j_2}^{k_1}$ are local $\mathbb{K}$-analytic functions of $(x,y,X,Y,X_x,X_y,Y_x,Y_y)$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution (X,Y)] are easily formulated:

$$\text{(2.22)} \qquad \begin{cases} (\Lambda_{0,0}^0)_y = (\Lambda_{0,1}^0)_x, & (\Lambda_{0,1}^0)_y = (\Lambda_{1,1}^0)_x, \\ (\Lambda_{0,0}^1)_y = (\Lambda_{0,1}^1)_x, & (\Lambda_{0,1}^1)_y = (\Lambda_{1,1}^1)_x. \end{cases}$$

Equivalently, we may express the compatibility conditions directly with the system (2.20), without transforming it to the form (2.21). This direct strategy will be more appropriate.

2.23. Compatibility conditions for the first auxiliary system. Indeed, to begin with, let us remind that the $\Delta(\cdot|\cdot)$ are determinant, hence we have the skew-symmetry relation $\Delta(x^ay^b|x^cy^d) = -\Delta(x^cy^d|x^ay^b)$ and the following two formulas for partial differentiation

$$\text{(2.24)} \qquad \begin{cases} \left[\Delta(x^ay^b|x^cy^d)\right]_x = \Delta(x^{a+1}y^b|x^cy^d) + \Delta(x^ay^b|x^{c+1}y^d), \\ \left[\Delta(x^ay^b|x^cy^d)\right]_y = \Delta(x^ay^{b+1}|x^cy^d) + \Delta(x^ay^b|x^cy^{d+1}). \end{cases}$$

With these formal rules at hand, as an exercise, let us compute for instance the following cross differentiation (remember that the lower index in the square functions is *not* a partial derivative):

$$
(2.25)\qquad
\begin{cases}
\left(\square^0_{xx}\right)_y - \left(\square^0_{xy}\right)_x = \frac{\partial}{\partial y}\left(\frac{\Delta(xx|y)}{\Delta(x|y)}\right) - \frac{\partial}{\partial x}\left(\frac{\Delta(xy|y)}{\Delta(x|y)}\right) = \\
= \frac{1}{[\Delta(x|y)]^2}\Big\{\underline{\Delta(xxy|y)\cdot\Delta(x|y)}_{\text{ⓐ}} + \Delta(xx|yy)\cdot\Delta(x|y) - \\
\qquad -\underline{\Delta(xy|y)\cdot\Delta(xx|y)}_{\text{ⓑ}} - \Delta(x|yy)\cdot\Delta(xx|y) - \\
\qquad -\underline{\Delta(xxy|y)\cdot\Delta(x|y)}_{\text{ⓐ}} - \underline{\Delta(xy|xy)\cdot\Delta(x|y)}_{\text{ⓒ}} + \\
\qquad +\underline{\Delta(xy|y)\cdot\Delta(xx|y)}_{\text{ⓑ}} + \Delta(xy|y)\cdot\Delta(x|xy)\Big\} = \\
= \frac{1}{[\Delta(x|y)]^2}\left\{\Delta(xx|yy)\cdot\Delta(x|y) - \Delta(x|yy)\cdot\Delta(xx|y) + \right. \\
\qquad \left. +\Delta(xy|y)\cdot\Delta(x|xy)\right\}.
\end{cases}
$$

Crucially, we observe that the third order derivatives kill each other and disappear, *see* the underlined terms with ⓐ appended. Also, two products of two determinants $\Delta(\cdot|\cdot)$ involving a second order derivative upon one column of each determinant kill each other: they are underlined with ⓑ appended. Finally, by antisymmetry of determinants, the term $\Delta(xy|xy)\cdot\Delta(x|y)$ vanishes gratuitously: it is underlined with ⓒ appended.

However, there still remains one term involving second order derivatives upon the *two* columns of a determinant: it is $\Delta(xx|yy)$.

We must transform this unpleasant term $\Delta(xx|yy)\cdot\Delta(x|y)$ and express it as a product of two determinants, each involving a second order derivative only in one column. To this aim, we have:

Lemma 2.26. *The following three relations between the differential determinants* $\Delta(\cdot|\cdot)$ *hold true*:

$$
(2.27)\qquad
\begin{cases}
\Delta(xx|xy)\cdot\Delta(x|y) = \Delta(xx|y)\cdot\Delta(x|xy) - \Delta(xy|y)\cdot\Delta(x|xx), \\
\Delta(xx|yy)\cdot\Delta(x|y) = \Delta(xx|y)\cdot\Delta(x|yy) - \Delta(yy|y)\cdot\Delta(x|xx), \\
\Delta(xy|yy)\cdot\Delta(x|y) = \Delta(xy|y)\cdot\Delta(x|yy) - \Delta(yy|y)\cdot\Delta(x|xy).
\end{cases}
$$

Proof. Each of these three formal identities is an immediate direct consequence of the following Plücker type identity, easily verified by developing all the determinants :

(2.28)

$$
\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & D_1 \\ C_2 & D_2 \end{vmatrix} = \begin{vmatrix} A_1 & D_1 \\ A_2 & D_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} - \begin{vmatrix} B_1 & D_1 \\ B_2 & D_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},
$$

where the variables $A_1,\ A_2,\ B_1,\ B_2,\ C_1,\ C_2,\ D_1,\ D_2 \in \mathbb{K}$ are arbitrary. □

Thanks to the second identity (2.27), we may therefore transform the result left above in the last two lines of (2.25); as desired, it will remain determinants having only one second order derivative per column, so that after division by $[\Delta(x|y)]^2$, we discover *a* *quadratic expression involving only the square functions themselves*:

(2.29)
$$\begin{cases} \left(\square^0_{xx}\right)_y - \left(\square^0_{xy}\right)_x = \dfrac{1}{[\Delta(x|y)]^2}\left\{\Delta(xx|y)\cdot\Delta(x|yy) - \Delta(yy|y)\cdot\Delta(x|xx) - \right. \\ \qquad\qquad \left. -\Delta(x|yy)\cdot\Delta(xx|y) + \Delta(xy|y)\cdot\Delta(x|xy)\right\} \\ \qquad = \dfrac{1}{[\Delta(x|y)]^2}\left\{-\Delta(yy|y)\cdot\Delta(x|xx) + \Delta(xy|y)\cdot\Delta(x|xy)\right\} \\ \qquad = -\square^0_{yy}\cdot\square^1_{xx} + \square^0_{xy}\cdot\square^1_{xy}. \end{cases}$$

In sum, the result of the cross differentiation $(\square^0_{xx})_y - (\square^0_{xy})_x$ is a quadratic expression in terms of the square functions themselves! Following the same recipe (with no surprise), one may establish the following relations, listing all the compatibility conditions (the first one is nothing else than (2.29)):

$$\text{(2.30)}\quad \begin{cases} \left(\square^0_{xx}\right)_y - \left(\square^0_{xy}\right)_x = -\square^1_{xx}\cdot\square^0_{yy} + \square^1_{xy}\cdot\square^0_{xy}, \\ \left(\square^0_{xy}\right)_y - \left(\square^0_{yy}\right)_x = -\square^0_{xy}\cdot\square^0_{yx} - \square^1_{xy}\cdot\square^0_{yy} + \square^0_{yy}\cdot\square^0_{xx} + \square^1_{yy}\cdot\square^0_{xy}, \\ \left(\square^1_{xx}\right)_y - \left(\square^1_{xy}\right)_x = -\square^0_{xx}\cdot\square^1_{yx} - \square^1_{xx}\cdot\square^1_{yy} + \square^0_{xy}\cdot\square^1_{xx} + \square^1_{xy}\cdot\square^1_{xy}, \\ \left(\square^1_{xy}\right)_y - \left(\square^1_{yy}\right)_x = -\square^0_{xy}\cdot\square^1_{yx} + \square^0_{yy}\cdot\square^1_{xx}. \end{cases}$$

Instead of checking patiently each of the remaining three cross above cross differentiation identities, it is better to establish directly the following general relation.

Lemma 2.31. *Remind from* (2.11) *that we identify* y^0 *with* x *and* y^1 *with* y *and let* $0 \leq j_1, j_2, j_3, k_1 \leq 1$. *Then*

(2.32)
$$\left(\square^{k_1}_{y^{j_1}y^{j_2}}\right)_{y^{j_3}} - \left(\square^{k_1}_{y^{j_1}y^{j_3}}\right)_{y^{j_2}} = -\sum_{k_2=0}^{1}\square^{k_2}_{y^{j_1}y^{j_2}}\cdot\square^{k_1}_{y^{j_3}y^{k_2}} + \sum_{k_2=0}^{1}\square^{k_2}_{y^{j_1}y^{j_3}}\cdot\square^{k_1}_{y^{j_2}y^{k_2}}.$$

This lemma is left to the reader; anyway, we shall complete the proof of a generalization of Lemma 2.31 to the case of $m \geq 1$ dependent variables $(y^1, \dots, y^m)$ in Section 2 below (Lemma 3.40).

Coming back to the first auxiliary system (2.20), we therefore have obtained a necessary and sufficient condition for the existence of (X, Y): the functions $\Pi^{k_1}_{j^1,j^2}$ should satisfy the following system of first order partial differential equations, just obtained from (2.30) by replacing the square functions by the Pi functions:

(2.33)
$$\begin{cases} \left(\Pi^0_{0,0}\right)_y - \left(\Pi^0_{0,1}\right)_x = -\Pi^1_{0,0}\cdot\Pi^0_{1,1} + \Pi^1_{0,1}\cdot\Pi^0_{0,1}, \\ \left(\Pi^0_{0,1}\right)_y - \left(\Pi^0_{1,1}\right)_x = -\Pi^0_{0,1}\cdot\Pi^0_{0,1} - \Pi^1_{0,1}\cdot\Pi^0_{1,1} + \Pi^0_{1,1}\cdot\Pi^0_{0,0} + \Pi^1_{1,1}\cdot\Pi^0_{0,1}, \\ \left(\Pi^1_{0,0}\right)_y - \left(\Pi^1_{0,1}\right)_x = -\Pi^0_{0,0}\cdot\Pi^1_{0,1} - \Pi^1_{0,0}\cdot\Pi^1_{1,1} + \Pi^0_{0,1}\cdot\Pi^1_{1,1} + \Pi^1_{0,1}\cdot\Pi^1_{0,1}, \\ \left(\Pi^1_{0,1}\right)_y - \left(\Pi^1_{1,1}\right)_x = -\Pi^0_{0,1}\cdot\Pi^1_{0,1} + \Pi^0_{1,1}\cdot\Pi^1_{0,0}. \end{cases}$$

2.34. Second auxiliary system. It is now time to come back to the functions G, H, L and M and to get rid of the auxiliary "Pi" functions. Unfortunately, we cannot invert directly the linear system (2.18), hence we must choose two specific square functions as *principal unknowns*, and the best, from a combinatorial point of view, is to choose $\square^0_{xx}$ and $\square^1_{yy}$. Remind that by (2.20), we have $\square^0_{xx} = \Pi^0_{0,0}$ and $\square^1_{yy} = \Pi^1_{1,1}$. For clarity, it will be useful to adopt the notational equivalences

$$\Theta^0 \equiv \Pi^0_{0,0} \qquad \text{and} \qquad \Theta^1 \equiv \Pi^1_{1,1}. \tag{2.35}$$

We may therefore quasi-inverse the linear system (2.18), obtaining that the four functions $\Pi^1_{0,0}$, $\Pi^1_{0,1}$, $\Pi^0_{0,1}$ and $\Pi^0_{1,1}$ may be expressed in terms of the functions G, H, L and M and in terms of the remaining two principal unknowns (2.35), which yields:

$$\begin{cases} \Pi^1_{0,0} = \square^1_{xx} = -G \\ \Pi^1_{0,1} = \square^1_{xy} = -\frac{1}{2}\,H + \frac{1}{2}\,\Theta^0, \\ \Pi^0_{0,1} = \square^0_{xy} = \frac{1}{2}\,L + \frac{1}{2}\,\Theta^1, \\ \Pi^0_{1,1} = \square^0_{yy} = M. \end{cases} \tag{2.36}$$

Replacing now each of these four expressions in the compatibility conditions of the first auxiliary system (2.33), solving the four equations with respect to Θ^1_y, Θ^0_x, Θ^1_x and Θ^0_y, we get after hygienic simplifications what we shall call the *second auxiliary system*, which is a complete system of first order partial derivatives in the remaining two principal unknowns Θ^0 and Θ^1:

$$\begin{cases} \Theta^1_y = -\,L_y + 2M_x + HM - \frac{1}{2}\,L^2 + M\,\Theta^0 + \frac{1}{2}\,\left(\Theta^1\right)^2, \\ \Theta^0_x = -\,2\,G_y + H_x + G\,L - \frac{1}{2}\,H^2 - G\,\Theta^1 + \frac{1}{2}\,\left(\Theta^0\right)^2, \\ \Theta^1_x = -\,\frac{2}{3}\,H_y + \frac{1}{3}\,L_x + 2\,G\,M - \frac{1}{2}\,H\,L - \frac{1}{2}\,H\,\Theta^1 + \frac{1}{2}\,L\,\Theta^0 + \frac{1}{2}\,\Theta^0\,\Theta^1, \\ \Theta^0_y = -\,\frac{1}{3}\,H_y + \frac{2}{3}\,L_x + 2\,G\,M - \frac{1}{2}\,H\,L - \frac{1}{2}\,H\,\Theta^1 + \frac{1}{2}\,L\,\Theta^0 + \frac{1}{2}\,\Theta^0\,\Theta^1. \end{cases} \tag{2.37}$$

We do not comment the intermediate computations, since they offer no new combinatorial discovery.

2.38. Precise rules for rigorous formal hygiene. Our formal hygiene implies that we respect the following rules: we group first order derivatives before zeroth order derivatives; in each group, we respect the lexicographic order of appearance given by the sequence G, H, L, M, Θ^0, Θ^1; we always put rational coefficient of every differential monomial in its left; consequently, we accept a minus sign just after an equality sign, as for instance in $(2.36)_1$ and in $(2.37)_2$; for clarity, we prefer to write a complicated differential equation as $0 = \Phi$, with 0 on the left, instead of $\Phi = 0$, since Φ may incoporate 10, 20 and up to 150 monomials, as will happen for instance in the next sections below.

2.39. Compatibility conditions for the second auxiliary system. Clearly, the necessary and sufficient condition for the existence of solutions (Θ^0, Θ^1) to the second auxiliary system (2.37) is that the two cross differentiations vanish:

$$\begin{cases} 0 = \left(\Theta^0_x\right)_y - \left(\Theta^0_y\right)_x, \\ 0 = \left(\Theta^1_x\right)_y - \left(\Theta^1_y\right)_x. \end{cases} \tag{2.40}$$

Using (2.37), we shall see that we exactly obtain the two second order partial differential equations written as condition **(ii)** in S. Lie's Theorem 1.2. For completeness, we shall perform completely the computation of the firt compatibility condition (2.40) and leave the second as an (easy) exercise.

First of all, inserting (2.37) and using the rule of Leibniz for the differentiation of a product, let us write the crude result, performing neither any simplification nor any reordering:

$$\text{(2.41)}\quad \begin{cases} 0 = \left(\Theta_x^0\right)_y - \left(\Theta_y^0\right)_x \\ \quad = -2\,G_{yy} + H_{xy} + G_y\,L + G\,L_y - H\,H_y - G_y\,\Theta^1 - G\,\Theta_y^1 + \Theta^0\,\Theta_y^0 + \\ \quad + \frac{1}{3}\,H_{xy} - \frac{2}{3}\,L_{xx} - 2\,G_x\,M - 2\,G\,M_x + \frac{1}{2}\,H_x\,L + \frac{1}{2}\,H\,L_x + \\ \quad + \frac{1}{2}\,H_x\,\Theta^1 + \frac{1}{2}\,H\,\Theta_x^1 - \frac{1}{2}\,L_x\,\Theta^0 - \frac{1}{2}\,L\,\Theta_x^0 - \frac{1}{2}\,\Theta_x^0\,\Theta^1 - \frac{1}{2}\,\Theta^0\,\Theta_x^1. \end{cases}$$

Next, replacing each first order derivative Θ_x^0, Θ_y^0, Θ_x^1 and Θ_y^1 occuring in (2.41) by its expression given in (2.37), we obtain (suffering a little) as a brute result, before any simplification (except that we put all second order derivatives in the beginning):

(2.42)

$$\begin{cases} 0 = -2\,G_{yy} + \frac{4}{3}\,H_{xy} - \frac{2}{3}\,L_{xx} + \\ \quad + G_y\,L + G\,L_y - H\,H_y - G_y\,\Theta^1 + G\,L_y - 2\,G\,M_x - G\,H\,M + \\ \quad + \frac{1}{2}\,G\,(L)^2 - G\,M\,\Theta^0 - \frac{1}{2}\,G\,(\Theta^1)^2 - \frac{1}{3}\,H_y\,\Theta^0 + \frac{2}{3}\,L_x\,\Theta^0 + \\ \quad + 2\,G\,M\,\Theta^0 - \frac{1}{2}\,H\,L\,\Theta^0 - \frac{1}{2}\,H\,\Theta^0\,\Theta^1 + \frac{1}{2}\,L\,(\Theta^0)^2 + \frac{1}{2}\,(\Theta^0)^2\,\Theta^1 - \\ \quad - 2\,G_x\,M - 2\,G\,M_x + \frac{1}{2}\,H_x\,L + \frac{1}{2}\,H\,L_x + \frac{1}{2}\,H_x\,\Theta^1 + \\ \quad + \frac{1}{6}\,H\,L_x - \frac{1}{3}\,H\,H_y + G\,H\,M - \frac{1}{4}\,(H)^2\,L - \frac{1}{4}\,(H)^2\,\Theta^1 + \\ \quad + \frac{1}{4}\,H\,L\,\Theta^0 + \frac{1}{4}\,H\,\Theta^0\,\Theta^1 - \frac{1}{2}\,L_x\,\Theta^0 + G_y\,L - \frac{1}{2}\,H_x\,L - \frac{1}{2}\,G\,(L)^2 + \\ \quad + \frac{1}{4}\,H^2\,L + \frac{1}{2}\,G\,L\,\Theta^1 - \frac{1}{4}\,L\,\left(\Theta^0\right)^2 + G_y\,\Theta^1 - \frac{1}{2}\,H_x\,\Theta^1 - \\ \quad - \frac{1}{2}\,G\,L\,\Theta^1 + \frac{1}{4}\,H^2\,\Theta^1 + \frac{1}{2}\,G\,(\Theta^1)^2 - \frac{1}{4}\,\left(\Theta^0\right)^2\,\Theta^1 - \frac{1}{6}\,L_x\,\Theta^0 + \\ \quad + \frac{1}{3}\,H_y\,\Theta^0 - G\,M\,\Theta^0 + \frac{1}{4}\,H\,L\,\Theta^0 + \frac{1}{4}\,H\,\Theta^0\,\Theta^1 - \frac{1}{4}\,L\,(\Theta^0)^2 - \\ \quad - \frac{1}{4}\,(\Theta^0)^2\,\Theta^1. \end{cases}$$

Now, we can simplify this brute expression by chasing very couple (or triple, or quadruple) of terms killing each other. After (patient) simplification and lexicographic ordering, we obtain the equation

$$\text{(2.43)}\quad \begin{cases} 0 = -2\,G_{yy} + \frac{4}{3}\,H_{xy} - \frac{2}{3}\,L_{xx} + \\ \qquad\qquad + 2(G\,L)_y - 2\,G_x\,M - 4\,G\,M_x + \frac{2}{3}\,H\,L_x - \frac{4}{3}\,H\,H_y, \end{cases}$$

which is exactly the first equation of (1.4). The treatment of the second one is totally similar. This completes the proof of Theorem 1.2. □

2.44. Interlude: about hand-computed formulas. In Section 4 below, when dealing with several dependant variables $y^1, \ldots, y^m$, many simplifications of identities wich are much more massive than (2.42) will occur several times. It is therefore welcome to explain how we manage to achieve such computations, without mistakes at the end and stricly by hand. One of the trick is to use colors, which, unfortunately, cannot be restituted in this printed document. Another trick is to *underline and to number the terms which disappear together*, by pair, by triple, by quadruple, *etc.* This trick is illustrated in the detailed identity (2.45) below, extracted from our manuscript, which is a copy of (2.42) together with the designation of terms which vanish together. Hence, *we keep a written track of each intermediate step of every computation and of every simplification*. Checking the correctness of a computation simply by reading is then the easiest way. On the contrary, in programming a digital computer, most intermediate steps are invisible; the chase of mistakes is by reading the program and by testing it on several instances, but all the finest intuitions which may awake in the extreme inside of a long computation are essentially absent, because the mind belives that the machine is stronger for such tasks. This last belief is in part true, in case straightforward known computations are concerned, and in part untrue, in case new hidden mathematical reality is concerned. For us, however, *the challenge is to control everything in a sea of signs*. Computations are to be organized like a living giant coral tree, all part of which should be clearly visible in a transparent fluid of thought, and permanently subject to corrections. Indeed, it often happens that going through a problem involving massive formal computations, some disharmony or some incoherency is discovered. Then one has to inspect every living atom in the preceding branches of the growing coral tree of computations until some very tiny or ridiculous mistake is found. In addition to making easy the reading, *a perfectly rigorous way of writing the formal identities which respects a large amount of virtual conventions facilitates to reorganize rapidly the coral tree after a mistake has been found*. The accumulation of new virtual conventions, all of which we cannot speak, constitute another coral meta-tree and another profound collection of trick. Finally, we use a blank fluid corrector to avoid copying to much.

Extracted from our hand manuscript, here is the identity (2.42) with the underlining-numbering of all the vanishing terms (without the original colours) until we get the final equation (2.43):

$$
\begin{aligned}
0 = & -2\,G_{yy} + \frac{4}{3}\,H_{xy} - \frac{2}{3}\,L_{xx} + \\
& + G_y\,L + G\,L_y - H\,H_y - \underline{G_y\,\Theta^1}_{\textcircled{o}} + G\,L_y - 2\,G\,M_x - \underline{G\,H\,M}_{\textcircled{k}} + \\
& + \underline{\frac{1}{2}\,G\,(L)^2}_{\textcircled{a}} - \underline{G\,M\,\Theta^0}_{\textcircled{b}} - \underline{\frac{1}{2}\,G\,(\Theta^1)^2}_{\textcircled{c}} - \underline{\frac{1}{3}\,H_y\,\Theta^0}_{\textcircled{d}} + \underline{\frac{2}{3}\,L_x\,\Theta^0}_{\textcircled{e}} + \\
& + \underline{2\,G\,M\,\Theta^0}_{\textcircled{b}} - \underline{\frac{1}{2}\,H\,L\,\Theta^0}_{\textcircled{f}} - \underline{\frac{1}{2}\,H\,\Theta^0\,\Theta^1}_{\textcircled{g}} + \underline{\frac{1}{2}\,L\,(\Theta^0)^2}_{\textcircled{h}} + \underline{\frac{1}{2}\,(\Theta^0)^2\,\Theta^1}_{\textcircled{i}} - \\
& - 2\,G_x\,M - 2\,G\,M_x + \frac{1}{2}\,H_x\,L + \frac{1}{2}\,H\,L_x + \underline{\frac{1}{2}\,H_x\,\Theta^1}_{\textcircled{j}} + \\
& + \frac{1}{6}\,H\,L_x - \frac{1}{3}\,H\,H_y + \underline{G\,H\,M}_{\textcircled{k}} - \underline{\frac{1}{4}\,(H)^2\,L}_{\textcircled{l}} - \underline{\frac{1}{4}\,(H)^2\,\Theta^1}_{\textcircled{m}} + \\
& + \underline{\frac{1}{4}\,H\,L\,\Theta^0}_{\textcircled{f}} + \underline{\frac{1}{4}\,H\,\Theta^0\,\Theta^1}_{\textcircled{g}} - \underline{\frac{1}{2}\,L_x\,\Theta^0}_{\textcircled{e}} + G_y\,L - \frac{1}{2}\,H_x\,L - \underline{\frac{1}{2}\,G\,(L)^2}_{\textcircled{a}} + \\
& + \underline{\frac{1}{4}\,H^2\,L}_{\textcircled{l}} + \underline{\frac{1}{2}\,G\,L\,\Theta^1}_{\textcircled{n}} - \underline{\frac{1}{4}\,L\,\left(\Theta^0\right)^2}_{\textcircled{h}} + \underline{G_y\,\Theta^1}_{\textcircled{o}} - \underline{\frac{1}{2}\,H_x\,\Theta^1}_{\textcircled{j}} - \\
& - \underline{\frac{1}{2}\,G\,L\,\Theta^1}_{\textcircled{n}} + \underline{\frac{1}{4}\,H^2\,\Theta^1}_{\textcircled{m}} + \underline{\frac{1}{2}\,G\,(\Theta^1)^2}_{\textcircled{c}} - \underline{\frac{1}{4}\,\left(\Theta^0\right)^2\,\Theta^1}_{\textcircled{i}} - \underline{\frac{1}{6}\,L_x\,\Theta^0}_{\textcircled{e}} +
\end{aligned}
\tag{2.45}
$$

$$+\underbrace{\frac{1}{3}\,H_y\,\Theta^0}_{\text{(d)}}-\underbrace{G\,M\,\Theta^0}_{\text{(b)}}+\underbrace{\frac{1}{4}\,H\,L\,\Theta^0}_{\text{(f)}}+\underbrace{\frac{1}{4}\,H\,\Theta^0\,\Theta^1}_{\text{(g)}}-\underbrace{\frac{1}{4}\,L\,(\Theta^0)^2}_{\text{(h)}}-$$
$$-\underbrace{\frac{1}{4}\,(\Theta^0)^2\,\Theta^1}_{\text{(i)}}.$$

As may be observed, the order in which we discover the terms which vanish is governed by chance. After some terms are underlined, they are automatically disregarded by the eyes, which lightens the chasing of other terms to be simplified. To collect the remaining terms in order to obtain the final expression (2.43), our method is similar: we underline the terms which may be summed together. However, whereas we use the red pencil to underline the vanishing terms, we use the green pencil to underline the remaining terms. This small trick is to avoid as much as possible to copy several times some long formal expressions. Finally, we reorder everything lexicographically, so as to get the conclusion (2.43). In order to obtain the final equation (2.43) as efficiently as possible, we read the remaining terms, picking them directly in lexicographic order. If, by lack of luck, one or two terms are forgotten by the eyes and not written in the right place, we copy once more the very final result in the right order, or we use the blank corrector.

Of course, such a refined methodology could seem to be essentially superfluous for such relatively accessible computations. However, when passing to several dependent variables, the current expressions will be approximatively five times more massive. So, we believe that a clever methodology of hand computations is helpful in this category.

§3. SYSTEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS EQUIVALENT TO FREE PARTICLES

3.1. Combinatorics of the second order prolongation of a point transformation. In this section, we endeavour to explain how S. Lie's theorem and proof may be generalized to the case of several dependent variables. As in the statement of Theorem 1.7, let us assume that the system $y_{xx}^j = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent under an invertible point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$ to the free particle system $Y_{XX}^j = 0$, $j = 1, \dots, m$. By assumption, the Jacobian determinant

$$\Delta(x|y^1|\dots|y^m) := \begin{vmatrix} X_x & X_{y^1} & \dots & X_{y^m} \\ Y_x^1 & Y_{y^1}^1 & \dots & Y_{y^m}^1 \\ \dots & \dots & \dots & \dots \\ Y_x^m & Y_{y^1}^m & \dots & Y_{y^m}^m \end{vmatrix} \tag{3.2}$$

does not vanish at the origin. As in the case $m = 1$, since the flat system $Y_{XX}^j = 0$ is left unchanged by any affine transformation, we can (and we shall) assume that the transformation is tangent to the identity at the origin, so that the above Jacobian matrix equals the identity matrix at $(x, y) = (0, 0)$, whence in a neighborhood of the origin it is close to the identity matrix, namely

$$X_x \cong 1, \qquad X_{y^j} \cong 0, \qquad Y_x^j \cong 0, \qquad Y_{y^{j_1}}^{j_2} \cong \delta_{j_1}^{j_2}. \tag{3.3}$$

Inductive formulas for the computation how the differential equation in the (X, Y) coordinates is related to the differential equation in the (x, y) coordinates may be found in [BK1989], [OL1995], [Su2002]; the explicit formulas are not achieved in these references. Let us recall the inductive formulas, just on the computational level (differential-geometric conceptional background about graph transformations may be found in [OL1986], Ch. 2).

First of all, we seek how the $Y_X^j := \frac{dY^j}{dX}$ are explicitely related to the y_x^l. It suffices to replace, in the identity

$$Y_X^j \cdot \left(X_x \, dx + \sum_{l=1}^m X_{y^l} \, dy^l \right) = Y_X^j \, dX = dY^j = Y_x^j \, dx + \sum_{l=1}^m Y_{y^l}^j \, dy^l \tag{3.4}$$

the differentials dy^l by $y_x^l \, dx$ and then to identify the coefficient of dx on both sides, which rapidly yields the formulas

$$Y_X^j = \frac{Y_x^j + \sum_{l=1}^m y_x^l \, Y_{y^l}^j}{X_x + \sum_{l=1}^m y_x^l \, X_{y^l}}, \tag{3.5}$$

for $j = 1, \dots, m$.

Next, we seek how the $Y_{XX}^j := \frac{d^2 Y^j}{dX^2} = \frac{dY_X^j}{dX}$ are related to the $y_x^{l_1}$, $y_{xx}^{l_2}$. It suffices to again replace each dy_l by $y_x^l \, dx$ and each dy_x^l by $y_{xx}^l \, dx$ in the identity

$$\begin{cases} Y_{XX}^j \cdot \left(X_x \, dx + \sum_{l=1}^m X_{y^l} \, dy^l \right) = Y_{XX}^j \cdot dX = dY_X^j \\ \qquad = \frac{\partial Y_X^j}{\partial x} \, dx + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y^l} \, dy^l + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y_x^l} \, dy_x^l \\ \qquad = \left(\frac{\partial Y_X^j}{\partial x} + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y^l} \, y_x^l + \sum_{l=1}^m \frac{\partial Y_X^j}{\partial y_x^l} \, y_{xx}^l \right) \cdot dx. \end{cases} \tag{3.6}$$

Before entering the precise combinatorics of the explicit expression of Y_{XX}^j, let us observe that the last term of (3.6) simply writes $D(Y_X^j) \, dx$, where D denotes the *total differentiation operator* (of order two) defined by

$$D := \frac{\partial}{\partial x} + \sum_{l=1}^m y_x^l \, \frac{\partial}{\partial y^l} + \sum_{l=1}^m y_{xx}^l \, \frac{\partial}{\partial y_x^l}. \tag{3.7}$$

Since $dX \equiv DX$ after replacing each dy^l by $y_x^l \, dx$, it follows that we may compactly rewrite (3.6) as

$$Y_{XX}^j \, DX \cdot dx = D \left(Y_X^j \right) \cdot dx \tag{3.8}$$

Consequently, the expressions of Y_X^j (obtained in (3.5)) and of Y_{XX}^j are

$$Y_X^j = \frac{DY^j}{DX} \qquad \text{and} \qquad Y_{XX}^j = \frac{D\left(Y_X^j \right)}{DX} = \frac{DDY^j \cdot DX - DDX \cdot DY^j}{[DX]^3}. \tag{3.9}$$

As, by assumption, the system $y_{xx}^j = F^j(x, y, y_x)$ transforms to the flat system $Y_{XX}^j = 0$, after erasing the denominator of (3.7), we come to the equations

$$0 = DDY^j \cdot DX - DDX \cdot DY^j, \tag{3.10}$$

for $j = 1, \dots, m$. However, this too simple and too compact expression of the system $y_{xx}^j = F^j(x, y, y_x)$ is of no use and we must develope (patiently!) the explicit expressions of DDY^j, of DX, of DDX and of DY^j, using the complete expression of D defined in (3.6).

At this point, we would like to stress that *it constitutes already a nontrivial computational and combinatorial task to obtain a complete explicit formula for the system*

$y_{xx}^j = F^j(x,y,y_x)$ *hidden in the compact form* (3.10), which would be the generalization of the nice formula (2.9) involving modifications of the Jacobian determinant. For general $m \geq 2$, the complete proofs are postponed to Section 5 below.

Since it would be intuitively unsatisfactory to provide directly the final simplified expression of the development of (3.10) in the general case $m \geq 2$, let us firstly describe step by step how one may guess what is the generalization of (2.9).

