The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems

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CONTENTS

Introduction

Historically it is the existence of a conservation law of higher order that attracted attention to the equations that have become classical now: the Korteweg-de Vries equation, the non-linear Schrödinger equation, the sine-Gordon equation, which were integrated subsequently by the method of the inverse scattering problem [1]–[3]. The problem of listing equations with one polynomial conservation law of fixed order n was considered, for example, in [4], [5]. This requirement, which is comparatively easily verifiable for a given equation for small values of n, leads to a large volume of hardly controllable calculations in the classification problem. Moreover, the approach mentioned does not guarantee the completeness of the list obtained: the result depends essentially on the order n of the conservation law.

Progress began after replacement of the conservation laws by symmetries of higher order. The first complete list of non-linear equations (Klein-Gordon models $u_{xy} = f(u)$) with higher symmetries was obtained in [6]. It consists of the three well-known equations:

$$(0.1) \qquad u_{xy} = e^u, \quad u_{xy} = e^u + e^{-u}, \quad u_{xy} = e^u + e^{-2u}.$$

Higher symmetries and conservation laws are important intrinsic properties of an equation. They are very useful both for the construction of exact solutions and for qualitative understanding of the behaviour of the solution as a whole. With appropriate refinements we take the existence of higher symmetries and conservation laws for the definition of integrability. Thanks to the symmetry approach it has proved possible to formulate simple and general necessary conditions for integrability (that is, conditions for the existence of higher symmetries and conservation laws). The simplest integrability conditions associated with the existence of a higher symmetry were established in [7]. It was shown later that the same conditions are necessary also for the existence of two local conservation laws of higher order [8].

The integrability conditions proved their effectiveness for the solution of classification problems; it turned out that the fulfilment of the first few conditions is sufficient for the complete determination of the right-hand side of the equations possessing higher symmetries and conservation laws. In other words, the integrability conditions turn out to be not only necessary but also sufficient, that is, they yield an integrability criterion. A definitive result of the symmetry approach consists in complete lists of integrable systems of the equations

$$(0.2) \qquad u_t = \Phi(x, u, \partial u/\partial x, \ldots, \partial^m u/\partial x^m), \quad m \geqslant 2$$

where $u = (u^1, \ldots, u^M)$, $\Phi = (\Phi^1, \ldots, \Phi^M)$, whose right-hand side has a special form, as a rule. The problem of classifying integrable equations led to an essential extension of the framework of the classical theory of invertible transformations which goes back to the work of Sophus Lie. New equivalence relations [9], [10] appeared, which made it possible to essentially reduce the lists of integrable systems and make them observable. Not only equations integrable by the method of the inverse scattering problem, but also the Burgers type equations $u_t = u_{xx} + uu_x$, satisfy the conditions that underlie the classification under the symmetry approach.

Classification is carried out modulo invertible transformations. In the scalar case any two equations of the first order $u_t = \Phi(x, u, u_x)$, $\Phi \neq 0$, are equivalent up to point and contact transformations. For $m = 2$ the classification of the scalar equations (0.2) leads to the list of three equations of Burgers type (see [11] and §5). These equations possess higher symmetries and though they do not reduce to the linear equations $u_t = u_{xx} + a(x)u_x + b(x)u + c(x)$ by invertible transformations, they linearize, like the Burgers equation, after introducing a potential.

As shown in [12] (compare §3), scalar equations (0.2) of even order $m = 2, 4, 6, \ldots$ have no infinite series of local conservation laws, so the equations of Korteweg-de Vries type, possessing both higher symmetries and conservation laws, correspond to odd values of $m = 3, 5, \ldots$. For $m = 3$ it is only the classification of quasilinear equations of the form (0.2) that has been completed ([13]-[15]). The general case requires an essential development of the classical theory of invertible transformations (compare §§4, 6). The main part of the list of equations (0.2) with $\Phi = u_{xxx} + F(u, u_x, u_{xx})$ consists of the four equations known from the inverse problem [16], [17]:

$$(0.3) \qquad u_t = u_{xxx} + (\alpha u^2 + \beta u)\, u_x,$$

$$(0.4) \qquad u_t = u_{xxx} - \frac{1}{2} u_x^3 + (\alpha e^{2u} + \beta e^{-2u})\, u_x,$$

$$(0.5) \qquad u_t = u_{xxx} - \frac{3}{2} [u_x^2 + Q(u)]^{-1} \left[u_x u_{xx}^2 + u_x u_{xx} Q'(u) + \frac{1}{4} Q'^2(u) \right] + \frac{1}{2} Q''(u)\, u_x, \quad d^5 Q(u)/du^5 = 0,$$

$$(0.6) \qquad u_t = u_{xxx} - \frac{3}{2} u_x^{-1} [u_{xx}^2 + Q(u)], \quad d^5 Q(u)/du^5 = 0.$$

The remaining equations from the complete list obtained in [8] can either be found from the equations (0.3), (0.4) by a substitution of the form $v_1 = \varphi(u)$ (transfer to a potential) or are Burgers type equations. The equation (0.5) is one of the first examples of equations whose integration is related to the Riemann problem on an elliptic curve [18].

We emphasize that the classification results mentioned above have a definitive character and the lists obtained are complete. In the scalar case a general algorithm for obtaining integrability conditions for equations of the form (0.2) is given in the survey [19] (see §3). On the basis of these conditions a number of classification results for equations of order $m > 3$ have been obtained (see [13], [20], [21]).

The question of whether a given equation is integrable or not can be solved either by comparing the equation with the lists obtained from the classification, or by directly verifying the conditions used for the classification. As a rule, the second method turns out to be more convenient, since it does not require searching for an invertible substitution that relates the equation under consideration with those from the list. The conditions concerned with conservation laws complete and strengthen the conditions for the existence of higher symmetries. In the scalar case the equations of Burgers type satisfy these strengthened conditions too for odd m, so the lists obtained by the purely symmetry approach do not differ in practice from the lists obtained by using the strengthened conditions.

A number of essentially new examples of equations, integrable by the method of the inverse scattering problem, have been obtained by classifying the chains of non-linear equations ([22], [23])

$$(0.7) \qquad \frac{du_n}{dt} = \Phi(u_{n-1}, u_n, u_{n+1}) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and the systems of two equations ([9], [24], [25]) of the form

(0.8) $u_t = A(u) u_{xx} + F(u, u_x)$, $\det A(u) \neq 0$.

We note that in these two cases the strengthened symmetry conditions
obtained by using conservation laws almost completely eliminate equations
of Burgers type.

The complete list of equations (0.7) obtained from three conditions (see
below) has much in common with the list of scalar equations (0.2) for
$m = 3$. The main part of the list consists of the four equations:

(0.9) $\dfrac{du_n}{dt} = P(u_n)(u_{n+1} - u_{n-1})$, $P''' = 0$,

(0.10) $\dfrac{du_n}{dt} = P(u_n^2)[(u_{n+1} + u_n)^{-1} - (u_n + u_{n-1})^{-1}]$, $P''' = 0$,

(0.11) $\dfrac{du_n}{dt} = Q(u_n)[(u_{n+1} - u_n)^{-1} + (u_n - u_{n-1})^{-1}]$, $Q^V = 0$,

(0.12) $\dfrac{du_n}{dt} = (u_{n+1} - u_{n-1})^{-1}[R(u_{n-1}, u_n, u_{n+1}) +$

$$+ \alpha \sqrt{R(u_{n-1}, u_n, u_{n-1}) R(u_{n+1}, u_n, u_{n+1})}],$$

where $R(u, v, w) = (w - v)(u - v)(\beta + Q''(v)/12) + (w - 2v + u)Q'(v)/4 + Q(v)$,
Q is an arbitrary polynomial of degree 4, $\beta \in \mathbf{C}$, $\alpha = 0, \pm 1$. The remaining
equations from the list can be obtained from (0.9)–(0.11) by the operations
$v_{n+1} \pm v_n = \varphi(u_n)$, which are analogous to introducing a potential. The
conditions on which the classification of chains of equations (0.7) is based can
be represented in the form

(0.13) $d\rho_1/dt$, ρ_1^0, $\rho_2^0 \in \mathrm{Im}(D - 1)$,

where D is the shift operator acting on the set of functions of finitely many
variables u_0, $u_{\pm 1}$, $u_{\pm 2}$, ... The functions ρ_1, ρ_1^0, ρ_2^0 are expressed in terms
of the right-hand side $\Phi = \Phi(u_{n-1}, u_n, u_{n+1})$ of (0.7) by the formulae

$$\rho_1 = \log \frac{\partial \Phi}{\partial u_{n+1}}, \quad \rho_1^0 = \log \left(-\frac{\partial \Phi}{\partial u_{n+1}} \Big/ \frac{\partial \Phi}{\partial u_{n-1}} \right),$$

$$\rho_2^0 = [(D - 1)^{-1}(\rho_1^0)]_t + 2\frac{\partial \Phi}{\partial u_n}.$$

The conditions (0.13) are necessary conditions for the existence of a pair of
local higher-order conservation laws. The derivation of these conditions in
the framework of the symmetry approach is based on a general scheme
presented in Chapter I (§3). It can be shown (see [22], [23]) that all the
equations (0.7) satisfying (0.13) have an infinite series of local conservation
laws and that the equations (0.10) and (0.12) for $\alpha = 0$ are difference
approximations of the equations (0.4) and (0.6) respectively. The limit
passage from (0.10), (0.12) to (0.4), (0.6) is carried out in the same way as
for the well-known equation (0.9), which is a difference analogue of the
modified Korteweg-de Vries equation (0.3).

Chapter IV is devoted to a description of the vast list of systems of equations (0.8). The classification is carried out over the enlarged module of invertible transformations described in Chapter III. This allows us not only to shorten the list and make it observable but also to reveal connections between equations that were assumed previously to be essentially distinct. As in the case of the chains of equations (0.7), the classification is based on the first few necessary conditions for the existence of a pair of local conservation laws. The question of whether the set of conditions under consideration is sufficient for the existence of an infinite series of conservation laws for most of the equations in the list can be solved by using the theory of transformations from Chapter III and by revealing connections with the following three classical equations, which are integrable by the method of the inverse scattering problem:

(0.14) $$iu_t = u_{xx} + |u|^2 u,$$

(0.15) $$u_{tt} = u_{xxxx} + (u^2)_{xx},$$

(0.16) $$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times I\mathbf{S}, \quad S_1^2 + S_2^2 + S_3^2 = 1, \quad I = \mathrm{diag}\,(I_1, I_2, I_3).$$

As already mentioned, in principle the equations of the lists satisfy only the first few necessary conditions for the existence of higher-order symmetries and conservation laws, and the question of their integrability and the so-called representations of zero curvature must be solved beyond the framework of the symmetry approach. The most familiar method of constructing the commutation representation

(0.17) $$A_t - B_x + [A, B] = 0$$

of a given non-linear equation is the direct method based on the definition (0.17). The problem then reduces to the construction of a matrix Lie algebra with a partially given multiplication table. It should be kept in mind, however, that the representation (0.17) itself without additional requirements on its structure does not guarantee the integrability of the equation under consideration. The system of equations found in [26]

$$u_t = u_{xx} + \alpha\,(u^{3/2}v^{1/2})_x + \beta u^2 v, \quad -v_t = v_{xx} - \alpha\,(v^{3/2}u^{1/2})_x + \beta v^2 u,$$

which does not satisfy the necessary conditions for the existence of local conservation laws, can serve as an example. In any case, commutation representations have been constructed by now for all systems (0.8) from the lists of Chapter IV except for the three systems (*c*), (*n*), (*p*), and it is lack of space that has not allowed us to include them in this survey. The systems (*n*) and (*p*) can be regarded as an illustration of the fact that the verification of integrability conditions is a much simpler problem than the problem of commutation representation.

As a matter of fact, the symmetry approach is simple. At the first stage a canonical series of conservation laws is constructed. It is proved that the local property of these laws is a necessary condition for the existence of

higher symmetries and conservation laws. The second, more difficult stage consists in the investigation of conditions that arise from the requirement of the local property of the first few canonical laws.

For integrable equations the algorithm for calculating the canonical series given in Chapter I is a convenient way of constructing higher conservation laws. We note that this algorithm is not unique; a close algorithm is contained in [27]. The series of conservation laws constructed by the method of the inverse problem from the scattering matrix coincides, as experience shows, with the canonical one.

We have not included results on classification of hyperbolic equations. An interesting and most investigated class is the exponential systems of Liouville type [28], [29], which are closely related to finite-dimensional Lie algebras. Though the analysis of hyperbolic systems follows the general scheme, there are specific difficulties in comparison with systems of the form (0.2) (see [30]).

The case of three or more independent variables differs essentially from the case of two independent variables. Strong results, related to the classification of integrable systems on the basis of analysis of series of perturbation theory, the S-matrix, and additional conservation laws, have been obtained recently in [31], [32].

CHAPTER I

INTEGRABILITY CONDITIONS

In this chapter we present a general scheme for deriving necessary conditions for the existence of higher symmetries and conservation laws for systems of equations of the form (0.2). In §1 we introduce the basic concepts of the symmetry approach. In §2 we present a technique related to formal series in inverse powers of the differentiation operator (compare [33], [34]). In §3 we state the main results (Theorems 3.2, 3.6) and give examples illustrating the general algorithm for evaluating canonical conservation laws. In many respects we follow [19]. For the sake of simplicity we assume throughout this chapter that the right-hand side of (0.2) does not depend on x.

§1. Formal symmetries and conservation laws

1. Under the symmetry approach the system of partial differential equations

$$(1.1) \qquad \mathbf{u}_t = \Phi\,(\mathbf{u},\ \partial\mathbf{u}/\partial x,\ \ldots, \partial^m\mathbf{u}/\partial x^m), \quad \mathbf{u} = (u^1,\ \ldots, u^M)$$

is replaced by a pair of infinite-dimensional dynamical systems of the form

$$(1.2) \qquad \frac{d\mathbf{u}_k}{dt} = \Phi_k\,(\mathbf{u}_0,\ \mathbf{u}_1,\ \ldots, \mathbf{u}_{m_k}) \quad (k = 0,\ 1,\ 2,\ \ldots),$$

$$(1.3) \qquad \frac{d\mathbf{u}_k}{dx} = \mathbf{u}_{k+1} \quad (k = 0,\ 1,\ 2,\ \ldots).$$

The compatibility condition for (1.2), (1.3)

$$\frac{d}{dx}\left(\frac{du_k}{dt}\right) = \frac{d}{dt}\left(\frac{du_k}{dx}\right) \quad (k = 0, 1, \ldots)$$

leads to the recursion relations

(1.4) $$\Phi_{k+1} = D(\Phi_k) \quad (k = 0, 1, \ldots),$$

where

(1.5) $$D = u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + \cdots$$

is the total differentiation operator with respect to x on the set of functions of $u_0, u_1, u_2, \ldots$ Setting

$$\Phi_0(u_0, u_1, \ldots, u_{m_0}) = \Phi(u_0, u_1, \ldots, u_m),$$

where Φ is the right-hand side of the system (1.1) under consideration, we obtain the infinite system of equations (1.2), (1.3), which is equivalent to (1.1) (with an appropriate change of the initial conditions).

Infinitesimal symmetries of an infinite-dimensional dynamical system are defined in the same way as in the finite-dimensional case. We recall that for a finite-dimensional dynamical system

(1.6) $$\frac{du_k}{dt} = \Phi_k(u_0, \ldots, u_N) \quad (k = 0, \ldots, N)$$

the infinitesimal symmetry transformations

$$\bar{u}_k = u_k + \tau f_k(u_0, \ldots, u_N)$$

are determined from the condition that $d\bar{u}_k/dt = \Phi_k(\bar{u}_0, \ldots, \bar{u}_N)$ for all k up to the first order with respect to the small parameter τ. This leads to the following equations for $f_0, \ldots, f_N$:

(1.7) $$\frac{df_k}{dt} = \sum \frac{\partial \Phi_k}{\partial u_i} f_i \quad (k = 0, \ldots, N),$$

where d/dt is the total differentiation operator with respect to t acting on functions of the dynamical variables $u_0, \ldots, u_N$:

(1.8) $$\frac{d}{dt} = \Phi_0 \frac{\partial}{\partial u_0} + \cdots + \Phi_N \frac{\partial}{\partial u_N}.$$

It is easy to check that the invariance condition (1.7) coincides with the condition of commutation of the operator (1.8) with the operator

(1.9) $$\frac{d}{d\tau} = f_0 \frac{\partial}{\partial u_0} + \cdots + f_N \frac{\partial}{\partial u_N}.$$

In other words, the condition (1.7) coincides with the condition $d/dt(d/d\tau) = d/d\tau(d/dt)$ of compatibility of the system (1.6) with the dynamical system

(1.10) $$\frac{du_k}{d\tau} = f_k(u_0, \ldots, u_N) \quad (k = 0, \ldots, N).$$

In the infinite-dimensional case the conditions of compatibility of the system (1.2), (1.3) with the dynamical system

(1.11) $\dfrac{du_k}{d\tau} = \mathbf{f}_k(\mathbf{u}_0, \ldots, \mathbf{u}_{n_k})$ $(k = 0, 1, \ldots)$

lead to the relations

(1.12) $\dfrac{d\mathbf{f}_k}{dt} = \sum \dfrac{\partial \Phi_k}{\partial \mathbf{u}_l} \mathbf{f}_l, \quad D(\mathbf{f}_k) = \mathbf{f}_{k+1}.$

The first of these relations is the condition that the operators

(1.13) $\dfrac{d}{dt} = \sum_{k=0}^{\infty} \Phi_k \dfrac{\partial}{\partial \mathbf{u}_k}, \quad \dfrac{d}{d\tau} = \sum_{k=0}^{\infty} \mathbf{f}_k \dfrac{\partial}{\partial \mathbf{u}_k}$

commute, while the second means that the form of the system (1.11) is completely determined by the first equation of this system. Since $\mathbf{f}_k = D^k(\mathbf{f}_0)$ and $\Phi_k = D^k(\Phi)$, the relations (1.12) can be written as the following equation for the vector-valued function $\mathbf{f}_0$ of the variables $\mathbf{u}_0, \ldots, \mathbf{u}_{n_0}$:

(1.14) $\dfrac{d\mathbf{f}_0}{dt} = \sum \dfrac{\partial \Phi}{\partial \mathbf{u}_l} D^l(\mathbf{f}_0).$

This equation can also be regarded as a condition of formal compatibility of the system of partial differential equations (1.1) with the system constructed from $\mathbf{f}_0$:

(1.15) $\partial \mathbf{u}/\partial \tau = \mathbf{f}_0(\mathbf{u}, \partial \mathbf{u}/\partial x, \ldots, \partial^{n_0}\mathbf{u}/\partial x^{n_0}).$

A vector-valued function $\mathbf{f}$ of a finite set of variables $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots$, satisfying (1.14) is called a *symmetry* (generator of an infinitesimal symmetry) *of the system of partial differential equations* (1.1). In contrast to the finite-dimensional case (1.6), the infinitesimal transformations $\overline{\mathbf{u}}_k = \mathbf{u}_k + \tau D^k \mathbf{f}(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ corresponding to higher symmetries $(n > 1)$ do not generate a one-parameter group of local transformations of the form $\overline{\mathbf{u}}_k = \psi_k(\tau, \mathbf{u}_0, \ldots, \mathbf{u}_{n_k})$ (see §4).

Example. For the Korteweg-de Vries equation $u_t = u_{xxx} + uu_x$ the function $f = u_5 + 5(2u_0u_3 + 4u_1u_2 + u_0^2 u_1)/6$ satisfies the equation (1.14)

$$\dfrac{df}{dt} = (D^3 + u_0 D + u_1) f$$

and so it is a symmetry. From the compatibility of the dynamical systems

$$\dfrac{du_k}{dt} = D^k(u_3 + u_0 u_1), \quad \dfrac{du_k}{d\tau} = D^k(f) \quad (k = 0, 1, \ldots)$$

it follows that the finite-dimensional dynamical system obtained from (1.2), (1.3) by adding the conditions $D^k(f) = 0$ $(k = 0, 1, \ldots)$ generates a finitely parametric family of solutions of the Korteweg-de Vries equation [16].

In contrast to symmetries, the notion of first integrals does not carry over to the case of an infinite-dimensional dynamical system; their place is

occupied by local conservation laws. A *local conservation law of the system* (1.2), (1.3) is by definition a relation of the form

$$(1.16) \qquad \frac{d\rho}{dt} = D(\sigma),$$

where ρ and σ are functions of finitely many dynamical variables $u_0, u_1, u_2, \ldots$ The function ρ will be called the *density*, while σ will be called the flow. If we return to the language of the partial differential equation (1.1), then (1.6) goes over to the equation of continuity $\rho_t = \sigma_x$, which is well-known in mechanics and physics. In the problem periodic in x (with period T) integration of the density leads to the motion constant

$$I = \int_0^T \rho dx, \ I_t = 0.$$

We introduce the following notation. We shall write the infinite-dimensional system of equations (1.2)–1.4) in abbreviated form as a single equation, omitting the subscript in u_0:

$$(1.17) \qquad u_t = \Phi(u, u_1, \ldots, u_m), m \geqslant 2.$$

(1) We define a linearization operation $*$ by

$$(1.18) \qquad f_*(v) = \frac{\partial}{\partial \varepsilon} f(u + \varepsilon v, D(u + \varepsilon v), \ldots, D^n(u + \varepsilon v))|_{\varepsilon=0}$$

and rewrite the equation (1.14) that defines symmetries in the form

$$(1.19) \qquad f_t = \Phi_*(f),$$

where by the definition (1.18)

$$(1.20) \qquad \Phi_* = \Phi_u + \Phi_{u_1}D + \ldots + \Phi_{u_n}D^n.$$

We recall that $u = (u^1, \ldots, u^M)^\tau$, $\Phi = (\Phi^1, \ldots, \Phi^M)^\tau$ and so Φ_{u_k} is an $M \times M$ matrix with entries $\partial\Phi^i/\partial u_k^j$.

(2) A^T denotes the differential operator formally adjoint to A,

$$(1.21) \qquad A = \sum A_h D^k, \quad A^T = \sum (-1)^k D^k A_k^\tau$$

(the matrix A_k^τ is obtained from A_k by transposition).

(3) We define the variational derivative $\delta/\delta u$ of a scalar function $h = h(u, u_1, \ldots, u_p)$ by the following rule:

$$(1.22) \qquad \frac{\delta h}{\delta u} = \sum_k (-1)^k D^k \left(\frac{\partial h}{\partial u_k} \right).$$

The variational derivative is the column vector $\left(\frac{\delta h}{\delta u^1}, \ldots, \frac{\delta h}{\delta u^M} \right)^\tau$.

