Localization and Representation Theory of Reductive Lie Groups

Dragan Miličić
Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

Version of April 1993

A very preliminary version of several chapters.

1. Sheaves of Differential Operators

1.1 Twisted Sheaves of Differential Operators

Let X be a smooth algebraic variety over an algebraically closed field k of characteristic zero, $\mathcal{O}_X$ the structure sheaf of X, $\mathcal{T}_X$ the tangent sheaf of X and $\mathcal{D}_X$ the sheaf of differential operators on X. We consider the category of pairs $(\mathcal{A}, i)$, where $\mathcal{A}$ is a sheaf of associative k-algebras with identity on X and $i : \mathcal{O}_X \longrightarrow \mathcal{A}$ a morphism of k-algebras with identity. The sheaf $\mathcal{D}_X$ with the natural inclusion $i_X : \mathcal{O}_X \longrightarrow \mathcal{D}_X$ is an object of this category. We say that a pair $(\mathcal{D}, i)$ is a *twisted sheaf of differential operators* on X if it is locally isomorphic to the pair $(\mathcal{D}_X, i_X)$, i. e. if X admits a cover by open sets U such that $(\mathcal{D}|U, i|U) \cong (\mathcal{D}_U, i_U)$.

Now we want to discuss the natural parametrization of twisted sheaves of differential operators on X. First we need some preparation.

Lemma 1. *Let ϕ be an endomorphism of $(\mathcal{D}_X, i_X)$. Then there exists a closed 1-form ω on X such that*

$$\phi(\xi) = \xi - \omega(\xi)$$

for any local vector field $\xi \in \mathcal{T}_X$, and ϕ is completely determined by ω. In particular, ϕ is an automorphism of $(\mathcal{D}_X, i_X)$.

Proof. Let $f \in \mathcal{O}_X$ and $\xi \in \mathcal{T}_X$. Then

$$[\phi(\xi), f] = [\phi(\xi), \phi(f)] = \phi([\xi, f]) = \phi(\xi(f)) = \xi(f).$$

Evaluating this on the function 1 we get

$$\phi(\xi)(f) = \xi(f) + f\phi(\xi)(1).$$

Therefore, we can put $\omega(\xi) = -\phi(\xi)(1)$. Obviously, ω is a 1-form on X. Also, we have

$$\begin{aligned}\omega([\xi, \eta]) &= -\phi([\xi, \eta])(1) = -(\phi(\xi)\phi(\eta) - \phi(\eta)\phi(\xi))(1) \\ &= \phi(\xi)(\omega(\eta)) - \phi(\eta)(\omega(\xi)) = \xi(\omega(\eta)) - \eta(\omega(\xi))\end{aligned}$$

for $\xi, \eta \in \mathcal{T}_X$. Therefore,

$$d\omega(\xi \wedge \eta) = \xi(\omega(\eta)) - \eta(\omega(\xi)) - \omega([\xi, \eta]) = 0$$

for $\xi, \eta \in \mathcal{T}_X$, i. e. $d\omega = 0$ and ω is closed. It is evident that ϕ is completely determined by ω. Also, ϕ preserves the filtration of $\mathcal{D}_X$ and the induced endomorphism $\operatorname{Gr}\phi$ of $\operatorname{Gr}\mathcal{D}_X$ is the identity morphism. This clearly implies that ϕ is an automorphism. □

By 1, every automorphism ϕ of $(\mathcal{D}_X, i_X)$ determines a closed 1-form ω on X. Evidently, this map is an monomorphism of the multiplicative group $\operatorname{Aut}(\mathcal{D}_X, i_X)$ into the additive group $\mathcal{Z}^1(X)$ of closed 1-forms on X.

Lemma 2. *The natural morphism of* $\operatorname{Aut}(\mathcal{D}_X, i_X)$ *into* $\mathcal{Z}^1(X)$ *is an isomorphism.*

Proof. It remains to show this morphism is surjective. Let ω be a closed 1-form on X. Then we can define a map ϕ of $\mathcal{T}_X$ into $\mathcal{D}_X$ by $\phi(\xi) = \xi - \omega(\xi)$, for $\xi \in \mathcal{T}_X$. Evidently, ϕ satisfies conditions (ii) and (iii) of (D.O.13). Also, since ω is closed, for $\xi, \eta \in \mathcal{T}_X$, we have

$$\begin{aligned}\phi([\xi,\eta]) &= [\xi,\eta] - \omega([\xi,\eta]) = [\xi,\eta] - \xi(\omega(\eta)) + \eta(\omega(\xi)) \\ &= [\xi - \omega(\xi), \eta - \omega(\eta)] = [\phi(\xi), \phi(\eta)];\end{aligned}$$

i. e. the condition (i) is also satisfied. Therefore, ϕ extends to an endomorphism of $(\mathcal{D}_X, i_X)$. By 1, ϕ is actually an automorphism. □

Let $(\mathcal{D}, i)$ be a twisted sheaf of differential operators on X. Then there exists an open cover $\mathcal{U} = (U_j; 1 \le j \le n)$ such that $(\mathcal{D}, i)|U_j$ is isomorphic to $(\mathcal{D}_{U_j}, i_{U_j})$ for $1 \le j \le n$. For each j fix an isomorphism $\psi_j : (\mathcal{D}, i)|U_j \longrightarrow (\mathcal{D}_{U_j}, i_{U_j})$. Then there exist an automorphism ϕ_{jk} of $(\mathcal{D}_{U_j\cap U_k}, i_{U_j\cap U_k})$ such that the diagram

$$\begin{array}{ccc} (\mathcal{D}, i)|U_j \cap U_k & = & (\mathcal{D}, i)|U_j \cap U_k \\ \psi_k \downarrow & & \psi_j \downarrow \\ (\mathcal{D}_{U_j\cap U_k}, i_{U_j\cap U_k}) & \xrightarrow{\phi_{jk}} & (\mathcal{D}_{U_j\cap U_k}, i_{U_j\cap U_k}) \end{array}$$

commutes, i. e. $\psi_j = \phi_{jk} \circ \psi_k$. By 2, there exists a closed 1-form ω_{jk} on $U_j \cap U_k$ which determines ϕ_{jk}. If $U_j \cap U_k \cap U_l \neq \emptyset$ we have on it

$$\psi_j = \phi_{jk} \circ \psi_k = \phi_{jk} \circ \phi_{kl} \circ \psi_l,$$

hence, $\phi_{jl} = \psi_{jk} \circ \psi_{kl}$ on $U_j \cap U_k \cap U_l$. This in turn implies that

$$\phi_{jl}(\xi) = \xi - \omega_{jl}(\xi) = (\phi_{jk} \circ \phi_{kl})(\xi) = \phi_{jk}(\xi - \omega_{kl}(\xi)) = \xi - \omega_{jk}(\xi) - \omega_{kl}(\xi)$$

for $\xi \in \mathcal{T}_{U_j\cap U_k\cap U_l}$, i. e.

$$\omega_{jl} = \omega_{jk} + \omega_{kl}$$

on $U_j \cap U_k \cap U_l$.

Let Z^1_X be the sheaf of closed 1-forms on X, and $C^\cdot(\mathcal{U}, \mathcal{Z}^1_X)$ the Čech complex of $\mathcal{Z}^1_X$ corresponding to the cover $\mathcal{U}$. Then $\omega = (\omega_{jk}; 1 \le j < k \le n)$ is an element of $C^1(\mathcal{U}, \mathcal{Z}^1_X)$ and $d\omega = 0$, i. e. $\omega \in Z^1(\mathcal{U}, \mathcal{Z}^1_X)$. Assume now that we take another set of local isomorphisms $\psi'_j : (\mathcal{D}, i)|U_j \longrightarrow (\mathcal{D}_{U_j}, i_{U_j})$, $1 \leq j \leq n$. This would lead to another set $(\phi'_{jk}; 1 \leq j < k \leq n)$ and another $\omega' \in Z^1(\mathcal{U}, \mathcal{Z}^1_X)$. Applying 2 again, we can get automorphisms σ_j of $(\mathcal{D}_{U_j}, i_{U_j})$, $1 \leq j \leq n$, such that $\psi'_j = \sigma_j \circ \psi_j$ for $1 \leq j \leq n$ and closed 1-forms ρ_j, $1 \leq j \leq n$, associated to them. Evidently, $\rho = (\rho_j; 1 \leq j \leq n)$ is an element of $C^0(\mathcal{U}, \mathcal{Z}^1_X)$. Now, we have

$$\sigma_j \circ \phi_{jk} \circ \psi_k = \sigma_j \circ \psi_j = \psi'_j = \phi'_{jk} \circ \psi'_k = \phi'_{jk} \circ \sigma_k \circ \psi_k$$

on $U_j \cap U_k$, hence $\sigma_j \circ \phi_{jk} = \phi'_{jk} \circ \sigma_k$. This leads to

$$\rho_j + \omega_{jk} = \omega'_{jk} + \rho_k$$

on $U_j \cap U_k$. It follows that $\omega' - \omega = d\rho$. Therefore, the twisted sheaf of differential operators $(\mathcal{D}, i)$ determines an element of $H^1(\mathcal{U}, \mathcal{Z}^1_X)$.

Therefore we have a well-defined map $t : (\mathcal{D}, i) \longmapsto t(\mathcal{D}, i) \in H^1(X, \mathcal{Z}^1_X)$ from the isomorphism classes of twisted sheaves of differential operators into the first Čech group of X with coefficients in $\mathcal{Z}^1_X$.

Theorem 3. *The map t defines a bijection between the isomorphism classes of twisted sheaves of differential operators on X and the elements of $H^1(X, \mathcal{Z}^1_X)$.*

Proof. First, we shall check that this map is injective. Let $\mathcal{D}$ and $\mathcal{D}'$ be two twisted sheaves of differential operators such that $t(\mathcal{D}) = t(\mathcal{D}')$. Then both of them determine an open cover $\mathcal{U} = (U_j; 1 \leq j \leq n)$ and $\omega, \omega' \in \mathcal{Z}^1(\mathcal{U}, \mathcal{Z}^1_X)$ such that they define the same element of $H^1(X, \mathcal{Z}^1_X)$; and families of local isomorphisms $\psi_j : \mathcal{D}|U_j \longrightarrow \mathcal{D}_{U_j}$, $1 \leq j \leq n$, and $\phi'_j : \mathcal{D}'|U_j \longrightarrow \mathcal{D}_{U_j}$, $1 \leq j \leq n$, as explained in the previous discussion. By taking possibly a refinement of $\mathcal{U}$, we can assume that $\omega - \omega' = d\rho$ for some $\rho = (\rho_j; l \leq j \leq n) \in C^0(\mathcal{U}, \mathcal{Z}^1_X)$. Let $\sigma_j : \mathcal{D}_{U_j} \longrightarrow \mathcal{D}_{U_j}$ be the automorphism determined by ρ_j, $1 \leq j \leq n$. Then $\phi''_j = \sigma_j \circ \phi'_j$, $1 \leq j \leq n$, is a family of local isomorphisms $\phi"_j : \mathcal{D}'|Uj \longrightarrow \mathcal{D}_{U_j}$ with the property that

$$\sigma_j \circ \phi'_{jk} \circ \psi'_k = \sigma_j \circ \psi'_j = \psi''_j = \phi''_{jk} \circ \psi''_k = \phi''_{jk} \circ \sigma_k \circ \psi'_k$$

and therefore $\sigma_j \circ \phi'_{jk} = \phi''_{jk} \circ \sigma_k$, i. e. $\phi''_{jk} = \sigma_j \circ \phi'_{jk} \circ \sigma_k^{-1}$ on $U_j \cap U_k$. This implies that $\omega''_{jk} = \omega'_{jk} + \rho_j - \rho_k$ on $U_j \cap U_k$, i. e. $\omega'' = \omega' + d\rho = \omega$. This finally implies that $\phi_{jk} = \phi''_{jk}$ on $U_j \cap U_k$. Define local isomorphisms $\theta_j : \mathcal{D}|U_j \longrightarrow \mathcal{D}'|U_j$ by $\theta_j = \psi''^{-1}_j \circ \psi_j$ for $1 \leq j \leq n$. Then, on $U_j \cap U_k$, we have

$$\theta_j = \psi''^{-1}_j \circ \psi_j = (\phi_{jk} \circ \psi''_k)^{-1} \circ \phi_{jk} \circ \psi_k = \psi''^{-1}_k \circ \psi_k = \theta_k,$$

and θ extends to a global isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$.

The proof of the surjectivity is the standard "recollement" argument using 2. □

Now we shall describe a construction of some twisted sheaves of differential operators on X. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X and $\mathcal{D}_{\mathcal{L}}$ the sheaf of all differential endomorphisms of $\mathcal{L}$. Because $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$, $\mathcal{D}_{\mathcal{L}}$ is a twisted sheaf of differential operators on X. Let $\mathcal{O}_X^*$ be the subsheaf of invertible elements in $\mathcal{O}_X$. Then, as it is well-known, the Picard group $\operatorname{Pic}(X)$ is equal to $H^1(X, \mathcal{O}_X^*)$. There exists a natural homomorphism $d\log : \mathcal{O}_X^* \longrightarrow \mathcal{Z}_X^1$ of sheaves of abelian groups given by the logarithmic derivative, i. e. $d\log f = f^{-1}df$, for any $f \in \mathcal{O}_X^*$. It induces morphisms $H^p(d\log) : H^p(X, \mathcal{O}_X^*) \longrightarrow H^p(X, \mathcal{Z}_X^1)$ of cohomology groups. Let $i(\mathcal{L})$ be the element of $H^1(X, \mathcal{O}_X^*)$ corresponding to $\mathcal{L}$. Then we have the following result.

Proposition 4. *For any invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X,*

$$t(\mathcal{D}_{\mathcal{L}}) = H^1(d\log)(i(\mathcal{L})).$$

Proof. Let $\mathcal{U} = (U_i; 1 \leq i \leq n)$ be an open cover of X such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ for all i, $1 \leq i \leq n$. Denote by $\alpha_i : \mathcal{L}|U_i \longrightarrow \mathcal{O}_X|U_i$, $1 \leq i \leq n$, the corresponding $\mathcal{O}_X$-module isomorphisms, and by s_{jk} the sections of $\mathcal{O}_X^*$ on $U_j \cap U_k$ defined by $s_{jk} = \alpha_j(\alpha_k^{-1}(1))$ for all $1 \leq j < k \leq n$. Then, for a section s of $\mathcal{L}|U_j \cap U_k$,

$$\alpha_j(s) = \alpha_j(\alpha_k^{-1}(\alpha_k(s))) = s_{jk}\alpha_k(s),$$

i. e. $s = (s_{jk}; 1 \leq j < k \leq n)$ is a 1-cocycle which represents $i(\mathcal{L})$. Also, α_i defines an isomorphism ψ_i of $\mathcal{D}_{\mathcal{L}}|U_i$ onto $\mathcal{D}_X|U_i$ by $\psi_i(u) = \alpha_i \circ u \circ \alpha_i^{-1}$ for all $1 \leq i \leq n$. This implies that

$$\begin{aligned} \psi_j(u)(f) &= (\alpha_j \circ u \circ \alpha_j^{-1})(f) = \alpha_j(u(\alpha_j^{-1}(f))) \\ &= s_{jk}\alpha_k(u(\alpha_k^{-1}(s_{jk}^{-1}f))) = s_{jk}\psi_k(u)(s_{jk}^{-1}f), \end{aligned}$$

for any $f \in \mathcal{O}_X|U_j \cap U_k$. Therefore, $\psi_j(u) = s_{jk}\psi_k(u)s_{jk}^{-1}$. It follows that

$$\phi_{jk}(D) = s_{jk}Ds_{jk}^{-1},$$

for any $D \in \mathcal{D}_X|U_j \cap U_k$. Let $\zeta \in \mathcal{T}_X|U_j \cap U_k$. Then,

$$\phi_{jk}(\zeta) = s_{jk}\,\zeta\,s_{jk}^{-1} = \zeta - s_{jk}^{-1}\,\zeta(s_{jk}),$$

i. e. $\omega_{jk} = s_{jk}^{-1}\,ds_{jk}$. Therefore, the 1-cocycle ω which represents $t(\mathcal{D}_{\mathcal{L}})$ is given by $(s_{jk}^{-1}\,ds_{jk}; 1 \leq i < j \leq n)$. □

Now we want to study the functoriality questions.

Let X and Y be two smooth algebraic varieties, $\varphi : X \longrightarrow Y$ a morphism of algebraic varieties and $\mathcal{D}$ a twisted sheaf of differential operators on Y. Then $\mathcal{D}_{X\to Y} = \varphi^*(\mathcal{D})$ is an $\mathcal{O}_X$-module for the left multiplication and a right $\varphi^{-1}(\mathcal{D})$-module for the right multiplication. We denote by $\mathcal{D}^\varphi$ the sheaf of all differential endomorphisms of the right $\varphi^{-1}(\mathcal{D})$-module $\mathcal{D}_{X\to Y}$. Evidently, $\mathcal{D}^\varphi$ is a sheaf of associative algebras on Y. There is also a natural morphism of sheaves of algebras $i_\varphi : O_X \longrightarrow \mathcal{D}^\varphi$. Hence, from (D.O.15) we know that the pair $(\mathcal{D}^\varphi, i_\varphi)$ is locally isomorphic to $(\mathcal{D}_X, i_X)$, i. e. $\mathcal{D}^\varphi$ is a twisted sheaf of differential operators on X.

By 3, $\mathcal{D}^\varphi$ determines an element $t(\mathcal{D}^\varphi)$ of $H^1(X, \mathcal{Z}_X^1)$. Now we want to calculate $t(\mathcal{D}^\varphi)$.

First we need a lifting result. Let $(\mathcal{C}, i)$ and $(\mathcal{D}, j)$ be two twisted sheaves of differential operators on Y, and $\psi : (\mathcal{C}, i) \longrightarrow (\mathcal{D}, j)$ an isomorphism. Therefore, ψ is an $\mathcal{O}_Y$-module isomorphism for the structures given by both left and right multiplication. Hence, ψ induces an $\mathcal{O}_X$-module isomorphism $\varphi^*(\psi) : \mathcal{C}_{X\to Y} \longrightarrow D_{X\to Y}$ of $\mathcal{O}_X$-modules. Also, if $u \in C_{X\to Y}$, $v \in \varphi^{-1}(\mathcal{C})$, we have

$$\varphi^*(\psi)(uv) = \varphi^*(\psi)(u)\varphi^{-1}(\psi)(v).$$

It follows that, for any $z \in \mathcal{C}^\varphi$, $u \in D_{X\to Y}$, $v \in \varphi^{-1}(\mathcal{D})$, we have

$$\begin{aligned}(\varphi^*(\psi)z\varphi^*(\psi^{-1}))(uv) &= \varphi^*(\psi)(z(\varphi^*(\psi^{-1})(u)\varphi^{-1}(\psi^{-1})(v))) \\ &= \varphi^*(\psi)((z\varphi^*(\psi^{-1})(u))\varphi^{-1}(\psi^{-1})(v)) \\ &= (\varphi^*(\psi)z\varphi^*(\psi^{-1}))(u)v,\end{aligned}$$

i. e. $\varphi^*(\psi)z\varphi^*(\psi^{-1}) \in \mathcal{D}^\varphi$. Hence, if we put

$$\varphi^{\#}(\psi)(z) = \varphi^*(\psi)z\phi^*(\psi^{-1}),$$

$\varphi^{\#}(\psi) : \mathcal{C}^\varphi \longrightarrow \mathcal{D}^\varphi$ is an isomorphism of sheaves of k-algebras on X. Evidently, $j_\varphi = \varphi^{\#}(\psi) \circ i_\varphi$. Therefore, $\varphi^{\#}(\psi)$ is an isomorphism of twisted sheaves of differential operators. We call it the *lifting* of ψ. Also, for any $z \in \mathcal{C}^\varphi$, $u \in \mathcal{C}_{X\to Y}$, we have

$$(\varphi^{\#}(\psi)(z))\varphi^*(\psi)(u) = \varphi^*(\psi)(zu).$$

Now, we consider the special case of an automorphism α of $\mathcal{D}_Y$. By 2, it is determined by a closed 1-form ω on Y. By D.O.15, there is a natural isomorphism δ of the pair $(\mathcal{D}_X, i_X)$ with $(\mathcal{D}_Y^\varphi, i_{Y,\varphi})$. We want to calculate the automorphism induced by the lifting $\varphi^{\#}(\alpha)$ of α on $\mathcal{D}_X$; more precisely, the closed 1-form on X it determines by 2.

Let $x \in X$ and U a small open neigborhood of $\varphi(x) \in Y$ such that we can find $f_i \in \mathcal{O}_Y(U)$, $1 \le i \le \dim Y$, such that df_i, $1 \le i \le \dim Y$, form a basis of the free $\mathcal{O}_U$-module $\mathcal{T}*_Y|U$. Let ∂_i, be the dual basis in $\mathcal{T}_Y|U$. Then, as we have seen in D.O, for a local vector field ξ around x, we have

$$\begin{aligned}\gamma(\varphi^{\#}(\alpha)(\xi))(1) &= \beta(\varphi^{\#}(\alpha)(\xi)(1\otimes 1)) = \beta(\varphi^*(\alpha)(\xi(1\otimes 1))) \\ &= \beta(\varphi^*(\alpha)(\sum \xi(f_i\circ\varphi)\otimes\varphi^{-1}\partial_i))) \\ &= \beta(\sum \xi(f_i\circ\varphi)\otimes\varphi^{-1}(\alpha\partial_i)) = -\sum \xi(f_i\circ\varphi)\omega(\partial_i) \\ &= -\omega(\sum \xi(f_i\circ\varphi)\partial_i) = -(\varphi^*\omega)(z),\end{aligned}$$

here we denoted by $\varphi^*\omega$ the 1-form on X induced by ω. Therefore $\varphi^*\omega$ is the closed 1-form associated to $\varphi^{\#}(\alpha)$.

Now we can apply this fact to the calculation of $t(\mathcal{D}^\varphi)$. Let $\mathcal{U} = (U_i; 1 \le i \le n)$ be an open cover of X and $\psi_i : (\mathcal{D}, i)|U_i \longrightarrow (\mathcal{D}_Y, i_Y)|U_i$ the corresponding isomorphisms. As before, for $1 \le j < k \le n$, denote by ϕ_{jk} the automorphisms of $(\mathcal{D}_Y, i_Y)|U_j \cap U_k$ such that $\psi_j = \phi_{jk} \circ \psi_k$. Let $\mathcal{V} = (V_i; 1 \le i \le n)$ be the open cover of X given by $V_i = \varphi^{-1}(U_i)$, $1 \le i \le n$. Then, the lifting $\varphi^\#(\psi_i)$ is an isomorphism of $(\mathcal{D}^\varphi, i_\varphi)|U_i$ onto $(\mathcal{D}_X, i_X)|U_i$, and the liftings $\varphi^\#(\phi_{jk})$ are automorphisms of $(\mathcal{D}_Y, i_Y)|U_j \cap U_k$ such that

$$\varphi^\#(\psi_j) = \varphi^*(\phi_{jk}) \circ \varphi^*(\psi_k),$$

for $1 \le j < k \le n$. From the previous discussion it follows that, if $\omega = (\omega_{jk}; 1 \le j < k \le n)$ is a 1-cocycle of closed 1-forms on Y corresponding to $\mathcal{D}$, then $(\varphi^*\omega_{jk}; 1 \le j < k \le n)$ is a 1-cocycle of closed 1-forms on X corresponding to $\mathcal{D}^\varphi$. The map $\omega \longrightarrow \varphi^*\omega$ of closed 1-forms on Y into closed 1-forms on X induces a morphism $\varphi^{-1}(\mathcal{Z}_Y^1) \longrightarrow \mathcal{Z}_X^1$ of the sheaves of vector spaces. This morphism, using ([**Tôhoku**], 3.2.2), induces linear maps $Z^p(\varphi) : H^p(Y, \mathcal{Z}_Y^1) \longrightarrow H^p(X, \mathcal{Z}_X^1)$ for each $p \in \mathbb{Z}_+$. Therefore, our previous discussion actually proves the following result.

Proposition 5. *Let $\varphi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties, and $\mathcal{D}$ a twisted sheaf of differential operators on Y. Then*

$$t(\mathcal{D}^\varphi) = Z^1(\varphi)(t(\mathcal{D})).$$

Moreover, the construction behaves well with respect to the composition of morphisms.

Proposition 6. *Let $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ be morphisms of smooth algebraic varieties and $\mathcal{D}$ a twisted sheaf of differential operators on Z. Then $\mathcal{D}^{\psi\circ\varphi}$ is naturally isomorphic to $(\mathcal{D}^\psi)^\varphi$.*

Proof. Evidently,

$$\begin{aligned}(\psi \circ \varphi)^*(\mathcal{D}) &= \varphi^*(\psi^*(\mathcal{D})) = \varphi^*(D_{Y\to Z}) = \mathcal{O}_X \otimes_{\varphi^{-1}(\mathcal{O}_Y)} \varphi^{-1}(D_{Y\to Z}) \\ &= \mathcal{O}_X \otimes_{\varphi^{-1}(\mathcal{O}_Y)} \varphi^{-1}(\mathcal{D}^\psi) \otimes_{\varphi^{-1}(\mathcal{D}^\psi)} \varphi^{-1}(\mathcal{D}_{Y\to Z}) \\ &= (\mathcal{D}^\psi)_{X\to Y} \otimes_{\varphi^{-1}(\mathcal{D}^\psi)} \varphi^{-1}(\mathcal{D}_{Y\to Z})\end{aligned}$$

as an $\mathcal{O}_X$-module and right $(\psi \circ \varphi)^{-1}(\mathcal{D})$-module. Also, the action of $(\mathcal{D}^\psi)^\varphi$ on the first factor in the last expression evidently commutes with the right action of $(\psi \circ \varphi)^{-1}(\mathcal{D})$. Therefore, there is a natural morphism of $(\mathcal{D}^\psi)^\varphi$ into $\mathcal{D}^{\psi\circ\varphi}$. By 1, this morphism is an isomorphism of twisted sheaves of differential operators. □

In the following we shall identify $(\mathcal{D}^\psi)^\varphi$ with $\mathcal{D}^{\psi\circ\varphi}$ using this isomorphism.

Another construction we want to discuss is the twist of a twisted sheaf of differential operators $\mathcal{D}$ on a smooth algebraic variety X by an invertible $\mathcal{O}_X$-module $\mathcal{L}$. If we consider $\mathcal{D}$ as an $\mathcal{O}_X$-module for the left multiplication, we can form the $\mathcal{O}_X$-module $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$. The sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ is a right $\mathcal{D}$-module for the right multiplication on the second factor. Therefore, we can consider the sheaf $\mathcal{D}^{\mathcal{L}}$ of local differential endomorphisms of the right $\mathcal{D}$-module $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$. It is obviously a sheaf of k-algebras on X. Also, because $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ is an $\mathcal{O}_X$-module, there is a natural homomorphism $i_{\mathcal{L}} : \mathcal{O}_X \longrightarrow \mathcal{D}^{\mathcal{L}}$. We claim that $(\mathcal{D}^{\mathcal{L}}, i_{\mathcal{L}})$ is a twisted sheaf of differential operators on X. Let $\mathcal{U} = (U_i; 1 \leq i \leq n)$ be an open cover of X such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ and $\mathcal{D}|U_i$ is isomorphic to $\mathcal{D}_X|U_i$. Therefore, as an $\mathcal{O}_X|U_i$-module, $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D})|U_i$ is isomorphic to $\mathcal{D}_X|U_i$. Also, under this isomorphism, the right $\mathcal{D}|U_i$-action on $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D})|U_i$ corresponds to the right $\mathcal{D}_X|U_i$-action on $\mathcal{D}_X|U_i$. This induces an isomorphism of the sheaves of differential endomorphisms, and therefore identifies $\mathcal{D}^{\mathcal{L}}|U_i$ with the sheaf of differential endomorphisms of $\mathcal{D}_X|U_i$ considered as a right $\mathcal{D}_X|U_i$-module. Evidently, this sheaf of algebras is naturally isomorphic to $\mathcal{D}_X|U_i$. Therefore, $\mathcal{D}^{\mathcal{L}}$ is a twisted sheaf of differential operators on X. It is called the *twist* of $\mathcal{D}$ by $\mathcal{L}$.

We start the study of twists with the following result.

Lemma 7. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on a smooth algebraic variety X. Then the twist $(\mathcal{D}_X)^{\mathcal{L}}$ of the sheaf of differential operators $\mathcal{D}_X$ is naturally isomorphic to $\mathcal{D}_{\mathcal{L}}$.*

Proof. Let $\mathcal{I}_X$ be the left ideal in $\mathcal{D}_X$ generated by $\mathcal{T}_X$. Then, we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{D}_X \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

and, by tensoring with $\mathcal{L}$,

$$0 \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_X \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L} \longrightarrow 0.$$

From the construction of $(\mathcal{D}_X)^{\mathcal{L}}$ is clear that this is an exact sequence of $(\mathcal{D}_X)^{\mathcal{L}}$-modules. Therefore there is a natural morphism of $(\mathcal{D}_X)^{\mathcal{L}}$ into $\mathcal{D}_{\mathcal{L}}$. By 1, it is an isomorphism of twisted sheaves of differential operators. □

In the following we shall identify $(\mathcal{D}_X)^{\mathcal{L}}$ with $\mathcal{D}_{\mathcal{L}}$ using this isomorphism.