For instance, in the case $m = 2$, by a direct and relatively short computation which consists in developing plainly (3.10), we obtain for $j = 1, 2$:

(3.11)
$$\left\{\begin{aligned}
0 = & -X_{xx}\,Y_x^j + Y_{xx}^j\,X_x + \\
& + y_x^1 \cdot \Big[-X_{xx}\,Y_{y^1}^j + Y_{xx}^j\,X_{y^1} - 2\,X_{xy^1}\,Y_x^j + 2\,Y_{xy^1}^j\,X_x\Big] + \\
& + y_x^2 \cdot \Big[-X_{xx}\,Y_{y^2}^j + Y_{xx}^j\,X_{y^2} - 2\,X_{xy^2}\,Y_x^j + 2Y_{xy^2}^j\,X_x\Big] + \\
& + y_x^1\,y_x^1 \cdot \Big[-2\,X_{xy^1}\,Y_{y^1}^j + 2\,Y_{xy^1}^j\,X_{y^1} - X_{y^1y^1}\,Y_x^j + Y_{y^1y^1}^j\,X_x\Big] + \\
& + y_x^1\,y_x^2 \cdot \Big[-2\,X_{xy^1}\,Y_{y^2}^j + 2\,Y_{xy^1}^j\,X_{y^2} - 2\,X_{xy^2}\,Y_{y^1}^j + 2\,Y_{xy^2}^j\,X_{y^1} - \\
& \qquad\qquad - 2\,X_{y^1y^2}\,Y_x^j + 2\,Y_{y^1y^2}^j\,X_x\Big] + \\
& + y_x^2\,y_x^2 \cdot \Big[-2\,X_{xy^2}\,Y_{y^2}^j + 2\,Y_{xy^2}^j\,X_{y^2} - X_{y^2y^2}\,Y_x^j + Y_{y^2y^2}^j\,X_x\Big] + \\
& + y_x^1\,y_x^1\,y_x^1 \cdot \Big[-X_{y^1y^1}\,Y_{y^1}^j + Y_{y^1y^1}^j\,X_{y^1}\Big] + \\
& + y_x^1\,y_x^1\,y_x^2 \cdot \Big[-X_{y^1y^1}\,Y_{y^2}^j + Y_{y^1y^1}^j\,X_{y^2} - 2\,X_{y^1y^2}\,Y_{y^1}^j + 2\,Y_{y^1y^2}^j\,X_{y^1}\Big] + \\
& + y_x^1\,y_x^2\,y_x^2 \cdot \Big[-X_{y^2y^2}\,Y_{y^1}^j + Y_{y^2y^2}^j\,X_{y^1} - 2\,X_{y^1y^2}\,Y_{y^2}^j + 2\,Y_{y^1y^2}^j\,X_{y^2}\Big] + \\
& + y_x^2\,y_x^2\,y_x^2 \cdot \Big[-X_{y^2y^2}\,Y_{y^2}^j + Y_{y^2y^2}^j\,X_{y^2}\Big] + \\
& + y_{xx}^1 \cdot \Big[-X_{y^1}\,Y_x^j + Y_{y^1}^j\,X_x + y_x^2 \cdot \Big\{-X_{y^1}\,Y_{y^2}^j + Y_{y^1}^j\,X_{y^2}\Big\}\Big] + \\
& + y_{xx}^2 \cdot \Big[-X_{y^2}\,Y_x^j + Y_{y^2}^j\,X_x + y_x^1 \cdot \Big\{-X_{y^2}\,Y_{y^1}^j + Y_{y^2}^j\,X_{y^1}\Big\}\Big].
\end{aligned}\right.$$

Unfortunately, *the above two equations are not solved with respect to* y_{xx}^1 *and to* y_{xx}^2. Consequently, if we abbreviate them as a linear system of the form

$$\text{(3.12)}\qquad \left\{\begin{aligned}
0 &= A^1 + y_{xx}^1 \cdot B_1^1 + y_{xx}^2 \cdot B_2^1, \\
0 &= A^2 + y_{xx}^1 \cdot B_1^2 + y_{xx}^2 \cdot B_2^2,
\end{aligned}\right.$$

we have to solve for y_{xx}^1 and for y_{xx}^2 by means of the classical rule of Cramer. Here, it is rather quick to check manually that the determinant of this system has the following nice expression:

$$\text{(3.13)}\qquad \left\{\begin{aligned}
\left|\begin{array}{cc} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{array}\right| &= \Delta(x|y^1|y^2) \cdot \big\{X_x + y_x^1\,X_{y^1} + y_x^2\,X_{y^2}\big\} \\
&= \Delta(x|y^1|y^2) \cdot DX.
\end{aligned}\right.$$

However, the complete solving for y_{xx}^1 and for y_{xx}^2 requires some more time. After a direct and rather long hand computation (or alternately, using Maple or Mathematica) one obtains formulas involving hidden 3×3 determinants, which have to be guessed by the intuition; the first equation that we obtain, namely for y^1_{xx} is as follows:

$$
\begin{aligned}
0 = y^1_{xx}\cdot
&\begin{vmatrix} X_x & X_{y^1} & X_{y^2} \\ Y^1_x & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_x & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
+
\begin{vmatrix} X_x & X_{xx} & X_{y^2} \\ Y^1_x & Y^1_{xx} & Y^1_{y^2} \\ Y^2_x & Y^2_{xx} & Y^2_{y^2} \end{vmatrix} + \\
&+ y^1_x\cdot\left\{ 2
\begin{vmatrix} X_x & X_{xy^1} & X_{y^2} \\ Y^1_x & Y^1_{xy^1} & Y^1_{y^2} \\ Y^2_x & Y^2_{xy^1} & Y^2_{y^2} \end{vmatrix}
-
\begin{vmatrix} X_{xx} & X_{y^1} & X_{y^2} \\ Y^1_{xx} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{xx} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^2_x\cdot\left\{ 2
\begin{vmatrix} X_x & X_{xy^2} & X_{y^2} \\ Y^1_x & Y^1_{xy^2} & Y^1_{y^2} \\ Y^2_x & Y^2_{xy^2} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^1_x\,y^1_x\cdot\left\{
\begin{vmatrix} X_x & X_{y^1y^1} & X_{y^2} \\ Y^1_x & Y^1_{y^1y^1} & Y^1_{y^2} \\ Y^2_x & Y^2_{y^1y^1} & Y^2_{y^2} \end{vmatrix}
- 2
\begin{vmatrix} X_{xy^1} & X_{y^1} & X_{y^2} \\ Y^1_{xy^1} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{xy^1} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^1_x\,y^2_x \left\{ 2
\begin{vmatrix} X_x & X_{y^1y^2} & X_{y^2} \\ Y^1_x & Y^1_{y^1y^2} & Y^1_{y^2} \\ Y^2_x & Y^2_{y^1y^2} & Y^2_{y^2} \end{vmatrix}
- 2
\begin{vmatrix} X_{xy^2} & X_{y^1} & X_{y^2} \\ Y^1_{xy^2} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{xy^2} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^2_x\,y^2_x\cdot\left\{
\begin{vmatrix} X_x & X_{y^2y^2} & X_{y^2} \\ Y^1_x & Y^1_{y^2y^2} & Y^1_{y^2} \\ Y^2_x & Y^2_{y^2y^2} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^1_x\,y^1_x\,y^1_x\cdot\left\{ -
\begin{vmatrix} X_{y^1y^1} & X_{y^1} & X_{y^2} \\ Y^1_{y^1y^1} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{y^1y^1} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} + y^1_x\,y^1_x\,y^2_x\cdot\left\{ -2
\begin{vmatrix} X_{y^1y^2} & X_{y^1} & X_{y^2} \\ Y^1_{y^1y^2} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{y^1y^2} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} + \\
&+ y^1_x\,y^2_x\,y^2_x\cdot\left\{ -
\begin{vmatrix} X_{y^2y^2} & X_{y^1} & X_{y^2} \\ Y^1_{y^2y^2} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{y^2y^2} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\}.
\end{aligned}
\tag{3.14}
$$

This formula and the next have been checked by Sylvain Neut and Michel Petitot with the help of Maple. We notice that the size is not negligible, but fortunately, there appears some combinatorics, much more visible than in (3.11). The second equation that we obtain, namely for y^2_{xx}, is as follows:

$$
\begin{aligned}
0 = y^2_{xx}\cdot
&\begin{vmatrix} X_x & X_{y^1} & X_{y^2} \\ Y^1_x & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_x & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
+
\begin{vmatrix} X_x & X_{y^1} & X_{xx} \\ Y^1_x & Y^1_{y^1} & Y^1_{xx} \\ Y^2_x & Y^2_{y^2} & Y^2_{xx} \end{vmatrix} + \\
&+ y^1_x\cdot\left\{ 2
\begin{vmatrix} X_x & X_{y^1} & X_{xy^1} \\ Y^1_x & Y^1_{y^1} & Y^1_{xy^1} \\ Y^2_x & Y^2_{y^1} & Y^2_{xy^1} \end{vmatrix}
\right\} + \\
&+ y^2_x\cdot\left\{ 2
\begin{vmatrix} X_x & X_{y^1} & X_{xy^2} \\ Y^1_x & Y^1_{y^1} & Y^1_{xy^2} \\ Y^2_x & Y^2_{y^1} & Y^2_{xy^2} \end{vmatrix}
-
\begin{vmatrix} X_{xx} & X_{y^1} & X_{y^2} \\ Y^1_{xx} & Y^1_{y^1} & Y^1_{y^2} \\ Y^2_{xx} & Y^2_{y^1} & Y^2_{y^2} \end{vmatrix}
\right\} +
\end{aligned}
\tag{3.15}
$$

$$
+ y_x^1\, y_x^1 \cdot \left\{ \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^1y^1} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^1y^1}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^1y^1}^2 \end{array} \right| \right\} +
$$

$$
+ y_x^1\, y_x^2 \left\{ 2 \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^1y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^1y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^1y^2}^2 \end{array} \right| - 2 \left| \begin{array}{ccc} X_{xy^1} & X_{y^1} & X_{y^2} \\ Y_{xy^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} +
$$

$$
+ y_x^2\, y_x^2 \cdot \left\{ \left| \begin{array}{ccc} X_x & X_{y^1} & X_{y^2y^2} \\ Y_x^1 & Y_{y^1}^1 & Y_{y^2y^2}^1 \\ Y_x^2 & Y_{y^1}^2 & Y_{y^2y^2}^2 \end{array} \right| - 2 \left| \begin{array}{ccc} X_{xy^2} & X_{y^1} & X_{y^2} \\ Y_{xy^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{xy^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} +
$$

$$
+ y_x^1\, y_x^1\, y_x^2 \cdot \left\{ - \left| \begin{array}{ccc} X_{y^1y^1} & X_{y^1} & X_{y^2} \\ Y_{y^1y^1}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1y^1}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} + y_x^1\, y_x^2\, y_x^2 \cdot \left\{ -2 \left| \begin{array}{ccc} X_{y^1y^2} & X_{y^1} & X_{y^2} \\ Y_{y^1y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^1y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\} +
$$

$$
+ y_x^2\, y_x^2\, y_x^2 \cdot \left\{ - \left| \begin{array}{ccc} X_{y^2y^2} & X_{y^1} & X_{y^2} \\ Y_{y^2y^2}^1 & Y_{y^1}^1 & Y_{y^2}^1 \\ Y_{y^2y^2}^2 & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \right\}.
$$

Importantly, the obtained formulas seem to be analogous to the formula (2.9), since we observe that the coefficients of the degree three polynomial in the y_x^l are modifications of the Jacobian determinant $\Delta(x|y^1|y^2)$.

To describe the underlying combinatorics, let us observe that there exist exactly six possible distinct second order derivatives: xx, xy^1, xy^2, y^1y^1, y^1y^2 and y^2y^2. There are also exactly three columns in the Jacobian determinant (3.2). By replacing each of the three columns of first order derivatives by a column of second order detivatives (leaving X, Y^1 and Y^2 unchanged), we may build exactly eighteen new determinants

$$
\left\{ \begin{array}{lll} \Delta(xx|y^1|y^2) & \Delta(x|xx|y^2) & \Delta(x|y^1|xx) \\ \Delta(xy^1|y^1|y^2) & \Delta(x|xy^1|y^2) & \Delta(x|y^1|xy^1) \\ \Delta(xy^2|y^1|y^2) & \Delta(x|xy^2|y^2) & \Delta(x|y^1|xy^2) \\ \Delta(y^1y^1|y^1|y^2) & \Delta(x|y^1y^1|y^2) & \Delta(x|y^1|y^1y^1) \\ \Delta(y^1y^2|y^1|y^2) & \Delta(x|y^1y^2|y^2) & \Delta(x|y^1|y^1y^2) \\ \Delta(y^2y^2|y^1|y^2) & \Delta(x|y^2y^2|y^2) & \Delta(x|y^1|y^2y^2), \end{array} \right. \tag{3.16}
$$

where for instance

$$
\left\{ \begin{array}{l} \Delta(\underline{y^1y^2}|y^1|y^2) := \left| \begin{array}{ccc} \underline{X_{y^1y^2}} & X_{y^1} & X_{y^2} \\ \underline{Y_{y^1y^2}^1} & Y_{y^1}^1 & Y_{y^2}^1 \\ \underline{Y_{y^1y^2}^2} & Y_{y^1}^2 & Y_{y^2}^2 \end{array} \right| \quad \text{and} \\ \Delta(x|y^1|\underline{xy^2}) := \left| \begin{array}{ccc} X_x & X_{y^1} & \underline{X_{xy^2}} \\ Y_x^1 & Y_{y^1}^1 & \underline{Y_{xy^2}^1} \\ Y_x^2 & Y_{y^1}^2 & \underline{Y_{xy^2}^2} \end{array} \right|. \end{array} \right. \tag{3.17}
$$

Hence, using the Δ-notation, we may rewrite the two equation (3.14) and (3.15) under a more compact form; after division by the Jacobian determinant $\Delta(x|y^1|y^2)$, the first equation becomes:

(3.18)

$$\begin{cases} 0 = y^1_{xx} + \frac{\Delta(x|xx|y^2)}{\Delta(x|y^1|y^2)} + y^1_x \cdot \left\{ 2\,\frac{\Delta(x|xy^1|y^2)}{\Delta(x|y^1|y^2)} - \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^2_x \cdot \left\{ 2\,\frac{\Delta(x|xy^2|y^2)}{\Delta(x|y^1|y^2)} \right\} + y^1_x\, y^1_x \cdot \left\{ \frac{\Delta(x|y^1y^1|y^2)}{\Delta(x|y^1|y^2)} - 2\,\frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^1_x\, y^2_x \cdot \left\{ 2\,\frac{\Delta(x|y^1y^2|y^2)}{\Delta(x|y^1|y^2)} - 2\,\frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y^2_x\, y^2_x \cdot \left\{ \frac{\Delta(x|y^2y^2|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^1_x\, y^1_x\, y^1_x \cdot \left\{ -\frac{\Delta(y^1y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y^1_x\, y^1_x\, y^2_x \cdot \left\{ -2\,\frac{\Delta(y^1y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^1_x\, y^2_x\, y^2_x \cdot \left\{ -\frac{\Delta(y^2y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\}. \end{cases}$$

Similarly, the second equation takes the form:

(3.19)

$$\begin{cases} 0 = y^2_{xx} + \frac{\Delta(x|y^1|xx)}{\Delta(x|y^1|y^2)} + y^1_x \cdot \left\{ 2\,\frac{\Delta(x|y^1|xy^1)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^2_x \cdot \left\{ 2\,\frac{\Delta(x|y^1|xy^2)}{\Delta(x|y^1|y^2)} - \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y^1_x\, y^1_x \cdot \left\{ \frac{\Delta(x|y^1|y^1y^1)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^1_x\, y^2_x \cdot \left\{ 2\,\frac{\Delta(x|y^1|y^1y^2)}{\Delta(x|y^1|y^2)} - 2\,\frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^2_x\, y^2_x \cdot \left\{ \frac{\Delta(x|y^2|y^2y^2)}{\Delta(x|y^1|y^2)} - 2\,\frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^1_x\, y^1_x\, y^2_x \cdot \left\{ -\frac{\Delta(y^1y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + y^1_x\, y^2_x\, y^2_x \cdot \left\{ -2\,\frac{\Delta(y^1y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\} + \\ \quad + y^2_x\, y^2_x\, y^2_x \cdot \left\{ -\frac{\Delta(y^2y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \right\}. \end{cases}$$

Since the formulas are still of a consequent size, analogously to what was achieved in Section 2, we shall introduce a new family of *square functions* as follows. We first index the coordinates $(x, y^1, \dots, y^m)$ as $(y^0, y^1, \dots, y^m)$, namely we introduce the notational equivalence

$$\boxed{y^0 \equiv x}, \tag{3.20}$$

which will be very convenient in the sequel, especially in order to write general formulas. With this convention at hand, our eighteen square functions $\square^{k_1}_{y^{l_1} y^{l_2}}$, defined for $0 \leq$

$j_1, j_2, k_1 \leq 2$ are defined by

(3.21)

$$\left\{\begin{aligned}
\square_{xx}^0 &:= \frac{\Delta(xx|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^0 &:= \frac{\Delta(xy^1|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^0 &:= \frac{\Delta(xy^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \\
\square_{y^1y^1}^0 &:= \frac{\Delta(y^1y^1|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^1y^2}^0 &:= \frac{\Delta(y^1y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2y^2}^0 &:= \frac{\Delta(y^2y^2|y^1|y^2)}{\Delta(x|y^1|y^2)} \\
\square_{xx}^1 &:= \frac{\Delta(x|xx|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^1 &:= \frac{\Delta(x|xy^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^1 &:= \frac{\Delta(x|xy^2|y^2)}{\Delta(x|y^1|y^2)} \\
\square_{y^1y^1}^1 &:= \frac{\Delta(x|y^1y^1|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^1y^2}^1 &:= \frac{\Delta(x|y^1y^2|y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2y^2}^1 &:= \frac{\Delta(x|y^2y^2|y^2)}{\Delta(x|y^1|y^2)} \\
\square_{xx}^2 &:= \frac{\Delta(x|y^1|xx)}{\Delta(x|y^1|y^2)} & \square_{xy^1}^2 &:= \frac{\Delta(x|y^1|xy^1)}{\Delta(x|y^1|y^2)} & \square_{xy^2}^2 &:= \frac{\Delta(x|y^1|xy^2)}{\Delta(x|y^1|y^2)} \\
\square_{y^1y^1}^2 &:= \frac{\Delta(x|y^1|y^1y^1)}{\Delta(x|y^1|y^2)} & \square_{y^1y^2}^2 &:= \frac{\Delta(x|y^1|y^1y^2)}{\Delta(x|y^1|y^2)} & \square_{y^2y^2}^2 &:= \frac{\Delta(x|y^1|y^2y^2)}{\Delta(x|y^1|y^2)}
\end{aligned}\right.$$

Obviously, the square functions are symmetric with respect to the lower indices: $\square_{y^{l_1}y^{l_2}}^{k_1} = \square_{y^{l_2}y^{l_1}}^{k_1}$. Here, the upper index designates the column upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that *the square functions are explicit rational expressions in terms of the second order jet of the transformation* $(x,y) \mapsto (X(x,y), Y(x,y))$. At this point, we may summarize what we have established so far.

Lemma 3.22. *The system of* two *second order ordinary differential equations* $y_{xx}^1 = F^1(x,y,y_x)$ *and* $y_{xx}^2 = F^2(x,y,y_x)$ *is equivalent, under a point transformation, to the flat system* $Y_{XX}^1 = 0$ *and* $Y_{XX}^2 = 0$ *if and only if there exist three local* $\mathbb{K}$*-analytic functions* $X(x,y)$, $Y^1(x,y)$ *and* $Y^2(x,y)$ *such that it may be written under the form*

(3.23)

$$\left\{\begin{aligned}
0 = y_{xx}^1 &+ \square_{xx}^1 + y_x^1 \cdot \left(2\,\square_{xy^1}^1 - \square_{xx}^0\right) + y_x^2 \cdot \left(2\,\square_{xy^2}^1\right) + \\
&+ y_x^1\,y_x^1 \cdot \left(\square_{y^1y^1}^1 - 2\,\square_{xy^1}^0\right) + y_x^1 y_x^2 \cdot \left(2\,\square_{y^1y^2}^1 - 2\,\square_{xy^2}^0\right) + y_x^2\,y_x^2 \cdot \left(\square_{y^2y^2}^1\right) + \\
&+ y_x^1\,y_x^1\,y_x^1 \cdot \left(-\square_{y^1y^1}^0\right) + y_x^1\,y_x^1\,y_x^2 \cdot \left(-2\,\square_{y^1y^2}^0\right) + y_x^1\,y_x^2\,y_x^2 \cdot \left(-\square_{y^2y^2}^0\right), \\
0 = y_{xx}^2 &+ \square_{xx}^2 + y_x^1 \cdot \left(2\,\square_{xy^1}^2\right) + y_x^2 \cdot \left(2\,\square_{xy^2}^2 - \square_{xx}^0\right) + \\
&+ y_x^1\,y_x^1 \cdot \left(\square_{y^1y^1}^2\right) + y_x^1 y_x^2 \cdot \left(2\,\square_{y^1y^2}^2 - 2\,\square_{xy^1}^0\right) + y_x^2\,y_x^2 \cdot \left(\square_{y^2y^2}^2 - 2\,\square_{xy^2}^0\right) + \\
&+ y_x^1\,y_x^1\,y_x^2 \cdot \left(-\square_{y^1y^1}^0\right) + y_x^1\,y_x^2\,y_x^2 \cdot \left(-2\,\square_{y^1y^2}^0\right) + y_x^2\,y_x^2\,y_x^2 \cdot \left(-\square_{y^2y^2}^0\right).
\end{aligned}\right.$$

3.24. Second Lie prolongation of a vector field. At this point, instead of proceeding further with the case $m = 2$, it is now time to pass to the general case $m \geq 2$. First of all, we would like to remind from [GM2003] the complete explicit expression of the point prolongation to the second order jet space of a general vector field of the form $L = X\,\frac{\partial}{\partial x} + \sum_{j=1}^m Y^j\,\frac{\partial}{\partial y^j}$: it is a vector field of the form

$$L^{(2)} = X\,\frac{\partial}{\partial x} + \sum_{j=1}^m Y^j\,\frac{\partial}{\partial y^j} + \sum_{j=1}^m \mathbf{R}_1^j\,\frac{\partial}{\partial y_x^j} + \sum_{j=1}^m \mathbf{R}_2^j\,\frac{\partial}{\partial y_{xx}^j}, \tag{3.25}$$

where the coefficients $\mathbf{R}_1^j$ and $\mathbf{R}_2^j$ are polynomials in the jet space variables having as coefficients certain specific linear combinations of first and second order derivatives of X

and of the Y^j:

$$
(3.26)\qquad
\left\{
\begin{aligned}
\mathbf{R}_1^j &= Y_x^j + \sum_{l_1=1}^m y_x^{l_1}\cdot\left[Y_{y^{l_1}}^j - \delta_{l_1}^j\,X_x\right] + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot\left[-\delta_{l_1}^j\,X_{y^{l_2}}\right],\\
\mathbf{R}_2^j &= Y_{xx}^j + \sum_{l_1=1}^m y_x^{l_1}\cdot\left[2\,Y_{xy^{l_1}}^j - \delta_{l_1}^j\,X_{xx}\right]+\\
&\quad + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot\left[Y_{y^{l_1}y^{l_2}}^j - \delta_{l_1}^j\,X_{xy^{l_2}} - \delta_{l_2}^j\,X_{xy^{l_1}}\right]+\\
&\quad + \sum_{l_1=1}^m\sum_{l_2=1}^m\sum_{l_3=1}^m y_x^{l_1}\,y_x^{l_2}\,y_x^{l_3}\cdot\left[-\delta_{l_1}^j\,X_{y^{l_2}y^{l_3}}\right]+\\
&\quad + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot\left[Y_{y^{l_1}}^j - 2\,\delta_{l_1}^j\,X_x\right]+\\
&\quad + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_{xx}^{l_2}\cdot\left[-\delta_{l_1}^j\,X_{y^{l_2}} - 2\,\delta_{l_2}^j\,X_{y^{l_1}}\right].
\end{aligned}
\right.
$$

However, since the notations in [GM2003] are different and since the general case of $n \geq 1$ independent variables and $m \geq 1$ dependent variables is considered there, it is certainly easier to reconstiture formulas (3.26) directly by means of the inductive formulas described in [OL1986], [BK1989]).

Analogously to the observation made in Section 2, we guess that there exists a formal correspondence between the terms of $\mathbf{R}_2^j$ not involving y_{xx}^l and the explicit form of the equation $y_{xx}^j = F^j(x,y,y_x)$ equivalent to $Y_{XX}^j = 0$. In the case $m = 2$, we claim that this formal correspondence also holds true. Indeed, it suffices to write formula (3.26) for $\mathbf{R}_2^j$ modulo the y_{xx}^l, which yields two expressions in total analogy with the two explicit polynomials appearing in the right hand side of (3.23):

(3.27)

$$
\left\{
\begin{aligned}
\mathbf{R}_2^1\ (\mathrm{mod}\ y_{xx}^l) &\equiv Y_{xx}^1 + y_x^1\cdot\left\{2\,Y_{xy^1}^1 - X_{xx}\right\} + y_x^2\cdot\left\{2\,Y_{xy^2}^1\right\} + y_x^1\,y_x^1\cdot\left\{Y_{y^1y^1}^1 - 2\,X_{xy^1}\right\}+\\
&\quad + y_x^1y_x^2\cdot\left\{2\,Y_{y^1y^2}^1 - 2\,X_{xy^2}\right\} + y_x^2\,y_x^2\cdot\left\{Y_{y^2y^2}^1\right\}+\\
&\quad + y_x^1\,y_x^1\,y_x^1\cdot\left\{-X_{y^1y^1}\right\} + y_x^1\,y_x^1\,y_x^2\cdot\left\{-2\,X_{y^1y^2}\right\} + y_x^1\,y_x^2\,y_x^2\cdot\left\{-X_{y^2y^2}\right\},\\
\mathbf{R}_2^2\ (\mathrm{mod}\ y_{xx}^l) &\equiv Y_{xx}^2 + y_x^1\cdot\left\{2\,Y_{xy^1}^2\right\} + y_x^2\cdot\left\{2\,Y_{xy^2}^2 - X_{xx}\right\} + y_x^1\,y_x^1\cdot\left\{Y_{y^1y^1}^2\right\}+\\
&\quad + y_x^1y_x^2\cdot\left\{2\,Y_{y^1y^2}^2 - 2\,X_{xy^1}\right\} + y_x^2\,y_x^2\cdot\left\{Y_{y^2y^2}^2 - 2\,X_{xy^2}\right\}+\\
&\quad + y_x^1\,y_x^1\,y_x^2\cdot\left\{-X_{y^1y^1}\right\} + y_x^1\,y_x^2\,y_x^2\cdot\left\{-2\,X_{y^1y^2}\right\} + y_x^2\,y_x^2\,y_x^2\cdot\left\{-X_{y^2y^2}\right\},
\end{aligned}
\right.
$$

Except for inductive inspiration (*see* the formulation of Lemma 3.32 below), this observation will not be used further. At this stage, it helps at least to maintain a strong intuitive control of the correctness of the underlying combinatorics.

3.28. System equivalent to the flat system. By induction, we therefore guess that the analogy holds for general $m \geq 2$, namely we guess the following combinatorics, which requires some preliminaries.

As in the beginning of §3.1, let $x \in \mathbb{K}$, let $y = (y^1, \dots, y^m) \in \mathbb{K}^m$, let $(x,y) \mapsto (X(x,y), Y(x,y))$ be a local $\mathbb{K}$-analytic transformation defined in a neighborhood of the origin in $\mathbb{K}^{m+1}$ and assume that the system $y_{xx}^j = F^j(x,y,y_x)$, $j = 1, \dots, m$, is equivalent to the flat system $Y_{XX}^j = 0$, $j = 1, \dots, m$. By assumption, the Jacobian matrix of the equivalence equals the identity matrix at the origin. Remind that we identify x with y^0. For all $k_1, l_1, l_2 = 0, \dots, m$, we define a modification

$$\Delta(x|\dots|^{k_1} y^{l_1} y^{l_2}|\dots|y^m) \tag{3.29}$$

of the Jacobian determinant as follows. We replace the k_1-th column of the determinant (3.2), which consists of first order derivatives $\cdot_{y^{k_1}}$, by a column which consists of second order derivatives $\cdot_{y^{l_1} y^{l_2}}$. In (3.29), the notation $|^{k_1}$ designates the k_1-th column, the first one being labelled by $k_1 = 0$ and the last one by $k_1 = m$. With this notation at hand, we may define the square functions

$$\square^{k_1}_{y^{l_1} y^{l_2}} := \frac{\Delta(x|\dots|^{k_1} y^{l_1} y^{l_2}|\dots|y^m)}{\Delta(x|\dots|^{k_1} y^{k_1}|\dots y^m)}, \tag{3.30}$$

which are rational expression in the second order jet of the transformation $(x, y) \mapsto (X(x, y), Y(x, y))$. As before, the denominator is the Jacobian determinant of the change of coordinates.

Since, according to (3.26), the expression of $\mathbf{R}^j_2 \ (\mathrm{mod}\ y^l_{xx})$ is

$$\begin{cases} \mathbf{R}^j_2 \ (\mathrm{mod}\ y^l_{xx}) = Y^j_{xx} \ + \sum_{l_1=1}^m y^{l_1}_x \cdot \left[2\, Y^j_{xy^{l_1}} - \delta^j_{l_1}\, X_{xx}\right] + \\ \qquad + \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x\, y^{l_2}_x \cdot \left[Y^j_{y^{l_1} y^{l_2}} - \delta^j_{l_1}\, X_{xy^{l_2}} - \delta^j_{l_2}\, X_{xy^{l_1}}\right] + \\ \qquad + \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y^{l_1}_x\, y^{l_2}_x\, y^{l_3}_x \cdot \left[-\delta^j_{l_1}\, X_{y^{l_2} y^{l_3}}\right], \end{cases} \tag{3.31}$$

and since, in the cases $m = 1$ and $m = 2$, we have already observed strong analogies between (3.31) and the complete explicit expression of the system $y^j_{xx} = F^j(x, y, y_x)$ equivalent to the flat system $Y^j_{XX} = 0$, we guess that the following lemma is formally true.

Lemma 3.32. *The system $y^j_{xx} = F^j(x, y, y_x)$, $j = 1, \dots, m$, is equivalent to the flat system $Y^j_{XX} = 0$, $j = 1, \dots, m$, if and only if there exist local $\mathbb{K}$-analytic functions $X(x, y)$ and $Y^j(x, y)$, $j = 1, \dots, m$, such that it may be written under the specific form*

$$\begin{cases} 0 = y^j_{xx} + \square^j_{xx} \ + \sum_{l_1=1}^m y^{l_1}_x \cdot \left[2\, \square^j_{xy^{l_1}} - \delta^j_{l_1}\, \square^0_{xx}\right] + \\ \qquad + \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x\, y^{l_2}_x \cdot \left[\square^j_{y^{l_1} y^{l_2}} - \delta^j_{l_1}\, \square^0_{xy^{l_2}} - \delta^j_{l_2}\, \square^0_{xy^{l_1}}\right] + \\ \qquad + y^j_x \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x\, y^{l_2}_x \cdot \left[-\square^0_{y^{l_1} y^{l_2}}\right]. \end{cases} \tag{3.33}$$

The complete proof of this lemma involves only linear algebra considerations, although with rather massive terms. This makes it rather lengthy. Consequently, we postpone it to the final Section 5 below.

3.34. First auxiliary system. Clearly, if we set

$$
\left\{
\begin{aligned}
G^j & := -\square^j_{xx}, \\
H^j_{l_1} & := -2\,\square^j_{xy^{l_1}} + \delta^j_{l_1}\,\square^0_{xx}, \\
L^j_{l_1,l_2} & := -\square^j_{y^{l_1}y^{l_2}} + \delta^j_{l_1}\,\square^0_{xy^{l_2}} + \delta^j_{l_2}\,\square^0_{xy^{l_1}}, \\
M_{l_1,l_2} & := \square^0_{y^{l_1}y^{l_2}},
\end{aligned}
\right.
\tag{3.35}
$$

we immediately see that condition **(i)** of Theorem 1.7 holds true. Moreover, we claim that there are $m+1$ more square functions than functions G^j, $H^j_{l_1}$, $L^j_{l_1,l_2}$ and M_{l_1,l_2}. Indeed, taking account of the symmetries, we denumber:

$$
\left\{
\begin{aligned}
& \#\{\square^j_{xx}\} = m, && \#\{\square^0_{xx}\} = 1, \\
& \#\{\square^j_{xy^{l_1}}\} = m^2, && \#\{\square^0_{xy^{l_1}}\} = m, \\
& \#\{\square^j_{y^{l_1}y^{l_2}}\} = \frac{m^2(m+1)}{2}, && \#\{\square^0_{y^{l_1}y^{l_2}}\} = \frac{m(m+1)}{2},
\end{aligned}
\right.
\tag{3.36}
$$

whereas

$$
\left\{
\begin{aligned}
& \#\{G^j\} = m, && \#\{H^j_{l_1}\} = m^2, \\
& \#\{L^j_{l_1,l_2}\} = \frac{m^2(m+1)}{2}, && \#\{M_{l_1,l_2}\} = \frac{m(m+1)}{2}.
\end{aligned}
\right.
\tag{3.37}
$$

Similarly as in Section 2, for $j, l_1, l_2 = 0, 1, \dots, m$, let us introduce functions $\Pi^j_{l_1,l_2}$ of $(x, y^1, \dots, y^m)$, symmetric with respect to the lower indices, and let us seek necessary and sufficient conditions in order that there exist solutions (X, Y) to the *first auxiliary system* defined precisely by:

$$
\left\{
\begin{aligned}
& \square^0_{xx} = \Pi^0_{0,0}, \quad && \square^0_{xy^{l_1}} = \Pi^0_{0,l_1}, \quad && \square^0_{y^{l_1}y^{l_2}} = \Pi^0_{l_1,l_2}, \\
& \square^j_{xx} = \Pi^j_{0,0}, \quad && \square^j_{xy^{l_1}} = \Pi^j_{0,l_1}, \quad && \square^j_{y^{l_1}y^{l_2}} = \Pi^j_{l_1,l_2}.
\end{aligned}
\right.
\tag{3.38}
$$

3.39. Compatibility conditions for the first auxiliary system. As in Section 2, the compatibility conditions for this system will simply be obtained by computing the cross differentiations. The following statement generalizes Lemma 2.31 and also provides a proof of it, in the case $m = 1$.

Lemma 3.40. *For all $j, l_1, l_2, l_3 = 0, 1, \dots, m$, we have the cross differentiation relations*

$$
\left(\square^j_{y^{l_1}y^{l_2}}\right)_{y^{l_3}} - \left(\square^j_{y^{l_1}y^{l_3}}\right)_{y^{l_2}} = -\sum_{k=0}^{m} \square^k_{y^{l_1}y^{l_2}} \cdot \square^j_{y^{l_3}y^k} + \sum_{k=0}^{m} \square^k_{y^{l_1}y^{l_3}} \cdot \square^j_{y^{l_2}y^k}.
\tag{3.41}
$$

Proof. To begin with, as a preliminary, let us generalize the Plücker identity (2.28). Let $C_1, C_2, \dots, C_m, D, E$ be $(m+2)$ column vectors in $\mathbb{K}^m$ and introduce the following notation for the $m \times (m+2)$ matrix consisting of these vectors:

$$
[C_1|C_2|\cdots|C_m|D|E]\,.
\tag{3.42}
$$

Extracting columns from this matrix, we shall construct $m \times m$ determinants which are modification of the following "fundamental" determinant

$$
\|C_1|\cdots|C_m\| \equiv \left\|C_1|\cdots|^{j_1}C_{j_1}|\cdots|^{j_2}C_{j_2}|\cdots|C_m\right\|.
\tag{3.43}
$$

Here and in the sequel, we use a double vertical line in the beginning and in the end to denote a determinant. Also, we emphasize two distinct columns, the j_1-th and the j_2-th, where $j_2 > j_1$, since we will modify them. For instance in this matrix, let us replace these two columns by the column D and by the column E, which yields the determinant

$$\big\| C_1 | \cdots |^{j_1} D | \cdots |^{j_2} E | \cdots | C_m \big\| . \tag{3.44}$$

In this notation, one should understand that *only* the j_1-th and the j_2-th columns are distinct from the columns of the fundamental $m \times m$ determinant (3.43). With this notation at hand, we can now formulate and prove a preliminary lemma that will be useful later.

