Applied and Computational Harmonic Analysis **23** (2007), 74-113

AN OVERVIEW OF THE RIEMANNIAN METRICS ON SPACES OF CURVES USING THE HAMILTONIAN APPROACH

PETER W. MICHOR, DAVID MUMFORD

Abstract. Here shape space is either the manifold of simple closed smooth unparameterized curves in $\mathbb{R}^2$ or is the orbifold of immersions from S^1 to $\mathbb{R}^2$ modulo the group of diffeomorphisms of S^1. We investige several Riemannian metrics on shape space: L^2-metrics weighted by expressions in length and curvature. These include a scale invariant metric and a Wasserstein type metric which is sandwiched between two length-weighted metrics. Sobolev metrics of order n on curves are described. Here the horizontal projection of a tangent field is given by a pseudo-differential operator. Finally the metric induced from the Sobolev metric on the group of diffeomorphisms on $\mathbb{R}^2$is treated. Although the quotient metrics are all given by pseudo-differential operators, their inverses are given by convolution with smooth kernels. We are able to prove local existence and uniqueness of solution to the geodesic equation for both kinds of Sobolev metrics.

We are interested in all conserved quantities, so the paper starts with the Hamiltonian setting and computes conserved momenta and geodesics in general on the space of immersions. For each metric we compute the geodesic equation on shape space. In the end we sketch in some examples the differences between these metrics.

1. Introduction — multiple Riemannian metrics on the space of curves

Both from a mathematical and a computer vision point of view, it is of great interest to understand the space of simple closed curves in the plane. Mathematically, this is arguably the simplest infinite-dimensional truly nonlinear space. From a vision perspective, one needs to make human judgements like 'such-and-such shapes are similar, but such-and-such are not' into precise statements. The common theory which links these two points of view is the study of the various ways in which the space of simple closed curves can be endowed with a Riemannian metric. From

---

Date: December 17, 2009.

2000 *Mathematics Subject Classification.* Primary 58B20, 58D15, 58E12.

Both authors were supported by NSF-Focused Research Group: The geometry, mechanics, and statistics of the infinite dimensional shape manifolds. PWM was supported by FWF Project P 17108.

a vision perspective, this converts the idea of similarity of two shapes into a quantitative metric. From a mathematical perspective, a Riemannian metric leads to geodesics, curvature and diffusion and, hopefully, to an understanding of the global geometry of the space. Much work has been done in this direction recently (see for example [10, 13, 15, 16, 24]). The purpose of the present paper is two-fold. On the one hand, we want to survey the spectrum of Riemannian metrics which have been proposed (omitting, however, the Weil-Peterson metric). On the other hand, we want to develop systematically the Hamiltonian approach to analyzing these metrics.

Next, we define the spaces which we will study and introduce the notation we will follow throughout this paper. To be precise, by a curve we mean a C^∞ simple closed curve in the plane. The space of these will be denoted B_e. We will consider two approaches to working with this space. In the first, we use parametrized curves and represent B_e as the quotient:

$$B_e \cong \operatorname{Emb}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$$

of the smooth Fréchet manifold of C^∞ embeddings of S^1 in the plane modulo the group of C^∞ diffeomorphisms of S^1. In this approach, it is natural to consider all possible *immersions* as well as embeddings, and thus introduce the larger space B_i as the quotient of the space of C^∞ immersions by the group of diffeomorphisms of S^1:

$$\begin{array}{ccccc}
\operatorname{Emb}(S^1,\mathbb{R}^2) & \longrightarrow & \operatorname{Emb}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1) & \cong & B_e \\
\cap & & \cap & & \cap \\
\operatorname{Imm}(S^1,\mathbb{R}^2) & \longrightarrow & \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1) & \cong & B_i
\end{array}$$

In the second approach, we use the group of diffeomorphisms $\operatorname{Diff}(\mathbb{R}^2)$ of the plane, where, more precisely, this is either the group of all diffeomorphisms equal to the identity outside a compact set or the group of all diffeomorphisms which decrease rapidly to the identity. Let Δ be the unit circle in the plane. This group has two subgroups, the normalizer and the centralizer of Δ in $\operatorname{Diff}(\mathbb{R}^2)$:

$$\begin{array}{ccccc}
\operatorname{Diff}^0(\mathbb{R}^2,\Delta) & \subset & \operatorname{Diff}(\mathbb{R}^2,\Delta) & \subset & \operatorname{Diff}(\mathbb{R}^2) \\
\| & & \| & & \\
\{\varphi \mid \varphi|_\Delta \equiv \mathrm{id}_\Delta\} & & \{\varphi \mid \varphi(\Delta)=\Delta\} & &
\end{array}$$

Let $i \in \operatorname{Emb}(S^1,\mathbb{R}^2)$ be the basepoint $i(\theta) = (\sin(\theta),\cos(\theta))$ carrying S^1 to the unit circle Δ. The group $\operatorname{Diff}(\mathbb{R}^2)$ acts on the space $\operatorname{Emb}(S^1,\mathbb{R}^2)$ of embeddings by composition on the left. The action on the space of embeddings is transitive (e.g., choose an isotopy between two embedded circles, transform and extend its velocity field into a time-dependent vector field with compact support on $\mathbb{R}^2$ and integrate it to a diffeomorphism). $\operatorname{Diff}^0(\mathbb{R}^2,\Delta)$ is the subgroup which fixes the base point i. Thus we can represent $\operatorname{Emb}(S^1,\mathbb{R}^2)$ as the coset space $\operatorname{Diff}(\mathbb{R}^2)/\operatorname{Diff}^0(\mathbb{R}^2,\Delta)$.

Furthermore $\operatorname{Diff}^0(\mathbb{R}^2,\Delta)$ is a *normal* subgroup of $\operatorname{Diff}(\mathbb{R}^2,\Delta)$, and the quotient of one by the other is nothing other than $\operatorname{Diff}(\Delta)$, the diffeomorphism group of the unit circle. So $\operatorname{Diff}(\Delta)$ acts on the coset space $\operatorname{Diff}(\mathbb{R}^2)/\operatorname{Diff}^0(\mathbb{R}^2,\Delta)$ with quotient the coset space $\operatorname{Diff}(\mathbb{R}^2)/\operatorname{Diff}(\mathbb{R}^2,\Delta)$. Finally, under the identification of $\operatorname{Diff}(\mathbb{R}^2)/\operatorname{Diff}^0(\mathbb{R}^2,\Delta)$ with $\operatorname{Emb}(S^1,\mathbb{R}^2)$, this action is the same as the previously defined one of $\operatorname{Diff}(S^1)$ on $\operatorname{Emb}(S^1,\mathbb{R}^2)$. This is because if $c = \varphi \circ i \in \operatorname{Emb}(S^1,\mathbb{R}^2)$, and $\psi \in \mathrm{Diff}(\mathbb{R}^2, \Delta)$ satisfies $\psi(i(\theta)) = i(h(\theta)), h \in \mathrm{Diff}(S^1)$, then the action of ψ carries φ to $\varphi \circ \psi$ and hence c to $\varphi \circ \psi \circ i = \varphi \circ i \circ h = c \circ h$.

All the spaces and maps we have introduced can be combined in one commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Diff}(\mathbb{R}^2) & & & & \\
\downarrow & & & & \\
\mathrm{Diff}(\mathbb{R}^2)/\,\mathrm{Diff}^0(\mathbb{R}^2, \Delta) & \xrightarrow{\approx} & \mathrm{Emb}(S^1, \mathbb{R}^2) & \subset & \mathrm{Imm}(S^1, \mathbb{R}^2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(\mathbb{R}^2)/\,\mathrm{Diff}(\mathbb{R}^2, \Delta) & \xrightarrow{\approx} & B_e & \subset & B_i
\end{array}$$

See [13] and [8] for the homotopy type of the spaces $\mathrm{Imm}(S^1, \mathbb{R}^2)$ and B_i.

What is the infinitesimal version of this? We will use the notation $\mathfrak{X}(\mathbb{R}^2)$ to denote the Lie algebra of $\mathrm{Diff}(\mathbb{R}^2)$, i.e., either the space of vector fields on $\mathbb{R}^2$ with compact support or the space of rapidly decreasing vector fields. As for any Lie group, the tangent bundle $T\,\mathrm{Diff}(\mathbb{R}^2)$ is the product $\mathrm{Diff}(\mathbb{R}^2) \times \mathfrak{X}(\mathbb{R}^2)$ by either right or left multiplication. We choose right so that a tangent vector to $\mathrm{Diff}(\mathbb{R}^2)$ at φ is given by a vector field X representing the infinitesimal curve $\varphi \mapsto \varphi(x, y) + \varepsilon X(\varphi(x, y))$.

Fix $\varphi \in \mathrm{Diff}(\mathbb{R}^2)$ and let it map to $c = \varphi \circ i \in \mathrm{Emb}(S^1, \mathbb{R}^2)$ and to the curve $C = \mathrm{Im}(c) \subset \mathbb{R}^2$ on the three levels of the above diagram. A tangent vector to $\mathrm{Emb}(S^1, \mathbb{R}^2)$ at c is given by a vector field Y to $\mathbb{R}^2$ along the map c, and the vertical map of tangent vectors simply takes the vector field X defined on all of $\mathbb{R}^2$ and restricts it to the map c, i.e. it takes the values $Y(\theta) = X(c(\theta))$. Note that if c is an embedding, a vector field along c is the same as a vector field on its image C. A tangent vector to B_e at the image curve C is given by a vector field Y along C *modulo* vector fields tangent to C itself. The vertical map on tangent vectors just takes the vector field X along c and treats it modulo vector fields tangent to c. However, it is convenient to represent a tangent vector to B_e or B_i at C not as an equivalence class of vector fields along C but by their unique representative which is everywhere *normal* to the curve C. This makes $T_C B_i$ the space of all normal vector fields to $C \subset \mathbb{R}^2$.

In both approaches, we will put a Riemannian metric on the top space, i.e. $\mathrm{Imm}(S^1, \mathbb{R}^2)$ or $\mathrm{Diff}(\mathbb{R}^2)$, which makes the map to the quotient B_i or to a coset space of $\mathrm{Diff}(\mathbb{R}^2)$ into a *Riemannian submersion*. In general, given a surjective mapping $f : A \to B$ with a surjective tangent map and a metric $G_a(h, k)$ on A, f is a submersion if it has the following property: first split the tangent bundle to A into the subbundle $TA^\top$ tangent to the fibres of f and its perpendicular $TA^\perp$ with respect to G (called the horizontal bundle). Then, under the isomorphisms $df : TA_a^\perp \xrightarrow{\approx} TB_{f(a)}$, the restriction of the A-metric to the horizontal subbundle is required to define a metric on TB_b, independent of the choice of the point $a \in f^{-1}(b)$ in the fiber. In this way we will define Riemannian metrics on all the spaces in our diagram above. Submersions have a very nice effect on geodesics: the geodesics on the quotient space B are exactly the images of the geodesics on the top space A which are perpendicular at one, and hence at all, points to the fibres of the map f (or, equivalently, their tangents are in the horizontal subbundle).

On $\mathrm{Diff}(\mathbb{R}^2)$, we will consider only right invariant metrics. These are given by putting a positive definite inner product $G(X,Y)$ on the vector space of vector fields to $\mathbb{R}^2$, and translating this to the tangent space above each diffeomorphism φ as above. That is, the length of the infinitesimal curve $\varphi \mapsto \varphi + \varepsilon X \circ \varphi$ is $\sqrt{G(X,X)}$. Then the map from $\mathrm{Diff}(\mathbb{R}^2)$ to any of its right coset spaces will be a Riemannian submersion, hence we get metrics on all these coset spaces.

A Riemannian metric on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ is just a family of positive definite inner products $G_c(h,k)$ where $c \in \mathrm{Imm}(S^1,\mathbb{R}^2)$ and $h,k \in C^\infty(S^1,\mathbb{R}^2)$ represent vector fields on $\mathbb{R}^2$ along c. We require that our metrics will be invariant under the action of $\mathrm{Diff}(S^1)$, hence the map dividing by this action will be a Riemannian submersion. Thus we will get Riemannian metrics on B_i: these are given by a family of inner products as above such that $G_c(h,k) \equiv 0$ if h is tangent to c, i.e., $\langle h(\theta), c_\theta(\theta)\rangle \equiv 0$ where $c_\theta := \partial_\theta c$.

When dealing with parametrized curves or, more generally, immersions, we will use the following terminology. Firstly, the immersion itself is usually denoted by:

$$c(\theta) : S^1 \to \mathbb{R}^2$$

or, when there is a family of such immersions:

$$c(\theta,t) : S^1 \times I \to \mathbb{R}^2.$$

The parametrization being usually irrelevant, we work mostly with arclength ds, arclength derivative D_s and the unit tangent vector v to the curve:

$$\begin{aligned} ds &= |c_\theta| d\theta \\ D_s &= \partial_\theta / |c_\theta| \\ v &= c_\theta / |c_\theta| \end{aligned}$$

An important caution is that when you have a family of curves $c(\theta,t)$, then ∂_θ and ∂_t commute but D_s and ∂_t don't because $|c_\theta|$ may have a t-derivative. Rotation through 90 degrees will be denoted by:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The unit normal vector to the image curve is thus

$$n = Jv.$$

Thus a Riemannian metric on B_e or B_i is given by inner products $G_C(a,b)$ where $a.n$ and $b.n$ are any two vector fields along C normal to C and $a,b \in C^\infty(C,\mathbb{R})$. Another important piece of notation that we will use concerns directional derivatives of functions which depend on several variables. Given a function $f(x,y)$ for instance, we will write:

$$D_{(x,h)} f \text{ or } df(x)(h) \text{ as shorthand for } \partial_t|_0 f(x+th, y).$$

Here the x in the subscript will indicate which variable is changing and the second argument h indicates the direction. This applies even if one of the variables is a curve $C \in B_i$ and h is a normal vector field.

The simplest inner product on the tangent bundle to $\mathrm{Imm}(S^1,\mathbb{R}^2)$ is:

$$G^0_c(h,k) = \int_{S^1} \langle h(\theta), k(\theta)\rangle \cdot ds.$$

Since the differential ds is invariant under the action of the group $\mathrm{Diff}(S^1)$, the map to the quotient B_i is a Riemannian submersion for this metric. A tangent vector h to $\mathrm{Imm}(S^1, \mathbb{R}^2)$ is perpendicular to the orbits of $\mathrm{Diff}(S^1)$ if and only if $\langle h(\theta), v(\theta)\rangle \equiv 0$, i.e. h is a multiple $a.n$ of the unit normal. This is the same subbundle as above, so that, for this metric, the *horizontal* subspace of the tangent space is the natural splitting. Finally, the quotient metric is given by

$$G_c^0(a \cdot n, b \cdot n) = \int_{S^1} a.b.ds.$$

All the metrics we will look at will be of the form:

$$G_c(h,k) = \int_{S^1} \langle Lh, k\rangle \cdot ds$$

where L is a positive definite operator on the vector-valued functions $h : S^1 \to \mathbb{R}^2$. The simplest such L is simply multiplication by some function $\Phi_c(\theta)$. However, it will turn out that most of the metrics involve L's which are differential or *pseudo-differential* operators. For these, the horizontal subspace is not the natural splitting, so the quotient metric on B_e and B_i involves restricting G_c to different sub-bundles and this makes these operators somewhat complicated. In fact, it is not guaranteed that the horizontal subspace is spanned by C^∞ vectors (in the sense that the full C^∞ tangent space is the direct sum of the vertical subspace and the horizontal C^∞ vectors). When dealing with metrics on $\mathrm{Diff}(\mathbb{R}^2)$ and vertical subspaces defined by the subgroups above, this does happen. In this case, the horizontal subspace must be taken using less smooth vectors.

In all our cases, L^{-1} will be a simpler operator than L: this is because the tangent spaces to B_e or B_i are *quotients* of the tangent spaces to the top spaces Diff or Imm where the metrics are most simply defined, whereas the cotangent spaces to B_e or B_i are subspaces of the cotangent spaces of the space 'above'. The dual inner product on the cotangent space is given by the inverse operator L^{-1} and in all our cases this will be an integral operator with a simple explicit kernel. A final point: we will use a constant A when terms with different physical 'dimensions' are being added in the operator L. Then A plays the role of fixing a scale relative to which different geometric phenomena can be expected.

Let us now describe in some detail the contents of this paper and the metrics. First, in section 2, we introduce the general Hamiltonian formalism. This is, unfortunately, more technical than the rest of the paper. First we consider general Riemannian metrics on the space of immersions which admit Christoffel symbols. We express this as the existence of two kinds of gradients. Since the energy function is not even defined on the whole cotangent bundle of the tangent bundle we pull back to the tangent bundle the canonical symplectic structure on the cotangent bundle. Then we determine the Hamiltonian vector field mapping and, as a special case, the geodesic equation. We detemine the equivariant moment mapping for several group actions on the space of immersions: the action of the reparametrization group $\mathrm{Diff}(S^1)$, of the motion group of $\mathbb{R}^2$, and also of the scaling group (if the metric is scale invariant). Finally the invariant momentum mapping on the group $\mathrm{Diff}(\mathbb{R}^2)$ is described.

Section 3 is then devoted to applying the Hamiltonian procedure to *almost local metrics*: these are the metrics in which L is multiplication by some function Φ. Let $\ell_c = \int_{S^1} ds$ be the length of the curve c and let

$$\kappa_c(\theta) = \langle n(\theta), D_s(v)(\theta)\rangle = \langle Jc_\theta, c_{\theta\theta}\rangle / |c_\theta|^3$$

be the curvature of c at $c(\theta)$. Then for any auxiliary function $\Phi(\ell,\kappa)$, we can define a weighted Riemannian metric:

$$G_c^\Phi(h,k) = \int_{S^1} \Phi(\ell_c, \kappa_c(\theta)) \cdot h(\theta)k(\theta) \cdot ds.$$

The motivation for introducing weights is simply that, for any 2 curves in B_i, the infimum of path lengths in the G^0 metric for paths joining them is zero, see [13, 14]. For all these metrics, the horizontal subspace is again the set of tangent vectors $a(\theta)n(\theta)$, so the metric on B_i is simply

$$G_c^\Phi(a\cdot n, b\cdot n) = \int_{S^1} \Phi(\ell_c, \kappa_c(\theta))a(\theta)b(\theta) \cdot ds.$$

We will determine the geodesic equation, the momenta and the sectional curvature for all these metrics. The formula for sectional curvature is rather complicated but for special Φ, it is quite usable.

We will look at several special cases. The weights

$$\Phi(\ell,\kappa) = 1 + A\kappa^2$$

were introduced and studied in [13]. As we shall see, this metric is also closely connected to the Wasserstein metric on probability measures (see [1]), if we assign to a curve C the probability measure given by scaled arc length. We show that it is sandwiched between the conformal metric $G^{\ell^{-1}}$ and G^{Φ_W} where $\Phi_W = \ell^{-1} + \frac{1}{12}\ell\kappa^2$. Weights of the form

$$\Phi(\ell,\kappa) = f(\ell)$$

were studied in [10] and independently by [19]. The latter are attractive because they give metrics which are conformally equivalent to G^0. These metrics are a borderline case between really stable metrics on B_e and the metric G^0 for which path length shrinks to 0: for them, the infimum of path lengths is positive but at least some paths seem to oscillate wildly when their length approaches this infimum. Another very interesting case is:

$$\Phi(\ell,\kappa) = \ell^{-3} + A|\kappa|^2\ell^{-1}$$

because this metric is scale-invariant.

A more standard approach to strengthening G^0 is to introduce higher derivatives. In section 4, we follow the Sobolev approach which puts a metric on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ by:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h,k) &= \int_{S^1} \sum_{i=0}^{n} \langle D_s^i h, D_s^i k\rangle ds \\ &= \int_{S^1} \langle Lh, k\rangle ds, \quad \text{where } L = \sum_{i=0}^{n} (-1)^i D_s^{2i} \end{aligned}$$

However, the formulas we get are substantially simpler and L^{-1} has an elegant expression is we take the equivalent metric:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h,k) &= \int_{S^1} \left(\langle h,k\rangle + A.\langle D_s^n h, D_s^n k\rangle\right) ds \\ &= \int_{S^1} \langle Lh,k\rangle ds, \quad \text{where } L = I + (-1)^n A D_s^{2n} \end{aligned}$$

We apply the Hamiltonian procedure to this metric. Here the horizontal space of all vectors in the tangent space $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ which are $G^{\mathrm{imm},n}$-orthogonal to the reparametrization orbits, is very different from the natural splitting in §3. The decomposition of a vector into horizontal and vertical parts involves pseudo differential operators, and thus also the horizontal geodesic equation is an integro-differential equation. However, its inverse L^{-1} is an integral operator whose kernel has a simple expression in terms of arc length distance between 2 points on the curve and their unit normal vectors.

For this metric, we work out the geodesic equation and prove that the geodesic flow is well posed in the sense that we have local existence and uniqueness of solutions in $\mathrm{Imm}(S^1,\mathbb{R}^2)$ and in B_i. Finally we discuss a little bit a scale invariant version of the metric $G^{\mathrm{imm},n}$. For the simplest of these metrics, the scaling invariant momentum along a geodesic turns out to be the time derivative of $\log(\ell)$. At this time, we do not know the sectional curvature for this metric. Sobolev-metrics of type H^n are also studied in [11], in particular in view of the completion of the space of curves.

In the next section 5, we start with the basic right invariant metrics on $\mathrm{Diff}(\mathbb{R}^2)$ which are given by the Sobolev H^n-inner product on $\mathfrak{X}(\mathbb{R}^2)$.

$$\begin{aligned} H^n(X,Y) &= \sum_{i,j\geq 0, i+j\leq n} \frac{A^{i+j} n!}{i!j!(n-i-j)!} \iint_{\mathbb{R}^2} \langle \partial_x^i \partial_y^j X, \partial_x^i \partial_y^j Y\rangle\, dx^1 dx^2 \\ &= \iint_{\mathbb{R}^2} \langle LX,Y\rangle dx.dy, \quad \text{where } L = (1-A\Delta)^n, \Delta = \partial_x^2 + \partial_y^2. \end{aligned}$$

These metrics have been extensively studied by Miller, Younes and Trouvé and their collaborators [5, 15, 16, 21]. Since these metrics are right invariant, all maps to coset spaces $\mathrm{Diff}(\mathbb{R}^2) \to \mathrm{Diff}(\mathbb{R}^2)/H$ are submersions. In particular, this metric gives a quotient metric on $\mathrm{Emb}(S^1,\mathbb{R}^2)$ and B_e which we will denote by $G_c^{\mathrm{diff},n}(h,k)$. In this case, the inverse L^{-1} of the operator defining the metric is an integral operator with a kernel given by a classical Bessel function applied to the distance in $\mathbb{R}^2$ between 2 points on the curve. We will derive the geodesic equations: they are all in the same family as fluid flow equations. We prove well posedness of the geodesic equation on $\mathrm{Emb}(S^1,\mathbb{R}^2)$ and on B_e. Although there is a formula of Arnold [4] for the sectional curvature of any right-invariant metric on a Lie group, we have not computed sectional curvatures for the quotient spaces.