Proposition 8. *Let $\mathcal{D}$ be a twisted sheaf of differential operators on a smooth algebraic variety X and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then*

$$t(\mathcal{D}^{\mathcal{L}}) = t(\mathcal{D}) + H^1(d\log)(i(\mathcal{L})).$$

Proof. Let $\mathcal{U} = (U_i; 1 \leq i \leq n)$ be an open cover of X, and $\alpha_i : \mathcal{L}|U_i \longrightarrow \mathcal{O}_X|U_i$ and $\psi_i : (\mathcal{D}, i)|Ui \longrightarrow (\mathcal{D}_X, i_X)|U_i$, $1 \leq i \leq n$, local isomorphisms. As in the proofs of 3. and 4, we denote by ϕ_{jk} the automorphisms of $(\mathcal{D}_X, i_X)|U_j \cap U_k$ such that $\psi_j = \phi_{jk} \circ \psi_k$ and $s_{jk} = \alpha_j(\alpha_k^{-1}(1)) \in \mathcal{O}_X^*$ for $1 \leq j < k \leq n$. For $1 \leq i \leq n$, $\sigma_i = \alpha_i \otimes \psi_i : (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D})|U_i \longrightarrow \mathcal{D}_X|U_i$ is an isomorphism of $\mathcal{O}_X$-modules, and

$$\sigma_i(s \otimes uv) = \alpha_i(s)\psi_i(uv) = \alpha_i(s)\psi_i(u)\psi_i(v) = \sigma_i(s \otimes u)\psi_i(v),$$

for $s \in \mathcal{L}|U_i$, $u, v \in \mathcal{D}|U_i$. Therefore, if we identify the differential endomorphisms of $\mathcal{D}_X$, considered as a right $\mathcal{D}_X$-module for the right multiplication, with $\mathcal{D}_X$ via the map $T \longrightarrow T(1)$, we have a natural isomorphism $\tau_i : \mathcal{D}^{\mathcal{L}}|U_i \longrightarrow \mathcal{D}_X|U_i$ given by

$$\tau_i(u) = (\sigma_i \circ u \circ \sigma_i^{-1})(1) = \sigma_i(u(\alpha_i^{-1}(1) \otimes 1)).$$

Also, for $1 \leq j < k \leq n$,

$$\sigma_j(s \otimes v) = \alpha_j(s)\psi_j(v) = s_{jk}\alpha_k(s)\phi_{jk}(\psi_k(v)) = s_{jk}\phi_{jk}(\sigma_k(s \otimes v)),$$

for any $s \in \mathcal{L}|U_j \cap U_k$ and $v \in \mathcal{D}|U_j \cap U_k$, what implies that

$$\begin{aligned}\tau_j(u) &= \sigma_j(u(\alpha_j^{-1}(1) \otimes 1)) = s_{jk}\phi_{jk}(\sigma_k(u(s_{jk}^{-1}\alpha_k^{-1}(1) \otimes 1))) \\ &= s_{jk}\phi_{jk}(\sigma_k(u(\alpha_k^{-1}(1) \otimes s_{jk}^{-1}))) = s_{jk}\phi_{jk}(\sigma_k(u(\alpha_k^{-1}(1) \otimes 1)s_{jk}^{-1})) \\ &= s_{jk}\phi_{jk}(\sigma_k(u(\alpha_k^{-1}(1) \otimes 1))s_{jk}^{-1} = s_{jk}\phi_{jk}(\tau_k(u))s_{jk}^{-1}\end{aligned}$$

for $u \in \mathcal{D}^{\mathcal{L}}|U_j \cap U_k$. If we put

$$\rho_{jk}(v) = s_{jk}\phi_{jk}(v)s_{jk}^{-1}$$

for $v \in \mathcal{D}_X|U_j \cap U_k$, we get an automorphism ρ_{jk} of $\mathcal{D}_X|U_j \cap U_k$ such that $\tau_j = \rho_{jk} \circ \tau_k$. As before, denote by $\omega = (\omega_{jk}; 1 \leq j < k \leq n)$ the element of $C^1(\mathcal{U}, \mathcal{Z}_X^1)$ corresponding to $(\phi_{jk}; 1 \leq j < k \leq n)$. Let $\xi \in \mathcal{T}_X|U_j \cap U_k$. Then

$$\rho_{jk}(\xi) = s_{jk}\phi_{jk}(\xi)s_{jk}^{-1} = s_{jk}(\xi - \omega_{jk}(\xi))s_{jk}^{-1} = \xi - s_{jk}^{-1}\, ds_{jk}(\xi) - \omega_{jk}(\xi),$$

hence, the element of $C^1(\mathcal{U}, \mathcal{Z}_X^1)$ corresponding to $(\rho_{jk}; 1 \leq j < k \leq n)$ is equal to $(\omega_{jk} + s_{jk}^{-1}\, ds_{jk}; 1 \leq j < k \leq n)$. □

Proposition 9. *Let $\mathcal{D}$ be a twisted sheaf of differential operators on X and $\mathcal{L}$ and $\mathcal{L}'$ two invertible $\mathcal{O}_X$-modules. Then the twisted sheaf of differential operators $(\mathcal{D}^{\mathcal{L}})^{\mathcal{L}'}$ is naturally isomorphic to $\mathcal{D}^{\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{L}}$.*

Proof. Evidently,

$$\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D} = (\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}^{\mathcal{L}}) \otimes_{\mathcal{D}^{\mathcal{L}}} (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}),$$

as an $\mathcal{O}_X$-module and right $\mathcal{D}$-module. Therefore, the right action of $(\mathcal{D}^{\mathcal{L}})^{\mathcal{L}'}$ on the first factor in the second expression commutes with the right $\mathcal{D}$-action. This gives a natural morphism of $(\mathcal{D}^{\mathcal{L}})^{\mathcal{L}'}$ into $\mathcal{D}^{\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{L}}$. By 1, this morphism is an isomorphism of twisted sheaves of differential operators. □

In the following we shall identify $(\mathcal{D}^{\mathcal{L}})^{\mathcal{L}'}$ with $\mathcal{D}^{\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{L}}$ using this isomorphism.

Proposition 10. *Let $\varphi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties, $\mathcal{D}$ a twisted sheaf of differential operators on Y and $\mathcal{L}$ an invertible*

$\mathcal{O}_Y$-module. Then the twisted sheaf of differential operators $(\mathcal{D}^{\mathcal{L}})^{\varphi}$ is naturally isomorphic to $(\mathcal{D}^{\varphi})^{\varphi^*(\mathcal{L})}$.

Proof. Evidently,

$$\begin{aligned}\mathcal{D}_{X\to Y} &= \varphi^*(\mathcal{D}) = \varphi^*(\mathcal{L}^{-1}\otimes_{\mathcal{O}_Y}\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{D}) = \varphi^*(\mathcal{L})^{-1}\otimes_{\mathcal{O}_X}\varphi^*(\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{D})\\ &= \varphi^*(\mathcal{L})^{-1}\otimes_{\mathcal{O}_X}(\mathcal{D}^{\mathcal{L}})_{X\to Y}\otimes_{\varphi^{-1}(\mathcal{D}^{\mathcal{L}})}\varphi^{-1}(\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{D})\\ &= (\varphi^*(\mathcal{L})^{-1}\otimes_{\mathcal{O}_X}(\mathcal{D}^{\mathcal{L}})^{\varphi})\otimes_{(\mathcal{D}^{\mathcal{L}})^{\varphi}}(\mathcal{D}^{\mathcal{L}})_{X\to Y}\otimes_{\varphi^{-1}(\mathcal{D}^{\mathcal{L}})}\varphi^{-1}(\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{D}),\end{aligned}$$

as a $\mathcal{O}_X$-module and right $\varphi^{-1}(\mathcal{D})$-module. It is clear that the left action of $((\mathcal{D}^{\mathcal{L}})^{\varphi})^{\phi^*(\mathcal{L})}$ on the first factor commutes with the right action of $\varphi^{-1}(\mathcal{D})$. Therefore there is a natural morphism of $((\mathcal{D}^{\mathcal{L}})^{\varphi})^{\varphi^*(\mathcal{L})}$ into $\mathcal{D}^{\varphi}$. By 1, this is an isomorphism of twisted sheaves of differential operators. By twisting this natural isomorphism by $\varphi^*(\mathcal{L})$ and using 9, we get that $(\mathcal{D}^{\mathcal{L}})^{\varphi}$ is naturally isomorphic to $(\mathcal{D}^{\varphi})^{\varphi^*(\mathcal{L})}$. □

If $\mathcal{A}$ is a sheaf of k-algebras on X, we denote by $\mathcal{A}^\circ$ the opposite sheaf of k-algebras on X.

Proposition 11. *Let $(\mathcal{D}, i)$ be a twisted sheaf of differential operators on a smooth algebraic variety X. Then $(\mathcal{D}^\circ, i)$ is a twisted sheaf of differential operators on X.*

Let $n = \dim X$ and ω_X be the sheaf of differential forms of degree n on X. Then there is a natural action of the sheaf of Lie algebras $\mathcal{T}_X$ on the sheaf ω_X; a vector field ξ acts by the corresponding Lie derivative L_ξ. In fact, if U is an open set in X, $\omega \in \omega_X(U)$, and $\xi, \eta_1, \eta_2, \ldots, \eta_n$ local vector fields on U, we have

$$(L_\xi\omega)(\xi_1\wedge\eta_2\wedge\ldots\wedge\eta_n) = \xi(\omega(\eta_1\wedge\eta_2\wedge\ldots\wedge\eta_n)) - \sum_{i=1}^{n}\omega(\eta_1\wedge\ldots\wedge[\xi,\eta_i]\wedge\ldots\wedge\eta_n).$$

Let ξ and ξ' be local vector fields on U. Then

$$\begin{aligned}([L_\xi, L_{\xi'}]\omega)(\eta_1\wedge\ldots\wedge\eta_n) &= \xi((L_{\xi'}\omega)(\eta_1\wedge\ldots\wedge\eta_n))\\ &\quad - \xi'((L_\xi\omega)(\eta_1\wedge\ldots\wedge\eta_n)) - \sum_{i=1}^{n}(L_{\xi'}\omega)(\eta_1\wedge\ldots\wedge[\xi,\eta_i]\wedge\ldots\wedge\eta_n)\\ &\quad + \sum_{i=1}^{n}(L_\xi\omega)(\eta_1\wedge\ldots\wedge[\xi',\eta_i]\wedge\ldots\wedge\eta_n) = [\xi,\xi'](\omega(\eta_1\wedge\ldots\wedge\eta_n))\\ &\quad + \sum_{i=1}^{n}\omega(\eta_1\wedge\ldots\wedge[\xi',[\xi,\eta_i]]\wedge\ldots\wedge\eta_n)\\ &\quad - \sum_{i=1}^{n}\omega(\eta_1\wedge\ldots\wedge[\xi,[\xi',\eta_i]]\wedge\ldots\wedge\eta_n)\end{aligned}$$

$$= [\xi, \xi'](\omega(\eta_1 \wedge \ldots \wedge \eta_n)) - \sum_{i=1}^{n} \omega(\eta_1 \wedge \ldots \wedge [[\xi, \xi'], \eta_i] \wedge \ldots \wedge \eta_n)$$
$$= (L_{[\xi,\xi']}\omega)(\eta_1 \wedge \ldots \wedge \eta_n),$$

showing that this is an action of a sheaf of Lie algebras.

Also, for a regular function f on U an a vector field ξ we have

$$L_\xi(f\omega)(\eta_1 \wedge \ldots \wedge \eta_n) = \xi(f\omega(\eta_1 \wedge \ldots \wedge \eta_n))$$
$$- \sum_{i=1}^{n} f\omega(\eta_1 \wedge \ldots \wedge [\xi, \eta_i] \wedge \ldots \wedge \eta_n) = (\xi(f)\omega + fL_\xi(\omega))(\eta_1 \wedge \ldots \wedge \eta_n).$$

This immediately implies that $[L_\xi, f] = \xi(f)$, i. e. L_ξ is a first order differential operator on ω_X.

Taking a small U, we can assume that $\mathcal{T}_U$ is a free $\mathcal{O}_U$-module, i. e. we can find local vector fields $\eta_1, \eta_2, \ldots, \eta_n$ on U which form a $\mathcal{O}_U$-basis of $\mathcal{T}_U$. Then we can represent ξ as $\xi = \sum_{i=1}^{n} g_i\eta_i$ for some $g_i \in \mathcal{O}_U$. This implies that

$$(L_{f\xi}\omega)(\eta_1 \wedge \ldots \wedge \eta_n) = f\xi(\omega(\eta_1 \wedge \ldots \wedge \eta_n))$$
$$- \sum_{i=1}^{n} \omega(\eta_1 \wedge \ldots \wedge [f\xi, \eta_i] \wedge \ldots \wedge \eta_n)$$
$$= f\xi(\omega(\eta_1 \wedge \ldots \wedge \eta_n)) - \sum_{i=1}^{n} f\omega(\eta_1 \wedge \cdots \wedge [\xi, \eta_i] \wedge \ldots \wedge \eta_n)$$
$$+ \sum_{i=1}^{n} \omega(\eta_1 \wedge \ldots \wedge \eta_i(f)\xi \wedge \ldots \wedge \eta_n) = fL_\xi(\omega)(\eta_1 \wedge \ldots \wedge \eta_n)$$
$$+ \sum_{i=1}^{n} \eta_i(f)g_i\omega(\eta_1 \wedge \ldots \wedge \eta_n) = (fL_\xi(\omega) + \xi(f)\omega)(\eta_1 \wedge \ldots \wedge \eta_n)$$
$$= L_\xi(f\omega)(\eta_1 \wedge \eta_2 \wedge \ldots \wedge \eta_n),$$

i. e. $L_{f\xi} = L_\xi f$. This implies that the map $\lambda : \xi \mapsto -L_\xi$ is an $\mathcal{O}_X$-module morphism from $\mathcal{T}_X$ into $\mathcal{D}_{\omega_X}$, considered as an $\mathcal{O}_X$-module for the right multiplication. It has the property that

$$\lambda([\xi, \eta]) = [\lambda(\eta), \lambda(\xi)]$$

for $\xi, \eta \in \mathcal{T}_X$. Therefore, it extends, by (D.O.13), to a morphism of the sheaf of k-algebras $\mathcal{D}_X$ into $\mathcal{D}^\circ_{\omega_X}$, which is the identity on $\mathcal{O}_X$. By 1, this implies that it is actually an isomorphism.

Hence, we have the following result.

Lemma 12. *Let X be a smooth algebraic variety of dimension n. Let ω_X be the invertible $\mathcal{O}_X$-module of differential n-forms on X. Then the pair $(\mathcal{D}^\circ_X, i_X)$ is naturally isomorphic to $(\mathcal{D}_{\omega_X}, i_{\omega_X})$.*

This result immediately implies 11. Therefore, we can calculate the isomorphism class of $\mathcal{D}^\circ$.

Proposition 13. *Let X be a smooth algebraic variety and $\mathcal{D}$ a twisted sheaf of differential operators on X. Then*

$$t(\mathcal{D}^\circ) = -t(\mathcal{D}) + H^1(d\log)(i(\omega_X)).$$

Proof. Let $\mathcal{U} = (U_i; 1 \leq i \leq n)$ be an open cover of X, and $\psi_i : (\mathcal{D}, i)|U_i \longrightarrow (\mathcal{D}_{U_i}, i_{U_i})$ corresponding isomorphisms. As before, for $1 \leq j < k \leq n$, denote by ϕ_{jk} the automorphisms of $(\mathcal{D}_{U_j \cap U_k}, i_{U_j \cap U_k})$ such that $\psi_j = \phi_{jk} \circ \psi_k$. Let ω_{jk} be the closed 1-form determined by ϕ_{jk} by 2, i. e. such that $\phi_{jk}(\xi) = \xi - \omega_{jk}(\xi)$ for any local vector field ξ on $U_j \cap U_k$. Then ψ_i, $1 \leq i \leq n$, are also isomorphisms of $(\mathcal{D}^\circ, i)|U_i$ onto $(\mathcal{D}_X^\circ, i_X)|U_i$. The composition with the map λ which we introduced in the proof of 11. gives us isomorphisms $\tau_i : (\mathcal{D}^\circ, i)|U_i \longrightarrow (\mathcal{D}_{\omega_X}, i_{\omega_X})|U_i$, $1 \leq i \leq n$. Also, the automorphisms ϕ_{jk} define, by $\sigma_{jk} = \lambda \circ \phi_{jk} \circ \lambda^{-1}$, automorphisms of $(\mathcal{D}_{\omega_X}, i_{\omega_X})|U_j \cap U_k$ such that $\tau_j = \sigma_{jk} \circ \tau_k$, $1 \leq j < k \leq n$. Evidently, σ_{jk} is determined by

$$\sigma_{jk}(\lambda(\xi)) = \lambda(\phi_{jk}(\xi)) = \lambda(\xi - \omega_{jk}(\xi)) = \lambda(\xi) - \omega_{jk}(\xi)$$

for any local vector field ξ on $U_j \cap U_k$. We can assume that, the open sets U_i are so small that there exist $\mathcal{O}_{U_i}$-module isomorphisms $\alpha_i : \omega_{U_i} \longrightarrow O_{U_i}$, $1 \leq i \leq n$. Then they define isomorphisms $\beta_i : (\mathcal{D}_{\omega_{U_i}}, i_{\omega_{U_i}}) \longrightarrow (\mathcal{D}_{U_i}, i_{U_i})$, $1 \leq i \leq n$, by $\beta_i(\eta) = \alpha_i \circ \eta \circ \alpha_i^{-1}$ for any $\eta \in \mathcal{D}_{\omega_{U_i}}$. Also, as in the proof of 4, we put $s_{jk} = \alpha_j(\alpha_k^{-1}(1))$ for all $1 \leq j < k \leq n$. The composition $\gamma_i = \beta_i \circ \tau_i$ is an isomorphism of $(\mathcal{D}^\circ, i)|U_i$ onto $(\mathcal{D}_{U_i}, i_{U_i})$ for $1 \leq i \leq n$. The automorphisms $\delta_{jk} = \beta_j \circ \sigma_{jk} \circ \beta_k^{-1}$, $1 \leq j < k \leq n$, satisfy

$$\gamma_j = \beta_j \circ \tau_j = \beta_j \circ \sigma_{jk} \circ \tau_k = \delta_j k \circ \gamma_k.$$

Let ξ be a local vector field on $U_j \cap U_k$. Then, for $f \in \mathcal{O}_X$,

$$\begin{aligned}\beta_i(\lambda(\xi))(f) &= \alpha_i(\lambda(\xi)\alpha_i^{-1}(f)) = \alpha_i(\lambda(\xi)(f\alpha_i^{-1}(1))) \\ &= -\alpha_i(L_\xi(f\alpha_i^{-1}(1))) = -\xi(f) + \beta_i(\lambda(\xi))(1),\end{aligned}$$

hence, we have

$$\beta_i^{-1}(\xi) = -\lambda(\xi) + \beta_i(\lambda(\xi))(1).$$

This leads to

$$\begin{aligned}\delta_{jk}(\xi) &= \beta_j(\sigma_{jk}(\beta_k^{-1}(\xi)) = -\beta_j(\sigma_{jk}(\lambda(\xi))) + \beta_k(\lambda(\xi))(1) \\ &= -\beta_j(\lambda(\phi_{jk}(\xi))) + \beta_k(\lambda(\xi))(1) = -\beta_j(\lambda(\xi)) + \omega_{jk}(\xi) + \beta_k(\lambda(\xi))(1) \\ &= \xi + \omega_{jk}(\xi) - \beta_j(\lambda(\xi))(1) + \beta_k(\lambda(\xi))(1).\end{aligned}$$

As in the proof of 4, we see that $\beta_j(\lambda(\xi)) = s_{jk}\beta_k(\lambda(\xi))s_{jk}^{-1}$, what implies that

$$\beta_j(\lambda(\xi))(1) = s_{jk}\beta_k(\lambda(\xi))(s_{jk}^{-1}) = -s_{jk}\xi(s_{jk}^{-1}) + \beta_k(\lambda(\xi))(1),$$

and finally

$$\begin{aligned}\delta_{jk}(\xi) &= \xi + \omega_{jk}(\xi) - \beta_j(\lambda(\xi))(1) + \beta_k(\lambda(\xi))(1) \\ &= \xi + \omega_{jk}(\xi) - s_{jk}^{-1}\, ds_{jk}(\xi) = \xi - (-\omega_{jk} + s_{jk}^{-1}\, ds_{jk})(\xi). \square\end{aligned}$$

1.2 Homogeneous Twisted Sheaves of Differential Operators

Let G be a connected algebraic group over an algebraically closed field k of characteristic zero and X its homogeneous space. By differentiation of the action of G on the structure sheaf $\mathcal{O}_X$ of X we get an algebra homomorphism $\tau : \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D}_X)$. Clearly, this map is G-equivariant.

Let $\mathcal{D}$ be a twisted sheaf of differential operators on X with an algebraic action γ of G and a morphism of algebras $\alpha : \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D})$ such that

(i) the multiplication in $\mathcal{D}$ is G-equivariant;

(ii) the differential of the G-action on $\mathcal{D}$ agrees with the action $T \longrightarrow [\alpha(\xi), T]$ for $\xi \in \mathfrak{g}$ and $T \in \mathcal{D}$.

(iii) the map $\alpha : \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D})$ is a morphism of G-modules.

Then the triple $(\mathcal{D}, \gamma, \alpha)$ is called a *homogeneous twisted sheaf of differential operators* on X. In this section we shall classify all homogeneous twisted sheaves of differential operators on X.

Clearly, $\mathcal{D}_X$ with the natural action of G and the homomorphism τ defines a homogeneous twisted sheaf of differential operators on X.

On the sheaf $\mathcal{U}^\circ = \mathcal{O}_X \otimes_k \mathcal{U}(\mathfrak{g})$ of vector spaces on X we can define a structure of the tensor product of $\mathcal{U}(\mathfrak{g})$-modules by putting

$$\xi(f \otimes \eta) = \tau(\xi) f \otimes \eta + f \otimes \xi\eta,$$

for $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $f \in \mathcal{O}_X$. On the other hand, $\mathcal{U}^\circ = \mathcal{O}_X \otimes_k \mathcal{U}(\mathfrak{g})$ has a structure of an $\mathcal{O}_X$-module, by multiplication on the first factor. Moreover,

$$[\xi, g](f \otimes h) = \xi(gf \otimes \eta) - g\xi(f \otimes \eta) = \tau(\xi)(g) f \otimes \eta = [\tau(\xi), g] f \otimes \eta$$

for $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $f, g \in \mathcal{O}_X$. This implies that $\mathcal{U}(\mathfrak{g})$ acts by differential operators on $\mathcal{U}^\circ$, and the corresponding homomorphism Ψ of $\mathcal{U}(\mathfrak{g})$ into the ring of differential operators $\mathrm{Diff}(\mathcal{U}^\circ)$ on $\mathcal{U}^\circ$ is compatible with the filtrations by degree. We can extend Ψ to a $\mathcal{O}_X$-module morphism of $\mathcal{O}_X \otimes_k \mathcal{U}(\mathfrak{g})$ into $\mathrm{Diff}(\mathcal{U}^\circ)$ which attaches to $f \otimes \xi$, $f \in \mathcal{O}_X$ and $\xi \in \mathcal{U}(\mathfrak{g})$, the differential operator $f\Psi(\xi) \in \mathrm{Diff}(\mathcal{U}^\circ)$. For $f \in \mathcal{O}_X$ and $\xi \in \mathcal{U}(\mathfrak{g})$ we have

$$\Psi(f \otimes \xi)(1 \otimes 1) = f\Psi(\xi)(1 \otimes 1) = f \otimes \xi,$$

what implies that $\Psi : \mathcal{O}_X \otimes_k \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{Diff}(\mathcal{U}^\circ)$ is injective. We claim that its image is a sheaf of subrings of $\mathrm{Diff}(\mathcal{U}^\circ)$. Clearly, it is an $\mathcal{O}_X$-module for the left multiplication and a right $\Psi(\mathcal{U}(\mathfrak{g}))$-module for the right multiplication.

Therefore, it remains to show that for any $f \in \mathcal{O}_X$, $\xi \in \mathfrak{g}$, the differential operator $\Psi(\xi)f$ is in the image of Ψ. On the other hand,

$$\Psi(\xi)f = [\Psi(\xi), f] + f\Psi(\xi) = \tau(\xi)f + f\Psi(\xi)$$

and the last expression is evidently in $\Psi(\mathcal{O}_X \otimes_k \mathcal{U}(\mathfrak{g}))$. This implies that $\mathcal{U}^\circ$ has a natural structure of a sheaf of rings such that $\Psi : \mathcal{U}^\circ \longrightarrow \mathrm{Diff}(\mathcal{U}^\circ)$ is a homomorphism. Moreover, the multiplication is given by

$$(f \otimes \xi)(g \otimes \eta) = f\tau(\xi)g \otimes \eta + fg \otimes \xi\eta$$

for any $f, g \in \mathcal{O}_X$, $\xi \in \mathfrak{g}$ and $\eta \in \mathcal{U}(\mathfrak{g})$. From this it follows that τ extends to a homomorphism of the sheaf of rings $\mathcal{U}^\circ$ into $\mathcal{D}_X$. Let $\mathfrak{g}^\circ = \mathcal{O}_X \otimes_k \mathfrak{g}$, considered as $\mathcal{O}_X$-submodule of $\mathcal{U}^\circ$. Then

$$[f \otimes \xi, g \otimes \eta] = f\tau(\xi)g \otimes \eta - g\tau(\eta)f \otimes \xi + fg \otimes [\xi, \eta]$$

for any $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$; what implies that $\mathfrak{g}^\circ$ is a sheaf of Lie algebras with this operation. By this calculation, we see that τ defines a homomorphism of $\mathfrak{g}^\circ$ into the sheaf of local vector fields $\mathcal{T}_X$ on X, which we denote by τ too.

Lemma 1. *The morphism* $\tau : \mathfrak{g}^\circ \longrightarrow \mathcal{T}_X$ *is an epimorphism.*

Proof. Both $\mathcal{O}_X$-modules $\mathfrak{g}^\circ$ and $\mathcal{T}_X$ are locally free, hence the statement follows from the fact that the linear map $T_x(\tau)$ the morphism τ induces on geometric fibres of $\mathfrak{g}^\circ$ and $\mathcal{T}_X$ at any $x \in X$ is surjective. □

We can define an increasing filtration on $\mathcal{U}^\circ$ by putting

$$\mathrm{F}_p\mathcal{U}^\circ = \mathcal{O}_X \otimes_k \mathrm{F}_p\mathcal{U}(\mathfrak{g}) \text{ for any } p \in \mathbb{Z}_+,$$

where $\mathrm{F}\,\mathcal{U}(\mathfrak{g})$ is the standard filtration of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. Clearly, this filtration is compatible with the algebra structure on $\mathcal{U}^\circ$ and with the homomorphism $\tau : \mathcal{U}^\circ \longrightarrow \mathcal{D}_X$. Also,

$$\mathrm{F}_0\mathcal{U}^\circ = \mathcal{O}_X \ , \ \mathrm{F}_1\mathcal{U}^\circ = \mathcal{O}_X \oplus \mathfrak{g}^\circ,$$

and $\mathrm{F}_1\mathcal{U}^\circ$ generates $\mathcal{U}^\circ$ as a sheaf of algebras.

Denote by $\mathfrak{b}^\circ$ the kernel of $\tau : \mathfrak{g}^\circ \longrightarrow \mathcal{T}_X$. Then $\mathfrak{b}^\circ$ is a sheaf of ideals in $\mathfrak{g}^\circ$. Moreover, if $\sum f_i \otimes \xi_i \in \mathfrak{b}^\circ$ and $g \in \mathcal{O}_X$, we have

$$\left[\sum f_i \otimes \xi_i, g \otimes 1\right] = \sum f_i\tau(\xi_i)g \otimes 1 = 0;$$

and this implies that $\mathcal{J}_0 = \mathfrak{b}^\circ\mathcal{U}^\circ$ is a sheaf of two-sided ideals in $\mathcal{U}^\circ$.

Proposition 2. *(i) The morphism* $\tau : \mathcal{U}^\circ \longrightarrow \mathcal{D}_X$ *is an epimorphism.*
(ii) The kernel of $\tau : \mathcal{U}^\circ \longrightarrow \mathcal{D}_X$ *is the sheaf of ideals* $\mathcal{J}_0$.

Proof. (i) Follows from 1. and the fact that $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and $\mathcal{T}_X$.

(ii) Clearly, $\mathcal{J}_0 = \mathfrak{b}^\circ \mathcal{U}^\circ$ is contained in the kernel of τ. Also, for any $x \in X$, the geometric fibre $T_x(\mathcal{J}_0) = \mathfrak{b}_x \mathcal{U}(\mathfrak{g})$ is the kernel of the linear map $T_x(\tau)$ from the geometric fibre $T_x(\mathcal{U}^\circ) = \mathcal{U}(\mathfrak{g})$ of $\mathcal{U}^\circ$ into the geometric fibre of $\mathcal{D}_X$ at x. □

Now we want to prove an analogous result for homogeneous twisted sheaves of differential operators on X.

Let $(\mathcal{D}, \gamma, \alpha)$ be a homogeneous twisted sheaf of differential operators on X. Then, by (ii), for any $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_X$,

$$[\alpha(\xi), f] = [\tau(\xi), f] = \tau(\xi) f.$$

In particular, we see that $[[\alpha(\xi), f], g] = 0$ for arbitrary $f, g \in \mathcal{O}_X$, hence $\alpha(\xi)$ is of degree ≤ 1 for any $\xi \in \mathfrak{g}$. We define a map $\mathcal{U}^\circ \longrightarrow \mathcal{D}$ by $f \otimes T \longmapsto f\alpha(T)$ for $f \in \mathcal{O}_X$ and $T \in \mathcal{U}(\mathfrak{g})$, and by abuse of notation, denote it by α again. Then

$$\begin{aligned}\alpha((f \otimes \xi)(g \otimes \eta)) &= \alpha(f\tau(\xi)g \otimes \eta + fg \otimes \xi\eta) = f\tau(\xi)(g)\,\alpha(\eta) + fg\alpha(\xi\eta) \\ &= f[\alpha(\xi), g]\alpha(\eta) + fg\alpha(\xi)\alpha(\eta) = f\alpha(\xi)g\alpha(\eta) \\ &= \alpha(f \otimes \xi)\alpha(g \otimes \eta),\end{aligned}$$

for any $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$. Therefore, α extends to a morphism of sheaves of rings which is compatible with the natural filtrations. Since $\operatorname{Gr} \mathcal{D} = S(\mathcal{T}_X) = \operatorname{Gr} \mathcal{D}_X$ and it is generated by $\operatorname{Gr}_1 \mathcal{D}$ as an $\mathcal{O}_X$-algebra, we immediately conclude that $\operatorname{Gr} \alpha = \operatorname{Gr} \tau$ and $\operatorname{Gr} \alpha : \operatorname{Gr} \mathcal{U}^\circ \longrightarrow \operatorname{Gr} \mathcal{D}$ is an epimorphism of sheaves of rings. This implies that $\alpha(\mathfrak{b}^\circ) \subset \mathcal{O}_X$, i. e. α defines a G-equivariant morphisms σ of the G-homogeneous $\mathcal{O}_X$-module $\mathfrak{b}^\circ$ into $\mathcal{O}_X$.

