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ORDINARY DIFFERENTIAL EQUATIONS

The Geometry of Point Extensions of a Class of Second-Order Differential Equations

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1. INTRODUCTION

The class of ordinary second-order differential equations

$$y'' = P(x,y) + 3Q(x,y)y' + 3R(x,y)y'^2 + S(x,y)y'^3 \tag{1.1}$$

with a third-order polynomial in y' on the right-hand side is closed under point transformations

$$\tilde{x} = \tilde{x}(x,y), \qquad \tilde{y} = \tilde{y}(x,y). \tag{1.2}$$

This means that the transformation (1.2) of Eq. (1.1) results in an equation of the same form:

$$\tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\tilde{Q}(\tilde{x},\tilde{y})\tilde{y}' + 3\tilde{R}(\tilde{x},\tilde{y})\tilde{y}'^2 + \tilde{S}(\tilde{x},\tilde{y})\tilde{y}'^3. \tag{1.3}$$

For two given equations (1.1) and (1.3), the problem on the existence of a point transformation (1.2) reducing one of the equations to the other is known as the *equivalence problem.*[1] Various aspects of this problem have been studied for a long time (see [1–22]). The paper [23] describes a scheme for the pointwise classification of equations of the form (1.1) and gives the complete list of possible cases appearing in the framework of that scheme.

Let us increase the degree of the polynomial on the right-hand side in (1.1) by one; then we obtain the class of equations of the form

$$y'' = P(x,y) + 4Q(x,y)y' + 6R(x,y)y'^2 + 4S(x,y)y'^3 + L(x,y)y'^4, \tag{1.4}$$

which is not closed under point transformations (1.2). Performing the transformation (1.2) in (1.4), we find that there appears a denominator; namely,

$$y'' = \frac{P(x,y) + 4Q(x,y)y' + 6R(x,y)y'^2 + 4S(x,y)y'^3 + L(x,y)y'^4}{Y(x,y) - X(x,y)y'}. \tag{1.5}$$

The class of equations of the form (1.5) is already closed under point transformations (1.2). It will be called the *point extension* of the class (1.4). The aim of the present paper is to describe some geometric structures induced by equations of the form (1.5) and playing an important role in the pointwise classification of such equations.

2. POINT TRANSFORMATIONS

The transformation (1.2) can be treated as the passage from some curvilinear coordinates on the plane to some other curvilinear coordinates. This interpretation leads to the geometric approach

[1] We mean the *local* equivalence of equations, since the class of transformations (1.2) is restricted only by the regularity (nondegeneracy) condition, which provides only the local invertibility of such transformations.

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to problems related to point transformations. The transformation (1.2) is assumed to be regular. By T and S we denote the direct and inverse Jacobi matrices of this transformation; namely,

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad T = \begin{Vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{Vmatrix}. \tag{2.1}$$

Following the notation in [23], in the forthcoming considerations, we denote partial derivatives by double subscripts. For example, if f is a function of two variables, then $f_{p.q}$ is the derivative of f of the order p with respect to the first argument and the order q with respect to the second argument.

The first derivatives are transformed under the point change of variables (1.2) according to the formula

$$y' = (y_{1.0} + y_{0.1}\tilde{y}') \,/\, (x_{1.0} + x_{0.1}\tilde{y}')\,. \tag{2.2}$$

For the second derivatives, the formula reads

$$\begin{aligned} y'' = \Big[&(x_{1.0} + x_{0.1}\tilde{y}')\left(y_{2.0} + 2y_{1.1}\tilde{y}' + y_{0.2}(\tilde{y}')^2 + y_{0.1}\tilde{y}''\right) \\ &- (y_{1.0} + y_{0.1}\tilde{y}')\left(x_{2.0} + 2x_{1.1}\tilde{y}' + x_{0.2}(\tilde{y}')^2 + x_{0.1}\tilde{y}''\right)\Big] \Big/ (x_{1.0} + x_{0.1}\tilde{y}')^3\,. \end{aligned} \tag{2.3}$$

The substitution of (2.2) and (2.3) into (1.5) determines the transformation rule for the coefficients of Eq. (1.5) under the point changes of variables (1.2). First, we consider the transformation rule for the functional parameters X and Y occurring in the denominator on the right-hand side in (1.5):

$$X = u\left(S^1_1\tilde{X} + S^1_2\tilde{Y}\right), \qquad Y = u\left(S^2_1\tilde{X} + S^2_2\tilde{Y}\right). \tag{2.4}$$