(4) We denote by Im D the linear space of the image of D ($h \in$ Im D means that there is a function $\tilde{h}$, depending on finitely many dynamical variables, such that $D\tilde{h} = h$). The following equality is a criterion for a function f to belong to the space Im $D +$ **C** (see [35]):

$$(1.23) \qquad \frac{\delta h}{\delta u} = 0 \iff h \in \text{Im } D + \textbf{C}.$$

We note that Ker D, the kernel of D, coincides with **C**.

We also state some easily verifiable properties of the operations introduced (see, for example, [36]):

(1.24) $(\alpha\beta)_* = \alpha_*\beta + \beta_*\alpha,$

(1.25) $(D(\alpha))_* = D \cdot \alpha_*,$

(1.26) $(\alpha_t)_* = \alpha_{*,t} + \alpha_*\Phi_*,$

(1.27) $\left(\dfrac{\delta\alpha}{\delta u}\right)_* = \left[\left(\dfrac{\delta\alpha}{\delta u}\right)_*\right]^T,$

(1.28) $\dfrac{d}{dt}\dfrac{\delta}{\delta u} = \dfrac{\delta}{\delta u}\dfrac{d}{dt} - \Phi_*^T\dfrac{\delta}{\delta u},$

where α and β are arbitrary scalar functions. The relation (1.28) follows from (1.26) and the definition (1.22).

The order of the differential operator f_* is called the *order of the symmetry* $f(u, u_1, \ldots)$. A symmetry is called *non-singular* if the leading coefficient of this operator is a non-singular matrix.

The conservation laws (1.16) are assumed to be *equivalent* if their densities differ by a total derivative. The *order of the conservation law* with density ρ is the order of the differential operator

(1.29) $\left(\dfrac{\delta\rho}{\delta u}\right)_*.$

Obviously, equivalent conservation laws have the same order. A conservation law is called *non-singular* if the matrix of the leading coefficient of this operator is non-singular. The variational derivative of the density of the conservation law of the dynamical system (1.17) satisfies the well-known equation [37]

(1.30) $\dfrac{d}{dt}\dfrac{\delta\rho}{\delta u} + \Phi_*^T\dfrac{\delta\rho}{\delta u} = 0,$

which follows directly from (1.16), (1.23), and (1.28).

2. Our aim is to obtain necessary conditions for the existence of symmetries and conservation laws of higher order. For this it is convenient to pass from the equations (1.19), (1.30) to

(1.31) $L_t - [\Phi_*, L] = 0,$

(1.32) $R_t + R\Phi_* + \Phi_*^T R = 0.$

Here L and R are formal series of the form

(1.33) $L = l_n D^n + l_{n-1}D^{n-1} + \ldots + l_0 + l_{-1}D^{-1} + l_{-2}D^{-2} + \ldots,$

(1.34) $R = r_p D^p + r_{p-1}D^{p-1} + \ldots + r_0 + r_{-1}D^{-1} + r_{-2}D^{-2} + \ldots,$

the matrix coefficients l_k and r_k being functions of a finite set of dynamical variables

$$l_k = l_k(u, \ldots, u_{n_k}), \quad r_k = r_k(u, \ldots, u_{p_k}).$$

The equations (1.31) (and similarly (1.32)) should be viewed as a compact way to write down the system of equations for the coefficients of the formal series L (respectively, R). The rules for multiplying formal series are given as follows:

(1.35) $[D^k, D^m] = 0,$

(1.36) $D^k a = a D^k + \binom{k}{1} D(a) D^{k-1} + \binom{k}{2} D^2(a) D^{k-2} + \ldots,$

where a is a function of the dynamical variables, $D^s(a)$ is the s-fold differentiation (1.5), $\binom{k}{n} = k(k-1) \cdot \ldots \cdot (k-n+1)/n!$. Formulae (1.35) and (1.36) are valid for any integers k and m.

The maximal degree of the symbol D is called the *degree of the formal series*.

For example, in (1.33) deg $L = n$, in (1.34) deg $R = p$. Unless otherwise specified, the leading coefficient of the series is assumed to be a non-singular matrix.

Definition. A *formal symmetry of order* N of the dynamical system (1.17) is by definition a formal series L of non-zero degree that satisfies the equation

(1.37) $L_t - [\Phi_*, L] = Q$

with 'remainder' Q such that deg $Q \leqslant$ deg $\Phi_* +$ deg $L - N$. deg L is called *the degree of the formal symmetry*.

Theorem 1.1. *If the dynamical system* (1.17) *possesses a symmetry f of order at least N or two conservation laws of orders* $N_1 > N_2 \geqslant N + m$ *and one of them is non-singular, then it has a formal symmetry of order N.*

Proof. We linearize the equation (1.19), using the operation $*$. Using the relations (1.24)–(1.26) and replacing $\sum_{s,k} \Phi_{u_s u_k} D^k(\mathsf{f}) D^s$ by $(\Phi_*)_\tau$ in accordance with (1.11), we obtain

(1.38) $L_t - [\Phi_*, L] = \Phi_{*,\tau},$

where $L = f_*$. Obviously, deg $Q =$ deg $\Phi_{*,\tau} \leqslant$ deg Φ_*, so it follows by definition that L is a formal symmetry of order N.

Suppose that there are two conservation laws with densities ρ_1 and ρ_2 of orders $N_1 > N_2$ (to be definite we shall suppose that a non-singular conservation law is associated with ρ_2). We linearize (1.30). Using (1.24)–(1.27), we obtain

(1.39) $R_{i,t} + R_i \Phi_* + \Phi_*^T R_i = Q_i \quad (i = 1, 2),$

where $R_1 = (\delta\rho_1/\delta\mathsf{u})_*$, deg $Q_1 \leqslant 2m$,

$$Q_i = \Phi_*^T \left(\frac{\delta\rho_i}{\delta\mathsf{u}} \right)_* - \left(\Phi_*^T \frac{\delta\rho_i}{\delta\mathsf{u}} \right)_* = \sum_{j,k} \frac{\partial \varphi_j}{\partial u_k} D^j \left(\frac{\delta\rho_i}{\delta\mathsf{u}} \right) D^k,$$

and φ_j is a coefficient of the operator $\Phi_*^T = \sum_j \varphi_j D^j$. From the operator R_2 with a non-singular leading coefficient we can evaluate uniquely the formal series R_2^{-1}, $\deg R_2^{-1} = -\deg R_2 = -N_2$, such that $R_2 R_2^{-1} = R_2^{-1} R_2 = E$ (see §2). The formal series $L = R_2^{-1} R_1$, $\deg L = N_1 - N_2$, satisfies the equation (1.37) with remainder $Q = R_2^{-1} Q_1 - R_2^{-1} Q_2 R_2^{-1} R_1$, $\deg Q \leqslant 2m - 2N_2 + N_1$. ∎

By analogy with formal symmetries we define formal conservation laws.

Definition. A *formal conservation law of order* $N \geqslant m$ of the dynamical system (1.17) is by definition a formal series R satisfying the equation

$$(1.40) \qquad\qquad R_t + R\Phi_* + \Phi_*^T R = Q$$

with remainder Q, where $\deg Q \leqslant \deg R + \deg \Phi_* - N$.

If a system possesses a conservation law with density ρ of order $N \geqslant m$, then it possesses a formal conservation law $R = \left(\dfrac{\delta\rho}{\delta u}\right)_*$ of the same order (see the proof of the theorem). The solubility of the equations (1.31), (1.32) means the existence of formal symmetries and conservation laws of an arbitrarily large order.

The conditions for the existence of formal symmetries and conservation laws are a criterion for the formation of the lists of equations discussed in the Introduction. For example, the list of equations of the form $u_t = u_3 + \Phi(u, u_1, u_2)$ consists exactly of those equations that possess a formal symmetry of order 7. The assertion of Theorem 1.1 means that these conditions are necessary for the existence of higher symmetries and local conservation laws.

As an example we obtain a condition for the existence of a formal conservation law of order $m + 1$. Substituting (1.34) in (1.32) and equating the coefficient of the highest power of the symbol D to zero, we obtain

$$(1.41) \qquad\qquad r_p \frac{\partial\Phi}{\partial u_m} + (-1)^m \left(\frac{\partial\Phi}{\partial u_m}\right)^\tau r_p = 0.$$

We consider first the scalar case. The relation (1.41) is valid only for odd m; this means the absence of local conservation laws (of order greater than m) for scalar evolution equations of the fourth order (compare [12]). For systems of two equations ($M = 2$) of even order with a non-singular principal differential part $\det \Phi_{u_m} \neq 0$ it follows from (1.41) that firstly trace $\Phi_{u_m} = 0$ and secondly the conservation laws are non-singular if they exist (compare [9]). For systems of M equations of even order with a non-singular matrix Φ_{u_m} possessing non-singular conservation laws it follows immediately from (1.41) that the eigenvalues of the matrix Φ_{u_m} occur in pairs:

$$(1.42) \qquad\qquad \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = \ldots = 0 \quad (M \text{ is even}).$$

§2. The technique of formal series

The procedure for finding conditions for the existence of formal symmetries is essentially simple, but it leads to a great volume of algebraic transformations. It is convenient to formulate the calculation algorithm in the language of formal series, to which this section is devoted. It is easy to explain this language to a computer; this language is general and has turned out to be useful in many problems arising in the theory of integrable systems. By now programs of analytical calculations on computers have already been created, which help to obtain and analyse criteria for integrability of non-linear equations.

Though integrability conditions admit an invariant matrix formulation, it is convenient to reduce the equation (1.31) connected with formal symmetries to the diagonal form and reduce the problem to a scalar one. We use the invariance of the equations (1.31), (1.32) under gauge transformations

$$(2.1) \qquad L \mapsto TLT^{-1}, \; R \mapsto T^T R T, \quad \Phi_* \mapsto T\Phi_* T^{-1} + T_t T^{-1},$$

where T is an arbitrary formal series.

Proposition 2.1. *Suppose that the matrix* Φ_{u_m} *(the leading coefficient of the operator* Φ_**) has no multiple eigenvalues and is reduced to the diagonal form by the conjugation* T_0:

$$\Phi_{u_m} = T_0^{-1} \Lambda T_0, \quad \Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_M).$$

Then there is a unique formal series

$$(2.2) \qquad T = T_0 (E + T_{-1} D^{-1} + T_{-2} D^{-2} + \ldots),$$

satisfying the condition $\mathrm{diag}\, T_k = 0$ $(k = -1, -2, \ldots)$ *such that all coefficients of the formal series*

$$(2.3) \qquad \hat{\Phi} = T\Phi_* T^{-1} + T_t T^{-1}$$

are diagonal.

Proof. We substitute (2.2) and $\Phi_* = \sum\limits_{k=0}^{m} \Phi_{m-k} D^{m-k}$, $\hat{\Phi} = \sum\limits_{k=0}^{m} \varphi_{m-k} D^{m-k}$ in (2.3). As a result of successively equating the coefficients of D^{m-i} $(i = 1, 2, \ldots)$ to zero we obtain a recursive sequence of relations of the form

$$(2.4) \qquad [\Lambda, \; T_{-i}] + \varphi_{m-i} = \Delta_i,$$

where $\Delta_i = \Delta(T_0^{-1}, \; T_0, \ldots; \; T_{1-i}; \; T_{0,t}, \ldots, T_{m-i,t}; \; \varphi_{m-1}, \ldots, \varphi_{m-i+1};$ $\Phi, \; \Phi_m, \; \Phi_{m-1}, \ldots, \Phi_{m-i})$ is a differential polynomial in the variables indicated. The relations (2.3), with regard for the condition $\mathrm{diag}\, T_k = 0$, allow us to successively calculate the coefficients

$$\varphi_{m-i} = \mathrm{diag}\, \Delta_i \quad \text{and} \quad T_{-i} = \mathrm{ad}_\Lambda^{-1} (\Delta_i - \varphi_{m-i}). \quad \blacksquare$$

The gauge transformation indicated in the proposition reduces L to the diagonal form

(2.5) $\hat{L} = TLT^{-1}$.

For it follows from (1.31), (2.3), (2.5) that

(2.6) $\hat{L}_t = [\hat{\Phi}, \hat{L}]$,

$\hat{\Phi}$ being a formal series with diagonal coefficients whose leading coefficient has no multiple eigenvalues. We substitute $\hat{L} = l_n D^n + l_{n-1} D^{n-1} + \ldots$ in (2.6). As a result of successively equating the coefficients of D^{m+n-k} ($k = 0, 1, \ldots$) we obtain a chain of relations, the first of which has the form $[\Lambda, l_n] = 0$ and means that l_n is diagonal. The fact that the next coefficients are diagonal can easily be obtained by induction.

The formal series $\hat{L}$ obtained from a formal symmetry L by the gauge transformation (2.5) will also be called a formal symmetry of the dynamical system (1.17). Obviously, the order of a formal symmetry is a gauge invariant characteristic.

For odd m the formal series $\hat{R} = T^T R T$ is diagonal (this can be proved in the same way as in the previous case of formal symmetry). If m is even and the conditions (1.42) are satisfied, it is easy to check that $\hat{R}$ is a block-diagonal matrix with (2×2)-blocks corresponding to pairs of eigenvalues with opposite signs. Every (2×2)-block is an antidiagonal matrix. Therefore the equation (1.32) splits into $M/2$ equations of the form

(2.7) $R_{k,t} + \hat{\Phi}_k^T R_k + R_k \hat{\Phi}_k = 0$ $(k = 1, 2, \ldots, M/2)$,

where

(2.8) $\hat{\Phi}_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & -\lambda_k \end{pmatrix} D^m + \begin{pmatrix} F_k & 0 \\ 0 & G_k \end{pmatrix}$, $R_k = \begin{pmatrix} 0 & r_k \\ s_k & 0 \end{pmatrix}$.

The following theorem shows that, in contrast to the order, the degree of a formal symmetry is not essential. We recall (see §1) that formal symmetries and conservation laws are assumed to be non-singular unless otherwise specified.

Theorem 2.1 (compare [19]). *Suppose that the matrix of the leading coefficient of the series Φ_* has neither zero nor multiple eigenvalues. Then*

1) *the existence of a formal symmetry of order N is equivalent to the existence of a formal symmetry of degree 1 of the same order;*

2) *the existence of a formal symmetry and a formal conservation law of order N is equivalent to the existence of a formal symmetry L and a formal conservation law R of order N of degree 1 satisfying*

(2.9) $L^T = -RLR^{-1}$, $R^T = -R$.

Proof. 1) Let L be a formal symmetry of order N, and $\hat{L}$ the formal symmetry obtained as a result of diagonalization of the gauge transformation (2.5) (Proposition 2.1). The series $L_d = \operatorname{diag} \hat{L}$ is also a formal symmetry of order N. Together with L_d the series $(L_d)^k$ $(k = \pm 1, \pm 2, \ldots)$ are formal symmetries of the same order, therefore without loss of generality we can assume that the degree of L_d is positive. The series $L_d = b_n D^n +$ $+ b_{n-1} D^{n-1} + \ldots$ has diagonal coefficients, and there is a simple method of taking the n-th root of it (compare [33], [34]). The coefficients of a series $A = a_1 D + a_0 + \ldots$ such that $A^n = L_d$ can be found from recursive relations of the form

$$(2.10) \qquad a_1^n = b_n, \ldots, n a_1^{n-1} a_{1-k} = b_{n-k} + \theta_k,$$

where $\theta_k = \theta_k(b_n, b_{n-1}, \ldots, b_{n-k+1}; a_1, a_0, \ldots, a_{2-k})$ is a differential polynomial in the variables indicated. The relations (2.10) allow us to successively define the coefficients a_k up to fixing $b_n^{1/n}$. It follows from the identity

$$(2.11) \qquad (L_d)_t - [\hat{\Phi}, L_d] = \sum_{k=0}^{n} A^k (A_t - [\hat{\Phi}, A]) A^{n-k-1}$$

that A is also a formal symmetry of order N and $\deg A = 1$.

2) Together with R the series RL and R^T are formal conservation laws of order N. It follows in particular that there is a formal conservation law R of order N such that $\deg R = 1$. The relations (2.9) are gauge invariant, and it is sufficient to prove them in the diagonal gauge (see Proposition 2.1). For odd $m = \deg \Phi_*$ the series $\hat{R}$ has diagonal coefficients. Therefore the series $\hat{R} - \hat{R}^T$ satisfies the second of the relations (2.9) and $\deg(\hat{R} - \hat{R}^T) = 1$. For even m it may happen that the series $\hat{R} - \hat{R}^T$ is singular or $\deg(\hat{R} - \hat{R}^T) < 1$. However, multiplying $\hat{R}$ by an appropriate constant diagonal matrix C, we can arrange that the series $\hat{R}C - C\hat{R}^T$ is a non-singular formal conservation law of degree 1. Together with L the series $R^{-1}L^T R$ is a formal symmetry. Let $R^T = -R$, $\deg R = 1$. For odd m the required formal symmetry is constructed by the formula $L - R^{-1}L^T R$. But if m is even, then as in the previous case we choose a constant diagonal matrix C such that the series $LC - R^{-1}(LC)^T R$ has non-singular leading coefficient and degree 1. ∎

The *residue* of a formal series

$$(2.12) \qquad A = \sum_{k \leqslant n} a_k D^k$$

with scalar coefficients a_k is by definition the coefficient of D^{-1} (compare [34]):

$$(2.13) \qquad \operatorname{res} A = a_{-1}.$$

The *logarithmic residue* of a formal series A of degree n is defined as follows:

(2.14) $$\text{res} \log A = a_{n-1}/a_n.$$

We note that for the logarithmic residue the following identities hold:

(2.15) $$\text{res} \log(AB) = \text{res} \log A + \text{res} \log B + \deg A \, D(\log b_n)$$

(b_n is the leading coefficient of the series B),

(2.16) $$D(\text{res} \log A) = \text{res}([D, A] A^{-1}).$$

For the commutator of formal series the following formula holds (compare [34]), which is easily verifiable for monomials:

(2.17) $$\text{res}[A, B] = D\sigma(A, B),$$

where $\sigma(A, B)$ is a differential polynomial in the coefficients a_k, b_k of the series A, B:

(2.18) $$\sigma(A, B) = \sum_{\substack{p \leqslant \deg B, \, q \leqslant \deg A}}^{p+q+1>0} \binom{q}{p+q+1} \sum_{s=0}^{p+q} (-1)^s D^s(a_q) D^{p+q-s}(b_p).$$

The logarithmic residue of the commutant of formal series is also a total derivative:

(2.19) $$\text{res} \log(ABA^{-1}B^{-1}) = Ds(A, B),$$

(2.20) $$s(A, B) = m \log b_n - n \log a_m,$$

where $m = \deg A$, $n = \deg B$. We give one final identity

(2.21) $$\text{res} \log A^{-1} = \text{res} \log A^T = -\text{res} \log A + nD(\log a_n).$$

The following obvious lemma holds.

Lemma 2.1. *The formal series A and B of degree 1 coincide if and only if*

$$\text{res} \log A = \text{res} \log B, \quad \text{res} A^k = \text{res} B^k \quad (k = -1, 1, 2, 3, \ldots).$$

Moreover, the first k coefficients $a_{-1}, a_0, \ldots, a_{k-2}$ of the series A ($\deg A = 1$) are in a one-to-one correspondence with the set of residues $\rho_{-1}, \rho_0, \ldots, \rho_{k-2}$, where $\rho_0 = \text{res} \log A$, $\rho_n = \text{res} A^n$, $n \neq 0$. For, evaluating the residues we obtain

(2.22) $$\begin{aligned} \rho_{-1} &= 1/a_1, \quad \rho_0 = a_0/a_1, \quad \rho_1 = a_{-1}, \ldots, \\ \rho_m &= ma_1^{m-1}a_{-m} + \Delta(a_1, a_0, \ldots, a_{1-m}), \quad m \geqslant 1, \end{aligned}$$

where Δ is the familiar differential polynomial in the variables indicated. This chain of relations can be inverted in the obvious way and allows us to express the coefficients of A in terms of differential polynomials of the residues.

§3. Canonical conservation laws and divergency conditions

1. As shown in the previous section, the equation (1.31) for formal symmetries of the infinite-dimensional dynamical system (1.17) reduces (Proposition 2.1, Theorem 2.1) to the problem of calculating the coefficients of the scalar series

$$(3.1) \qquad L = l_1 D + l_0 + l_{-1} D^{-1} + \ldots$$

from the commutation relation

$$(3.2) \qquad L_t = [F, \ L],$$

where the coefficients of the formal series

$$(3.3) \qquad F = \lambda D^m + f_{m-1} D^{m-1} + f_{m-2} D^{m-2} + \ldots$$

are expressed by explicit formulae in terms of the right-hand side $\Phi(u, \ldots, u_m)$ of the dynamical system under consideration. In particular, the first coefficient λ of the series (3.3) is one of the eigenvalues of Φ_{u_m}. We recall that the series (3.1) is called a formal symmetry of the dynamical system (1.17) of order N if

$$(3.4) \qquad \deg(L_t - [F, \ L]) \leqslant m + 1 - N.$$

Clearly, to construct a non-singular formal symmetry of the dynamical system (1.17) we have to construct M approximate solutions (compare the proof of Theorem 2.1) to equations of the form (3.2) corresponding to M distinct eigenvalues of Φ_{u_m}.

Substituting the series (3.1) in (3.4) and equating the coefficients of D^m, D^{m-1}, $\ldots$, D to zero, we arrive at the chain of equations for evaluating the coefficients of the series (3.1). The first equation of this chain yields

$$(3.5) \qquad m\lambda D(l_1) = l_1 D(\lambda) \Longleftrightarrow l_1 = \text{const } \lambda^{1/m}.$$

For $m > 2$ the first $m - 1$ coefficients of the series (3.1) can be found in a similar way, and without loss of generality we can assume that these coefficients coincide with the first $m - 1$ coefficients of the series

$$(3.6) \qquad F^{1/m} = \lambda^{1/m} D + \tilde{f}_0 + \tilde{f}_{-1} D^{-1} + \ldots,$$

that is,

$$(3.7) \qquad l_1 = \lambda^{1/m}, \ l_0 = \tilde{f}_0, \ \ldots, l_{3-m} = \tilde{f}_{3-m}.$$

For it follows from (3.5) that by multiplying L by a constant we can assume that $l_1 = \lambda^{1/m}$. It remains to prove that it is possible to choose constants $c_0, c_{-1}, \ldots$ such that the coefficients of the series

$$L + c_0 + c_{-1} L^{-1} + \ldots + c_{3-m} L^{3-m}$$

satisfy (3.7). In the proof (see [19]) we use the formula

(3.8) $\deg \{(F^{1/m})_t - [F, F^{1/m}]\} \leqslant 1,$

which means that for any dynamical system (1.17) the series (3.6) is a formal symmetry of order m. This formal symmetry, equivalent to $L = F$ (see (2.11)), is related to the invariance of a dynamical system of the form (1.17) under the infinitesimal transformations $\bar{t} = t + \tau$, $\bar{x} = x$, $\bar{u} = u$ (time shift).