In the final section 6, we study two examples to make clear the differences between the various metrics. The first example is the geodesic formed by the set of all concentric circles with fixed center. We will see how this geodesic is complete when the metric is reasonably strong, but incomplete in most 'borderline' cases.

The second example takes a fixed 'cigar-shaped' curve C and compares the unit balls in the tangent space $T_C B_e$ given by the different metrics.

2. The Hamiltonian approach

In our previous papers, we have derived the geodesic equation in our various metrics by setting the first variation of the energy of a path equal to 0. Alternately, the geodesic equation is the Hamiltonian flow associated to the first fundamental form (i.e. the length-squared function given by the metric on the tangent bundle). The Hamiltonian approach also provides a mechanism for converting symmetries of the underlying Riemannian manifold into conserved quantities, the momenta. We first need to be quite formal and lay out the basic definitions, esp. distinguishing between the tangent and cotangent bundles rather carefully: The former consists of smooth vector fields along immersions whereas the latter is comprised of 1-currents along immersions. Because of this we work on the tangent bundle and we pull back the symplectic form from the contantent bundle to $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$. We use the basics of symplectic geometry and momentum mappings on cotangent bundles in infinite dimensions, and we explain each step. See [12], section 2, for a detailed exposition in similar notation as used here.

2.1. **The setting.** Consider as above the smooth Fréchet manifold $\mathrm{Imm}(S^1,\mathbb{R}^2)$ of all immersions $S^1 \to \mathbb{R}^2$ which is an open subset of $C^\infty(S^1,\mathbb{R}^2)$. The tangent bundle is $T\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)$, and the cotangent bundle is $T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2)\times \mathcal{D}(S^1)^2$ where the second factor consists of pairs of periodic distributions.

We consider smooth Riemannian metrics on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, i.e., smooth mappings

$$\begin{aligned} &G : \mathrm{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)\to\mathbb{R}\\ &(c,h,k)\mapsto G_c(h,k),\qquad \text{bilinear in } h,k\\ &G_c(h,h)>0\qquad \text{for } h\neq 0.\end{aligned}$$

Each such metric is *weak* in the sense that G_c, viewed as bounded linear mapping

$$\begin{aligned} G_c : T_c\,\mathrm{Imm}(S^1,\mathbb{R}^2) = C^\infty(S^1,\mathbb{R}^2) &\to T_c^*\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathcal{D}(S^1)^2\\ G : T\,\mathrm{Imm}(S^1,\mathbb{R}^2) &\to T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2)\\ G(c,h) &= (c, G_c(h,\ .\ ))\end{aligned}$$

is injective, but can never be surjective. We shall need also its tangent mapping

$$TG : T(T\,\mathrm{Imm}(S^1,\mathbb{R}^2)) \to T(T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2))$$

We write a tangent vector to $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ in the form $(c,h;k,\ell)$ where $(c,h)\in T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ is its foot point, k is its vector component in the $\mathrm{Imm}(S^1,\mathbb{R}^2)$-direction and where ℓ is its component in the $C^\infty(S^1,\mathbb{R}^2)$-direction. Then TG is given by

$$TG(c,h;k,\ell) = (c, G_c(h,\ .\ ); k, D_{(c,k)}G_c(h,\ .\ ) + G_c(\ell,\ .\ ))$$

Moreover, if $X = (c,h;k,\ell)$ then we will write $X_1 = k$ for its first vector component and $X_2 = \ell$ for the second vector component. Note that only these smooth functions on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ whose derivative lies in the image of G in the cotangent bundle have G-gradients. This requirement has only to be satisfied for the first derivative, for the higher ones it follows (see [9]). We shall denote by $C^\infty_G(\operatorname{Imm}(S^1,\mathbb{R}^2))$ the space of such smooth functions.

We shall always assume that G is invariant under the reparametrization group $\operatorname{Diff}(S^1)$, hence each such metric induces a Riemann-metric on the quotient space $B_i(S^1,\mathbb{R}^2)=\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$.

In the sequel we shall further assume that that *the weak Riemannian metric G itself admits G-gradients with respect to the variable c in the following sense:*

$$\boxed{D_{c,m}G_c(h,k)=G_c(m,H_c(h,k))=G_c(K_c(m,h),k)} \quad \text{where}$$
$$H,K:\operatorname{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2)\to C^\infty(S^1,\mathbb{R}^2)$$
$$(c,h,k)\mapsto H_c(h,k),K_c(h,k)$$

are smooth and bilinear in h,k.

Note that H and K could be expressed in (abstract) index notation as $g_{ij,k}g^{kl}$ and $g_{ij,k}g^{il}$. We will check and compute these gradients for several concrete metrics below.

2.2. The fundamental symplectic form on $T\operatorname{Imm}(S^1,\mathbb{R}^2)$ induced by a weak Riemannian metric. The basis of Hamiltonian theory is the natural 1-form on the cotangent bundle $T^*\operatorname{Imm}(S^1,\mathbb{R}^2)$ given by:

$$\Theta:T(T^*\operatorname{Imm}(S^1,\mathbb{R}^2))=\operatorname{Imm}(S^1,\mathbb{R}^2)\times\mathcal{D}(S^1)^2\times C^\infty(S^1,\mathbb{R}^2)\times\mathcal{D}(S^1)^2\to\mathbb{R}$$
$$(c,\alpha;h,\beta)\mapsto\langle\alpha,h\rangle.$$

The pullback via the mapping $G:T\operatorname{Imm}(S^1,\mathbb{R}^2)\to T^*\operatorname{Imm}(S^1,\mathbb{R}^2)$ of the 1-form Θ is then:

$$(G^*\Theta)_{(c,h)}(c,h;k,\ell)=G_c(h,k).$$

Thus the symplectic form $\omega=-dG^*\Theta$ on $T\operatorname{Imm}(S^1,\mathbb{R}^2)$ can be computed as follows, where we use the constant vector fields $(c,h)\mapsto(c,h;k,\ell)$:

$$\begin{aligned}\omega_{(c,h)}((k_1,\ell_1),(k_2,\ell_2))&=-d(G^*\Theta)((k_1,\ell_1),(k_2,\ell_2))|_{(c,h)}\\&=-D_{(c,k_1)}G_c(h,k_2)-G_c(\ell_1,k_2)+D_{(c,k_2)}G_c(h,k_1)+G_c(\ell_2,k_1)\\&=G_c\big(k_2,H_c(h,k_1)-K_c(k_1,h)\big)+G_c(\ell_2,k_1)-G_c(\ell_1,k_2)\end{aligned}\tag{1}$$

2.3. The Hamiltonian vector field mapping. Here we compute the Hamiltonian vectorfield $\operatorname{grad}^\omega(f)$ associated to a smooth function f on the tangent space $T\operatorname{Imm}(S^1,\mathbb{R}^2)$, that is $f\in C^\infty_G(\operatorname{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2))$ assuming that it has smooth G-gradients in both factors. See [9], section 48. Using the explicit formulas in **2.2**, we have:

$$\begin{aligned}&\omega_{(c,h)}\left(\operatorname{grad}^\omega(f)(c,h),(k,\ell)\right)=\omega_{(c,h)}\left((\operatorname{grad}^\omega_1(f)(c,h),\operatorname{grad}^\omega_2(f)(c,h)),(k,\ell)\right)=\\&=G_c\big(k,H_c\big(h,\operatorname{grad}^\omega_1(f)(c,h)\big)\big)-G_c(K_c(\operatorname{grad}^\omega_1(f)(c,h),h),k)\\&\quad+G_c(\ell,\operatorname{grad}^\omega_1(f)(c,h))-G_c(\operatorname{grad}^\omega_2(f)(c,h),k)\end{aligned}$$

On the other hand, by the definition of the ω-gradient we have

$$\omega_{(c,h)}\left(\operatorname{grad}^\omega(f)(c,h),(k,\ell)\right)=df(c,h)(k,\ell)=D_{(c,k)}f(c,h)+D_{(h,\ell)}f(c,h)$$

$$= G_c(\mathrm{grad}_1^G(f)(c,h),k) + G_c(\mathrm{grad}_2^G(f)(c,h),\ell)$$

and we get the expression of the Hamiltonian vectorfield:

$$\boxed{\begin{aligned}&\mathrm{grad}_1^\omega(f)(c,h) = \mathrm{grad}_2^G(f)(c,h)\\ &\mathrm{grad}_2^\omega(f)(c,h) = -\mathrm{grad}_1^G(f)(c,h) + H_c\big(h,\mathrm{grad}_2^G(f)(c,h)\big) - K_c(\mathrm{grad}_2^G(f)(c,h),h)\end{aligned}}$$

Note that for a smooth function f on $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ the ω-gradient exists if and only if both G-gradients exist.

2.4. The geodesic equation. The geodesic flow is defined by a vector field on $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$. One way to define this vector field is as the Hamiltonian vector field of the energy function

$$E(c,h) = \frac{1}{2}G_c(h,h), \qquad E : \mathrm{Imm}(S^1,\mathbb{R}^2)\times C^\infty(S^1,\mathbb{R}^2) \to \mathbb{R}.$$

The two partial G-gradients are:

$$\begin{aligned}G_c(\mathrm{grad}_2^G(E)(c,h),\ell) &= d_2E(c,h)(\ell) = G_c(h,\ell)\\ \mathrm{grad}_2^G(E)(c,h) &= h\\ G_c(\mathrm{grad}_1^G(E)(c,h),k) &= d_1E(c,h)(k) = \tfrac{1}{2}D_{(c,k)}G_c(h,h)\\ &= \tfrac{1}{2}G_c(k,H_c(h,h))\\ \mathrm{grad}_1^G(E)(c,h) &= \tfrac{1}{2}H_c(h,h).\end{aligned}$$

Thus the geodesic vector field is

$$\begin{aligned}\mathrm{grad}_1^\omega(E)(c,h) &= h\\ \mathrm{grad}_2^\omega(E)(c,h) &= \tfrac{1}{2}H_c(h,h) - K_c(h,h)\end{aligned}$$

and the geodesic equation becomes:

$$\begin{cases} c_t &= h\\ h_t &= \tfrac{1}{2}H_c(h,h) - K_c(h,h)\end{cases} \qquad \text{or} \qquad \boxed{c_{tt} = \tfrac{1}{2}H_c(c_t,c_t) - K_c(c_t,c_t)}$$

This is nothing but the usual formula for the geodesic flow using the Christoffel symbols expanded out using the first derivatives of the metric tensor.

2.5. The momentum mapping for a G-isometric group action. We consider now a (possibly infinite dimensional regular) Lie group with Lie algebra $\mathfrak{g}$ with a right action $g \mapsto r^g$ by isometries on $\mathrm{Imm}(S^1,\mathbb{R}^2)$. If $\mathfrak{X}(\mathrm{Imm}(S^1,\mathbb{R}^2))$ denotes the set of vector fields on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, we can specify this action by the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(\mathrm{Imm}(S^1,\mathbb{R}^2))$, which will be a bounded Lie algebra homomorphism. The fundamental vector field $\zeta_X, X \in \mathfrak{g}$ is the infinitesimal action in the sense:

$$\zeta_X(c) = \partial_t|_0 r^{\exp(tX)}(c).$$

We also consider the tangent prolongation of this action on $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ where the fundamental vector field is given by

$$\zeta_X^{T\,\mathrm{Imm}} : (c,h) \mapsto (c,h;\zeta_X(c), D_{(c,h)}(\zeta_X)(c) =: \zeta'_X(c,h))$$

The basic assumption is that the action is by isometries,

$$G_c(h,k) = ((r^g)^*G)_c(h,k) = G_{r^g(c)}(T_c(r^g)h, T_c(r^g)k).$$

Differentiating this equation at $g=e$ in the direction $X\in\mathfrak g$ we get

(1) $$0 = D_{(c,\zeta_X(c))}G_c(h,k) + G_c(\zeta'_X(c,h),k) + G_c(h,\zeta'_X(c,k))$$

The key to the Hamiltonian approach is to define the group action by Hamiltonian flows. To do this, we define the *momentum map* $j:\mathfrak g\to C^\infty_G(T\operatorname{Imm}(S^1,\mathbb R^2),\mathbb R)$ by:

$$\boxed{j_X(c,h) = G_c(\zeta_X(c),h).}$$

Equivalently, since this map is linear, it is often written as a map

$$\mathcal J : T\operatorname{Imm}(S^1,\mathbb R^2)\to\mathfrak g', \qquad \langle \mathcal J(c,h),X\rangle = j_X(c,h).$$

The main property of the momentum map is that it fits into the following commutative diagram and is a homomorphism of Lie algebras:

$$H^0(T\operatorname{Imm}) \xrightarrow{\;i\;} C^\infty_G(T\operatorname{Imm},\mathbb R) \xrightarrow{\;\operatorname{grad}^\omega\;} \mathfrak X(T\operatorname{Imm},\omega) \longrightarrow H^1(T\operatorname{Imm})$$
$$\text{with } j:\mathfrak g\to C^\infty_G(T\operatorname{Imm},\mathbb R),\quad \zeta^{T\operatorname{Imm}}:\mathfrak g\to \mathfrak X(T\operatorname{Imm},\omega)$$

where $\mathfrak X(T\operatorname{Imm},\omega)$ is the space of vector fields on $T\operatorname{Imm}$ whose flow leaves ω fixed. We need to check that:

$$\zeta_X(c) = \operatorname{grad}^\omega_1(j_X)(c,h) = \operatorname{grad}^G_2(j_X)(c,h)$$
$$\zeta'_X(c,h) = \operatorname{grad}^\omega_2(j_X)(c,h) = -\operatorname{grad}^G_1(j_X)(c,h) + H_c(h,\zeta_X(c)) - K_c(\zeta_X(c),h)$$

The first equation is obvious. To verify the second equation, we take its inner product with some k and use:

$$\begin{aligned} G\big(k,\operatorname{grad}^G_1(j_X)(c,h)\big) &= D_{(c,k)}j_X(c,h) = D_{(c,k)}G_c(\zeta_X(c),h) + G_c(\zeta'_X(c,k),h)\\ &= G_c(k,H_c(\zeta_X(c),h)) + G_c(\zeta'_X(c,k),h). \end{aligned}$$

Combining this with (1), the second equation follows. Let us check that it is also a homomorphism of Lie algebras using the Poisson bracket:

$$\begin{aligned} \{j_X,j_Y\}(c,h) &= dj_Y(c,h)(\operatorname{grad}^\omega_1(j_X)(c,h),\operatorname{grad}^\omega_2(j_X)(c,h))\\ &= dj_Y(c,h)(\zeta_X(c),\zeta'_X(c,h))\\ &= D_{(c,\zeta_X(c))}G_c(\zeta_Y(c),h) + G_c(\zeta'_Y(c,\zeta_X(x)),h) + G_c(\zeta_Y(c),\zeta'_X(c,h))\\ &= G_c(\zeta'_Y(c,\zeta_X(c)) - \zeta'_X(c,\zeta_Y(c)),h) \qquad \text{by (1)}\\ &= G_c([\zeta_X,\zeta_Y](c),h) = G_c(\zeta_{[X,Y]}(c),h) = j_{[X,Y]}(c). \end{aligned}$$

Note also that $\mathcal J$ is equivariant for the group action, by the following arguments: For g in the Lie group let r^g be the right action on $\operatorname{Imm}(S^1,\mathbb R^2)$, then $T(r^g)\circ\zeta_X\circ(r^g)^{-1} = \zeta_{\operatorname{Ad}(g^{-1})X}$. Since r^g is an isometry the mapping $T(r^g)$ is a symplectomorphism for ω, thus $\operatorname{grad}^\omega$ is equivariant. Thus $j_X\circ T(r^g) = j_{\operatorname{Ad}(g)X}$ plus a possible constant which we can rule out since $j_X(c,h)$ is linear in h.

By Emmy Noether's theorem, along any geodesic $t\mapsto c(t,\ .\ )$ this momentum mapping is constant, thus for any $X\in\mathfrak g$ we have

$$\boxed{\langle\mathcal J(c,c_t),X\rangle = j_X(c,c_t) = G_c(\zeta_X(c),c_t) \quad \text{is constant in } t.}$$

We can apply this construction to the following group actions on $\operatorname{Imm}(S^1,\mathbb R^2)$.

- The smooth right action of the group $\mathrm{Diff}(S^1)$ on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, given by composition from the right: $c \mapsto c\circ\varphi$ for $\varphi\in\mathrm{Diff}(S^1)$. For $X\in\mathfrak{X}(S^1)$ the fundamental vector field is then given by
$$\zeta_X^{\mathrm{Diff}}(c) = \zeta_X(c) = \partial_t|_0(c\circ \mathrm{Fl}_t^X) = c_\theta.X$$
where Fl_t^X denotes the flow of X. The *reparametrization momentum*, for any vector field X on S^1 is thus:
$$j_X(c,h) = G_c(c_\theta.X,h).$$
Assuming the metric is reparametrization invariant, it follows that on any geodesic $c(\theta,t)$, the expression $G_c(c_\theta.X,c_t)$ is constant for all X.
- The left action of the Euclidean motion group $M(2)=\mathbb{R}^2\rtimes SO(2)$ on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ given by $c\mapsto e^{aJ}c+B$ for $(B,e^{aJ})\in\mathbb{R}^2\times SO(2)$. The fundamental vector field mapping is
$$\zeta_{(B,a)}(c) = aJc + B$$
The *linear momentum* is thus $G_c(B,h), B\in\mathbb{R}^2$ and if the metric is translation invariant, $G_c(B,c_t)$ will be constant along geodesics. The *angular momentum* is similarly $G_c(Jc,h)$ and if the metric is rotation invariant, then $G_c(Jc,c_t)$ will be constant along geodesics.
- The action of the scaling group of $\mathbb{R}$ given by $c\mapsto e^r c$, with fundamental vector field $\zeta_a(c)=a.c.$ If the metric is scale invariant, then the *scaling momentum* $G_c(c,c_t)$ will also be invariant along geodesics.

2.6. **Metrics and momenta on the group of diffeomorphisms.** Very similar things happen when we consider metrics on the group $\mathrm{Diff}(\mathbb{R}^2)$. As above, the tangent space to $\mathrm{Diff}(\mathbb{R}^2)$ at the identity is the vector space of vector fields $\mathfrak{X}(\mathbb{R}^2)$ on $\mathbb{R}^2$ and we can identify $T\,\mathrm{Diff}(\mathbb{R}^2)$ with the product $\mathrm{Diff}(\mathbb{R}^2)\times\mathfrak{X}(\mathbb{R}^2)$ using right multiplication in the group to identify the tangent at a point φ with that at the identity. The definition of this product decomposition means that right multiplication by ψ carries (φ,X) to $(\varphi\circ\psi,X)$. As usual, suppose that conjugation $\varphi\mapsto\psi\circ\varphi\circ\psi^{-1}$ has the derivative at the identity given by the linear operator Ad_ψ on the Lie algebra $\mathfrak{X}(\mathbb{R}^2)$. It is easy to calculate the explicit formula for Ad:
$$\mathrm{Ad}_\psi(X) = (D\psi\cdot X)\circ\psi^{-1}.$$
Then left multiplication by ψ on $\mathrm{Diff}(\mathbb{R}^2)\times\mathfrak{X}(\mathbb{R}^2)$ is given by $(\varphi,X)\mapsto(\psi\circ\varphi,\mathrm{Ad}_\psi(X))$. We now want to carry over the ideas of **2.5** replacing the space $\mathrm{Imm}(S^1,\mathbb{R}^2)$ by $\mathrm{Diff}(\mathbb{R}^2)$ and the group action there by the right action of $\mathrm{Diff}(\mathbb{R}^2)$ on itself. The Lie algebra $\mathfrak{g}$ is therefore $\mathfrak{X}(\mathbb{R}^2)$ and the fundamental vector field $\zeta_X(c)$ is now the vector field with value
$$\zeta_X(\varphi) = \partial_t|_0(\varphi\mapsto\varphi\circ\exp(tX)\circ\varphi^{-1}) = \mathrm{Ad}_\varphi(X)$$
at the point φ. We now assume we have a positive definite inner product $G(X,Y)$ on the Lie algebra $\mathfrak{X}(\mathbb{R}^2)$ and that we use right translation to extend it to a Riemannian metric on the full group $\mathrm{Diff}(\mathbb{R}^2)$. This metric being, by definition, invariant under the right group action, we have the setting for momentum. The theory of the last section tells us to define the momentum mapping by:
$$j_X(\varphi,Y) = G(\zeta_X(\varphi),Y).$$

Noether's theorem tells us that if $\varphi(t)$ is a geodesic in $\mathrm{Diff}(\mathbb{R}^2)$ for this metric, then this momentum will be constant along the lift of this geodesic to the tangent space. The lift of $\varphi(t)$, in the product decomposition of the tangent space is the curve:

$$t \mapsto (\varphi(t), \partial_t(\varphi) \circ \varphi^{-1}(t))$$

hence the theorem tells us that:

$$G(\mathrm{Ad}_{\varphi(t)}(X), \partial_t(\varphi) \circ \varphi^{-1}(t)) = \text{constant}$$

for all X. If we further assume that Ad has an adjoint with respect to G:

$$G(\mathrm{Ad}_\varphi(X), Y) \equiv G(X, \mathrm{Ad}^*_\varphi(Y))$$

then this invariance of momentum simplifies to:

$$\boxed{\mathrm{Ad}^*_{\varphi(t)}\left(\partial_t(\varphi) \circ \varphi^{-1}(t)\right) = \text{constant}}$$

This is a very strong invariance and it encodes an integrated form of the geodesic equations for the group.