Here $u = u(x, y)$ is an arbitrary function of two variables. The functional ambiguity in (2.4) is due to the fact that the numerator and denominator in (1.5) can be multiplied by the same function $u(x, y) \neq 0$ without changing the ratio and the character of dependence of the numerator and denominator on y'. To be definite, we set $u = 1$. Then the transformation rule for X and Y can be written out in matrix form via the matrix S given by (2.1):

$$\|X, Y\|^{\mathrm{T}} = \|S^i_j\|^2_{i,j=1}\left\|\tilde{X}, \tilde{Y}\right\|^{\mathrm{T}}. \tag{2.5}$$

The rule (2.5) is the transformation rule for the components of a vector under a change of curvilinear coordinates on the plane (for details, see [24]). Hence to X and Y we can assign the vector field $\boldsymbol{\alpha}$ with components $\alpha^1 = X$ and $\alpha^2 = Y$. Obviously, $\boldsymbol{\alpha} \neq 0$, since the simultaneous vanishing of the components of this field would imply the vanishing of the denominator in (1.5).

Along with traditional vector and tensor fields, the theory of point transformations of Eqs. (1.1) and (1.5) involves weighted pseudotensor fields [23, 25].

Definition 2.1. A *pseudotensor field of the type* (r, s) *and weight* m is an indexed array of quantities $F^{i_1\ldots i_r}_{j_1\ldots j_s}$ behaving under transformations of the form (1.2) according to the rule

$$F^{i_1\ldots i_r}_{j_1\ldots j_s} = (\det T)^m \sum_{\substack{p_1\ldots p_r \\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\, T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\, \tilde{F}^{p_1\ldots p_r}_{q_1\ldots q_s}\,.$$

We substitute $y' = z$ into the right-hand side of (1.5) and consider the resulting fraction on the right-hand side as a rational function of the parameter z:

$$f(z) = \left(P + 4Qz + 6Rz^2 + 4Sz^3 + Lz^4\right)/(Y - Xz). \tag{2.6}$$

The function (2.6) has a pole at the point $z_0 = Y/X$. We evaluate the residue of this function at the point z_0, and by $\Omega = \Omega(x, y)$ we denote the quantity

$$\Omega = -X^5 \operatorname*{Res}_{z=z_0} f(z). \tag{2.7}$$

We can readily obtain an explicit expression for the quantity Ω given by (2.7):

$$\Omega = PX^4 + 4QX^3Y + 6RX^2Y^2 + 4SXY^3 + LY^4. \tag{2.8}$$

Next, we can readily see that under point changes of variables (1.2) the quantity (2.8) is transformed as a pseudoscalar field of weight -2; namely,

$$\Omega = (\det T)^{-2}\tilde{\Omega}. \tag{2.9}$$

Relation (2.9) can be derived by straightforward computations with the use of transformation rules for the coefficients of Eq. (1.5). The transformation rules for P, Q, R, S, and L under the changes of variables (1.2) are quite cumbersome, and we do not write them out explicitly.

3. SPECIAL COORDINATES

The subsequent analysis of Eqs. (1.5) is based on the following well-known theorem on the rectification of a vector field [26, 27].

Theorem 3.1. *For any nonzero vector field* $\boldsymbol{\alpha}$ *on the plane with components* α^1 *and* α^2 *in the coordinates* x *and* y, *there exists a point transformation of the form* (1.2) *such that in the new coordinates* $\tilde{x}$ *and* $\tilde{y}$, *the field* $\boldsymbol{\alpha}$ *becomes the unit field*: $\tilde{\alpha}^1 = 0$ *and* $\tilde{\alpha}^2 = 1$.

Applying this theorem to Eq. (1.5), we obtain the following assertion.

Corollary. Every equation (1.5) can be reduced to the form (1.4) with the use of a point transformation (1.2).

Let x and y be special coordinates in which the vector field $\boldsymbol{\alpha}$ has the unit components

$$\tilde{\alpha}^1 = 0, \qquad \tilde{\alpha}^2 = 1. \tag{3.1}$$

Consider the changes of variables preserving condition (3.1). They form a special subclass in the class of general point transformations of the form (1.2) and are given by the formulas

$$\tilde{x} = h(x), \qquad \tilde{y} = y + g(x). \tag{3.2}$$

Differentiating (3.2), we can find the entries of the transition matrices S and T:

$$S = \frac{1}{h'}\left\| \begin{matrix} 1 & 0 \\ -g' & h' \end{matrix} \right\|, \qquad T = \left\| \begin{matrix} h' & 0 \\ g' & 1 \end{matrix} \right\|. \tag{3.3}$$

The nondegeneracy condition for these matrices implies that $\det T = h'(x) \neq 0$.