Lemma 3.45. *The following quadratic identity between determinants holds true*:

$$\left\{ \begin{aligned} & \big\| C_1 | \cdots |^{j_1} D | \cdots |^{j_2} E | \cdots | C_n \big\| \cdot \big\| C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} C_{j_2} | \cdots | C_n \big\| = \\ & = \big\| C_1 | \cdots |^{j_1} D | \cdots |^{j_2} C_{j_2} | \cdots | C_n \big\| \cdot \big\| C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} E | \cdots | C_n \big\| - \\ & - \big\| C_1 | \cdots |^{j_1} E | \cdots |^{j_2} C^{j_2} | \cdots | C_n \big\| \cdot \big\| C_1 | \cdots |^{j_1} C_{j_1} | \cdots |^{j_2} D | \cdots | C_n \big\| . \end{aligned} \right. \tag{3.46}$$

Proof. After some permutations of columns, this identity amounts to

$$\left\{ \begin{aligned} & \| C_1 | \cdots | C_{m-2} | D | E \| \cdot \| C_1 | \cdots | C_{m-2} | C_{m-1} | C_m \| = \\ & = \| C_1 | \cdots | C_{m-2} | D | C_m \| \cdot \| C_1 | \cdots | C_{m-2} | C_{m-1} | E \| - \\ & - \| C_1 | \cdots | C_{m-2} | E | C_m \| \cdot \| C_1 | \cdots | C_{m-2} | C_{m-1} | D \| . \end{aligned} \right. \tag{3.47}$$

To establish this identity, we introduce some notation. If A and B are vertical vectors in $\mathbb{K}^m$ and if $i_1, i_2 = 1, \ldots, m$ with $i_1 < i_2$, we denote

$$\Delta^2_{i_1, i_2}(A|B) := \left| \begin{array}{cc} A_{i_1} & B_{i_1} \\ A_{i_2} & B_{i_2} \end{array} \right| . \tag{3.48}$$

If $\| A_1 | A_2 | A_3 | \cdots | A_m \|$ is a $m \times m$ determinant, and if $i_1, i_2 = 1, \ldots, m$ with $i_1 < i_2$, we denote by $M^{m-2}_{i_1, i_2}(A_3 | \cdots | A_m)$ the $(m-2) \times (m-2)$ determinant obtained from the matrix $[A_3 | \cdots | A_m]$ by erasing the i_1-th line and the i_2-th line. Without proof, we recall an elementary classical formula

$$\left\{ \begin{aligned} & \| A_1 | A_2 | A_3 | \cdots | A_m \| = \\ & \quad = \sum_{1 \leq i_1 < i_2 \leq m} (-1)^{i_1 + i_2 - 1} \, \Delta^2_{i_1, i_2}(A_1 | A_2) \cdot M^{m-2}_{i_1, i_2}(A_3 | \cdots | A_m), \end{aligned} \right. \tag{3.49}$$

which may be established by developing the determinant $\| A_1 | A_2 | A_3 | \cdots | A_m \|$ with respect to its first column, and then re-developing all the obtained $(m-1) \times (m-1)$ determinants with respect to their first columns. To establish (3.47), we start with an equivalent version of the identity (2.28):

$$\left\{ \begin{aligned} \Delta^2_{i_1, i_2}(D|E) \cdot \Delta^2_{i_3, i_4}(C_1 | C_2) = \Delta^2_{i_1, i_2}(D | C_2) \cdot \Delta^2_{i_3, i_4}(C_1 | E) - & \\ - \Delta^2_{i_1, i_2}(E | C_2) \cdot \Delta^2_{i_3, i_4}(C_1 | D), & \end{aligned} \right. \tag{3.50}$$

where $1 \leq i_1 < i_2 \leq m$ and $1 \leq i_3 < i_4 \leq m$. Multiplying by $(-1)^{i_1 + i_2 + i_3 + i_4 - 2}$, multiplying by $M^{m-2}_{i_1, i_2}(C_3 | \cdots | C_m)$, and multiplying by $M^{m-2}_{i_3, i_4}(C_3 | \cdots | C_m)$ applying the double summation $\sum_{1\leq i_1<i_2\leq m}\sum_{1\leq i_3<i_4\leq m}$, we get

(3.51)

$$\begin{cases}\sum_{1\leq i_1<i_2\leq m}\sum_{1\leq i_3<i_4\leq m}(-1)^{i_1+i_2+i_3+i_4-2}\,\Delta^2_{i_1,i_2}(D|E)\cdot\Delta^2_{i_3,i_4}(C_1|C_2)\cdot \\ \qquad\cdot M^{m-2}_{i_1,i_2}(C_3|\cdots|C_m)\cdot M^{m-2}_{i_3,i_4}(C_3|\cdots|C_m)= \\ =\sum_{1\leq i_1<i_2\leq m}\sum_{1\leq i_3<i_4\leq m}(-1)^{i_1+i_2-1}\,(-1)^{i_3+i_4-1}\,\big[\Delta^2_{i_1,i_2}(D|C_2)\cdot\Delta^2_{i_3,i_4}(C_1|E)- \\ \qquad-\Delta^2_{i_1,i_2}(E|C_2)\cdot\Delta^2_{i_3,i_4}(C_1|D)\big]\cdot M^{m-2}_{i_1,i_2}(C_3|\cdots|C_m)\cdot M^{m-2}_{i_3,i_4}(C_3|\cdots|C_m).\end{cases}$$

Thanks to the relation (3.49), this last identity coincides exactly with the desired identity (3.47). The proof is complete. □

We can now establish Lemma 3.40. As a preliminary observation, by the Leibniz rule for the differentiation of a determinant, we must differentiate every column:

$$\begin{cases}\left[\Delta\left(y^{l_1}y^{k_1}|\cdots|y^{l_m}y^{k_m}\right)\right]_{y^j}=\Delta\left(y^jy^{l_1}y^{k_1}|\cdots|y^{l_m}y^{k_m}\right)+\cdots+ \\ \qquad\qquad+\Delta\left(y^{l_1}y^{k_1}|\cdots|y^jy^{l_m}y^{k_m}\right).\end{cases}\tag{3.52}$$

Using also the rule for the differentiation of a quotient, we may endeavour to compute the cross differentiations $\left(\square^j_{y^{l_1}y^{l_2}}\right)_{y^{l_3}}-\left(\square^j_{y^{l_1}y^{l_3}}\right)_{y^{l_2}}$ of the left hand side of (3.41). This will generalize (2.25). Sometimes in the computation, we shall abbreviate the Jacobian determinant $\Delta(y^0|\cdots|y^m)$ using the shorter notation Δ; as before, a product between two elements of $\mathbb{K}$ will often be denoted by the sign "$\cdot$", for clarity. Here is the computation:

(3.53)

$$\begin{cases}\left(\square^j_{y^{l_1}y^{l_2}}\right)_{y^{l_3}}-\left(\square^j_{y^{l_1}y^{l_3}}\right)_{y^{l_2}}= \\ =\frac{\partial}{\partial y^{l_3}}\left(\frac{\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_2}|\cdots|y^m\right)}{\Delta\left(y^0|\cdots|y^m\right)}\right)-\frac{\partial}{\partial y^{l_2}}\left(\frac{\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_3}|\cdots|y^m\right)}{\Delta\left(y^0|\cdots|y^m\right)}\right) \\ =\frac{1}{[\Delta]^2}\left[\begin{array}{l}\Delta\left(y^0y^{l_3}|\cdots|^jy^{l_1}y^{l_2}|\cdots|y^m\right)\cdot\Delta+\cdots+ \\ +\underline{\Delta\left(y^0|\cdots|^jy^{l_3}y^{l_1}y^{l_2}|\cdots|y^m\right)\cdot\Delta}_{\text{ⓐ}}+\cdots+ \\ +\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_2}|\cdots|y^{l_3}y^m\right)\cdot\Delta- \\ -\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_2}|\cdots|y^m\right)\cdot\big[\Delta\left(y^0y^{l_3}|\cdots|y^m\right)+\cdots+ \\ \qquad\qquad+\Delta\left(y^0|\cdots|y^{l_3}y^m\right)\big]\end{array}\right]- \\ \quad-\frac{1}{[\Delta]^2}\left[\begin{array}{l}\Delta\left(y^0y^{l_2}|\cdots|^jy^{l_1}y^{l_3}|\cdots|y^m\right)\cdot\Delta+\cdots+ \\ +\underline{\Delta\left(y^0|\cdots|^jy^{l_2}y^{l_1}y^{l_3}|\cdots|y^m\right)\cdot\Delta}_{\text{ⓐ}}+\cdots+ \\ +\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_3}|\cdots|y^{l_2}y^m\right)\cdot\Delta- \\ -\Delta\left(y^0|\cdots|^jy^{l_1}y^{l_3}|\cdots|y^m\right)\cdot\big[\Delta\left(y^0y^{l_2}|\cdots|y^m\right)+\cdots+ \\ \qquad\qquad+\Delta\left(y^0|\cdots|y^{l_2}y^m\right)\big]\end{array}\right].\end{cases}$$

Crucially, we observe that all the determinants involving a third order derivative upon one of their columns kill each other and disappear: we have underlined them with ⓐappended. However, it still remains plenty of determinants involving a second order derivative upon two different columns. We must transform all of them and express them in terms of determinants involving a second order derivative upon only one column. To this aim, as an application of our preliminary Lemma 3.45, we have the following relations, valid for $j_1, j_2, l_1, l_2, l_3, l_4 = 0, \ldots, m$ and $j_1 < j_2$:

(3.54)

$$\left\{\begin{aligned} & \Delta\left(y^0|\cdots|^{j_1} y^{l_1} y^{l_2}|\cdots|^{j_2} y^{l_3} y^{l_4}|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^{j_1} y^{j_1}|\cdots|^{j_2} y^{j_2}|\cdots|y^m\right) = \\ & = \Delta\left(y^0|\cdots|^{j_1} y^{l_1} y^{l_2}|\cdots|^{j_2} y^{j_2}|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^{j_1} y^{j_1}|\cdots|^{j_2} y^{l_3} y^{l_4}|\cdots|y^m\right) \\ & - \Delta\left(y^0|\cdots|^{j_1} y^{l_3} y^{l_4}|\cdots|^{j_2} y^{j_2}|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^{j_1} y^{j_1}|\cdots|^{j_2} y^{l_1} y^{l_2}|\cdots|y^m\right). \end{aligned}\right.$$

With these formulas, we may transform the lines number 3, 4, 5 and 8, 9, 10 of (3.53). Also, we observe that the lines 6, 7 and 11, 12 of (3.53) involve determinants having a single second order derivative. Taking account of the $\frac{1}{[\Delta]^2}$ factor, we deduce that the lines 6, 7 and 11, 12 of (3.53) may already be expressed as sums of square functions. Achieving all these transformations, we may rewrite (3.53) as follows

(3.55)

$$\left\{\begin{aligned} & \left(\square^j_{y^{l_1} y^{l_2}}\right)_{y^{l_3}} - \left(\square^j_{y^{l_1} y^{l_3}}\right)_{y^{l_2}} = \\ & = \frac{1}{[\Delta]^2}\left[\begin{aligned} & \Delta\left(y^0 y^{l_3}|\cdots|^j y^j|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^{l_1} y^{l_2}|\cdots|y^m\right) - \\ & - \Delta\left(y^{l_1} y^{l_2}|\cdots|^j y^j|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^0 y^{l_3}|\cdots|y^m\right) + \\ & + \cdots + \\ & + \Delta\left(y^0|\cdots|^j y^{l_1} y^{l_2}|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^j|\cdots|y^{l_3} y^m\right) - \\ & - \Delta\left(y^0|\cdots|^j y^{l_3} y^m|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^j|\cdots|y^{l_1} y^{l_2}\right) \end{aligned}\right] - \\ & - \sum_{k=0}^m \square^j_{y^{l_1} y^{l_2}} \square^k_{y^{l_3} y^k} - \\ & - \frac{1}{[\Delta]^2}\left[\begin{aligned} & \Delta\left(y^0 y^{l_2}|\cdots|^j y^j|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^{l_1} y^{l_3}|\cdots|y^m\right) - \\ & - \Delta\left(y^{l_1} y^{l_3}|\cdots|^j y^j|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^0 y^{l_2}|\cdots|y^m\right) + \\ & + \cdots + \\ & + \Delta\left(y^0|\cdots|^j y^{l_1} y^{l_3}|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^j|\cdots|y^{l_2} y^m\right) - \\ & - \Delta\left(y^0|\cdots|^j y^{l_2} y^m|\cdots|y^m\right) \cdot \Delta\left(y^0|\cdots|^j y^j|\cdots|y^{l_1} y^{l_3}\right) \end{aligned}\right] + \\ & + \sum_{k=0}^m \square^j_{y^{l_1} y^{l_3}} \square^k_{y^{l_2} y^k}. \end{aligned}\right.$$

Notice that two pairs of "cdots" terms $+\cdots+$ appearing in the lines 3, 4 and 8, 9 of (3.53) are replaced by a single "cdots" term $+\cdots+$ in the lines 4 and 9 of (3.55). Importantly, we point that in the "middle" of the two "cdots" terms $+\cdots+$ appearing in the lines 4 and 10 of (3.55) just above, there are two terms which do not occur: they simply correspond to the two underlined terms having ⓐ appended appearing in the lines 4 and 9 (3.53).

Now, taking account of the factor $\frac{1}{[\Delta]^2}$, we can re-express all the terms of (3.55) as sums of square functions:

$$
\left\{
\begin{aligned}
&\left(\square^j_{y^{l_1}y^{l_2}}\right)_{y^{l_3}} - \left(\square^j_{y^{l_1}y^{l_3}}\right)_{y^{l_2}} = \\
&\quad = \sum_{k=0;\, k\neq j}^m \square^k_{y^{l_3}y^k}\,\square^j_{y^{l_1}y^{l_2}} - \sum_{k=0;\, k\neq j}^m \square^k_{y^{l_1}y^{l_2}}\,\square^j_{y^{l_3}y^k} - \\
&\quad - \sum_{k=0}^m \square^j_{y^{l_1}y^{l_2}}\,\square^k_{y^{l_3}y^k} - \\
&\quad - \sum_{k=0;\, k\neq j}^m \square^k_{y^{l_2}y^k}\,\square^j_{y^{l_1}y^{l_3}} + \sum_{k=0;\, k\neq j}^m \square^k_{y^{l_1}y^{l_3}}\,\square^j_{y^{l_2}y^k} + \\
&\quad + \sum_{k=0}^m \square^j_{y^{l_1}y^{l_3}}\,\square^k_{y^{l_2}y^k}.
\end{aligned}
\right.
\tag{3.56}
$$

Finally, we observe that in the two pairs of sums having $k \neq j$ appearing in the lines 2 and 4 just above, we can include the term $k = j$ in each pair, because these two terms are immediately killed inside the corresponding pair. In conclusion, after a final obvious killing of four (among six) complete sums in this modification of (3.56), we obtain the desired formula (3.41), with two sums. This completes the proof of Lemma 3.40 and also at the same occasion, the proof of Lemma 2.31. □

3.57. Compatibility conditions for the first auxiliary system. According to the (approximate) identities (3.3), taking account of the explicit definitions (3.30) of the square functions, we have

$$
\left\{
\begin{aligned}
&\square^0_{xx} \cong X_{xx}, \quad \square^0_{xy^{l_1}} \cong X_{xy^{l_1}}, \quad \square^0_{y^{l_1}y^{l_2}} \cong X_{y^{l_1}y^{l_2}}, \\
&\square^j_{xx} \cong Y^j_{xx}, \quad \square^j_{xy^{l_1}} \cong Y^j_{xy^{l_1}}, \quad \square^j_{y^{l_1}y^{l_2}} \cong Y^j_{y^{l_1}y^{l_2}}.
\end{aligned}
\right.
\tag{3.58}
$$

Consequently, the first auxiliary system (3.38) looks approximatively like a complete second order system of partial differential equations in the $(m+1)$ independent variables (x,y) and in the $(m+1)$ dependent variables (X,Y). By means of elementary algebraic operations, one may transform this system in a true second order *complete* system, solved with respect to the top order derivatives, namely of the form

$$
\left\{
\begin{aligned}
&X_{xx} = \Lambda^0_{0,0}, \quad X_{xy^{l_1}} = \Lambda^0_{0,l_1}, \quad X_{y^{l_1}y^{l_2}} = \Lambda^0_{l_1,l_2}, \\
&Y^j_{xx} = \Lambda^j_{0,0}, \quad Y^j_{xy^{l_1}} = \Lambda^j_{0,l_1}, \quad Y_{y^{l_1}y^{l_2}} = \Lambda^j_{l_1,l_2},
\end{aligned}
\right.
\tag{3.59}
$$

where the $\Lambda^{k_1}_{j_1,j_2}$ are local $\mathbb{K}$-analytic functions of $(x, y^{l_1}, X, Y^j, X_x, X_{y^{l_1}}, Y^j_x, Y^j_{y^{l_1}})$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution (X,Y)] follow by obvious cross differentiation. Coming back to the system 3.38, these compatibility conditions amount to the quadratic-like compatibility conditions expressed in Lemma 3.40. In conclusion, we have proved the following intermediate statement.

Proposition 3.60. *There exist functions X, Y^j solving the first auxiliary system* (3.38) *of nonlinear second order partial differential equations* if and only if *the right hand side* *functions* $\Pi_{l_1,\,l_2}^j(x,\,y)$ *satisfy the quadratic compatibility conditions*

$$\frac{\partial \Pi_{l_1,l_2}^j}{\partial y^{l_3}} - \frac{\partial \Pi_{l_1,l_3}^j}{\partial y^{l_2}} = -\sum_{k=0}^m \Pi_{l_1,l_2}^k \cdot \Pi_{l_3,k}^j + \sum_{k=0}^m \Pi_{l_1,l_3}^k \cdot \Pi_{l_2,k}^j, \tag{3.61}$$

for $j, l_1, l_2, l_3 = 0, 1, \dots, m$.

3.62. Principal unknowns. As there are $(m+1)$ more square (or Pi) functions than the functions G^j, $H_{l_1}^j$, L_{l_1,l_2}^j and M_{l_1,l_2} defined by (3.35), we cannot invert directly the linear system (3.35) (which is of maximal rank). Hence we must choose $(m+1)$ specific square functions, calling them *principal unknowns*, and similarly as in §2.34, the best choice is to choose $\square_{xx}^0$ and $\square_{xx}^j$, for $j = 1, \dots, m$. For clarity, it will be useful to adopt the notational equivalences

$$\Theta^0 \equiv \Pi_{0,0}^0 \qquad \text{and} \qquad \Theta^j \equiv \Pi_{j,j}^j. \tag{3.63}$$

Then we may quasi-inverse the system (3.35), which yields :

(3.64)

$$\begin{cases} \Pi_{0,0}^j = \square_{xx}^j = -G^j, \\ \Pi_{0,l_1}^j = \square_{xy^{l_1}}^j = -\frac{1}{2}\, H_{l_1}^j + \frac{1}{2}\, \delta_{l_1}^j\, \Theta^0, \\ \Pi_{l_1,l_2}^j = \square_{y^{l_1}y^{l_2}}^j = -L_{l_1,l_2}^j + \frac{1}{2}\, \delta_{l_1}^j\, L_{l_2,l_2}^{l_2} + \frac{1}{2}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1} + \frac{1}{2}\, \delta_{l_1}^j\, \Theta^{l_2} + \frac{1}{2}\, \delta_{l_2}^j\, \Theta^{l_1}, \\ \Pi_{0,l_1}^0 = \square_{xy^{l_1}}^0 = \frac{1}{2}\, L_{l_1,l_1}^{l_1} + \frac{1}{2}\, \Theta^{l_1}, \\ \Pi_{l_1,l_2}^0 = \square_{y^{l_1}y^{l_2}}^0 = M_{l_1,l_2}. \end{cases}$$

Before replacing these new expressions of the functions $\Pi_{0,0}^j$, Π_{0,l_1}^j, Π_{l_1,l_2}^j, Π_{l_1,l_2}^0 and $\Pi_{0,l_1}^{l_1}$ into the compatibility conditions (3.61), it is necessary to expound first (3.61), taking account of the original splitting of the indices in the two sets $\{0\}$ and $\{1, 2, \dots, m\}$. This yields six families of compatibility conditions, totally equivalent to the compact identities (3.61):

(3.65)

$$\begin{cases} \left(\Pi_{0,0}^j\right)_{y^{l_1}} - \left(\Pi_{0,l_1}^j\right)_x = -\Pi_{0,0}^0\, \Pi_{l_1,0}^j - \sum_{k=1}^m \Pi_{0,0}^k\, \Pi_{l_1,k}^j + \Pi_{0,l_1}^0\, \Pi_{0,0}^j + \sum_{k=1}^m \Pi_{0,l_1}^k\, \Pi_{0,k}^j, \\ \left(\Pi_{l_1,l_2}^j\right)_x - \left(\Pi_{l_1,0}^j\right)_{y^{l_2}} = -\Pi_{l_1,l_2}^0\, \Pi_{0,0}^j - \sum_{k=1}^m \Pi_{l_1,l_2}^k\, \Pi_{0,k}^j + \Pi_{l_1,0}^0\, \Pi_{l_2,0}^j + \sum_{k=1}^m \Pi_{l_1,0}^k\, \Pi_{l_2,k}^j, \\ \left(\Pi_{l_1,l_2}^j\right)_{y^{l_3}} - \left(\Pi_{l_1,l_3}^j\right)_{y^{l_2}} = -\Pi_{l_1,l_2}^0\, \Pi_{l_3,0}^j - \sum_{k=1}^m \Pi_{l_1,l_2}^k\, \Pi_{l_3,k}^j + \Pi_{l_1,l_3}^0\, \Pi_{l_2,0}^j + \sum_{k=1}^m \Pi_{l_1,l_3}^k\, \Pi_{l_2,k}^j, \\ \left(\Pi_{0,0}^0\right)_{y^{l_1}} - \left(\Pi_{0,l_1}^0\right)_x = -\underline{\Pi_{0,0}^0\, \Pi_{l_1,0}^0}_{\textcircled{a}} - \sum_{k=1}^m \Pi_{0,0}^k\, \Pi_{l_1,k}^0 + \underline{\Pi_{0,l_1}^0\, \Pi_{0,0}^0}_{\textcircled{a}} + \sum_{k=1}^m \Pi_{0,l_1}^k\, \Pi_{0,k}^0, \\ \left(\Pi_{l_1,l_2}^0\right)_x - \left(\Pi_{l_1,0}^0\right)_{y^{l_2}} = -\Pi_{l_1,l_2}^0\, \Pi_{0,0}^0 - \sum_{k=1}^m \Pi_{l_1,l_2}^k\, \Pi_{0,k}^0 + \Pi_{l_1,0}^0\, \Pi_{l_2,0}^0 + \sum_{k=1}^m \Pi_{l_1,0}^k\, \Pi_{l_2,k}^0, \\ \left(\Pi_{l_1,l_2}^0\right)_{y^{l_3}} - \left(\Pi_{l_1,l_3}^0\right)_{y^{l_2}} = -\Pi_{l_1,l_2}^0\, \Pi_{l_3,0}^0 - \sum_{k=1}^m \Pi_{l_1,l_2}^k\, \Pi_{l_3,k}^0 + \Pi_{l_1,l_3}^0\, \Pi_{l_2,0}^0 + \sum_{k=1}^m \Pi_{l_1,l_3}^k\, \Pi_{l_2,k}^0. \end{cases}$$

3.66. Convention about sums. Up to the end of Section 4, we shall abbreviate any sum $\sum_{k=1}^{m}$ or $\sum_{p=1}^{m}$ as $\sum_k$ or $\sum_p$. Such sums will appear very frequently. For all other sums, we shall precisely write down the domain of variation of the summation index.

3.67. Continuation. Thus, we have to replace (3.64) in the six identities (3.65). Firstly, let us expose all the intermediate steps in dealing with the first identity $(3.65)_1$. Replacing plainly (3.64) in $(3.65)_1$, we get:

(3.68)

$$\left\{\begin{aligned}
&\left(\Pi_{0,0}^j\right)_{y^{l_1}}-\left(\Pi_{0,l_1}^j\right)_x=G_{y^{l_1}}^j+\frac{1}{2}\,H_{l_1,x}^j-\frac{1}{2}\,\delta_{l_1}^j\,\Theta_x^0=\\
&=\frac{1}{2}\,\Theta^0\,H_{l_1}^j-\frac{1}{2}\,\delta_{l_1}^j\,\Theta^0\,\Theta^0-\\
&\quad-\sum_k\left(-G^k\right)\left(-L_{l_1,k}^j+\frac{1}{2}\,\delta_{l_1}^j\,L_{k,k}^k+\frac{1}{2}\,\delta_k^j\,L_{l_1,l_1}^{l_1}+\frac{1}{2}\,\delta_{l_1}^j\,\Theta^k+\frac{1}{2}\,\delta_k^j\,\Theta^k\right)+\\
&\quad+\left(\frac{1}{2}\,L_{l_1,l_1}^{l_1}+\frac{1}{2}\,\Theta^{l_1}\right)\left(-G^j\right)+\\
&\quad+\sum_k\left(-\frac{1}{2}\,H_{l_1}^k+\frac{1}{2}\,\delta_{l_1}^k\,\Theta^0\right)\left(-\frac{1}{2}\,H_k^j+\frac{1}{2}\,\delta_k^j\,\Theta^0\right)=\\
&=\underline{\frac{1}{2}\,H_{l_1}^j\,\Theta^0}_{\textcircled{a}}-\frac{1}{2}\,\delta_{l_1}^j\,\Theta^0\,\Theta^0-\sum_k G^k\,L_{l_1,k}^j+\frac{1}{2}\,\delta_{l_1}^j\,\sum_k G^k\,L_{k,k}^k+\underline{\frac{1}{2}\,G^j\,L_{l_1,l_1}^{l_1}}_{\textcircled{b}}+\\
&\quad+\frac{1}{2}\,\delta_{l_1}^j\,\sum_k G^k\,\Theta^k+\underline{\frac{1}{2}\,G^j\,\Theta^{l_1}}_{\textcircled{c}}-\underline{\frac{1}{2}\,G^j\,L_{l_1,l_1}^{l_1}}_{\textcircled{b}}-\underline{\frac{1}{2}\,G^j\,\Theta^{l_1}}_{\textcircled{c}}+\frac{1}{4}\,\sum_k H_{l_1}^k\,H_k^j-\\
&\quad-\underline{\frac{1}{4}\,H_{l_1}^j\,\Theta^0}_{\textcircled{a}}-\underline{\frac{1}{4}\,H_{l_1}^j\,\Theta^0}_{\textcircled{a}}+\frac{1}{4}\,\delta_{l_1}^j\,\Theta^0\,\Theta^0.
\end{aligned}\right.$$

Eliminating the underlined vanishing terms with the letters a, b, c and d appended, multiplying by -2 and reorganizing the identity so as to put the term $\delta_{l_1}^j\,\Theta_x^0$ solely in the left hand side, we obtain the relation

$$\text{(3.69)}\qquad\left\{\begin{aligned}
\delta_{l_1}^j\,\Theta_x^0&=-2\,G_{y^{l_1}}^j+H_{l_1,x}^j+2\,\sum_k G^k\,L_{l_1,k}^j-\delta_{l_1}^j\,\sum_k G^k\,L_{k,k}^k-\\
&\qquad-\frac{1}{2}\,\sum_k H_{l_1}^k\,H_k^j-\delta_{l_1}^j\,\sum_k G^k\,\Theta^k+\frac{1}{2}\,\delta_{l_1}^j\,\Theta^0\,\Theta^0.
\end{aligned}\right.$$

3.70. Conventions for simplifications of formal expressions. Before proceeding further, let us explain how we will organize the computations with the formal expressions we shall encounter until the end of Section 4. *Our main goal is to devise a methodology of writing formal computations which enables to check every computation visually*, without being forced to rebuild any intermediate step. In fact, it would be unsatisfactory to just claim that Theorem 1.7 follows by hidden massive formal computations, so that we have to guide the rigorous and demanding reader until the very extremal branches of our coral tree of formal computations.

As an example, suppose that we have to simplify the equation $0=A_x+B_y+A-B+2\,C-\frac{1}{3}\,D-\frac{2}{3}\,A+\frac{1}{6}\,D+E+B-2\,C$. In the beginning, the terms A_x and B_y are differentiated once and they do not simplify with other terms. To distinguish them, we underline them plainly and we copy the nine remaining terms afterwards:

$$\text{(3.71)}\qquad\left\{\begin{aligned}
&0=\underline{A_x+B_y}+\\
&\qquad+\underline{A}_{\boxed{1}}-\underline{B}_{\textcircled{a}}+\underline{2\,C}_{\textcircled{b}}-\underline{\frac{1}{3}\,D}_{\boxed{2}}-\underline{\frac{2}{3}\,A}_{\boxed{1}}+\underline{\frac{1}{6}\,D}_{\boxed{2}}+\underline{E}_{\boxed{3}}+\underline{B}_{\textcircled{a}}-\underline{2\,C}_{\textcircled{b}}.
\end{aligned}\right.$$

Here, each remaining term is also underlined, with a number or with a letter appended. For reasons of typographical readability, we never underline the sign, $+$ or $-$ of each term; however, it should be understood that *every term always includes its* (not underlined) *sign*. Until the end of Section 4, we shall use the roman alphabetic letters a, b, c, *etc.* inside an octagon ◯ to exhibit the vanishing terms. As readily checked by the eyes, we indeed have $-\underline{B}_{ⓐ} + \underline{B}_{ⓐ} = 0$ and $\underline{2\,C}_{ⓑ} - \underline{2\,C}_{ⓑ} = 0$. Also, until the end of Section 4, we shall use the numbers 1, 2, 3, *etc.* inside a square ☐ to exhibit the remaining terms, collected in a certain order. The numbers have the following signification: after the simplifications, the equation (3.71) may be written

$$\left\{ \begin{aligned} 0 &= \underline{A_x + B_y} + \\ &+ \frac{1}{3}\,A - \frac{1}{6}\,D + E. \end{aligned} \right. \tag{3.72}$$

Here, the plainly underlined terms $\underline{A_x + B_y}$ do not count in the numbering (their number is zero, for instance) and *the first term of the second line* $\frac{1}{3}\,A$ *correspond to the addition of all terms* $\boxed{1}$ *in* (3.69). Analogously, the second term $-\frac{1}{6}\,D$ correspond to the addition of all terms $\boxed{2}$ in (3.69). Again, this guiding facilitates the checking of the correctness of the computation, using simply the eyes. No hidden delicate computational step is "left to the reader" for the convenience of the writer.

This principle will be constantly used until the end of Section 4; it has been systematically used in [M2003], [M2004b] and it could be applied in various other contexts. Again, the advantage is that it enables to check the correctness of all the formal computations just by reading, without having to write anything more. This is also useful for the author.

3.73. Choice of an ordering. Until the end of Section 4, we shall have to deal with terms G, H, L, M, Θ together with indices and partial derivatives up to order two. In order to organize the formal expressions in a way which provides an easier deciphering, it is convenient to introduce an order between these differential monomials. In a symbolic index-free notation, we choose:

$$G < H < L < M < \Theta. \tag{3.74}$$

It follows for instance that $G < GH < GL < HHL < HLM\Theta$. Also, if a sum appears, we choose: $GM < \sum GM$.

Here, we have only considered terms of order zero, without partial differentiation. The first order partial differentiations are $(\cdot)_x$ and $(\cdot)_y$, again in symbolic notation, dropping the indices. We choose:

$$G_x < G_y < H_x < H_y < L_x < L_y < M_x < M_y < \Theta_x < \Theta_y < G < \cdots . \tag{3.75}$$

For second order derivatives, we choose:

$$G_{xx} < G_{xy} < G_{yy} < H_{xx} < \cdots < \Theta_{yy} < G_x < \cdots . \tag{3.76}$$

As a final general example including indices we have the inequalities

$$H_{l_1,y^{l_2}}^j < L_{l_1,l_2,x}^j < G^j\,M_{l_1,l_2} < \sum_{k=1}^m G^k\,M_{l_2,k} < \sum_{k=1}^m H_k^{l_2}\,H_{l_2,l_2}^k, \tag{3.77}$$

extracted from (II) of Theorem 1.7.

In the sequel, we shall call

- terms *of order* 0 monomials like G, H, $L\,M$, $G\,H\,M$;
- terms *of order* 1 monomials like G_x, $G_x\,M$, $L_x\,\Theta$;

- terms *of order* 2 monomials like G_{xy}, L_{yy}, M_{xx} (our terms of order two will always be linear),

according to the top order partial derivatives.

3.78. A mean of checking intuitively the validity of partial differential relations. Before replacing (3.64) in the five remaining identities $(3.65)_2$, $(3.65)_3$, $(3.65)_4$, $(3.65)_5$ and $(3.65)_6$, let us observe that if we assume that $j \neq l_1$ in (3.69), then all the terms involving Θ vanish, so that we obtain the *nontrivial partial differential equations*:

$$0 = \underline{-2\, G^j_{y^{l_1}} + H^j_{l_1,x}} + 2 \sum_k G^k\, L^j_{l_1,k} - \frac{1}{2} \sum_k H^k_{l_1}\, H^j_k, \tag{3.79}$$

for $j \neq l_1$. Here, we have underlined the first order terms plainly, in order to distinguish them from the terms of order zero. These equations coincides with (I) of Theorem 1.7, again specialized with $j \neq l_1$. Importantly, we notice that the choice of indices $j \neq l_1$ is possible only if $m \geq 2$. Thus, we have derived *a subpart of (I) as a necessary condition for the point equivalence to* $Y^j_{XX} = 0$, $j = 1, \dots, m \geq 2$. These first order equations show at once that there is a strong difference with the case $m = 1$.

How can we confirm (at least informally) that the functions G^j, $H^j_{l_1}$ and $L^j_{l_1,l_2}$ given by (3.35) in terms of X and Y^j do indeed satisfy these equations for $j \neq l_1$? Dropping the zero order terms in (3.79) above, we obtain an approximated equation

$$0 \equiv -2\, G^j_{y^{l_1}} + H^j_{l_1,x}. \tag{3.80}$$

Here, the sign $\equiv$ precisely means: "*modulo zero order terms*". We claim that this approximated equation is a consequence of the existence of X, Y^j.