Fix a base point $x_0 \in X$. Its stabilizer B_0 acts in on the dual space $\mathfrak{b}_0^*$ of $\mathfrak{b}_0$. Denote by $I(\mathfrak{b}_0^*)$ the subspace of B_0-invariants in $\mathfrak{b}_0^*$. Then we have the natural linear isomorphism between $I(\mathfrak{b}_0^*)$ and the space of all G-equivariant morphisms σ of the G-homogeneous $\mathcal{O}_X$-module $\mathfrak{b}^\circ$ into $\mathcal{O}_X$. Therefore, $(\mathcal{D}, \gamma, \alpha)$ determines an element of $I(\mathfrak{b}_0^*)$.

To each $\lambda \in I(\mathfrak{b}_0^*)$ we can associate a G-equivariant morphism σ_λ of the G-homogeneous $\mathcal{O}_X$-module $\mathfrak{b}^\circ$ into $\mathcal{O}_X$. Let $\varphi_\lambda : \mathfrak{b}^\circ \longrightarrow \mathcal{U}^\circ$ given by $\varphi_\lambda(s) = s - \sigma_\lambda(s)$, $s \in \mathfrak{b}^\circ$. Then $\operatorname{im} \varphi_\lambda$ generates a sheaf of two-sided ideals $\mathcal{J}_\lambda$ in $\mathcal{U}^\circ$. We put

$$\mathcal{D}_{X,\lambda} = \mathcal{U}^\circ / \mathcal{J}_\lambda.$$

This is a sheaf of algebras on X.

Proposition 3. *The sheaf of algebras $\mathcal{D}_{X,\lambda}$ is a twisted sheaf of differential operators on X.*

We say that $\mathcal{D}_{X,\lambda}$ is the homogeneous twisted sheaf of differential operators on X associated to λ.

As a consequence of the preceding discussion and 3, we have the following result.

Theorem 4. *The map $\lambda \longmapsto \mathcal{D}_{X,\lambda}$ is an isomorphism of $I(\mathfrak{h}_0^*)$ onto the set of isomorphism classes of homogeneous twisted sheaves of differential operators on X.*

Proof. Let $(\mathcal{D}, \gamma, \alpha)$ be a homogeneous twisted sheaf of differential operators on X. Then, by the preceding discussion it determines a unique $\lambda \in I(\mathfrak{h}_0^*)$. Moreover, $\mathcal{J}_\lambda$ is in the kernel of the homomorphism $\alpha : \mathcal{U}^\circ \longrightarrow \mathcal{D}$. This implies that α induces a homomorphism $\beta : \mathcal{D}_{X,\lambda} \longrightarrow \mathcal{D}$ of sheaves of rings which is compatible with the filtrations of $\mathcal{D}_{X,\lambda}$ and $\mathcal{D}$, and with the natural maps of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_{X,\lambda})$ and $\Gamma(X, \mathcal{D})$ respectively. Also, $\operatorname{Gr} \beta$ is an isomorphism of graded sheaves of rings. This implies that β is an isomorphism too. □

To prove 3, by 2. and the G-homogeneity, it is enough to find a neighborhood U of the base point x_0 and a local automorphism Ψ_λ of $\mathcal{U}^\circ|U$ such that $\Psi_\lambda|\mathcal{O}_U = 1$ and $\Psi_\lambda(\mathcal{J}_0|U) = \mathcal{J}_\lambda|U$.

Let U be an open set in X. Now we want to describe some automorphisms ρ of $\mathcal{U}^\circ|U$ with the following properties:

(i) $\rho(f) = f$ for any $f \in \mathcal{O}_U$,

(ii) $\operatorname{Gr} \rho$ is the identity.

Clearly, ρ is completely determined by its values on $1 \otimes \xi$, $\xi \in \mathfrak{g}$. Moreover, (ii) implies that $\rho(1 \otimes \xi) = 1 \otimes \xi - \omega(\xi) \otimes 1$ where $\omega(\xi) \in \mathcal{O}(U)$. By (i) we also have

$$\rho(f \otimes \xi) = f \otimes \xi - f\omega(\xi) \otimes 1$$

for any $f \in \mathcal{O}_U$ and $\xi \in \mathfrak{g}$. To be an automorphism, ρ has to satisfy also

$$\rho([1 \otimes \xi, 1 \otimes \eta]) = [\rho(1 \otimes \xi), \rho(1 \otimes \eta)] = 1 \otimes [\xi, \eta] - \tau(\xi)\omega(\eta) \otimes 1 + \tau(\eta)\omega(\xi) \otimes 1,$$

i. e.

$$\omega([\xi, \eta]) = \tau(\xi)\omega(\eta) - \tau(\eta)\omega(\xi) \tag{1}$$

for any $\xi, \eta \in \mathfrak{g}$. Therefore, ω is a linear map from $\mathfrak{g}$ into $\mathcal{O}(U)$ which is annihilated by the differential of the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $\mathcal{O}(U)$. Moreover, we can extend ω to an $\mathcal{O}_U$-module morphism of $\mathfrak{g}^\circ|U$ into $\mathcal{O}_U$ given by

$$\omega(f \otimes \xi) = f\omega(\xi) \text{ for } f \in \mathcal{O}_U \text{ and } \xi \in \mathfrak{g}.$$

The relation (1) implies that

$$\begin{aligned} \omega([f \otimes \xi, g \otimes \eta]) &= fg\omega([\xi, \eta]) + f\tau(\xi)(g)\omega(\eta) - g\tau(\eta)(f)\omega(\xi) \\ &= f\tau(\xi)(g\omega(\eta)) - g\tau(\eta)(f\omega(\xi)) \\ &= \tau(f \otimes \xi)(\omega(g \otimes \eta)) - \tau(g \otimes \eta)(\omega(f \otimes \xi)); \end{aligned}$$

i. e. for any two sections $s, s' \in \mathfrak{g}^\circ|U$ we have

$$\omega([s, s']) = \tau(s)(\omega(s')) - \tau(s')(\omega(s)). \tag{2}$$

Also, we remark that ω is local, i. e. if $s \in \mathfrak{g}^\circ$ is such that $s(x) = 0$ for some $x \in U$ it follows that $\omega(s)(x) = 0$. Moreover, by (2), for $s \in \mathfrak{b}^\circ$ and $s' \in \mathfrak{g}^\circ|U$, we have

$$\omega([s', s]) = \tau(s')(\omega(s)),$$

what implies that the map ω from $\mathfrak{b}^\circ|U$ into $\mathcal{O}_U$ is $\mathfrak{g}^\circ|U$-module morphism.

We shall need the following result.

Lemma 5. *Let $\zeta : \mathfrak{b}^\circ|U \longrightarrow \mathcal{O}_U$ be a local $\mathfrak{g}^\circ|U$-module morphism and $x \in U$. If $\zeta(s)(x) = 0$ for any $s \in \mathfrak{b}^\circ$, there exists a neighborhood $V \subset U$ of x such that $\zeta|V = 0$.*

Proof. Let $\xi \in \mathfrak{g}$. Then,

$$\big(\tau(\xi)(\zeta(s))\big)(x) = \zeta([1 \otimes \xi, s])(x) = 0.$$

It follows that all derivatives of $\zeta(s)$ at x vanish, hence the germ of $\zeta(s)$ at x is zero. By the coherence of $\mathfrak{b}^\circ$ we see that ζ vanishes in a neighborhood of x. □

On the other hand, if we have a linear map $\omega : \mathfrak{g} \longrightarrow \mathcal{O}(U)$ satisfying the relation (1), it defines an automorphism ρ_ω of $\mathcal{U}^\circ|U$, which satisfies (i) and (ii), by

$$\rho_\omega(f \otimes \xi) = f \otimes \xi - f\omega(\xi) \otimes 1$$

for any $f \in \mathcal{O}_U$ and $\xi \in \mathfrak{g}$.

Clearly, all such ω form a vector space.

Now we want to construct some maps ω satisfying the above properties.

(I) Let χ be a character of B_0. Let s be a section of the homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\chi)$ over U. For $\xi \in \mathfrak{g}$ we put

$$\xi s = \omega(\xi)s.$$

Then

$$\begin{aligned}\omega([\xi,\eta])s &= [\xi,\eta]s = \xi(\eta s) - \eta(\xi s) = \xi(\omega(\eta)s) - \eta(\omega(\xi)s)\\ &= \tau(\xi)(\omega(\eta))s - \tau(\eta)(\omega(\xi))s + \omega(\eta)\xi s - \omega(\xi)\eta s\\ &= \big(\tau(\xi)(\omega(\eta)) - \tau(\eta)(\omega(\xi))\big)s,\end{aligned}$$

i. e. ω satisfies our conditions.

(II) Let

$$0 \longrightarrow k \longrightarrow V \longrightarrow k \longrightarrow 0$$

be an exact sequence of algebraic representations of B_0, where B_0 acts trivially on k. Let

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

be the corresponding exact sequence of G-homogeneous locally free $\mathcal{O}_X$-modules. Let s' be the section of $\mathcal{V}$ which is the image of the section 1 of $\mathcal{O}_X$, and s'' a local section of $\mathcal{V}$ such that its germ at x_0 maps into the germ of 1. Then there is a neighborhood U of x_0 such that s' and s'' form a basis of $\mathcal{V}|U$ as an $\mathcal{O}_U$-module, and

$$\xi s'' = \omega(\xi)s' \text{ and } \xi s' = 0$$

for any $\xi \in \mathfrak{g}$. Then

$$\omega([\xi,\eta])s' = [\xi,\eta]s'' = \xi(\omega(\eta)s') - \eta(\omega(\xi)s') = \big(\tau(\xi)(\omega(\eta)) - \tau(\eta)(\omega(\xi))\big)s'.$$

Hence, ω again has the required property.

Now, we want to prove 3. First, any B_0-invariant linear form λ on $\mathfrak{b}_0$ vanishes on $[\mathfrak{b}_0, \mathfrak{b}_0]$. Let C_0 be the identity component of the commutator subgroup of B_0. Then C_0 is a closed normal subgroup of B_0. The quotient group $D_0 = B_0/C_0$ is an algebraic group with commutative identity component, and Lie algebra $\mathfrak{d}_0 = \mathfrak{b}_0/[\mathfrak{b}_0, \mathfrak{b}_0]$. Let $D_0 = L_0U_0$ be a Levi decomposition of D_0. Then U_0 is an abelian unipotent subgroup, and the identity component of L_0 is the torus consisting of all semisimple elements in the identity component of D_0. Therefore, we have a direct sum decomposition $\mathfrak{d}_0 = \mathfrak{l}_0 \oplus \mathfrak{u}_0$, and both summands are D_0-invariant. This implies that the D_0-invariant linear form μ on $\mathfrak{d}_0$, defined by λ, can be written as a sum of two D_0-invariant linear forms μ_1 and μ_2 which vanish on $\mathfrak{l}_0$, resp. $\mathfrak{u}_0$. By composing these linear forms with the projection of $\mathfrak{b}_0$ onto $\mathfrak{d}_0$ we get the decomposition of λ into the sum of λ_1 and λ_2. We can define a representation of U_0 on k^2 such that $u \in U_0$ acts via the matrix

$$\begin{bmatrix} 1 & \mu_1(\log u) \\ 0 & 1 \end{bmatrix};$$

evidently it extends to a representation of D_0 in which L_0 acts trivially. Moreover, we can interpret it as a representation of B_0. Applying the construction from (II) we construct in a neighborhood U of x_0 a linear map ω_1 from $\mathfrak{g}$ into $\mathcal{O}(U)$ which satisfies (1) and such that $\omega_1|\mathfrak{b}_0 = \lambda_1$.

On the other hand, any linear form on $\mathfrak{l}_0$ is a linear combination of differentials of characters of the identity component of L_0. By averaging, using the component group of L_0, we conclude that every L_0-invariant linear form on $\mathfrak{l}_0$ is a linear combination of L_0-invariant characters, i. e. of differentials of one-dimensional representations of L_0. Applying the construction from (I) we get in a neighborhood U of x_0 a linear map ω_2 from $\mathfrak{g}$ into $\mathcal{O}(U)$ which satisfies (1) and such that $\omega_2|\mathfrak{b}_0 = \lambda_2$. Therefore, we get in a neighborhood U of x_0, a linear map ω from $\mathfrak{g}$ into $\mathcal{O}(U)$ which satisfies (1) and such that $\omega|\mathfrak{b}_0 = \lambda$. The corresponding $\mathfrak{g}^\circ|U$-morphism $\omega : \mathfrak{b}^\circ|U \longrightarrow \mathcal{O}_U$ agrees, by 5, with σ_λ on some smaller neighborhood V of x_0. This in turn implies that ρ_ω is an automorphism of $\mathcal{U}^\circ|V$ such that $\rho_\omega|\mathcal{O}_V = 1$ and $\rho_\omega(\mathcal{J}_0|V) = \mathcal{J}_\lambda|V$.

C. Cohomology of $\mathcal{D}_\lambda$-modules

C.1 Homogeneous Twisted Sheaves of Differential Operators on Flag Varieties

In this section we want to specialize our construction of homogeneous twisted sheaves of differential operators from ... to the case of a connected semisimple algebraic group G acting on its flag variety X.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the group of inner automorphisms of the Lie algebra $\mathfrak{g}$. Then the flag variety X of $\mathfrak{g}$ can be identified with the *variety of Borel subalgebras* of $\mathfrak{g}$. The group G acts naturally on the trivial vector bundle $X \times \mathfrak{g} \longrightarrow X$, and the tautological vector bundle $\mathcal{B}$ of Borel subalgebras is a homogeneous vector subbundle of it. We denote, for each $x \in X$, the corresponding Borel subalgebra of $\mathfrak{g}$ by $\mathfrak{b}_x$, and by $\mathfrak{n}_x$ the nilpotent radical of $\mathfrak{b}_x$. Hence, we have the homogeneous vector subbundle $\mathcal{N}$ of $\mathcal{B}$ of nilpotent radicals. Moreover, let B_x be the Borel subgroup of G corresponding to $\mathfrak{b}_x$. Then B_x is the stabilizer of x in G.

Let $\mathcal{H} = \mathcal{B}/\mathcal{N}$. Then $\mathcal{H}$ is a homogeneous vector bundle over X with the fiber $\mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$ over $x \in X$. The group B_x acts trivially on $\mathfrak{b}_x/\mathfrak{n}_x$, hence $\mathcal{H}$ is a trivial vector bundle over X with global sections $\mathfrak{h}$ naturally isomorphic to $\mathfrak{b}_x/\mathfrak{n}_x$ for any $x \in X$. We call the abelian Lie algebra $\mathfrak{h}$ the *Cartan algebra* for $\mathfrak{g}$.

Let $\mathcal{O}_X$ be the structure sheaf of the algebraic variety X. As in ..., let $\mathfrak{g}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ be the sheaf of local sections of the trivial bundle $X \times \mathfrak{g}$. Denote by $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ the corresponding subsheaves of local sections of $\mathcal{B}$ and $\mathcal{N}$, respectively. If we denote by τ the natural homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra of vector fields on X, we define a structure of a sheaf of complex Lie algebras on $\mathfrak{g}^\circ$ by putting

$$[f \otimes \xi, g \otimes \eta] = f\tau(\xi)g \otimes \eta - g\tau(\eta)f \otimes \xi + fg \otimes [\xi, \eta]$$

for $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$. If we extend τ to the natural homomorphism of $\mathfrak{g}^\circ$ into the sheaf of Lie algebras of local vector fields on X, $\ker \tau$ is exactly $\mathfrak{b}^\circ$. In addition, we have the following result.

Lemma 1. *(i) The sheaf $\mathfrak{b}^\circ$ is a sheaf of ideals in $\mathfrak{g}^\circ$. The commutator on $\mathfrak{b}^\circ$ is $\mathcal{O}_X$-linear.*

(ii) The sheaf $\mathfrak{n}^\circ$ is a sheaf of ideals in $\mathfrak{g}^\circ$.

Proof. The first assertion in (i) follows from the fact that $\mathfrak{b}^\circ = \ker\tau$. Moreover, if $\sum f_i \otimes \xi_i \in \mathfrak{b}^\circ$, $g \in \mathcal{O}_X$ and $\eta \in \mathfrak{g}$ we have

$$\begin{aligned}[\textstyle\sum f_i \otimes \xi_i, g \otimes \eta] &= \textstyle\sum f_i g \otimes [\xi_i, \eta] + \sum f_i \tau(\xi_i) g \otimes \eta - \sum g\tau(\eta) f_i \otimes \xi_i \\ &= g[\textstyle\sum f_i \otimes \xi_i, 1 \otimes \eta];\end{aligned}$$

this proves immediately the second assertion. Also, by this formula, to prove (ii) we need only to check that for any $\sum f_i \otimes \xi_i \in \mathfrak{n}^\circ$ and $\eta \in \mathfrak{g}$, the commutator $[\sum f_i \otimes \xi_i, 1 \otimes \eta]$ is in $\mathfrak{n}^\circ$. By the homogeneity, $g(\sum f_i \otimes \xi_i) \in \mathfrak{n}^\circ$ for any $g \in G$. By differentiation, this implies that

$$\textstyle\sum \tau(\eta) f_i \otimes \xi_i + \sum f_i \otimes [\eta, \xi_i] \in \mathfrak{n}^\circ$$

for any $\eta \in \mathfrak{g}$; and, by definition of the bracket, this expression is equal to $[1 \otimes \eta, \sum f_i \otimes \xi_i]$. □

The quotient sheaf $\mathfrak{h}^\circ = \mathfrak{b}^\circ/\mathfrak{n}^\circ$ is the sheaf of local sections of $\mathcal{H}$, and is therefore equal to the sheaf of abelian Lie algebras $\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{h}$.

Similarly, we defined in ... a multiplication in the sheaf $\mathcal{U}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ by

$$(f \otimes \xi)(g \otimes \eta) = f\tau(\xi) g \otimes \eta + fg \otimes \xi\eta$$

where $f, g \in \mathcal{O}_X$ and $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$. In this way $\mathcal{U}^\circ$ becomes a sheaf of complex associative algebras on X. Evidently, $\mathfrak{g}^\circ$ is a subsheaf of $\mathcal{U}^\circ$, and the natural commutator in $\mathcal{U}^\circ$ induces the bracket operation on $\mathfrak{g}^\circ$. It follows from 1. that the sheaf of right ideals $\mathfrak{n}^\circ\mathcal{U}^\circ$ generated by $\mathfrak{n}^\circ$ in $\mathcal{U}^\circ$ is a sheaf of two-sided ideals in $\mathcal{U}^\circ$. Therefore, the quotient $\mathcal{D}_{\mathfrak{h}} = \mathcal{U}^\circ/\mathfrak{n}^\circ\mathcal{U}^\circ$ is a sheaf of complex associative algebras on X.

The natural morphism of $\mathfrak{g}^\circ$ into $\mathcal{D}_{\mathfrak{h}}$ induces a morphism of the sheaf of Lie subalgebras $\mathfrak{h}^\circ$ into $\mathcal{D}_{\mathfrak{h}}$, hence there is a natural homomorphism ϕ of the enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ into the global sections of $\mathcal{D}_{\mathfrak{h}}$. The action of the group G on the structure sheaf $\mathcal{O}_X$ and $\mathcal{U}(\mathfrak{g})$ induces a natural G-action on $\mathcal{U}^\circ$ and $\mathcal{D}_{\mathfrak{h}}$. On the other hand, the triviality of $\mathcal{H}$ implies that the induced G-action on $\mathfrak{h}$ is trivial. It follows that ϕ maps $\mathcal{U}(\mathfrak{h})$ into the G-invariants of $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$.

Lemma 2. *(i) The natural morphism ϕ of $\mathcal{U}(\mathfrak{h})$ into the subalgebra of all G-invariants in $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$ is injective.*

(ii) The image of ϕ is in the center of $\mathcal{D}_{\mathfrak{h}}$.

Proof. (i) Let $x \in X$. Then the geometric fibre $T_x(\mathcal{D}_{\mathfrak{h}})$ of the $\mathcal{O}_X$-module $\mathcal{D}_{\mathfrak{h}}$ at x is equal to $\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$. The composition of ϕ with the evaluation of a section at x corresponds to the natural map

$$\mathcal{U}(\mathfrak{h}) \longrightarrow \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x\mathcal{U}(\mathfrak{b}_x) \longrightarrow \mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}),$$

which is injective by the Poincaré-Birkhoff-Witt theorem. Therefore, ϕ is injective.

(ii) Differentiating the G-action we see that elements of $\phi(\mathcal{U}(\mathfrak{h}))$ commute with the image of $\mathfrak{g}$ in $\mathcal{D}_\mathfrak{h}$. Since $\mathcal{D}_\mathfrak{h}$ is generated by $\mathcal{O}_X$ and the image of $\mathfrak{g}$, the assertion follows. □

Let $x \in X$ and $\mathfrak{b}_x$ the Borel subalgebra corresponding to x. Let $\bar{\mathfrak{n}}$ be the nilpotent radical of a Borel subalgebra opposite to $\mathfrak{b}_x$, and $\bar{N}$ the corresponding connected subgroup of G. Then, by Bruhat decomposition ([**Bo**], 14.11), the orbit map $\bar{N} \longrightarrow X$ defined by $\bar{n} \longrightarrow \bar{n}x$ is an isomorphism of the variety $\bar{N}$ onto an open neighborhood U of $x \in X$. Let $s : U \longrightarrow \bar{N}$ be the inverse map. Clearly, the inclusion of $\mathcal{U}(\bar{\mathfrak{n}})$ into $\mathcal{U}(\mathfrak{g})$ induces a injective morphism of the sheaf of algebras $\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})$ into $\mathcal{U}^\circ|U$. It follows that we have a natural morphism of the sheaf of algebras $\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})$ into $\mathcal{D}_\mathfrak{h}$. Moreover, if we consider the tensor product $(\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h})$ as a sheaf of algebras, by the previous discussion we have a natural morphism of sheaves of algebras ψ from $(\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h})$ into $\mathcal{D}_\mathfrak{h}|U$.

Lemma 3. *The morphism*

$$\psi : (\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h}) \longrightarrow \mathcal{D}_\mathfrak{h}|U$$

is an isomorphism of sheaves of algebras.

Proof. As in the proof 2.(i), we conclude that the composition of ψ with the evaluation map at $u \in U$ corresponds to the evaluation map of $(\mathcal{O}_U \otimes_\mathbb{C} \mathcal{U}(\bar{\mathfrak{n}})) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h})$ at u composed with the natural linear map of $\mathcal{U}(\bar{\mathfrak{n}}) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h})$ into $\mathcal{U}(\mathfrak{g})/\mathfrak{n}_u\mathcal{U}(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, the last map is an injection. This implies the injectivity of ψ. It remains to show its surjectivity. Clearly, $\mathcal{D}_\mathfrak{h}|U$ is generated, as a sheaf of algebras, by $\mathcal{O}_U$ and the image of $\mathfrak{g}$ in $\mathcal{D}_\mathfrak{h}|U$. On the other hand, as a vector space, $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{b}_u$ for any $u \in U$, hence we have a well-defined linear isomorphism $\zeta(u)$ of $\mathfrak{g}$ into itself, which is the identity on $\bar{\mathfrak{n}}$ and $\operatorname{Ad} s(u)$ on $\mathfrak{b}_x$. Therefore, any $\xi \in \mathfrak{g}$ determines a section $\zeta_\xi : u \longrightarrow \zeta(u)\xi$ of $\mathfrak{g}^\circ$ on U. It follows that $\mathcal{D}_\mathfrak{h}|U$ is generated by $\mathcal{O}_U$ and the images of the sections $u \longrightarrow \zeta(u)\xi$ in $\mathcal{D}_\mathfrak{h}|U$ for $\xi \in \mathfrak{g}$. But, if $\xi \in \bar{\mathfrak{n}}$, we have $\zeta(u)\xi = \xi$ for any $u \in U$, hence this section is in the image of ψ, and if $\xi \in \mathfrak{b}_x$, the corresponding section is in the image of ψ either. It follows that ψ is also surjective. □

In particular, if we view $\mathcal{D}_\mathfrak{h}$ as an $\mathcal{U}(\mathfrak{h})$-module, we have the following consequence.

Corollary 4. *The $\mathcal{U}(\mathfrak{h})$-module $\mathcal{D}_\mathfrak{h}$ is locally free.*

Also, we can improve 2.(i).

Lemma 5. *The natural morphism ϕ of $\mathcal{U}(\mathfrak{h})$ into the subalgebra of all G-invariants in $\Gamma(X, \mathcal{D}_\mathfrak{h})$ is an isomorphism.*

Proof. By 2.(i) we know that ϕ is injective. If s is a G-invariant global section of $\mathcal{D}_\mathfrak{h}$, its value at x must be B_x-invariant. This implies that, if we fix a

Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$, $s(x)$ must be of weight zero with respect to $\mathfrak{c}$ in $\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$. Therefore it is in the image of $\mathcal{U}(\mathfrak{h})$, i. e. there is a section t in $\phi(\mathcal{U}(\mathfrak{h}))$ such that $t - s$ is a G-invariant section which vanishes at x. By G-invariance, this implies that $t - s$ vanishes at any point of X. By 3, $\mathcal{D}_\mathfrak{h}$ is locally free as an $\mathcal{O}_X$-module for the left multiplication, hence this implies that $t - s = 0$, and $s = t \in \phi(\mathcal{U}(\mathfrak{h}))$. □

On the other hand, we have the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\mathcal{D}_\mathfrak{h}$, which induces a natural homomorphism of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_\mathfrak{h})$. Its image is contained in the subalgebra of G-invariants of $\Gamma(X, \mathcal{D}_\mathfrak{h})$, hence, by 5, it is in $\phi(\mathcal{U}(\mathfrak{h}))$. Finally, we have the canonical Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ ([**LG**], Ch. VIII, §6, no. 4), defined in the following way. First, for any $x \in X$, the center $\mathcal{Z}(\mathfrak{g})$ is contained in the sum of the subalgebra $\mathcal{U}(\mathfrak{b}_x)$ and the right ideal $\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. Therefore, we have the natural projection of $\mathcal{Z}(\mathfrak{g})$ into

$$\mathcal{U}(\mathfrak{b}_x)/(\mathfrak{n}_x\mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{b}_x)) = \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x\mathcal{U}(\mathfrak{b}_x) = \mathcal{U}(\mathfrak{h}_x).$$

Its composition with the natural isomorphism of $\mathcal{U}(\mathfrak{h}_x)$ with $\mathcal{U}(\mathfrak{h})$ is independent of x and, by definition, equal to γ.

Proposition 6. *The diagram*

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\gamma} & \mathcal{U}(\mathfrak{h}) \\ \| & & \downarrow{\scriptstyle\phi} \\ \mathcal{Z}(\mathfrak{g}) & \longrightarrow & \Gamma(X, \mathcal{D}_\mathfrak{h}) \end{array}$$

of natural algebra homomorphisms is commutative.

Proof. By 3, $\mathcal{D}_\mathfrak{h}$ is locally free as the $\mathcal{O}_X$-module for the left multiplication. Therefore it is enough to show that the compositions of $\phi \circ \gamma$ and the canonical homomorphism of $\mathcal{Z}(\mathfrak{g})$ into $\mathcal{D}_\mathfrak{h}$ with the evaluation map are equal for any $x \in X$. But this follows immediately from $T_x(\mathcal{D}_\mathfrak{h}) = \mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$. □

Let $x \in X$. Fix a Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$. Let R be the root system of $\mathfrak{g}$ in $\mathfrak{c}^*$ and

$$\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} \mid [\eta, \xi] = \alpha(\eta)\xi \text{ for } \eta \in \mathfrak{c}\}$$

the root subspace of $\mathfrak{g}$ determined by the root $\alpha \in R$. We define the ordering on R by choosing the set R^+ of positive roots by

$$R^+ = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_x\}.$$

Then the canonical isomorphism $\mathfrak{c} \longrightarrow \mathfrak{h}_x \longrightarrow \mathfrak{h}$ induces an isomorphism of the triple $(\mathfrak{c}^*, R, R^+)$ with the triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$, where Σ is a root system in $\mathfrak{h}^*$ and Σ^+ a set of positive roots in Σ. Clearly, Σ and Σ^+ are independent of the choice of $x \in X$. We call the triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ the *Cartan triple* of $\mathfrak{g}$; and the inverse isomorphism of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ onto $(\mathfrak{c}^*, R, R^+)$ a *specialization* at x.