It follows from (3.1) that in the special coordinates thus defined, Eq. (1.5) has the form (1.4). Let us consider the rule for the transformation of the parameters L and S in Eq. (1.4) under the change of variables (3.2) preserving the validity of condition (3.1):

$$L = h'^{-2}\tilde{L}, \qquad S = g'h'^{-2}\tilde{L} + h'^{-1}\tilde{S}. \tag{3.4}$$

The first of these relations implies that the parameter L is transformed as a pseudoscalar field of weight -2. This is not surprising, since if condition (3.1) is satisfied, then formula (2.8) for the field Ω implies that $\Omega = L$.

Conclusion. In the special coordinates, the parameter L determines a pseudoscalar field of weight -2, which can be continued to the field Ω in arbitrary coordinates.

Let us supplement the field $\boldsymbol{\alpha}$ by one more field $\boldsymbol{\psi}$ whose components in the special coordinates are defined as $\psi^1 = L$ and $\psi^2 = -S$. This allows one to rewrite formulas (3.4) in matrix form as $\|\psi^1, \psi^2\|^{\mathrm{T}} = h'^{-1}\, \|S^i_j\|^2_{i,j=1} \left\|\tilde{\psi}^1, \tilde{\psi}^2\right\|^{\mathrm{T}}$. Hence the field $\boldsymbol{\psi}$ defined in the special coordinates by the relations $\psi^1 = L$ and $\psi^2 = -S$ is a pseudovector field of weight -1. Note that the condition of the noncollinearity of $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ implies that $L \neq 0$ in special coordinates and $\Omega \neq 0$ in arbitrary coordinates. This conditions is necessarily satisfied for Eqs. (1.4) and (1.5), since its failure would reduce Eqs. (1.4) and (1.5) to the form (1.1), considered earlier in [23].

Having specified the field $\boldsymbol{\psi}$ in special coordinates, we naturally face the problem of computing its components in an arbitrary coordinate system. By M and N we denote the components of the field $\boldsymbol{\psi}$ in arbitrary coordinates, i.e., $\psi^1 = M$ and $\psi^2 = N$. We have the following formulas for M and N:

$$\begin{aligned} M &= QX^3 + 3RX^2Y + 3SXY^2 + LY^3 + (X/4)\left(X^2Y_{1.0} - XX_{1.0}Y + XYY_{0.1} - X_{0.1}Y^2\right), \\ N &= -PX^3 - 3QX^2Y - 3RXY^2 - SY^3 + (Y/4)\left(X^2Y_{1.0} - XX_{1.0}Y + XYY_{0.1} - X_{0.1}Y^2\right). \end{aligned} \tag{3.5}$$

To prove formulas (3.5), we show that in the special coordinate system with $X = 0$ and $Y = 1$ they become $M = L$ and $N = -S$. After that, we verify the transformation rule

$$\|M, N\|^{\mathrm{T}} = (\det T)^{-1} \left\|S_j^i\right\|_{i,j=1}^2 \left\|\tilde{M}, \tilde{N}\right\|^{\mathrm{T}} \tag{3.6}$$

under the passage from one arbitrary system of curvilinear coordinates to another. This can be performed by straightforward computation with regard for the transformation rules for the coefficients of Eq. (1.5) under such a transformation.

The vector field $\boldsymbol{\alpha}$, the pseudovector field $\boldsymbol{\psi}$, and the pseudoscalar field Ω are related by the formula

$$\Omega = \sum_{i=0}^{2} \sum_{j=0}^{2} d_{ij} \psi^i \alpha^j. \tag{3.7}$$

Here $d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$ is the unit skew-symmetric 2×2 matrix, which determines a twice covariant pseudotensor field of weight -1. Performing the summation on the right-hand side in (3.7), we can rewrite this relation as follows:

$$\Omega = MY - NX. \tag{3.8}$$

In this form, relation (3.8) can readily be verified by straightforward computation with regard for (2.8) and (3.5).