It follows from (3.8) that under the substitution of the series (3.1), (3.7) in (3.4) the coefficients of D^k, $k > 1$, cancel, and the chain of equations for evaluating the coefficients of a formal symmetry of order $N > m$ reduces to a system of equations of the form

(3.9)
$$\begin{cases} m\lambda D(l_{2-m}) + (m-2) D(\lambda) l_{2-m} = l_{1, t} + r_{2-m}, \\ m\lambda D(l_{1-m}) + (m-1) D(\lambda) l_{1-m} = l_{0, t} + r_{1-m}, \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ m\lambda D(l_{3-N}) + (3-N) D(\lambda) l_{3-N} = l_{m+2-N, t} + r_{3-N}, \end{cases}$$

where r_j, $j \leqslant 2 - m$, is a differential polynomial in the coefficients of the series (3.3) and the coefficients l_k of the series (3.1) with $k > j$. In particular, r_{2-m} is a differential polynomial in the coefficients (3.3), (3.6).

In the system (3.9) of $N - m$ equations the coefficients $l_{2-m}, \ldots, l_{3-N}$ are unknown. We pass to new variables $\sigma_{-1}, \sigma_0, \ldots, \sigma_{N-m-2}$. The substitution is given by the relations

(3.10) $\sigma_0 = \sigma (FL^{-1}; L), \quad \sigma_k = \sigma(F, L^k), \quad k \neq 0,$

where $\sigma(A, B)$ is a differential polynomial in the coefficients of the formal series A and B defined in (2.19). From (3.10), (3.7) we obtain

(3.11) $\sigma_0 = m\lambda^{(m-1)/m} l_{1-m} + \Delta_0, \quad \sigma_k = km\lambda^{(k+m-1)/m} l_{1-m-k} + \Delta_k, \quad k \neq 0,$

where the Δ_k are differential polynomials in the coefficients of the series (3.3) and the coefficients l_n, $n > 1 - m - k$. These formulae show that the substitution (3.10) has linear principal part, it is triangular, and so it is invertible. The residues ρ_k $(-1 \leqslant k \leqslant N - m - 2)$,

(3.12) $\rho_k = \operatorname{res} L^k, \quad k \neq 0, \quad \rho_0 = \operatorname{res} \log L,$

are expressed in terms of the coefficients of the series (3.6) and the variables $\sigma_{-1}, \ldots, \sigma_{N-m-2}$, which allows us to write down a system of equations for the σ_k.

Lemma 3.1. *The system of equations (3.9) is equivalent to the system of equations for the σ_k:*

(3.13) $D(\sigma_k) = \dfrac{d}{dt} \rho_k (\sigma_{k-m+1}, \sigma_{k-m}, \ldots), \quad -1 \leqslant k \leqslant N - m - 2,$

where the ρ_k are evaluated by (3.10), (3.12).

Proof. (3.9) implies (3.4), and so

$$\frac{d}{dt} \operatorname{res} L^k = \operatorname{res} [F, L^k], \quad -1 \leqslant k \leqslant N - m - 2,$$

(3.14) $$\frac{d}{dt} \operatorname{res} \log L = \operatorname{res} [FL^{-1}, L].$$

The assertion of the lemma is a consequence of (3.10), (3.12), (3.14), and the fact that the substitution (3.10) is invertible. ∎

The residues $\rho_k = \operatorname{res} L^k$, $k \neq 0$, $\rho_0 = \operatorname{res} \log L$ expressed in terms of the variables (3.11) are called in what follows the *densities of the canonical conservation laws* (3.13). For $k < m - 2$ these densities are differential polynomials in the coefficients of the series (3.6) and the variable $\lambda^{-1/m}$. For $k \geqslant m - 2$ the σ_j, $j < k$, are added to these variables. For example, $\rho_{-1} = \lambda^{-1/m}$, and as a consequence of Lemma 3.1 with $N = m + 1$ we find that the first equation of the system (3.9) after the substitution (3.10) takes the form

(3.15) $$D(\sigma_{-1}) = \frac{d}{dt}(\lambda^{-1/m}).$$

To distinct eigenvalues of Φ_{u_m} there correspond (see Proposition 2.1) M distinct equations of the form (3.2):

(3.16) $$\frac{d}{dt} L_j = [F_j, L_j] \quad (j = 1, \ldots, M)$$

and M series of canonical conservation laws

(3.17) $$\frac{d}{dt} \rho_{k,j} = D(\sigma_{k,j}) \quad (k = -1, 0, 1, 2, \ldots),$$

where in accordance with (3.10) $\sigma_{0,j} = \sigma(F_j L_j^{-1}, L_j)$, $\sigma_{k,j} = \sigma(F_j, L_j^k)$, $k \neq 0$, $\rho_{k,j} = \operatorname{res} L_j^k$. The following theorem is a direct consequence of Proposition 2.1, Theorem 2.1, and Lemma 3.1.

Theorem 3.1. *Suppose that the hypotheses of Proposition 2.1 are satisfied. Then the solubility of the system of equations (3.17) with $k \leqslant N - m - 2$ is a criterion for the existence of a formal symmetry of order $N > m$.*

Example 3.1. For systems of equations of the form (1.17) with $m = 2$ the question of the existence of a formal symmetry reduces to the question of the solubility of the scalar equation (3.2) with $F = \lambda D^2 + F_1 D + F_0 + + F_{-1} D^{-1} + \ldots$. We write down the canonical densities $\rho_{-1}, \rho_0, \rho_1$ connected with this equation. Clearly $\rho_{-1} = l_1^{-1} = \lambda^{-1/2}$ (see (3.12), (3.7)). It follows, in particular, that in the case of a system of two equations satisfying the condition trace $\Phi_{u_2} = 0$ (see (1.42)) we have

(3.18) $$\rho_{-1} = [-\det \Phi_{u_2}]^{-1/4}.$$

To find the next canonical density, in the formula

(3.19) $$\rho_0 = \operatorname{res} \log L = l_0 / l_1 \quad .$$

we need to express l_0 in terms of $\sigma_{-1} = \sigma(F, L^{-1})$. Formula (2.19) yields

$$\sigma_{-1} = -2l_0 + \lambda^{-1/2}F_1 - D(\lambda^{1/2}),$$

therefore (see (3.19))

(3.20) $-2\rho_0 = \lambda^{-1/2}\sigma_{-1} - \lambda^{-1}F_1 + D(\log \lambda^{1/2}).$

In exactly the same way we find $\rho_1 = \operatorname{res} L = l_{-1}$:

(3.21) $2\rho_1 = \lambda^{-1/2}(\sigma_0 - F_1\rho_0 + F_0) + \lambda^{1/2}(\rho_0^2 - D(\rho_0)).$

The resulting formulae are general and will be used for the classification of systems of two equations of the form (0.8). As already mentioned in the Introduction, the classification of the scalar equations $u_t = \Phi(u, u_1, u_2)$ is based on the existence of a formal symmetry of order 5. By Theorem 3.1 these conditions coincide with the conditions of solvability of the equations

$$\frac{d}{dt}\rho_k = D(\sigma_k) \quad (k = -1, 0, 1),$$

where $\rho_1 = (\Phi_{u_2})^{-1/2}$, and ρ_0, ρ_1 can be found from (3.20), (3.21) with $F_1 = \Phi_{u_1}$, $F_0 = \Phi_u$.

Example 3.2. The list (0.3)–(0.6) of equations of the form $u_t = u_3 + G(u, u_1, u_2)$ is obtained with the help of the canonical densities

(3.22)
$$\rho_0 = G_{u_2}, \quad \rho_1 = 3G_{u_1} - (G_{u_2})^2, \quad \rho_3 = \sigma_1,$$
$$\rho_2 = 27G_u - 9G_{u_1}G_{u_2} + 2(G_{u_2})^3 + 9\sigma_0.$$

In this example, in accordance with (3.7), $l_1 = 1$, $l_0 = G_{u_2}/3$, $\rho_{-1} = 1$.

Therefore the first canonical conservation law is trivial. The canonical conservation laws (3.17) can be simplified by replacing them by equivalent ones, using the rule

(3.23)
$$\bar{\rho}_{k, j} = \alpha_{k, j}\rho_{k, j} + D(r_{k, j}),$$
$$\bar{\sigma}_{k, j} = \alpha_{k, j}\sigma_{k, j} + (r_{k, j})_t + \beta_{k, j},$$

where the $\alpha_{k,j}$, $\beta_{k,j}$ are constants, while the $r_{k,j}$ are differential polynomials in the same variables as $\rho_{k,j}$. The conservation laws with densities (3.23) are equivalent to canonical ones.

In conclusion we note that by Theorem 3.1 for the existence of a formal symmetry of the system (1.17) of order $m + 1$ it is necessary and sufficient that the system under consideration has conservation laws of the form (3.15), where λ is an eigenvalue of Φ_{u_m}.

2. The conditions for the existence of formal conservation laws allow us to refine the form of the canonical conservation laws (3.17). At the end of §1 we showed that in the case of a system of odd order m the dimension M of the system is even and the eigenvalues of Φ_{u_m} split into pairs λ_j, $-\lambda_j$ $(j = 1, 2, ..., M/2)$. The gauge transformation (2.1) (see Proposition 2.1)

reduces the system (1.32) to the block-diagonal form (2.7), (2.8). Therefore for the study of formal conservation laws we can restrict ourselves to the examination of a system of two equations of the form (see Theorem 2.1)

$$R_t + \hat{\Phi}^T R + R\hat{\Phi} = 0,$$

$$R = \begin{pmatrix} 0 & r \\ -r^T & 0 \end{pmatrix}, \quad \hat{\Phi} = \begin{pmatrix} F & 0 \\ 0 & -G \end{pmatrix},$$

$$(3.24) \quad F = \lambda D^m + F_{m-1} D^{m-1} + \ldots, \quad G = \lambda D^m + G_{m-1} D^{m-1} + \ldots$$

Obviously, (3.24) is equivalent to a single scalar equation

$$(3.25) \qquad\qquad r_t + F^T r - rG = 0.$$

A series r (deg $r = 1$) satisfying the inequality $\deg(r_t + F^T r - rG) \leqslant m + 1 - N$ will also be called a formal conservation law of order N (compare (1.40)).

Lemma 3.2. *A formal series r satisfies* (3.25) *if and only if*

$$(3.26) \quad \text{res} \log G^{-1/m} (F^T + r_t r^{-1})^{1/m} = \text{res} \log (G^{-1/m} r G^{1/m} r^{-1}),$$

$$(3.27) \quad \text{res} \, [G^{n/m} - (F^T + r_t r^{-1})^{n/m}] = \text{res} \, [G^{n/m} r^{-1}, r], \quad n \geqslant 1.$$

Proof. We rewrite (3.25) as an equality of two series of degree 1

$$(F^T + r_t r^{-1})^{1/m} = r G^{1/m} r^{-1}.$$

The assertion of the lemma follows from Lemma 1.1, formula (2.15), and the additivity of residues. ∎

As we see from (2.18), the right-hand sides of (3.26) and (3.27) are total derivatives. This allows us to pass to a new set of variables

$$(3.28) \qquad\qquad \sigma_0^0 = s(G^{-1/m}, r),$$

$$(3.29) \qquad\qquad \sigma_k^0 = \sigma(G^{k/m} r^{-1}, r), \quad k \geqslant 1$$

(see (2.19), (2.21), in which the chain of equations for the coefficients of the formal series r is written in the form

$$(3.30) \quad \rho_k^0 (\sigma_{k-m+1}^0, \sigma_{k-m}^0, \ldots) = D(\sigma_k^0) \quad (k = 0, 1, \ldots),$$

where

$$(3.31) \qquad\qquad \rho_0^0 = \text{res} \log G^{-1/m} (F^T + r_t r^{-1})^{1/m},$$

$$(3.32) \qquad\qquad \rho_k^0 = \text{res} \, [G^{k/m} - (F^T + r_t r^{-1})^{k/m}], \quad k \geqslant 1.$$

It follows from (3.28) and (3.29) that σ_k^0 depends on the coefficients of G and the variables $r_1, r_0, \ldots, r_{1-k}$, the dependence on r_{1-k} being linear. Therefore the substitution $(r_1, r_0, \ldots, r_{-N}) \mapsto (\sigma_0^0, \sigma_1^0, \ldots, \sigma_{N+1}^0)$ is invertible. The residues (3.31), (3.32) for $0 \leqslant k \leqslant m - 2$ depend only on the coefficients of F and G:

$$(3.33) \qquad\qquad \rho_0^0 = -\frac{1}{m} \frac{F_{m-1} + G_{m-1}}{\lambda} + D \log \lambda,$$

$$(3.34) \qquad \rho_k^0 = \text{res} \, [G^{k/m} + (-1)^{k+1} F^{k/m}], \quad 1 \leqslant k \leqslant m - 2,$$

while for $k \geqslant m - 1$ they also depend on the variables $\sigma_0^0, \sigma_1^0, \ldots, \sigma_{k-m+1}^0$.
In the derivation of (3.33) we have used the identities (2.2) and

$$(3.35) \qquad \operatorname{res} \log A^{1/m} = \frac{1}{m} \operatorname{res} \log A - \frac{m-1}{2m} D \log a_m,$$

where $A = a_m D^m + \ldots$. For formal conservation laws the following theorem
is an analogue of Theorem 3.1.

Theorem 3.2. *A formal series r satisfies the inequality* $\deg(r_t + F^T r - rG) \leqslant$
$\leqslant m + 1 - N$ *if and only if the finite chain of equations* (3.30) *with*
$0 \leqslant k \leqslant N - 2$ *is soluble.*

Suppose that the formal series $\hat{L} = \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix}$ satisfies the equation
$\hat{L}_t = [\hat{\Phi}, \hat{L}]$, where $\hat{\Phi}$ is the same series as in (3.24). This equation splits
into two scalar ones:

$$(3.36) \qquad L_t = [F, L], \quad \bar{L}_t = -[G, \bar{L}].$$

Their solubility is equivalent to the existence of two series of canonical
conservation laws (3.17). If the equation (3.24) is soluble, then the
equations (3.36) are equivalent (since $\bar{L} = -r^{-1}L^T r$ satisfies the second of
the equations (3.36) (see Theorem 2.1)). Therefore one of the series of
canonical conservation laws must be replaced by the divergency conditions.

Example 3.3. On the example of the dynamical system

$$(3.37) \qquad u_t = u_2 + f(u, v, u_1, v_1), \quad -v_t = v_2 + g(u, v, u_1, v_1)$$

we obtain the first canonical conservation laws and divergency conditions.
The operator Φ_* for (3.37) has the form

$$(3.38) \quad \Phi_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D^2 + \begin{pmatrix} f_{u_1} & f_{v_1} \\ -g_{u_1} & -g_{v_1} \end{pmatrix} D + \begin{pmatrix} f_u & f_v \\ -g_u & -g_v \end{pmatrix}.$$

With the help of the gauge transformation (Proposition 2.1) we reduce the
first coefficients of this series to the diagonal form. Equating the coefficients
of D^k ($k = 0$ or 1) in the expression

$$\begin{pmatrix} F & 0 \\ 0 & -G \end{pmatrix} T = T\Phi_* + T_t,$$

where $T = E + T_{-1}D^{-1} + \ldots$, $\operatorname{diag} T_k = 0$, $F = D^2 + F_1 D + \ldots$,
$G = D^2 + G_1 D + \ldots$, we obtain

$$F_1 = f_{u_1}, \quad G_1 = g_{v_1}, \quad T_{-1} = \frac{1}{2} \begin{pmatrix} 0 & f_{v_1} \\ g_{u_1} & 0 \end{pmatrix},$$

$$F_0 = f_u - \frac{1}{2} f_{v_1} g_{u_1}, \quad G_0 = g_v - \frac{1}{2} f_{v_1} g_{u_1}.$$

The canonical density ρ_{-1} is equal to 1, therefore $\sigma_{-1} = \operatorname{const}$. When
evaluating the canonical densities ρ_0 and ρ_1 we can use the ready formulae
of Example 3.1, substituting in them the F_k obtained by diagonalization.

From (3.20) and (3.21) we obtain

(3.39) $\rho_0 = \frac{1}{2} f_{u_1}$,

(3.40) $\rho_1 = \frac{1}{2}\sigma_0 - \frac{1}{8} f_{u_1}^2 + \frac{1}{2} f_u - \frac{1}{4} g_{u_1} f_{v_1} - \frac{1}{4} D(f_{u_1})$.

Directly from (3.33), (3.34) we obtain

(3.41) $\rho_0^0 = -\frac{1}{2}(f_{u_1} + g_{v_1})$,

(3.42) $\rho_1^0 = \frac{1}{2}\left(\sigma_{0,1}^0 + \frac{1}{4} f_{u_1}^2 - \frac{1}{4} g_{v_1}^2 + g_v - f_u\right) + \frac{1}{4} D(f_{u_1} - g_{v_1})$.

If the system (3.37) possesses a formal symmetry and a conservation law of higher order, then $\rho_0^0 \in \mathrm{Im}\, D$, that is, there is a function $\varphi(u, v)$ such that $D(\varphi) = \rho_0^0$. It can be proved (see [9]) that the density of the conservation law

$$\rho(u, v, u_1, v_1, \ldots, u_n, v_n)$$

for $n \geqslant 2$ has the form

$$\rho = e^{\varphi}[\alpha u_n v_n + \beta(v_{n-1} u_n - u_{n-1} v_n)] + \alpha(Au_n + Bv_n) + C,$$

where α and β are numbers satisfying $\alpha\beta = 0$; A, B, C are functions of the variables $u, v, \ldots, u_{n-1}, v_{n-1}$, while for $n = 1$ it has the form

$$\rho = \alpha e^{\varphi} u_1 v_1 + a(u, v) u_1 + b(u, v) v_1 + c(u, v).$$

If the order m of the system (1.17) is odd, the question reduces to the investigation of the scalar equation (1.32).

Lemma 3.3. *In the scalar case the series* $R = r_1 D + r_0 + \ldots, R^T = -R$ (*see Theorem 2.1*) *satisfies* (1.32) *if and only if*

(3.43) $2r_{-2k} + \sum\limits_{i=1}^{2k+1} (-1)^i \binom{-2k+i}{i} D^i(r_{-2k+i}) = 0 \quad (k = 0, 1, \ldots)$,

(3.44) $2F_{m-1}/F_m = D(r_1^m F_m^{m-1})$,

(3.45) $\mathrm{res}\{F^{n/m} - (F + R^{-1}R_t)^{n/m}\} = \mathrm{res}[F^{n/m}R^{-1}, R] \quad (n = 2, 4, \ldots)$.

As before, we define a new set of variables $\sigma_n^0 = \sigma_n^0(r_1, r_0, \ldots, r_{2-n})$ by the relations

(3.46) $\sigma_0^0 = r_1^m F_m^{m-1}$,

(3.47) $\sigma_n^0 = \sigma(F^{n/m}R^{-1}, R) \quad (n = 2, 4, \ldots)$

and express the variables r_k in terms of the σ_k from (3.43), (3.46), (3.47). Using the formulae

$$\rho_0^0 = 2F_{m-1}/F_m,$$
$$\rho_n^0 = \mathrm{res}\{F^{n/m} - (F + R^{-1}R_t)^{n/m}\} \quad (n = 2, 4, \ldots),$$

we find $\rho_n^0 = \rho_n^0(\sigma_0, \sigma_2, \ldots, \sigma_{n-m+1})$. It follows from Lemma 3.3 that the solubility of (1.32) is equivalent to the solubility of the chain of relations

$$\rho_n^0 = D(\sigma_n^0).$$

as in the case of even order m of the system (1.17), passing to a self-adjoint solution of (3.2) (see Theorem 2.1); we remark that 'half' of the canonical conservation laws are replaced by the divergency conditions [19].

The formula (2.1) for rewriting formal symmetries under gauge transformations is closely connected with the general theory of transformations and to the question concerning a formula for rewriting a formal symmetry under the passage from the set of dynamical variables $u, u_1, \ldots$ to another set $\bar{u} = \psi(u, \ldots, u_n), \bar{u}_k = \psi_k(u, \ldots, u_{n_k})(k = 1, 2, \ldots)$ of dynamical variables. For example, the following simple assertion holds.

Lemma 3.4 [19]. *Suppose that the two dynamical systems*

$$u_t = \Phi(u, u_1, \ldots), \quad \bar{u}_{\bar{t}} = \bar{\Phi}(\bar{u}, \bar{u}_1, \ldots)$$

are related to each other by the substitution

(3.48) $$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = \psi(u, \ldots, u_n)$$

and that a formal symmetry $\bar{L} = \sum \bar{l}_k(\bar{u}, \ldots)\bar{D}^k$ and a formal conservation law $\bar{R} = \sum \bar{r}_k(\bar{u}, \ldots)\bar{D}^k$ of the dynamical system $\bar{u}_{\bar{t}} = \bar{\Phi}$ are given. Then the formulae

(3.49) $$L = \psi_*^{-1}\bar{L}\psi_*, \quad R = (\psi_-^T)^{-1}\bar{R}\psi_*^{-1}, \quad \bar{D} = D$$

define a formal symmetry $L = \sum l_k(u, \ldots)D^k$ and a formal conservation law $R = \sum r_k(u, \ldots)D^k$ of the system $u_t = \Phi$.

In (3.49) ψ_* is a differential polynomial of the form (1.20).

CHAPTER II

EVOLUTION EQUATIONS OF THE SECOND ORDER

In this chapter we present results of Svinolupov on the classification of scalar equations of the form

$$u_t = \Phi(x, u, u_x, u_{xx}).$$

The results of Chapter I are easily carried over to equations with right-hand side explicitly dependent on x. In particular, we associate with each equation of second order a collection of densities $\rho_{-1}, \rho_0, \rho_1$ of canonical conservation laws. These densities are calculated by the formulae given in Example 3.1, and if one includes x in the set of dynamical variables, the only thing that changes is the operator of total differentiation

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots$$

In §4 the classical Lie theory of contact transformations is presented, which serves as a model for the extension of the module of invertible transformations in Chapter III. We note that adding x to the collection of dynamical variables is necessary in a sense for the classification of second-order equations, since the rejection of contact transformations affecting the variable x considerably complicates the analysis of the conditions for the existence of symmetries and makes the list of integrable cases difficult to observe.