Let W be the Weyl group of Σ. Denote by ρ the half-sum of all positive roots in Σ. The enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ is naturally isomorphic to the algebra of polynomials on $\mathfrak{h}^*$, and therefore any $\lambda \in \mathfrak{h}^*$ determines a homomorphism of $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. Let I_λ be the kernel of the homomorphism $\varphi_\lambda : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathbb{C}$ determined by $\lambda + \rho$. Then $\gamma^{-1}(I_\lambda)$ is a maximal ideal in $\mathcal{Z}(\mathfrak{g})$, and, by a result of Harish-Chandra ([**LG**], Ch. VIII, §8, Cor. 1 of Th. 2), for $\lambda, \mu \in \mathfrak{h}^*$,

$$\gamma^{-1}(I_\lambda) = \gamma^{-1}(I_\mu) \text{ if and only if } w\lambda = \mu \text{ for some } w \in W.$$

For any $\lambda \in \mathfrak{h}^*$, by 3, the sheaf $I_\lambda \mathcal{D}_\mathfrak{h}$ is a sheaf of two-sided ideals in $\mathcal{D}_\mathfrak{h}$; therefore $\mathcal{D}_\lambda = \mathcal{D}_\mathfrak{h}/I_\lambda \mathcal{D}_\mathfrak{h}$ is a sheaf of complex associative algebras on X. In the case when $\lambda = -\rho$, we have $I_{-\rho} = \mathfrak{h}\mathcal{U}(\mathfrak{h})$, hence $\mathcal{D}_{-\rho} = \mathcal{U}^\circ/\mathfrak{b}^\circ\mathcal{U}^\circ$, i. e. it is the sheaf of local differential operators on X. In general $\mathcal{D}_\lambda$, $\lambda \in \mathfrak{h}^*$, are homogeneous twisted sheaves of differential operators on X. This follows from ... or directly from 3. In the parametrization of twisted sheaves of differential operators which we used in ... we have

$$\mathcal{D}_\lambda = \mathcal{D}_{X,\lambda+\rho}, \quad \lambda \in \mathfrak{h}^*.$$

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ and $\lambda \in \theta$. Denote by $J_\theta = \gamma^{-1}(I_\lambda)$ the maximal ideal in $\mathcal{Z}(\mathfrak{g})$ determined by θ. We denote by χ_λ the homomorphism of $\mathcal{Z}(\mathfrak{g})$ into $\mathbb{C}$ with $\ker \chi_\lambda = J_\theta$. As we remarked before, χ_λ depends only on the Weyl group orbit θ of λ. The elements of J_θ map into the zero section of $\mathcal{D}_\lambda$. Therefore, we have a canonical morphism of $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/J_\theta\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_\lambda)$. We shall see in §6 that this morphism is actually an isomorphism.

The objects of the category $\mathcal{M}(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$-modules can also be viewed as $\mathcal{U}(\mathfrak{g})$-modules with infinitesimal character χ_λ.

The category $\mathcal{M}(\mathcal{D}_\lambda)$ of all $\mathcal{D}_\lambda$-modules has enough injective objects ([**Hartshorne**], III.2.2). Moreover, injective $\mathcal{D}_\lambda$-modules are flasque (ibid., III.2.4). This implies that the cohomology modules $H^i(X, \mathcal{V})$ of a $\mathcal{D}_\lambda$-module $\mathcal{V}$ have natural structures of $\Gamma(X, \mathcal{D}_\lambda)$-modules. In particular, by the previous remark, they can be viewed as $\mathcal{U}_\theta$-modules. It follows that we have a family of functors

$$H^i(X, -) : \mathcal{M}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta) \quad \text{for} \quad 0 \le i \le \dim X.$$

In next few sections we shall study their basic properties.

C.2 Translation Principle for $\mathcal{D}_\lambda$-modules

In this section we collect certain technical results we need to study the cohomology of $\mathcal{D}_\lambda$-modules.

Let $Q(\Sigma)$ be the root lattice in $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, we denote by W_λ the subgroup of the Weyl group W given by

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q(\Sigma)\}.$$

Let $\Sigma\check{}$ be the root system in $\mathfrak{h}$ dual to Σ; and for any $\alpha \in \Sigma$, we denote by $\alpha\check{} \in \Sigma\check{}$ the dual root of α. Then, by ([**LG**], Ch. VI, §2, Ex. 2), we know that W_λ is the Weyl group of the root system

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) \in \mathbb{Z}\}.$$

We define the order on Σ_λ by putting $\Sigma_\lambda^+ = \Sigma^+ \cap \Sigma_\lambda$. This defines a set of simple roots Π_λ of Σ_λ, and the corresponding set of simple reflections S_λ. Let ℓ_λ be the length function on (W_λ, S_λ). We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha\check{}(\lambda)$ is different from zero for any $\alpha \in \Sigma$ and that λ is *antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$. We put

$$n(\lambda) = \min\{\ell_\lambda(w) \mid w \in W_\lambda,\ w\lambda \text{ is antidominant}\}.$$

In particular, $n(\lambda) = 0$ is equivalent to λ being antidominant. Let $P(\Sigma)$ be the weight lattice in $\mathfrak{h}^*$. Clearly, $\mu \in P(\Sigma)$ determines naturally a homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\mu)$ on X. If $\mathcal{V}$ is a $\mathcal{D}_\lambda$-module on X, then its twist $\mathcal{V}(\mu) = \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{O}(\mu)$ by the invertible $\mathcal{O}_X$-module $\mathcal{O}(\mu)$ is a $\mathcal{D}_{\lambda+\mu}$-module on X (...). This construction defines a covariant functor from the category $\mathcal{M}(\mathcal{D}_\lambda)$ into the category $\mathcal{M}(\mathcal{D}_{\lambda+\mu})$. We call this functor the *geometric translation functor*. It is evidently an equivalence of categories, and it induces also an equivalence of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$, resp. $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$, with $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\mu})$, resp. $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\mu})$.

Geometric translation is closely related to another construction. Let F be a finite-dimensional $\mathfrak{g}$-module. Then the sheaf $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ has a natural structure of a $\mathcal{U}^\circ$-module. We shall define its filtration which is related to the weight structure of the module F.

Fix a base point $x_0 \in X$. The $\mathfrak{b}_{x_0}$-module F has a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m,$$

where $m = \dim F$, such that

$$\dim(F_i/F_{i-1}) = 1 \quad \text{and} \quad \mathfrak{n}_{x_0} F_i \subset F_{i-1} \quad \text{for} \quad 1 \le i \le m.$$

Therefore, $\mathfrak{b}_{x_0}/\mathfrak{n}_{x_0}$ acts naturally on F_i/F_{i-1} and this action induces, by specialization, an action of the Cartan algebra $\mathfrak{h}$ on F_i/F_{i-1} given by a weight $\nu_i \in P(\Sigma)$. Clearly, $\nu_i < \nu_j$ implies that $i > j$. The sheaf $\mathcal{F}$ is the sheaf of local sections of the trivial homogeneous vector bundle $X \times F \longrightarrow X$. The filtration of F induces a filtration of this vector bundle by the homogeneous vector subbundles with fibres F_i, $1 \le i \le m$, at the base point x_0. Let $\mathcal{F}_i$, $1 \le i \le m$, be the sheaves of local sections of these subbundles. They are locally free coherent $\mathcal{O}_X$-modules and also $\mathcal{U}^\circ$-modules. On the other hand, $\mathcal{F}_i/\mathcal{F}_{i-1} = \mathcal{O}(\nu_i)$ as a $\mathcal{U}^\circ$-module, i. e. $\mathcal{F}_i/\mathcal{F}_{i-1}$ is naturally a $\mathcal{D}_{\nu_i-\rho}$-module. Let $\mathcal{V}$ be a quasi-coherent $\mathcal{D}_\lambda$-module on X. Then the $\mathcal{O}_X$-module $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ has a natural structure of a $\mathcal{U}^\circ$-module given by

$$\xi(v \otimes s) = \xi v \otimes s + v \otimes \xi s$$

for $\xi \in \mathfrak{g}$, and local sections v and s of $\mathcal{V}$ and $\mathcal{F}$, respectively. We can define its $\mathcal{U}^\circ$-module filtration by the submodules $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}_i$, $1 \leq i \leq m$. By the previous discussion, the corresponding graded module is the direct sum of $\mathcal{V}(\nu_i)$, $1 \leq i \leq m$. Therefore, for any $\xi \in \mathcal{Z}(\mathfrak{g})$, the product $\prod_{1 \leq i \leq m} (\xi - \chi_{\lambda+\nu_i}(\xi))$ annihilates $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$. By the elementary linear algebra, $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ decomposes into the direct sum of its generalized $\mathcal{Z}(\mathfrak{g})$-eigensheaves.

Let $\mathcal{V}$ be a $\mathcal{U}^\circ$-module and $\lambda \in \mathfrak{h}^*$. Denote by $\mathcal{V}_{[\lambda]}$ the generalized $\mathcal{Z}(\mathfrak{g})$-eigensheaf of $\mathcal{V}$ corresponding to χ_λ.

Lemma 1. *Let $\lambda \in \mathfrak{h}^*$, $\mu \in P(\Sigma)$ and $w \in W$ be such that $w\lambda$ and $-w\mu$ are antidominant. Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight $w\mu$. Then, $\mathcal{V} \longrightarrow (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}$ is a covariant functor from $\mathcal{M}(\mathcal{D}_\lambda)$ into itself, naturally equivalent to the identity functor.*

Proof. The filtration of $\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ has $\mathcal{V}(-\mu+\nu)$ as its composition factors, where ν ranges over the set of all weights of F. Therefore, $\mathcal{Z}(\mathfrak{g})$ acts on them with the infinitesimal character $\chi_{\lambda-\mu+\nu}$. Assume that

$$s\lambda = \lambda - \mu + \nu$$

for some $s \in W$. Then, if we put $s' = wsw^{-1}$ and $\lambda' = w\lambda$, we have

$$s'\lambda' - \lambda' = w\nu - w\mu,$$

and since $w\mu$ and $w\nu$ are weights of F, $s'\lambda' - \lambda' \in Q(\Sigma)$. Therefore, $s' \in W_{\lambda'}$. Now, since $w\mu$ is the highest weight of F, $w\nu - w\mu$ is a sum of negative roots. On the other hand, since λ' is antidominant, $s'\lambda' - \lambda'$ is a sum of roots from $\Sigma^+_\lambda \subset \Sigma^+$. Therefore, $s\lambda = \lambda$ and $\mu = \nu$, and the generalized eigensheaf of $\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ corresponding to χ_λ is isomorphic to $\mathcal{V}$. □

Lemma 2. *Let $\lambda \in \mathfrak{h}^*$, $\mu \in P(\Sigma)$ and $w \in W$ be such that $w\lambda$ and $-w\mu$ are antidominant. Assume that the stabilizers of λ and $\lambda - \mu$ in W are equal. Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with the lowest weight $-w\mu$. Then, $\mathcal{V} \longrightarrow (\mathcal{V}(\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda-\mu]}$ is a covariant functor from $\mathcal{M}(\mathcal{D}_{\lambda-\mu})$ into itself, naturally equivalent to the identity functor.*

Proof. The filtration of $\mathcal{V}(\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ has $\mathcal{V}(\mu + \nu)$ as its composition factors, where ν varies over the set of all weights of F. Therefore $\mathcal{Z}(\mathfrak{g})$ acts on them with the infinitesimal character $\chi_{\lambda+\nu}$. Assume that

$$\lambda - \mu = s(\lambda + \nu)$$

for some $s \in W$. Then, if we put $s' = wsw^{-1}$ and $\lambda' = w\lambda$, we have

$$\lambda' - s'\lambda' = s'w\nu + w\mu,$$

and, since $-w\mu$ is the lowest weight of F, $s'w\nu + w\mu$ is a sum of positive roots. Therefore, $s \in W_{\lambda'}$, and since $w\lambda$ is antidominant, it follows that $s\lambda = \lambda$. By our assumption, s stabilizes $\lambda - \mu$, what implies that $\nu = -\mu$. Therefore, the generalized eigensheaf of $\mathcal{V}(\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ corresponding to $\chi_{\lambda-\mu}$ is $\mathcal{V}$. □

Let $\lambda \in \mathfrak{h}^*$ be such that $n(\lambda) = k$, $k > 0$. Then there exists $w \in W_\lambda$ such that $\ell_\lambda(w) = k$ and $w\lambda$ is antidominant. Let

$$w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$$

be a reduced expression of w in (W_λ, S_λ). Let $\alpha = w^{-1}\beta_1$, and $w' = s_{\beta_1} w$. Then we have $w' = s_{\beta_2} \dots s_{\beta_k}$ and $\ell_\lambda(w') = k - 1$. It follows that

$$w' s_\alpha \lambda = s_{\beta_1} w s_\alpha \lambda = s_{\beta_1} s_{w\alpha} w\lambda = w\lambda$$

is antidominant, which implies that

$$n(s_\alpha \lambda) \leq \ell_\lambda(w') = k - 1.$$

Now, the antidominance of $w\lambda$ implies that $\beta_1\check{}(w\lambda) \in -\mathbb{Z}_+$; also, $\beta_1\check{}(w\lambda) = 0$ would imply that $w'\lambda = s_{\beta_1} w\lambda = w\lambda$ is antidominant, contradicting the choice of w. Therefore,

$$p = -\beta_1\check{}(w\lambda) \in \mathbb{N}.$$

Let C_λ be the Weyl chamber corresponding to Σ_λ^+, then the equation $\beta_1\check{}(\tau) = 0$ determines a wall of C_λ. Evidently, the Σ-regular points of C_λ are partitioned in finitely many Weyl chambers for Σ, and at least one of them shares this wall with C_λ. Let C be one of such Weyl chambers.

Let $\sigma \in P(\Sigma) \cap C$, such that $\beta_1\check{}(\sigma) = p$. Then $w\lambda - s_{\beta_1}\sigma$ is in the wall determined by $\beta_1\check{}$. Also, because of

$$\Sigma_\lambda = \Sigma_{w\lambda} = \Sigma_{w\lambda - s_{\beta_1}\sigma}$$

and

$$s_{\beta_1}(\Sigma_\lambda^+ - \{\beta_1\}) = \Sigma_\lambda^+ - \{\beta_1\},$$

we see that, for $\beta \in \Sigma_\lambda^+ - \{\beta_1\}$, we have

$$\beta\check{}(w\lambda - s_{\beta_1}\sigma) = \beta\check{}(w\lambda) - (s_{\beta_1}\beta)\check{}(\sigma) \in -\mathbb{Z}_+,$$

and $w\lambda - s_{\beta_1}\sigma$ is antidominant. Hence, because of

$$w'(\lambda - s_\alpha w^{-1}\sigma) = s_{\beta_1} w(\lambda - s_\alpha w^{-1}\sigma) = s_{\beta_1}(w\lambda - s_{\beta_1}\sigma) = w\lambda - s_{\beta_1}\sigma,$$

it follows that

$$n(\lambda - s_\alpha w^{-1}\sigma) \leq \ell_\lambda(w') = k - 1.$$

Now, let $\mathcal{V}$ be a $\mathcal{D}_\lambda$-module. Then its translation $\mathcal{V}(p\alpha)$ is a $\mathcal{D}_{\lambda+p\alpha}$-module. Also, we have

$$\lambda + p\alpha = \lambda - \beta_1\check{}(w\lambda)\alpha = \lambda - \alpha\check{}(\lambda)\alpha = s_\alpha\lambda.$$

Analogously, the translation $\mathcal{V}(-s_\alpha w^{-1}\sigma)$ is a $\mathcal{D}_{\lambda - s_\alpha w^{-1}\sigma}$-module.

Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with extremal weight σ. Let

$$\mathcal{G} = (\mathcal{V}(-s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

Then the filtration of $\mathcal{V}(-s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F}$ induces the filtration

$$\mathcal{G}_i = \mathcal{G} \cap (\mathcal{V}(-s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F}_i),\ 1 \leq i \leq m,$$

of $\mathcal{G}$. This filtration has the property that $\mathcal{G}_i = \mathcal{G}_{i-1}$, except in the case when λ and $\lambda - s_\alpha w^{-1}\sigma + \nu_i$ lie in the same Weyl group orbit θ. If this condition is satisfied, we have

$$\mathcal{G}_i/\mathcal{G}_{i-1} = \mathcal{V}(-s_\alpha w^{-1}\sigma + \nu_i).$$

Therefore, to get a better insight into the structure of $\mathcal{G}$ we have to find all weights ν of F such that

$$\lambda - s_\alpha w^{-1}\sigma + \nu = s\lambda$$

for some $s \in W$. This implies that

$$s\lambda - \lambda = \nu - s_\alpha w^{-1}\sigma \in Q(\Sigma),$$

hence $s \in W_\lambda$. Therefore $s' = wsw^{-1}$ satisfies

$$s'w\lambda - w\lambda = w(s\lambda - \lambda) = w(\nu - s_\alpha w^{-1}\sigma) = w\nu - s_{\beta_1}\sigma,$$

and since $w\lambda$ is antidominant,

$$s'(w\lambda) - w\lambda = \sum_{\beta \in \Pi_\lambda} m_\beta \beta,\quad m_\beta \in \mathbb{Z}_+.$$

It follows that

$$s_{\beta_1}\sigma - w\nu = -\sum_{\beta \in \Pi_\lambda} m_\beta \beta.$$

By the choice of C, the set of all positive roots $\Sigma^+(C)$ in Σ (with respect to the order defined by C) contains Σ^+_λ. Hence, if we denote by $\Pi(C)$ the set of simple roots in Σ determined by C, $w\nu - s_{\beta_1}\sigma$ is a sum of roots from $\Pi(C)$. Since the root β_1 is in $\Pi(C)$, $s_{\beta_1}w\nu - \sigma = s_{\beta_1}(w\nu - s_{\beta_1})$ is the difference of a sum of roots from $\Pi(C)$ and $r\beta_1$, $r \in \mathbb{Z}_+$. On the other hand, $s_{\beta_1}w\nu$ is a weight and σ the highest weight of F for the order defined by C, hence $\sigma - s_{\beta_1}w\nu$ is a sum of roots from $\Pi(C)$. This finally implies that $\sigma - s_{\beta_1}w\nu = q\beta_1$ for some $q \in \mathbb{Z}_+$. Therefore,

$$s'w\lambda - w\lambda = w\nu - s_{\beta_1}\sigma = -s_{\beta_1}(\sigma - s_{\beta_1}w\nu) = q\beta_1;$$

and, if we introduce the standard W-invariant bilinear form on $\mathfrak{h}^*$, we get

$$||\lambda||^2 = ||s'w\lambda||^2 = ||w\lambda + q\beta_1||^2 = ||\lambda||^2 + 2q(w\lambda|\beta_1) + q^2||\beta_1||^2;$$

what implies that either $q = 0$ or $q = -\check{\beta_1}(w\lambda) = p$.

In the first case, $\nu = s_\alpha w^{-1}\sigma$ is an extremal weight of F. It follows that $\mathcal{G}_i/\mathcal{G}_{i-1} = \mathcal{V}$ when $\nu_i = \nu$, and this happens for only one i, $1 \leq i \leq m$. In the second case,

$$w\nu = s_{\beta_1}\sigma + p\beta_1 = s_{\beta_1}(\sigma - p\beta_1) = \sigma,$$

hence $\nu = w^{-1}\sigma$ is an extremal weight of F again. Now

$$-s_\alpha w^{-1}\sigma + \nu = -s_\alpha w^{-1}\sigma + w^{-1}\sigma = \alpha\check{}(w^{-1}\sigma)\alpha = \check{\beta_1}(\sigma)\alpha = p\alpha,$$

what implies that $\mathcal{G}_j/\mathcal{G}_{j-1} = \mathcal{V}(p\alpha)$ when $\nu_j = \nu$, and this happens for only one j, $1 \leq j \leq m$. Therefore, the $\mathcal{U}^\circ$-module $\mathcal{G}$ has a composition series of length two, and the corresponding subquotients are $\mathcal{V}$ and $\mathcal{V}(p\alpha)$. Finally,

$$\alpha = w^{-1}\beta_1 = s_{\beta_k}\dots s_{\beta_1}\beta_1 = -s_{\beta_k}\dots s_{\beta_2}\beta_1 \in -\Sigma_\lambda^+ \subset -\Sigma^+,$$

by ([**LG**], Ch. VI, Cor. 2 of Prop. 17), what leads to

$$\begin{aligned}\nu_j = w^{-1}\sigma < w^{-1}\sigma - p\alpha &= w^{-1}\sigma - \check{\beta_1}(\sigma)\alpha \\ &= w^{-1}\sigma - \alpha\check{}(w^{-1}\sigma)\alpha = s_\alpha w^{-1}\sigma = \nu_i,\end{aligned}$$

and, by a previous remark, $i < j$. This gives us the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{G} \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0.$$

of $\mathcal{U}^\circ$-modules. Clearly, the whole construction is functorial, therefore we have the following result.

Lemma 3. *There exists a covariant functor from $\mathcal{M}(\mathcal{D}_\lambda)$ into the category of short exact sequences of $\mathcal{U}^\circ$-modules which maps any $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$ into*

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{G} \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0.$$

C.3 A Vanishing Theorem for Cohomology of $\mathcal{D}_\lambda$-modules

In this section we shall discuss some vanishing results for cohomology of quasi-coherent $\mathcal{D}_\lambda$-modules.

Theorem 1. *Let $\mathcal{V}$ be a quasi-coherent $\mathcal{D}_\lambda$-module on the flag variety X. Then the cohomology groups $H^i(X, \mathcal{V})$ vanish for $i > n(\lambda)$.*

This, in particular, includes the vanishing of all $H^i(X, \mathcal{V})$, $i > 0$, for antidominant $\lambda \in \mathfrak{h}^*$, i. e. we have the following consequence.

Corollary 2. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then the functor Γ is an exact functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{U}_\theta)$.*

First we shall prove 2, and later use the induction in $n(\lambda)$ to complete the proof of 1.

Let $\mathcal{G}$ be any $\mathcal{O}_X$-module, and $\mu \in P(\Sigma)$ a dominant weight. Denote by F the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight μ. In 2. we defined a filtration of the $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ by locally free $\mathcal{O}_X$-submodules $(\mathcal{F}_i; 0 \leq i \leq \dim F)$ such that $\mathcal{F}_1 = \mathcal{O}(\mu)$. Therefore, we have a monomorphism $i_{\mathcal{G}}$ of $\mathcal{G} = \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{O}(\mu) = \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}_1$ into $\mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$.

Let $\lambda \in \mathfrak{h}^*$ be antidominant, $\mathcal{V}$ a quasi-coherent $\mathcal{D}_\lambda$-module, and $\varphi : \mathcal{G} \longrightarrow \mathcal{V}$ a morphism of $\mathcal{O}_X$-modules. Then it induces the morphism $\varphi(-\mu) : \mathcal{G}(-\mu) \longrightarrow \mathcal{V}(-\mu)$, and also $\varphi(-\mu) \otimes 1 : \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$. Also, we have natural imbeddings $i_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ and $i_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\varphi} & \mathcal{V} \\ i_{\mathcal{G}} \big\downarrow & & i_{\mathcal{V}} \big\downarrow \quad j_{\mathcal{V}} \big\uparrow . \\ \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\varphi(-\mu)\otimes 1} & \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F} \end{array}$$

Therefore on the level of cohomology, we have

$$H^i(\varphi(-\mu) \otimes 1) \circ H^i(i_{\mathcal{G}}) = H^i(i_{\mathcal{V}}) \circ H^i(\varphi)$$

for $0 \leq i \leq \dim X$. Also,

$$H^i(X, \mathcal{G}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}) = H^i(X, \mathcal{G}(-\mu)) \otimes_{\mathbb{C}} F,$$

since $\mathcal{F}$ is a free $\mathcal{O}_X$-module. Assume, in addition, that $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module. The invertible $\mathcal{O}_X$-module $\mathcal{O}(-2\rho)$ is ample, hence, we can find a dominant weight μ such that $H^i(X, \mathcal{G}(-\mu)) = 0$ for $1 \leq i \leq \dim X$. It follows that for such $\mu \in P(\Sigma)$, we have $H^i(i_{\mathcal{V}}) \circ H^i(\varphi) = 0$ for $1 \leq i \leq \dim X$. By 2.1, $\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ is a direct sum of $\mathcal{V}$ and its $\mathcal{Z}(\mathfrak{g})$-invariant complement, i. e. $i_{\mathcal{V}}$ has a left inverse $j_{\mathcal{V}} : \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{V}$. Hence, we conclude that $H^i(\varphi) = H^i(j_{\mathcal{V}}) \circ H^i(i_{\mathcal{V}}) \circ H^i(\varphi) = 0$ for $1 \leq i \leq \dim X$.

Any quasi-coherent $\mathcal{O}_X$-module is a direct limit of its coherent submodules ([**EGA**], I.6.9.9), and the cohomology commutes with direct limits ([**Hartshorne**], III.2.9), what implies that $H^i(j) = 0$, $1 \leq i \leq \dim X$, for the identity morphism $j : \mathcal{V} \longrightarrow \mathcal{V}$. This finally implies that $H^i(X, \mathcal{V}) = 0$ for $1 \leq i \leq \dim X$, and finishes the proof of 2.

To prove 1. we use 2.3. Assume that $\lambda \in \mathfrak{h}^*$ and $n(\lambda) = k$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{G} \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0,$$

where

$$\mathcal{G} = (\mathcal{V}(-s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

As we have shown there, $n(\lambda + p\alpha) < k$ and $n(\lambda - s_\alpha w^{-1}\sigma) < k$. Therefore, by the induction assumption, we have

$$H^i(X, \mathcal{G}) = H^i(X, \mathcal{V}(s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} = (H^i(X, \mathcal{V}(s_\alpha w^{-1}\sigma)) \otimes_{\mathbb{C}} F)_{[\lambda]} = 0$$

and

$$H^i(X, \mathcal{V}(p\alpha)) = 0$$

for $i > k - 1$. The long exact sequence of cohomology, applied to the above short exact sequence, implies that $H^i(X, \mathcal{V}) = 0$ for $i > k$.

C.4 A Nonvanishing Theorem for Cohomology of $\mathcal{D}_\lambda$-modules

Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. The category of quasi-coherent $\mathcal{D}_\lambda$-modules $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is a thick subcategory in $\mathcal{M}(\mathcal{D}_\lambda)$, therefore we can consider the full subcategory $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ of the derived category $D(\mathcal{M}(\mathcal{D}_\lambda))$ of $\mathcal{D}_\lambda$-modules which consists of complexes with quasi-coherent cohomology. Let $D(\mathcal{U}_\theta)$ be the derived category of $\mathcal{U}_\theta$-modules. The category $\mathcal{M}(\mathcal{D}_\lambda)$ has sufficiently many injective objects, and they are flasque sheaves. Moreover, the right cohomological dimension of the functor Γ of global sections is less than or equal to $\dim X$. Therefore, one can define the derived functor $R\Gamma$ from $D(\mathcal{M}(\mathcal{D}_\lambda))$ into $D(\mathcal{U}_\theta)$.

Our main goal in this section is

Theorem 1. *Let $\lambda \in \mathfrak{h}^*$ be regular. Let $\mathcal{C}^\cdot, \mathcal{D}^\cdot \in D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ and $f : \mathcal{C}^\cdot \longrightarrow \mathcal{D}^\cdot$ a morphism. Then the following conditions are equivalent:*
(i) f is a quasi-isomorphism,
(ii) $R\Gamma(f)$ is a quasi-isomorphism.

Clearly, (i) implies (ii). The other implication follows from the following special case of 1.

Lemma 2. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\mathcal{C}^\cdot \in D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ be such that $R\Gamma(\mathcal{C}^\cdot) = 0$. Then $\mathcal{C}^\cdot = 0$.*

First, let's show that 2. implies 1. Let $\mathcal{C}_f^\cdot$ be the mapping cone of f. Then we have the standard triangle

$$\mathcal{C}^\cdot \longrightarrow \mathcal{D}^\cdot \longrightarrow \mathcal{C}_f^\cdot \longrightarrow \mathcal{C}^\cdot[1],$$

and the distinguished triangle

$$R\Gamma(\mathcal{C}^\cdot) \longrightarrow R\Gamma(\mathcal{D}^\cdot) \longrightarrow R\Gamma(\mathcal{C}_f^\cdot) \longrightarrow R\Gamma(\mathcal{C}^\cdot)[1].$$

If $R\Gamma(f)$ is a quasi-isomorphism, from the long exact sequence of cohomology we conclude that $H^i(R\Gamma(\mathcal{C}_f^\cdot)) = 0$ for $i \in \mathbb{Z}$, i. e. $R\Gamma(\mathcal{C}_f^\cdot) = 0$. By 2, we conclude that $\mathcal{C}_f^\cdot = 0$, and f is a quasi-isomorphism.

It remains to prove 2. The proof is by induction in $n(\lambda)$.

Assume that $n(\lambda) = 0$. Then, by 3.2, Γ is exact on $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Also we can assume that $\mathcal{C}^\cdot$ consists of Γ-acyclic $\mathcal{D}_\lambda$-modules. In this case $R\Gamma(\mathcal{C}^\cdot) =$

$\Gamma(\mathcal{C}^\cdot)$. Assume that $H^i(\mathcal{C}^\cdot)$ is a quasi-coherent $\mathcal{D}_\lambda$-module different from zero. Because $\mathcal{O}(-2\rho)$ is ample, we conclude that there is a dominant weight $\mu \in P(\Sigma)$ such that $H^i(\mathcal{C}^\cdot)(-\mu)$ has nontrivial global sections. By 2.2, if we denote by F the irreducible finite-dimensional $\mathfrak{g}$-module with lowest weight $-\mu$ and $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$, we see that

$$\Gamma(X, H^i(\mathcal{C}^\cdot)) \otimes_{\mathbb{C}} F = \Gamma(X, H^i(\mathcal{C}^\cdot) \otimes_{\mathcal{O}_X} \mathcal{F}) \neq 0.$$

Hence, $\Gamma(X, H^i(\mathcal{C}^\cdot)) \neq 0$. On the other hand, if we consider the short exact sequences

$$0 \longrightarrow \ker d^i \longrightarrow \mathcal{C}^i \longrightarrow \operatorname{im} d^i \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{im} d^{i-1} \longrightarrow \ker d^i \longrightarrow H^i(\mathcal{C}^\cdot) \longrightarrow 0,$$

by the long exact sequence of cohomology we conclude that

$$H^n(X, \operatorname{im} d^i) = H^{n+1}(X, \ker d^i) \quad \text{for} \quad n \geq 1,$$

and

$$H^n(X, \operatorname{im} d^{i-1}) = H^n(X, \ker d^i) \quad \text{for} \quad n \geq 2.$$

Hence, it follows that

$$H^n(X, \operatorname{im} d^i) = H^{n+1}(X, \operatorname{im} d^{i-1}) \quad \text{for} \quad n \geq 1,$$

and by finiteness of right cohomological dimension of Γ, $H^n(X, \operatorname{im} d^i) = 0$ for $n > 0$ and arbitrary $i \in \mathbb{Z}$. This in turn yields $H^n(X, \ker d^i) = 0$ for $n > 0$ and arbitrary $i \in \mathbb{Z}$. Finally we get

$$\Gamma(X, H^i(\mathcal{C}^\cdot)) = H^i(\Gamma(\mathcal{C}^\cdot)) \quad \text{for} \quad i \in \mathbb{Z},$$

what contradicts $H^i(\Gamma(\mathcal{C}^\cdot)) = 0$. This implies that $H^i(\mathcal{C}^\cdot) = 0$ for all $i \in \mathbb{Z}$.