3. Geodesic equations and conserved momenta for almost local Riemannian metrics

3.1. The general almost local metric G^Φ. We have introduced above the Φ-metrics:

$$G^\Phi_c(h,k) := \int_{S^1} \Phi(\ell_c, \kappa_c(\theta))\langle h(\theta), k(\theta)\rangle ds.$$

Since $\ell(c)$ is an integral operator the integrand is not a local operator, but the nonlocality is very mild. We call it *almost local.* The metric G^Φ is invariant under the reparametization group $\mathrm{Diff}(S^1)$ and under the Euclidean motion group. Note (see [13], 2.2) that

$$\begin{aligned}
D_{(c,h)}\ell_c &= \int_{S^1} \frac{\langle h_\theta, c_\theta\rangle}{|c_\theta|} d\theta = \int_{S^1} \langle D_s(h), v\rangle ds \\
&= -\int_{S^1} \langle h, D_s(v)\rangle ds = -\int_{S^1} \kappa(c)\langle h, n\rangle ds \\
D_{(c,h)}\kappa_c &= \frac{\langle Jh_\theta, c_{\theta\theta}\rangle}{|c_\theta|^3} + \frac{\langle Jc_\theta, h_{\theta\theta}\rangle}{|c_\theta|^3} - 3\kappa(c)\frac{\langle h_\theta, c_\theta\rangle}{|c_\theta|^2} \\
&= \langle D_s^2(h), n\rangle - 2\kappa\langle D_s(h), v\rangle.
\end{aligned}$$

We compute the G^Φ-gradients of $c \mapsto G^\Phi_c(h,k)$:

$$\begin{aligned}
D_{(c,m)}G^\Phi_c(h,k) &= \int_{S^1} \Big(\partial_1\Phi(\ell,\kappa).D_{(c,m)}\ell_c.\langle h,k\rangle + \partial_2\Phi(\ell,\kappa).D_{(c,m)}\kappa_c.\langle h,k\rangle \\
&\qquad + \Phi(\ell,\kappa).\langle h,k\rangle\langle .D_s(m), v\rangle\Big) ds \\
&= -\int_{S^1} \kappa_c\langle m, n\rangle ds \cdot \int_{S^1} \partial_1\Phi(\ell,\kappa)\langle h,k\rangle ds \\
&\quad + \int_{S^1} \Big(\partial_2\Phi(\ell,\kappa)(\langle D_s^2(m), n\rangle - 2\kappa\langle D_s(m), v\rangle) + \Phi(\ell,\kappa)\langle D_s(m), v\rangle\Big)\langle h,k\rangle ds \\
&= \int_{S^1} \Phi(\ell,\kappa)\Big\langle m\ ,\ \frac{1}{\Phi(\ell,\kappa)}\Big(-\kappa_c\Big(\int \partial_1\Phi(\ell,\kappa)\langle h,k\rangle ds\Big)n + D_s^2\Big(\partial_2\Phi(\ell,\kappa)\langle h,k\rangle n\Big)
\end{aligned}$$

$$+ 2D_s\Big(\partial_2\Phi(\ell,\kappa)\kappa\langle h,k\rangle v\Big) - D_s\Big(\Phi(\ell,\kappa)\langle h,k\rangle v\Big)\Big) \quad \Big\rangle ds$$

According to **2.1** we should rewrite this as

$$D_{(c,m)}G^\Phi_c(h,k) = G^\Phi_c(K^\Phi_c(m,h),k) = G^\Phi_c\big(m, H^\Phi_c(h,k)\big),$$

where the two G^Φ-gradients K^Φ and H^Φ of $c\mapsto G^\Phi_c(h,k)$ are given by:

$$\begin{aligned} K^\Phi_c(m,h) &= -\Big(\int_{S^1}\kappa_c\langle m,n\rangle ds\Big)\frac{\partial_1\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}h \\ &\qquad + \frac{\partial_2\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}\Big(\langle D_s^2(m),n\rangle - 2\kappa\langle D_s(m),v\rangle\Big)h + \langle D_s(m),v\rangle h \\ H^\Phi_c(h,k) &= \frac{1}{\Phi(\ell,\kappa)}\Big(- \Big(\kappa_c\int\partial_1\Phi(\ell,\kappa)\langle h,k\rangle ds\Big)n + D_s^2\Big(\partial_2\Phi(\ell,\kappa)\langle h,k\rangle n\Big) + \\ &\qquad + 2D_s\Big(\partial_2\Phi(\ell,\kappa)\kappa\langle h,k\rangle v\Big) - D_s\Big(\Phi(\ell,\kappa)\langle h,k\rangle v\Big)\Big) \end{aligned}$$

By substitution into the general formula of **2.4**, this gives the geodesic equation for G^Φ, but in a form which doesn't seem very revealing, hence we omit it. Below we shall give the equation for the special case of horizontal geodesics, i.e. geodesics in B_i.

3.2. Conserved momenta for G^Φ. According to **2.5** the momentum mappings for the reparametrization, translation and rotation group actions are conserved along any geodesic $t\mapsto c(t,\ .\ )$:

$\Phi(\ell_c,\kappa_c)\langle v,c_t\rangle\lvert c_\theta\rvert^2 \ \in \mathfrak{X}(S^1)$	reparametrization momentum
$\int_{S^1}\Phi(\ell_c,\kappa_c)c_t ds \ \in \mathbb{R}^2$	linear momentum
$\int_{S^1}\Phi(\ell_c,\kappa_c)\langle Jc,c_t\rangle ds \in \mathbb{R}$	angular momentum

Note that setting the reparametrization momentum to 0 and doing symplectic reduction there amounts exactly to investigating the quotient space $B_i(S^1,\mathbb{R}^2) = \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ and using horizontal geodesics for doing so; a horizontal geodesic is one for which $\langle v,c_t\rangle = 0$; or equivalently it is G^Φ-normal to the $\operatorname{Diff}(S^1)$-orbits. If it is normal at one time it is normal forever (since the reparametrization momentum is conserved). This was the approach taken in [13].

3.3. Horizontality for G^Φ. The tangent vectors to the $\operatorname{Diff}(S^1)$ orbit through c are $T_c(c\circ\operatorname{Diff}(S^1)) = \{X.c_\theta : X\in C^\infty(S^1,\mathbb{R})\}$. Thus the bundle of horizontal vectors is

$$\begin{aligned} \mathcal{N}_c &= \{h\in C^\infty(S^1,\mathbb{R}^2) : \langle h,v\rangle = 0\} \\ &= \{a.n\in C^\infty(S^1,\mathbb{R}^2) : a\in C^\infty(S^1,\mathbb{R})\} \end{aligned}$$

A tangent vector $h\in T_c\operatorname{Imm}(S^1,\mathbb{R}^2) = C^\infty(S^1,\mathbb{R}^2)$ has an orthonormal decomposition

$$h = h^\top + h^\perp \in T_c(c\circ\operatorname{Diff}^+(S^1))\oplus\mathcal{N}_c \qquad \text{where}$$

$$h^\top = \langle h, v\rangle v \in T_c(c \circ \mathrm{Diff}^+(S^1)),$$
$$h^\perp = \langle h, n\rangle n \in \mathcal{N}_c,$$

into smooth tangential and normal components, independent of the choice of $\Phi(\ell, \kappa)$. For the following result the proof given in [13], 2.5 works without any change:

Lemma. *For any smooth path c in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ there exists a smooth path φ in $\mathrm{Diff}(S^1)$ with $\varphi(0, \ .\ ) = \mathrm{Id}_{S^1}$ depending smoothly on c such that the path e given by $e(t,\theta) = c(t, \varphi(t,\theta))$ is horizontal: $e_t \perp e_\theta$.* □

Consider a path $t \mapsto c(\cdot, t)$ in the manifold $\mathrm{Imm}(S^1, \mathbb{R}^2)$. It projects to a path $\pi \circ c$ in $B_i(S^1, \mathbb{R}^2)$ whose energy is called the *horizontal energy* of c:

$$\begin{aligned} E^{\mathrm{hor}}_{G^\Phi}(c) &= E_{G^\Phi}(\pi \circ c) = \tfrac{1}{2} \int_a^b G^\Phi_{\pi(c)}(T_c\pi.c_t, T_c\pi.c_t)\, dt \\ &= \tfrac{1}{2} \int_a^b G^\Phi_c(c_t^\perp, c_t^\perp)\, dt = \tfrac{1}{2} \int_a^b \int_{S^1} \Phi(\ell_c, \kappa_c)\langle c_t^\perp, c_t^\perp\rangle ds\, dt \end{aligned}$$

$$\boxed{E^{\mathrm{hor}}_{G^\Phi}(c) = \tfrac{1}{2} \int_a^b \int_{S^1} \Phi(\ell_c, \kappa_c)\langle c_t, n\rangle^2\, d\theta dt}$$

For a horizontal path this is just the usual energy. As in [13], 3.12 we can express $E^{\mathrm{hor}}(c)$ as an integral over the graph S of c, the immersed surface $S \subset \mathbb{R}^3$ parameterized by $(t,\theta) \mapsto (t, c(t,\theta))$, in terms of the surface area $d\mu_S = |\Phi_t \times \Phi_\theta|\, d\theta\, dt$ and the unit normal $n_S = (n_S^0, n_S^1, n_S^2)$ of S:

$$E^{\mathrm{hor}}_{G^\Phi}(c) = \tfrac{1}{2} \int_{[a,b]\times S^1} \Phi(\ell_c, \kappa_c) \frac{|n_S^0|^2}{\sqrt{1 - |n_S^0|^2}}\, d\mu_S$$

Here the final expression is only in terms of the surface S and its fibration over the time axis, and is valid for any path c. This anisotropic area functional has to be minimized in order to prove that geodesics exists between arbitrary curves (of the same degree) in $B_i(S^1, \mathbb{R}^2)$.

3.4. **The horizontal geodesic equation.** Let $c(\theta, t)$ be a horizontal geodesic for the metric G^Φ. Then $c_t(\theta, t) = a(\theta, t).n(\theta, t)$. Denote the integral of a function over the curve with respect to arclength by a bar. Then the geodesic equation for horizontal geodesics is:

$$\boxed{\begin{aligned} a_t = \frac{-1}{2\Phi}\Big(&\big(-\kappa\Phi + \kappa^2\partial_2\Phi\big)\, a^2 - D_s^2\left(\partial_2\Phi \cdot a^2\right) + 2\partial_2\Phi \cdot a D_s^2(a) \\ &- 2\partial_1\Phi \cdot \overline{(\kappa a)} \cdot a + \overline{(\partial_1\Phi \cdot a^2)} \cdot \kappa\Big) \end{aligned}}$$

This comes immediately from the formulas for H and K in the metric G^Φ when you substitute $m = h = k = a.n$ and consider only the n-part. We obtain in this case:

$$\Phi \cdot \langle K, n\rangle = -\overline{(\kappa a)}.\partial_1\Phi.a + \partial_2\Phi.D_s^2(a).a + \partial_2\Phi.\kappa^2 a^2 - \Phi\kappa a^2$$
$$\Phi \cdot \langle H, n\rangle = -\overline{(\partial_1\Phi a^2)}.\kappa + D_s^2(\partial_2\Phi.a^2) + \partial_2\Phi.\kappa^2 a^2 - \Phi\kappa a^2.$$

and the geodesic formula follows by substitution.

3.5. Curvature on $B_{i,f}(S^1,\mathbb{R}^2)$ for G^Φ. We compute the curvature of $B_i(S^1,\mathbb{R}^2)$ in the general almost local metric G^Φ. We proceed as in [13], 2.4.3. We use the following chart near $C \in B_i(S^1,\mathbb{R}^2)$. Let $c \in \operatorname{Imm}_f(S^1,\mathbb{R}^2)$ be parametrized by arclength with $\pi(c) = C$ of length L, with unit normal n_c. We assume that the parameter θ runs in the scaled circle S^1_L below.

$$\begin{aligned}
&\psi : C^\infty(S^1_L,(-\varepsilon,\varepsilon)) \to \operatorname{Imm}_f(S^1_L,\mathbb{R}^2), \qquad \mathcal{Q}(c) := \psi(C^\infty(S^1_L,(-\varepsilon,\varepsilon)))\\
&\psi(f)(\theta) = c(\theta) + f(\theta)n_c(\theta) = c(\theta) + f(\theta)ic'(\theta),\\
&\pi\circ\psi : C^\infty(S^1_L,(-\varepsilon,\varepsilon)) \to B_{i,f}(S^1,\mathbb{R}^2),
\end{aligned}$$

where ε is so small that $\psi(f)$ is an embedding for each f. We have (see [13], 2.4.3)

$$\begin{aligned}
\psi(f)' &= c' + f'ic' + fic'' = (1-f\kappa_c)c' + f'ic'\\
\psi(f)'' &= c'' + f''ic' + 2f'ic'' + fic''' = -(2f'\kappa_c + f\kappa_c')c' + (\kappa_c + f'' - f\kappa_c^2)ic'\\
n_{\psi(f)} &= \frac{1}{\sqrt{(1-f\kappa_c)^2 + f'^2}}\Big((1-f\kappa_c)ic' - f'c'\Big),\\
T_f\psi.h &= h.ic' \qquad \in C^\infty(S^1,\mathbb{R}^2) = T_{\psi(f)}\operatorname{Imm}_f(S^1_L,\mathbb{R}^2)\\
&= \frac{h(1-f\kappa_c)}{\sqrt{(1-f\kappa_c)^2+f'^2}}n_{\psi(f)} + \frac{hf'}{(1-f\kappa_c)^2+f'^2}\psi(f)',\\
(T_f\psi.h)^\perp &= \frac{h(1-f\kappa_c)}{\sqrt{(1-f\kappa_c)^2+f'^2}}n_{\psi(f)} \qquad \in \mathcal{N}_{\psi(f)},\\
\kappa_{\psi(f)} &= \frac{1}{((1-f\kappa_c)^2+f'^2)^{3/2}}\langle i\psi(f)',\psi(f)''\rangle\\
&= \kappa_c + (f'' + f\kappa_c^2) + (f^2\kappa_c^3 + \tfrac12 f'^2\kappa_c + ff'\kappa_c' + 2ff''\kappa_c) + O(f^3)\\
\ell(\psi(f)) &= \int_{S^1_L}|\psi(f)|\,d\theta = \int_{S^1_L}(1 - 2f\kappa_c + f^2\kappa_c^2 + f'^2)^{1/2}\,d\theta\\
&= \int_{S^1_L}\Big(1 - f\kappa_c + \frac{f'^2}{2} + O(f^3)\Big)\,d\theta = L - \overline{f\kappa_c} + \tfrac12\overline{f'^2} + O(f^3)
\end{aligned}$$

where we use the shorthand $\overline{g} = \int_{S^1_L} g(\theta)\,d\theta = \int_{S^1_L} g(\theta)\,ds$. Let G^Φ denote also the induced metric on $B_{i,f}(S^1_L,\mathbb{R}^2)$. Since π is a Riemannian submersion, for $f \in C^\infty(S^1_L,(-\varepsilon,\varepsilon))$ and $h,k \in C^\infty(S^1_L,\mathbb{R})$ we have

$$\begin{aligned}
&((\pi\circ\psi)^*G^\Phi)_f(h,k) = G^\Phi_{\pi(\psi(f))}\Big(T_f(\pi\circ\psi)h, T_f(\pi\circ\psi)k\Big)\\
&= G^\Phi_{\psi(f)}\Big((T_f\psi.h)^\perp,(T_f\psi.k)^\perp\Big) = \int_{S^1_L}\Phi(\ell(\psi(f)),\kappa_{\psi(f)})\frac{hk(1-f\kappa_c)^2}{\sqrt{(1-f\kappa_c)^2+f'^2}}\,d\theta
\end{aligned}$$

We have to compute second derivatives in f of this. For that we expand the main contributing expressions in f to order 2:

$$\begin{aligned}
&(1-f\kappa)^2(1-2f\kappa+f^2\kappa^2+f'^2)^{-1/2} = 1 - f\kappa - \tfrac12 f'^2 + O(f^3)\\
&\Phi(\ell,\kappa) = \Phi(L,\kappa_c) + \partial_1\Phi(L,\kappa_c)(\ell - L) + \partial_2\Phi(L,\kappa_c)(\kappa-\kappa_c)
\end{aligned}$$

$$+\partial_1\partial_2\Phi(L,\kappa_c)(\ell-L)(\kappa-\kappa_c)$$
$$+\frac{\partial_1^2\Phi(L,\kappa_c)}{2}(\ell-L)^2+\frac{\partial_2^2\Phi(L,\kappa_c)}{2}(\kappa-\kappa_c)^2+O(3)$$

We simplify notation as $\kappa=\kappa_c$, $\Phi=\Phi(L,\kappa_c)$, $((\pi\circ\psi)^*G^\Phi)_f=G^\Phi_f$ etc. and expand the metric:

$$\begin{aligned}G^\Phi_f(h,k)=\int_{S^1_L} hk\Big(&\Phi-\partial_1\Phi.\overline{f\kappa}+\partial_2\Phi.(f''+f\kappa^2)-\Phi.f\kappa\\ &+\tfrac12\partial_1\Phi.\overline{f'^2}+\partial_2\Phi.(f^2\kappa^3+\tfrac12 f'^2\kappa+ff'\kappa'+2ff''\kappa)\\ &-\partial_1\partial_2\Phi.\overline{f\kappa}(f''+f\kappa^2)+\frac{\partial_1^2\Phi}{2}(\overline{f\kappa})^2+\frac{\partial_2^2\Phi}{2}(f''+f\kappa^2)^2\\ &+\partial_1\Phi.f\kappa.\overline{f\kappa}-\partial_2\Phi.f\kappa.(f''+f\kappa^2)-\Phi.\tfrac12 f'^2\Big)d\theta+O(f^3)\end{aligned}$$

Note that $G^\varphi_0(h,k)=\int_{S^1_L}hk\Phi\,d\theta$. We differentiate the metric and compute the Christoffel symbol at the center $f=0$

$$\begin{aligned}&-2G^A_0(\Gamma_0(h,k),l)=-dG^A(0)(l)(h,k)+dG^A(0)(h)(k,l)+dG^A(0)(k)(l,h)\\ &=\int_{S^1_L}\Big(-\partial_1\Phi.\overline{h\kappa}.kl-\partial_1\Phi.h.\overline{k\kappa}.l+\partial_1\Phi.hk\int l\kappa\,d\theta_1-\partial_2\Phi''.hkl\\ &\qquad-2\partial_2\Phi'.h'kl-2\partial_2\Phi'.hk'l-2\partial_2\Phi.h'k'l+\partial_2\Phi.hkl\kappa^2-\Phi.hkl\kappa\Big)d\theta\end{aligned}$$

Thus

$$\begin{aligned}\Gamma_0(h,k)=\frac{1}{2\Phi}\Big(&\partial_1\Phi.(\overline{h\kappa}.k+h.\overline{k\kappa})-\kappa\overline{\partial_1\Phi.hk}\\ &+\partial_2\Phi''.hk+2\partial_2\Phi'.h'k+2\partial_2\Phi'hk'+2\partial_2\Phi.h'k'\\ &-\partial_2\Phi.hk\kappa^2+\Phi.hk\kappa\Big)\end{aligned}$$

Letting $h=k=f_t=a$, this leads to the geodesic equation from **3.4**. For the sectional curvature we use the following formula which is valid in a chart:

$$\begin{aligned}&2R_f(m,h,m,h)=2G^A_f(R_f(m,h)m,h)=\\ &=-2d^2G^A(f)(m,h)(h,m)+d^2G^A(f)(m,m)(h,h)+d^2G^A(f)(h,h)(m,m)\\ &\quad-2G^A(\Gamma(h,m),\Gamma(m,h))+2G^A(\Gamma(m,m),\Gamma(h,h))\end{aligned}$$

The sectional curvature at the two-dimensional subspace $P_f(m,h)$ of the tangent space which is spanned by m and h is then given by:

$$k_f(P_f(m,h))=-\frac{G^\Phi_f(R(m,h)m,h)}{\|m\|^2\|h\|^2-G^\Phi_f(m,h)^2}.$$

We compute this directly for $f=0$, using the expansion up to order 2 of $G^A_f(h,k)$ and the Christoffels. We let $W(\theta_1,\theta_2)=h(\theta_1)m(\theta_2)-h(\theta_2)m(\theta_1)$ so that its second derivative $\partial_2W(\theta_1,\theta_1)=W_2(\theta_1,\theta_1)=h(\theta_1)m'(\theta_1)-h'(\theta_1)m(\theta_1)$ is the Wronskian of h and m. Then we have our final result for the main expression in the horizontal sectional curvature, where we use $\int=\int_{S^1_L}$, $\overline{g}=\int_{S^1_L}g\,ds$, and $\Phi_1=\partial_1\Phi$ etc. Also recall that the base curve is parametrized by arc-length.

$$\begin{aligned}
&R_0^\Phi(m,h,m,h) = G_0^\Phi(R_0(m,h)m,h) = \\
&= \int \Big(\frac{\kappa.\Phi_2 - \Phi}{2} + \frac{\Phi_2.\Phi_2'' - 2(\Phi_2')^2 - (\Phi_2\kappa)^2}{2\Phi}\Big)(\theta_1) W_2(\theta_1,\theta_1)^2\, d\theta_1 \\
&+ \int \frac{\Phi_{22}(\theta_1)}{2} W_{22}(\theta_1,\theta_1)^2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_1'\Phi_2}{\Phi} - \frac{\Phi_1\Phi_2\Phi_1'}{\Phi^2}\Big)(\theta_1) W_2(\theta_1,\theta_1) \int W(\theta_1,\theta_2)\kappa(\theta_2)\, d\theta_2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_1\Phi_2}{\Phi} - \Phi_{12}\Big)(\theta_1) W_{22}(\theta_1,\theta_1) \int W(\theta_1,\theta_2)\kappa(\theta_2)\, d\theta_2\, d\theta_1 \\
&+ \iint \frac{\Phi_1(\theta_1)}{2}\Big(1 - \frac{\Phi_2.\kappa}{\Phi}(\theta_2)\Big) W_1(\theta_1,\theta_2)^2\, d\theta_2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_2.\kappa^3 - \Phi_2''.\kappa}{4\Phi} - \frac{\kappa^2}{4} + \Big(\frac{\Phi_2'.\kappa}{2\Phi}\Big)' + \overline{\big(\frac{\kappa^2}{8\Phi}\big).\Phi_1}\Big)(\theta_1)\Phi_1(\theta_2) W(\theta_1,\theta_2)^2\, d\theta_2\, d\theta_1 \\
&+ \iiint \Big(\frac{\Phi_{11}}{2} - \frac{\Phi_1^2}{4\Phi}\big)(\theta_1) - \Phi_1(\theta_1)\frac{\Phi_1}{2\Phi}(\theta_2)\Big) \\
&\qquad\qquad \kappa(\theta_2)\kappa(\theta_3) W(\theta_1,\theta_2) W(\theta_1,\theta_3)\, d\theta_2\, d\theta_1\, d\theta_3
\end{aligned}$$

3.6. Special case: the metric G^A. If we choose $\Phi(\ell_c,\kappa_c) = 1 + A\kappa_c^2$ then we obtain the metric used in [13], given by

$$G_c^A(h,k) = \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle ds.$$

As shown in our earlier paper, $\sqrt{\ell}$ is Lipschitz in this metric and the metric dominates the Frechet metric.