In §5 we present a complete list of second-order equations possessing formal symmetries of order 5 (Theorem 5.1). The list consists of the linear equation

$$u_t = u_{xx} + q(x)u$$

and three Burgers type equations. Symmetries of the linear equation are given by the formula

$$u_\tau = [D^2 + q(x)]^k (u) \qquad (k = 2, \ 3, \ \ldots).$$

If the Schrödinger operator $L = D^2 + q(x)$ commutes with an operator M of odd order, then in addition to the symmetries specified of even order, generated by powers of L, there are symmetries of odd order generated by powers of the differential operator M. Burgers type equations reduce to a linear equation by differential substitutions that generalize the substitution connected with the introduction of a potential. Explicit formulae for the substitutions, specified at the end of §5, allow us to rewrite the symmetries of a linear equation. Therefore the conditions for the existence of a formal symmetry (the local property of the canonical conservation laws (3.17)) are not only necessary (see Theorem 1.1) but also sufficient for the existence of an infinite set of local symmetries.

§4. Invertible transformations

From the general point of view invertible transformations in the set of dynamical variables x, u, u_1, ... are given by the formulae

$$(4.1) \quad \bar{x} = \varphi(x, u, \ldots, u_m), \ \bar{u} = \psi(x, u, \ldots, u_n), \ \bar{u}_1 = \psi_1(x, u, \ldots, u_{n_1}), \ \ldots$$

Invertibility means the existence of functions $\bar{\varphi}$, $\bar{\psi}$, $\bar{\psi}_1$, ... such that

$$x = \bar{\varphi}(\bar{x}, \ \bar{u}, \ \ldots, \ \bar{u}_{\bar{m}}), \quad u = \bar{\psi}(\bar{x}, \ \bar{u}, \ \ldots, \ \bar{u}_{\bar{n}}), \quad u_1 = \bar{\psi}_1(\bar{x}, \ \bar{u}, \ \ldots, \ \bar{u}_{\bar{n}_1}), \ \ldots$$

In the classical Lie theory, whose simplified version we present in this section, an additional condition for invariance of the infinite-dimensional system (1.3) under the substitution (4.1) is introduced. This condition yields

$$\frac{d\bar{u}_k}{d\bar{x}} = \frac{d\psi_k}{dx} \frac{dx}{d\bar{x}} = \frac{D(\psi_k)}{D(\varphi)} = \bar{u}_{k+1}$$

or, more briefly,

(4.2) $\psi_{k+1} = D(\psi_k)/D(\varphi)$ $(k = 0, 1, 2, \ldots)$.

Theorem 4.1. *The invertible transformations (4.1) and (4.2) are exhausted by the point transformations* $\bar{x} = \varphi(x, u)$, $\bar{u} = \psi(x, u)$ *invertible in the collection of variables* (x, u), *and the contact transformations*

(4.3) $\bar{x} = \varphi(x, u, u_1)$, $\bar{u} = \psi(x, u, u_1)$, $\varphi_{u_1}\psi_{u_1} \neq 0$,

(4.4) $\dfrac{\varphi_x + \varphi_u u_1}{\varphi_{u_1}} = \dfrac{\psi_x + \psi_u u_1}{\psi_{u_1}}$, $\det \begin{pmatrix} \varphi_u & \varphi_{u_1} \\ \psi_u & \psi_{u_1} \end{pmatrix} \neq 0$,

invertible in the collection of variables (x, u, u_1).

We note that for the contact transformations from (4.2)–(4.4) it follows that $\bar{u}_1 = \psi_{u_1}/\varphi_{u_1}$ and that the functions φ and ψ are not arbitrary but are functionally independent solutions of the same equation of the form

$$\Phi_x + \Phi_u u_1 = A(x, u, u_1)\Phi_{u_1}.$$

The Legendre transformation $\bar{x} = u_1$, $\bar{u} = xu_1 - u$ is a classical example of a contact transformation.

Proof of Theorem 4.1. For any $m \geqslant 0$ and $0 \leqslant i \leqslant m$ we have $n_i \leqslant m$. For if $n_j > m$ for some $j \leqslant m$, then by (4.2) $n_{i+1} = 1 + n_i$ for any $i \geqslant j$ and the Jacobi matrix $\partial(\varphi, \psi, \psi_1, \ldots)/\partial(x, u, u_1, \ldots)$ has right inverse. Therefore for $m > 0$ we find from (4.2) that

(4.5) $\psi_{k+1} = \dfrac{D(\psi_k)}{D(\varphi)} = \dfrac{R(\psi_k) + u_{m+1}\psi_{k,u_m}}{R(\varphi) + u_{m+1}\varphi_{u_m}} = \dfrac{\psi_{k,u_m}}{\varphi_{u_m}}$ $(k = 0, \ldots, m-1)$,

where $R = \partial/\partial x + u_1\partial/\partial u + \ldots + u_m\partial/\partial u_{m-1}$. It follows from (4.5) that $\psi_{k,u_m} = \partial\psi_k/\partial u_m \neq 0$ (otherwise $\psi_{k+1} = 0$ and the substitution (4.1) is not invertible) and that

$$\frac{R(\varphi)}{\varphi_{u_m}} = \frac{R(\psi)}{\psi_{u_m}} = \cdots = \frac{R(\psi_{m-1})}{\psi_{m-1,u_m}} \overset{\text{def}}{=\!=} A.$$

Let us show that the assumption $m \geqslant 2$ leads to a contradiction. It is easy to check that $R(\varphi) = A\varphi_{u_m}$, $R(\psi) = A\psi_{u_m}$ implies the relation

$$\left(R - A\frac{\partial\psi}{\partial u_m}\right)\left(\frac{\psi_{u_m}}{\varphi_{u_m}}\right) = \frac{\varphi_{u_m}\psi_{u_{m-1}} - \psi_{u_m}\varphi_{u_{m-1}}}{\varphi_{u_m}^2} \ .$$

Taking into account that $\psi_1 = \psi_{u_m}/\varphi_{u_m}$ we obtain the equality $\psi_{u_m}\varphi_{u_{m-1}} - \varphi_{u_m}\psi_{u_{m-1}} = 0$, which means that ψ is expressed in terms of φ by the formula $\psi = K(x, u, \ldots, u_{m-2}, \varphi)$. It now follows from (4.5) that the last two columns of the matrix $\partial(\varphi, \psi, \ldots, \psi_m)/\partial(x, u, \ldots, u_m)$ are proportional. This contradicts the invertibility of the substitution.

The cases $m = 0, 1$ can be considered in a similar way and lead to contact and point transformations. ∎

A given function u of the variables t and x goes over under point and contact transformations of the form (4.1) to a function $\bar{u}(\bar{t}, \bar{x})$ ($\bar{t} = t$); also, $u_k = \partial^k u/\partial x^k$, $\partial_k \bar{u}/\partial \bar{x}_k$ ($k = 1, 2, ...$) (see (4.3)). The partial derivative of $\bar{u}(\bar{t}, \bar{x})$ with respect to the variable $\bar{t}$ (taken under the condition $d\bar{x} = 0$) is given by

$$(4.6) \qquad \bar{u}_{\bar{t}} = Ju_t, \quad J = \psi_u - \frac{D(\psi)}{D(\varphi)} \varphi_u.$$

In the case of point transformations $J = [D(\varphi)]^{-1}\partial(\varphi, \psi)/\partial(x, u)$, while in the case of contact transformations it follows from the formula $D(\psi)\varphi_{u_1} = \psi_{u_1}D(\varphi)$ that $J = (\varphi_{u_1})^{-1}\partial(\varphi, \psi)/\partial(u_1, u)$. Therefore the invertible transformations (4.1) and (4.2) allow us to consider the equations $u_t = \Phi(x, u, u_1, ...)$ and $\bar{u}_{\bar{t}} = \bar{\Phi}(\bar{x}, \bar{u}, \bar{u}_1, ...)$ related to each other by

$$(4.7) \qquad J\Phi(x, u, u_1, ...) = \bar{\Phi}(\varphi, \psi, \psi_1, ...),$$

as equivalent equations. It is easy to check that the conditions for the existence of formal symmetries are invariant under invertible transformations.

§5. The classification theorem

As mentioned in the beginning of the chapter, the classification of second-order equations is based on the criterion for the existence of a formal symmetry stated in Theorem 3.1. In the case of the infinite-dimensional dynamical system corresponding to the equation

$$(5.1) \qquad u_t = \Phi(x, u, u_1, u_2),$$

the criterion for the existence of a formal symmetry of order 5 consists in the solubility in the class of functions of x, u, u_1, of the system of three equations for σ_{-1}, σ_0, σ_1:

$$(5.2) \qquad \frac{d\rho_k}{dt} = D(\sigma_k) \quad (k = -1, 0, 1),$$

where the densities of the canonical conservation laws (5.2) are evaluated in Example 3.1.

Theorem 5.1 [11]. *The equation* (5.1) *possessing a formal symmetry of order 5 is reduced by the invertible transformations* (4.1), (4.2) *to one of the equations of the following list:*

$$(5.3) \qquad u_t = u_2 + q(x)u,$$
$$(5.4) \qquad u_t = u_2 + 2uu_1 + p(x),$$
$$(5.5) \qquad u_t = D(u^{-2}u_1 + \alpha xu + \beta u),$$
$$(5.6) \qquad u_t = D(u^{-2}u_1 - 2x).$$

Proof. We shall make use of the following assertion.

***Lemma* 5.1.** *The equation* (5.1) *possesses a formal symmetry of order* 3 *if and only if it can be written up to invertible transformations in the form*

(5.7) $u_t = u_2 + h(x,\ u,\ u_1)$,

(5.8) $u_t = D(u^{-2}u_1 + g(x,\ u))$.

The proof of the lemma will be given after completing the classification of the equations (5.7), (5.8). Let us consider equations of the form (5.7). In this case (see Example 3.1) $\rho_{-1} = 1$ and the following canonical densities are reduced by the transformations (3.23) to the form

(5.9) $\rho_0 = h_{u_1},\ \ \rho_1 = \sigma_0 + h_u + \frac{1}{2}\rho_0^2$.

We note first that equations of the form (5.7) have no local conservation laws of the first order. For if $\rho = \rho(x,\ u,\ u_1)$ and $\rho_{u_1 u_1} \neq 0$ is the density, then $\rho_t \sim \Phi\delta\rho/\delta u$ (ρ_t and $\Phi\delta\rho/\delta u$ are equivalent modulo Im D). Therefore the right-hand side Φ of the equation under consideration has the form

$$\Phi = \frac{Au_2 + B}{Cu_2 + D},$$

where A, B, C, D are functions of the variables x, u, u_1, and $C = \rho_{u_1 u_1} \neq 0$, which contradicts (5.7).

Since $d\rho_0/dt \in \text{Im } D$ and $\rho_0 = h_{u_1}$, we have

$$h_{u_1} = D(f(x,\ u)) + a(x,\ u).$$

This means that the equation (5.7) has the form

(5.10) $u_t = u_2 + Au_1^2 + Bu_1 + C$,

where A, B, C are functions of x and u. It is easy to check that by a substitution of the form $\bar{x} = x$, $\bar{u} = \psi(x,\ u)$ the equation (5.10) is reduced to one of the following:

(5.11) $u_t = u_2 + b(x,\ u)$,

(5.12) $u_t = u_2 + a(x,\ u)u_1 + b(x,\ u),\ \ a_u \neq 0$.

For the equation (5.11) the condition $\rho_{0,t} \in \text{Im } D$ holds automatically and $\rho_1 = b_u$ (see (5.9)). We have $b_{u,t} \sim -b_{uuu}u_1^2 + \varphi(x,\ u)$ and consequently we have $b_{uuu} = 0$, $b = p(x)u^2 + q(x)u + r(x)$. It can be established in a similar way that $b_{uu} = 2p = 0$. The substitution $\bar{x} = x$, $\bar{u} = u + y(x)$, $y'' + qy = r$ leads to the equation (5.3) from the list given in Theorem 5.1.

For the equation (5.2) $\rho_0 = -a$ (the next condition is not needed). We have $a_{uu} = 0$, that is, $a = \beta_1(x)u + \beta_2(x)$, $\beta_1 \neq 0$. The substitution $\bar{x} = x$, $\bar{u} = \alpha_1(x)u + \alpha_2(x)$, $2\alpha_1 = \beta_1$, $2\alpha_2 = \beta_2 - 2D\log\alpha_1$ reduces the equation to the form $u_t = u_2 + 2uu_1 + b(x,\ u)$. Since now $\rho_0 = -2u$, the right-hand side of the equation under consideration is a total derivative. This means that $b_u = 0$ and that the equation coincides with (5.4).

It remains to consider equations of the form (5.8), for the complete description of which the condition $\rho_{0,t} \in \text{Im } D$, $\rho_0 \sim ug - u^2g_u$ is sufficient.

If $\rho = \rho(x, u)$ is the density, then $\rho_t \sim -\rho_{uu}u^{-2}u_1^2 + \varphi(x, u)$. Therefore $\rho \sim b(x)u$, and

(5.13) $$u^{-1}b''(x) + b'(x) g(x, u) \in \mathrm{Im}\, D.$$

The canonical density ρ_0 is equivalent modulo $\mathrm{Im}\, D$ to the function $ug - u^2 g_u$. Hence $ug - u^2 g_u = b(x)u + c(x)$ and $g(x, u) = a(x)u + b(x) + c(x)/2u$. The substitution $\bar{x} = y(x)$, $\bar{u} = u/y'(x)$, $2y'' + cy' = 0$ leads to the equation (5.8) with $g = A(x)u + B(x)$. The condition (5.13) in conformity with the density $ug - u^2 g_u = B(x)u$ means that $B'' = AB' = 0$. In the case $B' \neq 0$ $(B'' = A = 0)$ we have $g = c_1 x + c_2$, c_1, $c_2 \in \mathbf{C}$. The resulting equation reduces to (5.6) by a dilatation of t, u. In the case $B = 0$ we have $g(x, u) = A(x)u$ and $\rho_1 \sim -A'(x)u$. The condition (5.13) yields $A''(x) = 0$. The resulting equation coincides with (5.5). ∎

Proof of the lemma stated at the beginning of the section. It follows from (1.32) (see Chapter I) that the equation (5.1) has no conservation laws of order greater than or equal to 2. This imposes the following restriction on the form of the density $\rho_{-1} = \Phi_{u_2}^{-1/2}$:

(5.14) $$\rho_{-1} = D[f(x, u, u_1)] + a(x, u, u_1).$$

It follows from (4.7) that the densities ρ_{-1} and $\bar{\rho}_{-1}$ of the canonical conservation laws, related to each other by the invertible transformation (4.1), (4.2), satisfy the relation

(5.15) $$\bar{\rho}_{-1}(\varphi, \psi, \psi_1, \psi_2) = \frac{1}{D(\varphi)} \rho_{-1}(x, u, u_1, u_2).$$

The substitution that reduces (5.1) to the form (5.7) or (5.8) is constructed from the functions f and a on the right-hand side of (5.14). In the case $a = 0$, which corresponds to the triviality of the conservation law under consideration, the desired substitution has the form (4.1), (4.2) with $\varphi = f$ and leads to an equation of the form (5.7). In the case $a_{u_1} = 0$, $a_u \neq 0$ corresponding to the conservation law of zero order we have $\Phi_{u_2 u_2} = 0$, that is, $f_{u_1} = 0$ (see (5.14)). We consider the point substitution $\bar{x} = \varphi(x, u)$, $\bar{u} = \psi(x, u)$ such that $\varphi_x = (f_x + a)\psi^{-1}$, $\varphi_u = f_u \psi^{-1}$. Since $D(\varphi) = \psi^{-1}\rho_{-1}$, it follows from (5.15) and the condition $(\rho_{-1})_t \in \mathrm{Im}\, D$ that the resulting equation has the form $u_t = u_2 u^{-2} + G(x, u, u_1)$. Now taking $\rho_{-1} = u$ into account, we obtain (5.8).

It remains to consider the case when the canonical density ρ_{-1} defines a conservation law of the second order, that is, when $a_{u_1 u_1} \neq 0$. Let us show that there is a contact transformation (4.3), (4.4) that reduces the density $a(x, u, u_1)$ to the density $\bar{u}$. Clearly, the equality for the variational derivatives has to be valid:

(5.16) $$\frac{\delta}{\delta u}[\psi D(\varphi)] = \frac{\delta a}{\delta u},$$

where $\delta/\delta u = \partial/\partial u - D\partial/\partial u_1 + D^2\partial/\partial u_2 + \ldots$, as well as the equality (see (4.44))

(5.17)
$$\frac{D(\psi)}{D(\varphi)} = \frac{\psi_{u_1}}{\varphi_{u_1}}.$$

It is easy to check that it follows from (5.17) that

(5.18)
$$\frac{\delta}{\delta u}[\psi D(\varphi)] = \psi_u D(\varphi) - \varphi_u D(\psi).$$

This formula implies in turn that (5.16) is equivalent to the pair of equations

$$a_{u_1 u_1} = \varphi_u \psi_{u_1} - \psi_u \varphi_{u_1},$$
$$a_{uu_1 u_1} + a_{xu_1} - a_u = \varphi_u \psi_x - \psi_u \varphi_x.$$

Adding (5.17) to these equations, we reduce the system (5.16), (5.17) to the following system equivalent to it:

$$\varphi_u \psi_{u_1} - \psi_u \varphi_{u_1} = a_{u_1 u_1},$$

(5.19)
$$\frac{\varphi_x + \varphi_u u_1}{\varphi_{u_1}} = \frac{\psi_x + \psi_u u_1}{\psi_{u_1}} = \frac{a_{x,u_1} + a_{uu_1} - a_u}{a_{u_1 u_1}}.$$

The solubility of this overdetermined system for the functions φ, ψ can be proved by the standard method of characteristics. The resulting contact transformation reduces the density $a(x, u, u_1)$ to the density $\bar{u}$, that is, it leads to an equation for which $\bar{\rho}_{-1} = \bar{u}$ (to check this one can use the formulae $\rho_{-1} = D(F) + \psi D(\varphi)$ and (5.15)). Therefore the contact transformation reduces the case $a_{u_1 u_1} \neq 0$ (a is the function on the right-hand side of (5.14)) to the case $a_{u_1} = 0$, $a_u \neq 0$ considered above. ∎

The equations (5.1) have symmetries of higher order only in exceptional cases. The classification theorem shows that not all equations possessing formal symmetries reduce to linear ones by invertible transformations. It will be shown that the question of their integration can be solved by a slight enlargement of the module of invertible transformations.

Actually it is the question of adding one (strictly speaking, non-invertible) transformation, which is often called the operation of introducing a potential. This transformation is possible if the initial equation has the form of a conservation law

(5.20)
$$u_t = D(F(x, u, u_1)).$$

In this case one can define a new collection of dynamical variables

(5.21)
$$\bar{x} = x, \quad \bar{u}_1 = u, \quad \bar{u}_2 = u_1, \ldots$$

and pass to the equation

(5.22)
$$\bar{u}_t = F(\bar{x}, \bar{u}_1, \bar{u}_2).$$

The connection between (5.20) and (5.22) is obvious.

As an example we consider the Burgers equation (5.4). It has the form (5.20) and as a result of the potentiation (5.21) is brought to the form (5.22) $(\overline{u}_t = \overline{u}_2 + \overline{u}_1^2 + q(\overline{x}))$. The point transformation $y = \overline{x}$, $v = \exp \overline{u}$ reduces the last equation to the linear one (5.3). The composition of these two transformations coincides with the well-known Cole-Hopf substitution [38], [39]:

$$(5.23) \qquad\qquad x = y, \quad u = v_1/v.$$

Another example of a substitution of this type is related to the equation (5.6). The potentiation and then the point substitution $y = \overline{u}$, $v = \overline{x}$ reduce it to the Burgers equation with $q(x) = 0$. The composition of the two substitutions has the form

$$(5.24) \qquad\qquad x = v, \quad u = 1/v_1.$$

We note that the same substitution reduces the equation (5.8) to the linear equation

$$(5.25) \qquad\qquad v_t = v_2 - \alpha v - \beta$$

with constant coefficients α, β. Therefore this insignificant enlargement of the module of invertible transformations allows us to relate all the equations listed in Theorem 5.1 to linear ones.

In the general case the potentiation operation is possible if the equation possesses a conservation law of order zero or two. Let $\rho = \rho(x, y)$ be a density of order zero; then a new collection of dynamical variables can be defined from the relations $\overline{x} = x$, $\overline{u}_1 = \rho(x, u)$. If $\rho = a(x, u, u_1)$ is a density of order two, then the potentiation operation is given by $\overline{x} = \varphi(x, u, u_1), \overline{u}_1 = \psi(x, u, u_1)$, where φ and ψ are functionally independent solutions of (5.9). Under potentiation of an equation the corresponding conservation law 'goes over' to a symmetry $(\overline{u}_\tau = 1)$. It is easy to show that the equation (5.1), which has a symmetry of zero or first order, is reduced to the form (5.22) by a point or contact transformation. Here the substitution $x = \overline{x}$, $u = \overline{u}_1$, $u_1 = \overline{u}_2$, which reduces (5.22) to (5.20), is admissible and will be called differentiation of the equation (5.22). It is the inverse to potentiation.

In contrast with invertible transformations, the potentiation and differentiation operations do not allow us in general to rewrite symmetries of equations. For example, the equation $u_t = D(u^{-2}u_1 - 2x - u)$ obtained from $\overline{u}_t = \overline{u}_1^{-2}\overline{u}_2 - \overline{u}_1 - 2\overline{x}$ by differentiation has no symmetries of the form $u_\tau = g(x, u, u_1, ..., u_n)$ for $n \geqslant 3$, while the initial equation has symmetries of arbitrarily large order and is related to the Burgers equation $v_t = v_2 + 2vv_1 + 1$ by the point transformation $\overline{x} = v$, $\overline{u} = y$. In this example the cause of the loss of symmetries is the fact that the variable t is not included in the collection of dynamical variables. It can be checked that the

equation under consideration possesses an infinite set of symmetries of the form $u_\tau = D^2(u^{-1}D + x - t)^k(x - t)$, since it reduces to (5.6) by the Galileo transformation.

Differentiation (potentiation) does not destroy symmetries if they do not depend on u (are total derivatives, respectively). This simple observation and the above substitutions allow us to construct symmetries of the equations (5.3)–(5.6) of arbitrarily large order. For example, the symmetries of (5.5)

$$u_\tau = D\left[u \sum_{k=1}^{N} c_k (u^{-1}D)^k (u^{-1}) + c_0 ux + cu\right], \quad c_k, \ c \in \mathbf{C},$$

are obtained from the symmetries

$$v_\tau = c + \sum_{k=0}^{N} c_k v_k$$

of the linear equation (5.25) by the substitution (5.24).