Assume now that $n(\lambda) = k > 0$. We can assume again that $\mathcal{C}^\cdot$ consists of Γ-acyclic $\mathcal{D}_\lambda$-modules. Then, $0 = R\Gamma(\mathcal{C}^\cdot) = \Gamma(\mathcal{C}^\cdot)$. Now, we shall use the notation from discussion preceding 2.3. Put $\mu = \lambda - s_\alpha w^{-1}\sigma$. What we have shown there is that $w\lambda$ and $w\mu$ are antidominant. Also, $w\lambda - w\mu = s_{\beta_1}\sigma \in P(\Sigma)$. Let τ be a regular dominant weight such that $\eta = \tau + w(\mu - \lambda)$ is dominant. Denote by $F_{-\tau}$ the finite-dimensional representation with lowest weight $-\tau$. Let $\mathcal{F}_{-\tau} = \mathcal{O}_X \otimes_{\mathbb{C}} F_{-\tau}$. Then, by 2.2, we have

$$\mathcal{C}^\cdot(-w^{-1}\tau) = (\mathcal{C}^\cdot \otimes_{\mathcal{O}_X} \mathcal{F}_{-\tau})_{[w\lambda-\tau]},$$

and

$$\begin{aligned} H^i(X, \mathcal{C}^j(-w^{-1}\tau)) &= H^i(X, \mathcal{C}^j \otimes_{\mathcal{O}_X} \mathcal{F}_{-\tau})_{[w\lambda-\tau]} \\ &= (H^i(X, \mathcal{C}^j) \otimes_{\mathbb{C}} F_{-\tau})_{[w\lambda-\tau]} = 0 \end{aligned}$$

for $i > 0$, i. e. the complex $\mathcal{C}^\cdot(-w^{-1}\tau)$ consists of Γ-acyclic $\mathcal{D}_{\lambda-w^{-1}\tau}$-modules. This implies that

$$\begin{aligned} R\Gamma(\mathcal{C}^\cdot(-w^{-1}\tau)) &= \Gamma(\mathcal{C}^\cdot(-w^{-1}\tau)) = \Gamma(\mathcal{C}^\cdot \otimes_{\mathcal{O}_X} \mathcal{F}_{-\tau})_{[w\lambda-\tau]} \\ &= (\Gamma(\mathcal{C}^\cdot) \otimes_{\mathbb{C}} F_{-\tau})_{[w\lambda-\tau]} = 0. \end{aligned}$$

On the other hand, if we take F^η to be the finite-dimensional representation with highest weight η and $\mathcal{F}^\eta = \mathcal{O}_X \otimes_{\mathbb{C}} F^\eta$, by 2.1. it follows that

$$(\mathcal{C}^{\cdot}(-w^{-1}\tau) \otimes_{\mathcal{O}_X} \mathcal{F}^\eta)_{[\mu]} = \mathcal{C}^{\cdot}(-w^{-1}(\tau - \eta)) = \mathcal{C}^{\cdot}(\mu - \lambda).$$

Applying the same argument as before we see that $\mathcal{C}^{\cdot}(\mu - \lambda)$ consists of Γ-acyclic $\mathcal{D}_\mu$-modules and get that

$$\Gamma(\mathcal{C}^{\cdot}(-s_\alpha w^{-1}\sigma)) = \Gamma(\mathcal{C}^{\cdot}(\mu - \lambda)) = 0.$$

Now, let

$$\mathcal{G}^{\cdot} = (\mathcal{C}^{\cdot}(-s_\alpha w^{-1}\sigma) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

Clearly, $\mathcal{G}^{\cdot}$ consists of Γ-acyclic $\mathcal{U}^\circ$-modules and $\Gamma(\mathcal{G}^{\cdot}) = 0$. Applying 2.3, we conclude that $\mathcal{C}^{\cdot}(p\alpha)$ consists of Γ-acyclic $\mathcal{D}_{s_\alpha\lambda}$-modules and $\Gamma(\mathcal{C}^{\cdot}(p\alpha)) = 0$. This implies that $R\Gamma(\mathcal{C}^{\cdot}(p\alpha)) = 0$. By our construction $n(s_\alpha\lambda) < k$, hence we can apply the induction assumption. It follows that $\mathcal{C}^{\cdot}(p\alpha) = 0$, and finally that $\mathcal{C}^{\cdot} = 0$. This completes the proof of 2.

Corollary 3. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ such that all its cohomology modules $H^i(X, \mathcal{V})$, $i \in \mathbb{Z}_+$, vanish. Then $\mathcal{V} = 0$.*

Corollary 4. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is generated by its global sections.*

Proof. Denote by $\mathcal{W}$ the $\mathcal{D}_\lambda$-submodule of $\mathcal{V}$ generated by all global sections. Then, by 3.2, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{W}) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(X, \mathcal{V}/\mathcal{W}) \longrightarrow 0,$$

of $\mathcal{U}_\theta$-modules, and therefore $\Gamma(X, \mathcal{V}/\mathcal{W}) = 0$. Hence, by 3, $\mathcal{V}/\mathcal{W} = 0$, and $\mathcal{V}$ is generated by its global sections. □

C.5 Borel-Weil-Bott Theorem

Let $n = \dim X$. Let $\lambda \in P(\Sigma)$ and $\mathcal{O}(\lambda)$ the corresponding invertible $\mathcal{O}_X$-module. Then $\mathcal{O}(\lambda)$ is a $\mathcal{D}_{\lambda-\rho}$-module, coherent as an $\mathcal{O}_X$-module. By the general results from algebraic geometry (see for example [**Hartshorne**], Ch. III), we know that

(i) $H^i(X, \mathcal{O}(\lambda))$ are finite-dimensional $\mathfrak{g}$-modules,

(ii) if we denote by $\mathcal{O}(\lambda)\check{}$ the dual of the invertible sheaf $\mathcal{O}(\lambda)$ and by ω_X the sheaf of local n-forms on X, then the Serre duality implies that the dual of the vector space $H^i(X, \mathcal{O}(\lambda))$ is isomorphic to the vector space $H^{n-i}(X, \mathcal{O}(\lambda)\check{} \otimes_{\mathcal{O}_X} \omega_X)$.

Of course, $\mathcal{O}(\lambda)\check{} \otimes_{\mathcal{O}_X} \omega_X = \mathcal{O}(-\lambda + 2\rho)$. Let w_0 be the longest element in W. Denote by w one of the longest elements in W such that $w(\lambda - \rho)$ is antidominant. Then $\ell(w) \geq n(\lambda - \rho)$ and the strict inequality holds if and only if $\lambda - \rho$ is not regular. On the other hand, $w_0 w(-\lambda + \rho)$ is antidominant too, hence

$$n(-\lambda + \rho) \leq \ell(w_0 w) = \ell(w^{-1} w_0^{-1}) = \ell(w^{-1} w_0) = \ell(w_0) - \ell(w) = n - \ell(w),$$

by ([**LG**], Ch. VI, Cor. 3 of Prop. 17). It follows that

$$n(\lambda - \rho) \leq \ell(w) \leq n - n(-\lambda + \rho). \tag{1}$$

Suppose that $H^i(X, \mathcal{O}(\lambda)) \neq 0$. Then, 3.1. applied to $\mathcal{O}(\lambda)$ implies that $i \leq n(\lambda - \rho)$; on the other hand, if we also use (ii), $n - i \leq n(-\lambda + \rho)$. Therefore,

$$n - n(-\lambda + \rho) \leq i \leq n(\lambda - \rho). \tag{2}$$

We see from (1) and (2) that $i = \ell(w) = n(\lambda - \rho)$. By the previous remark, this implies that $\lambda - \rho$ is regular.

It remains to study $H^{\ell(w)}(X, \mathcal{O}(\lambda))$ in the case of regular $\lambda - \rho$. Then $\mu = w(\lambda - \rho)$ is a regular antidominant weight. In the following we use the notation and results from 2.3. If we put $\mathcal{V} = \mathcal{O}(\lambda)$, then $\mathcal{V}(-s_\alpha w^{-1}\sigma) = \mathcal{O}(\lambda - s_\alpha w^{-1}\sigma)$ and $\lambda - s_\alpha w^{-1}\sigma - \rho$ is not regular. Therefore, all cohomology groups of it vanish. This implies that all cohomology groups of $\mathcal{G}$ vanish either. Therefore, the exact sequence

$$0 \longrightarrow \mathcal{O}(\lambda) \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\lambda + p\alpha) \longrightarrow 0$$

implies that $H^i(X, \mathcal{O}(\lambda + p\alpha)) = H^{i+1}(X, \mathcal{O}(\lambda))$ as a $\mathfrak{g}$-module for $i \in \mathbb{Z}_+$. Now $p = -\alpha\check{}(\lambda - \rho)$, so

$$\lambda + p\alpha = s_\alpha(\lambda - \rho) + \rho = w^{-1} s_{\beta_1} w(\lambda - \rho) + \rho = w^{-1} s_{\beta_1} \mu + \rho = w'^{-1} \mu + \rho.$$

It follows, by the induction in length of w, that

$$H^{\ell(w)}(X, \mathcal{O}(\lambda)) = \Gamma(X, \mathcal{O}(\mu + \rho)).$$

By 2.2, if $F_{\mu+\rho}$ is the irreducible $\mathfrak{g}$-module with lowest weight $\mu + \rho$ and we put $\mathcal{F}_{\mu+\rho} = \mathcal{O}_X \otimes_{\mathbb{C}} F_{\mu+\rho}$, we have

$$\mathcal{O}(\mu + \rho) = (\mathcal{F}_{\mu+\rho})_{[\mu]}.$$

Hence,

$$\begin{aligned}\Gamma(X, \mathcal{O}(\mu + \rho)) &= \Gamma(X, (\mathcal{F}_{\mu+\rho})_{[\mu]}) = \Gamma(X, \mathcal{O}_X \otimes_{\mathbb{C}} F_{\mu+\rho})_{[\mu]} \\ &= (\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} F_{\mu+\rho})_{[\mu]} = F_{\mu+\rho}.\end{aligned}$$

This ends the proof of the Borel-Weil-Bott theorem.

Theorem 1 (Borel-Weil-Bott). *Let $\lambda \in P(\Sigma)$. Then*

(i) if $\lambda - \rho$ is not regular, $H^i(X, \mathcal{O}(\lambda)) = 0$ for $i \in \mathbb{Z}_+$,

(ii) if $\lambda - \rho$ is regular and $\mu \in P(\Sigma)$ an antidominant weight such that $\lambda - \rho = w\mu$ for some $w \in W$, then $H^i(X, \mathcal{O}(\lambda)) = 0$ for $i \neq \ell(w)$

and $H^{\ell(w)}(X, \mathcal{O}(\lambda))$ *is the irreducible finite-dimensional* $\mathfrak{g}$*-module with lowest weight* $\mu + \rho$*.*

Denote, as in 1, by $\mathcal{N}$ the vector bundle of nilpotent radicals over the flag variety X and by $\mathfrak{n}^\circ$ the locally free $\mathcal{O}_X$-module of local sections of $\mathcal{N}$. Let

$$W(j) = \{w \in W \mid \ell(w) = j\}, \qquad 0 \le j \le \dim X.$$

Lemma 2. *Let* $0 \le j \le \dim X$*. Then*

$$H^i(X, \wedge^j \mathfrak{n}^\circ) = 0 \quad \textit{if} \quad i \neq j$$

and $H^j(X, \wedge^j \mathfrak{n}^\circ)$ *is the trivial* $\mathfrak{g}$*-module of dimension* $\operatorname{Card} W(j)$*.*

Proof. Choose a base point $x_0 \in X$. Evidently, $\wedge^j \mathfrak{n}^\circ$ is the $\mathcal{O}_X$-module of local sections of the G-homogeneous vector bundle $\wedge^j \mathcal{N}$ which is determined by the natural representation of the Borel subgroup B_{x_0} on $F_j = \wedge^j \mathfrak{n}_{x_0}$. The B_{x_0}-action on F_j defines a natural Jordan-Hölder filtration by B_{x_0}-invariant subspaces F_{jk}, $0 \le k \le \dim F_j$, such that $\dim F_{jk} = k$, $\mathfrak{n}_{x_0} F_{jk} \subset F_{jk-1}$ and the Cartan algebra $\mathfrak{h}$ acts on F_{jk}/F_{jk-1} by a weight ν_{jk} which is a sum of j different roots from Σ^+ for $0 \le k \le \dim F_j$. This filtration induces a filtration of the vector bundle $\wedge^j \mathcal{N}$ by G-homogeneous subbundles. We denote by $\mathcal{F}_{jk}$, $0 \le k \le \dim F_j$, the corresponding coherent $\mathcal{O}_X$-modules of local sections. It is evident that $\mathcal{F}_{jk}/\mathcal{F}_{jk-1} = \mathcal{O}(\nu_{jk})$ for $0 \le k \le \dim F_j$. To calculate the cohomology of $\wedge^j \mathfrak{n}^\circ$ we have to understand better the structure of the family $\{\nu_{jk} \mid 0 \le k \le \dim F_j,\ 0 \le j \le \dim X\}$ which is equal to the family of sums of roots from all subsets of Σ^+. For each $\Phi \subset \Sigma^+$ we denote by $\nu(\Phi)$ the sum of all roots from Φ. Let

$$S = \{\nu(\Phi) - \rho \mid \Phi \subset \Sigma^+\}.$$

Then, because $\nu(\Phi) - \rho$ is the difference of the half-sum of roots from Φ and half-sum of roots from $\Sigma^+ - \Phi$, it is evident that S is invariant under the action of the Weyl group W. Let S_- be the set of antidominant weights in S. Then, clearly $S = W \cdot S_-$. Let μ be a regular element of S_-. Denote by ω_α the fundamental weight corresponding to simple root $\alpha \in \Pi$. Then μ is a linear combination of ω_α, $\alpha \in \Pi$, with strictly negative integral coefficients. Therefore, $\mu + \rho$ is still antidominant. On the other hand it also must be a sum of positive roots. This implies that it is equal to 0, hence $\mu = -\rho$. It follows that the only regular elements of S are $-w\rho$, $w \in W$. In these cases

$$\rho - w\rho = \nu(\Sigma^+ \cap (-w(\Sigma^+))).$$

Also, we remark that $\ell(w) = \operatorname{Card}(\Sigma^+ \cap (-w(\Sigma^+)))$. From the Borel-Weil-Bott theorem we know that

$$H^i(X, \mathcal{O}(\nu_{jk})) = 0 \quad \text{for all} \quad 0 \le i \le \dim X$$

if $\nu_{jk} - \rho$ is singular. On the other hand, if $\nu_{jk} - \rho$ is regular, by previous discussion $\nu_{jk} = \rho - w\rho$ for some $w \in W(j)$. Hence, in this case, we have

$$H^i(X, \mathcal{O}(\nu_{jk})) = 0 \quad \text{for} \quad i \neq j$$

and

$$H^j(X, \mathcal{O}(\nu_{jk})) = H^j(X, \mathcal{O}(\rho - w\rho)) = \Gamma(X, \mathcal{O}_X) = \mathbb{C}.$$

Using this information and the long exact sequence of cohomology, the induction in k, $0 \leq k \leq \dim F_j$, applied to the short exact sequence

$$0 \longrightarrow \mathcal{F}_{jk-1} \longrightarrow \mathcal{F}_{jk} \longrightarrow \mathcal{O}(\nu_{jk}) \longrightarrow 0$$

implies easily our assertion. □

C.6 Cohomology of $\mathcal{D}_\lambda$

In this section we want to prove

Theorem 1. *(i) The natural map of $\mathcal{U}_\theta$ into $\mathcal{D}_\lambda$ induces an isomorphism of $\mathcal{U}_\theta$ onto $\Gamma(X, \mathcal{D}_\lambda)$.*

(ii) $H^i(X, \mathcal{D}_\lambda) = 0$ for $i > 0$.

Let $\mathcal{C}^\cdot$ be the graded module $\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge\mathfrak{n}^\circ$, i. e. $\mathcal{C}^i = \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^{-i}\mathfrak{n}^\circ$ for all $i \in \mathbb{Z}$. First we remark that $\mathcal{C}^\cdot$ has a structure of a left $\mathfrak{g}$-module, by left multiplication on the first factor. The exterior algebra $\wedge\mathfrak{n}^\circ$ has a natural structure of a left $\mathfrak{g}$-module. Also, $\mathcal{U}^\circ$ is a right $\mathfrak{g}$-module for right multiplication, so we can define another structure of a left $\mathfrak{g}$-module on $\mathcal{C}^\cdot$ by

$$\kappa(\xi)(u \otimes v) = -u\xi \otimes v + u \otimes \xi \cdot v,$$

for $\xi \in \mathfrak{g}$, $u \in \mathcal{U}^\circ$ and $v \in \wedge\mathfrak{n}^\circ$. To see that this definition makes sense, we remark that if we consider the biadditive map $\varphi(\xi)$, $\xi \in \mathfrak{g}$, from $\mathcal{U}^\circ \times \wedge\mathfrak{n}^\circ$ into $\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge\mathfrak{n}^\circ$ given by

$$\varphi(\xi)(u, v) = -u\xi \otimes v + u \otimes \xi \cdot v$$

for $u \in \mathcal{U}^\circ$ and $v \in \wedge\mathfrak{n}^\circ$, we have

$$\begin{aligned}\varphi(\xi)(uf, v) - \varphi(\xi)(u, fv) &= -(uf)\xi \otimes v + uf \otimes \xi \cdot v + u\xi \otimes fv - u \otimes \xi \cdot (fv) \\ &= u\xi(f) \otimes v + uf \otimes \xi \cdot v - u \otimes \xi(f)v - u \otimes f\xi \cdot v = 0\end{aligned}$$

for any $f \in \mathcal{O}_X$, $u \in \mathcal{U}^\circ$ and $v \in \wedge\mathfrak{n}^\circ$; hence it factors through $\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge\mathfrak{n}^\circ$ and induces $\kappa(\xi)$. By the construction it is evident that the two left $\mathfrak{g}$-module actions on $\mathcal{C}^\circ$ commute. Therefore, we can consider $\mathcal{C}^\cdot$ as a left $\mathfrak{g} \times \mathfrak{g}$-module via

$$(\xi, \eta)w = \xi w + \kappa(\eta)w$$

for $\xi, \eta \in \mathfrak{g}$ and $w \in \mathcal{C}^{\cdot}$. Also, the group G acts on $\mathcal{C}^{\cdot}$ with the tensor product of the adjoint action on $\mathcal{U}(\mathfrak{g})$ with the adjoint action on $\wedge\mathfrak{g}$. The differential of this action is equal to the restriction of the $\mathfrak{g} \times \mathfrak{g}$-action to the diagonal. Therefore. we can view $\mathcal{C}^{\cdot}$ as a $(\mathfrak{g} \times \mathfrak{g}, G)$-module. Consider map

$$d(u \otimes v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} u v_i \otimes v_1 \wedge v_2 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k + \sum_{1 \le i < j \le k} (-1)^{i+j} u \otimes [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k,$$

for $u \in \mathcal{U}^\circ$ and $v_1, v_2, \ldots, v_k \in \mathfrak{n}^\circ$. It is well-defined, because sections of $\mathcal{O}_X$ and $\mathfrak{n}^\circ$ commute in $\mathcal{U}^\circ$, and it maps $\mathcal{C}^i$ into $\mathcal{C}^{i+1}$. Also, by definition it commutes with the $\mathfrak{g}$-action given by left multiplication and the G-action. Since the difference of the differential of the G-action and the left multiplication action gives the action given by κ, d is a morphism of $(\mathfrak{g} \times \mathfrak{g}, G)$-modules. By calculation one also checks that $d^2 = 0$, i. e. $\mathcal{C}^{\cdot}$ is a complex of $(\mathfrak{g} \times \mathfrak{g}, G)$-modules.

Lemma 2. *The complex $\mathcal{C}^{\cdot}$ is acyclic.*

Proof. First we introduce a filtration of the complex $\mathcal{C}^{\cdot}$. The sheaf of algebras $\mathcal{U}^\circ$ has a natural filtration $(\mathrm{F}_p\, \mathcal{U}^\circ; p \in \mathbb{Z})$. We put $\mathrm{F}_p\, \mathcal{C}^{\cdot} = 0$ for $p < 0$ and

$$\mathrm{F}_p\, \mathcal{C}^{\cdot} = \sum_{q=-p}^{0} \mathrm{F}_{p+q}\, \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^{-q} \mathfrak{n}^\circ \quad \text{if} \quad p \in \mathbb{Z}_+.$$

The differential d maps $\mathrm{F}_p\, \mathcal{C}^{\cdot}$ into itself for any $p \in \mathbb{Z}$; hence, $\mathcal{C}^{\cdot}$ is a filtered complex. The corresponding graded bicomplex has the form

$$\mathrm{Gr}^{p,q}\, \mathcal{C}^{\cdot} = \mathrm{F}_p(\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^{-q} \mathfrak{n}^\circ) / \mathrm{F}_{p-1}(\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^{-q} \mathfrak{n}^\circ) = S^{p+q}(\mathfrak{g}^\circ) \otimes_{\mathcal{O}_X} \wedge^{-q} \mathfrak{n}^\circ,$$

with the differential $\delta = \mathrm{Gr}\, d$ of bidegree (0,1) given by the formula

$$\delta(u \otimes v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} u v_i \otimes v_1 \wedge v_2 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k,$$

for $u \in S^{p-k}(\mathfrak{g}^\circ)$ and $v_1, v_2, \ldots, v_k \in \mathfrak{n}^\circ$.

Let U be a sufficiently small affine open set, such that $\mathfrak{g}^\circ|U$ and $\mathfrak{n}^\circ|U$ are free $\mathcal{O}_U$-modules. Then, using the standard results on Koszul complexes ([**Alg**], Ch. X, §9, no. 3) it follows that $\mathrm{Gr}\, \mathcal{C}^{\cdot}|U$ is acyclic. Therefore, $\mathrm{Gr}\, \mathcal{C}^{\cdot}$ is acyclic.

It follows that for any $p \in \mathbb{Z}$ we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{F}_{p-1}\, \mathcal{C}^{\cdot} \longrightarrow \mathrm{F}_p\, \mathcal{C}^{\cdot} \longrightarrow \mathrm{Gr}^p\, \mathcal{C}^{\cdot} \longrightarrow 0$$

with $\mathrm{Gr}^p\, \mathcal{C}^{\cdot}$ acyclic. This implies that $H^k(\mathrm{F}_{p-1}\, \mathcal{C}^{\cdot}) = H^k(\mathrm{F}_p\, \mathcal{C}^{\cdot})$ for $k \in -\mathbb{N}$ and $p \in \mathbb{Z}$. Now, $\mathrm{F}_p\, \mathcal{C}^{\cdot} = 0$ for $p < 0$ implies that

$$H^k(\mathrm{F}_p\,\mathcal{C}^\cdot) = 0 \quad \text{for all} \quad k \in -\mathbb{N} \quad \text{and} \quad p \in \mathbb{Z},$$

i. e. all $\mathrm{F}_p\,\mathcal{C}^\cdot$ are acyclic.

Let $\xi \in \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^k \mathfrak{n}^\circ$, $k > 0$, be such that $d\xi = 0$. Since the filtration of $\mathcal{C}^\cdot$ is exhaustive, there exists $p \in \mathbb{Z}_+$ such that $\xi \in \mathrm{F}_p\,\mathcal{C}^\cdot$. Therefore, by the acyclicity of $\mathrm{F}_p\,\mathcal{C}^\cdot$, there exists $\eta \in \mathrm{F}_{p-k-1}\,\mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^{k+1}\mathfrak{n}^\circ$ such that $\xi = d\eta$. □

Putting everything together we get the following result.

Proposition 3. *The complex* $\mathcal{C}^\cdot = \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge \mathfrak{n}^\circ$ *is a left resolution of the* $(\mathfrak{g} \times \mathfrak{g}, G)$*-module* $\mathcal{D}_\mathfrak{h}$.

Clearly,

$$\begin{aligned} H^i(X, \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^j \mathfrak{n}^\circ) &= H^i(X, \mathcal{U}(\mathfrak{g}) \otimes_\mathbb{C} \wedge^j \mathfrak{n}^\circ) \\ &= \mathcal{U}(\mathfrak{g}) \otimes_\mathbb{C} H^i(X, \wedge^j \mathfrak{n}^\circ) \quad \text{for} \quad i, j \in \mathbb{Z}_+, \end{aligned}$$

as a $(\mathfrak{g} \times \mathfrak{g}, G)$-module. By 5.2, the action of $\mathfrak{g} \times \mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ is the natural action given by left and right multiplication, i. e.

$$(\xi_1, \xi_2)\eta = \xi_1\eta - \eta\xi_2, \quad \text{for} \quad \xi_1, \xi_2 \in \mathfrak{g}, \ \eta \in \mathcal{U}(\mathfrak{g}),$$

the group G acts on $\mathcal{U}(\mathfrak{g})$ by the adjoint action, and the actions on the second factors are trivial. Moreover, $H^i(X, \mathcal{U}^\circ \otimes_{\mathcal{O}_X} \wedge^j \mathfrak{n}^\circ)$ vanishes for $i \neq j$ and is a direct sum of $\operatorname{Card} W(j)$ copies of $\mathcal{U}(\mathfrak{g})$ for $i = j$. This implies that the spectral sequence ([**Tôhoku**], 2.4, Remark 3), which calculates the cohomology of $\mathcal{D}_\mathfrak{h}$ using this resolution, converges in its E^2-term, and we conclude that

(i) $H^i(X, \mathcal{D}_\mathfrak{h}) = 0$ for $i > 0$,

(ii) $\Gamma(X, \mathcal{D}_\mathfrak{h})$, considered as a $(\mathfrak{g} \times \mathfrak{g}, G)$-module, has a finite increasing filtration

$$0 = \mathrm{F}_0\,\Gamma(X, \mathcal{D}_\mathfrak{h}) \subset \mathrm{F}_1\,\Gamma(X, \mathcal{D}_\mathfrak{h}) \subset \cdots \subset \mathrm{F}_n\,\Gamma(X, \mathcal{D}_\mathfrak{h}) = \Gamma(X, \mathcal{D}_\mathfrak{h})$$

such that $\mathrm{F}_k\,\Gamma(X, \mathcal{D}_\mathfrak{h}) / \mathrm{F}_{k-1}\,\Gamma(X, \mathcal{D}_\mathfrak{h})$ is a direct sum of $\operatorname{Card} W(k)$ copies of $\mathcal{U}(\mathfrak{g})$ equipped with the $(\mathfrak{g} \times \mathfrak{g}, G)$-module structure described above.

By the construction, the filtration $\mathrm{F}\,\Gamma(X, \mathcal{D}_\mathfrak{h})$ is a G-module filtration for the natural G-module structure on $\mathcal{D}_\mathfrak{h}$ and it induces a filtration on the subalgebra of G-invariants of $\Gamma(X, \mathcal{D}_\mathfrak{h})$. By 1.5, this subalgebra is isomorphic to $\mathcal{U}(\mathfrak{h})$ via the map ϕ. Since the group G is semisimple and its action on $\Gamma(X, \mathcal{D}_\mathfrak{h})$ is algebraic, the G-module $\Gamma(X, \mathcal{D}_\mathfrak{h})$ is semisimple. This implies that the G-invariants of $\operatorname{Gr}\Gamma(X, \mathcal{D}_\mathfrak{h})$ are equal to $\operatorname{Gr}\phi(\mathcal{U}(\mathfrak{h}))$. By taking the G-invariants in the statement (ii) above, we see immediately that:

(iii) $\mathrm{F}_k\,\phi(\mathcal{U}(\mathfrak{h})) / \mathrm{F}_{k-1}\,\phi(\mathcal{U}(\mathfrak{h}))$ is a direct sum of $\operatorname{Card} W(k)$ copies of $\mathcal{Z}(\mathfrak{g})$.

We can view $\mathcal{U}(\mathfrak{h})$ as a $\mathcal{Z}(\mathfrak{g})$-module via the Harish-Chandra homomorphism. This immediately implies the following result.

Lemma 4. *The universal enveloping algebra* $\mathcal{U}(\mathfrak{h})$ *is a free* $\mathcal{Z}(\mathfrak{g})$*-module of rank* $\operatorname{Card} W$.

On the other hand, we can form $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$, which has a natural structure of an associative algebra. It has a natural G-action given by the adjoint action on the first factor and the trivial action on the second factor. Clearly, $\mathcal{U}(\mathfrak{h})$ is the subalgebra of G-invariants of this algebra. By 1.6, there exists a natural algebra homomorphism

$$\Psi : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h}) \longrightarrow \Gamma(X, \mathcal{D}_\mathfrak{h})$$

given by the tensor product of the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_\mathfrak{h})$ and ϕ.