The horizontal geodesic equation for the G^A-metric reduces to

$$a_t = \frac{\frac12\kappa_c a^2 - A\left(a^2(-D_s^2(\kappa_c) + \frac12\kappa_c^3) - 4D_s(\kappa_c)aD_s(a) - 2\kappa_c D_s(a)^2\right)}{1 + A\kappa_c^2}$$

as found in [13],4.2. Along a geodesic $t \mapsto c(t,\ .\ )$ we have the following conserved quantities:

$(1 + A\kappa_c^2)\langle v, c_t\rangle \lvert c_\theta\rvert^2\ \in \mathfrak{X}(S^1)$	reparametrization momentum
$\int_{S^1}(1 + A\kappa_c^2)c_t ds\ \in \mathbb{R}^2$	linear momentum
$\int_{S^1}(1 + A\kappa_c^2)\langle Jc, c_t\rangle ds\ \in \mathbb{R}$	angular momentum

For $\Phi(\ell,\kappa) = 1 + A\kappa^2$ we have $\partial_1\Phi = 0$, $\partial_2\Phi = 2A\kappa$, $\partial_2^2\Phi = 2A$, and the general curvature formula in **3.5** for the horizontal curvature specializes to the formula in [13], 4.6.4:

$$R_0^\Phi(m,h,m,h) = \int \Big(- \frac{1 + 3A^2\kappa^4 - 4A^2\kappa\kappa'' + 8A^2\kappa'^2}{2(1 + A\kappa^2)} W_2^2 + AW_{22}^2\Big) d\theta.$$

3.7. **Special case: the conformal metrics.** We put $\Phi(\ell(c),\kappa(c)) = \Phi(\ell(c))$ and obtain the metric proposed by Menucci and Yezzi and, for Φ linear, independently by Shah [19]:

$$G_c^\Phi(h,k) = \Phi(\ell_c)\int_{S^1}\langle h,k\rangle ds = \Phi(\ell_c)G_c^0(h,k).$$

All these metrics are conformally equivalent to the basic L^2-metric G^0. As they show, the infimum of path lengths in this metric is positive so long as Φ satifies an inequality $\Phi(\ell)\geq C.\ell$ for some $C>0$. This follows, as in [13], 3.4, by the inequality on area swept out by the curves in a horizontal path $c_t = a.n$:

$$\int |a|.ds \leq \left(\int a^2.ds\right)^{1/2}\cdot \ell^{1/2} \leq \left(\frac{\ell}{\Phi(\ell)}\right)^{1/2}\cdot (G^\Phi(a,a))^{1/2}$$

$$\text{Area swept out} \leq \max_t \left(\frac{\ell_{c(t,\cdot)}}{\Phi(\ell_{c(t,\cdot)})}\right)^{1/2}\cdot \big(G^\Phi\text{-path length}\big) \leq \frac{G^\Phi\text{-path length}}{\sqrt{C}}.$$

The horizontal geodesic equation reduces to:

$$a_t = \frac{\kappa}{2}a^2 - \frac{\partial_1\Phi}{\Phi}\cdot\left(\tfrac{1}{2}\left(\int a^2.ds\right)\kappa - \left(\int \kappa.a.ds\right)a\right)$$

If we change variables and write $b(s,t) = \Phi(\ell(t)).a(s,t)$, then this equation simplifies to:

$$\boxed{b_t = \frac{\kappa}{2\Phi}\left(b^2 - \frac{\partial_1\Phi}{\Phi}\int b^2\right)}$$

Along a geodesic $t\mapsto c(t,\ .\ )$ we have the following conserved quantities:

$\Phi(\ell_c)\langle v,c_t\rangle\lvert c'(\theta)\rvert^2 \ \in \mathfrak{X}(S^1)$	reparametrization momentum
$\Phi(\ell_c)\int_{S^1} c_t ds \ \in \mathbb{R}^2$	linear momentum
$\Phi(\ell_c)\int_{S^1}\langle Jc,c_t\rangle ds \in \mathbb{R}$	angular momentum

For the conformal metrics, sectional curvature has been computed by Shah [19] using the method of local charts from [13]. We specialize formula **3.5** to the case that $\Phi(\ell,\kappa) = \Phi(\ell)$ is independent of κ. Then $\partial_2\Phi = 0$. We also assume that h,m are orthonormal so that $\Phi\overline{h^2} = \Phi\overline{m^2} = 1$ and $\Phi\overline{hm} = 0$. Then the the sectional curvature at the two-dimensional subspace $P_0(m,h)$ of the tangent space which is spanned by m and h is then given by:

$$\begin{aligned}
k_0(P_0(m,h)) &= -\frac{G_0^\Phi(R_0(m,h)m,h)}{\|m\|^2\|h\|^2 - G_0^\Phi(m,h)^2} = \\
&= \tfrac{1}{2}\Phi.\overline{W(h,m)^2} + \frac{\partial_1\Phi}{4\Phi}.(\overline{m^2\kappa^2} + \overline{h^2\kappa^2}) + \frac{3(\partial_1\Phi)^2 - 2\Phi\partial_1^2\Phi}{4\Phi^2}(\overline{h\kappa}^2 + \overline{m\kappa}^2) \\
&\quad - \frac{\partial_1\Phi}{2\Phi}(\overline{m'^2} + \overline{h'^2}) - \frac{(\partial_1\Phi)^2}{4\Phi^3}\overline{\kappa^2}
\end{aligned}$$

which is the same as the equation (11) in [19]. Note that the first line is positive while the last line is negative. The first term is the curvature term for the H^0-metric. The key point about this formula is how many positive terms it has. This makes it very hard to get smooth geodesics in this metric. For example, in the case where $\Phi(\ell) = c.\ell$, the analysis of Shah [19] proves that the infimum of G^Φ path length between two embedded curves C and D is exactly the area of the symmetric difference of their interiors: $\text{Area}(\text{Int}(C)\Delta\,\text{Int}(D))$, but that this length is realized by a smooth path if and only if C and D can be connected by 'grassfire', i.e. a family in which the length $|c_t(\theta,t)| \equiv 1$.

3.8. Special case: the smooth scale invariant metric G^{SI}. Choosing the function $\Phi(\ell,\kappa) = \ell^{-3} + A\frac{\kappa^2}{\ell}$ we obtain the metric:

$$G_c^{SI}(h,k) = \int_{S^1} \Big(\frac{1}{\ell_c^3} + A\frac{\kappa_c^2}{\ell_c}\Big)\langle h,k\rangle ds.$$

The beauty of this metric is that (a) it is scale invariant and (b) $\log(\ell)$ is Lipschitz, hence the infimum of path lengths is always positive. Scale invariance is clear: changing c,h,k to $\lambda\cdot c, \lambda\cdot h, \lambda\cdot k$ changes ℓ to $\lambda\cdot\ell$ and κ to κ/λ so the λ's in G^{SI} cancel out. To see the second fact, take a horizontal path $c_t = a\cdot n$, $0\le t\le 1$, and abbreviate the lengths of the curves in this path, $\ell_{c(t,\cdot)}$, to $\ell(t)$. Then we have:

$$\begin{aligned}
\frac{\partial \log\ell(t)}{\partial t} &= \frac{1}{\ell(t)}\int_{S^1}\kappa_{c(t,\cdot)}(\theta)\cdot a(\theta,t)ds, \quad \text{hence}\\
\Big|\frac{\partial \log\ell(t)}{\partial t}\Big| &= \Big(\frac{\int\kappa^2a^2ds}{\ell(t)}\Big)^{1/2}\cdot\Big(\frac{\int 1\cdot ds}{\ell(t)}\Big)^{1/2}\\
&\le \frac{1}{\sqrt{A}}\big(G^{SI}(a,a)\big)^{1/2}, \quad \text{hence}\\
|\log(\ell(1)) - \log(\ell(0))| &\le \text{SI-path length}/\sqrt{A}.
\end{aligned}$$

Thus in a path whose length in this metric is K, the lengths of the individual curves can increase or decrease at most by a factor $e^{K/\sqrt{A}}$. Now use the same argument as above to control the area swept out by such a path:

$$\begin{aligned}
\int |a|ds &\le \Big(\int a^2ds\Big)^{1/2}\cdot\Big(\int 1\cdot ds\Big)^{1/2}\\
&\le \big(\ell^3G^{SI}(a,a)\big)^{1/2}\cdot\ell^{1/2} = \ell^2\cdot G^{SI}(a,a)^{1/2}, \quad \text{hence}\\
\text{Area-swept-out} &\le e^{K/\sqrt{A}}\ell(0)^2\cdot K
\end{aligned}$$

which verifies the second fact. We can readily calculate the geodesic equation for horizontal geodesics in this metric as another special case of the equation for G^Φ:

$$\begin{aligned}
a_t = \frac{-1}{1+A(\ell\kappa)^2}\bigg(&\big(-1+A(\ell\kappa)^2\big)\frac{\kappa a^2}{2} - A\ell^2D_s^2(\kappa)a^2 - 2A\ell^2\kappa D_s(a)^2\\
&- 4A\ell^2D_s(\kappa)aD_s(a) + \big(3+A(\ell\kappa)^2\big)\overline{(a\kappa)}\cdot a - \frac{3}{2}\overline{(a^2)}\cdot\kappa - \frac{A\ell^2}{2}\overline{(\kappa a)^2}\cdot\kappa\bigg)
\end{aligned}$$

where the "overline" stands now for the *average* of a function over the curve, i.e. $\int\cdots ds/\ell$. Since this metric is scale invariant, there are now *four* conserved quantities, instead of three:

$\Phi(\ell,\kappa)\langle v,c_t\rangle\lvert c'(\theta)\rvert^2 \in \mathfrak{X}(S^1)$	reparametrization momentum
$\int_{S^1}\Phi(\ell,\kappa)c_tds \in \mathbb{R}^2$	linear momentum

$$\int_{S^1} \Phi(\ell,\kappa)\langle Jc, c_t\rangle ds \in \mathbb{R} \qquad \text{angular momentum}$$

$$\int_{S^1} \Phi(\ell,\kappa)\langle c, c_t\rangle ds \in \mathbb{R} \qquad \text{scaling momentum}$$

It would be very interesting to compute and compare geodesics in these special metrics.

3.9. **The Wasserstein metric and a related G^Φ-metric.** The Wasserstein metric (also known as the Monge-Kantorovich metric) is a metric between probability measures on a common metric space, see [1], and [2] for more details. It has been studied for many years globally and is defined as follows: let μ and ν be 2 probability measures on a metric space (X,d). Consider all measures ρ on $X \times X$ whose marginals under the 2 projections are μ and ν. Then:

$$d_{\text{wass}}(\mu,\nu) = \inf_{\rho: p_{1,*}(\rho)=\mu, p_{2,*}(\rho)=\nu} \iint_{X\times X} d(x,y)d\rho(x,y).$$

It was discovered only recently by Benamou and Brenier [6] that, if $X = \mathbb{R}^n$, this is, in fact, path length for a Riemannian metric on the space of probability measures $\mathcal{P}$. In their theory, the tangent space at μ to the space of probability measures and the infinitesimal metric are defined by:

$$T_{\mu,\mathcal{P}} = \left\{ \text{vector fields } h = \nabla f \text{ completed in the norm } \int |h|^2 d\mu \right\}$$

where the tangent h to a family $t \mapsto \mu(t)$ is defined by the identity:

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(h.\mu) = 0.$$

In our case, we want to assign to an immersion c the scaled arc length measure $\mu_c = ds/\ell$. This maps B_i to $\mathcal{P}$. The claim is that the pull-back of the Wasserstein metric by this map is intermediate between $G^{\ell^{-1}}$ and G^{Φ_W}, where

$$\Phi_W(\ell,\kappa) = \ell^{-1} + \tfrac{1}{12}\ell\kappa^2.$$

This is not hard to work out.

(1) Because we are mod-ing out by vector fields of norm 0, the vector field h is defined only along the curve c and its norm is $\ell^{-1}.\int \|h\|^2 ds$.
(2) If we split $h = av + bn$, then the condition that $h = \nabla f$ means that $\int a.ds = 0$ and the norm is $\ell^{-1}.\int(a^2+b^2)ds$.
(3) But moving c infinitesimally by h, scaled arc length parametrization of c must still be scaled arc length. Let $c(\cdot,t) = c + th$. Then this means $|c_\theta|_t = \text{cnst.}\,|c_\theta|$ at $t=0$. Since $|c_\theta|_t = \langle c_{t\theta}, c_\theta\rangle/|c_\theta|$, this condition is the same as $\langle D_s(av+bn), v\rangle = \text{cnst.}$, or $D_s a - b\kappa_c = \text{cnst.}$.
(4) Combining the last 2 conditions on b, we get a formula for a in terms of b, namely $a = K * (b\kappa_c)$, where we convolve with respect to arc length using the kernel $K(x) = \operatorname{sign}(x)/2 - x/\ell, -\ell \le x \le \ell$.
(5) Finally, since $|K * f|(x) \le |K|.|f| = \sqrt{\ell/12}|f|$ for all f, it follows that

$$\ell^{-1}.\int b^2 ds \le \ell^{-1}.\int (a^2+b^2)ds \le \ell^{-1}.\int (b^2 + \frac{(\ell\kappa)^2}{12}.b^2)ds$$

which sandwiches the Wasserstein norm between $G^{\ell^{-1}}$ and G^{Φ_W} for $\Phi_W = \ell^{-1}.(1+(\ell\kappa)^2/12)$.

4. Immersion-Sobolev metrics on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ and on B_i

4.1. The $G^{\mathrm{imm},n}$-metric. We note first that the differential operator $D_s = \frac{\partial_\theta}{|c_\theta|}$ is anti self-adjoint for the metric G^0, i.e., for all $h,k \in C^\infty(S^1.\mathbb{R}^2)$ we have

$$\int_{S^1} \Big\langle D_s(h), k \Big\rangle ds = \int_{S^1} \Big\langle h, -D_s(k) \Big\rangle ds$$

We can define a Sobolev-type weak Riemannian metric[1] on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ which is invariant under the action of $\mathrm{Diff}(S^1)$ by:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h,k) &= \int_{S^1} \left(\langle h,k\rangle + A.\langle D_s^n h, D_s^n k\rangle\right).ds \qquad (1)\\ &= \int_{S^1} \langle L_n(h), k\rangle ds \qquad \text{where} \end{aligned}$$

$$L_n(h) \text{ or } L_{n,c}(h) = (I + (-1)^n A.D_s^{2n})(h) \qquad (2)$$

The interesting special case $n=1$ and $A \to \infty$ has been studied by Trouvé and Younes in [21, 24] and by Mio, Srivastava and Joshi in [17, 18]. In this case, the metric reduces to:

$$G_c^{\mathrm{imm},1,\infty}(h,k) = \int_{S^1} \langle D_s(h), D_s(k)\rangle.ds$$

which ignores translations, i.e. it is a metric on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ modulo translations. Now identify $\mathbb{R}^2$ with $\mathbb{C}$, so that this space embeds as follows:

$$\begin{aligned} \mathrm{Imm}(S^1,\mathbb{R}^2)/\mathrm{transl.} &\hookrightarrow C^\infty(S^1,\mathbb{C})\\ c &\longmapsto c_\theta. \end{aligned}$$

Then Trouvé and Younes use the new shape space coordinates $Z(\theta) = \sqrt{c_\theta(\theta)}$ and Mio et al use the coordinates $\Phi(\theta) = \log(c_\theta(\theta))$ – with *complex* square roots and logs. Both of these unfortunately require the introduction of a discontinuity, but this will drop out when you minimize path length with respect to reparametrizations. The wonderful fact about $Z(\theta)$ is that in a family $Z(t,\theta)$, we find:

$$Z_t = \frac{c_{t,\theta}}{2\sqrt{c_\theta}}, \quad \text{so} \quad \int_{S^1} |Z_t|^2 d\theta = \tfrac14 \int \frac{|c_{t,\theta}|^2}{|c_\theta|^2}|c_\theta| d\theta = \tfrac14 \int |D_s(c_t)|^2 ds$$

so the metric becomes a *constant* metric on the vector space of functions Z. With Φ, one has $\int |\Phi_t|^2 ds = \int |D_s(c_t)|^2 ds$, which is simple but not quite so nice. One can expect a very explicit representation of the space of curves in this metric.

Returning to the general case, for each fixed c of length ℓ, the differential operator $L_{n,c}$ is simply the constant coefficient ordinary differential operator $f \mapsto$

[1] There are other choices for the higher order terms, e.g. summing all the intermediate derivatives with or without binomial coefficients. These metrics are all equivalent and the one we use leads to the simplest equiations.

$f+(-1)^n A.f^{(2n)}$ on the s-line modulo $\ell.\mathbb{Z}$. Thus its Green's function is a linear combination of the exponentials $\exp(\lambda.x)$, where λ are the roots of $1+(-1)^n A.\lambda^{2n}=0$. A simple verification gives its Green's function (which we will not use below):

$$F_n(x) = \frac{1}{2n}\cdot \sum_{\lambda^{2n}=(-1)^{n+1}/A} \frac{\lambda}{1-e^{\lambda\ell}} e^{\lambda x}, \quad 0 \le x \le \ell.$$

This means that the dual metric $\check{G}_c^{\mathrm{imm},n} = (G_c^{\mathrm{imm},n})^{-1}$ on the *smooth cotangent space* $C^\infty(S^1,\mathbb{R}^2) \cong G_c^0(T_c \operatorname{Imm}(S^1,\mathbb{R}^2)) \subset T_c^* \operatorname{Imm}(S^1,\mathbb{R}^2) \cong \mathcal{D}(S^1)^2$ is given by the integral operator L^{-1} which is convolution by F_n with respect to arc length s:

$$\check{G}_c^{\mathrm{imm},n}(h,k) = \iint_{S^1\times S^1} F_n(s_1-s_2).\langle h(s_1), k(s_2)\rangle.ds_1.ds_2.$$

4.2. **Geodesics in the $G^{\mathrm{imm},n}$-metric.** Differentiating the operator $D_s = \frac{1}{|c_\theta|}\partial_\theta$ with respect to c in the direction m we get $-\frac{\langle m_\theta, c_\theta\rangle}{|c_\theta|^3}\partial_\theta$, or $-\langle D_s m, v\rangle D_s$. Thus differentiating the big operator $L_{n,c}$ with respect to c in the direction m, we get:

$$D_{(c,m)}L_{n,c}(h) = (-1)^{n+1}A.\sum_{j=0}^{2n-1} D_s^j\langle D_s(m),v\rangle D_s^{2n-j}(h) \tag{3}$$

Thus we have

$$\begin{aligned}
&D_{(c,m)}G_c^{\mathrm{imm},n}(h,k) = \\
&= A.\int_{S^1}(-1)^{n+1}\sum_{j=0}^{2n-1}\Big\langle D_s^j\langle D_s m,v\rangle D_s^{2n-j}(h),k\Big\rangle ds + \int_{S^1}\langle L_n(h),k\rangle\langle D_s m,v\rangle ds \\
&= A.\int_{S^1}\sum_{j=1}^{2n-1}(-1)^{n+j+1}\Big\langle \langle D_s m,v\rangle D_s^{2n-j}(h), D_s^j k\Big\rangle ds + \int_{S^1}\langle h,k\rangle\langle D_s m,v\rangle ds \\
&= \int_{S^1}\Big\langle m, A.\sum_{j=1}^{2n-1}(-1)^{n+j}D_s\Big(\langle D_s^{2n-j}h, D_s^j k\rangle v\Big) - D_s(\langle h,k\rangle v)\Big\rangle ds
\end{aligned}$$

According to **2.1** we should rewrite this as

$$D_{(c,m)}G_c^{\mathrm{imm},n}(h,k) = G_c^{\mathrm{imm},n}(K_c^n(m,h),k) = G_c^{\mathrm{imm},n}\big(m, H_c^n(h,k)\big),$$

and thus we find the two versions K^n and H^n of the G^n-gradient of $c\mapsto G_c^{\mathrm{imm},n}(h,k)$ are given by:

$$K_c^n(m,h) = L_n^{-1}\Big((-1)^{n+1}A.\sum_{j=1}^{2n-1} D_s^j\langle D_s m,v\rangle D_s^{2n-j}(h) + \langle D_s m,v\rangle h\Big) \tag{4}$$

and by

$$\begin{aligned}
H_c^n(h,k) &= L_n^{-1}\Big(A.\sum_{j=1}^{2n-1}(-1)^{n+j}D_s\Big(\langle D_s^{2n-j}h, D_s^j k\rangle v\Big) - D_s(\langle h,k\rangle v)\Big) \\
&= L_n^{-1}\Big(A.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j+1}h, D_s^j k\rangle v + A.\sum_{j=2}^{2i}(-1)^{n+j-1}\langle D_s^{2n-j+1}h, D_s^j k\rangle v
\end{aligned}$$

$$+ A.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}h, D_s^j k\rangle \kappa_c n - \langle D_s h, k\rangle v - \langle h, D_s k\rangle v - \langle h,k\rangle\kappa_c n\Big)$$

$$= L_n^{-1}\Big(- \langle L_n(h), D_s k\rangle v - \langle D_s h, L_n(k)\rangle v - \langle h,k\rangle \kappa(c) n$$

$$+ A.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}h, D_s^j k\rangle \kappa(c) n\Big) \tag{5}$$

since $D_s\big(v\big) = \kappa(c)n$. By **2.4** the geodesic equation for the metric G^n is

$$c_{tt} = \tfrac{1}{2}H_c^n(c_t,c_t) - K_c^n(c_t,c_t).$$

We expand it to get:

$$\boxed{\begin{aligned} L_n(c_{tt}) = &-\langle L_n(c_t), D_s(c_t)\rangle v - \frac{|c_t|^2\kappa(c)}{2}n - \langle D_s(c_t), v\rangle c_t \\ &+ \frac{A}{2}.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}c_t, D_s^j c_t\rangle\kappa(c)n \\ &+ (-1)^n A.\sum_{j=1}^{2n-1} D_s^j\big(\langle D_s(c_t), v\rangle D_s^{2n-j}(c_t)\big)\end{aligned}} \tag{6}$$

From (3) we see that

$$(L_n(c_t))_t - L_n(c_{tt}) = dL_n(c)(c_t)(c_t) = (-1)^{n+1}A.\sum_{j=0}^{2n-1} D_s^j\langle D_s(c_t), v\rangle D_s^{2n-j}(c_t).$$

so that a more compact form of the geodesic equation of the metric G^n is:

$$\boxed{\begin{aligned} (L_n(c_t))_t = &-\langle L_n(c_t), D_s(c_t)\rangle v - \frac{|c_t|^2\kappa(c)}{2}n - \langle D_s(c_t), v\rangle L_n c_t \\ &+ \frac{A}{2}.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}c_t, D_s^j c_t\rangle\kappa(c)n\end{aligned}} \tag{7}$$

For $n = 0$ this agrees with [13], 4.1.2.