CHAPTER III
ENLARGEMENT OF THE MODULE OF INVERTIBLE SUBSTITUTIONS

In the case of systems of partial differential equations there is no complete analogue of contact transformations. But here the module of invertible substitutions is not exhausted by the point substitutions. The transformation $\bar{x} = x$, $\bar{u} = u$, $\bar{v} = v + u_1$ and the transformation $\bar{x} = \varphi(x, u, u_1)$, $\bar{u} = \psi(x, u, u_1)$, $\bar{v} = \chi(x, u, v)$, where φ, ψ satisfy (4.3), (4.4), and $\chi_v \neq 0$, are examples of non-point invertible substitutions. In applications a situation often occurs when the system possesses a continuous point group of symmetries and it is sufficient to restrict oneself to a reduced collection of dynamical variables consisting of invariants of the group. For such systems the module of invertible transformations can essentially be enlarged by renouncing the requirement of the local property of a transformation of variables not belonging to the reduced collection of dynamical variables.

In §6 we consider systems of equations (1.17) with right-hand side independent of x. A complete description of invertible transformations that are local in the variables u, u_1, u_2, ... is given in the scalar case by Theorem 6.1. Some of these transformations generalize to the case of systems of equations and allow us, in particular, to reduce systems of the form (0.8) possessing a formal symmetry of order 3 to a simpler form (compare Lemma 5.1):

$$u_t = u_{xx} + f(u, v, u_x, v_x), \quad -v_t = v_{xx} + g(u, v, u_x, v_x).$$

In §§7, 8 systems of two equations of the above form are considered that are invariant under the infinitesimal transformations $\bar{u} = u + \tau p(u)$, $\bar{v} = v + \tau q(v)$. Substitutions in the corresponding reduced collection of dynamical variables are essentially used in Chapter IV for the classification of non-linear systems of Schrödinger type.

We note that the substitutions considered in this chapter are constructed by starting from classical symmetries and conservation laws. Substitutions of this sort are not bound by the framework of the theory of integrable systems and can be used for a broad class of partial differential equations.

§6. Generalized contact transformations

For equations invariant under space translations $\bar{x} = x + \tau$ we can restrict ourselves to a collection of dynamical variables $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \ldots$ not containing x. In this reduced collection of variables we define the substitution

$$(6.1) \qquad \bar{u}^i = \psi^i(\mathbf{u}, \ldots, \mathbf{u}_n), \quad \bar{u}^i_k = \psi^i_k(\mathbf{u}, \ldots, \mathbf{u}_{n_k}), \ldots$$
$$(i = 1, \ldots, M; \quad k = 1, 2, 3, \ldots).$$

We replace the requirement of the local property of transformations of the variable x (compare (4.1)) by a weaker condition of the existence of a function $\alpha = \alpha(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_m)$ such that

$$(6.2) \qquad d\bar{x} = \alpha(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_m)\, dx.$$

The condition of invariance of the dynamical system (1.3) under the substitution (6.1), (6.2) leads to the relations

$$(6.3) \qquad \psi^i_{k+1} = \frac{D(\psi^i_k)}{\alpha} \qquad (k = 0, 1, \ldots; \quad i = 1, \ldots, M),$$

which are equivalent to the formula

$$(6.4) \qquad \bar{D} = \alpha^{-1} D,$$

where $\bar{D} = \sum\limits_{k=0}^{\infty} \sum\limits_{i=1}^{M} \bar{u}^i_{k+1} \partial/\partial \bar{u}^i_k$.

The requirement of invertibility of the substitution (6.1)–(6.3) imposes certain restrictions on the functions $\alpha, \psi^1, \ldots, \psi^M$. In the case $M = 1$ the following theorem is an analogue of Theorem 4.1.

Theorem 6.1. *In the scalar case $(M = 1)$ the substitution (6.1)–(6.3) is invertible in the variables $u, u_1, u_2, \ldots$ if and only if α does not belong to Im D and one of the following cases holds:*

$$(6.5) \qquad 1) \quad \alpha = \alpha(u, u_1), \quad \psi = \psi(u);$$

$$(6.6) \qquad 2) \quad \alpha = D(a(u, u_1)) + b(u, u_1), \quad \psi = \psi(u, u_1),$$
$$a_{u_1} \neq 0, \quad a_{u_1}\psi_u = (a_u + u_1^{-1}b)\psi_{u_1};$$

$$(6.7) \qquad 3) \quad \alpha = D(a(u, u_1, u_2)) + b(u, u_1, u_2), \quad \psi = \psi(u, u_1, u_2),$$
$$a_{u_2} \neq 0, \quad b_{u_2 u_2} \neq 0, \quad \psi_1 = \psi_1(u, u_1, u_2), \quad \psi_{u_2} = a_{u_2}\psi_1,$$
$$\psi_{u_1} = (a_{u_1} + b_{u_2})\psi_1, \quad \psi_u = (a_u + u^{-1}b - u_1^{-1}b_{u_2 u_2})\psi_1.$$

Abbreviated proof. We consider the substitution

$$(6.8) \quad \hat{x} = \psi(u, u_1, \ldots), \quad \hat{u} = \psi_1(u, u_1, \ldots), \quad \hat{u}_k = \hat{D}^k(\psi_1), \quad k \geqslant 1,$$

where $\hat{D} = [D(\psi)]^{-1}D$, which is the composition of (6.1)–(6.3) with the transformation $\hat{x} = \bar{u}$, $\hat{u} = \bar{u}_1$, ... It can be checked that the proof of the theorem reduces to the proof of the following assertion.

Lemma 6.1. *The substitution* (6.8) *is an invertible transformation from the collection of variables* u, u_1, u_2, ... *to the collection of variables* $\hat{x}$, $\hat{u}$, $\hat{u}_1$, ... *if and only if one of the two following cases occurs*:

$$\hat{x} = \psi(u, u_1), \quad \hat{u} = \psi_1(u, u_1), \quad \partial(\psi, \psi_1)/\partial(u, u_1) \neq 0,$$

or

$$\hat{x} = \psi(u, u_1, u_2), \quad \hat{u} = \psi_1(u, u_1, u_2),$$

$$\psi_{u_2} \cdot \psi_{1,u_2} \neq 0, \quad \psi_{u_2}^{-1}(\psi_u u_1 + \psi_{u_1} u_2) = \psi_{1u_2}^{-1}(\psi_{1u} u_1 + \psi_{1u_1} u_2), \quad \partial(\psi, \psi_1)/\partial(u_1, u_2) \neq 0.$$

The proofs of Lemma 6.1 and Theorem 4.1 are almost the same. ∎

We note in addition to Theorem 6.1 that transformations of the form

$$(6.9) \quad d\bar{x} = \alpha(x, u, \ldots)dx, \quad \bar{u} = \psi(x, u, \ldots), \quad \bar{u}_1 = \psi_1(x, u, \ldots), \quad \ldots$$

being inverse to the transformations (6.8), are described by formulae similar to (6.5)–(6.7). For example, in the case similar to (6.5) we have

$$(6.10) \quad d\bar{x} = \alpha(x, u)dx, \quad \alpha_u \neq 0, \quad \bar{u} = \psi(u),$$

The transformation given by (6.5)

$$d\bar{x} = \alpha(u, u_1)dx, \quad \bar{u} = \psi(u), \quad \bar{u}_1 = \psi_1(u, u_1), \quad \ldots,$$

generalizes to the case of systems of equations. For the substitution

$$(6.11) \quad d\bar{x} = \alpha(\mathbf{u}, \mathbf{u}_1)dx, \quad \bar{u}^i = u^i, \quad \bar{u}_1^i = \alpha^{-1}u_1^i, \quad \ldots$$

with $\alpha \neq \sum u_1^s \, \partial\alpha/\partial u_1^s$ is invertible, since the Jacobi matrix

$$\frac{\partial(\bar{u}_1^1, \ldots, \bar{u}_1^M)}{\partial(u_1^1, \ldots, u_1^M)} = \frac{1}{\alpha}\delta_{ij} - \frac{u_1^i}{\alpha^2}\frac{\partial\alpha}{\partial u_1^j}$$

has an inverse with entries

$$a_{ij} = \alpha\left(\delta_{ij} + \frac{u_1^i \, \partial\alpha/\partial u_1^j}{\alpha - \sum_s u_1^s \partial\alpha/\partial u_1^s}\right).$$

The infinite-dimensional system (1.17) admits transformations (6.1)–(6.3) invertible in the collection $\mathbf{u}$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... if one can add to the set of dynamical variables $\mathbf{u}_k = \mathbf{u}_k(t, x)$ ($k = 0, 1, \ldots$) a new variable $\bar{x} = X(t, x)$ such that

$$(6.12) \quad \frac{\partial X}{\partial x} = \alpha(\mathbf{u}, \mathbf{u}_1, \ldots), \quad \frac{\partial X}{\partial t} = \beta(\mathbf{u}, \mathbf{u}_1, \ldots).$$

The condition of compatibility of the relations (6.12)

$$(6.13) \qquad \frac{d\alpha}{dt} = D(\beta)$$

means that the function α in (6.2) is the density of a conservation law of the initial dynamical system (1.17). The following substitution corresponds to the invertible semilocal transformation (6.1)–(6.3):

$$(6.14) \qquad \bar{t} = t, \ \bar{x} = X(t, x), \ \bar{u}^i = \psi^i(u, u_1, \ldots) \qquad (i = 1, \ldots, M).$$

The right-hand side of the transformed system

$$(6.15) \qquad d\bar{u}^i/d\bar{t} = \bar{\Phi}^i(\bar{u}, \bar{u}_1, \ldots) \qquad (i = 1, \ldots, M)$$

is evaluated by

$$(6.16) \qquad \bar{\Phi}^i = \sum_{k \geqslant 0} \sum_{j=1}^{M} \psi^i_{u^j_k} D^k(\Phi^j) - \frac{\beta}{\alpha} D(\psi^i).$$

In contrast with the semilocal transformation (6.1)–(6.3), invertible transformations of the form (6.8), whose complete description is given by Lemma 6.2, can be applied to arbitrary equations and reduce the dynamical system $u_t = \Phi(x, u, u_1, \ldots)$ to a system of the form

$$(6.17) \qquad \hat{u}_t = \hat{\Phi}(\hat{u}, \hat{u}_1, \ldots) = \left[\psi_{1, u_1} - \frac{D(\psi_1)}{D(\psi)} \psi_{u_1} \right] \left[D(\Phi) - \frac{u_2}{u_1} \Phi \right].$$

It is easy to check that the transformation (6.17) has a local conservation law with density $\alpha = D(\psi)/\psi_1$.

The transformations under consideration affecting the variable x essentially change the character of non-linearity of the initial equation. For example, it follows from (6.17) that the resulting equation is always quasilinear. To illustrate an application with the above theory of transformations we consider scalar equations of the second order $u_t = \Phi(u, u_1, u_2)$ satisfying the first integrability condition $(\Phi_{u_2}^{-1/2})_t \in \mathrm{Im}\, D$ (compare Lemma 5.1). One can check, starting from (6.16), that for $\alpha = \Phi_{u_2}^{-1/2} \notin \mathrm{Im}\, D$ the transformations indicated in Theorem 6.1 lead to an equation of the form $\bar{u}_t = \bar{u}_2 + f(\bar{u}, \bar{u}_1)$. For example, the equation $u_t = u^2 u_2$ is reduced to the linear one $\bar{u}_t = \bar{u}_2$ by the substitution $\bar{t} = t, \ d\bar{x} = u^{-1}dx - u_1 dt, \ \bar{u} = u$ (compare [40]).

We illustrate the general scheme of using the invertible transformations (6.1)–(6.3) for the classification of integrable equations by one more example. For the quasilinear system of two second-order equations

$$(6.18) \qquad \mathbf{u}_t = A(\mathbf{u}, \mathbf{u}_1)\mathbf{u}_2 + \mathbf{F}(\mathbf{u}, \mathbf{u}_1), \quad \det A \neq 0,$$

the first necessary conditions for the existence of conservation laws and symmetries (see Example 3.1) are written in the form

$$(6.19) \qquad \mathrm{trace}\, A = 0, \ \frac{d}{dt}(\det A)^{-1/4} \in \mathrm{Im}\, D.$$

In the case $A = A(\mathbf{u})$ the substitution (6.11) in composition with a point transformation allows us to reduce the system (6.18), (6.19) to the diagonal form [9]

(6.20) $u_t = u_2 + f(u, v, u_1, v_1), \quad -v_t = v_2 + g(u_1, v, u_1, v_1).$

In the general case $A = A(\mathbf{u}, \mathbf{u_1})$ the equations (6.18) are reduced to a system with constant eigenvalues equal to $+1$ and -1. The function α in (6.11) is chosen in both cases to be equal to $(\det A)^{-1/4}$. For example, the system of equations considered in [41]

$$u_t = i\sqrt{3}\, D^2 \frac{\partial}{\partial v}(uv)^{1/3}, \quad -v_t = i\sqrt{3}\, D^2 \frac{\partial}{\partial u}(uv)^{1/3}$$

is reduced by the transformation

$$-3\bar{u} = \exp(-2\pi i/3)\log u + \exp(2\pi i/3)\log v,$$

$$-3\bar{v} = \exp(2\pi i/3)\log u + \exp(-2\pi i/3)\log v$$

to the well-known (see [42], [43]) system of the form

$$\bar{u}_t = \bar{u}_2 + \bar{v}_1^2, \quad -\bar{v}_t = \bar{v}_2 + \bar{u}_1^2.$$

For the classification of the systems of equations (0.8) possessing an infinite series of conservation laws we shall use the easily verified assertion stated below, which shows that the local property of conservation laws is preserved under the invertible transformations (6.1)–(6.3).

Lemma 6.2. *Suppose that the two dynamical systems of the form* (1.17)

$$\bar{u}_{\bar{t}} = \bar{\Phi}(\bar{u}, \bar{u}_1, \ldots), \quad u_t = \Phi(u, u_1, \ldots)$$

are related to each other by the substitution

(6.21) $\bar{t} = t, \quad d\bar{x} = \alpha(u, u_1, \ldots)\, dx + \beta(u, u_1, \ldots)\, dt, \quad \bar{u} = \psi(u, u_1, \ldots)$

and that the conservation law

(6.22) $$\frac{d}{dt}\bar{\rho} = \bar{D}(\bar{\sigma})$$

is given in the dynamical variables $\bar{u}, \bar{u}_1, \bar{u}_2, \ldots$ *Then the formulae*

(6.23) $$\rho = \alpha\bar{\rho}, \quad \sigma = \bar{\sigma} + \beta\bar{\rho}$$

define the conservation law

(6.24) $$\frac{d}{dt}\rho = D(\sigma)$$

in the variables $u, u_1, u_2, \ldots$.

We emphasize that the general formula (6.23) also allows us to construct conservation laws (6.24) in the case of irreversible transformations of the form (6.21). The conservation law $d\alpha/dt = D(\beta)$, which generates the substitution $\bar{x} = X(t, x)$ corresponding to (6.21) (see (6.12)), corresponds by (6.23) to the trivial conservation law (6.22) with $\bar{\rho} = 1$.

The special cases of (6.21)

$$\alpha = c + D(a(u)), \quad c \neq 0; \quad \psi = \psi(u), \quad \bar{x} = cx + a(u),$$

$$\alpha = c + D(a(u, u_1)), \quad c \neq 0; \quad \psi = \psi(u, u_1), \quad \bar{x} = cx + a(u, u_1),$$

which satisfy the assumptions of Theorem 6.1, correspond to point and contact transformations admissible by the class of equations whose right-hand side does not depend on x.

To conclude this section we consider one more interesting example of the application of transformations of the form (6.1)–(6.3) connected with hyperbolic equations of the form

$$(6.25) \qquad au_{xx} + bu_{xy} + cu_{yy} + d = 0.$$

Here a, b, c, d are functions of u, u_x, u_y. In this case the following substitution is a natural generalization of the transformation (6.11):

$$(6.26) \qquad d\bar{x} = \alpha dx + \beta dy, \quad d\bar{y} = \gamma\, dx + \delta\, dy, \quad \bar{u} = u,$$

where α, β, γ, δ are functions of u, u_x, u_y satisfying

$$(6.27) \qquad \alpha_y = \beta_x, \quad \gamma_y = \delta_x, \quad \alpha\delta - \beta\gamma \neq 0.$$

If the local conservation laws in (6.27) are non-trivial, then the substitution (6.26) is invertible and the equation obtained by the substitution has the same form (6.25).

We apply the transformation (6.26)–(6.27) to the Klein–Gordon equation

$$(6.28) \qquad u_{xy} = dh(u)/du,$$

taking for α, β, γ, δ the components of the energy-impulse tensor:

$$\alpha = \frac{1}{2} u_x^2, \quad \beta = h(u), \quad \gamma = h(u), \quad \delta = \frac{1}{2} u_y^2.$$

It can be checked (see [44]) that the equation obtained by the substitution (6.25) is Lagrangian with Lagrangian density

$$\mathscr{L} = h^{-1}(1 + \sqrt{1 - 2h\overline{u_{\bar{x}}u_{\bar{y}}}}).$$

A formula for rewriting local conservation laws similar to (6.23) allows us to prove that the equations obtained by the above substitution from the equations of the list (0.1) possess an infinite series of local conservation laws.

§7. Partial differentiations and potentiations

The module of invertible substitutions that preserve the form of the system (6.20) is very meagre. It consists of compositions of elementary substitutions $\bar{x} = ax + bt + c$, $\bar{t} = \alpha^2 t + d$, *conformal* transformations $\bar{u} = U(u)$, $\bar{v} = V(v)$, and the involution

$$(7.1) \qquad \bar{t} = -t, \quad \bar{x} = -x, \quad \bar{u} = v, \quad \bar{v} = u.$$

In this section we shall enlarge this module by adding substitutions analogous to differentiation and potentiation of scalar equations.

A system of the form (6.20) invariant under a continuous group of point transformations can be reduced with the help of conformal transformations and the involution (7.1) to the form

$$(7.2) \qquad u_t = u_2 + f\,(\varepsilon u + v,\ u_1,\ v_1), \quad -v_t = v_2 + g(\varepsilon u + v,\ u_1,\ v_1)$$

with $\varepsilon = 0$ or 1. The substitution is chosen so that the symmetry $u_\tau = \varphi(u),\ v_\tau = \psi(v)$ of the initial equation in the new variables has the form $u_\tau = 1,\ v_\tau = -\varepsilon$. For example, the well-known Schrödinger equation (0.14) is reduced by an obvious substitution to the form (7.2) with $\varepsilon = 1$:

$$(7.3) \qquad u_t = u_2 + u_1^2 + \exp{(u+v)}, \quad -v_t = v_2 + v_1^2 + \exp{(u+v)}.$$

In what follows a maximal collection of dynamical variables invariant under a one-parameter point group will be called a *reduced collection of dynamical variables*. In the previous chapter we have already dealt with reductions of collections of dynamical variables: for example, the equation (5.22) is invariant under the point group $\bar{u} \to \bar{u} + \lambda$ ($\lambda = \text{const}$) and the corresponding collection $\bar{x},\ \bar{u}_1,\ \bar{u}_2,\ \ldots$ does not contain the variable $\bar{u}$. Other examples are connected with equations invariant under space translations; in this case the reduced collection of dynamical variables does not contain the variable x. In the case of systems of equations of the form (7.2) the reduced collection of dynamical variables has the form $\varepsilon u + v, u_1, v_1, \ldots$; it is generated by the action of the operator D on generators for which one can take $\varepsilon u + v,\ u_1$.

As an analogue of differentiation in the case of systems (7.2) we can take the substitution defined on the generators of collections of dynamical variables by

$$(7.4) \qquad \bar{u} = u_1, \quad \bar{v} = \varepsilon u + v.$$

However, this substitution in general does not leave invariant the class of equations (6.20). The condition

$$(7.5) \qquad f_{v_1 v_1} = 0$$

guarantees the independence of the coefficient of $\bar{v}_2$ on $\bar{u}_1,\ \bar{v}_1$ in the new system. It is easy to check that there is a simple point transformation reducing it to the form (6.20). The composition of these transformations can be represented as

$$(7.6) \qquad \begin{aligned} &\bar{u} = \varphi\,(\varepsilon u + v,\ u_1), \quad 2\varphi_v = f_{v_1}\varphi_{u_1}, \\ &\bar{v} = \psi\,(\varepsilon u + v), \quad\ \ \varphi_{v_1}\psi_v \neq 0. \end{aligned}$$

The substitution (7.6) is not defined uniquely, up to two functions of one variable. This non-uniqueness corresponds to an arbitrary conformal transformation in the new system. A composition of a point transformation and the substitution (7.4) will be called a *partial differentiation*. A partial differentiation is an invertible substitution on the reduced collection of dynamical variables of the initial system.

If the system (6.20) is invariant under an Abelian two-parameter group of conformal transformations, then it can be reduced to the form

$$(7.7) \qquad u_t = u_2 + f(u_1, v_1), \quad -v_t = v_2 + g(u_1, v_1).$$

The system (7.7) admits the substitution

$$(7.8) \qquad \bar{u} = u_1, \quad \bar{v} = v_1,$$

which leads to the system

$$(7.9) \qquad \bar{u}_t = D(\bar{u}_1 + f(\bar{u}, \bar{v})), \quad -\bar{v}_t = D(\bar{v}_1 + g(\bar{u}, \bar{v})).$$

The transformation (7.8), in contrast with (7.2), will be called *total differentiation*. Total differentiation is also an analogue of differentiation of scalar equations; it is invertible on the reduced collection of dynamical variables of the initial system $(u_1, v_1, u_2, v_2, \ldots)$.

The system of equations obtained as the result of the partial differentiation (7.4) possesses, as in the scalar case, a local zero-order conservation law with density $\rho = \bar{u}$. The total differentiation (7.8) leads to the system (7.9), which has the form of local conservation laws with densities $\bar{u}$ and $\bar{v}$.