We transfer, via the isomorphism ϕ, the filtration of $\phi(\mathcal{U}(\mathfrak{h}))$ to $\mathcal{U}(\mathfrak{h})$ and define a filtration on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$ by

$$\mathrm{F}_p(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h})) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathrm{F}_p(\mathcal{U}(\mathfrak{h})).$$

The map Ψ is evidently compatible with the filtrations. Consider the corresponding graded morphism $\mathrm{Gr}\,\Psi$ from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathrm{Gr}\,\mathcal{U}(\mathfrak{h})$ into $\mathrm{Gr}\,\Gamma(X, \mathcal{D}_\mathfrak{h})$. By the previous discussion we know that $\mathrm{Gr}\,\Gamma(X, \mathcal{D}_\mathfrak{h})$, considered as a $(\mathfrak{g} \times \mathfrak{g}, G)$-module, is the direct sum of $\mathrm{Card}\,W$ copies of $\mathcal{U}(\mathfrak{g})$. Hence, there exist G-invariant elements $e_1, e_2, \dots, e_q$, $q = \mathrm{Card}\ W$, such that $\mathrm{Gr}\,\Gamma(X, \mathcal{D}_\mathfrak{h}) = \oplus_{1 \le k \le q} \mathcal{U}(\mathfrak{g}) e_k$ and $\mathrm{Gr}\ \phi(\mathcal{U}(\mathfrak{h})) = \oplus_{1 \le k \le q} \mathcal{Z}(\mathfrak{g}) e_k$. Hence, $\mathrm{Gr}\,\Psi$ is evidently an isomorphism. This implies in turn that Ψ is also an isomorphism. Therefore, we proved the following result.

Theorem 5. *(i)* $\Gamma(X, \mathcal{D}_\mathfrak{h}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} \mathcal{U}(\mathfrak{h})$.
(ii) $H^i(X, \mathcal{D}_\mathfrak{h}) = 0$ *for* $i > 0$.

Now, let V a $\mathfrak{h}$-module. Then, by 1.6, we have the natural map from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} V$ into $\Gamma(X, \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V)$. Let

$$\cdots \longrightarrow F^{-p} \longrightarrow F^{-p+1} \longrightarrow \cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow V \longrightarrow 0$$

be a left free $\mathcal{U}(\mathfrak{h})$-module resolution of V. Then, by tensoring with $\mathcal{D}_\mathfrak{h}$ over $\mathcal{U}(\mathfrak{h})$ we get

$$\cdots \longrightarrow \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} F^{-p} \longrightarrow \cdots \longrightarrow \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} F^0 \longrightarrow \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V \longrightarrow 0.$$

By 1.4, $\mathcal{D}_\mathfrak{h}$ is locally $\mathcal{U}(\mathfrak{h})$-free, hence this is an exact sequence. Therefore, by 5.(ii), it is a left resolution of $\mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V$ by $\Gamma(X, -)$-acyclic sheaves. This implies first that all higher cohomologies of $\mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V$ vanish. Also, it gives, using 5.(i), the exact sequence

$$\cdots \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} F^{-p} \longrightarrow \cdots \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} F^0 \longrightarrow \Gamma(X, \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V) \longrightarrow 0,$$

which combined with 4, implies that $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} V = \Gamma(X, \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V)$ and $\mathrm{Tor}_p^{\mathcal{Z}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), V) = 0$ for $p \in \mathbb{N}$. Therefore, we have the following result.

Corollary 6. *Let* V *be an arbitrary* $\mathcal{U}(\mathfrak{h})$*-module. Then*
(i) $\Gamma(X, \mathcal{D}_\mathfrak{h} \otimes_{\mathcal{U}(\mathfrak{h})} V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}(\mathfrak{g})} V$;

(ii) $H^i(X, \mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}(\mathfrak{h})} V) = 0$ *for* $i > 0$.

In particular, if $\lambda \in \mathfrak{h}^*$ and $\mathbb{C}_{\lambda+\rho}$ the one dimensional $\mathfrak{h}$-module on which $\mathfrak{h}$ acts via $\lambda + \rho$, we finally get 1.

L. Localization of $\mathcal{U}_\theta$-modules

L.1 Localization of $\mathcal{U}_\theta$-modules

Let $\lambda \in \mathfrak{h}^*$ and θ the corresponding Weyl group orbit. Then we can define a right exact covariant functor Δ_λ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ by

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for any $V \in \mathcal{M}(\mathcal{U}_\theta)$. It is called the *localization functor*. Since

$$\Gamma(X, \mathcal{W}) = \mathrm{Hom}_{\mathcal{D}_\lambda}(\mathcal{D}_\lambda, \mathcal{W})$$

for any $\mathcal{W} \in \mathcal{M}(\mathcal{D}_\lambda)$, it follows that Δ_λ is a left adjoint functor to the functor of global sections Γ, i. e.

$$\mathrm{Hom}_{\mathcal{D}_\lambda}(\Delta_\lambda(V), \mathcal{W}) = \mathrm{Hom}_{\mathcal{U}_\theta}(V, \Gamma(X, \mathcal{W})),$$

for any $V \in \mathcal{M}(\mathcal{U}_\theta)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{D}_\lambda)$. In particular, there exists a functorial morphism φ from the identity functor into $\Gamma \circ \Delta_\lambda$. For any $V \in \mathcal{M}(\mathcal{U}_\theta)$, it is given by the natural morphism $\varphi_V : V \longrightarrow \Gamma(X, \Delta_\lambda(V))$.

Assume first that $\lambda \in \mathfrak{h}^*$ is antidominant.

Lemma 1. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then the natural map φ_V of V into $\Gamma(X, \Delta_\lambda(V))$ is an isomorphism of $\mathfrak{g}$-modules.*

Proof. If $V = \mathcal{U}_\theta$ this follows from C.6.1. Also, by C.3.2, we know that Γ is exact in this situation. This implies that $\Gamma \circ \Delta_\lambda$ is a right exact functor. Let

$$(\mathcal{U}_\theta)^{(J)} \longrightarrow (\mathcal{U}_\theta)^{(I)} \longrightarrow V \longrightarrow 0$$

be an exact sequence of $\mathfrak{g}$-modules. Then we have the commutative diagram

$$\begin{array}{ccccccc}
(\mathcal{U}_\theta)^{(J)} & \longrightarrow & (\mathcal{U}_\theta)^{(I)} & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma(X, \Delta_\lambda(\mathcal{U}_\theta))^{(J)} & \longrightarrow & \Gamma(X, \Delta_\lambda(\mathcal{U}_\theta))^{(I)} & \longrightarrow & \Gamma(X, \Delta_\lambda(V)) & \longrightarrow & 0
\end{array}$$

with exact rows, and the first two vertical arrows are isomorphisms. This implies that the third one is also an isomorphism. □

On the other hand, the adjointness gives also a functorial morphism ψ from $\Delta_\lambda \circ \Gamma$ into the identity functor. For any $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$, it is given by the natural morphism $\psi_\mathcal{V}$ of $\Delta_\lambda(\Gamma(X,\mathcal{V})) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} \Gamma(X,\mathcal{V})$ into $\mathcal{V}$. Assume that $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Let $\mathcal{K}$ be the kernel and $\mathcal{C}$ the cokernel of $\psi_\mathcal{V}$. Then we have the exact sequence of quasi-coherent $\mathcal{D}_\lambda$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta_\lambda(\Gamma(X,\mathcal{V})) \longrightarrow \mathcal{V} \longrightarrow \mathcal{C} \longrightarrow 0$$

and by applying Γ and using C.3.2. we get the exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{K}) \longrightarrow \Gamma(X,\Delta_\lambda(\Gamma(X,\mathcal{V}))) \longrightarrow \Gamma(X,\mathcal{V}) \longrightarrow \Gamma(X,\mathcal{C}) \longrightarrow 0.$$

Hence, by 1. we see that $\Gamma(X,\mathcal{K}) = 0$ and $\Gamma(X,\mathcal{C}) = 0$. This implies the following result.

Denote by $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$ the quotient category of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ with respect to the subcategory of all quasi-coherent $\mathcal{D}_\lambda$-modules with no global sections ([**Tôhoku**], 1.11). Let Q be the quotient functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$. Clearly, Γ induces an exact functor from $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{U}_\theta)$ which we also denote, by abuse of notation, by Γ.

Theorem 2. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then the functor $Q \circ \Delta_\lambda$ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{QM}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is Γ.*

If λ is antidominant and regular, by C.4.4, all objects in $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ are generated by their global sections. Therefore, in this case, $\mathcal{QM}_{qc}(\mathcal{D}_\lambda) = \mathcal{M}_{qc}(\mathcal{D}_\lambda)$.

Corollary 3. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functor Δ_λ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is Γ.*

As the first application of this equivalence of categories we shall prove a result on homological dimension of the ring $\mathcal{U}_\theta$.

Theorem 4. *Let θ be a Weyl group orbit in $\mathfrak{h}^*$ consisting of regular elements. Then the homological dimension $\mathrm{dh}(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$ is $\leq \dim X + \frac{1}{2} \operatorname{Card} \Sigma_\lambda$.*

Proof . Let $\lambda \in \theta$ be antidominant. By ([**BDM**], VI.1.10(ii)), we know that the homological dimension of $\mathcal{D}_{X,x}$ is equal to $\dim X$. Since $\mathcal{D}_\lambda$ is a twisted sheaf of differential operators, we conclude that $\mathrm{dh}(\mathcal{D}_{\lambda,x}) = \dim X$. Moreover, by ([**Tôhoku**], 4.2.2), we have

$$\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{V},\mathcal{U})_x = \mathrm{Ext}^i_{\mathcal{D}_{\lambda,x}}(\mathcal{V}_x,\mathcal{U}_x)$$

for any $i \in \mathbb{Z}_+$, $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ and $\mathcal{U} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. This implies that

$$\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{V},\mathcal{U}) = 0 \text{ for } i > \dim X.$$

On the other hand, we have the spectral sequence

$$H^p(X, \mathcal{E}xt^q_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{U})) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{U})$$

([**Tôhoku**], 4.2.1), and cohomology of any sheaf of abelian groups vanishes in all degrees above $\dim X$ ([**Hartshorne**], III.2.7). It follows that $\operatorname{Ext}^i_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{U}) = 0$ for $i > 2\dim X$. Now, by 3, $\mathcal{M}(\mathcal{U}_\theta)$ is equivalent to $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and localization of any finitely generated $\mathcal{U}_\theta$-module is in $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$. This implies that $\operatorname{Ext}^i_{\mathcal{U}_\theta}(V, U) = 0$ for $i > 2\dim X$ for any $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta)$ and $U \in \mathcal{M}(\mathcal{U}_\theta)$. By ([**Alg**], Ch. X, §8, no. 3, Cor. of Prop. 4) we see that $\operatorname{dh}(\mathcal{U}_\theta) \leq 2\dim X$.

If $\mathcal{V}$ is any coherent $\mathcal{D}_\lambda$-module, $\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{D}_\lambda)$, $i \in \mathbb{Z}_+$, are coherent $\mathcal{D}_{-\lambda}$-modules (...). Since λ is regular antidominant, $n(-\lambda) = \frac{1}{2}\operatorname{Card}\Sigma_\lambda \leq \dim X$, and by C.3.1. the cohomology of $\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{D}_\lambda)$ vanishes above $\frac{1}{2}\operatorname{Card}\Sigma_\lambda$. Therefore, as in the preceding argument, we conclude that $\operatorname{Ext}^i_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{D}_\lambda) = 0$ for $i > \dim X + \frac{1}{2}\operatorname{Card}\Sigma_\lambda$. By the equivalence of categories this immediately implies that $\operatorname{Ext}^i_{\mathcal{U}_\theta}(V, \mathcal{U}_\theta) = 0$ for any $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta)$ and $i > \dim X + \frac{1}{2}\operatorname{Card}\Sigma_\lambda$.

Assume now that V and U are in $\mathcal{M}_{fg}(\mathcal{U}_\theta)$. Since $\mathcal{U}_\theta$ is left nötherian, we have an exact sequence

$$0 \longrightarrow U' \longrightarrow \mathcal{U}_\theta^p \longrightarrow U \longrightarrow 0$$

with finitely generated $\mathcal{U}_\theta$-module U'. From the corresponding long exact sequence of $\operatorname{Ext}^{\cdot}_{\mathcal{U}_\theta}(V, -)$ we see that the connecting morphism $\operatorname{Ext}^j_{\mathcal{U}_\theta}(V, U) \to \operatorname{Ext}^{j+1}_{\mathcal{U}_\theta}(V, U')$ is an isomorphism for $j > \dim X + \frac{1}{2}\operatorname{Card}\Sigma_\lambda$. Since the homological dimension of $\mathcal{U}_\theta$ is finite, by downward induction in j we see that $\operatorname{Ext}^j_{\mathcal{U}_\theta}(V, U) = 0$ for $j > \dim X + \frac{1}{2}\operatorname{Card}\Sigma_\lambda$. By ([**Alg**], Ch. X, §8, no. 3, Cor. of Prop. 4) it follows that $\operatorname{dh}(\mathcal{U}_\theta) \leq \dim X + \frac{1}{2}\operatorname{Card}\Sigma_\lambda$. □

Remark 5. We shall see later in 2.8 that, contrary to 4, if θ contains singular elements of $\mathfrak{h}^*$, the homological dimension of $\mathcal{U}_\theta$ is infinite.

Also, for any W-orbit θ of an regular integral weight λ the preceding estimate of homological dimension of $\mathcal{U}_\theta$ is sharp. To see this, assume that $\lambda \in P(\Sigma)$ is regular antidominant and let F be the irreducible finite-dimensional $\mathfrak{g}$-module with lowest weight $\lambda + \rho$. Then, by the Borel-Weil-Bott theorem and 1.2, we know that $\Delta_\lambda(F) = \mathcal{O}(\lambda + \rho)$. Therefore, by ..., we have

$$\mathcal{E}xt^p_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda + \rho), \mathcal{D}_\lambda) = 0$$

for $p \neq \dim X$ and

$$\mathcal{E}xt^{\dim X}_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda + \rho), \mathcal{D}_\lambda) = \mathcal{O}(-\lambda + \rho)$$

as a left $\mathcal{D}_{-\lambda}$-module. Therefore, applying again the Borel-Weil-Bott theorem, we see that

$$H^p(X, \mathcal{E}xt^{\dim X}_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda + \rho), \mathcal{D}_\lambda)) = 0$$

for $p \neq \dim X$ and

$$H^{\dim X}(X, \mathcal{E}xt^{\dim X}_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda+\rho), \mathcal{D}_\lambda)) \neq 0.$$

By the Grothendieck spectral sequence relating $\mathcal{E}xt_{\mathcal{D}_\lambda}$ to $\operatorname{Ext}_{\mathcal{D}_\lambda}$, this implies that

$$\operatorname{Ext}^p_{\mathcal{U}_\theta}(F, \mathcal{U}_\theta) = \operatorname{Ext}^p_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda+\rho), \mathcal{D}_\lambda) = 0$$

if $p \neq 2\dim X$, and

$$\operatorname{Ext}^{2\dim X}_{\mathcal{U}_\theta}(F, \mathcal{U}_\theta) = \operatorname{Ext}^{2\dim X}_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda+\rho), \mathcal{D}_\lambda) \neq 0.$$

Hence, in this case $\operatorname{dh}(\mathcal{U}_\theta) = 2\dim X = \operatorname{Card}\Sigma$.

As a second application, we want to consider various derived categories of $\mathcal{D}_\lambda$-modules on X. As before, let $D(\mathcal{M}(\mathcal{D}_\lambda))$ be the derived category of $\mathcal{D}_\lambda$-modules, $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ its full subcategory consisting of complexes with quasi-coherent cohomology. Also, we can consider the derived category $D(\mathcal{D}_\lambda) = D(\mathcal{M}_{qc}(\mathcal{D}_\lambda))$ of quasi-coherent $\mathcal{D}_\lambda$-modules. As we remarked before, for any $\mu \in P(\Sigma)$, the geometric translation functor $\mathcal{V} \longrightarrow \mathcal{V}(\mu)$ is an equivalence of $\mathcal{M}(\mathcal{D}_\lambda)$ with $\mathcal{M}(\mathcal{D}_{\lambda+\mu})$, which also induces an equivalence of subcategories $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\mu})$. Moreover, it induces equivalencies of the corresponding derived categories $D(\mathcal{M}(\mathcal{D}_\lambda))$, $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ and $D(\mathcal{D}_\lambda)$ with $D(\mathcal{M}(\mathcal{D}_{\lambda+\mu}))$, $D_{qc}(\mathcal{M}(\mathcal{D}_{\lambda+\mu}))$ and $D(\mathcal{D}_{\lambda+\mu})$ respectively. In addition, the canonical functor Φ_λ from $D(\mathcal{D}_\lambda)$ into $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ satisfies the natural commutativity property with respect to these translation functors.

Theorem 6. *The functor $\Phi_\lambda : D(\mathcal{D}_\lambda) \longrightarrow D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ is an equivalence of categories.*

Proof. First, by the preceding discussion, by translation we can assume that λ is antidominant and regular. In this situation, the localization functor Δ_λ is exact. Therefore, it induces a functor $\Delta_\lambda : V^{\cdot} \longrightarrow \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V^{\cdot}$ from the derived category $D(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$-modules into the category $D(\mathcal{D}_\lambda)$. On the other hand, by C.3.1. every object in $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is Γ-acyclic, what in combination with 3. immediately implies that the functor $\Gamma : D(\mathcal{D}_\lambda) \longrightarrow D(\mathcal{U}_\theta)$ is an equivalence of categories and its inverse is the localization functor Δ_λ. Moreover, by the finiteness of right cohomological dimension of Γ on $\mathcal{M}(\mathcal{D}_\lambda)$, we have the derived functor $R\Gamma : D_{qc}(\mathcal{M}(\mathcal{D}_\lambda)) \longrightarrow D(\mathcal{U}_\theta)$, and clearly $R\Gamma \circ \Phi_\lambda = \Gamma$. Also, we can replace any $\mathcal{C}^{\cdot} \in D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ with a quasi-isomorphic complex $\mathcal{A}^{\cdot}$ consisting of Γ-acyclic objects from $\mathcal{M}(\mathcal{D}_\lambda)$. Therefore, $R\Gamma(\mathcal{C}^{\cdot}) = \Gamma(\mathcal{A}^{\cdot})$. Let $\mathcal{D}^{\cdot} = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} \Gamma(\mathcal{A}^{\cdot})$. We have the natural homomorphism $\phi : \mathcal{D}^{\cdot} \longrightarrow \mathcal{A}^{\cdot}$. We claim that it is a quasi-isomorphism. Clearly, by definition of ϕ and 1, $R\Gamma(\phi)$ is a quasi-isomorphism. Hence, by C.4.1. we see that ϕ is a quasi-isomorphism. It follows that $\Phi_\lambda \circ (\Delta_\lambda \circ R\Gamma)$ is isomorphic to the identity functor on $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$. On the other hand, $(\Delta_\lambda \circ R\Gamma) \circ \Phi_\lambda$ is isomorphic to the identity functor on $D(\mathcal{D}_\lambda)$. Therefore, Φ_λ is an equivalence of categories. □

Analogous statements hold for derived categories of complexes bounded above and below.

Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Denote by $D^-(\mathcal{U}_\theta)$ the derived category of $\mathcal{U}_\theta$-modules consisting of complexes bounded from above. We define the localization functor $L\Delta_\lambda$ from $D^-(\mathcal{U}_\theta)$ into $D(\mathcal{D}_\lambda)$ by

$$L\Delta_\lambda(V^\cdot) = \mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\theta} V^\cdot \text{ for } V^\cdot \in D^-(\mathcal{U}_\theta).$$

If λ is regular, 4. implies that the left cohomological dimension of the localization functor Δ_λ is $\leq 2 \dim X$. Therefore, one can extend $L\Delta_\lambda$ to a functor from $D(\mathcal{U}_\theta)$ into $D(\mathcal{D}_\lambda)$.

Lemma 7. *Let $P \in \mathcal{M}(\mathcal{U}_\theta)$ be projective. Then, its localization $\Delta_\lambda(P)$ is Γ-acyclic, and the morphism $\varphi_P : P \longrightarrow \Gamma(X, \Delta_\lambda(P))$ is an isomorphism.*

Proof. By C.6.1. we know that this statement is valid for free $\mathcal{U}_\theta$-modules, and any projective $\mathcal{U}_\theta$-module is a direct summand of a free $\mathcal{U}_\theta$-module. □

Let $V^\cdot \in D^-(\mathcal{U}_\theta)$, then there exists a complex $P^\cdot \in D(\mathcal{U}_\theta)$ of projective $\mathcal{U}_\theta$-modules, a quasi-isomorphism $\alpha_{V^\cdot} : P^\cdot \longrightarrow V^\cdot$ and $L\Delta_\lambda(V^\cdot) = \Delta_\lambda(P^\cdot)$. By 7. it follows that there is a natural isomorphism from $P^\cdot$ into $\Gamma(\Delta_\lambda(P^\cdot)) = R\Gamma(\Delta_\lambda(P^\cdot))$. This implies that the following result holds.

Lemma 8. *The functor $R\Gamma \circ L\Delta_\lambda$ from $D^-(\mathcal{U}_\theta)$ into itself is isomorphic to the identity functor on $D^-(\mathcal{U}_\theta)$.*

Let $D : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow D^-(\mathcal{U}_\theta)$ be the functor which maps any $V \in \mathcal{M}(\mathcal{U}_\theta)$ into the complex $D(V) \in D^-(\mathcal{U}_\theta)$ which is zero in all degrees except 0, where it is equal to V. If we denote, for any $V \in \mathcal{M}(\mathcal{U}_\theta)$, by $L^j\Delta_\lambda(V)$ the j^{th} cohomology of the complex $L\Delta_\lambda(D(V))$, we get the functor $V \longrightarrow L^j\Delta_\lambda(V)$ form $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ which is just the $(-j)^{\text{th}}$ left derived functor of Δ_λ. Therefore, 8. implies the following result.

Corollary 9. *Let $V \in \mathcal{M}(\mathcal{U}_\theta)$. Then there exists a cohomological spectral sequence with E_2-term*

$$E_2^{p,q} = H^p(X, L^q\Delta_\lambda(V))$$

which converges to V.

Corollary 10. *Let $F \in \mathcal{M}(\mathcal{U}_\theta)$ be a flat module. Then, its localization $\Delta_\lambda(F)$ is Γ-acyclic, and the morphism $\varphi_F : F \longrightarrow \Gamma(X, \Delta_\lambda(F))$ is an isomorphism.*

Proof. By definition, $L^q\Delta_\lambda(F) = 0$ for $q \neq 0$. Therefore the spectral sequence from 9. degenerates and we see that $H^i(X, \Delta_\lambda(F)) = 0$ for $i > 0$ and $\varphi_F : F \longrightarrow \Gamma(X, \Delta_\lambda(F))$ is an isomorphism. □

Assume now that $\lambda \in \mathfrak{h}^*$ is regular. Then the homological dimension of $\mathcal{U}_\theta$ is finite, hence any $V^\cdot \in D(\mathcal{U}_\theta)$ is quasi-isomorphic with a complex $P^\cdot \in D(\mathcal{U}_\theta)$ consisting of projective $\mathcal{U}_\theta$-modules (...). Now, using 7. again, we can prove the following version of 8.

Lemma 11. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then the functor $R\Gamma \circ L\Delta_\lambda$ from $D(\mathcal{U}_\theta)$ into itself is isomorphic to the identity functor on $D(\mathcal{U}_\theta)$.*

This finally leads to the following analogue of 3.

Theorem 12. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then the functor $L\Delta_\lambda$ from $D(\mathcal{U}_\theta)$ into $D(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is $R\Gamma$.*

Proof. Let $\mathcal{V}^\cdot \in D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$. Then, there exists a complex $\mathcal{C}^\cdot \in D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$ consisting of Γ-acyclic $\mathcal{D}_\lambda$-modules, a quasi-isomorphism $\beta_{\mathcal{V}^\cdot} : \mathcal{V}^\cdot \longrightarrow \mathcal{C}^\cdot$ and $R\Gamma(\mathcal{V}^\cdot) = \Gamma(\mathcal{C}^\cdot)$. Moreover, there exists a complex $P^\cdot \in D(\mathcal{U}_\theta)$ consisting of projective $\mathcal{U}_\theta$-modules and a quasi-isomorphism $\alpha_{\Gamma(\mathcal{C}^\cdot)} : P^\cdot \longrightarrow \Gamma(\mathcal{C}^\cdot)$ such that $L\Delta_\lambda(\Gamma(\mathcal{C}^\cdot)) = \Delta_\lambda(P^\cdot)$. Therefore, we get a natural morphism of $L\Delta_\lambda(R\Gamma(\mathcal{V}^\cdot))) = \Delta_\lambda(P^\cdot)$ into $\mathcal{C}^\cdot$. This gives a functorial morphism of $L\Delta_\lambda \circ R\Gamma$ into the identity functor on $D_{qc}(\mathcal{M}(\mathcal{D}_\lambda))$. By 6, the composition with Φ_λ gives a morphism ψ of functor $L\Delta_\lambda \circ R\Gamma$ into the identity functor on $D(\mathcal{D}_\lambda)$. It follows that, for any complex $\mathcal{V}^\cdot \in D(\mathcal{D}_\lambda)$, there exists a morphism $\psi_{\mathcal{V}^\cdot}$ of $L\Delta_\lambda(R\Gamma(\mathcal{V}^\cdot))$ into $\mathcal{V}^\cdot$, and by checking its definition and using 11, we see that $R\Gamma(\psi_{\mathcal{V}^\cdot})$ is an quasi-isomorphism. Now, C.4.1. implies that ψ is an isomorphism of functors. □

This implies, in particular, that $L\Delta_\lambda$ is an equivalence of category $D^b(\mathcal{U}_\theta)$ with $D^b(\mathcal{D}_\lambda)$ and $R\Gamma$ is its inverse.

Theorem 13. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then the left cohomological dimension of Δ_λ is $\leq n(\lambda)$.*

Proof. Let $\mathcal{V}^\cdot \in D(\mathcal{D}_\lambda)$ and $k \in \mathbb{Z}$. Then the truncated complex $\sigma_{\leq k}(\mathcal{V}^\cdot)$:

$$\dots \longrightarrow \mathcal{V}^p \longrightarrow \dots \longrightarrow \mathcal{V}^{k-2} \longrightarrow \mathcal{V}^{k-1} \longrightarrow \ker d^k \longrightarrow 0 \longrightarrow \dots$$

maps naturally into $\mathcal{V}^\cdot$ and this morphism of complexes induces isomorphisms $H^p(\sigma_{\leq k}(\mathcal{V}^\cdot)) \longrightarrow H^p(\mathcal{V}^\cdot)$ for $p \leq k$. Let $V \in \mathcal{M}(\mathcal{U}_\theta)$ and $\mathcal{V}^\cdot = L\Delta_\lambda(D(V))$. Assume that $-k > n(\lambda)$. Then we have a cohomological spectral sequence

$$H^p(X, H^q(\sigma_{\leq k}(\mathcal{V}^\cdot))) \Longrightarrow H^{p+q}(R\Gamma(\sigma_{\leq k}(\mathcal{V}^\cdot)))$$

([**Tôhoku**], II.2.4). By C.3.1, we conclude that $H^p(R\Gamma(\sigma_{\leq k}(\mathcal{V}^\cdot))) = 0$ for $p \in \mathbb{Z}_+$. Hence, by 12,

$$\begin{aligned}\operatorname{Hom}_{D(\mathcal{D}_\lambda)}(\sigma_{\leq k}(\mathcal{V}^\cdot), \mathcal{V}^\cdot) &= \operatorname{Hom}_{D(\mathcal{U}_\theta)}(R\Gamma(\sigma_{\leq k}(\mathcal{V}^\cdot)), R\Gamma(\mathcal{V}^\cdot)) \\ &= \operatorname{Hom}_{D(\mathcal{U}_\theta)}(R\Gamma(\sigma_{\leq k}(\mathcal{V}^\cdot)), D(V)) = 0.\end{aligned}$$

Therefore, $L^p\Delta_\lambda(V) = H^p(\mathcal{V}^\cdot) = 0$ for $p \leq -k$. □

Remark 14. On the contrary, we shall see in 2.7. that if λ is singular the left cohomological dimension of Δ_λ is infinite.

Let $\mathcal{A}$ be an abelian category and $D^b(\mathcal{A})$ its derived category of bounded complexes. For $s \in \mathbb{Z}$, we have the truncation functors $\tau_{\geq s}$ and $\tau_{\leq s}$ from $D^b(\mathcal{A})$ into itself. If $A^\cdot$ is a complex in $D^b(\mathcal{A})$, $\tau_{\geq s}(A^\cdot)$ is a complex which is zero in degrees less than s, $\tau_{\geq s}(A^\cdot)^s = \operatorname{coker} d^{s-1}$ and $\tau_{\geq s}(A^\cdot)^q = A^q$ for $q > s$, with the differentials induced by the differentials of $A^\cdot$. On the other hand, $\tau_{\leq s}(A^\cdot)$ is a complex which is zero in degrees greather than s, $\tau_{\leq s}(A^\cdot)^s = \ker d^s$ and $\tau_{\leq s}(A^\cdot)^q = A^q$ for $q < s$, with the differentials induced by the differentials of $A^\cdot$. The natural morphisms $\tau_{\leq s}(A^\cdot) \to A^\cdot$ and $A^\cdot \to \tau_{\geq s}(A^\cdot)$ induce isomorphisms on cohomology in degrees $\leq s$ and $\geq s$ respectively.

Lemma 15. *Let $C^\cdot$ and $D^\cdot$ be in $D^b(\mathcal{A})$. Assume that*

(i) $H^q(C^\cdot) = 0$ for $q > 0$,

(ii) $H^q(D^\cdot) = 0$ for $q < 0$.