4.3. Existence of geodesics.

Theorem. *Let $n \geq 1$. For each $k \geq 2n+1$ the geodesic equation **4.2** (6) has unique local solutions in the Sobolev space of H^k-immersions. The solutions depend C^∞ on t and on the initial conditions $c(0, \ . \ )$ and $c_t(0, \ . \ )$. The domain of existence (in t) is uniform in k and thus this also holds in $\operatorname{Imm}(S^1, \mathbb{R}^2)$.*

Proof. We consider the geodesic equation as the flow equation of a smooth (C^∞) vector field on the H^2-open set $U^k \times H^k(S^1,\mathbb{R}^2)$ in the Sobolev space $H^k(S^1,\mathbb{R}^2)\times H^k(S^1,\mathbb{R}^2)$ where $U^k = \{c \in H^k : |c_\theta| > 0\} \subset H^k$ is H^2-open. To see that this works we will use the following facts: By the Sobolev inequality we have a bounded linear embedding $H^k(S^1,\mathbb{R}^2) \subset C^m(S^1,\mathbb{R}^2)$ if $k > m + \frac{1}{2}$. The Sobolev space $H^k(S^1,\mathbb{R})$ is a Banach algebra under pointwise multiplication if $k > \frac{1}{2}$. For any fixed smooth mapping f the mapping $u \mapsto f \circ u$ is smooth $H^k \to H^k$ if $k > 0$.

The mapping $(c,u) \mapsto L_{n,c}u$ is smooth $U \times H^k \to H^{k-2n}$ and is a bibounded linear isomorphism $H^k \to H^{k-2n}$ for fixed c. This can be seen as follows (see **4.5** below): It is true if c is parametrized by arclength (look at it in the space of Fourier coefficients). The index is invariant under continuous deformations of elliptic operators of fixed degree, so the index of $L_{n,c}$ is zero in general. But $L_{n,c}$ is self-adjoint positive, so it is injective with vanishing index, thus surjective. By the open mapping theorem it is then bibounded. Moreover $(c,w) \mapsto L_{n,c}^{-1}(w)$ is smooth $U^k \times H^{k-2n} \to H^k$ (by the inverse function theorem on Banach spaces). The mapping $(c,f) \mapsto D_s f = \frac{1}{|c_\theta|}\partial_\theta f$ is smooth $H^k \times H^m \supset U \times H^m \to H^{m-1}$ for $k \geq m$, and is linear in f. Let us write $D_c f = D_s f$ just for the remainder of this proof to stress the dependence on c. We have $v = D_c c$ and $n = JD_c c$. The mapping $c \mapsto \kappa(c)$ is smooth on the H^2-open set $\{c : |c_\theta| > 0\} \subset H^k$ into H^{k-2}. Keeping all this in mind we now write the geodesic equation as follows:

$$
\begin{aligned}
c_t &= u =: X_1(c,u)\\
u_t &= L_{n,c}^{-1}\Big(-\langle L_{n,c}(u), D_c(u)\rangle D_c(c) - \frac{|c_t|^2\kappa(c)}{2} JD_c(c) - \langle D_c(u), D_c c\rangle u\\
&\qquad + \frac{A}{2}.\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_c^{2n-j}u, D_c^j u\rangle \kappa(c) JD_c(c)\\
&\qquad + (-1)^n A.\sum_{j=1}^{2n-1} D_c^j\left(\langle D_c(u), D_c(c)\rangle D_c^{2n-j}(u)\right)\Big)\\
&=: X_2(c,u)
\end{aligned}
$$

Now a term by term investigation of this shows that the expression in the brackets is smooth $U^k \times H^k \to H^{k-2n}$ since $k-2n \geq 1 > \frac{1}{2}$. The operator $L_{n,c}^{-1}$ then takes it smoothly back to H^k. So the vector field $X = (X_1, X_2)$ is smooth on $U^k \times H^k$. Thus the flow Fl^k exists on H^k and is smooth in t and the initial conditions for fixed k.

Now we consider smooth initial conditions $c_0 = c(0,\ .\ )$ and $u_0 = c_t(0,\ .\ ) = u(0,\ .\ )$ in $C^\infty(S^1,\mathbb{R}^2)$. Suppose the trajectory $\mathrm{Fl}_t^k(c_0,u_0)$ of X through these intial conditions in H^k maximally exists for $t \in (-a_k, b_k)$, and the trajectory $\mathrm{Fl}_t^{k+1}(c_0,u_0)$ in H^{k+1} maximally exists for $t \in (-a_{k+1}, b_{k+1})$ with $b_{k+1} < b_k$. By uniqueness we have $\mathrm{Fl}_t^{k+1}(c_0,u_0) = \mathrm{Fl}_t^k(c_0,u_0)$ for $t \in (-a_{k+1}, b_{k+1})$. We now apply ∂_θ to the equation $u_t = X_2(c,u) = L_{n,c}^{-1}(\dots)$, note that the commutator $[\partial_\theta, L_{n,c}^{-1}]$ is a pseudo differential operator of order $-2n$ again, and write $w = \partial_\theta u$. We obtain $w_t = \partial_\theta u_t = L_{n,c}^{-1}\partial_\theta(\dots) + [\partial_\theta, L_{n,c}^{-1}](\dots)$. In the term $\partial_\theta(\dots)$ we consider now only the terms $\partial_\theta^{2n+1}u$ and rename them $\partial_\theta^{2n}w$. Then we get an equation $w_t(t,\theta) = \tilde{X}_2(t, w(t,\theta))$ which is inhomogeneous bounded linear in $w \in H^k$ with coefficients bounded linear operators on H^k which are C^∞ functions of $c, u \in H^k$. These we already know on the intervall $(-a_k, b_k)$. This equation therefore has a solution $w(t,\ .\ )$ for all t for which the coefficients exists, thus for all $t \in (a_k, b_k)$. The limit $\lim_{t\nearrow b_{k+1}} w(t,\ .\ )$ exists in H^k and by continuity it equals $\partial_\theta u$ in H^k at $t = b_{k+1}$. Thus the H^{k+1}-flow was not maximal and can be continued. So $(-a_{k+1}, b_{k+1}) = (a_k, b_k)$. We can iterate this and conclude that the flow of X exists in $\bigcap_{m\geq k} H^m = C^\infty$. □

4.4. The conserved momenta of $G^{\text{imm},n}$. According to **2.5** the following momenta are preserved along any geodesic $t \mapsto c(t,\ .\ )$:

$\langle c_\theta, L_{n,c}(c_t)\rangle \lvert c_\theta(\theta)\rvert \ \in \mathfrak{X}(S^1)$	reparametrization momentum
$\int_{S^1} L_{n,c}(c_t)\, ds = \int_{S^1} c_t\, ds \ \in \mathbb{R}^2$	linear momentum
$\int_{S^1} \langle Jc, L_{n,c}(c_t)\rangle\, ds \in \mathbb{R}$	angular momentum

4.5. Horizontality for $G^{\text{imm},n}$. $h \in T_c \operatorname{Imm}(S^1,\mathbb{R}^2)$ is $G^{\text{imm},n}_c$-orthogonal to the $\operatorname{Diff}(S^1)$-orbit through c if and only if

$$0 = G^{\text{imm},n}_c(h, \zeta_X(c)) = G^{\text{imm},n}_c(h, c_\theta.X) = \int_{S^1} X.\langle L_{n,c}(h), c_\theta\rangle\, ds$$

for all $X \in \mathfrak{X}(S^1)$. So the $G^{\text{imm},n}$-normal bundle is given by

$$\mathcal{N}^n_c = \{h \in C^\infty(S,\mathbb{R}^2) : \langle L_{n,c}(h), v\rangle = 0\}.$$

The $G^{\text{imm},n}$-orthonormal projection $T_c \operatorname{Imm} \to \mathcal{N}^n_c$, denoted by $h \mapsto h^\perp = h^{\perp,G^n}$ and the complementary projection $h \mapsto h^\top \in T_c(c \circ \operatorname{Diff}(S^1))$ are determined as follows:

$$h^\top = X(h).v \qquad \text{where } \langle L_{n,c}(h), v\rangle = \langle L_{n,c}(X(h).v), v\rangle$$

Thus we are led to consider the linear differential operators associated to $L_{n.c}$

$$\begin{aligned} &L^\top_c, L^\perp_c : C^\infty(S^1) \to C^\infty(S^1),\\ &L^\top_c(f) = \langle L_{n,c}(f.v), v\rangle = \langle L_{n,c}(f.n), n\rangle,\\ &L^\perp_c(f) = \langle L_{n,c}(f.v), n\rangle = -\langle L_{n,c}(f.n), v\rangle. \end{aligned}$$

The operator $L^\top_c$ is of order $2n$ and also unbounded, self-adjoint and positive on $L^2(S^1, \lvert c_\theta\rvert\, d\theta)$ since

$$\begin{aligned} \int_{S^1} L^\top_c(f) g ds &= \int_{S^1} \langle L_{n,c}(fv), v\rangle g ds\\ &= \int_{S^1} \langle fv, L_{n,c}(gv)\rangle ds = \int_{S^1} f L^\top_c(g) ds,\\ \int_{S^1} L^\top_c(f) f ds &= \int_{S^1} \langle fv, L_{n,c}(fv)\rangle ds \quad > 0 \text{ if } f \neq 0. \end{aligned}$$

In particular, $L^\top_c$ is injective. $L^\perp_c$, on the other hand is of order $2n-1$ and a similar argument shows it is skew-adjoint. For example, if $n=1$, then one finds that:

$$\begin{aligned} L^\top_c &= -A.D_s^2 + (1 + A.\kappa^2).I\\ L^\perp_c &= -2A.\kappa.D_s - A.D_s(\kappa).I \end{aligned}$$

Lemma. *The operator $L^\top_c : C^\infty(S^1) \to C^\infty(S^1)$ is invertible.*

Proof. This is because its index vanishes, by the following argument: The index is invariant under continuous deformations of elliptic operators of degree $2n$. The operator

$$L^\top_c(f) = (-1)^n \frac{A}{\lvert c_\theta\rvert^{2n}} \partial_\theta^{2n}(f) + \text{ lower order terms}$$

is homotopic to $(1+(-1)^n\partial_\theta^{2n})(f)$ and thus has the same index which is zero since the operator $1+(-1)^n\partial_\theta^{2n}$ is invertible. This can be seen by expanding in Fourier series where the latter operator is given by $(\hat{f}(m)) \mapsto ((1+m^{2n})\hat{f}(m))$, a linear isomorphism of the space of rapidly decreasing sequences. Since $L_c^\top$ is injective, it is also surjective. □

To go back and forth between the 'natural' horizontal space of vector fields $a.n$ and the $G^{\text{imm},n}$-horizontal vector fields $\{h \mid \langle Lh, v\rangle = 0\}$, we only need to use these operators and the inverse of $L^\top$. Thus, given a, we want to find b and f such that $L(an+bv) = fn$, so that $an+bv$ is $G^{\text{imm},n}$-horizontal. But this implies that

$$L^\perp(a) = -\langle L(an), v\rangle = \langle L(bv), v\rangle = L^\top(b).$$

Thus if we define the operator $C_c : C^\infty(S^1) \to C^\infty(S^1)$ by

$$C_c := (L_c^\top)^{-1} \circ L_c^\perp,$$

we get a pseudo-differential operator of order -1 (which is an integral operator), so that $a.n + C(a).v$ is always $G^{\text{imm},n}$-horizontal. In particular, the restriction of the metric $G^{\text{imm},n}$ to horizontal vector fields $h_i = a_i.n + b_i.v$ can be computed like this:

$$\begin{aligned}
G_c^{\text{imm},n}(h_1,h_2) &= \int_{S^1} \langle Lh_1, h_2\rangle.ds \\
&= \int_{S^1} \langle L(a_1.n + b_1.v), n\rangle.a_2.ds \\
&= \int_{S^1} \left(L^\top(a_1) + L^\perp(b_1)\right).a_2.ds \\
&= \int_{S^1} \left(L^\top + L^\perp \circ C\right) a_1.a_2.ds.
\end{aligned}$$

Thus the metric restricted to horizontal vector fields is given by the pseudo differential operator $L^{\text{red}} = L^\top + L^\perp \circ (L^\top)^{-1} \circ L^\perp$. On the quotient space B_i, if we identify its tangent space at C with the space of normal vector fields $a.n$, then:

$$\boxed{G_C^{imm,n}(a_1,a_2) = \int_C (L^\top + L^\perp \circ (L^\top)^{-1} \circ L^\perp)a_1 \cdot a_2 \cdot ds}$$

Now, although this operator may be hard to analyze, its inverse, the metric on the cotangent space to B_i, is simple. The tangent space to B_i at a curve C is canonically the quotient of that of $\text{Imm}(S^1,\mathbb{R}^2)$ at a parametrization c of C, modulo the subspace of multiples of v. Hence the cotangent space to B_i at C injects into that of $\text{Imm}(S^1,\mathbb{R}^2)$ at c with image the linear functionals that vanish on v. In terms of the dual basis $\check{v}, \check{n}$, these are multiples of $\check{n}$. On the smooth cotangent space $C^\infty(S^1,\mathbb{R}^2) \cong G_c^0(T_c\,\text{Imm}(S^1,\mathbb{R}^2)) \subset T_c^*\,\text{Imm}(S^1,\mathbb{R}^2) \cong \mathcal{D}(S^1)^2$ the dual metric is given by convolution with the elementary kernel K_n which is a simple sum of exponentials. Thus we need only restrict this kernel to multiples $a(s).\check{n}_c(s)$ to obtain the dual metric on B_i. The result is that:

$$\check{G}_c^n(a_1,a_2) = \iint_{S^1\times S^1} K_n(s_1-s_2).\langle n_c(s_1), n_c(s_2)\rangle.a_1(s_1).a_2(s_2).ds_1 ds_2.$$

4.6. **Horizontal geodesics.** The normal bundle $\mathcal{N}_c$ mentioned in **4.5** is well defined and is a smooth vector subbundle of the tangent bundle. But $\mathrm{Imm}(S^1,\mathbb{R}^2)\to B_i(S^1,\mathbb{R}^2)=\mathrm{Imm}/\mathrm{Diff}(S^1)$ is *not* a principal bundle and thus there are no principal connections, but we can prove the main consequence, the existence of horizontal paths, directly:

Proposition. *For any smooth path c in $\mathrm{Imm}(S^1,\mathbb{R}^2)$ there exists a smooth path φ in $\mathrm{Diff}(S^1)$ with $\varphi(0,\ .\ )=\mathrm{Id}_{S^1}$ depending smoothly on c such that the path e given by $e(t,\theta)=c(t,\varphi(t,\theta))$ is horizontal: $\langle L_{n,e}(e_t),e_\theta\rangle=0$.*

Proof. Writing D_c instead of D_s we note that $D_{c\circ\varphi}(f\circ\varphi)=\frac{(f_\theta\circ\varphi)\varphi_\theta}{|c_\theta\circ\varphi|.|\varphi_\theta|}=(D_c(f))\circ\varphi$ for $\varphi\in\mathrm{Diff}^+(S^1)$. So we have $L_{n,c\circ\varphi}(f\circ\varphi)=(L_{n,c}f)\circ\varphi$.

Let us write $e=c\circ\varphi$ for $e(t,\theta)=c(t,\varphi(t,\theta))$, etc. We look for φ as the integral curve of a time dependent vector field $\xi(t,\theta)$ on S^1, given by $\varphi_t=\xi\circ\varphi$. We want the following expression to vanish:

$$\begin{aligned}\langle L_{n,c\circ\varphi}(\partial_t(c\circ\varphi)),\partial_\theta(c\circ\varphi)\rangle&=\langle L_{n,c\circ\varphi}(c_t\circ\varphi+(c_\theta\circ\varphi)\,\varphi_t),(c_\theta\circ\varphi)\,\varphi_\theta\rangle\\&=\langle L_{n,c}(c_t)\circ\varphi+L_{n,c}(c_\theta.\xi)\circ\varphi,c_\theta\circ\varphi\rangle\varphi_\theta\\&=\big((\langle L_{n,c}(c_t),c_\theta\rangle+\langle L_{n,c}(\xi.c_\theta),c_\theta\rangle)\circ\varphi\big)\,\varphi_\theta.\end{aligned}$$

Using the time dependent vector field $\xi=-\frac{1}{|c_\theta|}(L_c^\top)^{-1}(\langle L_{n,c}(c_t),v\rangle)$ and its flow φ achieves this. $\square$

If we write

$$c_t=na+vb=\big(n,v\big)\begin{pmatrix}a\\b\end{pmatrix}$$

then we can expand the condition for horizontality as follows:

$$\begin{aligned}D_s(c_t)&=\big(D_sa+\kappa(c)b\big)n+\big(D_sb-\kappa(c)a\big)v.\\&=(n,v)\begin{pmatrix}D_s&\kappa\\-\kappa&D_s\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}\\L_n^c(c_t)&=c_t+(-1)^nA(n,v)\begin{pmatrix}D_s&\kappa\\-\kappa&D_s\end{pmatrix}^{2n}\begin{pmatrix}a\\b\end{pmatrix}\\&=c_t+(-1)^nA(n,v)\begin{pmatrix}D_s^2-\kappa^2&D_s\kappa+\kappa D_s\\-D_s\kappa-\kappa D_s&D_s^2-\kappa^2\end{pmatrix}^n\begin{pmatrix}a\\b\end{pmatrix}\end{aligned}$$

so that horizontality becomes

$$0=\langle L_{n,c}(c_t),v\rangle=\langle c_t,v\rangle+(-1)^nA(0,1)\begin{pmatrix}D_s^2-\kappa^2&D_s\kappa+\kappa D_s\\-D_s\kappa-\kappa D_s&D_s^2-\kappa^2\end{pmatrix}^n\begin{pmatrix}a\\b\end{pmatrix}$$

We may specialize the general geodesic equation to horizontal paths and then take the v and n parts of the geodesic equation. For a horizontal path we may write $L_{n,c}(c_t)=\tilde{a}n$ for $\tilde{a}(t,\theta)=\langle L_{n,c}(c_t),n\rangle$. The v part of the equation turns out to vanish identically and then n part gives us (because n_t is a multiple of v):

$$\boxed{\tilde{a}_t=-\frac{|c_t|^2\kappa(c)}{2}-\langle D_sc_t,v\rangle\tilde{a}+\frac{A\kappa(c)}{2}\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}c_t,D_s^jc_t\rangle}$$

Note that applying **4.3** with horizontal initial vectors gives us local existence and uniqueness for solutions of this horizontal geodesic equation.