We proceed now to the discussion of substitutions analogous to the potentiation operation (5.21) defined in the scalar case. Suppose that the system (6.20) possesses a local conservation law with density $\rho = \rho(u, v)$, $\rho_u \neq 0$. We define a new collection of dynamical variables by the rule

$$(7.10) \qquad \bar{u}_1 = \rho(u, v) \quad \bar{v} = v.$$

The substitution (7.10) in (6.10) defines a time evolution of the reduced collection of dynamical variables. The fact that ρ is the density of the local conservation law $\rho_t = D(\sigma)$ allows us to extend the time evolution to the total collection by adding the equation $\bar{u}_t = \sigma$. The transformation (7.10) may fail to leave invariant the class of equations (6.20) under consideration. If the right-hand side of the initial equation satisfies

$$(7.11) \qquad g_{u_1 u_1} = g_{u_1 v_1} = (g_{u_1}/\rho_u)_u = 0,$$

then the system obtained by the substitution (7.10) can be reduced by a point transformation to the form (6.20). The composition of these transformations can be represented as follows:

$$\bar{u}_1 = \rho(u, v), \quad \varepsilon \bar{u} + \bar{v} = \chi(v),$$
$$(7.12) \qquad \varepsilon = 0, \quad \chi(v) = v \quad \text{for} \quad g_{u_1} = 0,$$
$$\varepsilon = 1, \quad \chi'(v) = -g_{u_1}/\rho_u \quad \text{for} \quad g_{u_1} \neq 0.$$

The substitutions (7.12) transform equations of the form (6.20) possessing local conservation laws to equations of the form (7.2), while the substitutions (7.6) act in the opposite direction. The composition of the substitution (7.10) with a point transformation will be called a *partial potentiation*.

The total potentiation is possible under the existence of two conservation laws of order zero with functionally independent densities[1]. As a result a system is obtained that is invariant under an Abelian two-parameter group of point symmetries.

We illustrate the substitutions under consideration by the example of the following systems of equations:

(7.13) $u_t = u_2 + u^2 v,$ $-v_t = v_2 + v^2 u,$

(7.14) $u_t = u_2 + (u + v) u_1,$ $-v_t = v_2 - (u + v) v_1,$

(7.15) $u_t = u_2 + D(u^2 + v),$ $-v_t = v_2 - 2D(uv),$

(7.16) $u_t = u_2 + u_1^2 + v_1,$ $-v_t = v_2 - 2u_1 v_1.$

With the help of the operations of differentiation and potentiation these systems of equations can be constructed starting from any of them. We take for a base the non-linear Schrödinger equation (7.13). In it one can perform the partial differentiation (compare (7.3)):

(7.17) $\bar{u} = u_1/u,$ $\bar{v} = uv,$

which leads to the well-known equation (7.15).

We shall represent substitutions on diagrams as follows: vertices correspond to systems of equations, arrows correspond to transformations. Transformations in (against) the direction of a solid arrow denote a partial differentiation (potentiation). Dotted arrows oriented in a similar way correspond to total differentiations and potentiations. For example, the substitution (7.17) is shown in the diagram of Figure 1 by the arrow joining the lower vertices.
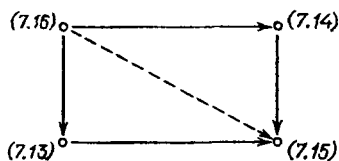

Fig. 1

The Schrödinger equation has a conservation law with density uv. Therefore one can perform the partial differentiation

(7.18) $\bar{u} = \log u,$ $\bar{v}_1 = uv,$

giving the system (7.16). It can also be obtained from the system (7.17) by the operation of total differentiation $\bar{u}_1 = u$, $\bar{v}_1 = v$, which is the composition of the substitutions (7.17), (7.18) (compare Figure 1). The last of the four systems, the system (7.14), is obtained by the partial differentiation

(7.19) $\bar{u} = 2u_1 + v,$ $\bar{v} = -v$

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[1]In the case of systems (6.20) any two linearly independent densities are functionally independent and there are at most three linearly independent densities [9].

from (7.16) or by the partial potentiation

$$(7.20) \qquad\qquad \bar{u} + \bar{v} = 2u, \quad \bar{v}_1 = -v$$

from (7.15). All our substitutions are shown in the diagram of Figure 1. We note that this diagram is commutative. The composition of the substitutions (7.17), (7.20) is an almost invertible transformation

$$(7.21) \qquad\qquad \bar{u}_1 = uv, \quad \bar{u} + \bar{v} = 2u_1/u$$

and reduces the non-linear Schrödinger equation (7.13) to (7.14). We note finally that the system (7.14) is related to the well-known Kaup system [45]

$$\pi_t = \varphi_4 + \varphi_2 - 2D(\varphi_1 \pi), \quad \varphi_t = \pi - \varphi_1^2$$

by the almost invertible substitution

$$u + v = -2\varphi_1, \quad v_1 - u_1 = 2\pi - 1.$$

In contrast to substitutions invertible on total collections of dynamical variables discussed in the previous chapter, the differentiation operation destroys the structure of conservation laws in general. For example, the integrable system

$$(7.22) \qquad u_t = u_2 + 2vv_1, \quad -v_t = v_2 - u_1,$$

simply related to the Boussinesq equation by the partial differentiation

$$\bar{u} = u_1 - v^2/2, \quad \bar{v} = v$$

is reduced to the system

$$\bar{u}_t = \bar{u}_2 + \bar{v}_1^2 + \bar{v}(\bar{u} - \bar{v}^2/2), \quad -\bar{v}_t = \bar{v}_2 - \bar{u} + \bar{v}^2/2,$$

without local conservation laws: it does not satisfy the integrability conditions (see Chapters I, IV). Local conservation laws do not disappear under potentiations. This follows from Lemma 6.2.

§8. Transformations of symmetric systems

Among systems of the form (7.2) a special place is occupied by the systems invariant under the involution (7.1)

$$(8.1) \quad u_t = u_2 + f(u + v, u_1, v_1), \quad -v_t = v_2 + g(u + v, u_1, v_1), \quad g = f^*$$

(where f^* denotes the result of the action of the involution on f). This system will be called *symmetric* in what follows. We shall show that for the system (8.1) not only potentiation but also partial differentiation does not destroy the local structure of conservation laws. Moreover, in this section we define substitutions invertible on reduced collections of dynamical variables and acting in the class of symmetric systems. Such substitutions will not destroy the local conservation laws either.

In spite of the fact that the form of (8.1) is not invariant under conformal transformations, conformally equivalent symmetric systems exist. They occur in those cases when the system possesses a continuous non-Abelian group of conformal symmetries. For example, the well-known Heisenberg model

$$(8.2) \qquad \mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \quad S_1^2 + S_2^2 + S_3^2 = 1$$

can be written in the symmetric form in two ways:

$$(8.3) \qquad u_t = u_2 - 2\,\frac{u_1^2}{u+v}\,, \quad -v_t = v_2 - 2\,\frac{v_1^2}{u+v}\,,$$

$$(8.4) \qquad u_t = u_2 - 2\text{th}\,(u+v)\,u_1^2, \quad -v_t = v_2 - 2\text{th}\,(u+v)\,v_1^2.$$

The equation (8.3) is obtained from (8.2) by the point transformation

$$(8.5) \quad S_1 + iS_2 = \frac{2}{u+v}\,, \quad S_1 - iS_2 = \frac{2uv}{u+v}\,, \quad S_3 = \frac{u-v}{u+v}\,, \quad t \to it,$$

and the equation (8.4) is obtained from (8.3) by the conformal substitution

$$\bar{u} = \frac{1}{2}\,\log u, \quad \bar{v} = -\frac{1}{2}\,\log v.$$

We shall rely on the lemma on invariant densities. A function h will be called *invariant* if $h_u = h_v$, $h^* = h$.

Lemma 8.1. *If the density ρ of the local conservation law*

$$(8.6) \qquad \frac{d\rho}{dt} = D(\sigma)$$

is an invariant function, then the function σ is also invariant.

Proof. Obviously, for any function g

$$(8.7) \qquad (Dg)^* = -D(g^*), \quad \left[\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)g\right]^* = -\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)g^*$$

and also for symmetric systems $(g_t)^* = -(g^*)_t$. Applying the involution to (8.6) we obtain $D(\sigma - \sigma^*) = 0$, whence it follows that σ is invariant under the involution, that is, $\sigma^* = \sigma$. Applying the operator $\partial/\partial u - \partial/\partial v$ to (8.6), we obtain $D(\sigma_u - \sigma_v) = 0$. It follows from the invariance of σ under the involution that $\sigma_u = \sigma_v$. ∎

We consider the substitution

$$(8.8) \qquad \bar{u} + \bar{v} = p\,(u+v), \quad \bar{u}_1 = \varphi\,(u+v,\ u_1), \quad p' \cdot \varphi_{u_1} \neq 0$$

for systems of the form (8.1). Obviously, it is invertible on reduced collections of dynamical variables. A dynamical system on the total collection of new variables can be defined if and only if $\varphi(u+v,\ u_1)$ is the density of a local conservation law of the initial system. If φ is a density, then $\varphi(u+v,\ u_1) = R(u+v)u_1 + Q(u+v)$ (see Example 3.3 of §3). It is easy to check that the substitution (8.8) preserves the form (6.20) if and only if $R(u+v) = p'(u+v)$.

Therefore a substitution (8.8) that preserves the form (6.20) necessarily has the form

(8.9) $\bar{u} + \bar{v} = p\,(u + v),\ \bar{u}_1 = p'\,(u + v)\,u_1 + q(u + v),\ \ p' \neq 0,$

where $p'u_1 + q$ is the density of a local conservation law of the initial system (8.1). Such substitutions will be called *symmetric*. The reason for this is the fact that symmetric substitutions preserve the class of symmetric systems. For since the function $p'u_1 + q$ is a density, it follows that $[p'(u_1 - v_1) + 2q]_t = D(\sigma)$, and by the lemma on invariant densities σ is an invariant function. The evolution of the total collection of new variables is defined from the relations

(8.10) $\bar{u}_t + \bar{v}_t = p_t,\ \bar{u}_t - \bar{v}_t = \sigma.$

With the help of these formulae it is easy to check that the new system is symmetric.

We note that the function σ is defined up to an additive constant. This means that each symmetric substitution (including the 'identical' substitution $\bar{u} + \bar{v} = u + v$, $\bar{u}_1 = u_1$) leads to a family of symmetric systems which reduce to each other by transformations of the form $\bar{u} = u + ct$, $\bar{v} = v - ct$, $c \in \mathbf{C}$.

Proposition **8.1**. *The symmetric substitution* (8.9) *defines an equivalence relation on the set of symmetric systems* (8.1).

The fact that the equivalence relation defined by (8.9) is reflexive and symmetric is obvious, and the fact that it is transitive is easily verified. We note that symmetrically equivalent systems are not conformally equivalent, and conversely in those cases when symmetric systems are conformally equivalent they are not symmetrically equivalent (see (8.3), (8.4)).

The equivalence relation defined above splits the set of symmetric systems into classes. Systems belonging to the same class will be called *symmetrically equivalent*. To construct an equivalence class from the system (8.1) one has to find all densities of the form $p'(u + v)u_1 + q(u + v)$ of local conservation laws of this system. If there are no non-trivial densities, then the equivalence class is defined by the substitution $\bar{u} + \bar{v} = u + v$, $\bar{u}_1 = u_1 + \lambda$, where λ is an arbitrary constant.

Example **8.1**. We consider two examples. The system of the form (6.20) with

(8.11) $f = 2\alpha uvu_1 + \beta u^2 v_1 + \frac{1}{2}\,\beta\,(\alpha - \beta)\,u^3 v^2 + \gamma u^2 v,\ g = f^*$

is reduced by the conformal substitution $\bar{u} = \log u$, $\bar{v} = \log v$ to the form (8.1) with

(8.12) $f = u_1^2 + (2\alpha u_1 + \beta v_1 + \gamma)\exp(u + v) - \frac{1}{2}\,\beta\,(\alpha - \beta)\exp(2u + 2v).$

Using the conservation law of the system (8.12) with density $au_1 + b \exp(u+v)$, where $a, b \in \mathbf{C}$, one can easily check that for $\beta = 2\alpha$, $\gamma = 0$ this system is symmetrically equivalent to a linear system, for $\beta = 2\alpha$, $\gamma \neq 0$ it is equivalent to the non-linear Schrödinger equation (7.3), and for $\beta \neq 2\alpha$ it is equivalent to the non-linear Schrödinger equation with a derivative (that is, to the system (8.1), (8.12) with $\alpha = \beta = 1$, $\gamma = 0$). Some special cases of the system (6.20), (8.11) have been investigated by many authors from the point of view of the method of the inverse scattering problem. Here gauge transformations were discovered which led to symmetric substitutions. Under our approach the substitutions are constructed directly from the system of equations and are not connected with the method of the inverse problem.

As the second example we indicate links between the familiar systems (8.3) and

$$(8.13) \qquad u_t = u_2 - u_1^2 + 2u_1 v_1, \quad -v_t = v_2 - v_1^2 + 2u_1 v_1.$$

The system (8.13) has been investigated with the help of the inverse problem by many authors. The symmetric substitution $\bar{u} + \bar{v} = \log(u+v)$, $\bar{u}_1 = (u+v)^{-1} u_1$ reduces the system (8.3) to (8.13).

These examples show that systems of equations not related to each other at first sight can be symmetrically equivalent. The following obvious assertion gives a criterion for equivalence.

***Proposition* 8.2.** *Two symmetric systems of equations (on u, v and $\bar{u}, \bar{v}$) are related to each other by the substitution* (8.9) *if and only if after the partial differentiation* (7.4) *with $\varepsilon = 1$ ($U = u_1$, $V = u+v$, $\bar{U} = \bar{u}_1$, $\bar{V} = \bar{u}+\bar{v}$) they are related to each other by the point transformation*

$$\bar{U} = p'(V)U + q(V), \quad \bar{V} = p(V).$$

Let us refine this proposition in reference to the important class of symmetric systems satisfying $f_{v_1 v_1} = 0$ (obviously, this condition is symmetrically invariant).

Corollary. *Two systems of equations of the form* (8.1) *with $f_{v_1 v_1} = 0$ are symmetrically equivalent if and only if they become conformally equivalent after the partial differentiation* (7.6).

In other words, a symmetric substitution for this class of equations can be regarded as the composition of a partial differentiation, a conformal substitution, and potentiation.

It follows from Lemma 8.1 that symmetric transformations and partial differentiations (7.6) with $\varepsilon = 1$ allow us to rewrite local conservation laws with an invariant density, the local property being preserved. The following theorem shows that conservation laws of higher order can essentially be assumed to be invariant.

Theorem 8.1. *Suppose that a symmetric system has at least two local conservation laws of higher order and that ρ is the density of a conservation law of order $N \geqslant 2$. Then $\rho + \rho^* = A + B$, where A and B are densities of local conservation laws of order N and $m \leqslant 1$ respectively, and A is invariant.*

Corollary. *If a symmetric system has local conservation laws of higher order, then the system obtained by the partial differentiation (7.6) with $\varepsilon = 1$ or by the symmetric substitution (8.9) also possesses this property.*

For the proof of the theorem we make use of the following lemma.

Lemma 8.2. *Suppose that the function f on the right-hand side of the symmetric system (8.1) has the form*

$$(8.14) \qquad f = v_1^2 + pv_1 + q,$$

or

$$(8.15) \qquad f = Au_1^2 v_1 + Bu_1^2 + Cu_1 v_1 + ru_1 + pv_1 + q,$$

where A, B, C, r, p, q are functions of the variable $u + v$. Then for any density ρ of a local conservation law of order 0 or 1 there is a density $\bar{\rho}$ of order 0 or 1 respectively satisfying the condition

$$(8.16) \qquad \rho - \rho^* = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \bar{\rho}.$$

We have not found a brief proof of this lemma. Our proof reduces to listing all densities of orders 0, 1 of the systems of equations mentioned and directly verifying the assertion. For example, in the case of systems (8.1), (8.14) there are no conservation laws of the first order, and the general form of the density of order zero is the following:

$$(8.17) \quad \rho = \alpha \exp(-u - v) + \beta \exp(-\lambda u - \bar{\lambda} v) + \gamma \exp(-\bar{\lambda} u - \lambda v),$$

where α, β, γ are arbitrary constants, $\lambda = \exp(2\pi i/3)$, $\bar{\lambda} = \exp(-2\pi i/3)$. A density $\bar{\rho}$ satisfying (8.16) exists and has the form

$$\bar{\rho} = \frac{\beta - \gamma}{\bar{\lambda} - \lambda} \left(\exp(-\lambda u - \bar{\lambda} v) - \exp(-\bar{\lambda} u - \lambda v) \right).$$

Proof of the theorem. Together with ρ the function $\rho + \rho^*$ is the density of a local conservation law of a symmetric system. As indicated in Example 3.3,

$$(8.18) \qquad \rho = \begin{cases} \Phi(u + v) u_n v_n + R + \operatorname{Im} D, & N = 2n, \\ \Phi(u + v)(u_n v_{n-1} - v_n u_{n-1}) + Q + \operatorname{Im} D, & N = 2n + 1, \end{cases}$$

where $Q = Q(u, v, \ldots, u_{n-1}, v_{n-1})$, $R = R(u, v, \ldots, u_n, v_n)$, R being a linear function of u_n, v_n. Therefore the density $\rho + \rho^*$ has the same order N. This density can be represented as $\rho + \rho^* = A + B$, where A is an invariant function and $B = B(u, v, u_1, v_1)$ is a function linear in u_1, v_1. For it is evident that the fucntion $\tilde{\rho} = (\partial/\partial u - \partial/\partial v)(\rho + \rho^*)$ is also the density of a local conservation law of a symmetric system, and $\tilde{\rho}^* = -\tilde{\rho}$. It follows from (8.18) that any anti-involutory density has order at most 1, therefore there

is a function $B(u, v, u_1, v_1)$ satisfying $\widetilde{p} = (\partial/\partial u - \partial/\partial v)B$, $B^* = B$. Also, the function $A = p + p^* - B$ is invariant. To complete the proof of the theorem it remains to check that one may take for B a density of a local conservation law of our system. In the next chapter it will be shown that a symmetric system satisfying the hypotheses of the theorem necessarily has the form (8.1), (8.14). Therefore the existence of the necessary density B follows from Lemma 8.2. ∎

Chapter IV
INTEGRABLE SYSTEMS OF SCHRÖDINGER TYPE

In this chapter we present results on the classification of systems of equations possessing local conservation laws of higher order. We consider systems of two equations of the form (6.20), which contain the well-known Schrödinger and Landau-Lifshits equations. The problem of composing a complete list of this type has a rather long history. Here different approaches are used, which are related mainly to the search for $L - A$ pairs; however, before [9], [24] even the general structure of non-linearity in integrable cases had not been established.

The symmetry approach allows us to state an explicit (in a certain sense) criterion for integrability of systems of Schrödinger type and to obtain an exhaustive list of equations possessing local conservation laws of higher order. The theory of transformations developed in Chapter III essentially simplifies the classification of integrable cases. We note that the equations of the list satisfy a priori only a few necessary integrability conditions. Nevertheless, we can prove that all systems except two for which the proof is still lacking possess an infinite series of local conservation laws. The proof is based on the construction of $L - A$ pairs for the key equations of the list.

§9. The list of integrable equations

Considering the systems of equations $u_t = A(u)u_2 + F(u, u_1)$, $\det A \neq 0$, $u = (u^1, u^2)$, we can restrict ourselves as shown in §6 to systems of the form (6.20) whose principal part is diagonal:

$$(9.1) \qquad u_t = u_2 + f(u, v, u_1, v_1), \qquad -v_t = v_2 + g(u, v, u_1, v_1).$$

The main result of [9], [24], [25] is the lists of equations of the form (9.1) given below. These lists are obtained as a result of thorough and difficult work related to the refinement of the form of the functions f and g in (9.1) with the help of the necessary conditions for the existence of local conservation laws and the integrability conditions stated in §3. Starting with these conditions, we prove that f and g are polynomials in u_1, v_1 and

have the following form:

(9.2)
$$f = Au_1^2 v_1 + B_u u_1^2 + C_v u_1 v_1 + ru_1 + F(u, v, v_1),$$
$$g = -Av_1^2 u_1 + B_v v_1^2 + C_u u_1 v_1 - rv_1 + G(u, v, u_1),$$

where A, B, C, r depend only on u and v, while F and G are quadratic polynomials in u_1 and v_1 respectively.

The classification problem is simplified if we restrict ourselves to systems (9.1) that are reduced by conformal transformations $\bar{u} = \varphi(u), \bar{v} = \psi(v)$ to the symmetric form

(9.3) $u_t = u_2 + f(u + v, u_1, v_1),$ $-v_t = v_2 + f(u + v, -u_1, -v_1).$

The theory of transformations developed in §8 is used essentially in the proof of Theorem 10.1 on the completeness of the list of symmetric systems (9.3) (List I). The classification in the remaining cases is carried out modulo the conformal substitutions

(9.4) $\bar{u} = \varphi(u), \; \bar{v} = \psi(v)$

and the scale transformations

(9.5) $\bar{t} = t, \; \bar{x} = x + \alpha t,$

(9.6) $\bar{t} = \alpha^2 t, \; \bar{x} = \alpha x,$

(9.7) $\bar{t} = -t, \; \bar{u} = v, \; \bar{v} = u.$

If the system (9.1) does not reduce to the symmetric system (9.6) by a conformal substitution, then it is related by a composition of transformations (9.4)-(9.7) to one of the systems of Lists II, III or to a splitting system ($f = f(u, u_1)$), or to a linear system. A criterion for the composition of Lists I, II, III consists in conditions that are necessary and sufficient for the existence of a formal symmetry and a formal conservation law of order 6 (see §10). We note that List III consists of equations that satisfy these conditions and have the form

(9.8) $u_t = u_2 + f(u_1, v_1),$ $-v_t = v_2 + g(u_1, v_1),$

(9.9) $u_t = u_2 + D(p(u, v)),$ $-v_t = v_2 + D(q(u, v)).$

List I

(a) $u_t = u_2 + (u + v)^2, \; -v_t = v_2 + (u + v)^2;$

(b) $u_t = u_2 + (u + v) u_1, \; -v_t = v_2 - (u + v) v_1;$

(c) $u_t = u_2 + (u + v) v_1 - \frac{1}{6}(u + v)^3, \; -v_t = v_2 - (u + v) u_1 - \frac{1}{6}(u + v)^3;$

(d) $u_t = u_2 + u_1^2 + \exp(u + v), \; -v_t = v_2 + v_1^2 + \exp(u + v);$

(e) $u_t = u_2 + u_1^2 + (2u_1 + v_1) \exp(u + v),$
$-v_t = v_2 + v_1^2 - (2v_1 + u_1) \exp(u + v);$

(f) $u_t = u_2 + (u_1^2 + \alpha)\, y, \quad -v_t = v_2 + (v_1^2 + \alpha)\, y$

(where $y = y\,(u+v),\ 2y' = y^2 - 4\beta,\ \alpha,\ \beta \in \{0,\ 1\}$);

(g) $u_t = u_2 + v_1^2 + q, \quad -v_t = v_2 + u_1^2 + q$

(where $q = \alpha \exp (u + v) + \beta \exp (-2u - 2v),\ \alpha,\ \beta,\ \in \{0,\ 1\}$);

(h)
$$u_t = u_2 + R\,(y)\, u_1^2 v_1 + R'\,(y)\, u_1^2 - \frac{2}{3}\,[R''\,(y) - 2\gamma]\, u_1 + \frac{1}{3}\, R'''\,(y),$$
$$-v_t = v_2 - R\,(y)\, v_1^2 u_1 + R'\,(y)\, v_1^2 + \frac{2}{3}\,[R''\,(y) - 2\gamma]\, v_1 + \frac{1}{3}\, R'''\,(y)$$

(where $y = y\,(u+v),\ y' = R\,(y),\ R\,(y) = \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \varepsilon$).