Indeed, according to the approximation (3.58), together with the definition (3.35) of the functions G^j and $H^j_{l_1}$, we have

$$\begin{cases} G^j & = -\square^j_{xx} \cong -Y^j_{xx}, \\ H^j_{l_1} & = -2\, \square^j_{xy^{l_1}} \cong -2\, Y^j_{xy^{l_1}}. \end{cases} \tag{3.81}$$

Differentiation of the first line with respect to y^{l_1} and of the second line with respect to x yields:

$$G^j_{y^{l_1}} \cong -Y^j_{xxy^{l_1}} \qquad \text{and} \qquad H^j_{l_1,x} \cong -2\, Y^j_{xy^{l_1}x}, \tag{3.82}$$

so that we indeed have $0 \equiv -2\, G^j_{y^{l_1}} + H^j_{l_1,x}$, approximatively and modulo the derivatives of order 0, 1 and 2 of the functions X, Y^j.

Similar verifications have been effected constantly in the manuscript [M2003] in order to control the truth of the formal computations that we shall expose until the end of Section 4.

3.83. Continuation. From now on and up to the end of Section 4, the hardest computational core of the proof may — at last — be developed. Further amazing computational obstacles will be encountered.

Replacing plainly (3.64) in $(3.65)_2$, we get:

(3.84)
$$
\begin{aligned}
\Big(\Pi^j_{l_1,l_2}\Big)_x - \Big(\Pi^j_{l_1,0}\Big)_{y^{l_2}} &= -L^j_{l_1,l_2,x} + \frac{1}{2}\,\delta^j_{l_1}\,L^{l_2}_{l_2,l_2,x} + \frac{1}{2}\,\delta^j_{l_2}\,L^{l_1}_{l_1,l_1,x} + \frac{1}{2}\,\delta^j_{l_1}\,\Theta^{l_2}_x + \\
&\quad + \frac{1}{2}\,\delta^j_{l_2}\,\Theta^{l_1}_x + \frac{1}{2}\,H^j_{l_1,y^{l_2}} - \frac{1}{2}\,\delta^j_{l_1}\,\Theta^0_{y^{l_2}} = \\
&= -\Pi^0_{l_1,l_2}\cdot\Pi^j_{0,0} - \sum_k \Pi^k_{l_1,l_2}\cdot\Pi^j_{0,k} + \Pi^0_{l_1,0}\cdot\Pi^j_{l_2,0} + \sum_k \Pi^k_{l_1,0}\cdot\Pi^j_{l_2,k} = \\
&= M_{l_1,l_2}\,G^j - \sum_k \left(-L^k_{l_1,l_2} + \frac{1}{2}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2} + \frac{1}{2}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1} + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^{l_2} + \frac{1}{2}\,\delta^k_{l_2}\,\Theta^{l_1}\right)\cdot \\
&\qquad \cdot\left(-\frac{1}{2}\,H^j_k + \frac{1}{2}\,\delta^j_k\,\Theta^0\right) + \\
&\quad + \left(\frac{1}{2}\,L^{l_1}_{l_1,l_1} + \frac{1}{2}\,\Theta^{l_1}\right)\cdot\left(-\frac{1}{2}\,H^j_{l_2} + \frac{1}{2}\,\delta^j_{l_2}\,\Theta^0\right) + \\
&\quad + \sum_k \left(-\frac{1}{2}\,H^k_{l_1} + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^0\right)\cdot \\
&\qquad \cdot\left(-L^j_{l_2,k} + \frac{1}{2}\,\delta^j_{l_2}\,L^k_{k,k} + \frac{1}{2}\,\delta^j_k\,L^{l_2}_{l_2,l_2} + \frac{1}{2}\,\delta^j_{l_2}\,\Theta^k + \frac{1}{2}\,\delta^j_k\,\Theta^{l_2}\right).
\end{aligned}
$$

Developing the products and ordering each monomial, we get:

(3.85)
$$
\begin{aligned}
&= \underline{G^j\,M_{l_1,l_2}}_{\boxed{1}} - \underline{\frac{1}{2}\sum_k H^j_k\,L^k_{l_1,l_2}}_{\boxed{2}} + \underline{\frac{1}{4}\,H^j_{l_1}\,L^{l_2}_{l_2,l_2}}_{\textcircled{a}} + \underline{\frac{1}{4}\,H^j_{l_2}\,L^{l_1}_{l_1,l_1}}_{\textcircled{b}} + \\
&\quad + \underline{\frac{1}{4}\,H^j_{l_1}\,\Theta^{l_2}}_{\textcircled{c}} + \underline{\frac{1}{4}\,H^j_{l_2}\,\Theta^{l_1}}_{\textcircled{d}} + \underline{\frac{1}{2}\,L^j_{l_1,l_2}\,\Theta^0}_{\textcircled{e}} - \underline{\frac{1}{4}\,\delta^j_{l_1}\,L^{l_2}_{l_2,l_2}\,\Theta^0}_{\textcircled{f}} - \\
&\quad - \underline{\frac{1}{4}\,\delta^j_{l_2}\,L^{l_1}_{l_1,l_1}\,\Theta^0}_{\textcircled{g}} - \underline{\frac{1}{4}\,\delta^j_{l_1}\,\Theta^0\,\Theta^{l_2}}_{\textcircled{h}} - \underline{\frac{1}{4}\,\delta^j_{l_2}\,\Theta^0\,\Theta^{l_1}}_{\textcircled{i}} - \underline{\frac{1}{4}\,H^j_{l_2}\,L^{l_1}_{l_1,l_1}}_{\textcircled{b}} + \\
&\quad + \underline{\frac{1}{4}\,\delta^j_{l_2}\,L^{l_1}_{l_1,l_1}\,\Theta^0}_{\boxed{6}} - \underline{\frac{1}{4}\,H^j_{l_2}\,\Theta^{l_1}}_{\textcircled{d}} + \underline{\frac{1}{4}\,\delta^j_{l_2}\,\Theta^0\,\Theta^{l_1}}_{\textcircled{i}} + \underline{\frac{1}{2}\sum_k H^k_{l_1}\,L^j_{l_2,k}}_{\boxed{3}} - \\
&\quad - \underline{\frac{1}{4}\sum_k H^k_{l_1}\,L^k_{k,k}}_{\boxed{4}} - \underline{\frac{1}{4}\,H^j_{l_1}\,L^{l_2}_{l_2,l_2}}_{\textcircled{a}} - \underline{\frac{1}{4}\,\delta^j_{l_2}\sum_k H^k_{l_1}\,\Theta^k}_{\boxed{5}} - \underline{\frac{1}{4}\,H^j_{l_1}\,\Theta^{l_2}}_{\textcircled{c}} - \\
&\quad - \underline{\frac{1}{2}\,L^j_{l_2,l_1}\,\Theta^0}_{\textcircled{e}} + \underline{\frac{1}{4}\,\delta^j_{l_2}\,L^{l_1}_{l_1,l_1}\,\Theta^0}_{\textcircled{g}} + \underline{\frac{1}{4}\,\delta^j_{l_1}\,L^{l_2}_{l_2,l_2}\,\Theta^0}_{\textcircled{f}} + \underline{\frac{1}{4}\,\delta^j_{l_2}\,\Theta^0\,\Theta^{l_1}}_{\boxed{7}} + \\
&\quad + \underline{\frac{1}{4}\,\delta^j_{l_1}\,\Theta^0\,\Theta^{l_2}}_{\textcircled{h}}.
\end{aligned}
$$

We simplify according to our general principles and we reorganize the equality between the first two lines of (3.84) and (3.85) so as to put all terms Θ_x in the left hand side of the equality and to put all remaining terms in the right hand side, respecting the order of

§3.73. We get:

$$
\begin{aligned}
&\frac{1}{2}\,\delta_{l_1}^j\,\Theta_x^{l_2}+\frac{1}{2}\,\delta_{l_2}^j\,\Theta_x^{l_1}-\frac{1}{2}\,\delta_{l_1}^j\,\Theta_{y^{l_2}}^0=\\
&\qquad=\underline{-\frac{1}{2}\,H_{l_1,y^{l_2}}^j+L_{l_1,l_2,x}^j-\frac{1}{2}\,\delta_{l_1}^j\,L_{l_2,l_2,x}^{l_2}-\frac{1}{2}\,\delta_{l_2}^j\,L_{l_1,l_1,x}^{l_1}}+\\
&\qquad\quad+G^j\,M_{l_1,l_2}-\frac{1}{2}\sum_k H_k^j\,L_{l_1,l_2}^k+\frac{1}{2}\sum_k H_{l_1}^k\,L_{l_2,k}^j-\\
&\qquad\quad-\frac{1}{4}\,\delta_{l_2}^j\sum_k H_{l_1}^k\,L_{k,k}^k-\frac{1}{4}\,\delta_{l_2}^j\sum_k H_{l_1}^k\,\Theta^k+\frac{1}{4}\,\delta_{l_2}^j\,L_{l_1,l_1}^{l_1}\,\Theta^0+\\
&\qquad\quad+\frac{1}{4}\,\delta_{l_2}^j\,\Theta^0\,\Theta^{l_1}.
\end{aligned}
\tag{3.86}
$$

Here, we have underlined plainly the four first order terms appearing in the second line.

Next, replacing plainly (3.64) in $(3.65)_3$, we get:

$$
\begin{aligned}
&\Big(\Pi_{l_1,l_2}^j\Big)_{y^{l_3}}-\Big(\Pi_{l_1,l_3}^j\Big)_{y^{l_2}}=\\
&=-L_{l_1,l_2,y^{l_3}}^j+\frac{1}{2}\,\delta_{l_1}^j\,L_{l_2,l_2,y^{l_3}}^{l_2}+\frac{1}{2}\,\delta_{l_2}^j\,L_{l_1,l_1,y^{l_3}}^{l_1}+\frac{1}{2}\,\delta_{l_1}^j\,\Theta_{y^{l_3}}^{l_2}+\frac{1}{2}\,\delta_{l_2}^j\,\Theta_{y^{l_3}}^{l_1}+\\
&\quad+L_{l_1,l_3,y^{l_2}}^j-\frac{1}{2}\,\delta_{l_1}^j\,L_{l_3,l_3,y^{l_2}}^{l_3}-\frac{1}{2}\,\delta_{l_3}^j\,L_{l_1,l_1,y^{l_2}}^{l_1}-\frac{1}{2}\,\delta_{l_1}^j\,\Theta_{y^{l_2}}^{l_3}-\frac{1}{2}\,\delta_{l_3}^j\,\Theta_{y^{l_2}}^{l_1}=\\
&=-\Pi_{l_1,l_2}^0\cdot\Pi_{l_3,0}^j-\sum_k\Pi_{l_1,l_2}^k\cdot\Pi_{l_3,k}^j+\Pi_{l_1,l_3}^0\cdot\Pi_{l_2,0}^j+\sum_k\Pi_{l_1,l_3}^k\cdot\Pi_{l_2,k}^j=\\
&=-M_{l_1,l_2}\cdot\left(-\frac{1}{2}\,H_{l_3}^j+\frac{1}{2}\,\delta_{l_3}^j\,\Theta^0\right)-\sum_k\Bigg(-L_{l_1,l_2}^k+\frac{1}{2}\,\delta_{l_1}^k\,L_{l_2,l_2}^{l_2}+\\
&\quad+\frac{1}{2}\,\delta_{l_2}^k\,L_{l_1,l_1}^{l_1}+\frac{1}{2}\,\delta_{l_1}^k\,\Theta^{l_2}+\frac{1}{2}\,\delta_{l_2}^k\,\Theta^{l_1}\Bigg)\cdot\Bigg(-L_{l_3,k}^j+\frac{1}{2}\,\delta_{l_3}^j\,L_{k,k}^k+\\
&\quad+\frac{1}{2}\,\delta_k^j\,L_{l_3,l_3}^{l_3}+\frac{1}{2}\,\delta_{l_3}^j\,\Theta^k+\frac{1}{2}\,\delta_k^j\,\Theta^{l_3}\Bigg)+\\
&\quad+M_{l_1,l_3}\cdot\left(-\frac{1}{2}\,H_{l_2}^j+\frac{1}{2}\,\delta_{l_2}^j\,\Theta^0\right)+\sum_k\Bigg(-L_{l_1,l_3}^k+\frac{1}{2}\,\delta_{l_1}^k\,L_{l_3,l_3}^{l_3}+\\
&\quad+\frac{1}{2}\,\delta_{l_3}^k\,L_{l_1,l_1}^{l_1}+\frac{1}{2}\,\delta_{l_1}^k\,\Theta^{l_3}+\frac{1}{2}\,\delta_{l_3}^k\,\Theta^{l_1}\Bigg)\cdot\Bigg(-L_{l_2,k}^j+\frac{1}{2}\,\delta_{l_2}^j\,L_{k,k}^k+\\
&\quad+\frac{1}{2}\,\delta_k^j\,L_{l_2,l_2}^{l_2}+\frac{1}{2}\,\delta_{l_2}^j\,\Theta^k+\frac{1}{2}\,\delta_k^j\,\Theta^{l_2}\Bigg).
\end{aligned}
\tag{3.87}
$$

Developing the products and ordering each monomial, we get:

(3.88)

$$
\begin{aligned}
&= \underbrace{\frac{1}{2}\, H_{l_3}^j\, M_{l_1,l_2}}_{\boxed{1}} - \underbrace{\frac{1}{2}\, \delta_{l_3}^j\, M_{l_1,l_2}\, \Theta^0}_{\boxed{16}} - \underbrace{\sum_k L_{l_1,l_2}^k\, L_{l_3,k}^j}_{\boxed{6}} + \underbrace{\frac{1}{2}\, \delta_{l_3}^j \sum_k L_{l_1,l_2}^k\, L_{k,k}^k}_{\boxed{7}} + \\
&\quad + \underbrace{\frac{1}{2}\, L_{l_1,l_2}^j\, L_{l_3,l_3}^{l_3}}_{(\mathrm{a})} + \underbrace{\frac{1}{2}\, \delta_{l_3}^j \sum_k L_{l_1,l_2}^k\, \Theta^k}_{\boxed{13}} + \underbrace{\frac{1}{2}\, L_{l_1,l_2}^j\, \Theta^{l_3}}_{(\mathrm{b})} + \underbrace{\frac{1}{2}\, L_{l_2,l_2}^{l_2}\, L_{l_3,l_1}^j}_{(\mathrm{c})} - \\
&\quad - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_2,l_2}^{l_2}\, L_{l_1,l_1}^{l_1}}_{\boxed{4}} - \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_2,l_2}^{l_2}\, L_{l_3,l_3}^{l_3}}_{(\mathrm{d})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_2,l_2}^{l_2}\, \Theta^{l_1}}_{(\mathrm{e})} - \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_2,l_2}^{l_2}\, \Theta^{l_3}}_{(\mathrm{f})} + \\
&\quad + \underbrace{\frac{1}{2}\, L_{l_1,l_1}^{l_1}\, L_{l_3,l_2}^j}_{(\mathrm{g})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_1,l_1}^{l_1}\, L_{l_2,l_2}^{l_2}}_{(\mathrm{h})} - \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1}\, L_{l_3,l_3}^{l_3}}_{(\mathrm{i})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_2}}_{(\mathrm{j})} - \\
&\quad - \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_3}}_{(\mathrm{k})} + \underbrace{\frac{1}{2}\, L_{l_3,l_1}^j\, \Theta^{l_2}}_{(\mathrm{l})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_2}}_{\boxed{10}} - \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_3,l_3}^{l_3}\, \Theta^{l_2}}_{(\mathrm{m})} - \\
&\quad - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, \Theta^{l_2}\, \Theta^{l_1}}_{(\mathrm{n})} - \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, \Theta^{l_2}\, \Theta^{l_3}}_{(\mathrm{o})} + \underbrace{\frac{1}{2}\, L_{l_3,l_2}^j\, \Theta^{l_1}}_{(\mathrm{p})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_2,l_2}^{l_2}\, \Theta^{l_1}}_{\boxed{12}} - \\
&\quad - \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_3,l_3}^{l_3}\, \Theta^{l_1}}_{(\mathrm{q})} - \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, \Theta^{l_1}\, \Theta^{l_2}}_{\boxed{18}} - \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, \Theta^{l_1}\, \Theta^{l_3}}_{(\mathrm{r})} -
\end{aligned}
$$

$$
\begin{aligned}
&- \underbrace{\frac{1}{2}\, H_{l_2}^j\, M_{l_1,l_3}}_{\boxed{2}} + \underbrace{\frac{1}{2}\, \delta_{l_2}^j\, M_{l_1,l_3}\, \Theta^0}_{\boxed{15}} + \underbrace{\sum_k L_{l_1,l_3}^k\, L_{l_2,k}^j}_{\boxed{5}} - \underbrace{\frac{1}{2}\, \delta_{l_2}^j \sum_k L_{l_1,l_3}^k\, L_{k,k}^k}_{\boxed{8}} - \\
&- \underbrace{\frac{1}{2}\, L_{l_1,l_3}^j\, L_{l_2,l_2}^{l_2}}_{(\mathrm{c})} - \underbrace{\frac{1}{2}\, \delta_{l_2}^j \sum_k L_{l_1,l_3}^k\, \Theta^k}_{\boxed{14}} - \underbrace{\frac{1}{2}\, L_{l_1,l_3}^j\, \Theta^{l_2}}_{(\mathrm{l})} - \underbrace{\frac{1}{2}\, L_{l_3,l_3}^{l_3}\, L_{l_2,l_1}^j}_{(\mathrm{a})} + \\
&+ \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_3,l_3}^{l_3}\, L_{l_1,l_1}^{l_1}}_{\boxed{3}} + \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_3,l_3}^{l_3}\, L_{l_2,l_2}^{l_2}}_{(\mathrm{d})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_3,l_3}^{l_3}\, \Theta^{l_1}}_{(\mathrm{q})} + \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_3,l_3}^{l_3}\, \Theta^{l_2}}_{(\mathrm{m})} - \\
&- \underbrace{\frac{1}{2}\, L_{l_1,l_1}^{l_1}\, L_{l_2,l_3}^j}_{(\mathrm{g})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1}\, L_{l_3,l_3}^{l_3}}_{(\mathrm{i})} + \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_1,l_1}^{l_1}\, L_{l_2,l_2}^{l_2}}_{(\mathrm{h})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_3}}_{\boxed{9}} + \\
&+ \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_2}}_{(\mathrm{j})} - \underbrace{\frac{1}{2}\, L_{l_2,l_1}^j\, \Theta^{l_3}}_{(\mathrm{b})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_1,l_1}^{l_1}\, \Theta^{l_3}}_{(\mathrm{k})} + \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, L_{l_2,l_2}^{l_2}\, \Theta^{l_3}}_{(\mathrm{f})} + \\
&+ \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, \Theta^{l_3}\, \Theta^{l_1}}_{\boxed{17}} + \underbrace{\frac{1}{4}\, \delta_{l_1}^j\, \Theta^{l_3}\, \Theta^{l_2}}_{(\mathrm{o})} - \underbrace{\frac{1}{2}\, L_{l_2,l_3}^j\, \Theta^{l_1}}_{(\mathrm{p})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, L_{l_3,l_3}^{l_3}\, \Theta^{l_1}}_{\boxed{11}} + \\
&+ \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, L_{l_2,l_2}^{l_2}\, \Theta^{l_1}}_{(\mathrm{e})} + \underbrace{\frac{1}{4}\, \delta_{l_2}^j\, \Theta^{l_1}\, \Theta^{l_3}}_{(\mathrm{r})} + \underbrace{\frac{1}{4}\, \delta_{l_3}^j\, \Theta^{l_1}\, \Theta^{l_2}}_{(\mathrm{n})}.
\end{aligned}
$$

We simplify and we reorganize the equality between the second and third lines of (3.88) and (3.85) so as to put all terms Θ_y in the left hand side of the equality and to put all remaining terms in the right hand side, respecting the order of §3.73. We get:

(3.89)

$$\begin{aligned}
&\frac{1}{2}\,\delta_{l_1}^j\,\Theta_{y^{l_3}}^{l_2}-\frac{1}{2}\,\delta_{l_1}^j\,\Theta_{y^{l_2}}^{l_3}+\frac{1}{2}\,\delta_{l_2}^j\,\Theta_{y^{l_3}}^{l_1}-\frac{1}{2}\,\delta_{l_3}^j\,\Theta_{y^{l_2}}^{l_1}=\\
&\quad=\underline{L_{l_1,l_2,y^{l_3}}^j-L_{l_1,l_3,y^{l_2}}^j+\frac{1}{2}\,\delta_{l_1}^j\,L_{l_3,l_3,y^{l_2}}^{l_3}-\frac{1}{2}\,\delta_{l_1}^j\,L_{l_2,l_2,y^{l_3}}^{l_2}}+\\
&\qquad+\underline{\frac{1}{2}\,\delta_{l_3}^j\,L_{l_1,l_1,y^{l_2}}^{l_1}-\frac{1}{2}\,\delta_{l_2}^j\,L_{l_1,l_1,y^{l_3}}^{l_1}}+\\
&\qquad+\frac{1}{2}\,H_{l_3}^j\,M_{l_1,l_2}-\frac{1}{2}\,H_{l_2}^j\,M_{l_1,l_3}+\frac{1}{4}\,\delta_{l_2}^j\,L_{l_3,l_3}^{l_3}\,L_{l_1,l_1}^{l_1}-\frac{1}{4}\,\delta_{l_3}^j\,L_{l_2,l_2}^{l_2}\,L_{l_1,l_1}^{l_1}+\\
&\qquad+\sum_k L_{l_1,l_3}^k\,L_{l_2,k}^j-\sum_k L_{l_1,l_2}^k\,L_{l_3,k}^j+\\
&\qquad+\frac{1}{2}\,\delta_{l_3}^j\,\sum_k L_{l_1,l_2}^k\,L_{k,k}^k-\frac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,L_{k,k}^k+\\
&\qquad+\frac{1}{4}\,\delta_{l_2}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_3}-\frac{1}{4}\,\delta_{l_3}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_2}+\frac{1}{4}\,\delta_{l_2}^j\,L_{l_3,l_3}^{l_3}\,\Theta^{l_1}-\frac{1}{4}\,\delta_{l_3}^j\,L_{l_2,l_2}^{l_2}\,\Theta^{l_1}+\\
&\qquad+\frac{1}{2}\,\delta_{l_3}^j\,\sum_k L_{l_1,l_2}^k\,\Theta^k-\frac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,\Theta^k+\\
&\qquad+\frac{1}{2}\,\delta_{l_2}^j\,M_{l_1,l_3}\,\Theta^0-\frac{1}{2}\,\delta_{l_3}^j\,M_{l_1,l_2}\,\Theta^0+\frac{1}{4}\,\delta_{l_2}^j\,\Theta^{l_3}\,\Theta^{l_1}-\frac{1}{4}\,\delta_{l_3}^j\,\Theta^{l_2}\,\Theta^{l_1}.
\end{aligned}$$

Next, replacing plainly (3.64) in $(3.65)_4$, we get:

$$\begin{aligned}
&\left(\Pi_{0,0}^0\right)_{y^{l_1}}-\left(\Pi_{0,l_1}^0\right)_x=\\
&\quad=\Theta_{y^{l_1}}^0-\frac{1}{2}\,L_{l_1,l_1,x}^{l_1}-\frac{1}{2}\,\Theta_x^{l_1}=\\
&\quad=-\underline{\Pi_{0,0}^0\cdot\Pi_{l_1,0}^0}_{\textcircled{a}}-\sum_k\Pi_{0,0}^k\cdot\Pi_{l_1,k}^0+\underline{\Pi_{0,l_1}^0\cdot\Pi_{0,0}^0}_{\textcircled{a}}+\sum_k\Pi_{0,l_1}^k\cdot\Pi_{0,k}^0=\\
&\quad=-\sum_k\left(-G^k\right)\cdot\left(M_{l_1,k}\right)+\sum_k\left(-\frac{1}{2}\,H_{l_1}^k+\frac{1}{2}\,\delta_{l_1}^k\,\Theta^0\right)\cdot\\
&\qquad\cdot\left(\frac{1}{2}\,L_{k,k}^k+\frac{1}{2}\,\Theta^k\right)=\\
&\quad=\sum_k G^k\,M_{l_1,k}-\frac{1}{4}\sum_k H_{l_1}^k\,L_{k,k}^k-\frac{1}{4}\sum_k H_{l_1}^k\,\Theta^k+\frac{1}{4}\,L_{l_1,l_1}^{l_1}\,\Theta^0+\frac{1}{4}\,\Theta^0\,\Theta^{l_1}.
\end{aligned}\tag{3.90}$$

Reorganizing the equality so as to put the terms Θ_x and Θ_y alone in the left hand side, we get:

$$\begin{aligned}
-\frac{1}{2}\,\Theta_x^{l_1}+\Theta_{y^{l_1}}^0&=\underline{\frac{1}{2}\,L_{l_1,l_1,x}^{l_1}}+\\
&\quad+\sum_k G^k\,M_{l_1,k}-\frac{1}{4}\sum_k H_{l_1}^k\,L_{k,k}^k-\frac{1}{4}\sum_k H_{l_1}^k\,\Theta^k+\\
&\quad+\frac{1}{4}\,L_{l_1,l_1}^{l_1}\,\Theta^0+\frac{1}{4}\,\Theta^0\,\Theta^{l_1}.
\end{aligned}\tag{3.91}$$

Next, replacing plainly (3.64) in $(3.65)_5$, we get:

$$
\begin{aligned}
\big(\Pi^0_{l_1,l_2}\big)_x - \big(\Pi^0_{l_1,0}\big)_{y^{l_2}} &= \\
&= M_{l_1,l_2,x} - \frac{1}{2}\,L^{l_1}_{l_1,l_1,y^{l_2}} - \frac{1}{2}\,\Theta^{l_1}_{y^{l_2}} = \\
&= -\Pi_{l_1,l_2}\cdot\Pi^0_{0,0} - \sum_k \Pi^k_{l_1,l_2}\cdot\Pi^0_{0,k} + \Pi^0_{l_1,0}\cdot\Pi^0_{l_2,0} + \sum_k \Pi^k_{l_1,0}\cdot\Pi^0_{l_2,k} = \\
&= -M_{l_1,l_2}\,\Theta^0 - \sum_k \bigg(-L^k_{l_1,l_2} + \frac{1}{2}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2} + \frac{1}{2}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1} + \\
&\quad + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^{l_2} + \frac{1}{2}\,\delta^k_{l_2}\,\Theta^{l_1}\bigg)\cdot\bigg(\frac{1}{2}\,L^k_{k,k} + \frac{1}{2}\,\Theta^k\bigg) + \\
&\quad + \bigg(\frac{1}{2}\,L^{l_1}_{l_1,l_1} + \frac{1}{2}\,\Theta^{l_1}\bigg)\cdot\bigg(\frac{1}{2}\,L^{l_2}_{l_2,l_2} + \frac{1}{2}\,\Theta^{l_2}\bigg) + \\
&\quad + \sum_k \bigg(-\frac{1}{2}\,H^k_{l_1} + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^0\bigg)\cdot M_{l_2,k} =
\end{aligned}
\tag{3.92}
$$

$$
\begin{aligned}
&= -\underline{M_{l_1,l_2}\,\Theta^0}_{\boxed{7}} + \underline{\frac{1}{2}\sum_k L^k_{l_1,l_2}\,L^k_{k,k}}_{\boxed{3}} + \underline{\frac{1}{2}\sum_k L^k_{l_1,l_2}\,\Theta^k}_{\boxed{6}} - \underline{\frac{1}{4}\,L^{l_2}_{l_2,l_2}\,L^{l_1}_{l_1,l_1}}_{\boxed{2}} - \\
&\quad - \underline{\frac{1}{4}\,L^{l_2}_{l_2,l_2}\,\Theta^{l_1}}_{(a)} - \underline{\frac{1}{4}\,L^{l_1}_{l_1,l_1}\,L^{l_2}_{l_2,l_2}}_{(b)} - \underline{\frac{1}{4}\,L^{l_1}_{l_1,l_1}\,\Theta^{l_2}}_{(c)} - \underline{\frac{1}{4}\,L^{l_1}_{l_1,l_1}\,\Theta^{l_2}}_{\boxed{4}} - \\
&\quad - \underline{\frac{1}{4}\,\Theta^{l_1}\,\Theta^{l_2}}_{(d)} - \underline{\frac{1}{4}\,L^{l_2}_{l_2,l_2}\,\Theta^{l_1}}_{\boxed{5}} - \underline{\frac{1}{4}\,\Theta^{l_1}\,\Theta^{l_2}}_{\boxed{8}} + \underline{\frac{1}{4}\,L^{l_1}_{l_1,l_1}\,L^{l_2}_{l_2,l_2}}_{(b)} + \\
&\quad + \underline{\frac{1}{4}\,L^{l_1}_{l_1,l_1}\,\Theta^{l_2}}_{(c)} + \underline{\frac{1}{4}\,L^{l_2}_{l_2,l_2}\,\Theta^{l_1}}_{(a)} + \underline{\frac{1}{4}\,\Theta^{l_1}\,\Theta^{l_2}}_{(d)} - \underline{\frac{1}{2}\sum_k H^k_{l_1}\,M_{l_2,k}}_{\boxed{1}} + \\
&\quad + \underline{\frac{1}{2}\,M_{l_2,l_1}\,\Theta^0}_{\boxed{7}}.
\end{aligned}
$$

Multiplying by -2 and reorganizing the equality, we get:

$$
\begin{aligned}
\Theta^{l_1}_{y^{l_2}} = -&\underline{L^{l_1}_{l_1,l_1,y^{l_2}} + 2\,M_{l_1,l_2,x}} + \\
&+ \sum_k H^k_{l_1}\,M_{l_2,k} + \frac{1}{2}\,L^{l_1}_{l_1,l_1}\,L^{l_2}_{l_2,l_2} - \sum_k L^k_{l_1,l_2}\,L^k_{k,k} + \\
&+ \frac{1}{2}\,L^{l_1}_{l_1,l_1}\,\Theta^{l_2} + \frac{1}{2}\,L^{l_2}_{l_2,l_2}\,\Theta^{l_1} - \sum_k L^k_{l_1,l_2}\,\Theta^k + \\
&+ M_{l_1,l_2}\,\Theta^0 + \frac{1}{2}\,\Theta^{l_1}\,\Theta^{l_2}.
\end{aligned}
\tag{3.93}
$$

Next, replacing plainly (3.64) in $(3.65)_4$, we get:

$$
\begin{aligned}
&\big(\Pi^0_{l_1,l_2}\big)_{y^{l_3}} - \big(\Pi^0_{l_1,l_3}\big)_{y^{l_2}} = \\
&\quad = M_{l_1,l_2,y^{l_3}} - M_{l_1,l_3,y^{l_2}} = \\
&\quad = -\Pi^0_{l_1,l_2}\cdot\Pi^0_{l_3,0} - \sum_k \Pi^k_{l_1,l_2}\cdot\Pi^0_{l_3,k} + \Pi^0_{l_1,l_3}\cdot\Pi^0_{l_2,0} + \sum_k \Pi^k_{l_1,l_3}\cdot\Pi^0_{l_2,k} = \\
&\quad = -M_{l_1,l_2}\left(\frac{1}{2}\,L^{l_3}_{l_3,l_3} + \frac{1}{2}\,\Theta^{l_3}\right) - \sum_k M_{l_3,k}\bigg(-L^k_{l_1,l_2} + \frac{1}{2}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2} + \\
&\qquad + \frac{1}{2}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1} + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^{l_2} + \frac{1}{2}\,\delta^k_{l_2}\,\Theta^{l_1}\bigg) + \\
&\qquad + M_{l_1,l_3}\left(\frac{1}{2}\,L^{l_2}_{l_2,l_2} + \frac{1}{2}\,\Theta^{l_2}\right) + \sum_k M_{l_2,k}\bigg(-L^k_{l_1,l_3} + \frac{1}{2}\,\delta^k_{l_1}\,L^{l_3}_{l_3,l_3} + \\
&\qquad + \frac{1}{2}\,\delta^k_{l_3}\,L^{l_1}_{l_1,l_1} + \frac{1}{2}\,\delta^k_{l_1}\,\Theta^{l_3} + \frac{1}{2}\,\delta^k_{l_3}\,\Theta^{l_1}\bigg).
\end{aligned}
\tag{3.94}
$$

Developing the products and ordering each monomial, we get:

$$
\begin{aligned}
&= \underbrace{-\frac{1}{2}\,L^{l_3}_{l_3,l_3}\,M_{l_1,l_2}}_{\text{(a)}} \underbrace{-\frac{1}{2}\,M_{l_1,l_2}\,\Theta^{l_3}}_{\text{(b)}} \underbrace{+\sum_k L^k_{l_1,l_2}\,M_{l_3,k}}_{\boxed{1}} \underbrace{-\frac{1}{2}\,L^{l_2}_{l_2,l_2}\,M_{l_3,l_1}}_{\text{(c)}} - \\
&\quad \underbrace{-\frac{1}{2}\,L^{l_1}_{l_1,l_1}\,M_{l_3,l_2}}_{\text{(d)}} \underbrace{-\frac{1}{2}\,M_{l_3,l_1}\,\Theta^{l_2}}_{\text{(e)}} \underbrace{-\frac{1}{2}\,M_{l_3,l_2}\,\Theta^{l_1}}_{\text{(f)}} \underbrace{+\frac{1}{2}\,L^{l_2}_{l_2,l_2}\,M_{l_3,l_1}}_{\text{(c)}} + \\
&\quad \underbrace{+\frac{1}{2}\,M_{l_1,l_3}\,\Theta^{l_2}}_{\text{(e)}} \underbrace{-\sum_k L^k_{l_1,l_3}\,M_{l_2,k}}_{\boxed{2}} \underbrace{+\frac{1}{2}\,L^{l_3}_{l_3,l_3}\,M_{l_2,l_1}}_{\text{(a)}} \underbrace{+\frac{1}{2}\,L^{l_1}_{l_1,l_1}\,M_{l_2,l_3}}_{\text{(d)}} + \\
&\quad \underbrace{+\frac{1}{2}\,M_{l_2,l_1}\,\Theta^{l_3}}_{\text{(b)}} \underbrace{+\frac{1}{2}\,M_{l_2,l_3}\,\Theta^{l_1}}_{\text{(f)}}.
\end{aligned}
\tag{3.95}
$$

Simplifying, we obtain the family (IV) in the statement of Theorem 1.7:

$$
0 = \underline{M_{l_1,l_2,y^{l_3}} - M_{l_1,l_3,y^{l_2}}} - \sum_k L^k_{l_1,l_2}\,M_{l_3,k} + \sum_k L^k_{l_1,l_3}\,M_{l_2,k}.
\tag{3.96}
$$

3.97. Solving Θ^0_x, $\Theta^0_{y^{l_1}}$, $\Theta^{l_1}_x$ and $\Theta^{l_1}_{y^{l_2}}$. It is now easy to solve all first order partial derivatives of the functions Θ^0 and Θ^l. Equation (3.93) already provides the solution for $\Theta^{l_1}_{y^{l_2}}$. We state the result as an independent proposition.