4.7. A Lipschitz bound for arclength in $G^{\text{imm},n}$. We apply the inequality of Cauchy-Schwarz to the derivative of the length function $\ell(c) = \int |c_\theta| d\theta$ along a path $t \mapsto c(t,\ .\ )$:

$$\begin{aligned}
\partial_t \ell(c) = d\ell(c)(c_t) &= \int_{S^1} \frac{\langle c_{t\theta}, c_\theta\rangle}{|c_\theta|}\, d\theta = \int_{S^1} \langle D_s(c_t), v\rangle\, ds \\
&\le \Big(\int_{S^1} |D_s(c_t)|^2 ds\Big)^{\frac12} \cdot \Big(\int_{S^1} 1^2 ds\Big)^{\frac12} \le \sqrt{\ell(c)}\frac{1}{A}\|c_t\|_{G^1}, \\
&\le \sqrt{\ell(c)} C(A,n) \|c_t\|_{G^n}, \\
\partial_t \sqrt{\ell(c)} = \frac{\partial_t \ell(c)}{2\sqrt{\ell(c)}} &\le \frac{C(A,n)}{2} \|c_t\|_{G^n}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
|\sqrt{\ell(c(1,\ .\ ))} - \sqrt{\ell(c(0,\ .\ ))}| \le \int_0^1 |\partial_t \sqrt{\ell(c)}|\, dt &\le \frac{C(A,n)}{2} \int_0^1 \|c_t\|_{G^n}\, dt \\
&= \frac{C(A,n)}{2} L_{G^n}(c).
\end{aligned}$$

Taking the infinimum of this over all paths $t \mapsto c(t,\ .\ )$ from c_0 to c_1 we see that for $n \ge 1$ we have the Lipschitz estimate:

$$|\sqrt{\ell(c_1)} - \sqrt{\ell(c_0)}| \le \frac12 \operatorname{dist}^{\text{Imm}}_{G^n}(c_1, c_0)$$

Since we have $L^{\text{hor}}_{G^n}(c) \le L_{G^n}(c)$ with equality for horizontal curves we also have:

$$\boxed{\textit{If } n \ge 1, \quad |\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)}| \le \frac12 \operatorname{dist}^{B_i}_{G^n}(C_1, C_0)}$$

4.8. Scale invariant immersion Sobolev metrics. Let us mention in passing that we may use the length of the curve to modify the immersion Sobolev metric so that it becomes scale invariant:

$$\begin{aligned}
G^{\text{imm,scal},n}_c(h,k) &= \int_{S^1} \big(\ell(c)^{-3}\langle h,k\rangle + \ell(c)^{2n-3} A \langle D^n_s(h), D^n_s(k)\rangle\big)\, ds \\
&= \int_{S^1} \big\langle (\ell(c)^{-3} + (-1)^n \ell(c)^{2n-3} A D_s^{2n}) h, k\big\rangle\, ds
\end{aligned}$$

This metric can easily be analyzed using the methods described above. In particular we note that the geodesic equation on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ for this metric is built in a similar way than that for $G^{\text{imm},n}$ and that the existence theorem in **4.3** holds for it. Note the conserved momenta along a geodesic $t \mapsto c(t,\ .\ )$ are:

$$\begin{aligned}
&\frac{1}{\ell(c)^3} \int_{S^1} c_t\, ds + (-1)^n \ell(c)^{2n-3} A \int_{S^1} D_s^{2n}(c_t)\, ds \\
&\quad = \frac{1}{\ell(c)^3} \int_{S^1} c_t\, ds \in \mathbb{R}^2 && \text{linear momentum} \\
&\frac{1}{\ell(c)^3} \int_{S^1} \langle Jc, c_t\rangle\, ds + (-1)^n \ell(c)^{2n-3} A \int_{S^1} \langle Jc, D_s^{2n}(c_t)\rangle\, ds && \text{angular momentum}
\end{aligned}$$

$$\frac{1}{\ell(c)^3}\int_{S^1}\langle c,c_t\rangle\,ds+(-1)^n\ell(c)^{2n-3}A\int_{S^1}\langle c,D_s^{2n}(c_t)\rangle\,ds \qquad \text{scaling momentum}$$

As in the work of Trouvé and Younes [21, 24], we may consider the following variant.

$$\begin{aligned}G_c^{\text{imm,scal},n,\infty}(h,k)&=\lim_{A\to\infty}\frac{1}{A}\int_{S^1}\big\langle(\ell(c)^{-3}+(-1)^n\ell(c)^{2n-3}AD_s^{2n})h,k\big\rangle\,ds\\&=(-1)^n\ell(c)^{2n-3}\int_{S^1}\big\langle D_s^{2n}h,k\big\rangle\,ds\end{aligned}$$

It is degenerate with kernel the constant tangent vectors. The interesting fact is that the scaling momentum for $G^{\text{imm,scal},1,\infty}$ is given by

$$-\frac{1}{\ell(c)}\int_{S^1}\langle c,D_s^2(c_t)\rangle ds=\partial_t\log\ell(c).$$

5. Sobolev metrics on Diff($\mathbb{R}^2$) and on its quotients

5.1. The metric on Diff($\mathbb{R}^2$)**.** We consider the regular Lie group $\mathrm{Diff}(\mathbb{R}^2)$ which is either the group $\mathrm{Diff}_c(\mathbb{R}^2)$ of all diffeomorphisms with compact supports of $\mathbb{R}^2$ or the group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^2)$ of all diffeomorphisms which decrease rapidly to the identity. The Lie algebra is $\mathfrak{X}(\mathbb{R}^2)$, by which we denote either the Lie algebra $\mathfrak{X}_c(\mathbb{R}^2)$ of vector fields with compact support or the Lie algebra $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^2)$ of rapidly decreasing vector fields, with the negative of the usual Lie bracket. For any $n\ge 0$, we equip $\mathrm{Diff}(\mathbb{R}^2)$ with the right invariant weak Riemannian metric $G^{\mathrm{Diff},n}$given by the Sobolev H^n-inner product on $\mathfrak{X}_c(\mathbb{R}^2)$.

$$\begin{aligned}H^n(X,Y)&=\sum_{\substack{0\le i,j\le n\\ i+j\le n}}\frac{A^{i+j}n!}{i!j!(n-i-j)!}\int_{\mathbb{R}^2}\langle\partial_{x^1}^i\partial_{x^2}^jX,\partial_{x^1}^i\partial_{x^2}^jY\rangle\,dx\\&=\sum_{\substack{0\le i,j\le n\\ i+j\le n}}(-A)^{i+j}\frac{n!}{i!j!(n-i-j)!}\int_{\mathbb{R}^2}\langle\partial_{x^1}^{2i}\partial_{x^2}^{2j}X,Y\rangle\,dx\\&=\int_{\mathbb{R}^2}\langle LX,Y\rangle\,dx\qquad\text{where}\\L&=L_{A,n}=(1-A\Delta)^n,\qquad\Delta=\partial_{x^1}^2+\partial_{x^2}^2.\end{aligned}$$

(We will write out the full subscript of L only where it helps clarify the meaning.) The completion of $\mathfrak{X}_c(\mathbb{R}^2)$ is the Sobolev space $H^n(\mathbb{R}^2)^2$. With the usual L^2-inner product we can identify the dual of $H^n(\mathbb{R}^2)^2$ with $H^{-n}(\mathbb{R}^2)^2$ (in the space of tempered distributions). Note that the operator $L:H^n(\mathbb{R}^2)^2\to H^{-n}(\mathbb{R}^2)^2$ is a bounded linear operator. On $L^2(\mathbb{R}^2)$ the operator L is unbounded selfadjoint and positive. In terms of Fourier transform we have $\widehat{L_{A,n}u}(\xi)=(1+A|\xi|^2)^n\hat{u}$. Let $F_{A,n}$ in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$ be the fundamental solution (or Green's function: note that we use the letter 'F' for 'fundamental' because 'G' has been used as the metric) of $L_{A,n}$ satisfying $L_{A,n}(F_{A,n})=\delta_0$ which is given by

$$F_{A,n}(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{i\langle x,\xi\rangle}\frac{1}{(1+A|\xi|^2)^n}\,d\xi.$$

The functions $F_{A,n}$ are given by the classical modified Bessel functions K_r (in the notation, e.g., of Abramowitz and Stegun [7] or of Matlab) by the formula:

$$F_{A,n}(x) = \frac{1}{2^n \pi (n-1)! A} \cdot \left(\frac{|x|}{\sqrt{A}}\right)^{n-1} K_{n-1}\left(\frac{|x|}{\sqrt{A}}\right).$$

and it satisfies $(L^{-1}u)(x) = \int_{\mathbb{R}^2} F(x-y)u(y)\,dy$ for each tempered distribution u. The function $F_{A,n}$ is C^{n-1} except that $F_{A,1}$ has a log-pole at zero. At infinity, $F_{A,n}(x)$ is asymptotically a constant times $x^{n-3/2}e^{-x}$: these facts plus much much more can be found in [7].

5.2. Strong conservation of momentum and 'EPDiff'. What is the form of the conservation of momentum for a geodesic $\varphi(t)$ in this metric, that is to say, a flow $x \mapsto \varphi(x,t)$ on $\mathbb{R}^2$? We need to work out Ad_φ^* first. Using the definition, we see:

$$\begin{aligned}\int_{\mathbb{R}^2} \langle LX, \mathrm{Ad}_\varphi^*(Y)\rangle &:= \int_{\mathbb{R}^2} \langle L\,\mathrm{Ad}_\varphi(X), Y\rangle = \int_{\mathbb{R}^2} \langle (d\varphi.X)\circ\varphi^{-1}, LY\rangle \\ &= \int_{\mathbb{R}^2} \det(d\varphi)\langle d\varphi.X, LY\circ\varphi\rangle = \int_{\mathbb{R}^2} \langle X, \det(d\varphi).d\varphi^T.(LY\circ\varphi)\rangle\end{aligned}$$

hence:

$$\mathrm{Ad}_\varphi^*(Y) = L^{-1}\left(\det(d\varphi).d\varphi^T.(LY\circ\varphi)\right).$$

Now the conservation of momentum for geodesics $\varphi(t)$ of right invariant metrics on groups says that:

$$L^{-1}\left(\det(d\varphi)(t).d\varphi(t)^t.\left(L(\frac{\partial\varphi}{\partial t}\circ\varphi^{-1})\circ\varphi\right)\right)$$

is independent of t. This can be put in a much more transparent form. First, L doesn't depend on t, so we cross out the outer L^{-1}. Now let $v(t) = \frac{\partial\varphi}{\partial t}\circ\varphi^{-1} \in \mathfrak{X}(\mathbb{R}^2)$ be the tangent vector to the geodesic. Let $u(t) = Lv(t)$, so that:

$$\det(d\varphi)(t).d\varphi(t)^t.(u(t)\circ\varphi(t))$$

is independent of t. We should *not* think of $u(t)$ as a vector field on $\mathbb{R}^2$: this is because we want $\langle u, v\rangle$ to make invariant sense in any coordinates whatsoever. This means we should think of u as expanding to the differential form:

$$\omega(t) = (u_1.dx^1 + u_2.dx^2)\otimes\mu$$

where $\mu = dx^1\wedge dx^2$, the area form. But then:

$$\varphi(t)^*(\omega(t)) = \langle d\varphi^t.(u\circ\varphi(t)), dx\rangle \otimes \det(d\varphi)(t).\mu$$

so conservation of momentum says simply:

$$\boxed{\varphi(t)^*\omega(t) \text{ is independent of } t}$$

This motivates calling $\omega(t)$ the momentum of the geodesic flow. As we mentioned above, conservation of momentum for a Riemannian metric on a group is very strong and is an integrated form of the geodesic equation. To see this, we need only take the differential form of this conservation law. $v(t)$ is the infinitesimal flow, so the infinitesimal form of the conservation is:

$$\frac{\partial}{\partial t}\omega(t) + \mathcal{L}_{v(t)}(\omega(t)) = 0$$

where $\mathcal{L}_{v(t)}$ is the Lie derivative. We can expand this term by term:

$$\mathcal{L}_{v(t)}(u_i) = \sum_j v^j.\frac{\partial u_i}{\partial x^j}$$

$$\mathcal{L}_{v(t)}(dx^i) = dv^i = \sum_j \frac{\partial v^i}{\partial x^j}.dx^j$$

$$\mathcal{L}_{v(t)}(\mu) = \operatorname{div} v(t)\mu$$

$$\mathcal{L}_{v(t)}(\omega(t)) = \left(\sum_{i,j}\left(v^j.\frac{\partial u_i}{\partial x^j}.dx^i + u^j.\frac{\partial v^j}{\partial x^i}.dx^i\right) + \operatorname{div} v.\sum_i u_i dx^i\right)\otimes\mu.$$

The resulting differential equation for geodesics has been named *EPDiff*:

$$\boxed{\begin{aligned} &v = \frac{\partial\varphi}{\partial t}\circ\varphi^{-1}, \qquad u = L(v)\\ &\frac{\partial u_i}{\partial t} + \sum_j\left(v^j.\frac{\partial u_i}{\partial x^j} + u^j.\frac{\partial v^j}{\partial x^i}\right) + \operatorname{div} v.u_i = 0.\end{aligned}}$$

Note that this is a special case of the general equation of Arnold: $\partial_t u = -\operatorname{ad}(u)^* u$ for geodesics on any Lie group in any right (or left) invariant metric. The name 'EPDiff' was coined by Holm and Marsden and stands for 'Euler-Poincaré', although it takes a leap of faith to see it in the reference they give to Poincaré.

5.3. **The quotient metric on** $\operatorname{Emb}(S^1,\mathbb{R}^2)$**.** We now consider the quotient mapping $\operatorname{Diff}(\mathbb{R}^2)\to\operatorname{Emb}(S^1,\mathbb{R}^2)$ given by $\varphi\mapsto\varphi\circ i$ as in the section 1. Since this identifies $\operatorname{Emb}(S^1,\mathbb{R}^2)$ with a right coset space of $\operatorname{Diff}(\mathbb{R}^2)$, and since the metric G^n_{diff} is right invariant, we can put a quotient metric on $\operatorname{Emb}(S^1,\mathbb{R}^2)$ for which this map is a Riemannian submersion. Our next step is to identify this metric. Let $\varphi\in\operatorname{Diff}(\mathbb{R}^2)$ and let $c=\varphi\circ i\in\operatorname{Emb}(S^1,\mathbb{R}^2)$. The fibre of this map through φ is the coset

$$\varphi.\operatorname{Diff}^0(S^1,\mathbb{R}^2) = \{\psi \mid \psi\circ c\equiv c\}.\varphi.$$

whose tangent space is (the right translate by φ of) the vector space of vector fields $X\in\mathfrak{X}(\mathbb{R}^2)$ with $X\circ c\equiv 0$. This is the vertical subspace. Thus the horizontal subspace is

$$\left\{Y\,\middle|\,\int_{\mathbb{R}^2}\langle LY,X\rangle dx = 0,\ \text{if } X\circ c\equiv 0\right\}.$$

If we want $Y\in\mathfrak{X}(\mathbb{R}^2)$ then the horizontal subspace is 0. But we can also search for Y in a bigger space of vector fields on $\mathbb{R}^2$. What we need is that $LY = c_*(p(\theta).ds)$, where p is a function from S^1 to $\mathbb{R}^2$ and ds is arc-length measure supported on C. To make $c_*(p(\theta).ds)$ pair with smooth vector fields $\mathfrak{X}(\mathbb{R}^2)$ in a coordinate invariant way, we should interpret the values of p as 1-forms. Solving for Y, we have:

$$Y(x) = \int_{S^1} F(x-c(\theta)).p(\theta)ds$$

(where, to make Y a vector field, the values of p are now interpreted as vectors, using the standard metric on $\mathbb{R}^2$ to convert 1-forms to vectors). Because F is not C^∞, we have a case here where the horizontal subspace is not given by C^∞ vector fields. However, we can still identify the set of vector fields in this horizontal subspace which map bijectively to the C^∞ tangent space to $\mathrm{Emb}(S^1,\mathbb{R}^2)$ at c. Mapped to $T_c\,\mathrm{Emb}(S^1,\mathbb{R}^2)$, the above Y goes to:

$$(1)\quad \begin{aligned} (Y\circ c)(\theta) &= \int_{S^1} F(c(\theta)-c(\theta_1)).p(\theta_1).|c'(\theta_1)|d\theta_1 \\ &=: (F_c * p)(\theta) \quad \text{where} \\ F_c(\theta_1,\theta_2) = F(c(\theta_1)-c(\theta_2)) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\langle c(\theta_1)-c(\theta_2),\xi\rangle}\frac{1}{(1+A|\xi|^2)^n}\,d\xi. \end{aligned}$$

Note that here, convolution on S^1 uses the metric $L^2(S^1,|c'(\theta|d\theta)$ and it defines a self-adjoint operator for this Hilbert space. Moreover, it is covariant with respect to change in parametrization:

$$F_{c\circ\varphi} * (f\circ\varphi) = (F_c * f)\circ\varphi.$$

What are the properties of the kernel F_c? From the properties of F, we see that F_c is C^{n-1} kernel (except for log poles at the diagonal when $n=1$). It is also a pseudo-differential operator of order $-2n+1$ on S^1. To see that let us assume for the moment that each function of θ is a periodic function on $\mathbb{R}$. Then

$$\begin{aligned} c(\theta_1)-c(\theta_2) &= \int_0^1 c_\theta(\theta_2+t(\theta_1-\theta_2))dt.(\theta_1-\theta_2) =: \tilde c(\theta_1,\theta_2)(\theta_1-\theta_2) \\ F_c(\theta_1,\theta_2) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(\theta_1-\theta_2)\langle\tilde c(\theta_1,\theta_2),\xi\rangle}\frac{1}{(1+A|\xi|^2)^n}\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\theta_1-\theta_2)\eta_1}\Big(\int_{\mathbb{R}}\frac{|\tilde c(\theta_1,\theta_2)|^{-2}}{(1+\frac{A}{|\tilde c(\theta_1,\theta_2)|^2}(|\eta_1|^2+|\eta_2|^2))^n}\,d\eta_2\Big)\,d\eta_1 \\ &=: \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\theta_1-\theta_2)\eta_1}\tilde F_c(\theta_1,\theta_2,\eta_1)\,d\eta_1 \end{aligned}$$

where we changed variables as $\eta_1 = \langle\tilde c(\theta_1,\theta_2),\xi\rangle$ and $\eta_2=\langle J\tilde c(\theta_1,\theta_2),\xi\rangle$. So we see that $F_c(\theta_1,\theta_2)$ is an elliptic pseudo differential operator kernel of degree $-2n+1$ (the loss comes from integrating with respect to η_2). The symbol $\tilde F_c$ is real and positive, so the operator $p\mapsto F_c*p$ is self-adjoint and positive. Thus it is injective, and by an index argument similar to the one in **4.5** it is invertible. The inverse operator to the integral operator F_c is a pseudo-differential operator L_c of order $2n-1$ given by the distribution kernel $L_c(\theta,\theta_1)$ which satisfies

$$(2)\quad \begin{aligned} &L_c * F_c * f = F_c * L_c * f = f \\ &L_{c\circ\varphi} * (h\circ\varphi) = ((L_c * h)\circ\varphi) \quad \text{for all } \varphi\in\mathrm{Diff}^+(S^1) \end{aligned}$$

If we write $h = Y\circ c$, then we want to express the horizontal lift Y in terms of h and write Y_h for it. The set of all these Y_h spans the horizontal subspace which maps isomorphically to $T_c\,\mathrm{Emb}(S^1,\mathbb{R}^2)$. Now:

$$h = Y\circ c = (F*(c_*(p.ds)))\circ c = F_c * p.$$

Therefore, using the inverse operator, we get $p = L_c * h$ and:

$$\begin{aligned} Y_h &= F*(c_*(p.ds)) = F*(c_*((L_c*h).ds)) \quad \text{or} \\ Y_h(x) &= \int_{S^1} F(x-c(\theta))\int_{S^1} L_c(\theta,\theta_1)h(\theta_1)|c'(\theta_1)|d\theta_1|c'(\theta)|d\theta \end{aligned}$$

and $LY_h = c_*((L_c * h).ds)$. Thus we can finally write down the quotient metric

$$\begin{aligned} G_c^{\mathrm{diff},n}(h,k) &= \int_{\mathbb{R}^2} \langle LY_h, Y_k\rangle dx \\ &= \int_{S^1} \Big\langle L_c * h(\theta), \int_{S^1} F(c(\theta)-c(\theta_1)) \int_{S^1} L_c(\theta_1,\theta_2)k(\theta_2)ds_2\, ds_1 \Big\rangle ds \qquad (3)\\ &= \int_{S^1} \langle L_c * h(\theta), k(\theta)\rangle ds = \iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle h(\theta_1), k(\theta)\rangle ds_1\, ds. \end{aligned}$$

The dual metric on the associated smooth cotangent space $L_c * C^\infty(S^1,\mathbb{R}^2)$ is similarly:

$$\check{G}_c^{\mathrm{diff},n}(p,q) = \iint_{S^1\times S^1} F_c(\theta,\theta_1)\langle p(\theta_1), q(\theta)\rangle ds_1\, ds.$$

5.4. The geodesic equation on $\mathrm{Emb}(S^1,\mathbb{R}^2)$ via conservation of momentum. A quite convincing but not rigorous derivation of this equation can be given using the fact that under a submersion, geodesics on the quotient space are the projections of those geodesics on the total space which are horizontal at one and hence every point. In our case, the geodesics on $\mathrm{Diff}(\mathbb{R}^2)$ can be characterized by the strong conservation of momentum we found above: $\varphi(t)^*\omega(t)$ is independent of t. If $X(t)$ is the tangent vector to the geodesic, i.e. the velocity $X(t) = \partial_t\varphi\circ\varphi^{-1}(t)$, then $\omega(t)$ is just $LX(t) = c_*(p(\theta,t).ds) = c_*(p(\theta,t).|c_\theta(\theta,t)|.d\theta)$ considered as a measure valued 1-form instead of a vector field.

When we pass to the quotient $\mathrm{Emb}(S^1,\mathbb{R}^2)$, a horizontal geodesic of diffeomorphisms $\varphi(t)$ with $\varphi(0) =$ identity gives a geodesic path of embeddings $c(\theta,t) = \varphi(t)\circ c(0,\theta)$. For these geodesic equations, it will be most convenient to take as the momentum the 1-form $\tilde{p}(\theta,t) = p(\theta,t).|c_\theta(\theta,t)|$, the measure factor $d\theta$ being constant along the flow. We must take the velocity to be the horizontal vector field $X(t) = F * c(\cdot,t)_*(\tilde{p}(\theta,t).d\theta)$. For this to be the velocity of the path of maps c, we must have $c_t(\theta,t) = X(c(\theta),t)$ because the global vector field X must extend c_t. To pair $\tilde{p}$ and c_t, we regard $\tilde{p}$ as a 1-form along c (the area factor having been replaced by the measure $d\theta$ supported on C). The geodesic equation must be the differential form of the conservation equation:

$$\boxed{\varphi(t)^*\tilde{p}(\cdot,t) \text{ is independent of } t.}$$

More explicitly, if d_x stands for differentiating with respect to the spatial coordinates x,y, then this means:

$$d_x\varphi(t)^T|_{c(\theta,t)}\tilde{p}(\theta,t) = \text{ cnst.}$$

We differentiate this with respect to t, using the identity:

$$\partial_t d_x\varphi(t) = d_x(\varphi_t(t)) = d_x(X\circ\varphi(t)) = (d_x(X)\circ\varphi(t))\cdot d_x\varphi(t),$$

we get

$$0 = d_x\varphi(t)^T\cdot\big((d_x(X)^T\circ c(\theta,t))\cdot\tilde{p}(\theta,t) + \tilde{p}_t(\theta,t)\big)\,.$$

Writing this out and putting the discussion together, we get the following form for the geodesic equation on $\mathrm{Emb}(S^1, \mathbb{R}^2)$:

$$\begin{aligned} c_t(\theta,t) &= X(t) \circ c(\theta,t) \\ \tilde{p}_t(\theta,t) &= -\operatorname{grad} X^t(c(\theta,t),t) \cdot \tilde{p}(\theta,t) \\ X(t) &= F * c(\cdot,t)_* \left(\tilde{p}(\theta,t).d\theta\right) \end{aligned}$$

Note that X is a vector field on the plane: these are not closed equations if we restrict X to the curves. The gradient of X requires that we know the normal derivative of X to the curves. Alternatively, we may introduce a second *vector-valued* kernel on S^1 depending on c by:

$$F'_c(\theta_1,\theta_2) = \operatorname{grad} F(c(\theta_1) - c(\theta_2)).$$

Then the geodesic equations may be written:

$$\boxed{\begin{aligned} c_t(\theta,t) &= (F_c * \tilde{p})(\theta,t) \\ \tilde{p}_t(\theta,t) &= -\langle \tilde{p}(\theta,t), (F'_c * \tilde{p})(\theta,t)\rangle. \end{aligned}}$$

where, in the second formula, the dot product is between the two $\tilde{p}$'s and the vector value is given by F'_c.

The problem with this approach is that we need to enlarge the space $\mathrm{Diff}(\mathbb{R}^2)$ to include diffeomorphisms which are not C^∞ along some C^∞ curve but have a mild singularity normal to the curve. Then we would have to develop differential geometry and the theory of geodesics on this space, etc. It seems more straightforward to outline the direct derivation of the above geodesic equation, along the lines used above.