List II

(i) $u_t = u_2 + D\,(u^2 + v^{-1}), \quad -v_t = v_2 - 2D\,(uv) - 1$;

(j) $u_t = u_2 + v_1, \quad -v_t = v_2 - u_1^2 - \left(v + \frac{1}{2}\, u^2\right) u_1$;

(k)
$$u_t = u_2 - \frac{1}{2}\,(u+v)^{-1}\,(u_1^2 + 2u_1 v_1) + \alpha\,(u+v),$$
$$-v_t = v_2 - \frac{1}{2}\,(u+v)^{-1}\,(v_1^2 + 2u_1 v_1) + \beta\,(u+v);$$

(l)
$$u_t = u_2 - 2\,(u+v)^{-1}\,[u_1^2 + R\,(u)] + \frac{1}{2}\, R'\,(u),$$
$$-v_t = v_2 - 2\,(u+v)^{-1}\,[v_1^2 + R\,(-v)] - \frac{1}{2}\, R'\,(-v);$$

(m)
$$u_t = u_2 - 2\,(u+v)^{-1}\, u_1^2 - 4\,[P\,(u,\ v)\, u_1 + R\,(u)\, v_1]\,(u+v)^{-2},$$
$$-v_t = v_2 - 2\,(u+v)^{-1}\, v_1^2 + 4\,[P\,(u,\ v)\, v_1 + R\,(-v)\, u_1]\,(u+v)^{-2}$$

(in the equations (l), (m) $R\,(z) = \alpha z^4 + \beta z^3 + \gamma z^2 + \delta z + \varepsilon,\ P\,(u,\ v) =$
$$= 2\alpha u^2 v^2 + \beta\,(uv^2 - vu^2) - 2\gamma uv + \delta\,(u-v) + 2\varepsilon);$$

(n)
$$u_t = u_2 + \exp (\varphi)\,(u_1^2 + 1)\, v_1 + \varphi_u\,(u_1^2 + 1),$$
$$-v_t = v_2 - \exp (\varphi)\,(v_1^2 + 1)\, u_1 + \varphi_v\,(v_1^2 + 1);$$

(p)
$$u_t = u_2 + \exp (\varphi)\,(u_1^2 + 1)\, v_1 + \varphi_u u_1^2 + 2r u_1,$$
$$-v_t = v_2 - \exp (\varphi)\,(v_1^2 + 1)\, u_1 + \varphi_v v_1^2 - 2r v_1$$

(in the equations (n), (p) $\exp (\varphi) = y\,(u+v) - y\,(u-v),\ r = y\,(u+v) - y\,(u-v)$, in the case (n) $(y')^2 = -y^4 + \alpha y^3 + \beta y^2 + \gamma y + \delta$, in the case (p) $(y')^2 = -4y^4 + \alpha y^3 + \beta y^2 + \gamma y + \delta$;

(q) $u_t = u_2 + v_1^2 + \chi_u v_1, \quad -v_t = v_2 + u_1^2 - \chi_v u_1$

(where $\chi = \alpha \exp (-u - v) + \exp (-\lambda u - \bar{\lambda} v) + \exp (-\bar{\lambda} u - \lambda v)$);

(r) $u_t = u_2 + v_1^2 + z_v, \quad -v_t = v_2 + u_1^2 + z_u$

(where $z = \alpha \exp (u + v) + \exp (\lambda u + \bar{\lambda} v) + \exp (\bar{\lambda} u + \lambda v)$);

(s) $u_t = u_2 + v_1^2 + z_v, \quad -v_t = v_2 + u_1^2 + z_u$

(where $z = \alpha \exp(-2u - 2v) + \exp(-2\lambda u - 2\bar{\lambda}v) + \exp(-2\bar{\lambda}u - 2\lambda v)$);

(t) $u_t = u_2 + v_1^2 + \chi_u v_1 + z_v, \quad -v_t = v_2 + u_1^2 - \chi_v u_1 + z_u$

(where $\chi = \alpha \exp(-u - v) + \exp(-\lambda u - \bar{\lambda}v) + \exp(-\bar{\lambda}u - \lambda v)$,

$z = -\dfrac{1}{6}[\exp(u + v) + \alpha \exp(\lambda u + \bar{\lambda}v) + \alpha \exp(\bar{\lambda}u + \lambda v)] +$

$\qquad + \dfrac{1}{12}\left[\alpha^2 \exp(-2u - 2v) + \exp(-2\lambda u - 2\bar{\lambda}v) + \exp(-2\bar{\lambda}u - 2\lambda v)\right]$).

In the equations (q) — (t) $\alpha \in \{0, 1\}$, and λ, $\bar{\lambda}$ denote the cube roots of
1: $\lambda = \exp(2\pi i/3)$, $\bar{\lambda} = \exp(-2\pi i/3)$.

List III

(w$_1$) $u_t = u_2 + v_1 + \alpha, \quad -v_t = v_2 - u_1^2 + \beta u_1 + \gamma$;

(w$_2$) $u_t = u_2 + v_1^2 + \alpha v_1 + \beta, \quad -v_t = v_2 + u_1^2 + \gamma u_1 + \delta$;

(w$_3$) $u_t = u_2 + \varepsilon u_1^2 v_1 + \alpha u_1^2 - 2\beta u_1 v_1 + \gamma v_1 + \lambda,$
$\qquad -v_t = v_2 - \varepsilon v_1^2 u_1 + \beta v_1^2 - 2\alpha u_1 v_1 + \delta u_1 + \mu$;

(z$_1$) $u_t = u_2 + D(v), \quad -v_t = v_2 - D(u^2)$;

(z$_2$) $u_t = u_2 + D(v^2), \quad -v_t = v_2 + D(u^2)$;

(z$_3$) $u_t = u_2 + D(\varepsilon u^2 v + \alpha u^2 - 2\beta uv + \gamma v),$
$\qquad -v_t = v_2 + D(-\varepsilon v^2 u + \beta v^2 - 2\alpha uv + \delta u)$.

As we showed in §§7, 8, the partial differentiation (7.6) and the integration (7.12) allow us to establish hidden connections between integrable equations. First of all we note that the system (i) is obtained by partial differentiation from the splitting system

(i) $\leftarrow u_t = u_2 + v^{-1}, \quad -v_t = v_2,$

while the system (k) is reduced by partial differentiation to the linear system

(k) $\rightarrow u_t = u_2 + v_1 + (\alpha - \beta)u/2, \quad -v_t = v_2 - 2\beta u_1 + (\alpha - \beta)v/2.$

The remaining equations of the lists can be split into three groups. The systems close to the Schrödinger equation (0.14)

(9.10)   (b), (d), (e), (f), (h), (z$_3$), (w$_3$),

the systems of the type of the Boussinesq equation (0.15)

(9.11)   (a), (c), (g), (j), (q) — (t), (z$_1$), (z$_2$), (w$_1$), (w$_2$)

and the systems of the type of the Landau-Lifshits equation (0.16), whose integration is related to the spectral problem with a parameter on an elliptic curve[1]:

(9.12)   (l), (m), (n), (p).

---

[1]For the two last systems of (9.12) this is a conjecture.

We note also that each of the systems of List I is a representative of the class of equations related to each other by symmetric substitutions (compare Theorem 10.1).

In §§7, 8 a number of examples of transformations of integrable systems have been considered in detail. In particular, the interconnection between the systems (d), (b) and the special cases of (z_3), (w_3) corresponding to $\varepsilon = \beta = \delta = 0$, $\alpha = \gamma = 1$, is shown in Figure 1. In the general case the system (z_3) is reduced by the substitutions (9.4)–(9.7) to one of the equations of the following list:

$$(z_4) \qquad u_t = u_2 + D\,(u^2 + v), \quad -v_t = v_2 - 2D\,(uv);$$
$$(z_5) \qquad u_t = u_2 + D\,(2uv - u^2), \quad -v_t = v_2 + D(2uv - v^2);$$
$$(z_6) \qquad u_t = u_2 + D\,(u^2 v), \quad -v_t = v_2 - D\,(v^2 u);$$
$$(z_7) \qquad u_t = u_2 + D\,(u^2 v), \quad -v_t = v_2 - D\,(v^2 u + u);$$
$$(z_8) \qquad u_t = u_2 + D\,(u^2 v + v), \quad -v_t = v_2 - D\,(v^2 u + u).$$

Point substitutions and partial differentiations allow us to indicate the links between the equations (9.10) shown in Figure 2.
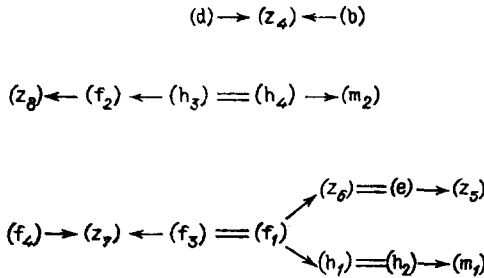

Fig. 2

In Figure 2 two bars join conformally equivalent systems of equations; as in Chapter III (Figure 1), arrows denote partial differentiations (7.6) with $\varepsilon = 1$; (f_1)–(f_4) denote systems of the type (f) with coefficients (α, β) equal to (0, 0), (1, 1), (0, 1), and (1, 0) respectively; (h_1)–(h_4) are special cases of a system of the type (h) with $y = u + v$, $y = \exp(u + v)$, $y = (u + v)^{-1}$, and $y = \mathrm{th}(u + v)$ respectively; (m_1) and (m_2) are degenerations of the system of the type (m) with $R(z) = z$ and $R(z) = 1 - z^2$.

Additional relations between the above equations can be obtained by introducing transformations of Miura type (compare [14]). Transformations of this sort allow us to connect the three components of the graph shown in Figure 2. To this end one has to pass from the systems (f_2), (f_4) to the systems, which are equivalent to them up to symmetric substitutions (8.9), of the form

$$(9.13) \qquad \begin{aligned} u_t &= u_2 + Au_1^2 + Bu_1 v_1 + \exp\,(u + v), \\ -v_t &= v_2 + Av_1^2 + Bu_1 v_1 + \exp\,(u + v), \end{aligned}$$

where

(9.14)
$$A = \frac{\varepsilon - \exp(u+v)}{\varepsilon - 2\exp(u+v)} , \quad B = \frac{2\exp(u+v)}{\varepsilon - 2\exp(u+v)}$$

and $\varepsilon = 0$ for (f_4), $\varepsilon \neq 0$ for (f_2). The substitution

(9.15)
$$\bar{u} = e^u [(\varepsilon - 2\exp(u+v))^{1/2} u_1 + 1],$$
$$\bar{v} = e^v [(\varepsilon - 2\exp(u+v))^{1/2} v_1 + 1]$$

connects the Schrödinger equation (7.13) written in the dynamical variables $\bar{u}, \bar{v}, \bar{u}_1, \bar{v}_1, \ldots$ with the system (9.13), (9.14).

The above connections between equations of Schrödinger type allow us to rewrite (see §8 and Lemma 6.2) infinite collections of local conservation laws, which are obtained by the method of the inverse scattering problem as a result of asymptotic expansion of the diagonal entries of the scattering matrix [16]. Taking as supporting equations the systems (d), (f_1), and (f_2), that is, the non-linear Schrödinger equation [2], the Heisenberg model (8.2), and the uniaxial Landau-Lifshits model (0.16) with $I_1 = I_2 \neq I_3$ [47], one can easily check that the connections shown in Figure 2 allow us to cover the systems (z_4)–(z_8) from the above list.

We also mention the following connections between the equations (9.11). The systems (a) and (w_2) are related to each other by a transformation of Miura type [42], and the systems (z_1) and (j) are obtained from (a) and (c) by partial differentiation. One can construct a transformation of Miura type from the system (c) to (a).

§10. A description of symmetric systems. Tests for integrability

The main advantage of the symmetry approach to the classification of systems of equations (9.1) is the possibility of formulating explicit (in a certain sense) necessary conditions for the existence of local conservation laws of higher order. It is surprising that in the final analysis the first few conditions turn out to be sufficient for the existence of local conservation laws of arbitrarily large order. This collection of necessary conditions can serve as a criterion for the verification of whether a given concrete system of equations of the form (9.1) reduces to one of the systems of Lists I–III considered in the previous section.

1. The algorithm for obtaining integrability conditions stated in §3 is associated with cumbersome calculations, which we omit. The result of these calculations is the table (10.3)–(10.8) of canonical densities ρ_k, ρ_k^0 ($k = 1, 2, 3, 4$) (see Example 3.3 and formulae (3.39)–(3.41)) which define the canonical conservation laws

(10.1)
$$\frac{d\rho_k}{dt} = D(\sigma_k) \quad (k = 1, 2, 3, 4)$$

and the divergency conditions

(10.2) $$\rho_k^0 = D(\sigma_k^0) \quad (k = 1,\ 2,\ 3,\ 4).$$

(10.3) $$\rho_1^0 = \frac{1}{2}(f_{u_1} + g_{v_1}), \quad \rho_1 = \frac{1}{2}(f_{u_1} - g_{v_1}),$$

(10.4) $$\rho_2^0 = \sigma_{1,t}^0 - \rho_1^0 \rho_1 + f_u - g_v,$$

(10.5) $$\rho_3^0 = \sigma_{2,t}^0 + 2\rho_1^0 f_{v_1} g_{u_1} - 2(f_{v_1} g_u + g_{u_1} f_v),$$

(10.6) $$\rho_2 = \sigma_1 - \frac{1}{2}[(\rho_1^0)^2 + \rho_1^2] - f_{v_1} g_{u_1} + f_u + g_v, \quad \rho_3 = \sigma_2,$$

(10.7) $$\rho_4^0 = \sigma_{3,t}^0 + \rho_2^0 \rho_2 + \rho_1[\rho_3^0 - \sigma_{2,t}^0] + (f_u + g_v)_t + D(\rho_1^0)D(\rho_1) +$$
$$+ \rho_1^0[D(f_{v_1})g_{u_1} - D(g_{u_1})f_{v_1}] + g_v D(g_{v_1}) - f_u D(f_{u_1}) + 2f_v D(g_{u_1}) - 2g_u D(f_{v_1}),$$

(10.8) $$\rho_4 = \sigma_3 + \frac{1}{2}[(\rho_2^0)^2 + \rho_2^2] - \rho_1^0[\rho_3^0 - \sigma_{2,t}^0] + \frac{1}{2}[(D\rho_1^0)^2 + (D\rho_1)^2] +$$
$$+ \rho_1[g_{u_1}D(f_{v_1}) - f_{v_1}D(g_{u_1})] + f_{v_1}(g_{u_1})_t - g_{u_1}(f_{v_1})_t -$$
$$- f_u D(f_{u_1}) - g_v D(g_{v_1}) + 2f_v D(g_{u_1}) + 2g_u D(f_{v_1}) -$$
$$- f_{v_1}^2 g_{u_1}^2 + 2f_{v_1} g_{u_1}(f_u + g_v) + (f_u - g_v)_t - 2D(f_{v_1})D(g_{u_1}) - 4f_t g_u.$$

The solubility of the system of equations (10.1)–(10.2) for the functions σ_k, σ_k^0 ($k \leqslant 4$) of the variables u, v, u_1, v_1, ... is by Theorems 3.1, 3.2 of §3 a criterion for the existence of a non-singular formal symmetry and a non-singular conservation law of order 6. We recall that the existence of a pair of conservation laws of higher order guarantees the existence of both a formal symmetry and a formal conservation law satisfying the non-singularity condition (see §1 and Theorem 1.1). Therefore the solubility conditions

(10.9) $$\frac{d\rho_k}{dt}, \quad \rho_k^0 \in \text{Im } D \quad (k = 1,\ 2,\ 3,\ 4)$$

of the system (10.1), (10.2) are necessary conditions for the existence of conservation laws.

For $k = 1$ the conditions (10.9) are explicit. For $k = k_0 > 1$ we suppose that the conditions (10.9) are satisfied for smaller values of k and that local functions σ_k, σ_k^0 ($k < k_0$) have been found that are a solution to the truncated system (10.1), (10.2) with $k = 1, ..., k_0 - 1$.

Lemma 10.1. *Suppose that the system of equations (9.1) satisfies (10.9) with $k = 1, 2$. Then there are functions φ, ψ, r of the variables u, v and a constant $\varepsilon \in \mathbf{C}$ such that*

(10.10)
$$f = \varepsilon e^{\varphi} u_1^2 v_1 + \frac{1}{2}(\varphi + \psi)_u u_1^2 + (\varphi - \psi)_v u_1 v_1 + r u_1 + F(u,\ v,\ v_1),$$
$$g = -\varepsilon e^{\varphi} v_1^2 u_1 + \frac{1}{2}(\varphi + \psi)_v v_1^2 + (\varphi - \psi)_u u_1 v_1 - r v_1 + G(u, v, u_1).$$

Proof. The condition (10.9) with $k = 1$ means (see Example 3.3) that there are functions φ, a, b, r of the variables u, v such that

$$\frac{1}{2}(f_{u_1} + g_{v_1}) = \varphi_u u_1 + \varphi_v v_1,$$

$$\frac{1}{2}(f_{u_1} - g_{v_1}) = 2\varepsilon e^\varphi u_1 v_1 + a u_1 - b v_1 + r.$$

Adding and subtracting these equalities, we see that the functions have the form

$$f = \varepsilon v^\varphi u_1^2 v_1 + A u_1^2 + C u_1 v_1 + r u_1 + F,$$

$$g = -\varepsilon e^\varphi v_1^2 u_1 + B v_1^2 + D u_1 v_1 - r v_1 + G,$$

where $A = a + \varphi_u/2$, $B = b + \varphi_v/2$, $C = \varphi_v - b$, $D = \varphi_u - a$.
A solution of the first of the equations (10.2) is $\sigma_1^0 = \varphi$, and we can proceed to the condition $\rho_2^0 = \varphi_t - \rho_1^0 \rho_1 + f_u - g_v \in \mathrm{Im}\, D$ (see (10.4)).
Differentiating with respect to t, we obtain

$$(10.11)\quad \rho_3^0 = D(\varphi_u u_1 - \varphi_v v_1) + \varphi_{vv} v_1^2 - \varphi_{uu} u_1^2 + \varphi_u f - \varphi_v g - \rho_1^0 \rho_1 + f_u - g_v.$$

The vanishing of the coefficient of $u_1 v_1$ in (10.11) yields $a_v = b_u$, as is easy to check. Therefore there is a function $\psi(u, v)$ such that $\psi_u = a$, $\psi_v = b$. These relations and the above expressions for f and g yield (10.10). ∎

The expressions for f and g given in Lemma 10.1 guarantee the validity of the condition $\rho_1^0 \in \mathrm{Im}\, D$ and make explicit the condition $\rho_2^0 \in \mathrm{Im}\, D$ (see (10.11)). The following list of identities simplifies the investigation of the equations (10.1) and allows us to refine the form of σ_k:

$$(10.12)\quad (au_1 + bv_1 + c)_t - D[au_2 - bv_2 + (b_u - a_v) u_1 v_1 + af - bg +$$
$$+ c_u u_1 - c_v v_1] = (b_u - a_v)(v_1 f + u_1 g) - u_1 v_1 D(b_u - a_v) + c_u f - c_v g + c_{vv} v_1^2 - c_{uu} u_1^2,$$

$$(10.13)\quad [H(u, v, u_1\, v_1)]_t = H_{v_1 v_1} v_2^2 - H_{u_1 u_1} u_2^2 + f[H_u - D_0(H_{u_1})] +$$
$$+ g[-H_v + D_0(H_{v_1})] + u_2[H_{u_1 v_1} g - H_{u_1 u_1} f + H_u - D_0(H_{u_1})] +$$
$$+ v_2[-H_{u_1 r_1} f + H_{v_1 v_1} g - H_r + D_0(H_{v_1})] + \mathrm{Im}\, D,$$

$$(10.14)\quad [e^\varphi(v_1 u_2 - u_1 v_2)]_t = 2e^\varphi(\varphi_u v_1 - g_{u_1}) u_2^2 + 2e^\varphi(\varphi_v u_1 - f_{v_1}) v_2^2 +$$
$$+ 2e^\varphi[(v_1 f + u_1 g)\varphi_u - D_0(g)] u_2 + 2e^\varphi[(v_1 f + u_1 g)\varphi_v - D_0(f)] v_2 +$$
$$+ e^\varphi[(D\varphi)^2 + u_1 D\varphi_u + v_1 D\varphi_v](v_1 u_2 + u_1 v_2 + v_1 f + u_1 g) + \mathrm{Im}\, D,$$

$$(10.15)\quad (e^\varphi u_1 v_1)_t - D(\theta) = e^\varphi[u_1 v_1 D(r) - v_1^2 \varphi_v F + u_1^2 \varphi_u G] +$$
$$+ \frac{1}{2} e^\varphi\{u_1^2 v_1[\varphi_u(\varphi - \psi)_u - (\varphi - \psi)_{uu}] + v_1^2 u_1[(\varphi - \psi)_{vv} - \varphi_v(\varphi - \psi)_v]\} + D_0(P - Q),$$

$$\partial P(u, v, v_1)/\partial v_1 = e^\varphi F, \quad \partial Q(u, v, u_1)/\partial u_1 = e^\varphi G,$$

$$(10.16)\quad \theta = e^\varphi\left[v_1 u_2 - u_1 v_2 + \frac{3}{2}\varepsilon e^\varphi u_1^2 v_1^2 + \psi_u u_1^2 v_1 - \psi_v v_1^2 u_1\right] +$$
$$+ e^\varphi[r u_1 v_1 + v_1 F - u_1 G] - P + Q.$$

The identities (10.12) and (10.13) are valid for any system of the form (9.1), while (10.14) and (10.15) are valid for any system of the form (9.1), (10.10), where $D_0 \overset{\text{def}}{=\joinrel=} u_1 \partial/\partial u + v_1 \partial/\partial v$.