Proposition 3.98. *As a consequence of the six families of equations* (3.69), (3.86), (3.89), (3.91), (3.93) *and* (3.96) *the first order derivatives* Θ^0_x, $\Theta^0_{y^{l_1}}$, $\Theta^{l_1}_x$ *and* $\Theta^{l_1}_{y^{l_2}}$ *of the principal unknowns are given by*:

$$
\left\{
\begin{aligned}
\Theta^0_x &= \underline{-2\,G^{l_1}_{y^{l_1}} + H^{l_1}_{l_1,x}} + \\
&\quad + 2\sum_k G^k\,L^{l_1}_{l_1,k} - \sum_k G^k\,L^k_{k,k} - \frac{1}{2}\sum_k H^k_{l_1}\,H^{l_1}_k - \\
&\quad - \sum_k G^k\,\Theta^k + \frac{1}{2}\,\Theta^0\,\Theta^0.
\end{aligned}
\right.
\tag{3.99}
$$

$$
(3.100) \qquad \left\{
\begin{aligned}
\Theta_{y^{l_1}}^0 = & \underline{\frac{2}{3}\,L_{l_1,l_1,x}^{l_1} - \frac{1}{3}\,H_{l_1,y^{l_1}}^{l_1}} + \\
& + \frac{2}{3}\,G^{l_1}\,M_{l_1,l_1} + \frac{4}{3}\sum_k G^k\,M_{l_1,k} - \frac{1}{3}\sum_k H_k^{l_1}\,L_{l_1,l_1}^k + \\
& + \frac{1}{3}\sum_k H_{l_1}^k\,L_{l_1,k}^{l_1} - \frac{1}{2}\sum_k H_{l_1}^k\,L_{k,k}^k - \frac{1}{2}\sum_k H_{l_1}^k\,\Theta^k + \\
& + \frac{1}{2}\,L_{l_1,l_1}^{l_1}\,\Theta^0 + \frac{1}{2}\,\Theta^0\,\Theta^{l_1}.
\end{aligned}
\right.
$$

$$
(3.101) \qquad \left\{
\begin{aligned}
\Theta_x^{l_1} = & \underline{-\frac{2}{3}\,H_{l_1,y^{l_1}}^{l_1} + \frac{1}{3}\,L_{l_1,l_1,x}^{l_1}} + \\
& + \frac{4}{3}\,G^{l_1}\,M_{l_1,l_1} + \frac{2}{3}\sum_k G^k\,M_{l_1,k} - \frac{2}{3}\sum_k H_k^{l_1}\,L_{l_1,l_1}^k + \\
& + \frac{2}{3}\sum_k H_{l_1}^k\,L_{l_1,k}^{l_1} - \frac{1}{2}\sum_k H_{l_1}^k\,L_{k,k}^k - \frac{1}{2}\sum_k H_{l_1}^k\,\Theta^k + \\
& + \frac{1}{2}\,L_{l_1,l_1}^{l_1}\,\Theta^0 + \frac{1}{2}\,\Theta^0\,\Theta^{l_1}.
\end{aligned}
\right.
$$

$$
(3.102) \qquad \left\{
\begin{aligned}
\Theta_{y^{l_2}}^{l_1} = & \underline{-L_{l_1,l_1,y^{l_2}}^{l_1} + 2\,M_{l_1,l_2,x}} + \\
& + \sum_k H_{l_1}^k\,M_{l_2,k} + \frac{1}{2}\,L_{l_1,l_1}^{l_1}\,L_{l_2,l_2}^{l_2} - \sum_k L_{l_1,l_2}^k\,L_{k,k}^k + \\
& + \frac{1}{2}\,L_{l_1,l_1}^{l_1}\,\Theta^{l_2} + \frac{1}{2}\,L_{l_2,l_2}^{l_2}\,\Theta^{l_1} - \sum_k L_{l_1,l_2}^k\,\Theta^k + \\
& + M_{l_1,l_2}\,\Theta^0 + \frac{1}{2}\,\Theta^{l_1}\,\Theta^{l_2}.
\end{aligned}
\right.
$$

We notice that the right hand side of (3.99) should be independent of l_1; this phenomenon will be explained in a while.

Proof. For Θ_x^0 in (3.99), it suffices to put $j := l_1$ in (3.69).

To obtain $\Theta_{y^{l_1}}^0$, we put $j := l_2$ and $l_2 := l_1$ in (3.86), which yields:

$$
(3.103) \qquad
\begin{aligned}
\Theta_x^{l_1} - \frac{1}{2}\,\Theta_{y^{l_1}}^0 = & \underline{-\frac{1}{2}\,H_{l_1,y^{l_1}}^{l_1}} + \\
& + G^{l_1}\,M_{l_1,l_1} - \frac{1}{2}\sum_k H_k^{l_1}\,L_{l_1,l_1}^k + \\
& + \frac{1}{2}\sum_k H_{l_1}^k\,L_{l_1,k}^{l_1} - \frac{1}{4}\sum_k H_{l_1}^k\,L_{k,k}^k - \\
& - \frac{1}{4}\sum_k H_{l_1}^k\,\Theta^k + \frac{1}{4}\,L_{l_1,l_1}^{l_1}\,\Theta^0 + \frac{1}{4}\,\Theta^0\,\Theta^{l_1}.
\end{aligned}
$$

We may easily solve $\Theta_{y^{l_1}}^0$ and $\Theta_x^{l_1}$ thanks to this equation (3.103) and thanks to (3.91): indeed, to obtain (3.100), it suffices to compute $\frac{4}{3}\cdot(3.91)+\frac{2}{3}\cdot(3.103)$; to obtain (3.101), it suffices to compute $\frac{2}{3}\cdot(3.91)+\frac{4}{3}\cdot(3.103)$. Finally, (3.102) is a copy of (3.93). This completes the proof. □

3.104. Appearance of the crucial four families of first order partial differential relations (I), (II), (III) and (IV) of Theorem 1.7. However, in solving Θ_x^0, $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$ from our six families of equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96), only a subpart of these equations has been used. We notice that the two families of equations (3.91) and (3.93) have been used completely and that the family of equations (3.96), which does not involve Θ, coincides precisely with the system (IV) of Theorem 1.7. To insure that Θ_x^0, $\Theta_{y^{l_1}}^0$, $\Theta_x^{l_1}$ and $\Theta_{y^{l_2}}^{l_1}$ as written in Proposition 3.98 are true solutions, it is necessary and sufficient that they satisfy the remaining equations. Thus, we have to replace these solutions (3.99), (3.100), (3.101) and (3.102) in the three remaining families (3.69), (3.86) and (3.89).

Firstly, let us insert inside (3.69) the value of Θ_x^0 given by the equation (3.99), in which the index l_1 is replaced in advance by an arbitrary index l_2. We get:

$$
\begin{aligned}
0 = & \underline{-2\, G_{y^{l_1}}^j + 2\, \delta_{l_1}^j\, G_{y^{l_2}}^{l_2} + H_{l_1,x}^j - \delta_{l_1}^j\, H_{l_2,x}^{l_2}} + \\
& + 2 \sum_k G^k\, L_{l_1,k}^j - 2\, \delta_{l_1}^j \sum_k G^k\, L_{l_2,k}^{l_2} \underbrace{- \delta_{l_1}^j \sum_k G^k\, L_{k,k}^k}_{\text{(a)}} + \\
& \underbrace{+ \delta_{l_1}^j \sum_k G^k\, L_{k,k}^k}_{\text{(a)}} - \frac{1}{2} \sum_k H_{l_1}^k\, H_k^j + \frac{1}{2}\, \delta_{l_1}^j \sum_k H_{l_2}^k\, H_k^{l_2} - \\
& \underbrace{- \delta_{l_1}^j \sum_k G^k\, \Theta^k}_{\text{(b)}} \underbrace{+ \delta_{l_1}^j \sum_k G^k\, \Theta^k}_{\text{(b)}} \underbrace{+ \frac{1}{2}\, \delta_{l_1}^j\, \Theta^0\, \Theta^0}_{\text{(c)}} \underbrace{- \frac{1}{2}\, \delta_{l_1}^j\, \Theta^0\, \Theta^0}_{\text{(c)}}.
\end{aligned}
\tag{3.105}
$$

We simplify, which yields the family (I) of partial differential relations of Theorem 1.7:

$$
\begin{aligned}
0 = & \underline{-2\, G_{y^{l_1}}^j + 2\, \delta_{l_1}^j\, G_{y^{l_2}}^{l_2} + H_{l_1,x}^j - \delta_{l_1}^j\, H_{l_2,x}^{l_2}} + \\
& + 2 \sum_k G^k\, L_{l_1,k}^j - 2\, \delta_{l_1}^j \sum_k G^k\, L_{l_2,k}^{l_2} + \\
& - \frac{1}{2} \sum_k H_{l_1}^k\, H_k^j + \frac{1}{2}\, \delta_{l_1}^j \sum_k H_{l_2}^k\, H_k^{l_2}.
\end{aligned}
\tag{3.106}
$$

Secondly, let us insert inside (3.86) the values of $\Theta_x^{l_1}$, $\Theta_x^{l_2}$ given by (3.101) and the value of $\Theta_{y^{l_1}}^0$ given by (3.100). We place all the terms in the right hand side of the equality and we place the first order terms in the beginning (first three lines just below).

We obtain:

(3.107)

$$
\begin{aligned}
0 = & -\underbrace{\frac{1}{2} H^j_{l_1,y^{l_2}}}_{\boxed{1}} + \underbrace{L^j_{l_1,l_2,x}}_{\boxed{4}} - \underbrace{\frac{1}{2}\delta^j_{l_1} L^{l_2}_{l_2,l_2,x}}_{\boxed{5}} - \underbrace{\frac{1}{2}\delta^j_{l_1} L^{l_1}_{l_1,l_1,x}}_{\boxed{6}} + \\
& + \underbrace{\frac{1}{3}\delta^j_{l_1} H^{l_2}_{l_2,y^{l_2}}}_{\boxed{2}} - \underbrace{\frac{1}{6}\delta^j_{l_1} L^{l_2}_{l_2,l_2,x}}_{\boxed{5}} + \underbrace{\frac{1}{3}\delta^j_{l_2} H^{l_1}_{l_1,y^{l_1}}}_{\boxed{3}} - \underbrace{\frac{1}{6}\delta^j_{l_2} L^{l_1}_{l_1,l_1,x}}_{\boxed{6}} + \\
& + \underbrace{\frac{1}{3}\delta^j_{l_1} L^{l_2}_{l_2,l_2,x}}_{\boxed{5}} - \underbrace{\frac{1}{6}\delta^j_{l_1} H^{l_2}_{l_2,y^{l_2}}}_{\boxed{2}} + \\
& + \underbrace{G^j M_{l_1,l_2}}_{\boxed{7}} - \underbrace{\frac{1}{2}\sum_k H^j_k L^k_{l_1,l_2}}_{\boxed{12}} + \underbrace{\frac{1}{2}\sum_k H^k_{l_1} L^j_{l_2,k}}_{\boxed{13}} - \underbrace{\frac{1}{4}\delta^j_{l_2}\sum_k H^k_{l_1} L^k_{k,k}}_{ⓐ} - \\
& - \underbrace{\frac{1}{4}\delta^j_{l_2}\sum_k H^k_{l_1}\Theta^k}_{ⓑ} + \underbrace{\frac{1}{4}\delta^j_{l_2} L^{l_1}_{l_1,l_1}\Theta^0}_{ⓒ} + \underbrace{\frac{1}{4}\delta^j_{l_2}\Theta^0\Theta^{l_1}}_{ⓓ} - \\
& - \underbrace{\frac{2}{3}\delta^j_{l_1} G^{l_2} M_{l_2,l_2}}_{\boxed{8}} - \underbrace{\frac{1}{3}\delta^j_{l_1}\sum_k G^k M_{l_2,k}}_{\boxed{10}} + \underbrace{\frac{1}{3}\delta^j_{l_1}\sum_k H^{l_2}_k L^k_{l_2,l_2}}_{\boxed{14}} - \underbrace{\frac{1}{3}\delta^j_{l_1}\sum_k H^k_{l_2} L^{l_2}_{l_2,k}}_{\boxed{15}} + \\
& + \underbrace{\frac{1}{4}\delta^j_{l_1}\sum_k H^k_{l_2} L^k_{k,k}}_{ⓔ} + \underbrace{\frac{1}{4}\delta^j_{l_1}\sum_k H^k_{l_2}\Theta^k}_{ⓕ} - \underbrace{\frac{1}{4}\delta^j_{l_1} L^{l_2}_{l_2,l_2}\Theta^0}_{ⓖ} - \underbrace{\frac{1}{4}\delta^j_{l_1}\Theta^0\Theta^{l_2}}_{ⓗ} - \\
& - \underbrace{\frac{2}{3}\delta^j_{l_2} G^{l_1} M_{l_1,l_1}}_{\boxed{9}} - \underbrace{\frac{1}{3}\delta^j_{l_2}\sum_k G^k M_{l_1,k}}_{\boxed{11}} + \underbrace{\frac{1}{3}\delta^j_{l_2}\sum_k H^{l_1}_k L^k_{l_1,l_1}}_{\boxed{16}} - \underbrace{\frac{1}{3}\delta^j_{l_2}\sum_k H^k_{l_1} L^{l_1}_{l_1,k}}_{\boxed{17}} + \\
& + \underbrace{\frac{1}{4}\delta^j_{l_2}\sum_k H^k_{l_1} L^k_{k,k}}_{ⓐ} + \underbrace{\frac{1}{4}\delta^j_{l_2}\sum_k H^k_{l_1}\Theta^k}_{ⓑ} - \underbrace{\frac{1}{4}\delta^j_{l_2} L^{l_1}_{l_1,l_1}\Theta^0}_{ⓒ} - \underbrace{\frac{1}{4}\delta^j_{l_2}\Theta^0\Theta^0}_{ⓓ} + \\
& + \underbrace{\frac{1}{3}\delta^j_{l_1} G^{l_2} M_{l_2,l_2}}_{\boxed{8}} + \underbrace{\frac{2}{3}\delta^j_{l_1}\sum_k G^k M_{l_2,k}}_{\boxed{10}} - \underbrace{\frac{1}{6}\delta^j_{l_1}\sum_k H^{l_2}_k L^k_{l_2,l_2}}_{\boxed{14}} + \underbrace{\frac{1}{6}\delta^j_{l_1}\sum_k H^k_{l_2} L^{l_2}_{l_2,k}}_{\boxed{15}} - \\
& - \underbrace{\frac{1}{4}\delta^j_{l_1}\sum_k H^k_{l_2} L^k_{k,k}}_{ⓔ} - \underbrace{\frac{1}{4}\delta^j_{l_1}\sum_k H^k_{l_2}\Theta^k}_{ⓕ} + \underbrace{\frac{1}{4}\delta^j_{l_1} L^{l_2}_{l_2,l_2}\Theta^0}_{ⓖ} + \underbrace{\frac{1}{4}\delta^j_{l_1}\Theta^0\Theta^{l_2}}_{ⓗ}.
\end{aligned}
$$

Simplifying and ordering, we obtain the family (II) of partial differential relations of Theorem 1.7:

(3.108)

$$
\begin{aligned}
0 = & \underline{-\frac{1}{2} H^j_{l_1,y^{l_2}} + \frac{1}{6}\delta^j_{l_1} H^{l_2}_{l_2,y^{l_2}} + \frac{1}{3}\delta^j_{l_2} H^{l_1}_{l_1,y^{l_1}}} + \\
& \underline{+ L^j_{l_1,l_2,x} - \frac{1}{3}\delta^j_{l_1} L^{l_2}_{l_2,l_2,x} - \frac{2}{3}\delta^j_{l_2} L^{l_1}_{l_1,l_1,x}} + \\
& + G^j M_{l_1,l_2} - \frac{1}{3}\delta^j_{l_1} G^{l_2} M_{l_2,l_2} - \frac{2}{3}\delta^j_{l_2} G^{l_1} M_{l_1,l_1} + \frac{1}{3}\delta^j_{l_1}\sum_k G^k M_{l_2,k} - \\
& - \frac{1}{3}\delta^j_{l_2}\sum_k G^k M_{l_1,k} - \frac{1}{2}\sum_k H^j_k L^k_{l_1,l_2} + \frac{1}{2}\sum_k H^k_{l_1} L^j_{l_2,k} + \\
& + \frac{1}{6}\delta^j_{l_1}\sum_k H^{l_2}_k L^k_{l_2,l_2} - \frac{1}{6}\delta^j_{l_1}\sum_k H^k_{l_2} L^{l_2}_{l_2,k} + \\
& + \frac{1}{3}\delta^j_{l_2}\sum_k H^{l_1}_k L^k_{l_1,l_1} - \frac{1}{3}\delta^j_{l_2}\sum_k H^k_{l_1} L^{l_1}_{l_1,k}.
\end{aligned}
$$

Thirdly, let us insert inside (3.89) the values of $\Theta_{y^{l_3}}^{l_2}$, of $\Theta_{y^{l_2}}^{l_3}$, of $\Theta_{y^{l_3}}^{l_1}$ and of $\Theta_{y^{l_2}}^{l_1}$ given by (3.102). We place all the terms in the right hand side of the equality and we place the first order terms in the beginning (first four lines just below). We obtain:

$$
\begin{aligned}
0 = {} & \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,L_{l_2,l_2,y^{l_3}}^{l_2}}_{\text{(a)}} - \underline{\delta_{l_1}^j\,M_{l_2,l_3,x}}_{\text{(b)}} + \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,L_{l_1,l_1,y^{l_3}}^{l_1}}_{\text{(c)}} - \underline{\delta_{l_2}^j\,M_{l_1,l_3,x}}_{\boxed{4}} - \\
& - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,L_{l_3,l_3,y^{l_2}}^{l_3}}_{\text{(d)}} + \underline{\delta_{l_1}^j\,M_{l_3,l_2,x}}_{\text{(b)}} - \underline{\tfrac{1}{3}\,\delta_{l_3}^j\,L_{l_1,l_1,y^{l_2}}^{l_1}}_{\text{(e)}} + \underline{\delta_{l_3}^j\,M_{l_1,l_2,x}}_{\boxed{3}} + \\
& + \underline{L_{l_1,l_2,y^{l_3}}^j}_{\boxed{1}} - \underline{L_{l_1,l_3,y^{l_2}}^j}_{\boxed{2}} + \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,L_{l_3,l_3,y^{l_2}}^{l_3}}_{\text{(d)}} - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,L_{l_2,l_2,y^{l_3}}^{l_2}}_{\text{(a)}} + \\
& + \underline{\tfrac{1}{3}\,\delta_{l_3}^j\,L_{l_1,l_1,y^{l_2}}^{l_1}}_{\text{(e)}} - \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,L_{l_1,l_1,y^{l_3}}^{l_1}}_{\text{(c)}} + \\
& + \underline{\tfrac{1}{2}\,H_{l_3}^j\,M_{l_1,l_2}}_{\boxed{5}} - \underline{\tfrac{1}{2}\,H_{l_2}^j\,M_{l_1,l_3}}_{\boxed{6}} + \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_3,l_3}^{l_3}\,L_{l_1,l_1}^{l_1}}_{\text{(f)}} - \underline{\tfrac{1}{4}\,\delta_{l_3}^j\,L_{l_2,l_2}^{l_2}\,L_{l_1,l_1}^{l_1}}_{\text{(g)}} + \\
& + \underline{\sum_k L_{l_1,l_3}^k\,L_{l_2,k}^j}_{\boxed{11}} - \underline{\sum_k L_{l_1,l_2}^k\,L_{l_3,k}^j}_{\boxed{12}} + \\
& + \underline{\tfrac{1}{2}\,\delta_{l_3}^j\,\sum_k L_{l_1,l_2}^k\,L_{k,k}^k}_{\text{(h)}} - \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,L_{k,k}^k}_{\text{(i)}} + \\
& + \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_3}}_{\text{(j)}} - \underline{\tfrac{1}{4}\,\delta_{l_3}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_2}}_{\text{(k)}} + \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_3,l_3}^{l_3}\,\Theta^{l_1}}_{\text{(l)}} - \underline{\tfrac{1}{4}\,\delta_{l_3}^j\,L_{l_2,l_2}^{l_2}\,\Theta^{l_1}}_{\text{(m)}} + \\
& + \underline{\tfrac{1}{2}\,\delta_{l_3}^j\,\sum_k L_{l_1,l_2}^k\,\Theta^k}_{\text{(n)}} - \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,\Theta^k}_{\text{(o)}} + \\
& + \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,M_{l_1,l_3}\,\Theta^0}_{\text{(p)}} - \underline{\tfrac{1}{2}\,\delta_{l_3}^j\,M_{l_1,l_2}\,\Theta^0}_{\text{(q)}} + \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,\Theta^{l_3}\,\Theta^{l_1}}_{\text{(r)}} - \underline{\tfrac{1}{4}\,\delta_{l_3}^j\,\Theta^{l_2}\,\Theta^{l_1}}_{\text{(s)}} - \\
& \quad - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k H_{l_2}^k\,M_{l_3,k}}_{\boxed{8}} - \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_2,l_2}^{l_2}\,L_{l_3,l_3}^{l_3}}_{\text{(t)}} + \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k L_{l_2,l_3}\,L_{k,k}^k}_{\text{(u)}} - \\
& \quad - \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_2,l_2}^{l_2}\,\Theta^{l_3}}_{\text{(v)}} - \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_3,l_3}^{l_3}\,\Theta^{l_2}}_{\text{(w)}} + \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k L_{l_2,l_3}^k\,\Theta^k}_{\text{(x)}} - \\
& \quad - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,M_{l_2,l_3}\,\Theta^0}_{\text{(y)}} - \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,\Theta^{l_2}\,\Theta^{l_3}}_{\text{(z)}} - \\
& \quad - \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,\sum_k H_{l_1}^k\,M_{l_3,k}}_{\boxed{10}} - \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_1,l_1}^{l_1}\,L_{l_3,l_3}^{l_3}}_{\text{(f)}} + \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,L_{k,k}^k}_{\text{(i)}} - \\
& \quad - \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_3}}_{\text{(j)}} - \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,L_{l_3,l_3}^{l_3}\,\Theta^{l_1}}_{\text{(l)}} + \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,\sum_k L_{l_1,l_3}^k\,\Theta^k}_{\text{(o)}} - \\
& \quad - \underline{\tfrac{1}{2}\,\delta_{l_2}^j\,M_{l_1,l_3}\,\Theta^0}_{\text{(p)}} - \underline{\tfrac{1}{4}\,\delta_{l_2}^j\,\Theta^{l_3}\,\Theta^{l_1}}_{\text{(r)}} + \\
& \quad + \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k H_{l_3}^k\,M_{l_2,k}}_{\boxed{7}} + \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_3,l_3}^{l_3}\,L_{l_2,l_2}^{l_2}}_{\text{(t)}} - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k L_{l_3,l_2}\,L_{k,k}^k}_{\text{(u)}} + \\
& \quad + \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_3,l_3}^{l_3}\,\Theta^{l_2}}_{\text{(w)}} + \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,L_{l_2,l_2}^{l_2}\,\Theta^{l_3}}_{\text{(v)}} - \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,\sum_k L_{l_3,l_2}^k\,\Theta^k}_{\text{(x)}} + \\
& \quad + \underline{\tfrac{1}{2}\,\delta_{l_1}^j\,M_{l_3,l_2}\,\Theta^0}_{\text{(y)}} + \underline{\tfrac{1}{4}\,\delta_{l_1}^j\,\Theta^{l_3}\,\Theta^{l_2}}_{\text{(z)}} +
\end{aligned}
\tag{3.109}
$$

$$+\underbrace{\frac{1}{2}\,\delta_{l_3}^j\sum_k H_{l_1}^k\,M_{l_2,k}}_{\boxed{9}}+\underbrace{\frac{1}{4}\,\delta_{l_3}^j\,L_{l_1,l_1}^{l_1}\,L_{l_2,l_2}^{l_2}}_{(g)}-\underbrace{\frac{1}{2}\,\delta_{l_3}^j\sum_k L_{l_1,l_2}^k\,L_{k,k}^k}_{(h)}+$$
$$+\underbrace{\frac{1}{4}\,\delta_{l_3}^j\,L_{l_1,l_1}^{l_1}\,\Theta^{l_2}}_{(k)}+\underbrace{\frac{1}{4}\,\delta_{l_3}^j\,L_{l_2,l_2}^{l_2}\,\Theta^{l_1}}_{(m)}-\underbrace{\frac{1}{2}\,\delta_{l_3}^j\sum_k L_{l_1,l_2}^k\,\Theta^k}_{(n)}+$$
$$+\underbrace{\frac{1}{2}\,\delta_{l_3}^j\,M_{l_1,l_2}\,\Theta^0}_{(q)}+\underbrace{\frac{1}{4}\,\delta_{l_3}^j\,\Theta^{l_1}\,\Theta^{l_2}}_{(s)}.$$

Simplifying and ordering, we obtain the family (III) of partial differential relations of Theorem 1.7:

$$\begin{aligned}
0 &= \underline{L_{l_1,l_2,y^{l_3}}^j - L_{l_1,l_3,y^{l_2}}^j + \delta_{l_3}^j\,M_{l_1,l_2,x} - \delta_{l_2}^j\,M_{l_1,l_3,x}}+\\
&+\frac{1}{2}\,H_{l_3}^j\,M_{l_1,l_2}-\frac{1}{2}\,H_{l_2}^j\,M_{l_1,l_3}+\\
&+\frac{1}{2}\,\delta_{l_1}^j\sum_k H_{l_3}^k\,M_{l_2,k}-\frac{1}{2}\,\delta_{l_1}^j\sum_k H_{l_2}^k\,M_{l_3,k}+\\
&+\frac{1}{2}\,\delta_{l_3}^j\sum_k H_{l_1}^k\,M_{l_2,k}-\frac{1}{2}\,\delta_{l_2}^j\sum_k H_{l_1}^k\,M_{l_3,k}+\\
&+\sum_k L_{l_1,l_3}^k\,L_{l_2,k}^j-\sum_k L_{l_1,l_2}^k\,L_{l_3,k}^j.
\end{aligned}\tag{3.110}$$

3.111. Arguments for the proof of Theorem 1.7: necessity and sufficiency of (I), (II), (III), (IV). Let us summarize the implications that have been established so far, from the beginning of Section 3. Recall that $m \geq 2$.

- There exist functions X, Y^j of (x,y) transforming the system $y_{xx}^j = F^j(x,y,y_x)$, $j=1,\dots,m$, to the free particle system $Y_{XX}^j = 0$, $j=1,\dots,m$.

$\Downarrow$

- There exist functions Π_{l_1,l_2}^j of (x,y), $0 \leq j,l_1,l_2 \leq m$, satisfying the first auxiliary system (3.38) of partial differential equations.

$\Downarrow$

- There exist (principal unknowns) functions Θ^0, Θ^j satisfying the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).

$\Downarrow$

- The functions G^j, $H_{l_1}^j$, L_{l_1,l_2}^j and M_{l_1,l_2} satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7.

The four families of first order partial differential equations (3.99), (3.100), (3.101) and (3.102) satisfied by the principal unknowns will be called the *second auxiliary system*. It is a complete system.

To achieve the proof of Theorem 1.7, we have to establish the reverse implications. More precisely:

- Some given functions G^j, $H_{l_1}^j$, $L_{l_1,l_2}^j = L_{l_2,l_1}^j$ and $M_{l_1,l_2} = M_{l_2,l_1}$ of (x,y) satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7, or equivalently, the partial differential equations (3.106), (3.108), (3.110) and (3.96).

$\Downarrow$

- There exist functions Θ^0, Θ^j satisfying the second auxiliary system (3.99), (3.100), (3.101) and (3.102).

$\Downarrow$

- These solution functions Θ^0, Θ^j satisfy the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).

$\Downarrow$

- There exist functions $\Pi^j_{l_1,l_2}$ of (x,y), $0 \leq j, l_1, l_2 \leq m$, satisfying the first auxiliary system (3.38) of partial differential equations.

$\Downarrow$

- There exist functions X, Y^j of (x,y) transforming the system $y^j_{xx} = F^j(x,y,y_x)$, $j = 1,\dots,m$, to the free particle system $Y^j_{XX} = 0$, $j = 1,\dots,m$.

The above last three implications have been already implicitely established in the preceding paragraphs, as may be checked by inspecting Lemma 3.40 and the formal computations after §3.62.

Thus, *it remains only to establish the first implication in the above reverse list.* Since the second auxiliary system (3.99), (3.100), (3.101) and (3.102) is complete and of first order, a necessary and sufficient condition for the existence of solutions follows by writing out the following four families of cross-differentiations:

$$\tag{3.112} \left\{ \begin{aligned} 0 &= \left(\Theta^0_x\right)_{y^{l_1}} - \left(\Theta^0_{y^{l_1}}\right)_x, \\ 0 &= \left(\Theta^0_{y^{l_1}}\right)_{y^{l_2}} - \left(\Theta^0_{y^{l_2}}\right)_{y^{l_1}}, \\ 0 &= \left(\Theta^{l_1}_x\right)_{y^{l_2}} - \left(\Theta^{l_1}_{y^{l_2}}\right)_x, \\ 0 &= \left(\Theta^{l_1}_{y^{l_2}}\right)_{y^{l_3}} - \left(\Theta^{l_1}_{y^{l_3}}\right)_{y^{l_2}}. \end{aligned} \right.$$

In the hardest techical part of this paper (Section 4 below), we verify that these four families of compatibility conditions are a consequence of (I), (II), (III) and (IV). For reasons of space, we shall in fact only study the first family of compatibility conditions, *i.e.* the first line of (3.112). In the manuscript [M2003], we have treated the remaining three families of compatibility conditions similarly and completely, up to the very end of every branch of the coral tree of computations. However, we would like to mention that typesetting the remaining three cases would add at least fifty pages of Latex to Section 4. Thus, we prefer to expose thoroughly the treatment of the first family of compatibility conditions, explaining implicitely how to guess the treatment of the remaining three.