5.5. The geodesic equation on $\mathrm{Emb}(S^1, \mathbb{R}^2)$, direct approach. The space of invertible pseudo differential operators on a compact manifold is a regular Lie group (see [3]), so we can use the usual formula $d(A^{-1}) = -A^{-1}.dA.A^{-1}$ for computing the derivative of L_c with respect to c. Note that we have a simple expression for $D_{c,h}F_c$, namely

$$D_{c,h}F_c(\theta_1,\theta_2) = dF(c(\theta_1) - c(\theta_2))(h(\theta_1) - h(\theta_2)) = \langle F'_c(\theta_1,\theta_2), h(\theta_1) - h(\theta_2)\rangle$$

hence

$$\begin{aligned} D_{c,\ell}L_c(\theta_1,\theta_2) &= -\int_{(S^1)^2} L_c(\theta_1,\theta_3)\, D_{c,h}F_c(\theta_3,\theta_4)\, L_c(\theta_4,\theta_2)\, d\theta_3\, d\theta_4 \\ &= -\int_{(S^1)^2} L_c(\theta_1,\theta_3)\, \langle (F'_c(\theta_3,\theta_4), \ell(\theta_3)\rangle\, L_c(\theta_4,\theta_2)\, d\theta_3\, d\theta_4 \\ &\quad + \int_{(S^1)^2} L_c(\theta_1,\theta_4)\, \langle (F'_c(\theta_4,\theta_3), \ell(\theta_3)\rangle\, L_c(\theta_3,\theta_2)\, d\theta_3\, d\theta_4 \end{aligned}$$

We can now differentiate the metric where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ is the variable on $(S^1)^n$:

$$D_{c,\ell}G^{\mathrm{diff},n}(h,k) = \int_{(S^1)^2} D_{c,\ell}L_c(\theta_1,\theta_2)\langle h(\theta_2), k(\theta_1)\rangle d\theta$$

$$= \int_{(S^1)^4} \Big\langle - L_c(\theta_1,\theta_3)F'_c(\theta_3,\theta_4)L_c(\theta_4,\theta_2)$$

$$+ L_c(\theta_1,\theta_4)F_c'(\theta_4,\theta_3)\,L_c(\theta_3,\theta_2),\ell(\theta_3)\Big\rangle\langle h(\theta_2),k(\theta_1)\rangle\,d\theta$$

We have to write this in the form

$$D_{c,\ell}G_c^{\text{diff},n}(h,k) = G_c^{\text{diff},n}(\ell, H_c(h,k)) = G_c^{\text{diff},n}(K_c(\ell,h),k)$$

For H_c we use $\delta(\theta_5-\theta_3) = \int L_c(\theta_5,\theta_6)F_c(\theta_6,\theta_3)d\theta_6 = (L_c * F_c)(\theta_5,\theta_3)$ as follows:

$$D_{c,\ell}G^{\text{diff},n}(h,k) = \int_{(S^1)^6} L_c(\theta_5,\theta_6)\Big\langle \ell(\theta_5), \Big(- L_c(\theta_1,\theta_3)F_c'(\theta_3,\theta_4)L_c(\theta_4,\theta_2)$$
$$+ L_c(\theta_1,\theta_4)F_c'(\theta_4,\theta_3)\,L_c(\theta_3,\theta_2)\Big)F_c(\theta_6,\theta_3)\langle h(\theta_2),k(\theta_1)\rangle\Big\rangle\,d\theta$$

Thus

$$H_c(h,k)(\theta_0) = \int_{(S^1)^4}\Big(- L_c(\theta_1,\theta_3)F_c'(\theta_3,\theta_4)L_c(\theta_4,\theta_2)$$
$$+ L_c(\theta_1,\theta_4)F_c'(\theta_4,\theta_3)\,L_c(\theta_3,\theta_2)\Big)F_c(\theta_0,\theta_3)\langle h(\theta_2),k(\theta_1)\rangle\,d\theta$$

Similarly we get

$$D_{c,\ell}G^{\text{diff},n}(h,k) = \int_{(S^1)^6} L_c(\theta_6,\theta_5)\Big\langle F_c(\theta_1,\theta_6)\big\langle - L_c(\theta_1,\theta_3)F_c'(\theta_3,\theta_4)L_c(\theta_4,\theta_2)$$
$$+ L_c(\theta_1,\theta_4)F_c'(\theta_4,\theta_3)\,L_c(\theta_3,\theta_2),\ell(\theta_3)\big\rangle h(\theta_2),k(\theta_5)\Big\rangle\,d\theta$$

so that

$$K_c(\ell,h)(\theta_0) =$$
$$= \int_{(S^1)^2}\Big(- \langle F_c'(\theta_0,\theta_1)L_c(\theta_1,\theta_2),\ell(\theta_0)\rangle + \langle F_c'(\theta_0,\theta_1)\,L_c(\theta_1,\theta_2),\ell(\theta_1)\rangle\Big)h(\theta_2)\,d\theta$$

By **2.4** the geodesic equation is given by

$$c_{tt}(\theta_0) = \tfrac12 H_c(c_t,c_t)(\theta_0) - K_c(c_t,c_t)(\theta_0)$$

Let us rewrite the geodesic equation in terms of $L_c * c_t$. We have (suppressing the variable t and collecting all terms)

$$\begin{aligned}(L_c * c_t)_t(\theta_0) &= \int_{S^1} D_{c,c_t}L_c(\theta_0,\theta_1)c_t(\theta_1)\,d\theta_1 + L_c * c_{tt}\\ &= \tfrac12\int_{S^1}\Big(F_c'(\theta_1,\theta_0) - F_c'(\theta_0,\theta_1)\Big)\langle L_c * c_t(\theta_0), L_c * c_t(\theta_1)\rangle\,d\theta_1\end{aligned}$$

Since the kernel F is an even function we get the same geodesic equation as above for the momentum $\tilde p(\theta,t) = L_c * c_t = p(\theta,t).|c_\theta|$:

$$\tilde p_t(\theta_0) = -\int_{S^1} F_c'(\theta_0,\theta_1)\langle \tilde p(\theta_0),\tilde p(\theta_1)\rangle\,d\theta_1 \tag{1}$$

5.6. **Existence of geodesics.**

Theorem. *Let $n \geq 1$. For each $k > 2n - \frac12$ the geodesic equation **5.5** (1) has unique local solutions in the Sobolev space of H^k-embeddings. The solutions are C^∞ in t and in the initial conditions $c(0,\,.\,)$ and $c_t(0,\,.\,)$. The domain of existence (in t) is uniform in k and thus this also holds in $\operatorname{Emb}(S^1,\mathbb{R}^2)$.*

An even stronger theorem, proving *global* existence on the level of H^k-diffeomorphisms on $\mathbb{R}^2$, has been proved by [20, 22, 23].

Proof. Let $c \in H^k$. We begin by checking that F'_c is a pseudo differential operator kernel of order $-2n+2$ as we did for F_c in **5.3**.

$$
\begin{aligned}
c(\theta_1) - c(\theta_2) &=: \tilde c(\theta_1,\theta_2)(\theta_1-\theta_2)\\
\operatorname{grad} F(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\langle x,\xi\rangle}\frac{J\xi}{(1+A|\xi|^2)^n}\,d\xi\\
F'_c(\theta_1,\theta_2) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(\theta_1-\theta_2)\langle \tilde c(\theta_1,\theta_2),\xi\rangle}\frac{J\xi}{(1+A|\xi|^2)^n}\,d\xi\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\theta_1-\theta_2)\eta_1}\Big(\int_{\mathbb{R}}\frac{|\tilde c(\theta_1,\theta_2)|^{-3}.J\eta}{(1+\frac{A}{|\tilde c(\theta_1,\theta_2)|^2}(|\eta_1|^2+|\eta_2|^2))^n}\,d\eta_2\Big)\,d\eta_1\\
&=: \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\theta_1-\theta_2)\eta_1}\tilde F_c(\theta_1,\theta_2,\eta_1)\,d\eta_1
\end{aligned}
$$

where we changed variables as $\eta_1 = \langle \tilde c(\theta_1,\theta_2),\xi\rangle$ and $\eta_2 = \langle J\tilde c(\theta_1,\theta_2),\xi\rangle$. So we see that $F'_c(\theta_1,\theta_2)$ is an elliptic pseudo differential operator kernel of degree $-2n+2$ (the loss of 1 comes from integrating with respect to η_2). We write the geodesic equation in the following way:

$$
\begin{aligned}
c_t &= F_c * q =: Y_1(c,q)\\
q_t &= \big\langle q, F'_c * q\big\rangle = \int F'_c(\ .\ ,\theta)\langle q(\theta), q(\ .\ )\rangle\,d\theta =: Y_2(c,q)
\end{aligned}
$$

We start with $c \in H^k$ where $k > 2n - \frac{1}{2}$, in the H^2-open set $U^k := \{c : |c_\theta| > 0\} \subset H^k$. Then $q = L_c * c_t \in H^{k-2n+1}$ and $F'_c * q \in H^{k-1} \subset H^{k-2n+1}$. By the Banach algebra property of the Sobolev space H^{k-2n+1} the expression (with missuse of notation) $Y_2(c,q) = \langle q, F'_c * q\rangle \in H^{k-2n+1}$. Since the kernel F is not smooth only at 0, all appearing pseudo differential operators kernels are C^∞ off the diagonal, thus are smooth mappings in c with values in the space of operators between the relevant Sobolev spaces. Let us make this more precise. We claim that $c \mapsto F'_c*(\ .\ ) \in L(H^k, H^{k+2n-2})$ is C^∞. Since the Sobolev spaces are convenient, we can (a) use the smooth uniform boundedness theorem [9], 5.18, so that it suffices to check that for each fixed $q \in H^k$ the mapping $c \mapsto F'_c * q$ is smooth into H^{k+2n-2}. Moreover, by [9], 2.14 it suffices (b) to check that this is weakly smooth: Using the L^2-duality between H^{k+2n-2} and $H^{-k-2n+2}$ it suffices to check, that for each $p \in H^{-k-2n+2}$ the expression

$$
\begin{aligned}
&\int p(\theta_1)(F'_c * q)(\theta_1)\,d\theta_1 =\\
&= \iint \frac{p(\theta_1)}{2\pi}\int_{\mathbb{R}} e^{i(\theta_1-\theta_2)\eta_1}\Big(\int_{\mathbb{R}}\frac{|\tilde c(\theta_1,\theta_2)|^{-3}.J\eta}{(1+\frac{A}{|\tilde c(\theta_1,\theta_2)|^2}(|\eta_1|^2+|\eta_2|^2))^n}\,d\eta_2\Big)\,d\eta_1 q(\theta_2)\,d\theta_1\,d\theta_2
\end{aligned}
$$

is a smooth mapping $\operatorname{Emb}(S^1,\mathbb{R}^2) \to \mathbb{R}^2$. For that we may assume that c depends on a further smooth variable s. Convergence of this integral depends on the highest order term in the asymptotic expansion in η, which does not change if we differentiate with respect to s.

Thus the geodesic equation is the flow equation of a smooth vector field $Y = (Y_1, Y_2)$ on $U^k \times H^{k-2n+1}$. We thus have local existence and uniqueness of the flow Fl^k on $U^k \times H^{k-2n+1}$.

Now we consider smooth initial conditions $c_0 = c(0,\ .\ )$ and $q_0 = q(0,\ .\ ) = (L_c * c_t)(0,\ .\ )$ in $C^\infty(S^1, \mathbb{R}^2)$. Suppose the trajectory $\mathrm{Fl}^k_t(c_0, q_0)$ of Y through these intial conditions in $U^k \times H^{k+1-2n}$ maximally exists for $t \in (-a_k, b_k)$, and the trajectory $\mathrm{Fl}^{k+1}_t(c_0, u_0)$ in $U^{k+1} \times H^{k+2-2n}$ maximally exists for $t \in (-a_{k+1}, b_{k+1})$ with $b_{k+1} < b_k$. By uniqueness we have $\mathrm{Fl}^{k+1}_t(c_0, u_0) = \mathrm{Fl}^k_t(c_0, u_0)$ for $t \in (-a_{k+1,} b_{k+1})$. We now apply ∂_θ to the equation $q_t = Y_2(c, q)$, note that the commutator $q \mapsto [F'_c, \partial_\theta] * q = \partial_t h(F'_c * q) - F'_c * (\partial_\theta q)$ is a pseudo differential operator of order $-2n+2$ again, and obtain

$$\partial_\theta q_t = \int [F'_c, \partial_\theta](\ .\ , \theta)\langle q(\theta), q(\ .\ )\rangle\, d\theta + \int F'_c(\ .\ , \theta)\langle \partial_\theta q(\theta), q(\ .\ )\rangle\, d\theta$$
$$+ \int F'_c(\ .\ , \theta)\langle q(\theta), \partial_\theta q(\ .\ )\rangle\, d\theta$$

which is an inhomogeneous linear equation for $w = \partial_\theta q$ in $U^k \times H^{k+1-2n}$. By the variation of constant method one sees that the solution w exists in H^k for as long as (c, q) exists in $U^k \times H^{k+1-2n}$, i.e., for all $t \in (-a_k, b_k)$. By continuity we can conclude that $w = \partial_\theta q$ is the derivative in H^{k+2-2n} for $t = b_{k+1}$, and thus the domain of definition was not maximal. Iterating this argument we can conclude that the solution (c, q) lies in C^∞ for $t \in (-a_k, b_k)$. □

5.7. **Horizontality for $G^{\mathrm{diff},n}$.** The tangent vector $h \in T_c \operatorname{Emb}(S^1, \mathbb{R}^2)$ is $G^{\mathrm{diff},n}_c$-orthogonal to the $\operatorname{Diff}(S^1)$-orbit through c if and only if

$$0 = G^{\mathrm{diff},n}_c(h, \zeta_X(c)) = \int_{(S^1)^2} L_c(\theta_1, \theta_2)\langle h(\theta_2), c_\theta(\theta_1)\rangle X(\theta_1)\, ds_1\, ds_2$$

for all $X \in \mathfrak{X}(S^1)$. So the $G^{\mathrm{diff},n}$-normal bundle is given by

$$\mathcal{N}^{\mathrm{diff},n}_c = \{h \in C^\infty(S^1, \mathbb{R}^2) : \langle L_c * h, v\rangle = 0\}.$$

Working exactly as in section 4, we want to split any tangent vector into vertical and horizontal parts as $h = h^\top + h^\perp$ where $h^\top = X(h).v$ for $X(h) \in \mathfrak{X}(S^1)$ and where $h^\perp$ is horizontal, $\langle L_c * h^\perp, v\rangle = 0$. Then $\langle L_c * h, v\rangle = \langle L_c * (X(h)v), v\rangle$ and we are led to consider the following operators:

$$L^\top_c, L^\perp_c : C^\infty(S^1) \to C^\infty(S^1),$$
$$L^\top_c(f) = \langle L_c * (f.v), v\rangle = \langle L_c * (f.n), n\rangle,$$
$$L^\perp_c(f) = \langle L_c * (f.v), n\rangle = -\langle L_c * (f.n), v\rangle.$$

The pseudo differential operator $L^\top_c$ is unbounded, selfadjoint and positive on $L^2(S^1, d\theta)$ since we have

$$\int_{S^1} L^\top_c(f).f\, d\theta = \int_{(S^1)^2} \langle L_c(\theta_1, \theta_2) f(\theta_2) v(\theta_2), f(\theta_1).v(\theta_1)\rangle\, d\theta = \|f.v\|^2_{G^{\mathrm{diff},n}} > 0.$$

Thus $L^\top_c$ is injective and by an index argument as in **4.5** the operator $L^\top_c$ is invertible. Moreover, the operator $L^\perp_c$ is skew-adjoint. To go back and forth between the natural horizontal space of vector fields $a.n$ and the $G^{\text{diff},n}$-horizontal vectors, we have to find b such that $L_c * (a.n + b.v) = f.n$ for some f. But then

$$L_c^\perp(a) = -\langle L_c * (a.n), v\rangle = \langle L_c * (b.v), v\rangle = L_c^\top(b) \quad \text{thus} \quad b = (L_c^\top)^{-1} L_c^\perp(a).$$

Thus $a.n + (L_c^\top)^{-1}L_c^\perp(a).v$ is always $G^{\text{diff},n}$-horizontal and is the horizontal projection of $a.n + b.v$ for any b.

Proposition. *For any smooth path c in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ there exists a smooth path φ in $\operatorname{Diff}(S^1)$ with $\varphi(0, \,.\,) = \operatorname{Id}_{S^1}$ depending smoothly on c such that the path e given by $e(t,\theta) = c(t, \varphi(t,\theta))$ is $G^{\text{diff},n}$-horizontal: $\langle L_c * e_t, e_\theta\rangle = 0$.*

Proof. Let us write $e = c \circ \varphi$ for $e(t,\theta) = c(t,\varphi(t,\theta))$, etc. We look for φ as the integral curve of a time dependent vector field $\xi(t,\theta)$ on S^1, given by $\varphi_t = \xi \circ \varphi$. We want the following expression to vanish. In its computation the equivariance of L_c under $\varphi \in \operatorname{Diff}^+(S^1)$ from **5.3**(2) will play an important role.

$$\begin{aligned}\big\langle L_{c\circ\varphi} * (\partial_t(c\circ\varphi)), \partial_\theta(c\circ\varphi)\big\rangle &= \langle L_{c\circ\varphi} * (c_t \circ \varphi + (c_\theta\circ\varphi)\,\varphi_t), (c_\theta\circ\varphi)\,\varphi_\theta\rangle \\ &= \big\langle ((L_c * c_t)\circ\varphi) \ + ((L_c * (c_\theta.\xi))\circ\varphi), (c_\theta\circ\varphi)\varphi_\theta\big\rangle \\ &= \big((\langle L_c * c_t, c_\theta\rangle + \langle L_c * (\xi.c_\theta), c_\theta\rangle)\circ\varphi\big)\,\varphi_\theta.\end{aligned}$$

Using the time dependent vector field $\xi = -(L_c^\top)^{-1}\langle L_c * c_t, c_\theta\rangle$ and its flow φ achieves this. □

To write the quotient metric on B_e, we want to lift normal vector fields $a.n$ to a curve C to horizontal vector fields on $\operatorname{Emb}(S^1,\mathbb{R}^2)$. Substituting $h = a.n + (L_c^\top)^{-1}L_c^\perp(a).v,\ k = b.n + (L^\top)^{-1}L^\perp(b).v$ in **5.3**(3), we get as above:

$$G_C^{\text{diff},n}(a,b) = \int_C \left(L_c^\top + L_c^\perp (L_c^\top)^{-1} L_c^\perp\right)(a).bds.$$

The dual metric on the cotangent space is just the restriction of the dual metric on $\operatorname{Emb}(S^1,\mathbb{R}^2)$ to the cotangent space to B_e and is much simpler. We simply set $p = f.n,\ q = g.n$ and get:

$$\check{G}_C^{\text{diff},n}(f,g) = \iint_{C^2} F(x(s) - x(s_1)).\langle n(s), n(s_1)\rangle.f(s)g(s_1).dsds_1$$

where $x(s) \in \mathbb{R}^2$ stands for the point in the plane with arc length coordinate s and F is the Bessel kernel. Since these are dual inner products, we find that the two operators, (a) convolution with the kernel $F(x(s)-x(s_1)).\langle n(s), n(s_1)\rangle$ and (b) $L_c^\top + L_c^\perp(L_c^\top)^{-1}L_c^\perp$ are inverses of each other.

5.8. The geodesic equation on B_e via conservation of momentum. The simplest way to find the geodesic equation on B_e is again to specialize the general rule $\varphi(t)^*\omega(t) =$ cnst. to the horizontal geodesics. Now horizontal in the present context, that is for B_e, requires more of the momentum $\omega(t)$. As well as being given by $c_*(p(s).ds)$, we require the 1-form p to kill the tangent vectors v to the curve. If we identify 1-forms and vectors using the Euclidean metric, then we may say simply $p(s) = a(s).n$, where a is a scalar function on C. But note that if you take the momentum as $c_*(a(s)n(s)ds)$ and integrate it against a vector field X, then you find:

$$\langle X, c_*(a(s)n(s)ds)\rangle = \int_C a(s)\langle X, n(s)\rangle ds = \int_C a(s).i_X(dx\wedge dy)$$

where i_X is the 'interior product' or contraction with X taking a 2-form to a 1-form. Noting that 1-forms can be integrated along curves without using any metric, we see that the 2-form along c defined by $\{a(s).(dx\wedge dy)_{c(s)}\}$ can be naturally paired with vector fields so it defines a canonical measure valued 1-form. Therefore, the momentum for horizontal geodesics can be identified with this 2-form.

If $\varphi(x,t)$ is a horizontal geodesic in $\mathrm{Diff}(\mathbb{R}^2)$, then the curves $C_t = \mathrm{image}(c(\cdot,t))$ are given by $C_t = \varphi(C_0,t)$ and the momentum is given by $a(\theta,t).(dx\wedge dy)$, where $c(\theta,t)$ parametrizes the curves C_t. Note that in order to differentiate a with respect to t, we need to assign parameters on the curves C_t simultaneously. We do this in the same way we did for almost local metrics: assume c_θ is a multiple of the normal vector n_C. But $\theta_0 \mapsto \varphi(\theta_0,t)$ gives a second map from C_0 to C_t: in terms of θ, assume this is $\theta = \bar\varphi(\theta_0,t)$. Then the conservation of momentum means simply:

$$a(\bar\varphi(\theta_0,t),t).\det(D_x\varphi)(c(\theta_0,0),t) \text{ is independent of } t.$$

Let X be the global vector field giving this geodesic, so that $\varphi_t = X\circ\varphi$. Note that $\bar\varphi_t = (\langle X\circ c, v\rangle/|c_\theta|)\circ\bar\varphi$. Using this fact, we can differentiate the displayed identity. Recalling the definition of the flow from its momentum and the identifying $T_C B_e$ with normal vector fields along C, we get the full equations for the geodesic:

$$\begin{aligned}
C_t &= \langle X, n\rangle\cdot n\\
a_t &= -\langle X, v\rangle D_s(a) - \mathrm{div}(X).a\\
X &= F * c_*(a(s)n(s)ds)
\end{aligned}$$

Note, as in the geodesic equations in **5.4**, that we must use F to extend X to the whole plane. In this case, we only need (a) the normal component of X along C, (b) its tangential component along C and (c) the divergence of X along C. These are obtained by convolving $a(s)$ with the kernels (which we give now in terms of arc-length):

$$\begin{aligned}
F_c^{nn}(s_1,s_2) &= F(c(s_1)-c(s_2))\langle n(s_1), n(s_2)\rangle ds_2\\
F_c^{vn}(s_1,s_2) &= F(c(s_1)-c(s_2))\langle v(s_1), n(s_2)\rangle ds_2\\
F_c^{\mathrm{div}}(s_1,s_2) &= \langle \mathrm{grad}\, F(c(s_1)-c(s_2)), n(s_1)\rangle ds_2
\end{aligned}$$

Then the geodesic equations become:

$$\begin{aligned}
C_t &= (F_c^{nn} * a).n\\
a_t &= -(F_c^{vn} * a)D_s(a) - (F_c^{\mathrm{div}} * a).a
\end{aligned}$$

Alternately, we may specialize the geodesic equation in **5.4** to horizontal paths. Then the v part vanishes identically and the n part gives the last equation above. We omit this calculation.