***Lemma* 10.2.** *Under the hypotheses of Lemma* 10.1 *the following equalities hold*:

$$(10.17) \quad \begin{aligned} [F_{v_1} + (\varphi + \psi)_v\, u_1 + \varepsilon e^\varphi u_1^2]\, G_{u_1 u_1 u_1} + [3\,(\varphi - \psi)_v + 6\varepsilon e^\varphi u_1]\, G_{u_1 u_1} &= 0, \\ [G_{u_1} + (\varphi - \psi)_u\, v_1 - \varepsilon e^\varphi v_1^2]\, F_{v_1 v_1 v_1} + [3\,(\varphi - \psi)_u - 6\varepsilon e^\varphi v_1]\, F_{v_1 v_1} &= 0. \end{aligned}$$

Proof. By (10.10) we have

$$(10.18) \qquad \rho_1 = 2\varepsilon e^\varphi u_1 v_1 + \psi_u u_1 - \psi_v v_1 + r.$$

Applying formulae (10.12) and (10.16) we see from the equations $D(\sigma_1) = \rho_{1,t}$ that

$$(10.19) \quad \sigma_1 = 2\varepsilon\theta + \psi_u f + \psi_v g - 4\psi_{uv} u_1 v_1 - \psi_{uu} u_1^2 - \psi_{vv} v_1^2 + D\,(r) + D^2(\psi) + \chi.$$

The form of the function $\chi = \chi(u, v)$ in (10.19) is defined only after satisfying the condition $\rho_{1,t} \in \operatorname{Im} D$. Substituting (10.19) in (10.6), we find that

$$(10.20) \qquad \rho_2 = 2\varepsilon e^\varphi\,(v_1 u_2 - u_1 v_2) + H\,(u,\ v,\ u_1,\ v_1) + D^2\,(\psi).$$

The condition $\rho_{2,t} \in \operatorname{Im} D$ yields by (10.13), (10.14)

$$(10.21) \quad H_{u_1 u_1} = 4\varepsilon e^\varphi\,(\varphi_u v_1 - g_{u_1}), \qquad H_{v_1 v_1} = 4\varepsilon e^\varphi\,(\varphi_v u_1 - f_{v_1}).$$

The equivalence of (10.17) and (10.21) can be verified by a straightforward calculation. ∎

The proof of the general classification theorem, that is, the theorem of the completeness of the Lists I–III of §9, consists in the examination of all variants of the validity of (10.9). Besides a great number of a priori possible cases, the complexity of the problem is connected with the necessity of successive refinement of the form of the canonical conservation laws (10.1) with $k = 2, 3, 4$. This refinement is based on the identities (10.10)–(10.16) and the divergency conditions (10.2). It becomes clear that the validity of the canonical conservation law with density ρ_k equivalent to a function of the form

$$(10.22) \qquad e^\varphi\,(v_1 u_2 - u_1 v_2) + H\,(u,\ v,\ u_1,\ v_1)$$

completely defines the form of the system under consideration. For example, for the system (10.10) with $\varepsilon \neq 0$ the canonical density ρ_2 has the form (10.21), by (10.20). Thus in the classification of systems with $\varepsilon \neq 0$ we can restrict ourselves to the first two canonical conservation laws (10.1). For systems of the form

$$(10.23) \qquad u_t = u_2 + p(u, v), \qquad -v_t = v_2 + q(u, v)$$

the densities $\rho_1 = \rho_1^0 = 0$, $\rho_2^0 = p_u - q_v$, $\rho_2 = p_u + q_v$. The condition $\rho_2^0 \in \mathrm{Im}\, D$ yields $p_u = q_v$. Introducing the potential $p = z_v$, $q = z_u$, we obtain, using (10.12),

$$\rho_2 = 2z_{uv}, \quad \rho_3 = 2\,(z_{uvu}u_1 - z_{uvv}v_1), \quad \rho_4 = \sigma_3 + z_{uv}^2 - 4z_{uu}z_{vv}.$$

For $z_{uu} = 0$ (or $z_{vv} = 0$) the system (10.23) splits. For $z_{uu}z_{vv} \neq 0$ we can check by using the table of identities (10.12)–(10.16) that the conditions (10.9) are equivalent to the equations

$$z_{uvuu} = z_{uvvv} = z_{uuuu} = z_{vvvv} = 0, \quad z_v(z_{uu}z_{vv})_u = z_u(z_{uu}z_{vv})_v.$$

From this for $z_{uvuv} = 0$ we obtain the system (a), while for $z_{uvuv} \neq 0$ we obtain the system equivalent to (d). Therefore for systems of the form (10.23) canonical conservation laws of order at least 3 are not used at all in the classification.

For the proof of the general classification theorem the systems of equations (10.10) are split into classes according to their properties. Systems possessing an Abelian two-parameter subgroup (the systems (w_1), (w_2), (w_3) of List III) belong to the simplest of these classes. We shall present a proof of the classification theorem for symmetric systems and dwell briefly on the properties that single out the systems (n), (p) and (q)–(t) not occuring in the literature. We note that both the systems of List I and the systems reduced to them by symmetric transformations are integrable symmetric systems. Therefore the class of systems reduced to the systems of List I is bigger than the remaining classes of integrable systems contained in Lists II and III.

Theorem 10.1. *Any symmetric system* (9.3) *satisfying* (10.9) *is reduced by a symmetric transformation* (8.9) *and scale transformations* (9.5), (9.6) *either to one of the systems* (a)–(h) *of List* I *of §9 or to the systems*

$$u_t = u_2 + u_1^2, \quad -v_t = v_2 + v_1^2; \quad u_t = u_2 + \alpha\,(u+v), \quad -v_t = v_2 + \alpha\,(u+v).$$

The equations under consideration split into two classes in accordance with the following lemma.

Lemma 10.3. *For the systems* (9.3), (10.10) *the following alternative holds*: *either* $F = pv_1 + q$ *or* $F = sv_1^2 + pv_1 + q$ *and* $s' = r' = 0$, $\varepsilon = \varphi' = \psi' = 0$.

Proof of the lemma. Differentiating the second of the equalities (10.17), we obtain $G_{u_1u_1}F_{v_1v_1v_1} = 0$. Since $G(u, v, u_1) = F(u+v, -u_1)$, it follows that $F = sv_1^2 + pv_1 + q$. The case $s \neq 0$ requires a detailed analysis. The second of the relations (10.17) yields $\varepsilon = 0$, $\varphi' = \psi'$. We consider the condition $\rho_{1,t} \in \mathrm{Im}\, D$, where by (10.18) $\rho_1 = \psi'(u_1 - v_1) + r$. The term with u_1^3 in $\rho_{1,t}$ (see (10.12)) yields $\psi'' = 0$. Taking $\varphi' = \psi'$ into account, we obtain $\varphi' = \psi' = \mathrm{const}$. We find from (10.19), (10.6) that the density ρ_2 can be represented as

$$\rho_2 = -4s^2u_1v_1 - (s' + \varphi's)\,(u_1^2 - v_1^2) + A\,(u+v)\,(u_1 - v_1) + B\,(u+v).$$

The condition $\rho_{2t} \in \text{Im } D$ now yields

$$s^2 = \text{const } e^{\varphi}, \quad s' + \varphi's = 0 \Rightarrow s' = \varphi' = 0.$$

It remains to show that $r' = 0$.

We find from (10.11) that $\sigma_2^0 = r + p$. Collecting the terms with u_1^2 in ρ_3^0 (10.5), we find that $p'' + sp' + 4sr' = 0$. The vanishing of the coefficient of u_1^2 in $\rho_{1,t}$ yields the equation $r'' + sr' = 0$. Now the condition $\rho_{1,t} \in \text{Im } D$ holds and

$$\rho_2 = -4s^2 u_1 v_1 + (2r' - p' - 2sp)(u_1 - v_1) + b(u + v).$$

The vanishing of the coefficient of u_1^3 in $\rho_{2,t}$ yields the third condition on the coefficients $r, p : sp' + (2r' - p' - 2sp)' = 0$. Comparing the three resulting equations, we find that $r' = 0$ and

(10.24) $$p'' + sp' = 0.$$

Proof of Theorem 10.1 *for systems of the form*

(10.25) $$u_t = u_2 + v_1^2 + pv_1 + q, \quad -v_t = v_2 + u_1^2 - pu_1 + q.$$

We find from (10.24) that $p = \alpha \exp(-u - v) + \beta$. The symmetric substitution[1] $\bar{u} + \bar{v} = u + v$, $\bar{u}_1 = u_1 - (\alpha/4) \exp(-u - v) - \beta/2$ leads to a system of the form (10.25) with $p = 0$. In this case

$$\rho_1 = \rho_1^0 = \rho_2^0 = 0, \quad \rho_3^0 = -4D(q), \quad \rho_4^0 = (2q' - 4q)_t + 4q'(u_2 - v_2).$$

The condition $\rho_4^0 \in \text{Im } D$ is equivalent to the equation $q''' + q'' - 2q' = 0$. The resulting system coincides with the system (g) of List I.

Proof of Theorem 10.1 *for systems of the form*

(10.26) $$\begin{aligned} u_t &= u_2 + \lambda u_1^2 + ru_1 + pv_1 + q, \\ -v_t &= v_2 + \lambda v_1^2 - rv_1 - pu_1 + q, \quad \lambda = 0,1. \end{aligned}$$

Systems of the form (10.26) admit the symmetric substitutions

$$\bar{u} + \bar{v} = u + v, \quad \bar{u}_1 = u_1 + \rho, \quad \rho'' = \lambda \rho'.$$

For the transformed system

(10.27) $$\bar{r} = r - 2\lambda \rho, \quad \bar{p} = p - 2\rho', \quad \bar{\lambda} = \lambda.$$

For any system (10.26) the conditions ρ_1^0, $\rho_2^0 \in \text{Im } D$ are satisfied and

$$\sigma_2^0 = \lambda(u_1 - v_1) + r + p + \lambda \int p(u + v) \, d(u + v).$$

The conditions $\rho_{1,t}$, $\rho_3^0 \in \text{Im } D$ are equivalent to the equations

(10.28) $$r'' = \lambda r', \quad p'' = \lambda p.$$

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[1]Under symmetric substitutions formal symmetries and conservation laws are rewritten by formulae analogous to (3.49). The orders of formal symmetries and conservation laws do not change. Therefore the transformed system also satisfies (10.9).

Finding r and p from (10.28), we check with the help of (10.27) that the transformations mentioned in the statement of the theorem allow us to reduce the problem to the following two cases:

$$\lambda = 0, \; r = \alpha(u + v), \qquad p = \beta(u + v),$$
$$\lambda = 1, \; r = \alpha \exp(u + v), \qquad p = \beta \exp(-u - v),$$

where $\alpha, \beta \in \mathbf{C}$, $\alpha = 0, 1$. Next, we check that $\rho_4^0 - \rho_{2t} \in \operatorname{Im} D$ if and only if

$$\lambda = 0, \; \alpha\beta = 0 \quad \text{or} \quad \lambda = 1, \; \beta = 0.$$

The condition $\rho_{2t} \in \operatorname{Im} D$ is equivalent to the relations

$$q''' + \beta^2 = 0 \quad (\lambda = 0), \qquad q''' = q' \quad (\lambda = 1).$$

Simple transformations (both symmetric and point) reduce the resulting systems either to the systems (a), (c) from List I or to the following three cases:

(10.29) $\lambda = \beta = 0, \quad \alpha = 1, \quad q = \gamma(u+v)^2, \quad r = (u+v), \quad p = 0,$

(10.30) $\lambda = \alpha = 1, \quad \beta = 0, \quad q = \gamma \exp(-u-v), \quad r = \exp(u+v), \quad p = 0,$

(10.31) $\lambda = 1, \quad \alpha = \beta = 0, \quad q = \gamma \exp(-u-v) + \delta \exp(u+v), \quad r = p = 0.$

In the cases (10.29), (10.30) the condition $\rho_{3t} \in \operatorname{Im} D$ yields $\gamma = 0$. The resulting systems coincide with (b) and the system

$$u_t = u_2 + u_1^2 + u_1 \exp(u+v), \quad -v_t = v_2 + v_1^2 - v_1 \exp(u+v),$$

which is symmetrically equivalent to the system (e) (see Example 8.1). In the case (10.31) $\rho_{3,t} \in \operatorname{Im} D$, while the condition $\rho_{4,t} \in \operatorname{Im} D$ is equivalent to the equality $\gamma = 0$. The resulting system is conformally equivalent either to a linear system or to (d).

Proof of Theorem 10.1 *in the case*

(10.32) $F = pv_1 + q, \quad |\varepsilon| + |\psi''| + |(\varphi - \psi)'| \neq 0.$

Admissible symmetric substitutions have the form

$$\bar{u} + \bar{v} = \int a(u+v) \, d(u+v), \quad \bar{u}_1 = au_1 + b,$$

(10.33)

$$a'' = \frac{1}{2}(3\varphi - \psi)' a' - \varepsilon e^{\varphi} b', \quad b'' = \frac{1}{2}(\varphi + \psi)' b' + pa',$$

the coefficients of $\bar{u}_1\bar{v}_1$ and $\bar{v}_1$ in the transformed system being

(10.34) $\bar{\varphi}' - \bar{\psi}' = [(\varphi - \psi)' a - 2a' - 2\varepsilon e^{\varphi} b] a^{-2},$

(10.35) $\bar{p} = p - 2b'a^{-1} + [\varepsilon e^{\varphi} b + 2a' - (\varphi - \psi)' a] b a^{-2}.$

The condition $\rho_2^0 \in \operatorname{Im} D$ (see (10.11)) is equivalent to the equation

(10.36) $(\varphi - \psi)'' = \varphi'(\varphi - \psi)'.$

It is easy to check that if (10.36) is satisfied, the system consisting of the equation $(\varphi - \psi)'a = 2a' + 2\varepsilon e^{\varphi}b$ (compare (10.34)) and the equations (10.33) is compatible. Therefore the functions a and b can be chosen so that the coefficient of $\bar{u}_1\bar{v}_1$ in the transformed system is equal to zero. The equation

(10.37) $$p' = \varphi'p$$

for $\varepsilon \neq 0$ follows from the condition $\rho_{1,t} \in \text{Im } D$ and for $\varepsilon = 0$ from $\rho_{2,t} \in \text{Im } D$ (one has to equate to zero the coefficient of u_1^3 in $\rho_{1,t}$ and $\rho_{2,t}$). If $\varphi' - \psi' = 0$ and (10.37) is valid, then (10.33) follows from the fact that the right-hand sides of (10.34) and (10.35) are equal to zero. Therefore the functions a and b can be chosen so that in the new system $\bar{p} = 0$.

Next, we consider systems with $p = (\varphi - \psi)' = 0$. Such systems are completely described by the following equations:

(10.38) $$2\varphi''' - 2\varphi'\varphi'' + 3\varepsilon r'e^{\varphi} = 0,$$
(10.39) $$r'' - \varphi'r' + 2\varepsilon q' = 0,$$
(10.40) $$q'' - \varphi'q' = 0.$$

The equations (10.38) and (10.39) follow from $\rho_{1,t} \in \text{Im } D$. The equation (10.40) is obtained from $\rho_{2,t} \in \text{Im } D$ for $\varepsilon \neq 0$ and from $\rho_{3,t} \in \text{Im } D$ (we equate the coefficient of u_1^3 to zero) for $\varepsilon = 0$. If $\varepsilon = 0$, then $\varphi'' = \exp(\varphi)$ (see (10.38)) and $r = \lambda\varphi' + \mu$ (see (10.39)). The symmetric substitution $\bar{u} + \bar{v} = u + v$, $\bar{u}_1 = u_1 + \lambda/2$ and the transformation (9.5) lead to a system with $r = 0$. The system with $\varepsilon = r = 0$ is conformally equivalent to the system (f) of List I. It remains to consider the case $\varepsilon \neq 0$, $\varphi' - \psi' = p = 0$. In this case the system described by the equations (10.29)–(10.40) coincides with the system (h) up to the transformation (9.5).

To complete the proof of Theorem 10.1 it remains to check that we have considered all the cases listed in Lemma 10.3. If for $F = pv_1 + q$ the condition (10.32) is broken, then $\varepsilon = (\varphi - \psi') = \psi'' = 0$ and the system has the form (10.26). In the second case $F = sv_1^2 + pv_1 + q$ the system is reduced by Lemma 10.3 to the form (10.25) by scale transformations. ■

We proceed to the examination of the systems (q)–(t) of List II. These systems belong to the class of equations (9.1) satisfying the condition

(10.41) $$f_{v_1v_1} \neq 0, \quad g_{u_1u_1} \neq 0.$$

As in the proof of Lemma 10.3, we can check that in the case (10.41) the functions (10.10) satisfy

(10.42)
$$\varepsilon = 0, \quad (\varphi - \psi)_u(\varphi - \psi)_v = 0, \quad \psi_{uv} = 0, \quad r = \text{const},$$
$$F = \tilde{s}v_1^2 + \tilde{p}v_1 + \tilde{q}, \quad G = su_1^2 + pu_1 + q.$$

An essential feature is the choice of an appropriate conformal system of coordinates in which the analysis of the solubility conditions of the system of equations (10.1), (10.2) is carried out. The conformal substitution

$\bar{u} = U(u)$, $\bar{v} = V(v)$ transforms the coefficient functions φ, ψ in (10.10) as follows:

(10.43)
$$\bar{\varphi}(\bar{u}, \bar{v}) = \varphi(u, v) - \log U'(u) - \log V'(v),$$
$$\bar{\psi}(\bar{u}, \bar{v}) = \psi(u, v) - \log U'(u) - \log V'(v).$$

It follows from (10.42) that $\varphi_{uv} = 0$, and so we can choose a conformal system of coordinates in which $\varphi = 0$. In these coordinates (compare the proof of Lemma 10.3) not only r but also s and $\tilde{s}$ are constants, and the systems under consideration are reduced by scale transformations to the form

(10.44) $u_t = u_2 + v_1^2 + \chi_u v_1 + z_v, \quad -v_t = v_2 + u_1^2 - \chi_v u_1 + z_u.$

If the system (10.44) is not symmetric (see (g) from List I) and does not coincide with the system (w$_2$) from List III, then the coefficient functions χ, z in (10.44) have to satisfy the equations

(10.45) $\chi_{uu} + \chi_v = \chi_{vv} + \chi_u = 0, \quad z_{uuu} = z_{vvv},$

(10.46) $2\chi_u z_{uu} + z_u \chi_{uu} = 2\chi_v z_{vv} + z_v \chi_{vv},$

(10.47) $z_{uvvv} + z_{uvu} - 2z_{vv} = z_{uvuu} + z_{uvv} - 2z_{uu} = 0,$

(10.48) $z_u(\chi_u \chi_v + 2z_{uv})_v = z_v(\chi_u \chi_v + 2z_{uv})_u.$

The equations (10.45)–(10.48) guarantee the validity of the conditions $\rho_k^0 \in \operatorname{Im} D$ ($k \leqslant 4$); $\rho_{k,1} \in \operatorname{Im} D$ ($k = 1, 2$). The analysis of the resulting equations leads to the cases (q)–(t) from List II.

In conclusion we give a description of the system (n) and (p) that is invariant under conformal transformations. For $\varepsilon = 1$ the equations (9.1), (10.10) are either reduced by conformal transformations to a symmetric system or to the system (w$_3$) of List III, or $\varphi = \psi$ and the system under consideration has the form

(10.49)
$$u_t = u_2 + e^\varphi u_1^2 v_1 + \varphi_u u_1^2 + ru_1 + \tilde{p}v_1 + e^{-\varphi}z_v,$$
$$-v_t = v_2 - e^\varphi v_1^2 u_1 + \varphi_v v_1^2 - rv_1 - pu_1 + e^{-\varphi}z_u,$$

where

(10.50) $\tilde{p}_v = \varphi_v \tilde{p}, \quad p_u = \varphi_u p, \quad p\tilde{p} \neq 0,$

(10.51) $\tilde{p}_u + \varphi_u \tilde{p} = \chi_v, \quad p_v + \varphi_v p = \chi_u,$

(10.52) $r - \chi + 2z = 0, \quad 2\varphi_{uv}e^{-\varphi} + 3r + \chi = 0,$

(10.53) $(\tilde{p}z_u)_u = (pz_v)_v, \quad z_u r_v = z_v r_u,$

(10.54) $(e^{-\varphi}z_v)_v + \tilde{p}z = (e^{-\varphi}z_u)_u + pz = 0.$

The conditions (10.50)–(10.54) guarantee that the system (10.49) is reduced by a conformal transformation to the systems (n) and (p). The system of coordinates in (n), (p) is chosen so that

(10.55) $p = \tilde{p} = e^\varphi.$

In the general case we find from (10.50) that $p = e^{\varphi}B(v)$, $\widetilde{p} = e^{\varphi}A(u)$.
Under the conformal transformation $\bar{u} = U(u)$, $\bar{v} = V(v)$ the functions p and $\widetilde{p}$ in (10.49) are transformed by the formulae

$$\bar{p}\,(\bar{u},\ \bar{v}) = p\,(u,\ v)\,\frac{V'(v)}{U'(u)}\,, \quad \bar{\widetilde{p}}\,(\bar{u},\ \bar{v}) = \widetilde{p}\,(u,\ v)\,\frac{U'(u)}{V'(v)}\,.$$

Taking (10.43) into account, we find that for $U' = A^{-1/2}$, $V' = B^{-1/2}$ the functions p and $\widetilde{p}$ in the new coordinate system have the form (10.55).

§ 11. A brief bibliographical comment

Many of the systems of equations of Lists I–III are familiar in publications on the method of the inverse problem. Without claiming completeness, we give below the necessary references. For the following systems of Boussinesq type (9.11) (a), (z_1), (z_2), (w_2), and (s) commutation relations were found in [48], [49], [43], [42], and [40]. Using the method of reduction groups [43], [50], we have found commutation representations for the systems (w_2), (g), (s), (q), and (t); it turned out that they are related to the groups Z_3, two representations of the dihedral group D_3, the tetrahedral group, and the octahedral group. For the system (r) a commutation representation was found directly from the definition (0.17). We do not give these new commutation representations here because of lack of space. Commutation representations for systems close to the non-linear Schrödinger equation (9.10) have been investigated best of all. For the systems (b), (d), (e) = (z_6), (f_1) = (f_3), (f_2), (z_4), (z_5) they can be extracted from [45], [2], [51], [52], [47], [53], [54] respectively. Finally, for the systems (l) and (m) commutation representations were found in [55], [56], and [57]. Interesting problems arise in the construction and classification of representations of zero curvature [43], [45] and r-matrices [58], [59] connected with the equations of the lists.

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