§4. COMPATIBILITY CONDITIONS FOR THE SECOND AUXILIARY SYSTEM

So, we have to develope the first line of (3.112): we replace Θ^0_x by its expression (3.99), we differentiate it with respect to y^{l_1}, we replace $\Theta^0_{y^{l_1}}$ by its expression (3.100), we differentiate it with respect to x and we substract. We get:

(4.1)

$$
\begin{aligned}
0 &= \left(\Theta_x^0\right)_{y^{l_1}} - \left(\Theta_{y^{l_1}}^0\right)_x \\
&= \underline{\underline{-2\, G_{y^{l_1} y^{l_1}}^{l_1} + H_{l_1, x y^{l_1}}^{l_1}}} + \\
&\quad + 2 \sum_k G_{y^{l_1}}^k\, L_{l_1,k}^{l_1} + 2 \sum_k G^k\, L_{l_1,k,y^{l_1}}^{l_1} - \sum_k G_{y^{l_1}}^k\, L_{k,k}^k - \sum_k G^k\, L_{k,k,y^{l_1}}^k - \\
&\quad - \frac{1}{2} \sum_k H_{l_1,y^{l_1}}^k\, H_k^{l_1} - \frac{1}{2} \sum_k H_{l_1}^k\, H_{k,y^{l_1}}^{l_1} - \sum_k G_{y^{l_1}}^k\, \Theta^k - \sum_k G^k\, \underline{\Theta_{y^{l_1}}^k} + \\
&\quad + \Theta^0\, \underline{\Theta_{y^{l_1}}^0} - \\
&\quad \underline{\underline{- \frac{2}{3}\, L_{l_1,l_1,xx}^{l_1} + \frac{1}{3}\, H_{l_1,y^{l_1}x}^{l_1}}} - \\
&\quad - \frac{2}{3}\, G_x^{l_1}\, M_{l_1,l_1} - \frac{2}{3}\, G^{l_1}\, M_{l_1,l_1,x} - \frac{4}{3} \sum_k G_x^k\, M_{l_1,k} - \frac{4}{3} \sum_k G^k\, M_{l_1,k,x} + \\
&\quad + \frac{1}{3} \sum_k H_{k,x}^{l_1}\, L_{l_1,l_1}^k + \frac{1}{3} \sum_k H_k^{l_1}\, L_{l_1,l_1,x}^k - \frac{1}{3} \sum_k H_{l_1,x}^k\, L_{l_1,k}^{l_1} - \frac{1}{3} \sum_k H_{l_1}^k\, L_{l_1,k,x}^{l_1} + \\
&\quad + \frac{1}{2} \sum_k H_{l_1,x}^k\, L_{k,k}^k + \frac{1}{2} \sum_k H_{l_1}^k\, L_{k,k,x}^k + \frac{1}{2} \sum_k H_{l_1,x}^k\, \Theta^k + \frac{1}{2} \sum_k H_{l_1}^k\, \underline{\Theta_x^k} - \\
&\quad - \frac{1}{2}\, L_{l_1,l_1,x}\, \Theta^0 - \frac{1}{2}\, L_{l_1,l_1}\, \underline{\Theta_x^0} - \frac{1}{2}\, \underline{\Theta_x^0}\, \Theta^{l_1} - \frac{1}{2}\, \Theta^0\, \underline{\Theta_x^{l_1}}.
\end{aligned}
$$

Here, we underline twice the second order terms. Also, we have underlined once the six terms: $\underline{\Theta_{y^{l_1}}^k}$, $\underline{\Theta_{y^{l_1}}^0}$, $\underline{\Theta_x^k}$, $\underline{\Theta_x^0}$, $\underline{\Theta_x^0}$ and $\underline{\Theta_x^{l_1}}$. They must be replaced by their values given in (3.99), (3.100), (3.101) and (3.102). In this replacement, some double sums appear. As before, we use the first index $k = 1, \ldots, m$ for single summation and then the second index $p = 1, \ldots, m$ for double summation. Finally, we put all the second order terms in the first line, not disturbing the order of appearance of the 73 remaining terms. We get:

(4.2)

$$
\begin{aligned}
0 = & \underline{\underline{-2\, G_{y^{l_1} y^{l_1}}^{l_1} + \frac{4}{3}\, H_{l_1, x y^{l_1}}^{l_1} - \frac{2}{3}\, L_{l_1,l_1,xx}^{l_1}}} + \\
& + \underbrace{2 \sum_k G_{y^{l_1}}^k\, L_{l_1,k}^{l_1}}_{\boxed{4}} + \underbrace{2 \sum_k G^k\, L_{l_1,k,y^{l_1}}^{l_1}}_{\boxed{16}} - \underbrace{\sum_k G_{y^{l_1}}^k\, L_{k,k}^k}_{\boxed{5}} - \underbrace{\sum_k G^k\, L_{k,k,y^{l_1}}^k}_{ⓐ} - \\
& - \underbrace{\frac{1}{2} \sum_k H_{l_1,y^{l_1}}^k\, H_k^{l_1}}_{\boxed{10}} - \underbrace{\frac{1}{2} \sum_k H_{l_1}^k\, H_{k,y^{l_1}}^{l_1}}_{\boxed{11}} - \underbrace{\sum_k G_{y^{l_1}}^k\, \Theta^k}_{(\alpha)} + \\
& + \underbrace{\sum_k G^k\, L_{k,k,y^{l_1}}^k}_{ⓐ} - \underbrace{2 \sum_k G^k\, M_{k,l_1,x}}_{\boxed{18}} - \\
& - \underbrace{\sum_k \sum_p G^k\, H_k^p\, M_{l_1,p}}_{\boxed{19}} - \underbrace{\frac{1}{2} \sum_k G^k\, L_{k,k}^k\, L_{l_1,l_1}^{l_1}}_{ⓑ} + \underbrace{\sum_k \sum_p G^k\, L_{k,l_1}^p\, L_{p,p}^p}_{\boxed{23}} - \underbrace{\frac{1}{2} \sum_k G^k\, L_{k,k}^k\, \Theta^{l_1}}_{ⓒ} - \\
& - \underbrace{\frac{1}{2} \sum_k G^k\, L_{l_1,l_1}^{l_1}\, \Theta^k}_{ⓓ} + \underbrace{\sum_k \sum_p G^k\, L_{k,l_1}^p\, \Theta^p}_{(\varepsilon)} - \underbrace{\sum_k G^k\, M_{k,l_1}\, \Theta^0}_{ⓔ} - \underbrace{\frac{1}{2} \sum_k G^k\, \Theta^k\, \Theta^{l_1}}_{ⓕ} +
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\tfrac{2}{3}\,L^{l_1}_{l_1,l_1,x}\,\Theta^0}_{\textcircled{g}}-\underbrace{\tfrac{1}{3}\,H^{l_1}_{l_1,y^{l_1}}\,\Theta^0}_{\textcircled{h}}+\\
&+\underbrace{\tfrac{2}{3}\,G^{l_1}\,M_{l_1,l_1}\,\Theta^0}_{\textcircled{i}}+\underbrace{\tfrac{4}{3}\sum_k G^k\,M_{l_1,k}\,\Theta^0}_{\textcircled{e}}-\underbrace{\tfrac{1}{3}\sum_k H^{l_1}_k\,L^{l_1}_{l_1,l_1}\,\Theta^0}_{\textcircled{j}}+\underbrace{\tfrac{1}{3}\sum_k H^k_{l_1}\,L^{l_1}_{l_1,k}\,\Theta^0}_{\textcircled{k}}-\\
&-\underbrace{\tfrac{1}{2}\sum_k H^k_{l_1}\,L^k_{k,k}\,\Theta^0}_{\textcircled{l}}-\underbrace{\tfrac{1}{2}\sum_k H^k_{l_1}\,\Theta^k\,\Theta^0}_{\textcircled{m}}+\underbrace{\tfrac{1}{2}\,L^{l_1}_{l_1,l_1}\,\Theta^0\,\Theta^0}_{\textcircled{n}}+\underbrace{\tfrac{1}{2}\,\Theta^0\,\Theta^0\,\Theta^{l_1}}_{\textcircled{o}}-\\
&-\underbrace{\tfrac{2}{3}\,G^{l_1}_x\,M_{l_1,l_1}}_{\boxed{1}}-\underbrace{\tfrac{2}{3}\,G^{l_1}\,M_{l_1,l_1,x}}_{\boxed{17}}-\underbrace{\tfrac{4}{3}\sum_k G^k_x\,M_{l_1,k}}_{\boxed{2}}-\underbrace{\tfrac{4}{3}\sum_k G^k\,M_{l_1,k,x}}_{\boxed{18}}+\\
&+\underbrace{\tfrac{1}{3}\sum_k H^{l_1}_{k,x}\,L^k_{l_1,l_1}}_{\boxed{9}}+\underbrace{\tfrac{1}{3}\sum_k H^{l_1}_k\,L^k_{l_1,l_1,x}}_{\boxed{14}}-\underbrace{\tfrac{1}{3}\sum_k H^k_{l_1,x}\,L^{l_1}_{l_1,k}}_{\boxed{7}}-\underbrace{\tfrac{1}{3}\sum_k H^k_{l_1}\,L^{l_1}_{l_1,k,x}}_{\boxed{13}}+\\
&+\underbrace{\tfrac{1}{2}\sum_k H^k_{l_1,x}\,L^k_{k,k}}_{\boxed{8}}+\underbrace{\tfrac{1}{2}\sum_k H^k_{l_1}\,L^k_{k,k,x}}_{\boxed{15}}+\underbrace{\tfrac{1}{2}\sum_k H^k_{l_1,x}\,\Theta^k}_{\textcircled{\gamma}}-\\
&-\underbrace{\tfrac{1}{3}\sum_k H^k_{l_1}\,H^k_{k,y^k}}_{\boxed{12}}+\underbrace{\tfrac{1}{6}\sum_k H^k_{l_1}\,L^k_{k,k,x}}_{\boxed{15}}+\\
&+\underbrace{\tfrac{2}{3}\sum_k H^k_{l_1}\,G^k\,M_{k,k}}_{\boxed{20}}+\underbrace{\tfrac{1}{3}\sum_k\sum_p H^k_{l_1}\,G^p\,M_{k,p}}_{\boxed{21}}-\underbrace{\tfrac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^k_p\,L^p_{k,k}}_{\boxed{24}}+\underbrace{\tfrac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^k_{k,p}}_{\boxed{25}}-\\
&-\underbrace{\tfrac{1}{4}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^p_{p,p}}_{\boxed{26}}-\underbrace{\tfrac{1}{4}\sum_k\sum_p H^k_{l_1}\,H^p_k\,\Theta^p}_{\textcircled{\eta}}+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,L^k_{k,k}\,\Theta^0}_{\textcircled{l}}+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,\Theta^0\,\Theta^k}_{\textcircled{m}}-\\
&-\underbrace{\tfrac{1}{2}\,L^{l_1}_{l_1,l_1,x}\,\Theta^0}_{\textcircled{g}}+\\
&+\underbrace{G^{l_1}_{y^{l_1}}\,L^{l_1}_{l_1,l_1}}_{\boxed{3}}-\underbrace{\tfrac{1}{2}\,H^{l_1}_{l_1,x}\,L^{l_1}_{l_1,l_1}}_{\boxed{6}}-\\
&-\underbrace{\sum_k G^k\,L^{l_1}_{l_1,l_1}\,L^{l_1}_{l_1,k}}_{\boxed{22}}+\underbrace{\tfrac{1}{2}\sum_k G^k\,L^{l_1}_{l_1,l_1}\,L^k_{k,k}}_{\textcircled{b}}+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,H^{l_1}_k\,L^{l_1}_{l_1,l_1}}_{\boxed{27}}+\underbrace{\tfrac{1}{2}\sum_k G^k\,L^{l_1}_{l_1,l_1}\,\Theta^k}_{\textcircled{d}}-\\
&-\underbrace{\tfrac{1}{4}\,L^{l_1}_{l_1,l_1}\,\Theta^0\,\Theta^0}_{\textcircled{n}}+\\
&+\underbrace{G^{l_1}_{y^{l_1}}\,\Theta^{l_1}}_{\textcircled{\beta}}-\underbrace{\tfrac{1}{2}\,H^{l_1}_{l_1,x}\,\Theta^{l_1}}_{\textcircled{\delta}}-\\
&-\underbrace{\sum_k G^k\,L^{l_1}_{l_1,k}\,\Theta^{l_1}}_{\textcircled{\zeta}}+\underbrace{\tfrac{1}{2}\sum_k G^k\,L^k_{k,k}\,\Theta^{l_1}}_{\textcircled{c}}+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,H^{l_1}_k\,\Theta^{l_1}}_{\textcircled{\theta}}+\underbrace{\tfrac{1}{2}\sum_k G^k\,\Theta^k\,\Theta^{l_1}}_{\textcircled{f}}-\\
&-\underbrace{\tfrac{1}{4}\,\Theta^0\,\Theta^0\,\Theta^{l_1}}_{\textcircled{o}}+\\
&+\underbrace{\tfrac{1}{3}\,H^{l_1}_{l_1,y^{l_1}}\,\Theta^0}_{\textcircled{h}}-\underbrace{\tfrac{1}{6}\,L^{l_1}_{l_1,l_1,x}\,\Theta^0}_{\textcircled{g}}-\\
&-\underbrace{\tfrac{2}{3}\,G^{l_1}\,M_{l_1,l_1}\,\Theta^0}_{\textcircled{i}}-\underbrace{\tfrac{1}{3}\sum_k G^k\,M_{l_1,k}\,\Theta^0}_{\textcircled{e}}+\underbrace{\tfrac{1}{3}\sum_k H^{l_1}_k\,L^k_{l_1,l_1}\,\Theta^0}_{\textcircled{j}}-\underbrace{\tfrac{1}{3}\sum_k H^k_{l_1}\,L^{l_1}_{l_1,k}\,\Theta^0}_{\textcircled{k}}+\\
&+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,L^k_{k,k}\,\Theta^0}_{\textcircled{l}}+\underbrace{\tfrac{1}{4}\sum_k H^k_{l_1}\,\Theta^0\,\Theta^k}_{\textcircled{m}}-\underbrace{\tfrac{1}{4}\,L^{l_1}_{l_1,l_1}\,\Theta^0\,\Theta^0}_{\textcircled{n}}-\underbrace{\tfrac{1}{4}\,\Theta^0\,\Theta^0\,\Theta^{l_1}}_{\textcircled{o}}.
\end{aligned}
$$

As usual, all the terms underlined with the 15 roman alphabetic letters $\mathsf{a}, \mathsf{b}, \dots, \mathsf{n}, \mathsf{o}$ appended vanish evidently. Furthermore, we claim that the eight terms underlined with the 8 Greek alphabetic letters α, β, γ, δ, ε, ζ, η and θ also vanish:

(4.3)
$$0 =?= -\sum_k G^k_{y^{l_1}}\,\Theta^k + G^{l_1}_{y^{l_1}}\,\Theta^{l_1} + \frac{1}{2}\sum_k H^k_{l_1,x}\,\Theta^k - \frac{1}{2}\,H^{l_1}_{l_1,x}\,\Theta^{l_1} + \\ + \sum_k\sum_p G^k\,L^p_{k,l_1}\,\Theta^p - \sum_k G^k\,L^{l_1}_{l_1,k}\,\Theta^{l_1} - \frac{1}{4}\sum_k\sum_p H^k_{l_1}\,H^p_k\,\Theta^p + \frac{1}{4}\sum_k H^k_{l_1}\,H^{l_1}_k\,\Theta^{l_1}.$$

Indeed, it suffices to observe that this identity coincides with

$$0 = \frac{1}{2}\sum_k \Theta^k\left((3.106)\big|_{j:=k;\ l_1:=l_1;\ l_2:=l_1}\right). \tag{4.4}$$

Simplifying then (4.2), we get the explicit formulation of the first family of compatibility conditions for the second auxiliary system:

(4.5)
$$\begin{aligned} 0 =?= \underline{\underline{2\,G^{l_1}_{y^{l_1}y^{l_1}} + \frac{4}{3}\,H^{l_1}_{l_1,xy^{l_1}} - \frac{2}{3}\,L^{l_1}_{l_1,l_1,xx}}} - \\ - \frac{2}{3}\,G^{l_1}_x\,M_{l_1,l_1} - \frac{4}{3}\sum_k G^k_x\,M_{l_1,k} + G^{l_1}_{y^{l_1}}\,L^{l_1}_{l_1,l_1} + \\ + 2\sum_k G^k_{y^{l_1}}\,L^{l_1}_{l_1,k} - \sum_k G^k_{y^{l_1}}\,L^k_{k,k} - \\ - \frac{1}{2}\,H^{l_1}_{l_1,x}\,L^{l_1}_{l_1,l_1} - \frac{1}{3}\sum_k H^k_{l_1,x}\,L^{l_1}_{l_1,k} + \frac{1}{2}\sum_k H^k_{l_1,x}\,L^k_{k,k} + \\ + \frac{1}{3}\sum_k H^{l_1}_{k,x}\,L^k_{l_1,l_1} - \frac{1}{2}\sum_k H^k_{l_1,y^{l_1}}\,H^{l_1}_k - \frac{1}{2}\sum_k H^{l_1}_{k,y^{l_1}}\,H^k_{l_1} - \\ - \frac{1}{3}\sum_k H^k_{k,y^k}\,H^k_{l_1} - \\ - \frac{1}{3}\sum_k L^{l_1}_{l_1,k,x}\,H^k_{l_1} + \frac{1}{3}\sum_k L^k_{l_1,l_1,x}\,H^{l_1}_k + \frac{2}{3}\sum_k L^k_{k,k,x}\,H^k_{l_1} + \\ + 2\sum_k L^{l_1}_{l_1,k,y^{l_1}}\,G^k - \\ - \frac{2}{3}\,M_{l_1,l_1,x}\,G^{l_1} - \frac{10}{3}\sum_k M_{k,l_1,x}\,G^k - \\ - \sum_k\sum_p G^k\,H^p_k\,M_{l_1,p} + \frac{2}{3}\sum_k G^k\,H^k_{l_1}\,M_{k,k} + \frac{1}{3}\sum_k\sum_p G^p\,H^k_{l_1}\,M_{k,p} - \\ - \sum_k G^k\,L^{l_1}_{l_1,k}\,L^{l_1}_{l_1,l_1} + \sum_k\sum_p G^k\,L^p_{k,l_1}\,L^p_{p,p} - \\ - \frac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^k_p\,L^p_{k,k} + \frac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^k_{k,p} - \\ - \frac{1}{4}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^p_{p,p} + \frac{1}{4}\sum_k H^k_{l_1}\,H^{l_1}_k\,L^{l_1}_{l_1,l_1}. \end{aligned}$$

We can now state the main technical lemma of this section and of this paper.

Lemma 4.6. *The second order partial differential relations* (4.5) *hold true for* $l = 1, \dots, m$*, and they are a consequence, by differentiations and by linear combinations, of the fundamental first order partial differential equations* (3.106), (3.108), (3.110) *and* (3.96).

4.7. Reconstitution of the appropriate linear combinations. The remaining of Section 4 is entirely devoted to the proof of this statement. From the manual computational point of view, the difficulty of the task is due to the fact that one has to manipulate formal expressions having from 10 to 50 terms. So the real question is: *how can we reconstitute the linear combinations and the differentiations which lead to the goal* (4.5) *from the data* (3.106), (3.108), (3.110) *and* (3.96)?.

The main trick is to first neglect the first order and the zero order terms in the goal (4.5). Using the symbol "$\equiv$" to denote "*modulo first order and the zero order terms*", we formulate the following sub-goal:

$$0 \equiv ?\equiv \underline{\underline{-2\,G_{y^{l_1}y^{l_1}}^{l_1} + \frac{4}{3}\,H_{l_1,xy^{l_1}}^{l_1} - \frac{2}{3}\,L_{l_1,l_1,xx}^{l_1}}}, \tag{4.8}$$

for $l_1 = 1,\dots,m$. Before estabilishing that these partial differential relations are a consequence of the data (3.106), (3.108), (3.110) and (3.96) (written with a similar sign $\equiv$), let us check that they are a consequence of the existence of the change of coordinates $(x,y)\mapsto(X,Y)$ (however, recall that, in establishing the reverse implications of §3.111, we still do not know that such a change of coordinates really exists); this will confirm the coherence and the validity of our computations. Importantly, we have been able to achieve systematic corrections of our computations by always checking them alongside with the existence of the change of coordinates $(x,y)\mapsto(X,Y)$.

Coming back to the definition (3.35) and to the approximation (3.58), we have:

$$\begin{aligned} G^{l_1} &= -\square_{xx}^{l_1} \cong -Y_{xx}^{l_1}, \\ H_{l_1}^{l_1} &= -2\square_{xy^{l_1}}^{l_1} + \square_{xx}^{0} \cong -2\,Y_{xy^{l_1}}^{l_1} + X_{xx}, \\ L_{l_1,l_1}^{l_1} &= -\square_{y^{l_1}y^{l_1}}^{l_1} + 2\,\square_{xy^{l_1}}^{0} \cong -Y_{y^{l_1}y^{l_1}}^{l_1} + 2\,X_{xy^{l_1}}. \end{aligned} \tag{4.9}$$

Differentiating the first two lines with respect to y^{l_1} and the third line with respect to x, and replacing the sign $\cong$ by the sign $\equiv$ (in a non-rigorous way, this corresponds essentially to neglecting the derivatives of order 0, 1, 2 and 3 of X, Y^j and to neglecting the difference between the Jacobian matrix of the transformation and the identity matrix), we get:

$$\begin{aligned} \underline{\underline{G_{y^{l_1}y^{l_1}}^{l_1}}} &\equiv -Y_{xxy^{l_1}y^{l_1}}^{l_1}, \\ \underline{\underline{H_{l_1,xy^{l_1}}^{l_1}}} &\equiv -2\,Y_{xy^{l_1}xy^{l_1}}^{l_1} + X_{xxxy^{l_1}}, \\ \underline{\underline{L_{l_1,l_1,xx}^{l_1}}} &\equiv -Y_{y^{l_1}y^{l_1}xx}^{l_1} + 2\,X_{xy^{l_1}xx}. \end{aligned} \tag{4.10}$$

Hence the linear combination $-2\cdot(4.10)_1 + \frac{4}{3}\cdot(4.10)_2 - \frac{2}{3}(4.10)_3$ yields the desired result:

$$\begin{aligned} 0 &\equiv ?\equiv \underline{\underline{-2\,G_{y^{l_1}y^{l_1}}^{l_1} + \frac{4}{3}\,H_{l_1,xy^{l_1}}^{l_1} - \frac{2}{3}\,L_{l_1,l_1,xx}^{l_1}}} \\ &\equiv \underline{2\,Y_{xxy^{l_1}y^{l_1}}^{l_1}}_{ⓐ} - \underline{\frac{8}{3}\,Y_{xy^{l_1}xy^{l_1}}^{l_1}}_{ⓐ} + \underline{\frac{4}{3}\,X_{xxxy^{l_1}}}_{ⓑ} + \underline{\frac{2}{3}\,Y_{y^{l_1}y^{l_1}xx}^{l_1}}_{ⓐ} - \underline{\frac{4}{3}\,X_{xy^{l_1}xx}}_{ⓑ} \\ &\equiv 0, \quad \text{indeed!} \end{aligned} \tag{4.11}$$

Thanks to this straightforward computation, we guess that the approximate partial differential relations (4.8) are a consequence of the approximate relations (3.106), (3.108), (3.110) and (3.96), namely:

$$
\begin{aligned}
(3.106)^{\rm mod}: \quad & 0 \equiv -2\,G_{y^{l_1}}^j + 2\,\delta_{l_1}^j\,G_{y^{l_2}}^{l_2} + H_{l_1,x}^j - \delta_{l_1}^j\,H_{l_2,x}^{l_2}, \\
(3.108)^{\rm mod}: \quad & 0 \equiv -\frac{1}{2}\,H_{l_1,y^{l_2}}^j + \frac{1}{6}\,\delta_{l_1}^j\,H_{l_2,y^{l_2}}^{l_2} + \frac{1}{3}\,\delta_{l_2}^j\,H_{l_1,y^{l_1}}^{l_1} + \\
& \qquad + L_{l_1,l_2,x}^j - \frac{1}{3}\,\delta_{l_1}^j\,L_{l_2,l_2,x}^{l_2} - \frac{2}{3}\,\delta_{l_2}^j\,L_{l_1,l_1,x}^{l_1}, \\
(3.110)^{\rm mod}: \quad & 0 \equiv L_{l_1,l_2,y^{l_3}}^j - L_{l_1,l_3,y^{l_2}}^j + \delta_{l_3}^j\,M_{l_1,l_2,x} - \delta_{l_2}^j\,M_{l_1,l_3,x}, \\
(3.96)^{\rm mod}: \quad & 0 \equiv M_{l_1,l_2,y^{l_3}} - M_{l_1,l_3,y^{l_2}}.
\end{aligned}
\tag{4.12}
$$

Here, the sign $\equiv$ means "*modulo zero order terms*". Before proceding further, recall the correspondence between partial differential relations:

$$
\begin{aligned}
&(\mathrm{I}) = (3.106), \\
&(\mathrm{II}) = (3.108), \\
&(\mathrm{III}) = (3.110), \\
&(\mathrm{IV}) = (3.96).
\end{aligned}
\tag{4.13}
$$

However, these couples of equivalent identities are written slightly differently, as may be read by comparison. To fix ideas and to facilitate the eye-checking of our subsequent computations, *we shall only use and refer to the exact writing of* (3.106), *of* (3.108), *of* (3.110) *and of* (3.96).

4.14. Construction of a guide. So we want to show that the approximate relation (4.8) is a consequence, by differentiations and by linear combinations, of the approximate identities (4.12). The interest of working with approximate identities is that formal computations are lightened substantially. After having discovered which linear combinations and which differentiations are appropriate, *i.e.* after having constructed a "guide", in §4.22 below, we shall write down the complete computations, including all zero order terms, following our guide.

We shall use two indices l_1 and l_2 with $1 \leq l_1, l_2 \leq m$ and, crucially, $l_2 \neq l_1$. Again, the assumption $m \geq 2$ is used strongly.

Firstly, put $j := l_1$ in $(3.106)^{\rm mod}$ with $l_2 \neq l_1$ and differentiate with respect to y^{l_1}:

$$
0 \equiv -2\,G_{y^{l_1}y^{l_1}}^{l_1} + 2\,G_{y^{l_2}y^{l_1}}^{l_2} + H_{l_1,xy^{l_1}}^{l_1} - H_{l_2,xy^{l_1}}^{l_2}.
\tag{4.15}
$$

Secondly, put $j := l_2$ in $(3.106)^{\rm mod}$ with $l_1 \neq l_2$ and differentiate with respect to y^{l_2}:

$$
0 \equiv -2\,G_{y^{l_1}y^{l_2}}^{l_2} + H_{l_1,xy^{l_2}}^{l_2}.
\tag{4.16}
$$

Thirdly, put $j := l_2$ in $(3.108)^{\rm mod}$ with $l_1 \neq l_2$ and differentiate with respect to x:

$$
0 \equiv -\frac{1}{2}\,H_{l_1,y^{l_2}x}^{l_2} + \frac{1}{3}\,H_{l_1,y^{l_1}x}^{l_1} + L_{l_1,l_2,xx}^{l_2} - \frac{2}{3}\,L_{l_1,l_1,xx}^{l_1}.
\tag{4.17}
$$

Fourthly, put $j := l_1$ in $(3.108)^{\rm mod}$ with $l_2 \neq l_1$ and differentiate with respect to x:

$$
0 \equiv -\frac{1}{2}\,H_{l_1,y^{l_2}x}^{l_1} + \frac{1}{6}\,H_{l_2,y^{l_2}x}^{l_2} + L_{l_1,l_2,xx}^{l_1} - \frac{1}{3}\,L_{l_2,l_2,xx}^{l_2}.
\tag{4.18}
$$

Fithly, permute the indices $(l_1, l_2) \mapsto (l_2, l_1)$:

$$
0 \equiv -\frac{1}{2}\,H_{l_2,y^{l_1}x}^{l_2} + \frac{1}{6}\,H_{l_1,y^{l_1}x}^{l_1} + L_{l_2,l_1,xx}^{l_2} - \frac{1}{3}\,L_{l_1,l_1,xx}^{l_1}.
\tag{4.19}
$$

Finally, compute the linear combination $(4.15) + (4.16) + 2\cdot(4.17) - 2\cdot(4.19)$:

$$
\begin{aligned}
0 \equiv & -\underline{2\,G^{l_1}_{y^{l_1}y^{l_1}}}_{\boxed{1}} + \underline{2\,G^{l_2}_{y^{l_1}y^{l_2}}}_{\text{ⓐ}} + \underline{H^{l_1}_{l_1,xy^{l_1}}}_{\boxed{2}} - \underline{H^{l_2}_{l_2,xy^{l_1}}}_{\text{ⓑ}} - \\
& - \underline{2\,G^{l_2}_{y^{l_1}y^{l_2}}}_{\text{ⓐ}} + \underline{H^{l_2}_{l_1,xy^{l_2}}}_{\text{ⓒ}} - \\
& - \underline{H^{l_2}_{l_1,xy^{l_2}}}_{\text{ⓒ}} + \underline{\frac{2}{3}\,H^{l_1}_{l_1,l_1,xy^{l_1}}}_{\boxed{2}} + \underline{2\,L^{l_2}_{l_1,l_2,xx}}_{\text{ⓓ}} - \underline{\frac{4}{3}\,L^{l_1}_{l_1,l_1,xx}}_{\boxed{3}} + \\
& + \underline{H^{l_2}_{l_2,xy^{l_1}}}_{\text{ⓑ}} - \underline{\frac{1}{3}\,H^{l_1}_{l_1,xy^{l_1}}}_{\boxed{2}} - \underline{2\,L^{l_2}_{l_2,l_1,xx}}_{\text{ⓓ}} + \underline{\frac{2}{3}\,L^{l_1}_{l_1,l_1,xx}}_{\boxed{3}}.
\end{aligned}
\tag{4.20}
$$

We indeed get the desired approximate identity:

$$
0 \equiv -2\,G^{l_1}_{y^{l_1}y^{l_1}} + \frac{4}{3}\,H^{l_1}_{l_1,xy^{l_1}} - \frac{2}{3}\,L^{l_1}_{l_1,l_1,xx}.
\tag{4.21}
$$

4.22. Complete computation. Now that the guide is constructed, we can achieve the complete computations.

Firstly, put $j := l_1$ in (3.106) with $l_2 \neq l_1$ and differentiate with respect to y^{l_1}:

$$
\begin{aligned}
0 \equiv & \underline{\underline{-2\,G^{l_1}_{y^{l_1}y^{l_1}} + 2\,G^{l_2}_{y^{l_2}y^{l_1}} + H^{l_1}_{l_1,xy^{l_1}} - H^{l_2}_{l_2,xy^{l_1}}}} + \\
& + 2\sum_k G^k_{y^{l_1}}\,L^{l_1}_{l_1,k} + 2\sum_k G^k\,L^{l_1}_{l_1,k,y^{l_1}} - 2\sum_k G^k_{y^{l_1}}\,L^{l_2}_{l_2,k} - 2\sum_k G^k\,L^{l_2}_{l_2,k,y^{l_1}} - \\
& - \frac{1}{2}\sum_k H^k_{l_1,y^{l_1}}\,H^{l_1}_k - \frac{1}{2}\sum_k H^k_{l_1}\,H^{l_1}_{k,y^{l_1}} + \frac{1}{2}\sum_k H^k_{l_2,y^{l_1}}\,H^{l_2}_k + \frac{1}{2}\sum_k H^k_{l_2}\,H^{l_2}_{k,y^{l_1}}.
\end{aligned}
\tag{4.23}
$$

Secondly, put $j := l_2$ in (3.106) with $l_1 \neq l_2$ and differentiate with respect to y^{l_2}:

$$
\begin{aligned}
0 \equiv & \underline{\underline{-2\,G^{l_2}_{y^{l_1}y^{l_2}} + H^{l_2}_{l_1,xy^{l_2}}}} + \\
& 2\sum_k G^k_{y^{l_2}}\,L^{l_2}_{l_1,k} + 2\sum_k G^k\,L^{l_2}_{l_1,k,y^{l_2}} - \frac{1}{2}\sum_k H^k_{l_1,y^{l_2}}\,H^{l_2}_k - \frac{1}{2}\sum_k H^k_{l_1}\,H^{l_2}_{k,y^{l_2}}.
\end{aligned}
\tag{4.24}
$$

Thirdly, put $j := l_2$ in (3.108) with $l_1 \neq l_2$ and differentiate with respect to x:

$$
\begin{aligned}
0 \equiv & \underline{\underline{-\frac{1}{2}\,H^{l_2}_{l_1,y^{l_2}x} + \frac{1}{3}\,H^{l_1}_{l_1,y^{l_1}x} + L^{l_2}_{l_1,l_2,xx} - \frac{2}{3}\,L^{l_1}_{l_1,l_1,xx}}} + \\
& + G^{l_2}_x\,M_{l_1,l_2} + G^{l_2}_x\,M_{l_1,l_2,x} - \frac{2}{3}\,G^{l_1}_x\,M_{l_1,l_1} - \frac{2}{3}\,G^{l_1}\,M_{l_1,l_1,x} - \\
& - \frac{1}{3}\sum_k G^k_x\,M_{l_1,k} - \frac{1}{3}\sum_k G^k\,M_{l_1,k,x} - \frac{1}{2}\sum_k H^{l_2}_{k,x}\,L^k_{l_1,l_2} - \frac{1}{2}\sum_k H^{l_2}_k\,L^k_{l_1,l_2,x} + \\
& + \frac{1}{2}\sum_k H^k_{l_1,x}\,L^{l_2}_{l_2,k} + \frac{1}{2}\sum_k H^k_{l_1}\,L^{l_2}_{l_2,k,x} + \frac{1}{3}\sum_k H^{l_1}_{k,x}\,L^k_{l_1,l_1} + \frac{1}{3}\sum_k H^{l_1}_k\,L^k_{l_1,l_1,x} - \\
& - \frac{1}{3}\sum_k H^k_{l_1,x}\,L^{l_1}_{l_1,k} - \frac{1}{3}\sum_k H^k_{l_1}\,L^{l_1}_{l_1,k,x}.
\end{aligned}
\tag{4.25}
$$

Fourthly, put $j := l_1$ in (3.108) with $l_2 \neq l_1$, differentiate with respect to x and permute the indices $(l_1, l_2) \mapsto (l_2, l_1)$:

(4.26)

$$
\begin{aligned}
0 \equiv & -\underline{\underline{\frac{1}{2}\, H_{l_2,y^{l_1}x}^{l_2} + \frac{1}{6}\, H_{l_1,y^{l_1}x}^{l_1} + L_{l_2,l_1,xx}^{l_2} - \frac{1}{3}\, L_{l_1,l_1,xx}^{l_1}}} + \\
& + G_x^{l_2}\, M_{l_2,l_1} + G^{l_2}\, M_{l_2,l_1,x} - \frac{1}{3}\, G_x^{l_1}\, M_{l_1,l_1} - \frac{1}{3}\, G^{l_1}\, M_{l_1,l_1,x} + \\
& + \frac{1}{3}\sum_k G_x^k\, M_{l_1,k} + \frac{1}{3}\sum_k G^k\, M_{l_1,k,x} - \frac{1}{2}\sum_k H_{k,x}^{l_2}\, L_{l_2,l_1}^k - \frac{1}{2}\sum_k H_k^{l_2}\, L_{l_2,l_1,x}^k + \\
& + \frac{1}{2}\sum_k H_{l_2,x}^k\, L_{l_1,k}^{l_2} + \frac{1}{2}\sum_k H_{l_2}^k\, L_{l_1,k,x}^{l_2} + \frac{1}{6}\sum_k H_{k,x}^{l_1}\, L_{l_1,l_1}^k + \frac{1}{6}\sum_k H_k^{l_1}\, L_{l_1,l_1,x}^k - \\
& - \frac{1}{6}\sum_k H_{l_1,x}^k\, L_{l_1,k}^{l_1} - \frac{1}{6}\sum_k H_{l_1}^k\, L_{l_1,k,x}^{l_1}.
\end{aligned}
$$

Finally, compute the linear combination $(4.23) + (4.24) + 2 \cdot (4.25) - 2 \cdot (4.26)$:

(4.27)

$$
\begin{aligned}
0 = & \underline{\underline{-2\, G_{y^{l_1}y^{l_1}}^{l_1} + \frac{4}{3}\, H_{l_1,xy^{l_1}}^{l_1} - \frac{2}{3}\, L_{l_1,l_1,xx}^{l_1}}} - \\
& - \frac{2}{3}\, G_x^{l_1}\, M_{l_1,l_1} - \frac{4}{3}\sum_k G_x^k\, M_{l_1,k} + \\
& + 2\sum_k G_{y^{l_1}}^k\, L_{l_1,k}^{l_1} - 2\sum_k G_{y^{l_1}}^k\, L_{l_2,k}^{l_2} + 2\sum_k G_{y^{l_2}}^k\, L_{l_1,k}^{l_2} + \\
& + \sum_k H_{l_1,x}^k\, L_{l_2,k}^{l_2} + \frac{1}{3}\sum_k H_{k,x}^{l_1}\, L_{l_1,l_1}^k - \frac{1}{3}\sum_k H_{l_1,x}^k\, L_{l_1,k}^{l_1} - \sum_k H_{l_2,x}^k\, L_{l_1,k}^{l_2} + \\
& + \frac{1}{2}\sum_k H_{l_2,y^{l_1}}^k\, H_k^{l_2} + \frac{1}{2}\sum_k H_{k,y^{l_1}}^{l_2}\, H_{l_2}^k - \frac{1}{2}\sum_k H_{l_1,y^{l_1}}^k\, H_k^{l_1} - \\
& - \frac{1}{2}\sum_k H_{l_1}^k\, H_{k,y^{l_1}}^{l_1} - \frac{1}{2}\sum_k H_{l_1,y^{l_2}}^k\, H_k^{l_1} - \frac{1}{2}\sum_k H_{l_1}^k\, H_{k,y^{l_2}}^{l_2} + \\
& + \sum_k L_{l_2,k,x}^{l_2}\, H_{l_1}^k + \frac{1}{3}\sum_k L_{l_1,l_1,x}^k\, H_k^{l_1} - \frac{1}{3}\sum_k L_{l_1,k,x}^{l_1}\, H_{l_1}^k - \sum_k L_{l_1,k,x}^{l_2}\, H_{l_2}^k + \\
& + 2\sum_k L_{l_1,k,y^{l_1}}^{l_1}\, G^k + 2\sum_k L_{l_1,k,y^{l_2}}^{l_2}\, G^k - 2\sum_k L_{l_2,k,y^{l_1}}^{l_2}\, G^k - \\
& - \frac{2}{3}\, M_{l_1,l_1,x}\, G^{l_1} - \frac{4}{3}\sum_k M_{l_1,k,x}\, G^k.
\end{aligned}
$$

In this partial differential relation, importantly, the second order terms are exactly the same as in our goal (4.5). Unfortunately, the first order and the zero order terms are not the same.