Then $H^0 : \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D^\cdot) \to \operatorname{Hom}_{\mathcal{A}}(H^0(C^\cdot), H^0(D^\cdot))$ is an isomorphism.

Proof. By hypothesis, $\tau_{\leq 0}(C^\cdot) \to C^\cdot$ and $D^\cdot \to \tau_{\geq 0}(D^\cdot)$ are quasiisomorphisms, and by composing them with ϕ we can assume that $C^q = 0$ for $q > 0$ and $D^q = 0$ for $q < 0$. Therefore, each element of $\operatorname{Hom}_{\mathcal{A}}(H^0(C^\cdot), H^(D^\cdot))$ defines a morphism of the complex $C^\cdot$ into the complex $D^\cdot$ and our mapping is surjective.

To prove injectivity, consider a morphism $\phi \in \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D^\cdot)$ such that $H^0(\phi) = 0$. By the definition of a morphism in derived categories, there exist a complex $B^\cdot \in D^b(\mathcal{A})$ and morphisms of complexes $q : B^\cdot \to C^\cdot$, $f : B^\cdot \to D^\cdot$, where q is a quasiisomorphism, which represent ϕ. By composing them with the truncation morphism $\tau_{\leq 0}(B^\cdot) \to B^\cdot$, we see that we can assume in addition that $B^\cdot$ satisfies $B^q = 0$ for $q > 0$. But this implies that $f^q = 0$ for $q \neq 0$, $\operatorname{im} f^0 \subset \ker d^0$ and $\operatorname{im} d^{-1} \subset \ker f^0$. Hence $H^0(\phi) = 0$ implies $f^0 = 0$. □

The next result is a weak generalization of 1. to arbitrary regular $\lambda \in \mathfrak{h}^*$.

Lemma 16. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\theta = W \cdot \lambda$. Let V be a $\mathcal{U}_\theta$-module and $p = \min\{q \in \mathbb{Z} \mid L^{-q}\Delta_\lambda(V) \neq 0\}$. Assume that $H^q(X, L^{-p}\Delta_\lambda(V)) = 0$ for $q < p$. Then there exists a nontrivial morphism of V into $H^p(X, L^{-p}\Delta_\lambda(V))$.*

Proof. Consider the truncation morphism

$$L\Delta_\lambda(D(V)) \to \tau_{\geq -p}(L\Delta_\lambda(D(V))) = D(L^{-p}\Delta_\lambda(V))[p].$$

By equivalence of derived categories, it leads to a nontrivial morphism ϕ of $D(V)$ into $R\Gamma(D(L^{-p}\Delta_\lambda(V)[p]) = R\Gamma(D(L^{-p}\Delta_\lambda(V))[p]$. It induces zero morphisms between the cohomology modules of both complexes, except in degree zero where we get a morphism of V into $H^p(X, L^{-p}\Delta_\lambda(V))$. Since cohomology modules of $L^{-p}\Delta_\lambda(V)$ vanish below degree p, the complex $R\Gamma(D(L^{-p}\Delta_\lambda(V))[p]$ satisfies the condition (ii). Hence, by 15, the morphism $H^0(\phi)$ of V into $R\Gamma(D(L^{-p}\Delta_\lambda(V)))[p]^0 = H^p(X, L^{-p}\Delta_\lambda(V))$ is nonzero. □

Now we want to study some finiteness results.

Let $\mathcal{M}_{fg}(\mathcal{U}_\theta)$ be the full subcategory of $\mathcal{M}(\mathcal{U}_\theta)$ consisting of finitely generated $\mathcal{U}_\theta$-modules. Clearly, for any $\lambda \in \theta$, the localization of $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta)$ is a coherent $\mathcal{D}_\lambda$-module. Conversely, we have the following result.

Lemma 17. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then for any $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$, the $\mathcal{U}_\theta$-module $\Gamma(X, \mathcal{V})$ is finitely generated.*

Proof. Let $V = \Gamma(X, \mathcal{V})$ for a coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$. Assume that $(V_n; n \in \mathbb{N})$ is an increasing sequence of finitely generated $\mathcal{U}_\theta$-submodules of V. By localizing it, we get an increasing sequence $(\mathcal{V}_n = \Delta_\lambda(V_n);\ n \in \mathbb{N})$ of coherent $\mathcal{D}_\lambda$-submodules of $\mathcal{V}$. Since $\mathcal{D}_\lambda$ is a nötherian sheaf of rings, it follows that the sequence $(\mathcal{V}_n;\ n \in \mathbb{N})$ stabilizes. By applying Γ and 3. we get the same conclusion for the original sequence $(V_n;\ n \in \mathbb{N})$. This implies that V is finitely generated. □

This implies the following ramification of 3.

Proposition 18. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functor Δ_λ from $\mathcal{M}_{fg}(\mathcal{U}_\theta)$ into $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is Γ.*

Now we want to extend these results to regular $\lambda \in \theta$. First, $\mathcal{M}_{fg}(\mathcal{U}_\theta)$ is a thick subcategory of $\mathcal{M}(\mathcal{U}_\theta)$, therefore we can consider the full subcategory $D^b_{fg}(\mathcal{U}_\theta)$ of $D^b(\mathcal{U}_\theta)$ consisting of all bounded complexes with finitely generated cohomology modules.

Lemma 19. *The natural functor i from $D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$ into $D^b_{fg}(\mathcal{U}_\theta)$ is an equivalence of categories.*

Proof. First we claim that $i : D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta)) \longrightarrow D^b_{fg}(\mathcal{U}_\theta)$ is fully faithful. Let $A^\cdot, B^\cdot \in D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$ and $\varphi \in \operatorname{Hom}_{D^b(\mathcal{U}_\theta)}(A^\cdot, B^\cdot)$. Then there exists a complex $C^\cdot \in D^b_{fg}(\mathcal{U}_\theta)$, a quasi-isomorphism $s : C^\cdot \longrightarrow A^\cdot$ and a morphism of complexes $f : C^\cdot \longrightarrow B^\cdot$ which define φ. By ... we can find a complex $D^\cdot \in D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$ and a quasi-isomorphism $s' : D^\cdot \longrightarrow C^\cdot$. It follows that $D^\cdot$, $s \circ s'$ and $f \circ s'$ define also φ, what implies that φ is a morphism in $D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$. This proves that i is fully faithful. Also, by ... it is essentially surjective. □

This result in particular implies that for any $V^\cdot \in D^b_{fg}(\mathcal{U}_\theta)$ its localization $L\Delta_\lambda(V^\cdot) \in D^-(\mathcal{D}_\lambda)$ is a complex with coherent cohomology. To discuss this more precisely we first introduce several subcategories of $D^b(\mathcal{D}_\lambda)$. Since the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ is a thick subcategory of $\mathcal{M}(\mathcal{D}_\lambda)$, we can define the category $D^b_{coh}(\mathcal{D}_\lambda)$ which is the full subcategory of $D^b(\mathcal{D}_\lambda)$ consisting of all bounded complexes with coherent cohomology and the derived category $D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda))$ of coherent $\mathcal{D}_\lambda$-modules. There is a natural functor Ψ_λ from $D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda))$ into $D^b_{coh}(\mathcal{D}_\lambda)$, and it satisfies the natural commutativity property with respect to the translation functors.

Lemma 20. *The natural functor Ψ_λ from $D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda))$ into $D^b_{coh}(\mathcal{D}_\lambda)$ is an equivalence of categories.*

Proof. Using translation functors we can assume that λ is antidominant and regular. By 3. and 6, this result follows from 19. □

Therefore, for regular $\lambda \in \mathfrak{h}^*$, we can view the functor $L\Delta_\lambda$ as the functor from $D^b_{fg}(\mathcal{U}_\theta)$ into $D^b_{coh}(\mathcal{D}_\lambda)$. Moreover, we have the following result.

Theorem 21. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then the functor $L\Delta_\lambda$ from $D^b_{fg}(\mathcal{U}_\theta)$ into $D^b_{coh}(\mathcal{D}_\lambda)$ is an equivalence of categories. Its inverse is $R\Gamma$.*

To prove this statement we only need to show that $R\Gamma(\mathcal{V}^\cdot) \in D^b_{fg}(\mathcal{U}_\theta)$ for any $\mathcal{V}^\cdot \in D^b_{coh}(\mathcal{D}_\lambda)$. This is a consequence of the following generalization of 17. First we need a simple lemma.

Lemma 22. *Let V be a finitely generated $\mathfrak{g}$-module and F a finite-dimensional $\mathfrak{g}$-module. Then $V \otimes_{\mathbb{C}} F$ is a finitely generated $\mathfrak{g}$-module.*

Proof. Let U be a finite-dimensional subspace of V which generates V as an $\mathfrak{g}$-module. We claim that $U \otimes_{\mathbb{C}} F$ generates $V \otimes_{\mathbb{C}} F$ as a $\mathfrak{g}$-module. Let W be the $\mathfrak{g}$-submodule of $V \otimes_{\mathbb{C}} F$ generated by $U \otimes_{\mathbb{C}} F$. Define V' to be the subset of V consisting of all v such that $v \otimes f \in W$ for all $f \in F$. Clearly, V' is a linear subspace of V. Moreover, for any $\xi \in \mathfrak{g}$ and $v \in V'$,

$$\xi v \otimes f = \xi(v \otimes f) - v \otimes \xi f \in W,$$

for all $f \in F$, what implies that $\xi v \in V'$, and V' is a $\mathfrak{g}$-submodule of V. On the other hand, V' contains U, hence it is equals V. This in turn gives $W = V \otimes_{\mathbb{C}} F$. □

Lemma 23. *Let $\lambda \in \mathfrak{h}^*$. Then for any $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$, $\mathcal{U}_\theta$-modules $H^i(X, \mathcal{V})$, $i \in \mathbb{Z}_+$, are finitely generated.*

Proof. The proof is by induction in i, $0 \leq i \leq \dim X$. We can find a dominant weight μ such that $\lambda - \mu$ is antidominant and regular. Then the translation $\mathcal{V}(-\mu)$ of $\mathcal{V}$ is a coherent $\mathcal{D}_{\lambda-\mu}$-module, and, by 17, $\Gamma(X, \mathcal{V}(-\mu))$ is a finitely generated $\mathfrak{g}$-module. Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight μ. Now

$$H^i(X, \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}) = H^i(X, \mathcal{V}(-\mu)) \otimes_{\mathbb{C}} F$$

for all i, $0 \leq i \leq \dim X$; therefore it vanishes for $i > 0$. On the other hand, the filtration of $\mathcal{F}$ studied in C.2. gives an injection of $\mathcal{F}_1 = \mathcal{O}(\mu)$ into $\mathcal{F}$. It follows that, by tensoring with $\mathcal{V}$, we get the exact sequence of $\mathcal{U}^\circ$-modules

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Applying Γ to this exact sequence we see that $\Gamma(X,\mathcal{V})$ is a $\mathfrak{g}$-submodule of the tensor product $\Gamma(X,\mathcal{V}(-\mu))\otimes_{\mathbb{C}} F$, which is finitely generated by 20. This proves our assertion for $i=0$. Assume that the assertion holds for $k-1$, $k\geq 1$. Then the long exact sequence of cohomology implies that $H^k(X,\mathcal{V})$ is a quotient of $H^{k-1}(X,\mathcal{K})$. On the other hand, from the definition of the filtration of $\mathcal{F}$, it follows that $\mathcal{K}$ has a natural $\mathcal{U}^\circ$-module filtration such that the corresponding graded module $\operatorname{Gr}\mathcal{K}$ is equal to $\oplus\mathcal{V}(-\mu+\nu)$, where the sum is taken over all weights ν of F different from μ. By the induction assumption, $H^{k-1}(X,\mathcal{V}(-\mu+\nu))$ are finitely generated $\mathfrak{g}$-modules. An induction in the length of the filtration of $\mathcal{K}$ implies that $H^{k-1}(X,\mathcal{K})$ is a finitely generated $\mathfrak{g}$-module. □

Finally, the equivalence of derived categories (12.) and the Borel-Weil-Bott theorem (C.5.1.) have the following imediate consequence.

Proposition 24. *Let F be the finite-dimensional irreducible $\mathfrak{g}$-module with lowest weight λ. Then, for any $\mu=w(\lambda-\rho)$, $w\in W$, we have $L^p\Delta_\mu(F)=0$ for $p\neq-\ell(w)$ and $L^{-\ell(w)}\Delta_\mu(F)=\mathcal{O}(\mu+\rho)$.*

L.2 Localization and $\mathfrak{n}$-homology

Let $\mathcal{M}(\mathcal{U}(\mathfrak{g}))$ be the category of $\mathcal{U}(\mathfrak{g})$-modules. Fix a point $x\in X$. For $V\in\mathcal{M}(\mathcal{U}(\mathfrak{g}))$, put

$$V_{\mathfrak{n}_x}=V/\mathfrak{n}_xV=\mathbb{C}\otimes_{\mathcal{U}(\mathfrak{n}_x)}V,$$

where we view $\mathbb{C}$ as a module with the trivial action of $\mathfrak{n}_x$. We say that $V_{\mathfrak{n}_x}$ is the module of $\mathfrak{n}_x$-coinvariants in V. It has a natural structure of an $\mathfrak{h}_x$-module. Therefore, we can view it as an $\mathfrak{h}$-module. It follows that $V\longrightarrow V_{\mathfrak{n}_x}$ is a right exact covariant functor from the category $\mathcal{M}(\mathcal{U}(\mathfrak{g}))$ into the category $\mathcal{M}(\mathcal{U}(\mathfrak{h}))$ of $\mathcal{U}(\mathfrak{h})$-modules. If we compose it with the forgetful functor from $\mathcal{M}(\mathcal{U}(\mathfrak{h}))$ into the category of vector spaces, we get the functor $H_0(\mathfrak{n}_x,-)$ of zeroth $\mathfrak{n}_x$-homology. By the Poincaré-Birkhoff-Witt theorem, free $\mathcal{U}(\mathfrak{g})$-modules are also $\mathcal{U}(\mathfrak{n}_x)$-free, what implies the equality for the left derived functors. Therefore, with some abuse of language, we shall call the $(-p)^{\text{th}}$ left derived functor of $V\longrightarrow V_{\mathfrak{n}_x}$ the p^{th} $\mathfrak{n}_x$-homology functor and denote it by $H_p(\mathfrak{n}_x,-)=\operatorname{Tor}_p^{\mathcal{U}(\mathfrak{n}_x)}(\mathbb{C},-)$. There is a simple relationship between these functors and the localization functors which we shall explain in the following.

First we need a technical result.

Lemma 1. *$\mathcal{U}_\theta$ is free as $\mathcal{U}(\mathfrak{n}_x)$-module.*

Proof. Fix a specialization $\mathfrak{c}$ of $\mathfrak{h}$ and a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. Then we have $\mathfrak{g}=\mathfrak{n}_x\oplus\mathfrak{c}\oplus\bar{\mathfrak{n}}$, and by the Poincaré-Birkhoff-Witt theorem it follows that $\mathcal{U}(\mathfrak{g})=\mathcal{U}(\mathfrak{n}_x)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{c})\otimes_{\mathbb{C}}\mathcal{U}(\bar{\mathfrak{n}})$ as a left $\mathcal{U}(\mathfrak{n}_x)$-module for left multiplication. Let $\mathrm{F}_p\,\mathcal{U}(\mathfrak{c})$, $p \in \mathbb{Z}_+$, be the degree filtration of $\mathcal{U}(\mathfrak{c})$. Then we define a filtration $\mathrm{F}_p\,\mathcal{U}(\mathfrak{g})$, $p \in \mathbb{Z}_+$, of $\mathcal{U}(\mathfrak{g})$ via

$$\mathrm{F}_p\,\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathrm{F}_p\,\mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

This is clearly a $\mathcal{U}(\mathfrak{n}_x)$-module filtration. The corresponding graded module is

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

This filtration induces a filtration on the submodule $J_\theta\mathcal{U}(\mathfrak{g})$ and the quotient module $\mathcal{U}_\theta$. The Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ is compatible with the degree filtrations and the homomorphism $\mathrm{Gr}\,\gamma$ is an isomorphism of $\mathrm{Gr}\,\mathcal{Z}(\mathfrak{g})$ onto the subalgebra $I(\mathfrak{h})$ of all W-invariants in $S(\mathfrak{h})$ ([**LG**], Ch. VIII, §8, no. 5). Denote by $I_+(\mathfrak{h})$ the homogeneous ideal spanned by the elements of strictly positive degree in $I(\mathfrak{h})$. Then

$$\mathrm{Gr}\,J_\theta\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

By ([**CA**], Ch. III, §2, no. 4, Prop. 2) it follows that

$$\begin{aligned}\mathrm{Gr}\,\mathcal{U}_\theta &= (\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}))/(\mathrm{Gr}\,J_\theta\mathcal{U}(\mathfrak{g}))\\ &= (\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}))/(\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}))\\ &= \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} S(\mathfrak{c})/(I_+(\mathfrak{c})S(\mathfrak{c})) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}),\end{aligned}$$

i. e. it is a free $\mathcal{U}(\mathfrak{n}_x)$-module. Moreover, by ([**LG**], Ch. V, §5, no. 2, Th. 1) we know that the dimension of the complex vector space $S(\mathfrak{h})/(I_+(\mathfrak{h})S(\mathfrak{h}))$ is $\mathrm{Card}\,W$. It follows that $\mathcal{U}_\theta$ has a finite filtration by $\mathcal{U}(\mathfrak{n}_x)$-submodules such that $\mathrm{Gr}\,\mathcal{U}_\theta$ is a free $\mathcal{U}(\mathfrak{n}_x)$-module. By induction in length, this implies that $\mathcal{U}_\theta$ is a free $\mathcal{U}(\mathfrak{n}_x)$-module. □

Let $\varphi : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathcal{U}(\mathfrak{h})$ be the automorphism given by $\varphi(\xi) = \xi + \rho(\xi)$ for $\xi \in \mathfrak{h}$. Then, by ([**LG**], Ch. VIII, §8, no. 5, Th. 2), $\varphi(\gamma(\mathcal{Z}(\mathfrak{g})))$ is the algebra of W-invariants in $\mathcal{U}(\mathfrak{h})$. In addition, by ([**LG**], Ch. V, §5, no. 2, Th. 1), the dimension of the vector space $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ is equal to $\mathrm{Card}\,W$. This implies that $V_\theta = \mathcal{U}(\mathfrak{h})/\gamma(J_\theta)\mathcal{U}(\mathfrak{h})$ is an $\mathcal{U}(\mathfrak{h})$-module of dimension $\mathrm{Card}\,W$.

Lemma 2. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Then:*

(i) V_θ is a $\mathcal{U}(\mathfrak{h})$-module of dimension $\mathrm{Card}\,W$*,*

(ii) the characteristic polynomial of the action of $\xi \in \mathfrak{h}$ on V_θ is

$$P(\xi) = \prod_{w \in W} (\xi - (w\lambda + \rho)(\xi));$$

(iii) $H_0(\mathfrak{n}_x, \mathcal{U}_\theta)$ is a direct sum of countably many copies of V_θ.

Proof. We already proved (i). Clearly, $I_\mu \supset \varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ is equivalent to $\mu = w\lambda$ for some $w \in W$. Hence the linear transformation of $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ induced by multiplication by ξ has eigenvalues $(w\lambda)(\xi)$, $w \in W$, and by symmetry they all have the same multiplicity. This in turn implies that

$$\varphi(P(\xi)) = \prod_{w\in W} \varphi(\xi - (w\lambda+\rho)(\xi)) = \prod_{w\in W} (\xi - (w\lambda)(\xi))$$

is the characteristic polynomial for the action of ξ on $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\,\mathcal{U}(\mathfrak{h})$. This proves (ii).

(iii) As in the proof of 1, we fix a specialization $\mathfrak{c}$ of $\mathfrak{h}$ and choose a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. By Poincaré-Birkhoff-Witt theorem, it follows that as a vector space $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})$. Moreover,

$$H_0(\mathfrak{n}_x, \mathcal{U}_\theta) = \mathcal{U}(\mathfrak{g})/(J_\theta\,\mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\,\mathcal{U}(\mathfrak{g})).$$

Denote by $\gamma_x : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{c})$ the composition of the specialization map with the Harish-Chandra homomorphism γ. Then

$$J_\theta\,\mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\,\mathcal{U}(\mathfrak{g}) = J_\theta\,\mathcal{U}(\mathfrak{c})\,\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\,\mathcal{U}(\mathfrak{g}) = \gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c})\,\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\,\mathcal{U}(\mathfrak{g}),$$

which implies that under the above isomorphism,

$$J_\theta\,\mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\,\mathcal{U}(\mathfrak{g}) = \big(\mathbb{C} \otimes_{\mathbb{C}} \gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})\big) \oplus \big(\mathfrak{n}_x\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})\big).$$

This yields

$$H_0(\mathfrak{n}_x, \mathcal{U}_\theta) = \mathcal{U}(\mathfrak{c})/(\gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c})) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}) = V_\theta \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})$$

and the action of $\mathfrak{h}$ is given by multiplication in the first factor. □

Corollary 3. *Let $\lambda \in \mathfrak{h}^*$, $\theta = W \cdot \lambda$ and $V \in \mathcal{M}(\mathcal{U}_\theta)$. If we put*

$$P(\xi) = \prod_{w\in W} (\xi - (w\lambda+\rho)(\xi)) \text{ for } \xi \in \mathfrak{h},$$

$P(\xi)$ annihilates $H_p(\mathfrak{n}_x, V)$ for any $\xi \in \mathfrak{h}$ and $p \in \mathbb{Z}_+$.

Proof. By 1, we can calculate $\mathfrak{n}_x$-homology of V using a left resolution of V by free $\mathcal{U}_\theta$-modules. The assertion follows from 2. □

In particular, if $V \in \mathcal{M}(\mathcal{U}_\theta)$, $H_p(\mathfrak{n}_x, V)$ is a direct sum of generalized $\mathcal{U}(\mathfrak{h})$-eigenspaces corresponding to $w\lambda+\rho$, $w \in W$. If U is a $\mathcal{U}(\mathfrak{h})$-module, we denote by $U_{(\lambda)}$ the eigenspace corresponding to $\lambda \in \mathfrak{h}^*$.

Corollary 4. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then, for $V \in \mathcal{M}(\mathcal{U}_\theta)$, the $\mathfrak{n}_x$-homology modules $H_p(\mathfrak{n}_x, V)$ are semisimple as $\mathcal{U}(\mathfrak{h})$-modules. More precisely,*

$$H_p(\mathfrak{n}_x, V) = \sum_{w\in W} H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)}$$

for any $p \in \mathbb{Z}_+$.

This implies, in particular, that for regular $\lambda \in \mathfrak{h}^*$, we can view the functor $H_p(\mathfrak{n}_x, -)_{(w\lambda+\rho)}$ as the p^{th} left derived functor of the right exact functor $H_0(\mathfrak{n}_x, -)_{(w\lambda+\rho)}$ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}(\mathcal{U}(\mathfrak{h}))$.

In general situation, we can view V_θ as a semilocal ring and $H_p(\mathfrak{n}_x, V)$ as V_θ-modules. Also, for any $\lambda \in \theta$, $\mathbb{C}_{\lambda+\rho}$ is a V_θ-module.

For any $\mathcal{O}_X$-module $\mathcal{F}$ on X we denote by $T_x(\mathcal{F})$ its geometric fibre, i. e.

$$T_x(\mathcal{F}) = \mathcal{F}_x/\mathbf{m}_x\mathcal{F}_x.$$

Then T_x is a right exact covariant functor from $\mathcal{M}(\mathcal{O}_X)$ into complex vector spaces. If $\mathcal{F}$ is a $\mathcal{D}_\lambda$-module, we can view $T_x(\mathcal{F})$ as the inverse image of $\mathcal{F}$ for the inclusion i of the one-point space $\{x\}$ into X.

Theorem 5. *Let $\lambda \in \mathfrak{h}^*$, $\theta = W \cdot \lambda$ and $x \in X$. Then the functors $LT_x \circ L\Delta_\lambda$ and $D(\mathbb{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbb{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} -)$ from $D^-(\mathcal{U}_\theta)$ into the derived category of complexes of complex vector spaces are isomorphic.*

Proof. By 1, we know that $\mathcal{U}_\theta$ is acyclic for the functor $H_0(\mathfrak{n}_x, -) = \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} -$. By 2, we also know that $\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} \mathcal{U}_\theta$ is acyclic for the functor $\mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} -$. Let $F^\cdot$ be a complex isomorphic to $V^\cdot$ consisting of free $\mathcal{U}_\theta$-modules. Then, since the functors commute with infinite direct sums, we get

$$D(\mathbb{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbb{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} V^\cdot) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} F^\cdot).$$

On the other hand, the localization $\Delta_\lambda(\mathcal{U}_\theta) = \mathcal{D}_\lambda$ is a locally free $\mathcal{O}_X$-module, and therefore acyclic for T_x. This implies that

$$LT_x(L\Delta_\lambda(V^\cdot)) = T_x(\Delta_\lambda(F^\cdot)).$$

Hence, to complete the proof it is enough to establish the following identity

$$T_x(\Delta_\lambda(\mathcal{U}_\theta)) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} \mathcal{U}_\theta).$$

First, we have $T_x(\Delta_\lambda(\mathcal{U}_\theta)) = T_x(\mathcal{D}_\lambda)$. Moreover, from the construction of $\mathcal{D}_\mathfrak{h}$ it follows that

$$T_x(\mathcal{D}_\mathfrak{h}) = \mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}),$$

what yields, by using the properties of the Harish-Chandra homomorphism,

$$\begin{aligned} T_x(\mathcal{D}_\lambda) = T_x(\mathcal{D}_\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} &= (\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))/(I_{\lambda+\rho}(\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) \\ &= \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))/(\gamma(J_\theta)(\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) \\ &= \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathcal{U}(\mathfrak{g})/(J_\theta\mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} H_0(\mathfrak{n}_x, \mathcal{U}_\theta). \square \end{aligned}$$

Corollary 6. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\theta = W \cdot \lambda$. Then for any $V \in \mathcal{M}(\mathcal{U}_\theta)$ we have the spectral sequence*

$$L^pT_x(L^q\Delta_\lambda(V)) \Longrightarrow H_{-(p+q)}(\mathfrak{n}_x, V)_{(\lambda+\rho)}.$$

Proof. As we remarked before, in this case all V_θ-modules are semisimple and $\mathbb{C}_\lambda \otimes_{V_\theta} -$ is an exact functor. Therefore, the spectral sequence corresponding to the second functor in 5. collapses, and we get

$$
\begin{aligned}
&H^{-p}(D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V)))\\
&\qquad = H^{-p}(\mathbb{C}_{\lambda+\rho}\otimes_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V)))\\
&\qquad = H^{-p}((D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V))_{(\lambda+\rho)}) = H^{-p}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V))_{(\lambda+\rho)}\\
&\qquad = H_p(\mathfrak{n}_x, V)_{(\lambda+\rho)}
\end{aligned}
$$

for $p \in \mathbb{Z}_+$. Therefore, the asserted spectral sequence is just the Grothendieck spectral sequence attached to the composition of LT_x and $L\Delta_\lambda$. □

The behavior at singular λ is more obscure as we see from the following result.

Proposition 7. *Let $\lambda \in \mathfrak{h}^*$ be singular. Then there exists $V \in \mathcal{M}(\mathcal{U}_\theta)$ such that $L\Delta_\lambda(D(V))$ is not a bounded complex.*

In particular, the left cohomological dimension of Δ_λ is infinite.

Proof. Since the functor T_x has finite left cohomological dimension, it is enough to find a $\mathcal{U}_\theta$-module V such that $LT_x(L\Delta_\lambda(V))$ is not a bounded complex for some $x \in X$. By 5, this is equivalent to the fact that

$$D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V))$$

is not a bounded complex.

To finish the proof we use some elementary results about Verma modules which are discussed later in V.1. Fix a Borel subalgebra $\mathfrak{b}_0$, put $\mathfrak{n}_0 = [\mathfrak{b}_0, \mathfrak{b}_0]$ and consider the Verma module

$$M(w_0\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{w_0\lambda-\rho}.$$

Pick x so that $\mathfrak{n}_x$ is opposite to $\mathfrak{n}_0$. Then, by Poincaré-Birkhoff-Witt theorem, $M(w_0\lambda)$ is, as $\mathcal{U}(\mathfrak{n}_x)$-module, isomorphic to $\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathbb{C}_{w_0\lambda-\rho}$. This implies, since $\mathfrak{n}_x$ is opposite to $\mathfrak{n}_0$, that

$$H_0(\mathfrak{n}_x, M(w_0\lambda)) = \mathbb{C}_{\lambda+\rho},$$

and $H_p(\mathfrak{n}_x, M(w_0\lambda)) = 0$ for $p \in \mathbb{N}$. Therefore,

$$D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(M(w_0\lambda)) = D(\mathbb{C}_{\lambda+\rho}),$$

and

$$D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(M(w_0\lambda))) = D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}D(\mathbb{C}_{\lambda+\rho}).$$

Clearly, we have

$$H^{-p}(D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}D(\mathbb{C}_{\lambda+\rho})) = \operatorname{Tor}_p^{V_\theta}(\mathbb{C}_{\lambda+\rho}, \mathbb{C}_{\lambda+\rho}),\ p \in \mathbb{Z}_+.$$

Let $W(\lambda)$ be the stabilizer of λ in W. By 2, the maximal ideals in V_θ are the projections of the ideals $I_{w\lambda+\rho}$, $w \in W/W(\lambda)$. Since V_θ is an artinian ring, it is the product of local rings $R_{w\lambda}$, $w \in W/W(\lambda)$, obtained by localizing V_θ at $I_{w\lambda+\rho}$ ([**CA**], Ch. IV, §2, no. 5, Cor. 1 of Prop. 9). This implies that

$$\mathrm{Tor}_p^{V_\theta}(\mathbb{C}_{\lambda+\rho}, \mathbb{C}_{\lambda+\rho}) = \mathrm{Tor}_p^{R_\lambda}(\mathbb{C}, \mathbb{C}), \ \ p \in \mathbb{Z}_+.$$

Since $R_{w\lambda}$ are mutualy isomorphic,

$$\operatorname{Card} W = \dim_{\mathbb{C}} V_\theta = \sum_{w \in W/W(\lambda)} \dim_{\mathbb{C}} R_{w\lambda} = \operatorname{Card}(W/W(\lambda)) \dim_{\mathbb{C}} R_\lambda,$$

i. e. $\dim_{\mathbb{C}} R_\lambda = \operatorname{Card} W(\lambda) \neq 1$. Therefore, R_λ is not a regular local ring, its homological dimension is infinite ([**EGA**], 17.3.1) and $\mathrm{Tor}_p^{R_\lambda}(\mathbb{C}, \mathbb{C}) \neq 0$ for $p \in \mathbb{Z}_+$ ([**EGA**], 17.2.11). □

This immediately implies the following result.