4. THE ASSOCIATED EQUATION

We use the components of the field $\boldsymbol{\alpha}$, the components of the field $\boldsymbol{\psi}$, and the pseudoscalar field Ω to define the function

$$k(z) = (N - Mz)^4 / \left(\Omega^3 (Y - Xz)\right). \tag{4.1}$$

The functions $k(z)$ and $f(z)$ occurring in (4.1) and (2.6) have simple poles at the same point $z_0 = Y/X$. By virtue of (3.8) and (2.7), the residues of $f(z)$ and $k(z)$ at z_0 also coincide; therefore, the difference of these functions is a polynomial, i.e.,

$$f(z) - k(z) = P^*(x, y) + 3Q^*(x, y)z + 3R^*(x, y)z^2 + S^*(x, y)z^3. \tag{4.2}$$

The simplest way is to find the coefficients of the polynomial (4.2) in the special coordinate system in which $X = 0$ and $Y = 1$. In this case, we have $P^* = P - S^4/L^3$, $Q^* = 4Q/3 - 4S^3/L^2$, $R^* = 2R - 2S^2/L$, and $S^* = 0$. The coefficients of the polynomial (4.2) in an arbitrary coordinate system can be expressed in explicit form via the parameters P, Q, R, S, L, X, and Y. However, the corresponding expressions are quite cumbersome; thus we do not write them out but prove the following theorem.

Theorem 4.1. *The coefficients of the polynomial* (4.2) *are uniquely determined by the differential equation* (1.5).

Proof. We have already proved that the coefficients of the polynomial (4.2) are uniquely determined by the parameters P, Q, R, S, L, X, and Y. However, these parameters themselves are not uniquely determined by Eq. (1.5). Their choice admits the following gauge freedom related to the fact that the numerator and denominator of the fraction (1.5) can be multiplied by the same function $\varphi(x, y)$:

$$\begin{array}{llll} P \to \varphi(x,y)P, & Q \to \varphi(x,y)Q, & R \to \varphi(x,y)R, & S \to \varphi(x,y)S, \\ L \to \varphi(x,y)L, & X \to \varphi(x,y)X, & Y \to \varphi(x,y)Y. & \end{array} \tag{4.3}$$

The substitution of (4.3) into formulas (3.5) for the components of the field ψ determines gauge transformations of the parameters M and N:

$$M \to \varphi(x,y)^4 M, \qquad N \to \varphi(x,y)^4 N. \tag{4.4}$$

Substituting them, together with (4.3), into (3.8) and (4.1), we obtain the following transformation rule for the field Ω: $\Omega \to \varphi(x,y)^5\Omega$. Taking account of this rule and relation (4.4), we find that the function $k(z)$ is invariant under gauge transformations (4.3). Consequently, the polynomial (4.2) is also invariant under such transformations. The proof of the theorem is complete.

Let us derive the transformation rules for P^*, Q^*, R^*, and S^* under point transformations (1.2). We substitute $z = y'$ into the fraction (4.1) and use formula (2.2) for the computation of the derivative. Taking into account the transformation rules (2.5), (2.9), and (3.6), we obtain the relation

$$k\,(y') = \det S\,(x_{1.0} + x_{0.1}\tilde{y}')^{-3}\,\tilde{k}\,(\tilde{y}'). \tag{4.5}$$

Here $\tilde{k}(z)$ is a fraction of the form (4.1), where X, Y, M, N, and Ω are replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{M}$, $\tilde{N}$, and $\tilde{\Omega}$. To derive transformation rules for the quantities P^*, Q^*, R^*, and S^* under point transformations (1.2), we rewrite Eq. (1.5) in the form

$$y'' = P^*(x,y) + 3Q^*(x,y)y' + 3R^*(x,y)y'^2 + S^*(x,y)y'^3 + k\,(y').$$

Substituting (2.2) and (2.3) into the last equation, taking into account relation (4.5), and performing simple manipulations, we obtain the following main result of the present research.

Theorem 4.2. *Under the point transformations* (1.2), *the coefficients* $P^*(x,y)$, $Q^*(x,y)$, $R^*(x,y)$, *and* $S^*(x,y)$ *of the polynomial* (4.2) *are transformed in the same way as the coefficients of the differential equation* $y'' = P^*(x,y) + 3Q^*(x,y)y' + 3R^*(x,y)y'^2 + S^*(x,y)y'^3$.

The equation given in Theorem 4.2 is referred to as a *canonically associated equation of the form* (1.1) for Eq. (1.5). Theorems 4.1 and 4.2 imply the following assertion.

Theorem 4.3. *If two equations of the form* (1.5) *are pointwise equivalent* [*i.e., one of them is obtained from the other by a change of variables of the form* (1.2)], *then the corresponding associated equations are also pointwise equivalent.*

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