4.28. Formulation of a new goal. Thus, in order to get rid of the second order expression $2\, G_{y^{l_1}y^{l_1}}^{l_1} + \frac{4}{3}\, H_{l_1,xy^{l_1}}^{l_1} - \frac{2}{3}\, L_{l_1,l_1,xx}^{l_1}$, we substract: $(4.5) - (4.27)$. In the result, we write the first order terms in a certain way, adapted in advance to our subsequent computations. For this substraction yielding (4.29) just below, we have not underlined the terms in (4.5) and in (4.27). However, they may be underlined with a pencil to check that the result (4.29) is correct. We get:

(4.29)

$$
\begin{aligned}
0 &=?= \\
&= \underline{-\sum_k G_{y^{l_1}}^k\, L_{k,k}^k + G_{y^{l_1}}^{l_1}\, L_{l_1,l_1}^{l_1} + \frac{1}{2}\sum_k H_{l_1,x}^k\, L_{k,k}^k - \frac{1}{2}\, H_{l_1,x}^{l_1}\, L_{l_1,l_1}^{l_1}} + \\
&\quad \underline{+ 2\sum_k G_{y^{l_1}}^k\, L_{l_2,k}^{l_2} - \sum_k H_{l_1,x}^k\, L_{l_2,k}^{l_2}} - \\
&\quad \underline{- 2\sum_k G_{y^{l_2}}^k\, L_{l_1,k}^{l_2} + \sum_k H_{l_2,x}^k\, L_{l_1,k}^{l_2}} + \\
&\quad \underline{+ \frac{1}{2}\sum_k H_{k,y^{l_2}}^{l_2}\, H_{l_1}^k - \frac{1}{3}\sum_k H_{k,y^k}^k\, H_{l_1}^k - \sum_k L_{l_2,k,x}^{l_2}\, H_{l_1}^k + \frac{2}{3}\sum_k L_{k,k,x}^k\, H_{l_1}^k} - \\
&\quad \underline{- \frac{1}{2}\sum_k H_{k,y^{l_1}}^{l_2}\, H_{l_2}^k + \sum_k L_{l_1,k,x}^{l_2}\, H_{l_2}^k} + \\
&\quad \underline{+ \frac{1}{2}\sum_k H_{l_1,y^{l_2}}^k\, H_k^{l_2} - \frac{1}{2}\sum_k H_{l_2,y^{l_1}}^k\, H_k^{l_2}} + \\
&\quad \underline{+ 2\sum_k L_{l_2,k,y^{l_1}}^{l_2}\, G^k - 2\sum_k L_{l_1,k,y^{l_2}}^{l_2}\, G^k - 2\sum_k M_{k,l_1,x}\, G^k} - \\
&\quad - \sum_k\sum_p G^k\, H_k^p\, M_{l_1,p} + \frac{2}{3}\sum_k G^k\, H_{l_1}^k\, M_{k,k} + \frac{1}{3}\sum_k\sum_p G^p\, H_{l_1}^k\, M_{k,p} - \\
&\quad - \sum_k G^k\, L_{l_1,l_1}^{l_1}\, L_{l_1,k}^{l_1} + \sum_k\sum_p G^k\, L_{k,l_1}^p\, L_{p,p}^p - \frac{1}{3}\sum_k\sum_p H_{l_1}^k\, H_p^k\, L_{k,k}^p + \\
&\quad + \frac{1}{3}\sum_k\sum_p H_{l_1}^k\, H_k^p\, L_{k,p}^k - \frac{1}{4}\sum_k\sum_p H_{l_1}^k\, H_k^p\, L_{p,p}^p + \frac{1}{4}\sum_k H_{l_1}^k\, H_k^{l_1}\, L_{l_1,l_1}^{l_1}.
\end{aligned}
$$

We have underlined plainly the first order terms appearing in lines 1, 2, 3, 4, 5, 6 and 7.

4.30. Reconstitution of the subgoal (4.29) **from** (3.106)**, from** (3.108) **and from** (3.110)**.** Now, it suffices to establish that the first order partial differential relations (4.29) for $1 \leq l_1, l_2 \leq m$ and $l_2 \neq l_1$ (crucial assumption) are a consequence of (3.106), of (3.108) and of (3.110) by linear combinations. The auxiliary index l_2, which is absent in the goal (4.5), will disappear at the end. Differentiations will not be applied anymore. Also, the partial differential relations (3.96), which were not used above, will neither be used in the sequel. However, they are strongly used in the treatment of the remaining three compatibility conditions $(3.112)_2$, $(3.112)_3$ and $(3.112)_4$, the detail of which we do not copy in the typesetted paper ([M2003]). Finally, the construction of a guide for the subgoal (4.29) may be guessed similarly as in §4.14 above. We shall provide the final computations directly, without any guide: they consists of the *seven* partial differential relations (4.32), (4.33), (4.35), (4.37), (4.39), (4.43) and (4.45) below. At the end, we shall make the addition (4.47) below, producing the desired subgoal (4.29) := (4.32) + (4.33) + (4.35) + (4.37) + (4.39) + (4.43) + (4.45), with the numerotation of terms corresponding to the order of appearance of the terms of (4.29), as usual.

Firstly, put $j := k$, $l_1 := l_1$ and $l_2 := l_1$ in (3.106):

$$
\begin{aligned}
0 &= \underline{-2\, G_{y^{l_1}}^k + 2\,\delta_{l_1}^k\, G_{y^{l_1}}^{l_1} + H_{l_1,x}^k - \delta_{l_1}^k\, H_{l_1,x}^{l_1}} + \\
&\quad + 2\sum_p G^p\, L_{l_1,p}^k - 2\,\delta_{l_1}^k \sum_p G^p\, L_{l_1,p}^{l_1} - \frac{1}{2}\sum_p H_{l_1}^p\, H_p^k + \frac{1}{2}\,\delta_{l_1}^k \sum_p H_{l_1}^p\, H_p^{l_1}.
\end{aligned}
\tag{4.31}
$$

Apply the operator $\frac{1}{2}\sum_k L^k_{k,k}(\cdot)$ to the preceding equality, namely compute $\frac{1}{2}\sum_k L^k_{k,k}\cdot(4.31)$. This yields:

$$
\begin{aligned}
0 = & \underline{-\sum_k G^k_{y^{l_1}}\, L^k_{k,k} + G^{l_1}_{y^{l_1}} + \frac{1}{2}\sum_k H^k_{l_1,x}\, L^k_{k,k} - \frac{1}{2}\, H^{l_1}_{l_1,x}\, L^{l_1}_{l_1,l_1}} + \\
& + \sum_k \sum_p G^p\, L^k_{k,k}\, L^k_{l_1,p} - \sum_p G^p\, L^{l_1}_{l_1,l_1}\, L^k_{l_1,p} - \frac{1}{4}\sum_k\sum_p H^p_{l_1}\, H^k_p\, L^k_{k,k} + \\
& + \frac{1}{4}\sum_p H^p_{l_1}\, H^{l_1}_p\, L^{l_1}_{l_1,l_1}.
\end{aligned}
\tag{4.32}
$$

Secondly, apply the operator $-\sum_k L^{l_2}_{l_2,k}(\cdot)$ to (4.32), namely compute $-\sum_k L^{l_2}_{l_2,k}\cdot(4.32)$. This yields:

$$
\begin{aligned}
0 = & \underline{2\sum_k G^k_{y^{l_1}}\, L^{l_2}_{l_2,k} - 2\, G^{l_1}_{y^{l_1}}\, L^{l_2}_{l_2,l_1} - \sum_k H^k_{l_1,x}\, L^{l_2}_{l_2,k} + H^{l_1}_{l_1,x}\, L^{l_2}_{l_2,l_1}} - \\
& - 2\sum_k\sum_p G^p\, L^{l_2}_{l_2,k}\, L^k_{l_1,p} + 2\sum_p G^p\, L^{l_2}_{l_2,l_1}\, L^{l_1}_{l_1,p} + \frac{1}{2}\sum_k\sum_p H^p_{l_1}\, H^k_p\, L^{l_2}_{l_2,k} - \\
& - \frac{1}{2}\sum_p H^p_{l_1}\, H^{l_1}_p\, L^{l_2}_{l_2,l_1}.
\end{aligned}
\tag{4.33}
$$

Thirdly, put $j := k$, $l_1 := l_2$ and $l_2 := l_1$ with $l_2 \neq l_1$ in (3.106):

$$
\begin{aligned}
0 = & \underline{-2\, G^k_{y^{l_2}} + 2\,\delta^k_{l_2}\, G^{l_1}_{y^{l_1}} + H^k_{l_2,x} - \delta^k_{l_2}\, H^{l_1}_{l_1,x}} + \\
& + 2\sum_p G^p\, L^k_{l_2,p} - 2\,\delta^k_{l_2}\sum_p G^p\, L^{l_1}_{l_1,p} - \frac{1}{2}\sum_p H^p_{l_2}\, H^k_p + \frac{1}{2}\,\delta^k_{l_2}\sum_p H^p_{l_1}\, H^{l_1}_p.
\end{aligned}
\tag{4.34}
$$

Next, apply the operator $\sum_k L^{l_2}_{l_1,k}(\cdot)$ to (4.34), namely compute $\sum_k L^{l_2}_{l_1,k}\cdot(4.34)$. This yields:

$$
\begin{aligned}
0 = & \underline{-2\sum_k G^k_{y^{l_2}}\, L^{l_2}_{l_1,k} + 2\, G^{l_1}_{y^{l_1}}\, L^{l_2}_{l_1,l_2} + \sum_k H^k_{l_2,x}\, L^{l_2}_{l_1,k} - H^{l_1}_{l_1,x}\, L^{l_2}_{l_1,l_2}} + \\
& + 2\sum_k\sum_p G^p\, L^k_{l_2,p}\, L^{l_2}_{l_1,k} - 2\sum_p G^p\, L^{l_1}_{l_1,p}\, L^{l_2}_{l_1,l_2} - \frac{1}{2}\sum_k\sum_p H^p_{l_2}\, H^k_p\, L^{l_2}_{l_1,k} + \\
& + \frac{1}{2}\sum_p H^p_{l_1}\, H^{l_1}_p\, L^{l_2}_{l_1,l_2}.
\end{aligned}
\tag{4.35}
$$

Fourthly, put $j := l_2$, $l_1 := k$ and $l_2 := l_2$ in (3.108):

$$
\begin{aligned}
0 = & \underline{-\frac{1}{2}\, H^{l_2}_{k,y^{l_2}} + \frac{1}{6}\,\delta^{l_2}_k\, H^{l_2}_{l_2,y^{l_2}} + \frac{1}{3}\, H^k_{k,y^k} + L^{l_2}_{k,l_2,x}} - \\
& \underline{- \frac{1}{3}\,\delta^k_{l_2}\, L^{l_2}_{l_2,l_2,x} - \frac{2}{3}\, L^k_{k,k,x}} + \\
& + G^{l_2}\, M_{k,l_2} - \frac{1}{3}\,\delta^k_{l_2}\, G^{l_2}\, M_{l_2,l_2} - \frac{2}{3}\, G^k\, M_{k,k} + \frac{1}{3}\,\delta^{l_2}_k\sum_p G^p\, M_{l_2,p} - \\
& - \frac{1}{3}\sum_p G^p\, M_{k,p} - \frac{1}{2}\sum_p H^{l_2}_p\, L^p_{k,l_2} + \frac{1}{2}\sum_p H^p_k\, L^{l_2}_{l_2,p} + \frac{1}{6}\,\delta^{l_2}_k\sum_p H^{l_2}_p\, L^p_{l_2,l_2} - \\
& - \frac{1}{6}\,\delta^{l_2}_k\sum_p H^p_{l_2}\, L^{l_2}_{l_2,p} + \frac{1}{3}\sum_p H^k_p\, L^p_{k,k} - \frac{1}{3}\sum_p H^p_k\, L^k_{k,p}.
\end{aligned}
\tag{4.36}
$$

Next, apply the operator $-\sum_k H_{l_1}^k(\cdot)$ to (4.), namely compute $-\sum_k H_{l_1}^k \cdot (4.36)$. This yields:

$$
\begin{aligned}
0 = & \underline{\frac{1}{2}\sum_k H^{l_2}_{k,y^{l_2}}\, H^k_{l_1} - \frac{1}{6} H^{l_2}_{l_2,y^{l_2}}\, H^{l_2}_{l_1} - \frac{1}{3}\sum_k H^k_{k,y^k}\, H^k_{l_1} -} \\
& \underline{-\sum_k L^{l_2}_{k,l_2,x}\, H^k_{l_1} + \frac{1}{3} L^{l_2}_{l_2,l_2,x}\, H^{l_2}_{l_1} + \frac{2}{3}\sum_k L^k_{k,k,x}\, H^k_{l_1} -} \\
& -\sum_k G^{l_2}\, H^k_{l_1}\, M_{k,l_2} + \frac{1}{3} G^{l_2}\, H^{l_2}_{l_1}\, M_{l_2,l_2} + \frac{2}{3}\sum_k G^k\, H^k_{l_1}\, M_{k,k} - \\
& -\frac{1}{3}\sum_p G^p\, H^{l_2}_{l_1}\, M_{l_2,p} + \frac{1}{3}\sum_k\sum_p G^p H^k_{l_1}\, M_{k,p} + \frac{1}{2}\sum_k\sum_p H^k_{l_1}\, H^{l_2}_p\, L^p_{k,l_2} - \\
& -\frac{1}{2}\sum_k\sum_p H^k_{l_1}\, H^p_k\, L^{l_2}_{l_2,p} - \frac{1}{6}\sum_p H^{l_2}_{l_1}\, H^{l_2}_p\, L^p_{l_2,l_2} + \frac{1}{6}\sum_p H^{l_2}_{l_1}\, H^p_{l_2}\, L^{l_2}_{l_2,p} - \\
& -\frac{1}{3}\sum_k\sum_p H^k_{l_1}\, H^k_p\, L^p_{k,k} + \frac{1}{3}\sum_k\sum_p H^k_{l_1}\, H^p_k\, L^k_{k,p}.
\end{aligned}
\tag{4.37}
$$

Fithly, put $j := l_2$, $l_1 := k$ and $l_2 := l_1$ in (3.108):

$$
\begin{aligned}
0 = & \underline{-\frac{1}{2} H^{l_2}_{k,y^{l_1}} + \frac{1}{6}\delta^{l_2}_k\, H^{l_1}_{l_1,y^{l_1}} + L^{l_2}_{k,l_1,x} - \frac{1}{3}\delta^{l_2}_k\, L^{l_1}_{l_1,l_1,x} +} \\
& + G^{l_2}\, M_{k,l_1} - \frac{1}{3}\delta^{l_2}_k\, G^{l_1}\, M_{l_1,l_1} + \frac{1}{3}\delta^{l_2}_k \sum_p G^p\, M_{l_1,p} - \frac{1}{2}\sum_p H^{l_2}_p\, L^p_{k,l_1} + \\
& + \frac{1}{2}\sum_p H^p_k\, L^{l_2}_{l_1,p} + \frac{1}{6}\delta^{l_2}_k \sum_p H^{l_1}_p\, L^p_{l_1,l_1} - \frac{1}{6}\delta^{l_2}_k \sum_p H^p_{l_1}\, L^{l_1}_{l_1,p}.
\end{aligned}
\tag{4.38}
$$

Next, apply the operator $\sum_k H_{l_2}^k(\cdot)$ to (4.38), namely compute $\sum_k H_{l_2}^k \cdot (4.38)$. This yields:

$$
\begin{aligned}
0 = & \underline{-\frac{1}{2}\sum_k H^{l_2}_{k,y^{l_1}}\, H^k_{l_2} + \frac{1}{6} H^{l_1}_{l_1,y^{l_1}}\, H^{l_2}_{l_2} + \sum_k L^{l_2}_{k,l_1,x}\, H^k_{l_2} -} \\
& \underline{-\frac{1}{3} L^{l_1}_{l_1,l_1,x}\, H^{l_2}_{l_2} +} \\
& + \sum_k G^{l_2}\, H^k_{l_2}\, M_{k,l_1} - \frac{1}{3} G^{l_1}\, H^{l_2}_{l_2}\, M_{l_1,l_1} + \frac{1}{3}\sum_p G^p\, H^{l_2}_{l_2}\, M_{l_1,p} - \\
& -\frac{1}{2}\sum_k\sum_p H^k_{l_2}\, H^{l_2}_p\, L^p_{k,l_1} + \frac{1}{2}\sum_k\sum_p H^k_{l_2}\, H^p_k\, L^{l_2}_{l_1,p} + \frac{1}{6}\sum_p H^{l_2}_{l_2}\, H^{l_1}_p\, L^p_{l_1,l_1} - \\
& -\frac{1}{6}\sum_p H^{l_2}_{l_2}\, H^p_{l_1}\, L^{l_1}_{l_1,p}.
\end{aligned}
\tag{4.39}
$$

Sixthly, we form the expression:

$$
(3.108)\big|_{j:=k;\ l_1:=l_1;\ l_2:=l_2} - (3.108)\big|_{j:=k;\ l_1:=l_2;\ l_2:=l_1}.
\tag{4.40}
$$

Writing term by term the substractions, we get:

(4.41)

$$
\begin{aligned}
0 = &-\underline{\frac{1}{2}\,H^k_{l_1,y^{l_2}}}\,\boxed{1} + \underline{\frac{1}{2}\,H^k_{l_2,y^{l_1}}}\,\boxed{2} + \underline{\frac{1}{6}\,\delta^k_{l_1}\,H^{l_2}_{l_2,y^{l_2}}}\,\boxed{3} - \underline{\frac{1}{6}\,\delta^k_{l_2}\,H^{l_1}_{l_1,y^{l_1}}}\,\boxed{4} + \\
&+ \underline{\frac{1}{3}\,\delta^k_{l_2}\,H^{l_1}_{l_1,y^{l_1}}}\,\boxed{4} - \underline{\frac{1}{3}\,\delta^k_{l_1}\,H^{l_2}_{l_2,y^{l_2}}}\,\boxed{3} + \underline{L^k_{l_1,l_2,x}}\,\textcircled{a} - \underline{L^k_{l_2,l_1,x}}\,\textcircled{a} - \\
&- \underline{\frac{1}{3}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2,x}}\,\boxed{5} + \underline{\frac{1}{3}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1,x}}\,\boxed{6} - \underline{\frac{2}{3}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1,x}}\,\boxed{6} + \underline{\frac{2}{3}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2,x}}\,\boxed{5} + \\
&+ \underline{G^k\,M_{l_1,l_2}}\,\textcircled{b} - \underline{G^k\,M_{l_2,l_1}}\,\textcircled{b} - \underline{\frac{1}{3}\,\delta^k_{l_1}\,G^{l_2}\,M_{l_2,l_2}}\,\boxed{7} + \underline{\frac{1}{3}\,\delta^k_{l_2}\,G^{l_1}\,M_{l_1,l_1}}\,\boxed{8} - \\
&- \underline{\frac{2}{3}\,\delta^k_{l_2}\,G^{l_1}\,M_{l_1,l_1}}\,\boxed{8} + \underline{\frac{2}{3}\,\delta^k_{l_1}\,G^{l_2}\,M_{l_2,l_2}}\,\boxed{7} + \underline{\frac{1}{3}\,\delta^k_{l_1}\sum_p G^p\,M_{l_2,p}}\,\boxed{9} - \underline{\frac{1}{3}\,\delta^k_{l_2}\sum_p G^p\,M_{l_1,p}}\,\boxed{10} - \\
&- \underline{\frac{1}{3}\,\delta^k_{l_2}\sum_p G^p\,M_{l_1,p}}\,\boxed{10} + \underline{\frac{1}{3}\,\delta^k_{l_1}\sum_p G^p\,M_{l_2,p}}\,\boxed{9} - \underline{\frac{1}{2}\sum_p H^k_p\,L^p_{l_1,l_2}}\,\textcircled{c} + \underline{\frac{1}{2}\sum_p H^k_p\,L^p_{l_2,l_1}}\,\textcircled{c} + \\
&+ \underline{\frac{1}{2}\sum_p H^p_{l_1}\,L^k_{l_2,p}}\,\boxed{11} - \underline{\frac{1}{2}\sum_p H^p_{l_2}\,L^k_{l_1,p}}\,\boxed{12} + \underline{\frac{1}{6}\,\delta^k_{l_1}\sum_p H^{l_2}_p\,L^p_{l_2,l_2}}\,\boxed{13} - \underline{\frac{1}{6}\,\delta^k_{l_2}\sum_p H^{l_1}_p\,L^p_{l_1,l_1}}\,\boxed{14} - \\
&- \underline{\frac{1}{6}\,\delta^k_{l_1}\sum_p H^p_{l_2}\,L^{l_2}_{l_2,p}}\,\boxed{15} + \underline{\frac{1}{6}\,\delta^k_{l_2}\sum_p H^p_{l_1}\,L^{l_1}_{l_1,p}}\,\boxed{16} + \underline{\frac{1}{3}\,\delta^k_{l_2}\sum_p H^{l_1}_p\,L^p_{l_1,l_1}}\,\boxed{14} - \underline{\frac{1}{3}\,\delta^k_{l_1}\sum_p H^{l_2}_p\,L^p_{l_2,l_2}}\,\boxed{13} - \\
&- \underline{\frac{1}{3}\,\delta^k_{l_2}\sum_p H^p_{l_1}\,L^{l_1}_{l_1,p}}\,\boxed{16} + \underline{\frac{1}{3}\,\delta^k_{l_1}\sum_p H^p_{l_2}\,L^{l_2}_{l_2,p}}\,\boxed{15}.
\end{aligned}
$$

Simplifying, we get:

(4.42)

$$
\begin{aligned}
0 = &\underline{-\frac{1}{2}\,H^k_{l_1,y^{l_2}} + \frac{1}{2}\,H^k_{l_2,y^{l_1}} - \frac{1}{6}\,\delta^k_{l_1}\,H^{l_2}_{l_2,y^{l_2}} + \frac{1}{6}\,\delta^k_{l_2}\,H^{l_1}_{l_1,y^{l_1}}} + \\
&+ \underline{\frac{1}{3}\,\delta^k_{l_1}\,L^{l_2}_{l_2,l_2,x} - \frac{1}{3}\,\delta^k_{l_2}\,L^{l_1}_{l_1,l_1,x}} + \\
&+ \frac{1}{3}\,\delta^k_{l_1}\,G^{l_2}\,M_{l_2,l_2} - \frac{1}{3}\,\delta^k_{l_2}\,G^{l_1}\,M_{l_1,l_1} + \frac{2}{3}\,\delta^k_{l_1}\sum_p G^p\,M_{l_2,p} - \frac{2}{3}\,\delta^k_{l_2}\sum_p G^p\,M_{l_1,p} + \\
&+ \frac{1}{2}\sum_p H^p_{l_1}\,L^k_{l_2,p} - \frac{1}{2}\sum_p H^p_{l_2}\,L^k_{l_1,p} - \frac{1}{6}\,\delta^k_{l_1}\sum_p H^{l_2}_p\,L^p_{l_2,l_2} + \frac{1}{6}\,\delta^k_{l_2}\sum_p H^{l_1}_p\,L^p_{l_1,l_1} + \\
&+ \frac{1}{6}\,\delta^k_{l_1}\sum_p H^p_{l_2}\,L^{l_2}_{l_2,p} - \frac{1}{6}\,\delta^k_{l_2}\sum_p H^p_{l_1}\,L^{l_1}_{l_1,p}.
\end{aligned}
$$

Next, apply the operator $-\sum_k H_k^{l_2}(\cdot)$ to (4.42), namely compute $-\sum_k H_k^{l_2} \cdot (4.42)$. This yields:

(4.43)
$$
\begin{aligned}
0 = & \underline{\frac{1}{2}\sum_k H^k_{l_1,y^{l_2}}\,H^{l_2}_k - \frac{1}{2}\sum_k H^k_{l_2,y^{l_1}}\,H^{l_2}_k + \frac{1}{6}\,H^{l_2}_{l_2,y^{l_2}}\,H^{l_2}_{l_1}} - \\
& - \underline{\frac{1}{6}\,H^{l_1}_{l_1,y^{l_1}}\,H^{l_2}_{l_2} - \frac{1}{3}\,L^{l_2}_{l_2,l_2,x}\,H^{l_2}_{l_1} + \frac{1}{3}\,L^{l_1}_{l_1,l_1,x}\,H^{l_2}_{l_2}} - \\
& - \frac{1}{3}\,G^{l_2}\,H^{l_2}_{l_1}\,M_{l_2,l_2} + \frac{1}{3}\,G^{l_1}\,H^{l_2}_{l_2}\,M_{l_1,l_1} - \frac{2}{3}\sum_p G^p\,H^{l_2}_{l_1}\,M_{l_2,p} + \\
& + \frac{2}{3}\sum_p G^p\,H^{l_2}_{l_2}\,M_{l_1,p} - \frac{1}{2}\sum_k\sum_p H^{l_2}_k\,H^p_{l_1}\,L^k_{l_2,p} + \frac{1}{2}\sum_k\sum_p H^{l_2}_k\,H^p_{l_2}\,L^k_{l_1,p} + \\
& + \frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^{l_2}_p\,L^p_{l_2,l_2} - \frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^{l_1}_p\,L^p_{l_1,l_1} - \frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^p_{l_2}\,L^{l_2}_{l_2,p} + \\
& + \frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^p_{l_1}\,L^{l_1}_{l_1,p}.
\end{aligned}
$$

Seventhly, put $j := l_2$, $l_1 := k$, $l_2 := l2$ and $l_3 := l_1$ in (3.108):

$$
\begin{aligned}
0 = & \underline{L^{l_2}_{k,l_2,y^{l_1}} - L^{l_2}_{k,l_1,y^{l_2}} - M_{k,l_1,x}} + \\
& + \frac{1}{2}\,H^{l_2}_{l_1}\,M_{k,l_2} - \frac{1}{2}\,H^{l_2}_{l_2}\,M_{k,l_1} + \frac{1}{2}\,\delta^{l_2}_k \sum_p H^p_{l_1}\,M_{l_2,p} - \frac{1}{2}\,\delta^{l_2}_k \sum_p H^p_{l_2}\,M_{l_1,p} - \\
& - \frac{1}{2}\sum_p H^p_k\,M_{l_1,p} + \sum_p L^p_{k,l_1}\,L^{l_2}_{l_2,p} - \sum_p L^p_{k,l_2}\,L^{l_2}_{l_1,p},
\end{aligned}
\tag{4.44}
$$

and then apply the operator $2\sum_k G^k(\cdot)$:

$$
\begin{aligned}
0 = & \, 2\sum_k L^{l_2}_{k,l_2,y^{l_1}}\,G^k - 2\sum_k L^{l_2}_{k,l_1,y^{l_2}}\,G^k - 2\sum_k M_{k,l_1,x}\,G^k + \\
& + \sum_k G^k\,H^{l_2}_{l_1}\,M_{l_1,l_2} - \sum_k G^k\,H^{l_2}_{l_2}\,M_{k,l_1} + \sum_p G^{l_2}\,H^p_{l_1}\,M_{l_2,p} - \\
& - \sum_p G^{l_2}\,H^p_{l_2}\,M_{l_1,p} - \sum_k\sum_p G^k\,H^p_k\,M_{l_1,p} + 2\sum_k\sum_p G^k\,L^p_{k,l_1}\,L^{l_2}_{l_2,p} - \\
& - 2\sum_k\sum_p G^k\,L^p_{k,l_2}\,L^{l_2}_{l_1,p}.
\end{aligned}
\tag{4.45}
$$

Finally, achieve the addition

$$
(4.32) + (4.33) + (4.35) + (4.37) + (4.39) + (4.43) + (4.45).
\tag{4.46}
$$

We copy these seven formal expression, we underline the vanishing terms and we number the remaining terms so as to respect the order of appearance of the terms of the sub-goal (4.29):

(4.47)
$$
\begin{aligned}
0 = & -\underbrace{\sum_k G^k_{y^{l_1}}\,L^k_{k,k}}_{\boxed{1}} + \underbrace{G^{l_1}_{y^{l_1}}}_{\boxed{2}} + \underbrace{\frac{1}{2}\sum_k H^k_{l_1,x}\,L^k_{k,k}}_{\boxed{3}} - \underbrace{\frac{1}{2}\,H^{l_1}_{l_1,x}\,L^{l_1}_{l_1,l_1}}_{\boxed{4}} + \\
& + \underbrace{\sum_k\sum_p G^p\,L^k_{k,k}\,L^k_{l_1,p}}_{\boxed{24}} - \underbrace{\sum_p G^p\,L^{l_1}_{l_1,l_1}\,L^k_{l_1,p}}_{\boxed{23}} - \underbrace{\frac{1}{4}\sum_k\sum_p H^p_{l_1}\,H^k_p\,L^k_{k,k}}_{\boxed{27}} + \\
& + \underbrace{\frac{1}{4}\sum_p H^p_{l_1}\,H^{l_1}_p\,L^{l_1}_{l_1,l_1}}_{\boxed{28}} +
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{+2\sum_k G^k_{y^{l_1}}\,L^{l_2}_{l_2,k}}_{\boxed{5}} \underbrace{-2\,G^{l_1}_{y^{l_1}}\,L^{l_2}_{l_2,l_1}}_{\text{(a)}} \underbrace{-\sum_k H^k_{l_1,x}\,L^{l_2}_{l_2,k}}_{\boxed{6}} \underbrace{+H^{l_1}_{l_1,x}\,L^{l_2}_{l_2,l_1}}_{\text{(b)}} - \\
&\quad \underbrace{-2\sum_k\sum_p G^p\,L^{l_2}_{l_2,k}\,L^k_{l_1,p}}_{\text{(g)}} \underbrace{+2\sum_p G^p\,L^{l_2}_{l_2,l_1}\,L^{l_1}_{l_1,p}}_{\text{(h)}} \underbrace{+\frac{1}{2}\sum_k\sum_p H^p_{l_1}\,H^k_p\,L^{l_2}_{l_2,k}}_{\text{(i)}} - \\
&\quad \underbrace{-\frac{1}{2}\sum_p H^p_{l_1}\,H^{l_1}_p\,L^{l_2}_{l_2,l_1}}_{\text{(j)}} -
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{-2\sum_k G^k_{y^{l_2}}\,L^{l_2}_{l_1,k}}_{\boxed{7}} \underbrace{+2\,G^{l_1}_{y^{l_1}}\,L^{l_2}_{l_1,l_2}}_{\text{(a)}} \underbrace{+\sum_k H^k_{l_2,x}\,L^{l_2}_{l_1,k}}_{\boxed{8}} \underbrace{-H^{l_1}_{l_1,x}\,L^{l_2}_{l_1,l_2}}_{\text{(b)}} + \\
&\quad \underbrace{+2\sum_k\sum_p G^p\,L^k_{l_2,p}\,L^{l_2}_{l_1,k}}_{\text{(k)}} \underbrace{-2\sum_p G^p\,L^{l_1}_{l_1,p}\,L^{l_2}_{l_1,l_2}}_{\text{(h)}} \underbrace{-\frac{1}{2}\sum_k\sum_p H^p_{l_2}\,H^k_p\,L^{l_2}_{l_1,k}}_{\text{(l)}} + \\
&\quad \underbrace{+\frac{1}{2}\sum_p H^p_{l_1}\,H^{l_1}_p\,L^{l_2}_{l_1,l_2}}_{\text{(j)}} +
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{+\frac{1}{2}\sum_k H^{l_2}_{k,y^{l_2}}\,H^k_{l_1}}_{\boxed{9}} \underbrace{-\frac{1}{6}\,H^{l_2}_{l_2,y^{l_2}}\,H^{l_2}_{l_1}}_{\text{(c)}} \underbrace{-\frac{1}{3}\sum_k H^k_{k,y^k}\,H^k_{l_1}}_{\boxed{10}} - \\
&\quad \underbrace{-\sum_k L^{l_2}_{k,l_2,x}\,H^k_{l_1}}_{\boxed{11}} \underbrace{+\frac{1}{3}\,L^{l_2}_{l_2,l_2,x}\,H^{l_2}_{l_1}}_{\text{(d)}} \underbrace{+\frac{2}{3}\sum_k L^k_{k,k,x}\,H^k_{l_1}}_{\boxed{12}} - \\
&\quad \underbrace{-\sum_k G^{l_2}\,H^k_{l_1}\,M_{k,l_2}}_{\text{(m)}} \underbrace{+\frac{1}{3}\,G^{l_2}\,H^{l_2}_{l_1}\,M_{l_2,l_2}}_{\text{(n)}} \underbrace{+\frac{2}{3}\sum_k G^k\,H^k_{l_1}\,M_{k,k}}_{\boxed{21}} - \\
&\quad \underbrace{-\frac{1}{3}\sum_p G^p\,H^{l_2}_{l_1}\,M_{l_2,p}}_{\text{(o)}} \underbrace{+\frac{1}{3}\sum_k\sum_p G^p H^k_{l_1}\,M_{k,p}}_{\boxed{22}} \underbrace{+\frac{1}{2}\sum_k\sum_p H^k_{l_1}\,H^{l_2}_p\,L^p_{k,l_2}}_{\text{(p)}} - \\
&\quad \underbrace{-\frac{1}{2}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^{l_2}_{l_2,p}}_{\text{(i)}} \underbrace{-\frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^{l_2}_p\,L^p_{l_2,l_2}}_{\text{(q)}} \underbrace{+\frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^p_{l_2}\,L^{l_2}_{l_2,p}}_{\text{(r)}} - \\
&\quad \underbrace{-\frac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^k_p\,L^p_{k,k}}_{\boxed{25}} \underbrace{+\frac{1}{3}\sum_k\sum_p H^k_{l_1}\,H^p_k\,L^k_{k,p}}_{\boxed{26}} -
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{-\frac{1}{2}\sum_k H^{l_2}_{k,y^{l_1}}\,H^k_{l_2}}_{\boxed{13}} \underbrace{+\frac{1}{6}\,H^{l_1}_{l_1,y^{l_1}}\,H^{l_2}_{l_2}}_{\text{(e)}} \underbrace{+\sum_k L^{l_2}_{k,l_1,x}\,H^k_{l_2}}_{\boxed{14}} - \\
&\quad \underbrace{-\frac{1}{3}\,L^{l_1}_{l_1,l_1,x}\,H^{l_2}_{l_2}}_{\text{(f)}} + \\
&\quad \underbrace{+\sum_k G^{l_2}\,H^k_{l_2}\,M_{k,l_1}}_{\text{(s)}} \underbrace{-\frac{1}{3}\,G^{l_1}\,H^{l_2}_{l_2}\,M_{l_1,l_1}}_{\text{(t)}} \underbrace{+\frac{1}{3}\sum_p G^p\,H^{l_2}_{l_2}\,M_{l_1,p}}_{\text{(u)}} - \\
&\quad \underbrace{-\frac{1}{2}\sum_k\sum_p H^k_{l_2}\,H^{l_2}_p\,L^p_{k,l_1}}_{\text{(v)}} \underbrace{+\frac{1}{2}\sum_k\sum_p H^k_{l_2}\,H^p_k\,L^{l_2}_{l_1,p}}_{\text{(l)}} \underbrace{+\frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^{l_1}_p\,L^p_{l_1,l_1}}_{\text{(w)}} - \\
&\quad \underbrace{-\frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^p_{l_1}\,L^{l_1}_{l_1,p}}_{\text{(x)}} +
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\frac{1}{2}\sum_k H^k_{l_1,y^{l_2}}\,H^{l_2}_k}_{\boxed{15}}
-\underbrace{\frac{1}{2}\sum_k H^k_{l_2,y^{l_1}}\,H^{l_2}_k}_{\boxed{16}}
+\underbrace{\frac{1}{6}H^{l_2}_{l_2,y^{l_2}}\,H^{l_2}_{l_1}}_{\text{(c)}}-\\
&-\underbrace{\frac{1}{6}H^{l_1}_{l_1,y^{l_1}}\,H^{l_2}_{l_2}}_{\text{(e)}}
-\underbrace{\frac{1}{3}L^{l_2}_{l_2,l_2,x}\,H^{l_2}_{l_1}}_{\text{(d)}}
+\underbrace{\frac{1}{3}L^{l_1}_{l_1,l_1,x}\,H^{l_2}_{l_2}}_{\text{(f)}}-\\
&-\underbrace{\frac{1}{3}G^{l_2}\,H^{l_2}_{l_1}\,M_{l_2,l_2}}_{\text{(n)}}
+\underbrace{\frac{1}{3}G^{l_1}\,H^{l_2}_{l_2}\,M_{l_1,l_1}}_{\text{(t)}}
-\underbrace{\frac{2}{3}\sum_p G^p\,H^{l_2}_{l_1}\,M_{l_2,p}}_{\text{(o)}}+\\
&+\underbrace{\frac{2}{3}\sum_p G^p\,H^{l_2}_{l_2}\,M_{l_1,p}}_{\text{(u)}}
-\underbrace{\frac{1}{2}\sum_k\sum_p H^{l_2}_k\,H^p_{l_1}\,L^k_{l_2,p}}_{\text{(p)}}
+\underbrace{\frac{1}{2}\sum_k\sum_p H^{l_2}_k\,H^p_{l_2}\,L^k_{l_1,p}}_{\text{(v)}}+\\
&+\underbrace{\frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^{l_2}_p\,L^p_{l_2,l_2}}_{\text{(q)}}
-\underbrace{\frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^{l_1}_p\,L^p_{l_1,l_1}}_{\text{(w)}}
-\underbrace{\frac{1}{6}\sum_p H^{l_2}_{l_1}\,H^p_{l_2}\,L^{l_2}_{l_2,p}}_{\text{(r)}}+\\
&+\underbrace{\frac{1}{6}\sum_p H^{l_2}_{l_2}\,H^p_{l_1}\,L^{l_1}_{l_1,p}}_{\text{(x)}}+
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{2\sum_k L^{l_2}_{k,l_2,y^{l_1}}\,G^k}_{\boxed{17}}
-\underbrace{2\sum_k L^{l_2}_{k,l_1,y^{l_2}}\,G^k}_{\boxed{18}}
-\underbrace{2\sum_k M_{k,l_1,x}\,G^k}_{\boxed{19}}+\\
&+\underbrace{\sum_k G^k\,H^{l_2}_{l_1}\,M_{l_1,l_2}}_{\text{(o)}}
-\underbrace{\sum_k G^k\,H^{l_2}_{l_2}\,M_{k,l_1}}_{\text{(u)}}
+\underbrace{\sum_p G^{l_2}\,H^p_{l_1}\,M_{l_2,p}}_{\text{(m)}}-\\
&-\underbrace{\sum_p G^{l_2}\,H^p_{l_2}\,M_{l_1,p}}_{\text{(s)}}
-\underbrace{\sum_k\sum_p G^k\,H^p_k\,M_{l_1,p}}_{\boxed{20}}
+\underbrace{2\sum_k\sum_p G^k\,L^p_{k,l_1}\,L^{l_2}_{l_2,p}}_{\text{(g)}}-\\
&-\underbrace{2\sum_k\sum_p G^k\,L^p_{k,l_2}\,L^{l_2}_{l_1,p}}_{\text{(k)}}.
\end{aligned}
$$