6. Examples

6.1. **The Geodesic of concentric circles.** All the metrics that we have studied are invariant under the motion group, thus the 1-dimensional submanifold of B_e consisting of all concentric circles centered at the origin is the fixed point set of the group of rotations around the center. Therefore it is a geodesic in all our metrics. It is given by the map $c(t,\theta) = r(t)e^{i\theta}$. Then $c_\theta = ire^{i\theta}$, $v_c = ie^{i\theta}$, $n_c = -e^{i\theta}$, $\ell(c) = 2\pi r(t)$, $\kappa_c = \frac{1}{r}$ and $c_t = r_t e^{i\theta} = -r_t.n_c$.

The parametrization $r(t)$ can be determined by requiring constant speed σ, i.e. if the metric is $G(h,k)$, then we require $G_c(c_t,c_t) = r_t^2 G_c(n_c,n_c) = \sigma^2$, which leads to $\sqrt{G_c(c_t,c_t)}\,dt = \pm\sqrt{G_c(n_c,n_c)}\,dr$. To determine when the geodesic is complete as $r \to 0$ and $r \to \infty$, we merely need to look at its length which is given by:

$$\int_0^\infty \sqrt{G_c(c_t,c_t)}dt = \int_0^\infty \sqrt{G_c(n_c,n_c)}dr,$$

and we need to ask whether this integral converges or diverges at its two limits. Let's consider this case by case.

The metric G^Φ: The geodesic is determined by the equation:

$$G^\Phi(c_t,c_t) = 2\pi r \cdot \Phi\left(2\pi r(t), \frac{1}{r(t)}\right) \cdot r_t^2 = \sigma^2.$$

Differentiating this with respect to t leads to the geodesic equation in the standard form $r_{tt} = r_t^2 f(r)$. It is easily checked that all three invariant momentum mappings vanish: the reparameterization, linear and angular momentum.

Theorem. *If $\Phi(2\pi r, 1/r) \approx C_1 r^a$ (resp. $C_2 r^b$) as $r \to 0$ (resp. ∞), then the geodesic of concentric circles is complete for $r \to 0$ if and only $a \le -3$ and is complete for $r \to \infty$ if and only if $b \ge -3$. In particular, for $\varphi = \ell^k$, we find $k = a = b$ and the geodesic is given by $r(t) = cnst.t^{2/(k+3}$. For the scale invariant case $\Phi(\ell,\kappa) = \frac{4\pi^2}{\ell^3} + \frac{\kappa^2}{\ell}$, we find $a = b = -3$, the geodesic is given by $r(t) = e^{\sqrt{2}\sigma t}$ and is complete. Moreover, in this case, the scaling momentum $\frac{2r_t}{r}$ is constant in t along the geodesic.*

The proof is straightforward.

The metric $G^{\mathbf{imm},n}$ Recall from **4.1** the operator $L_{n,c} = I + (-1)^n A.D_s^{2n}$. For $c(t,\theta) = r(t)e^{i\theta}$ **6.1** we have

$$L_{n,c}(c_t) = \big(1 + (-1)^n \frac{A}{r^{2n}} \partial_\theta^{2n}\big)(r_t e^{i\theta}) = r_t\big(1 + \frac{A}{r^{2n}}\big)e^{i\theta}$$

which is still normal to c_θ. So $t \mapsto c(t,\ .\ )$ is a horizontal path for any choice of $r(t)$. Thus its speed is the square root of:

$$G^{\mathrm{imm},n}(c_t,c_t) = 2\pi r \cdot \left(1 + \frac{A}{r^{2n}}\right) \cdot r_t^2 = \sigma^2.$$

For $n = 1$ this is the same as the identity for the metric with $\Phi(\ell,\kappa) = 1 + A\kappa^2$ which was computed in [13], 5.1. An explanation of this phenomenon is in [13], 3.2.

Theorem. *The geodesic of concentric cirles is complete in the $G^{\mathrm{imm},n}$ metric if $n \ge 2$. For $n = 1$, it is incomplete as $r \to 0$ but complete if $r \to \infty$.*

The metric $G^{\text{diff},n}$ To evaluate the norm of a path of concentric circles, we now need to find the vector field X on $\mathbb{R}^2$ gotten by convolving the Bessel kernel with the unit normal vector field along a circle. Using circular symmetry, we find that:

$$X(x,y) = f(r)\left(\frac{x}{r}, \frac{y}{r}\right) r_t$$
$$\left(I - A(\partial_{rr} + \frac{1}{r}\partial_r)\right)^n f = 0 \text{ except on the circle } r = r_0$$
$$f \in C^{2n-2} \text{ everywhere}, f(r_0) = 1$$

For $n = 1$, we can solve this and the result is the vector field on $\mathbb{R}^2$given by the Bessel functions I_1 and K_1:

$$X(x,y) = \begin{cases} \frac{I_1(r/\sqrt{A})}{I_1(r_0/\sqrt{A})} \text{ if } r \le r_0 \\ \frac{K_1(r/\sqrt{A})}{K_1(r_0/\sqrt{A})} \text{ if } r \ge r_0 \end{cases}$$

Using the fact that the Wronskian of I_1, K_1 is $1/r$, we find:

$$G^{\text{diff},1}(r_t n, r_t n) = \int \langle (I - A\triangle)X, X\rangle r_t^2 = \frac{2\pi r_t^2}{K_1(r/\sqrt{A}).I_1(r/\sqrt{A})}.$$

Using the asymptotic laws for Bessel functions, one finds that the geodesic of concentric circles has finite length to $r = 0$ but infinite length to $r = \infty$.

For $n > 1$, it gets harder to solve for X. But lower bounds are not hard:

$$G^{\text{diff},n}(n,n) = \inf_{X, \langle X,n\rangle \equiv 1 \text{ on } C_r} \int \langle (I - A\triangle)^n X, X\rangle$$
$$\ge A^n . \inf_{\begin{pmatrix} X, \langle X,n\rangle \equiv 1 \text{ on } C_r \\ X \to 0, \text{ when } x \to \infty \end{pmatrix}} \int \langle \triangle^n(X), X\rangle \underset{\text{def}}{=} M(r)$$

Then $M(r)$ scales with r: $M(r) = M(1)/r^{2n-2}$, hence the length of the path when the radius shrinks to 0 is bounded below by $\int_0 dr/r^{n-1}$ which is infinite if $n > 1$. On the other hand, the metric $G^{\text{diff},n}$ dominates the metric $G^{\text{diff},1}$ so the length of the path when the radius grows to infinity is always infinite. Thus:

Theorem. *The geodesic of concentric cirles is complete in the $G^{\text{diff},n}$ metric if $n \ge 2$. For $n = 1$, it is incomplete as $r \to 0$ but complete if $r \to \infty$.*

6.2. **Unit balls in five metrics at a 'cigar'-like shape.** It is useful to get a sense of how our various metrics differ. One way to do this is to take one simple shape C and examine the unit balls in the tangent space $T_C B_e$ for various metrics. All of our metrics (except the simple L^2 metric) involve a constant A whose dimension is length-squared. We take as our base shape C a strip of length L, where $L \gg \sqrt{A}$, and width w, where $w \ll \sqrt{A}$. We round the two ends with semi-circles, as shown in on the top in figure 1.

As functions of a normal vector field $a.n$ along C, the metrics we want to compare are:

(1) $G_C^A(a,a) = \int_C (1 + A\kappa^2)a^2.ds$
(2) $G_C^{\text{imm},1}(a,a) = \inf_b \int_C \Big(|a \cdot n + b \cdot v)|^2 + A|D_s(a \cdot n + b \cdot v)|^2\Big) ds$

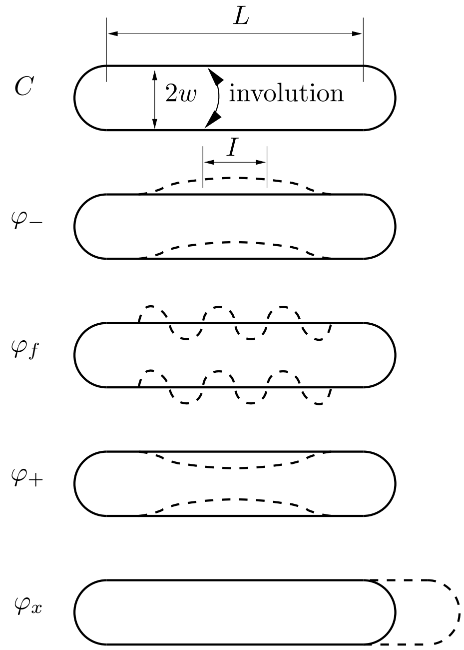

FIGURE 1. The cigarlike shape and their deformations

(3) $G_C^{\text{diff},1}(a,a) = \frac{1}{\sqrt{A}} \inf_{\left(\substack{\mathbb{R}^2\text{-vec.flds.}X \\ \langle X,n\rangle = a}\right)} \iint_{\mathbb{R}^2} \left(|X|^2 + A|DX|^2\right) dx\,dy,$

(4) $G_C^{\text{diff},2}(a,a) = \frac{1}{\sqrt{A}} \inf_{\left(\substack{\mathbb{R}^2\text{-vec.flds.}X \\ \langle X,n\rangle = a}\right)} \iint_{\mathbb{R}^2} \left(|X|^2 + 2A|DX|^2 + A^2|D^2X|^2\right) dx\,dy$

The term $\frac{1}{\sqrt{A}}$ in the last 2 metrics is put there so that the double integrals have the same 'dimension' as the single integrals. In this way, all the metrics will be comparable.

To compare the 4 metrics, we don't take all normal vector fields $a.n$ along C. Note that C has an involution, which flips the top and bottom edges. Thus we have even normal vector fields and odd normal vector fields. Examples are shown in figure 1. We will consider two even and two odd normal vector field, described below, and normalize each of them so that $\int_C a^2 ds = 1$. They are also shown in figure 1. They involve some interval I along the long axis of the shape of length $\lambda \gg w$. The interval determines a part I_t of the top part of C and I_b of the bottom.

(1) Let $a \equiv +1/\sqrt{2\lambda}$ along I_t and $a \equiv -1/\sqrt{2\lambda}$ along I_b, a zero elsewhere except we smooth it at the endpoints of I. Call this odd vector field φ_-.
(2) Fix a high frequency f and, on the same intervals, let $a(x) = \pm\sin(f.x)/\sqrt{\lambda}$. Call this odd vector field φ_f.
(3) The third vector field is even and is defined by $a(x) = \sqrt{\frac{2}{\pi w}}\langle n, \frac{\partial}{\partial x}\rangle$ at the right end of the curve, being zero along top, bottom and left end. Call this

φ_x. The factor in front normalizes this vector field so that its L^2 norm is 1.

(4) Finally, we define another even vector field by $a = +1/\sqrt{2\lambda}$ on both I_t and I_b, zero elsewhere except for being smoothed at the ends of I. Call this φ_+.

The following table shows the approximate leading term in the norm of each of these normal vector fields a in each of the metrics. By approximate, we mean the exact norm is bounded above and below by the entry times a constant depending only on C and by leading term, we mean we ignore terms in the small ratios $w/\sqrt{A}$, $\lambda/\sqrt{A}$:

function	G^A	$G^{\text{imm},1}$	$G^{\text{diff},1}$	$G^{\text{diff},2}$
φ_-	1	1	1	1
φ_f	1	$(\sqrt{A}f)^2$	$\sqrt{A}f$	$(\sqrt{A}f)^3$
φ_+	1	1	$\sqrt{A}/w$	$(\sqrt{A}/w)^3$
φ_x	A/w^2	$\sqrt{A}/w$	$\frac{\sqrt{A}}{w\log(A/w^2)}$	$\sqrt{A}/w$

Thus, for instance:

$$\begin{aligned} G^A(\varphi_x,\varphi_x) &= \frac{2}{\pi w}\int_{\text{right end}}(1+A\kappa^2)\langle n,\tfrac{\partial}{\partial x}\rangle^2 ds \\ &= \tfrac{2}{\pi w}(1+Aw^{-2})\ell(\text{right end})\text{Ave}(\langle n,\tfrac{\partial}{\partial x}\rangle^2) \\ &= (1+Aw^{-2}) \approx A/w^2 \end{aligned}$$

The values of all the other entries under G^A are clear because $\kappa \equiv 0$ in their support.

To estimate the other entries, we need to estimate the horizontal lift, i.e., the functions b or v. To estimate the norms for $G^{\text{imm},1}$, we take $b = 0$ in all cases except φ_x and then get

$$G^{\text{imm},1}_C(a,a) = \|a\|^2_{H^1_A}$$

the first Sobolev norm. For $a = \varphi_f$, we simplify this, replacing the full norm by the leading term $A(D_s(a))^2$ and working this out. To compute $G^{\text{imm},1}(\varphi_x,\varphi_x)$, let $k = \sqrt{\frac{2}{\pi w}}$ be the normalizing factor and lift $a.n$ along the right end of C to the $\mathbb{R}^2$ vector field $k.\frac{\partial}{\partial x}$. This adds a tangential component which we taper to zero on the top and bottom of C like $k.e^{-x/\sqrt{A}}$. This gives the estimate in the table.

Finally, consider the 2 metrics $G^{\text{diff},k}$, $k = 1,2$. For these, we need to lift the normal vector fields along C to vector fields on all of $\mathbb{R}^2$. For the two odd vector fields $f = \varphi_-$ and $f = \varphi_f$, we take v to be constant along the small vertical lines inside C and zero in the extended strip $-w \le y \le w, x \notin I$ and we define v outside $-w \le y \le w$ by:

$$\begin{aligned} v(x,y+w) = v(x,-y-w) &= F(x,y)\tfrac{\partial}{\partial y}, \\ \widehat{F}(\xi,\eta) &= \frac{k\sqrt{A}(1+A\xi^2)^{k-1/2}.\hat{f}(\xi)}{\pi(1+A(\xi^2+\eta^2))^k} \end{aligned}$$

We check the following:

(a) $\big((I-A\triangle)^k F\big)^\wedge = \frac{k}{\pi}\sqrt{A}(1+A\xi^2)^{k-1/2}\hat{f}(\xi)$ is indep. of η hence support$(I-A\triangle)^k F) \subset \{y=0\}$

(b) $\displaystyle\int \hat{F} d\eta = \hat{f}, \text{ hence } F|_{y=0} = f.$

Thus:

$$\begin{aligned} G_C^{\text{diff},k}(f,f) &\approx \tfrac{1}{\sqrt{A}} \iint_{\mathbb{R}^2} \langle (I - A\triangle)^k F, F\rangle dxdy \\ &= \tfrac{k}{\pi} \int (1 + A\xi^2)^{k-1/2} |\hat{f}(\xi)|^2 d\xi = \tfrac{k}{\pi} \|f\|^2_{H_A^{k-1/2}} \end{aligned}$$

The leading term in the k^{th} Sobolev norm of φ_f is $(\sqrt{A}f)^{2k}$, which gives these entries in the table.

To estimate $G_C^{\text{diff},1}(\varphi_+,\varphi_+)$, we define v by extending φ_+ linearly across the vertical line segments $-w \le y \le w, x \in I$, i.e. to $\varphi_+(x)y/w$. This gives the leading term now, as the derivative there is φ_+/w. In fact for any odd vector field a of L^2-norm 1 and for which the derivatives are sufficiently small compared to w, the norm has the same leading term:

$$G_C^{\text{diff},1}(a,a) \approx \sqrt{A}/w.$$

To estimate $G_C^{\text{diff},2}(\varphi_+,\varphi_+)$, we need a smoother extension across the interior of C. We can take $\varphi_+(x).\frac{3}{2}(\frac{y}{w} - \frac{1}{3}(\frac{y}{w})^3)$. Computing the square integral of the second derivative, we get the table entry $G_C^{\text{diff},2}(a,a) \approx (\sqrt{A}/w)^3$.

To estimate $G_C^{\text{diff},k+1}(\varphi_x,\varphi_x)$, we now take v to be

$$v = c(k,A,w)\left[\left(\tfrac{|x|}{\sqrt{A}}\right)^k K_k\left(\tfrac{|x|}{\sqrt{A}}\right) * \chi_D\right]\frac{\partial}{\partial x}.$$

where D is the disk of radius w containing the arc making up the right hand end of C, and where $c(k,A,w)$ is a constant to be specified later. The function $\frac{1}{2\pi k! A}\left(\frac{|x|}{\sqrt{A}}\right)^k K_k\left(\frac{|x|}{\sqrt{A}}\right)$ is the fundamental solution of $(I - A\triangle)^{k+1}$ and is C^1 for $k > 0$ but with a log pole at 0 for $k = 0$. Thus:

$$(I - A\triangle)^{k+1} v = 2\pi k! A c(k,A,w)\chi_D.\frac{\partial}{\partial x}$$

while, up to upper and lower bounds depending only on k, the restriction of v to the disk D itself is equal to $\log(\sqrt{A}/w)c(0,A,w)w^2$ if $k = 0$ and simply $c(k,A,w)w^2$ for $k > 0$. By symmetry v is also constant on the boundary of D and thus v extends φ_x if we take $c(0,A,w) = c_0/\log(\sqrt{A}/w)w^{5/2}$ if $k = 0$ and $c(k,A,w) = c_k/w^{5/2}$ if $k > 0$ (constants c_k depending only on k). Computing the H^k-norm of v, we get the last table entries.

Summarizing, we can say that the large norm of φ_x is what characterizes G^A; the large norms of φ_+ characterize G^{diff}; and the rate of growth in frequency of the norm of φ_f distinguishes all 4 norms.

References

[1] L. Ambrosio, N. Gigli, G. Savaré. Gradient flows with metric and differentiable structures, and applications to the Wasserstein space. *Rend. Mat. Acc. Lincei* s.9, 15 (2004), 327–343.

[2] L. Ambrosio, N. Gigli, G. Savaré. *Gradient Flows In Metric Spaces and in the Space of Probability Measures.* Lectures in Mathematics. ETH Zürich. Birkhäuser, Basel, 2005.

[3] M. Adams, T. Ratiu, R. Schmid. A Lie group structure for pseudo differential operators. *Math. Ann.* 273 (1986), 529–551.

[4] V.Arnold. Sur la gomtrie diffrentielle des groupes de Lie de dimension infinie et ses applications l'hydrodynamique des fluides parfaits. *Annales de l'institut Fourier*, 16 (1966), 319–361.

[5] M.F. Beg, M.I. Miller, A. Trouve, L. Younes. Computing large deformation metric mappings via geodesic flows of diffeomorphisms. *International Journal of Computer Vision* 61,2 (2005), 139-157.

[6] J.-D. Benamou, Y. Brenier. A computational fluid mechanics solution to the Monge Kantorovich mass transfer problem. *Numer. Math.* 84 (2000), 375–393

[7] *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Edited by Milton Abramowitz and Irene A. Stegun. Reprint of the 1972 edition. Dover Publications, Inc., New York, 1992

[8] H. Kodama, P. W. Michor. The homotopy type of the space of degree 0 immersed curves. *Revista Matemática Complutense* 19 (2006), 227-234. arXiv:math/0509694.

[9] A. Kriegl and P. W. Michor. *The Convenient Setting for Global Analysis.* AMS, Providence, 1997. Surveys and Monographs 53.

[10] A. Mennucci, A. Yezzi. Metrics in the space of curves. arXiv:math.DG/0412454

[11] A. Mennucci, A. Yezzi, G. Sundaramoorthi. Sobolev–type metrics in the space of curves. arXiv:math.DG/0605017.

[12] P. W. Michor. Some Geometric Evolution Equations Arising as Geodesic Equations on Groups of Diffeomorphism, Including the Hamiltonian Approach. In: Phase Space Analysis of Partial Differential Equations. Progress in Non Linear Differential Equations and Their Applications, Vol. 69. Birkhäuser Boston, 2006.

[13] P. W. Michor and D. Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)* 8 (2006), 1-48, arXiv:math.DG/0312384.

[14] P. W. Michor and D. Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Documenta Math.* 10 (2005), 217-245. arXiv:math.DG/0409303.

[15] M.I. Miller and L. Younes. Group actions, homeomorphisms, and matching: A general framework. *International Journal of Computer Vision* 41, No 1/2 (2001), 61-84.

[16] M.I. Miller and A. Trouve and L. Younes On the metrics and Euler-Lagrange equations of computational anatomy. *Annual Review of biomedical Engineering* 4 (2002), 375-405.

[17] W. Mio and A. Srivastava. Elastic-String Models for Representation and Analysis of Planar Shapes. *Proceedings of the IEEE Computer Society International Conference on Computer Vision and Pattern Recognition (CVPR)*, Washington, DC, 2004.

[18] W. Mio, A. Srivastava and S. Joshi. On the Shape of Plane Elastic Curves. *preprint*, Florida State University (Math/Stat/CS departments), 2005.

[19] J. Shah. H^0-type Riemannian metrics on the space of planar curves. arXiv:math.DG/0510192

[20] A. Trouvé. Infinite dimensional group actions and pattern recognition. Technical report, DMI, Ecole Normale Superieure, 1995.

[21] A. Trouvé and L. Younes. Diffeomorphic Matching in 1D. *Proc. ECCV 2000*, D. Vernon editor, 2000.

[22] A. Trouvé and L. Younes. Local Analysis on a Shape Manifold. preprint, Univ. de Paris 13, 2002. available at the URL "citeseer.ist.psu.edu/608827.html".

[23] A. Trouv and L. Younes. Local Geometry of Deformable Templates *SIAM Journal of Mathematical Analysis* 37,1 (2005), 17–59.

[24] L. Younes. Computable elastic distances between shapes. *SIAM J. Appl. Math.* 58 (1998), 565–586.

Peter W. Michor: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria; *and:* Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse 9, A-1090 Wien, Austria

E-mail address: `Peter.Michor@univie.ac.at`

David Mumford: Division of Applied Mathematics, Brown University, Box F, Providence, RI 02912, USA

E-mail address: David_Mumford@brown.edu