In conclusion, there is exact coincidence with the subgoal (4.29). The proof that the first family $(3.112)_1$ of compatibility conditions of the second auxiliary system (3.99), (3.100), (3.101) and (3.102) are a consequence of (I), (II), (III) and (IV) of Theorem 1.7 is complete. Granted that the treatment of the other three families of compatibility conditions $(3.112)_2$, $(3.112)_3$ and $(3.112)_4$ is similar (and as well painful), we consider that the proof of Theorem 1.7 is complete, now. □

§5. General form of the point transformation of the free particle system

This section is devoted to the exposition of a complete proof of Lemma 3.32. To start with, we must develope the fundamental equations (3.10), for $j = 1, \dots, m$. Recalling that the total differentiation operator is given by $D = \frac{\partial}{\partial x} + \sum_{l_1=1}^m y_x^{l_1} \cdot \frac{\partial}{\partial y^{l_1}} + \sum_{l_1=1}^m y_{xx}^{l_1} \cdot \frac{\partial}{\partial y_x^{l_1}}$, we compute first

(5.1)

$$
\left\{
\begin{aligned}
DDX &= D\left[X_x + \sum_{l_1=1}^m y_x^{l_1}\cdot X_{y^{l_1}}\right]\\
&= X_{xx} + 2\sum_{l_1=1}^m y_x^{l_1}\cdot X_{xy^{l_1}} + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot X_{y^{l_1}y^{l_2}} + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot X_{y^{l_1}},
\end{aligned}
\right.
$$

and

(5.2)
$$\begin{cases} DDY^j & = D\left[Y_x^j + \sum_{l_1=1}^m y_x^{l_1}\cdot Y_{y^{l_1}}^j\right] \\ & = Y_{xx}^j + 2\sum_{l_1=1}^m y_x^{l_1}\cdot Y_{xy^{l_1}}^j + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot Y_{y^{l_1}y^{l_2}}^j + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot Y_{y^{l_1}}^j. \end{cases}$$

Now, we can develope the equation $0 = -DY^j\cdot DDX + DX\cdot DDY^j$, which yields

(5.3)
$$\begin{aligned} 0 = & -\left[Y_x^j + \sum_{l_1=1}^m y_x^{l_1}\cdot Y_{y^{l_1}}^j\right]\cdot\left[X_{xx} + 2\sum_{l_1=1}^m y_x^{l_1}\cdot X_{xy^{l_1}} + \right. \\ & \left. + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot X_{y^{l_1}y^{l_2}} + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot X_{y^{l_1}}\right] + \\ & + \left[X_x + \sum_{l_1=1}^m y_x^{l_1}\cdot X_{y^{l_1}}\right]\cdot\left[Y_{xx}^j + 2\sum_{l_1=1}^m y_x^{l_1}\cdot Y_{xy^{l_1}}^j + \right. \\ & \left. + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}y_x^{l_2}\cdot Y_{y^{l_1}y^{l_2}}^j + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot Y_{y^{l_1}}^j\right] = \end{aligned}$$

$$\begin{aligned} = & -X_{xx}\,Y_x^j + Y_{xx}^j\,X_x + \\ & + \sum_{l_1=1}^m y_x^{l_1}\cdot\left[-2\,X_{xy^{l_1}}\,Y_x^j + 2\,Y_{xy^{l_1}}^j\,X_x - \right. \\ & \left. \qquad - X_{xx}\,Y_{y^{l_1}}^j + Y_{xx}^j\,X_{y^{l_1}}\right] + \\ & + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot\left[-X_{y^{l_1}y^{l_2}}\,Y_x^j + Y_{y^{l_1}y^{l_2}}^j\,X_x - \right. \\ & \left. \qquad - 2\,X_{xy^{l_2}}\,Y_{y^{l_1}}^j + 2\,Y_{xy^{l_2}}^j\,X_{y^{l_1}}\right] + \\ & + \sum_{l_1=1}^m\sum_{l_2=1}^m\sum_{l_3=1}^m y_x^{l_1}\,y_x^{l_2}y_x^{l_3}\cdot\left[-X_{y^{l_2}y^{l_3}}\,Y_{y^{l_1}}^j + Y_{y^{l_2}y^{l_3}}^j\,X_{y^{l_1}}\right] + \\ & + \sum_{l_1=1}^m y_{xx}^{l_1}\cdot\left[-X_{y^{l_1}}\,Y_x^j + Y_{y^{l_1}}^j\,X_x\right] + \\ & + \sum_{l_1=1}^m\sum_{l_2=1}^m y_{xx}^{l_1}\,y_x^{l_2}\cdot\left[-X_{y^{l_1}}\,Y_{y^{l_2}}^j + Y_{y^{l_1}}^j\,X_{y^{l_2}}\right]. \end{aligned}$$

The goal is to show that after solving these m equations for $j = 1,\dots,m$ with respect to the y_{xx}^l, $l = 1,\dots,m$, one obtains the expression (3.33) of Lemma 3.32, or equivalently, using the Δ notation instead of the square notation, one obtains

$$(5.4)\qquad \begin{cases} 0 = y^j_{xx} \cdot \Delta\left(x|y^1|\cdots|y^m\right) + \Delta\left(x|y^1|\cdots|^j xx|\cdots|y^m\right) + \\ \quad + \sum_{l_1=1}^m y^{l_1}_x \cdot \left[2\,\Delta\left(x|y^1|\cdots|^j xy^{l_1}|\cdots|y^m\right) - \right. \\ \qquad\qquad \left. -\delta^j_{l_1}\,\Delta\left(xx|y^1|\cdots|y^m\right)\right] + \\ \quad + \sum_{l_1=1}^m \sum_{l_2=1}^m y^{l_1}_x\, y^{l_2}_x \cdot \left[\Delta\left(x|y^1|\cdots|^j y^{l_1}y^{l_2}|\cdots|y^m\right) - \right. \\ \qquad\qquad \left. -2\,\delta^j_{l_1}\,\Delta\left(xy^{l_2}|y^1|\cdots|y^m\right)\right] + \\ \quad + \sum_{l_1=1}^m \sum_{l_2=1}^m \sum_{l_3=1}^m y^{l_1}_x\, y^{l_2}_x\, y^{l_3}_x \left[-\delta^j_{l_1}\,\Delta\left(y^{l_2}y^{l_3}|y^1|\cdots|y^m\right)\right]. \end{cases}$$

Unfortunately, the equations (5.3) are not solved with respect to the y^j_{xx}, because in its last line, we notice that the $y^{l_2}_{xx}$ are mixed with the $y^{l_1}_x$. Consequently, we have to solve a linear system of m equations with the unknowns y^j_{xx} of the form

$$(5.5)\qquad \left\{ 0 = A^j + \sum_{l_1=1}^m y^{l_1}_{xx} \cdot \left[-X_{y^{l_1}}\, Y^j_x + Y^j_{y^{l_1}}\, X_x + \sum_{l_2=1}^m y^{l_2}_x \cdot \left[-X_{y^{l_1}}\, Y^j_{y^{l_2}} + Y^j_{y^{l_1}}\, X_{y^{l_2}} \right], \right], \right.$$

for $j = 1, \dots, m$, where A^j is an abbreviation for the terms appearing in the lines 5, 6, 7, 8, 9 and 10 of (5.3), or even more compactly, changing the index j to the index k

$$(5.6)\qquad \left\{ 0 = A^k + \sum_{l_1=1}^m y^{l_1}_{xx} \cdot B^k_{l_1}, \right.$$

for $k = 1, \dots, m$, where $B^k_{l_1}$ is an abbreviation for the terms in the brackets in (5.5).

Thanks to the assumption that the determinant (3.2) is the identity determinant at $(x, y) = (0, 0)$, we deduce that the determinant of the $m \times m$ matrix $(B^k_{l_1})^{1\leq k\leq m}_{1\leq l_1\leq m}$ is also the identity determinant at $(x, y, y_x) = (0, 0, 0)$. It follows that the determinant of the $m \times m$ matrix $(B^k_{l_1})^{1\leq k\leq m}_{1\leq l_1\leq m}$ is nonvanishing in a neighborhood of the origin in the first order jet space. Consequently, we can apply the rule of Cramer to solve the y^j_{xx} explicitely interms of the A^k and of the $B^k_{l_1}$ as follows

$$(5.7)\qquad \left\{ y^j_{xx} = -\frac{\begin{vmatrix} B^1_1 & \cdots & A^1 & \cdots & B^1_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B^m_1 & \cdots & A^m & \cdots & B^m_m \end{vmatrix}}{\begin{vmatrix} B^1_1 & \cdots & B^1 & \cdots & B^1_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B^m_1 & \cdots & B^m & \cdots & B^m_m \end{vmatrix}} \right.$$

where on the numerator, the only modification of the determinant of the matrix $(B^k_{l_1})^{1\leq k\leq m}_{1\leq l_1\leq m}$ is the replacement of its j-th column by the column vector A. We have to show that after replacing the A^k and the $B^k_{l_1}$ by their complete expressions, one indeed obtains the desired equation (5.4). As in (3.43), we shall introduce a notation for the two

$m \times m$ determinants appearing in (5.6): we write this quotient under the form

$$
\left\{ y_{xx}^j = -\frac{\left\| B_1^k | \cdots |^j A^k | \cdots | B_m^k \right\|}{\left\| B_1^k | \cdots |^j B_j^k | \cdots | B_m^k \right\|}, \right. \tag{5.8}
$$

where it is understood that $B_1^k, \ldots, B_j^k, \ldots, B_m^k$ and A^k are column vectors whose index k (for their lines) varies from 1 to m. This representation of determinants emphasizing only its columns will be appropriate for later manipulations.

Our first task is to compute the determinant in the denominator of (5.8). Recalling that we make the notational identification $y^0 \equiv x$, it will be convenient to reexpress the $B_{l_1}^k$ in a slightly compacter form, using the total differentiation operator D:

$$
\left\{ \begin{aligned} B_{l_1}^k & = -X_{y^{l_1}} Y_x^k + Y_{y^{l_1}}^k X_x + \sum_{l_2=1}^m y_x^{l_2} \cdot \left[-X_{y^{l_1}} Y_{y^{l_2}}^k + Y_{y^{l_1}}^k X_{y^{l_2}} \right] = \\ & = Y_{y^{l_1}}^k \cdot DX - X_{y^{l_1}} \cdot DY^k. \end{aligned} \right. \tag{5.9}
$$

Lemma 5.10. *We have the following expression for the determinant of the matrix* $(B_{l_1}^k)_{1 \leq l_1 \leq m}^{1 \leq k \leq m}$:

$$
\left\{ \begin{aligned} & \left\| Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k | \cdots | Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k \right\| = \\ & = [DX]^{m-1} \cdot \Delta \left(x | y^1 | \cdots | y^m \right). \end{aligned} \right. \tag{5.11}
$$

Proof. By multilinearity, we may develope the determinant written in the first line of (5.11). Since it contains two terms in each columns, we should obtain a sum of 2^m determinants. However, since the obtained determinants vanish as soon as the column DY^k (multiplied by various factors X_{y^l}) appears at least two different places, it remains only $(m+1)$ nonvanishing determinants, those for which the column DY^k appears at most once:

$$
\left\{ \begin{aligned} & \left\| Y_{y^1}^k \cdot DX - X_{y^1} \cdot DY^k | \cdots | Y_{y^m}^k \cdot DX - X_{y^m} \cdot DY^k \right\| = \\ & = [DX]^m \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^m}^k \right\| - [DX]^{m-1} X_{y^1} \cdot \left\| DY^k | Y_{y^2}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\ & \quad - [DX]^{m-1} X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | DY^k \right\|. \end{aligned} \right. \tag{5.12}
$$

To establish the desired expression appearing in the second line of (5.11), we factor out by $[DX]^{m-1}$ and we develope all the remaining total differentiation operators D. Since $y^0 \equiv x$, we have $y_x^0 = 1$, and this enables us to contract $X_x + \sum_{l_1}^m y_x^{l_1} X_{y^{l_1}}$ as $\sum_{l_1=0}^m y_x^{l_1} X_{y^{l_1}}$. So, we achieve the following computation (further explanations and comments just afterwards):

(5.13)

$$\begin{cases} = [DX]^{m-1} \cdot \left\{ \sum_{l_1=0}^{m} y_x^{l_1}\, X_{y^{l_1}} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^m}^k \right\| - \right. \\ \qquad - \sum_{l_1=0}^{m} y_x^{l_1}\, X_{y^1} \cdot \left\| Y_{y^{l_1}}^k | Y_{y^2}^k | \cdots | Y_{y^m}^k \right\| - \\ \qquad \left. - \cdots - \sum_{l_1=0}^{m} y_x^{l_1}\, X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_{y^{l_1}}^k \right\| \right\} \\ = [DX]^{m-1} \cdot \left\{ \sum_{l_1=0}^{m} y_x^{l_1}\, X_{y^{l_1}} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^m}^k \right\| - X_{y^1} \cdot \left\| Y_x^k | Y_{y^2}^k | \cdots | Y_{y^m}^k \right\| - \right. \\ \qquad - y_x^1\, X_{y^1} \cdot \left\| Y_{y^1}^k | Y_{y^2}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\ \qquad \left. - X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_x^k \right\| - y_x^m\, X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_{y^m}^k \right\| \right\} \\ = [DX]^{m-1} \cdot \left\{ X_x \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^m}^k \right\| - X_{y^1} \cdot \left\| Y_x^k | Y_{y^2}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \right. \\ \qquad \left. - X_{y^m} \cdot \left\| Y_{y^1}^k | \cdots | Y_{y^{m-1}}^k | Y_x^k \right\| \right\} \\ = [DX]^{m-1} \cdot \left\{ \Delta(x|y^1|\cdots|y^m) \right\}. \end{cases}$$

For the passage to the equality of line 4, using the fact that a determinant having two identical columns vanishes, we observe that in each of the m sums $\sum_{l_1=0}^{m}$ appearing in lines 2 and 3 (including the `cdots`), there remains only two non-vanishing determinants. For the passage to the equality of line 7, we just sum up all the linear combinations of determinants appearing in lines 4, 5 and 6. Finally, for the passage to the equality of line 9, we recognize the development of the fundamental Jacobian determinant (3.2) along its first line $(X_x, X_{y^1}, \dots, X_{y^m})$, modulo some permutations of columns in the $m \times m$ minors. The proof is complete. □

Our second task, similar but computationnally more heavy, is to compute the determinant in the numerator of (5.8). First of all, we have to re-express the A^k defined implicitely between (5.3) and (5.5) using the total differentiation operator to contract them as follows

(5.14)

$$\begin{cases} A^k = DX \cdot Y_{xx}^k - DY^k \cdot X_{xx} + 2 \sum_{l_1=1}^{m} y_x^{l_1} \cdot \left[DX \cdot Y_{xy^{l_1}}^k - DY^k \cdot X_{xy^{l_1}} \right] + \\ \qquad + \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2} \cdot \left[DX \cdot Y_{y^{l_1} y^{l_2}}^k - DY^k \cdot X_{y^{l_1} y^{l_2}} \right]. \end{cases}$$

Replacing this expression of A^k in (5.8), taking account of the expression of the denominator already obtained in the second line of (5.11) and abbreviating $\Delta(x|y^1|\cdots|y^m)$ as

Δ, we may write (5.8) in length and then develope it by linearity as follows
(5.15)

$$\left\{\begin{aligned} y_{xx}^j &= \frac{-1}{[DX]^{m-1}\cdot\Delta}\cdot\left[\begin{array}{l} \big\|Y_{y^1}^k\cdot DX - X_{y^1}\cdot DY^k\big|\cdots \\ \quad\cdots\big|^j\, DX\cdot Y_{xx}^k - DY^k\cdot Y_{xx}^k + \\ \qquad +2\sum_{l_1=1}^m y_x^{l_1}\cdot\Big[DX\cdot Y_{xy^{l_1}}^k - DY^k\cdot X_{xy^{l_1}}\Big] + \\ \qquad +\sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot\Big[DX\cdot Y_{y^{l_1}y^{l_2}}^k - DY^k\cdot X_{y^{l_1}y^{l_2}}\Big]\big|\cdots \\ \qquad\qquad\cdots\big|Y_{y^m}^k\cdot DX - X_{y^m}\cdot DY^k\big\| \end{array}\right] \\ &= \frac{-1}{[DX]^{m-1}\cdot\Delta}\cdot\left[\begin{array}{l} \big\|Y_{y^1}^k\cdot DX - X_{y^1}\cdot DY^k\big|\cdots \\ \quad\cdots\big|^j\, DX\cdot Y_{xx}^k - DY^k\cdot X_{xx}\big|\cdots \\ \qquad\cdots\big|Y_{y^m}^k\cdot DX - X_{y^m}\cdot DY^k\big\| + \\ +\,2\sum_{l_1=1}^m y_x^{l_1}\cdot\big\|Y_{y^1}^k\cdot DX - X_{y^1}\cdot DY^k\big|\cdots \\ \quad\cdots\big|^j\, DX\cdot Y_{xy^{l_1}}^k - DY^k\cdot X_{xy^{l_1}}\big|\cdots \\ \qquad\cdots\big|Y_{y^m}^k\cdot DX - X_{y^m}\cdot DY^k\big\| + \\ +\sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\cdot\big\|Y_{y^1}^k\cdot DX - X_{y^1}\cdot DY^k\big|\cdots \\ \quad\cdots\big|^j\, DX\cdot Y_{y^{l_1}y^{l_2}}^k - DY^k\cdot X_{y^{l_1}y^{l_2}}\big|\cdots \\ \qquad\cdots\big|Y_{y^m}^k\cdot DX - X_{y^m}\cdot DY^k\big\| \end{array}\right]. \end{aligned}\right.$$

As it is delicate to read, let us say that lines 2, 3 and 4 just express the j-th colum $|^j\, A^k|$ of the determinant $\|B_1 k|\cdots|j\, A^k|\cdots|B_m^k\|$, after replacement of A^k by its complete expression (5.14).

In lines 6, 7, 8; in lines 9, 10, 11; and in lines 12, 13, 14, there are three families of $m\times m$ determinants containing a linear combination (soustraction) having exactly two terms in each column. As in the proof of Lemma 5.10, by multilinarity, we have to develope each such determinant. In principle, for each development, we should get 2^m terms, but since the obtained determinants vanish as soon as the column DY^k (modulo a multiplication by some factor) appears at least twice, it remains only $(m+1)$ nonvanishing determinants, those for which the column DY^k appears at most once. In addition, for each of the obtained determinant, the factor $[DX]^{m-1}$ appears (sometimes even the factor $[DX]^m$), so that this factor compensates the factor $[DX]^{m-1}$ in the numerator. In sum, the continuation of the huge computation yields:

(5.16)

$$y_{xx}^j = -\frac{1}{\Delta}\cdot\left[\begin{array}{l} DX\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{xx}^k\right|\cdots\left|Y_{y^m}^k\right\| - \\ - X_{y^1}\cdot\left\|DY^k\right|\cdots\left|^j\,Y_{xx}^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - X_{xx}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,DY^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - X_{y^m}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{xx}^k\right|\cdots\left|DY^k\right\| + \\ + 2\sum_{l_1=1}^m y_x^{l_1}\,DX\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{xy^{l_1}}^k\right|\cdots\left|Y_{y^m}^k\right\| - \\ - 2\sum_{l_1=1}^m y_x^{l_1}\,X_{y^1}\cdot\left\|DY^k\right|\cdots\left|^j\,Y_{xy^{l_1}}^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - 2\sum_{l_1=1}^m y_x^{l_1}\,X_{xy^{l_1}}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,DY^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - 2\sum_{l_1=1}^m y_x^{l_1}\,X_{y^m}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{xy^{l_1}}^k\right|\cdots\left|DY^k\right\| + \\ + \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\,DX\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{y^{l_1}y^{l_2}}^k\right|\cdots\left|Y_{y^m}^k\right\| - \\ - \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\,X_{y^1}\cdot\left\|DY^k\right|\cdots\left|^j\,Y_{y^{l_1}y^{l_2}}^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\,X_{y^{l_1}y^{l_2}}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,DY^k\right|\cdots\left|Y_{y^m}^k\right\| - \cdots - \\ - \sum_{l_1=1}^m\sum_{l_2=1}^m y_x^{l_1}\,y_x^{l_2}\,X_{y^m}\cdot\left\|Y_{y^1}^k\right|\cdots\left|^j\,Y_{y^{l_1}y^{l_2}}^k\right|\cdots\left|DY^k\right\| \end{array}\right].$$

To establish the desired expression (5.4), we must develope all the total differentiation operators D of the terms DX placed as factor and of the terms DY^k placed in various columns of determinants. We notice that in developing DY^k, we obtain columns $Y_{y^l}^k$ (multiplied by the factor y_x^l) and for *only* three (or two) values of $l = 0, 1, \ldots, m$, this column does not already appear in the corresponding determinant, so that $(m-1)$ determinants vanish and only 3 (or 2) remain nonzero. Taking account of these simplifications, we have the continuation

$$-y_{xx}^j\cdot\Delta = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{5.17}$$

where the term I is the development of lines 1, 2, 3, 4 of (5.16); the term II is the development of lines 5, 6, 7, 8 of (5.16); and the term III is the development of lines 9, 10, 11, 12 of (5.16). So we get firstly (further explanations follows):

$$
(5.18)\qquad \mathrm{I} := \begin{aligned}
&\underline{\sum_{l=0}^{m} y_x^l\, X_{y^l} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{1}} \quad - \\
&- X_{y^1} \cdot \left\| Y_x^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^m}^k \right\| - \\
&- \underline{y_x^1\, X_{y^1} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{1}} - \\
&- y_x^j\, X_{y^j} \cdot \left\| Y_{y^j}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
&- X_{xx} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
&- y_x^j\, X_{xx} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{y^j}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
&- X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_x^k \right\| - \\
&- \underline{y_x^m\, X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{1}} - \\
&- y_x^j\, X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xx}^k | \cdots | Y_{y^j}^k \right\|,
\end{aligned}
$$

and secondly (we discuss afterwards the annihilation of the underlined terms):

$$
(5.19)\qquad \mathrm{II} := \begin{aligned}
&\underline{2 \sum_{l_1=1}^{m} \sum_{l=0}^{m} y_x^{l_1}\, y_x^l\, X_{y^l} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{2}} \quad - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, X_{y^1} \cdot \left\| Y_x^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| - \\
&- \underline{2 \sum_{l_1=1}^{m} y_x^{l_1}\, y_x^1\, X_{y^1} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{2}} \quad - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, y_x^j\, X_{y^1} \cdot \left\| Y_{y^j}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_x^k | \cdots | Y_{y^m}^k \right\| - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, y_x^j\, X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{y^j}^k | \cdots | Y_{y^m}^k \right\| - \cdots - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_x^k \right\| - \\
&- \underline{2 \sum_{l_1=1}^{m} y_x^{l_1}\, y_x^m\, X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^m}^k \right\|}_{\boxed{2}} \quad - \\
&- 2 \sum_{l_1=1}^{m} y_x^{l_1}\, y_x^j\, X_{y^m} \cdot \left\| Y_{y^1}^k \right| \cdots |^j\, Y_{xy^{l_1}}^k | \cdots | Y_{y^j}^k \right\|,
\end{aligned}
$$

and where thirdly (we are nearly the end of the proof):

$$
\text{(5.20)} \qquad \text{III} := \begin{aligned}
&\underline{\sum_{l_1=1}^{m}\sum_{l_2=1}^{m}\sum_{l=0}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{l}\, X_{y^l}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^m}^k \right\|}_{\boxed{3}} \quad - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, X_{y^1}\cdot \left\| Y_{x}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^m}^k \right\| - \\
&\underline{-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{1}\, X_{y^1}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^m}^k \right\|}_{\boxed{3}} \quad - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{j}\, X_{y^1}\cdot \left\| Y_{y^j}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^m}^k \right\| - \cdots - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, X_{y^{l_1}y^{l_2}}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{x}^k \,|\cdots|\, Y_{y^m}^k \right\| - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{j}\, X_{y^{l_1}y^{l_2}}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^j}^k \,|\cdots|\, Y_{y^m}^k \right\| - \cdots - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, X_{y^m}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{x}^k \right\| - \\
&\underline{-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{m}\, X_{y^m}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^m}^k \right\|}_{\boxed{3}} \quad - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^{j}\, X_{y^m}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^{l_1}y^{l_2}}^k \,|\cdots|\, Y_{y^j}^k \right\|.
\end{aligned}
$$

Now, we explain the annihilation of the underlined terms. Consider I: in the first sum $\sum_{l=0}^{m}$, all the terms except only the two corresponding to $l = 0$ and to $l = j$ are annihilated by the other terms with $\boxed{1}$ appended: indeed, one must take account of the fact that in the expression of I, we have two sums represented by some `cdots`, the nature of which was defined without ambiguity in the passage from (5.15) to (5.16).

Similar simplifications occur for II and for III. Consequently, we obtain firstly:

$$
\text{(5.21)} \qquad \text{I} := \begin{aligned}
&X_x\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{y^m}^k \right\| + \\
&+ y_x^j\, X_{y^j}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{y^m}^k \right\| - \\
&- X_{y^1}\cdot \left\| Y_{x}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{y^m}^k \right\| - \\
&- y_x^j\, X_{y^j}\cdot \left\| Y_{y^j}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{y^m}^k \right\| - \cdots - \\
&- X_{xx}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{x}^k \,|\cdots|\, Y_{y^m}^k \right\| - \\
&- y_x^j\, X_{xx}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{y^j}^k \,|\cdots|\, Y_{y^m}^k \right\| - \cdots - \\
&- X_{y^m}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{x}^k \right\| - \\
&- y_x^j\, X_{y^m}\cdot \left\| Y_{y^1}^k \,|\cdots|^j\, Y_{xx}^k \,|\cdots|\, Y_{y^j}^k \right\|;
\end{aligned}
$$

just above, the first two lines consist of the two terms in the sum underlined at the first line of (5.18) which are not annihilated; secondly we obtain:

$$
\text{(5.22)} \qquad \text{II} := \begin{aligned}
& 2 \sum_{l_1=1}^{m} y_x^{l_1} \, X_x \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| + \\
& + 2 \sum_{l_1=1}^{m} y_x^{l_1} \, y_x^j \, X_{y^j} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, X_{y^1} \cdot \left\| Y_x^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, y_x^j \, X_{y^1} \cdot \left\| Y_{y^j}^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \cdots - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_x^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, y_x^j \, X_{xy^{l_1}} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_x^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \cdots - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, X_{y^m} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_x^k \right\| - \\
& - 2 \sum_{l_1=1}^{m} y_x^{l_1} \, y_x^j \, X_{y^m} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{xy^{l_1}}^k \middle| \cdots \middle| Y_{y^j}^k \right\| ;
\end{aligned}
$$

similarly, the first two lines above consist of the two terms in the sum underlined at the first line of (5.19) which are not annihilated; and thirdly we obtain:

$$
\text{(5.23)} \qquad \text{III} := \begin{aligned}
& \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, X_x \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{y^{l_1} y^{l_2}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| + \\
& + \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, y_x^j \, X_{y^j} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{y^{l_1} y^{l_2}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, X_{y^1} \cdot \left\| Y_x^k \middle| \cdots \middle|^j \, Y_{y^{l_1} y^{l_2}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, y_x^j \, X_{y^1} \cdot \left\| Y_{y^j}^k \middle| \cdots \middle|^j \, Y_{y^{l_1} y^{l_2}}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \cdots - \\
& - \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, X_{y^{l_1} y^{l_2}} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_x^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \\
& - \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} y_x^{l_1} \, y_x^{l_2} \, y_x^j \, X_{y^{l_1} y^{l_2}} \cdot \left\| Y_{y^1}^k \middle| \cdots \middle|^j \, Y_{y^j}^k \middle| \cdots \middle| Y_{y^m}^k \right\| - \cdots -
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, X_{y^m}\cdot\left\| Y_{y^1}^k \right| \cdots |^j\, Y_{y^{l_1}y^{l_2}}^k | \cdots \left| Y_x^k \right\| - \\
&-\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\, y_x^j\, X_{y^m}\cdot\left\| Y_{y^1}^k \right| \cdots |^j\, Y_{y^{l_1}y^{l_2}}^k | \cdots \left| Y_{y^j}^k \right\| .
\end{aligned}
$$

Collecting the odd lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $\Delta(x|\cdots|^j\, xx|\cdots|y^m)$ along its first line, modulo permutations of columns of the associated $m\times m$ minors; collecting the even lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $-y_x^j\cdot\Delta(xx|y^1|\cdots|y^m)$ along its first lines, modulo permutations of columns of the associated $m\times m$ minors. Similar observations hold about II and III.

In , we may rewrite the final expressions of these three terms: firstly

$$
(5.24)\qquad \left\{
\begin{aligned}
\mathrm{I} &= \Delta(x|\cdots|^j\, xx|\cdots|y^m) - y_x^j\cdot\Delta(xx|y^1|\cdots|y^m),\\
\mathrm{II} &= 2\sum_{l_1=1}^{m} y_x^{l_1}\cdot\Delta(x|\cdots|^j\, xy^{l_1}|\cdots|y^m) - \\
&\qquad - 2\, y_x^j \sum_{l_1=1}^{m} y_x^{l_1}\cdot\Delta(xy^{l_1}|y^1|\cdots|y^m),\\
\mathrm{III} &= \sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\cdot\Delta(x|\cdots|^j\, y^{l_1}y^{l_2}|\cdots|y^m) - \\
&\qquad - y_x^j\sum_{l_1=1}^{m}\sum_{l_2=1}^{m} y_x^{l_1}\, y_x^{l_2}\cdot\Delta(y^{l_1}y^{l_2}|y^1|\cdots|y^m).
\end{aligned}
\right.
$$

Coming back to (5.17), we obtain the desired expression (5.4).

The proof of the — awfully technical, though involving only linear algebra — Lemma 3.32 is complete. □

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