Proposition 8. *Let θ be a Weyl group orbit in $\mathfrak{h}^*$ consisting of singular elements. Then the homological dimension of $\mathcal{U}_\theta$ is infinite.*

From 1.22. we can deduce the following consequence. As before, we put $W(p) = \{w \in W \mid \ell(w) = p\}$.

Proposition 9. *Let F be a finite-dimensional irreducible $\mathfrak{g}$-module with lowest weight λ. Then*

$$H_p(\mathfrak{n}_x, F) = \sum_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}$$

for any $p \in \mathbb{Z}_+$.

Proof. Clearly, $\lambda - \rho$ is regular, hence we can apply 6. From 1.22. we know that the localizations of F are locally free $\mathcal{O}_X$-modules. Therefore, all higher geometric fibres vanish on them and the spectral sequence degenerates. The formula follows immediately from 1.22. □

L.3 Intertwining Functors

Let θ be a Weyl group orbit in $\mathfrak{h}^*$. If θ consists of regular elements, by 1.12, the category $D(\mathcal{U}_\theta)$ is equivalent to the category $D(\mathcal{D}_\lambda)$. This implies in particular, that for any two $\lambda, \mu \in \theta$, the categories $D(\mathcal{D}_\lambda)$ and $D(\mathcal{D}_\mu)$ are equivalent. This equivalence is given by the functor $L\Delta_\mu \circ R\Gamma$ from $D(\mathcal{D}_\lambda)$ into $D(\mathcal{D}_\mu)$. In this section we want to construct, in geometric terms, a functor isomorphic to this functor.

We start with some geometric preliminaries. Define the action of G on $X \times X$ by

$$g \cdot (x, x') = (g \cdot x, g \cdot x')$$

for $g \in G$ and $(x, x') \in X \times X$. The G-orbits in $X \times X$ can be parametrized in the following way.

First we need to introduce a relation between Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two Borel subalgebras in $\mathfrak{g}$, $\mathfrak{n}$ and $\mathfrak{n}'$ their nilpotent radicals and N and N' the corresponding subgroups of G. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \mathfrak{b}'$. Denote by R the root system of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$ and by R^+ the set of positive roots determined by $\mathfrak{b}$. This determines a specialization of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R^+)$. On the other hand, $\mathfrak{b}'$ determines another set of positive roots in R, which corresponds via this specialization to $w(\Sigma^+)$ for some uniquely determined $w \in W$. Since $\mathfrak{c}$ is a Levi subalgebra of $\mathfrak{b} \cap \mathfrak{b}'$, all Cartan subalgebras in $\mathfrak{b} \cap \mathfrak{b}'$ are conjugate by elements of $N \cap N'$. This implies that the element $w \in W$ doesn't depend on the choice of $\mathfrak{c}$, and we say that $\mathfrak{b}'$ is in *relative position* w with respect to $\mathfrak{b}$. Let $s : \mathfrak{h}^* \longrightarrow \mathfrak{c}^*$ be the specialization determined by $\mathfrak{b}$. Then the specialization s' determined by $\mathfrak{b}'$ is equal to $s' = s \circ w$. This implies that $\mathfrak{b}$ is in relative position w^{-1} with respect to $\mathfrak{b}'$.

Let

$$Z_w = \{(x, x') \in X \times X \mid \mathfrak{b}_{x'} \text{ is in the relative position } w \text{ with respect to } \mathfrak{b}_x\}$$

for $w \in W$.

Lemma 1. *(i) Sets Z_w, $w \in W$, are smooth subvarieties of $X \times X$.*

(ii) The map $w \longrightarrow Z_w$ is a bijection of W onto the set of G-orbits in $X \times X$.

Proof. Fix $w \in W$. The set Z_w is G-invariant, hence it contains a G-orbit O. Let $x \in X$. Since G acts transitively on X, every G-orbit in $X \times X$ intersects $\{x\} \times X$, hence there exists $x' \in X$ such that $(x, x') \in O$. Let $(x, x'') \in Z_w$. Fix a Cartan subalgebra $\mathfrak{c}'$ in $\mathfrak{b}_x \cap \mathfrak{b}_{x'}$, and $\mathfrak{c}''$ in $\mathfrak{b}_x \cap \mathfrak{b}_{x''}$. Then, there exists $n \in N_x$ such that $(\operatorname{Ad} n)(\mathfrak{c}') = \mathfrak{c}''$. Since both (x, x') and (x, x'') are in Z_w, we have $(\operatorname{Ad} n)(\mathfrak{b}_{x'}) = \mathfrak{b}_{x''}$. Hence x' and x'' are in the same B_x-orbit in X. This in turn implies that $O = Z_w$. □

Denote by p_1 and p_2 the projections of Z_w onto the first and second factor in $X \times X$, respectively.

Lemma 2. *The fibrations $p_i : Z_w \longrightarrow X$, $i = 1, 2$, are locally trivial with fibres isomorphic to $\ell(w)$-dimensional affine spaces. The projections p_i, $i = 1, 2$, are affine morphisms.*

Proof. It is enough to discuss p_1. Let $(x, x') \in Z_w$ and denote by B, resp. B', the stabilizers of x, resp. x', in G. Then, by 2, instead of $p_1 : Z_w \longrightarrow X$ we can consider the the projection $p_1 : G/(B \cap B') \longrightarrow G/B$. Let $\bar{N}$ be the unipotent radical of a Borel subgroup of G opposite to B. Then, by the Bruhat decomposition, the natural map of $\bar{N} \times B$ into G is an isomorphism onto an open neighborhood of the identity ([**Bo**], 14.13). This implies that the orbit map $g \longrightarrow g \cdot x$ induces an isomorphism of $\bar{N}$ onto an open neighborhood U of $x \in X$. Moreover, the orbit map $g \longrightarrow g \cdot (x, x')$ induces an isomorphism of $\bar{N} \times (B/(B \cap B'))$ onto $p_1^{-1}(U)$ such that the diagram

$$\begin{array}{ccc} \bar{N} \times (B/(B \cap B')) & \longrightarrow & p_1^{-1}(U) \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle p_1}\downarrow \\ \bar{N} & \longrightarrow & U \end{array}$$

commutes. The fibres of p_1 are isomorphic to $B/(B \cap B') = N/(N \cap N')$, and this is an affine space of dimension

$$\mathrm{Card}(\Sigma^+) - \mathrm{Card}(\Sigma^+ \cap w(\Sigma^+)) = \mathrm{Card}(\Sigma^+ \cap (-w(\Sigma^+)) = \ell(w)$$

([**LG**], Ch. VI, §1, no. 6, Cor. 2. of Prop. 17). □

Let $\Omega_{Z_w|X}$ be the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \longrightarrow X$. Let $\mathcal{T}_w$ be its inverse. Since the tangent space at $(x, x') \in Z_w$ to the fibre of p_1 can be identified with $\mathfrak{n}_x/(\mathfrak{n}_x \cap \mathfrak{n}_{x'})$, and $\rho - w\rho$ is the sum of roots in $\Sigma^+ \cap (-w(\Sigma^+))$, we conclude that

$$\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho)).$$

Lemma 3. *(i) Let $\nu \in P(\Sigma)$. Then*

$$p_1^*(\mathcal{O}(w\nu)) = p_2^*(\mathcal{O}(\nu)).$$

(ii) Let $\lambda \in \mathfrak{h}^*$. Then

$$(\mathcal{D}_{w\lambda})^{p_1} = (\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}.$$

Proof. Let $(x, x') \in Z_w$. The stabilizer $B_x \cap B_{x'}$ of (x, x') in G contains a Cartan subgroup T of G, which is therefore its Levi factor. It follows that $B_x \cap B_{x'}$ is a connected group. Evidently, we have canonical morphisms

$$\mathfrak{b}_x \cap \mathfrak{b}_{x'} \longrightarrow (\mathfrak{b}_x \cap \mathfrak{b}_{x'})/(\mathfrak{n}_x \cap \mathfrak{b}_{x'}) \longrightarrow \mathfrak{b}_x/\mathfrak{n}_x$$

and

$$\mathfrak{b}_x \cap \mathfrak{b}_{x'} \longrightarrow (\mathfrak{b}_x \cap \mathfrak{b}_{x'})/(\mathfrak{b}_x \cap \mathfrak{n}_{x'}) \longrightarrow \mathfrak{b}_{x'}/\mathfrak{n}_{x'};$$

and $\mathfrak{b}_x/\mathfrak{n}_x$ and $\mathfrak{b}_{x'}/\mathfrak{n}_{x'}$ are naturally isomorphic to $\mathfrak{h}$. These two natural morphisms of $\mathfrak{b}_x \cap \mathfrak{b}_{x'}$ onto $\mathfrak{h}$ differ by the action of w^{-1}. The homogeneous invertible $\mathcal{O}_{Z_w}$-modules $p_1^*(\mathcal{O}(w\nu))$ and $p_2^*(\mathcal{O}(\nu))$ correspond therefore to the same character of the stabilizer of (x, x'). This proves (i).

By definition $(\mathcal{D}_{w\lambda})^{p_1}$ and $(\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}$ are G-homogeneous twisted sheaves of differential operators on the G-homogeneous space Z_w. Then, as we know from ..., the G-homogeneous twisted sheaves of differential operators on Z_w are parametrized by $(B_x \cap B_{x'})$-invariant linear forms on $\mathfrak{b}_x \cap \mathfrak{b}_{x'}$. The twisted sheaf of differential operators $(\mathcal{D}_{w\lambda})^{p_1}$ corresponds to the linear form on $\mathfrak{b}_x \cap \mathfrak{b}_{x'}$ induced by $w\lambda + \rho \in \mathfrak{h}^*$ under the first morphism, and the twisted sheaf of differential operators $\mathcal{D}_\lambda^{p_2}$ corresponds to the linear form on $\mathfrak{b}_x \cap \mathfrak{b}_{x'}$ induced by $\lambda + \rho \in \mathfrak{h}^*$ under the second isomorphism. Hence to get $(\mathcal{D}_{w\lambda})^{p_1}$ from $\mathcal{D}_\lambda^{p_2}$, we have to twist it by a homogeneous invertible $\mathcal{O}_{Z_w}$-module corresponding to the weight $w\lambda + \rho - w(\lambda + \rho) = \rho - w\rho$ under the first isomorphism. □

Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. The morphism $p_2 : Z_w \longrightarrow X$ is a surjective submersion, hence p_2^+ is an exact functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}((\mathcal{D}_\lambda)^{p_2})$ ([**Hartshorne**], III.10.4.). By 3, twisting by $\mathcal{T}_w$ defines an exact functor $\mathcal{V} \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V})$ from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}((\mathcal{D}_{w\lambda})^{p_1})$. Therefore, we have a functor $\mathcal{V}^\cdot \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)$ from $D^b(\mathcal{D}_\lambda)$ into $D^b((\mathcal{D}_{w\lambda})^{p_1})$. By composing it with the direct image functor $Rp_{1+} : D^b((\mathcal{D}_{w\lambda})^{p_1}) \longrightarrow D^b(\mathcal{D}_{w\lambda})$, we get the functor $J_w : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda})$ by the formula

$$J_w(\mathcal{V}^\cdot) = Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot))$$

for any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$. Let $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Since p_1 is an affine morphism with $\ell(w)$-dimensional fibres by 2, we see that $H^i(J_w(D(\mathcal{V})))$ vanish for $i < -\ell(w)$ and $i > 0$ (...). Moreover, the functor

$$I_w(\mathcal{V}) = R^0p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}))$$

from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is right exact. We call it the *intertwining functor* attached to $w \in W$ between $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $\mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$. The reason for this will become apparent later.

Lemma 4. *The category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ has enough projective objects.*

Proof. By twisting we can clearly assume that λ is antidominant and regular. But in this situation, $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is equivalent to $\mathcal{M}(\mathcal{U}_\theta)$, by 1.3. □

Therefore, for any $w \in W$, we can define the left derived functor

$$LI_w : D^-(\mathcal{D}_\lambda) \longrightarrow D^-(\mathcal{D}_{w\lambda})$$

of I_w. We shall see later that this functor, restricted to $D^b(\mathcal{D}_\lambda)$, agrees with J_w.

Now we study some basic properties of these functors. We start with an analysis of their behavior under geometric translation.

Lemma 5. *Let $w \in W$, $\lambda \in \mathfrak{h}^*$ and $\nu \in P(\Sigma)$. Then*

(i)

$$J_w(\mathcal{V}^\cdot(\nu)) = J_w(\mathcal{V}^\cdot)(w\nu)$$

for any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$;

(ii)

$$LI_w(\mathcal{V}^\cdot(\nu)) = LI_w(\mathcal{V}^\cdot)(w\nu)$$

for any $\mathcal{V}^\cdot \in D^-(\mathcal{D}_\lambda)$.

Proof. We start with the proof of the first relation. By 3.(i), for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$,

$$p_2^+(\mathcal{V}(\nu)) = p_2^+(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{O}(\nu)) = p_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Z_w}} p_2^*(\mathcal{O}(\nu)) = p_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Z_w}} p_1^*(\mathcal{O}(w\nu))$$

as $(\mathcal{D}_{\lambda+\nu})^{p_2}$-module. Since the direct image functor is local with respect to the target variety,

$$\begin{aligned}
J_w(\mathcal{V}^\cdot(\nu)) &= Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot(\nu))) \\
&= Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot) \otimes_{\mathcal{O}_{Z_w}} p_1^*(\mathcal{O}(w\nu))) \\
&= Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)) \otimes_{\mathcal{O}_X} \mathcal{O}(w\nu) = J_w(\mathcal{V}^\cdot)(w\nu),
\end{aligned}$$

for any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$. This, in particular, implies

$$I_w(\mathcal{V}(\nu)) = I_w(\mathcal{V})(w\nu)$$

for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Since twists preserve projective objects, the lemma follows. □

The next step is a "product formula" for functors J_w, $w \in W$. First, we need some additional geometric information on G-orbits in $X \times X$. Let $w, w' \in W$. Denote by p_1 and p_2 the projections of Z_w into X, and by p_1' and p_2' the corresponding projections of $Z_{w'}$ into X. Let $Z_{w'} \times_X Z_w$ be the fibre product of $Z_{w'}$ and Z_w with respect to the morphisms p_2' and p_1. Denote by $q' : Z_{w'} \times_X Z_w \longrightarrow Z_{w'}$ and $q : Z_{w'} \times_X Z_w \longrightarrow Z_w$ the corresponding projections to the first, resp. second factor. Then, by 2, q and q' are affine morphisms. Finally, the morphisms $p_1' \circ q' : Z_{w'} \times_X Z_w \longrightarrow X$ and $p_2 \circ q : Z_{w'} \times_X Z_w \longrightarrow X$ determine a morphism $r : Z_{w'} \times_X Z_w \longrightarrow X \times X$. Therefore, we have the following commutative diagram.

$$\begin{array}{ccccccc}
X \times X & \xleftarrow{\ r\ } & Z_{w'} \times_X Z_w & \xrightarrow{\ q\ } & Z_w & \xrightarrow{\ p_2\ } & X \\
 & & {\scriptstyle q'}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \\
 & & Z_{w'} & \xrightarrow{\ p'_2\ } & X & & \\
 & & {\scriptstyle p'_1}\big\downarrow & & & & \\
 & & X & & & &
\end{array}$$

Moreover, all morphisms in the diagram are G-equivariant. From the construction it follows that the image of r is contained in $Z_{w'w}$. Hence by the G-equivariance of r, it is a surjection of $Z_{w'} \times_X Z_w$ onto $Z_{w'w}$.

Lemma 6. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then $r : Z_{w'} \times_X Z_w \longrightarrow Z_{w'w}$ is an isomorphism.*

Proof. By 2. we know that

$$\dim(Z_{w'} \times_X Z_w) = \dim X + \ell(w) + \ell(w') = \dim X + \ell(w'w),$$

and

$$\dim Z_{w'w} = \dim X + \ell(w'w).$$

By the G-equivariance of r any G-orbit O in $Z_{w'} \times_X Z_w$ projects onto $Z_{w'w}$. Hence, $\dim O = \dim X + \ell(w'w) = \dim(Z_{w'} \times_X Z_w)$, and O is open in $Z_{w'} \times_X Z_w$. On the other hand, $Z_{w'} \times_X Z_w$ is irreducible, and it follows that $O = Z_{w'} \times_X Z_w$. This implies that $Z_{w'} \times_X Z_w$ is a G-homogeneous space covering $Z_{w'w}$. Since the stabilizer of a base point in $Z_{w'w}$ is connected, r is an isomorphism. □

Therefore, if we assume that $w, w', w'' \in W$ satisfy $w'' = w'w$ and $\ell(w'') = \ell(w') + \ell(w)$, we can identify $Z_{w''}$ and $Z_{w'} \times_X Z_w$. Under this identification the projections p_1'' and p_2'' of $Z_{w''}$ into X correspond to the maps $p_1' \circ q'$ and $p_2 \circ q$. Moreover, we have the following result.

Lemma 7. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then*

$$J_{w'} \circ J_w = J_{w'w}.$$

Proof. Let $w'' = w'w$. By 3.(i), we have

$$\begin{aligned} &{q'}^*(\mathcal{T}_{w'}) \otimes_{\mathcal{O}_{Z_{w'} \times_X Z_w}} q^*(\mathcal{T}_w) \\ &\qquad = (p_1' \circ q')^*(\mathcal{O}(\rho - w'\rho)) \otimes_{\mathcal{O}_{Z_{w'} \times_X Z_w}} (p_1 \circ q)^*(\mathcal{O}(\rho - w\rho)) \\ &\qquad = (p_1' \circ q')^*(\mathcal{O}(\rho - w'\rho)) \otimes_{\mathcal{O}_{Z_{w'} \times_X Z_w}} (p_2' \circ q')^*(\mathcal{O}(\rho - w\rho)) \\ &\qquad = {q'}^*({p_1'}^*(\mathcal{O}(\rho - w'\rho)) \otimes_{\mathcal{O}_{Z_{w'}}} {p_2'}^*(\mathcal{O}(\rho - w\rho))) \\ &\qquad = {q'}^*({p_1'}^*(\mathcal{O}(\rho - w'\rho)) \otimes_{\mathcal{O}_{Z_{w'}}} {p_1'}^*(\mathcal{O}(w'\rho - w'w\rho))) \\ &\qquad = {q'}^*({p_1'}^*(\mathcal{O}(\rho - w''\rho))) = (p_1' \circ q')^*(\mathcal{O}(\rho - w''\rho)) = \mathcal{T}_{w''}, \end{aligned}$$

under the identification of $Zw' \times_X Z_w$ with $Z_{w''}$. Then, by the base change (...),

$$\begin{aligned} J_{w'}(J_w(\mathcal{V}^\cdot)) &= Rp'_{1+}(\mathcal{T}_{w'} \otimes_{\mathcal{O}_{Z_{w'}}} {p'}_2^+(Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)))) \\ &= Rp'_{1+}(\mathcal{T}_{w'} \otimes_{\mathcal{O}_{Z_{w'}}} Rq'_+(q^+(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)))) \\ &= R(p_1' \circ q')_+({q'}^*(\mathcal{T}_{w'}) \otimes_{\mathcal{O}_{Z_{w'} \times_X Z_w}} (p_2 \circ q)^+(\mathcal{V}^\cdot)) = J_{w''}(\mathcal{V}^\cdot) \end{aligned}$$

for any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$. □

Corollary 8. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then*

$$I_{w'} \circ I_w = I_{w'w}.$$

Now we want to analyze in more details functors attached to simple reflections. Fix a simple root $\alpha \in \Pi$. To simplify the notation in the following, we put $Z = Z_{s_\alpha}$ and $\mathcal{T} = \mathcal{T}_{s_\alpha}$. In this situation, by 2, the fibres of the projection $p_1 : Z \longrightarrow X$ are affine lines. Hence one can view $\mathcal{T}$ as the invertible $\mathcal{O}_Z$-module of local vector fields tangent to the fibres of p_1.

Lemma 9. *Let $\alpha \in \Pi$ and $\lambda \in \mathfrak{h}^*$. Then*

$$H^i(J_{s_\alpha}(D(\mathcal{D}_\lambda))) = 0 \textit{ for } i \neq 0.$$

The proof of 9. will be a consequence of the following discussion, which will also lead to more detailed information about the action of the intertwining functor I_{s_α}. The basic idea is to reduce the analysis to the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. This reduction is based on a local trivialization result. Let X_α be the generalized flag variety of all parabolic subalgebras of type α. Any Borel subalgebra $\mathfrak{b}_x$ in $\mathfrak{g}$ is contained in a unique parabolic subalgebra $\mathfrak{p}_y$ of type α, $y \in X_\alpha$; hence we have the canonical projection $p_\alpha : X \longrightarrow X_\alpha$. For any $x \in X$, the fibre $p_\alpha^{-1}(p_\alpha(x))$ consists of x and all $x' \in X$ such that $\mathfrak{b}_{x'}$ is in relative position s_α wuth respect to $\mathfrak{b}_x$.

Lemma 10. *The projection $p_\alpha : X \longrightarrow X_\alpha$ is locally trivial. Its fibers are isomorphic to the projective line $\mathbb{P}^1$.*

Proof. Fix points $y \in X_\alpha$ and $x \in X$ such that $p_\alpha(x) = y$. Denote by B, resp. P, the stabilizers of x, resp. y, in G. Let P' be a parabolic subgroup opposite to P and N' its unipotent radical. Then the natural map $N' \times P \longrightarrow G$ is an isomorphism onto an open neighborhood of the identity in G ([**BT**], 4.2). Therefore, the natural morphism of N' into X_α induced by the orbit map $g \longrightarrow g \cdot y$ is an isomorphism of N' onto an open neighborhood U of y. Moreover, the orbit map $g \longrightarrow g \cdot x$ induces an isomorphism of $N' \times (P/B)$ with $p_\alpha^{-1}(U)$ such that the diagram

$$\begin{array}{ccc} N' \times (P/B) & \longrightarrow & p_\alpha^{-1}(U) \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle p_\alpha}\downarrow \\ N' & \longrightarrow & U \end{array}$$

commutes. This implies that the fibres are isomorphic to P/B. Let R be the radical of P. Then $R \subset B$, hence $P/B = (P/R)/(B/R)$. Since P/R is a cover of $\mathrm{PSL}(2, \mathbb{C})$ and B/R is its Borel subgroup, P/B is isomorphic to $\mathbb{P}^1$. □

We remark that $p_\alpha^{-1}(U)$ is a homogeneous space for P' and, if we denote by L the common Levi factor of P' and P, we see that $p_\alpha^{-1}(U)$ is identified with $N' \times (L/(L \cap B))$. Let M be the quotient of L with respect to its center, $\mathfrak{m}$ its Lie algebra and $X_\mathfrak{m}$ the corresponding flag variety. Clearly, M is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$ and $X_\mathfrak{m}$ is isomorphic to $\mathbb{P}^1$. Choosing base points $\mathfrak{b}$ and $\mathfrak{b} \cap \mathfrak{m}$ in X resp. $X_\mathfrak{m}$ determines a canonical inclusion of the Cartan algebra $\mathfrak{h}_\mathfrak{m}$ into $\mathfrak{h}$ which identifies the root system $\Sigma_\mathfrak{m}$ in $\mathfrak{h}_\mathfrak{m}^*$ with the restrictions of α and $-\alpha$, and the positive root β in $\Sigma_\mathfrak{m}$ corresponds to α. We can identify the dual space of $\mathfrak{h}_\mathfrak{m}$ with $\mathbb{C}$ via the map $\mu \longrightarrow \beta\check{\ }(\mu)$. From the discussion of homogeneous twisted sheaves (...), we see that, for any $\lambda \in \mathfrak{h}^*$,

$$\mathcal{D}_\lambda | p_\alpha^{-1}(U) = \mathcal{D}_{N'} \boxtimes \mathcal{D}_{\alpha\check{\ }(\lambda)};$$

here we denoted by $\mathcal{D}_{\alpha\check{\ }(\lambda)}$ the homogeneous twisted sheaf on $\mathbb{P}^1 = X_\mathfrak{m}$ determined by $\alpha\check{\ }(\lambda) \in \mathbb{C}$ under the above correspondence. By definition, $p_\alpha \circ p_1 = p_\alpha \circ p_2$, hence

$$p_1^{-1}(p_\alpha^{-1}(U)) = p_2^{-1}(p_\alpha^{-1}(U)),$$

as an open subset of Z. Moreover, under the above identifications, it is isomorphic to $N' \times ((X_\mathfrak{m} \times X_\mathfrak{m}) - \Delta_\mathfrak{m})$, where we denoted by $\Delta_\mathfrak{m}$ the diagonal in $X_\mathfrak{m} \times X_\mathfrak{m}$. Let q_1, resp. q_2, be the morphism of the variety $Z_\mathfrak{m} = (X_\mathfrak{m} \times X_\mathfrak{m}) - \Delta_\mathfrak{m}$ into $X_\mathfrak{m}$ induced by the projection to the first, resp. second factor. Then, using the above identifications, we have

$$R^i p_{1+}(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))|p_\alpha^{-1}(U) = \mathcal{D}_{N'} \boxtimes R^i q_{1+}(\mathcal{T}_\mathfrak{m} \otimes_{\mathcal{O}_{Z_\mathfrak{m}}} q_2^+(\mathcal{D}_{\alpha\check{}(\lambda)})),$$

where we denoted by $\mathcal{T}_\mathfrak{m}$ the invertible $\mathcal{O}_{Z_\mathfrak{m}}$-module of local vector fields on $Z_\mathfrak{m}$ tangent to the fibres of q_1. Now we can prove 9. The preceding discussion reduces the calculations to the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Hence, we assume this in the following discussion. Clearly, $\mathcal{D}_\lambda$ is a G-homogeneous $\mathcal{O}_X$-module, what implies that $\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)$ is a G-homogeneous $\mathcal{O}_Z$-module. Its direct images under p_1 are G-homogeneous $\mathcal{O}_X$-modules. Hence they are completely determined, as $\mathcal{O}_X$-modules, by their geometric fibres at the base point $x \in X$, and their higher geometric fibres vanish. Let $F = X - \{x\}$. Then $p_1^{-1}(x)$ is a smooth closed subvariety of Z equal to $\{x\} \times F$. Let $i_F : \{x\} \times F \longrightarrow Z$ and $i_x : \{x\} \longrightarrow X$ be the natural inclusions and r the projection of $\{x\} \times F$ into x. Then we have the following commutative diagram:

$$\begin{array}{ccc} \{x\} \times F & \xrightarrow{\ i_F\ } & Z \\ {\scriptstyle r}\big\downarrow & & {\scriptstyle p_1}\big\downarrow \\ \{x\} & \xrightarrow{\ i_x\ } & X \end{array}$$

By the base change ([**BDM**], 8.4),

$$\begin{aligned} T_x(R^i p_{1+}(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))) &= i_x^+(R^i p_{1+}(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))) \\ &= R^i r_+(i_F^+(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))) = R^i r_+(i_F^+(\mathcal{T}) \otimes_{\mathcal{O}_F} (p_2 \circ i_F)^+(\mathcal{D}_\lambda)) \\ &= R^i r_+(\mathcal{T}_F \otimes_{\mathcal{O}_F} \mathcal{D}_\lambda|F). \end{aligned}$$

In addition, F is the orbit of an one-dimensional unipotent subgroup of G, hence the homogeneous invertible $\mathcal{O}_F$-module $\mathcal{T}_F$ is isomorphic to $\mathcal{O}_F$ and $\mathcal{D}_\lambda|F$ is isomorphic to $\mathcal{D}_F$ as a homogeneous twisted sheaf of differential operators. Finally, $R^i r_+(\mathcal{D}_F) = 0$ for $i < 0$. This ends the proof of 9.

Now we return to the general situation. The next step is critical for our analysis of the intertwining functor attached to a simple root $\alpha \in \Pi$. Since the morphism p_1 is an affine surjective submersion, we can use the de Rham complex (...) to calculate $I_{s_\alpha}(\mathcal{V})$ as $\mathcal{O}_X$- and $\mathcal{U}(\mathfrak{g})$-module. For $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, $I_{s_\alpha}(\mathcal{V})$ is the 0^{th} cohomology of the complex $p_{1*}(C_{Z|X}(\mathcal{V}))$, where $C_{Z|X}(\mathcal{V})$ denotes the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{V}) \longrightarrow p_2^+(\mathcal{V}) \longrightarrow 0 \longrightarrow \ldots.$$

In particular, $I_{s_\alpha}(\mathcal{V})$ is a quotient of $p_{1*}(p_2^*(\mathcal{V}))$. Therefore, there is a natural $\mathcal{U}(\mathfrak{g})$-module morphism of the global sections of $p_{1*}(p_2^*(\mathcal{V}))$ into $\Gamma(X, I_{s_\alpha}(\mathcal{V}))$. Since

$$\Gamma(X, p_{1*}(p_2^*(\mathcal{V}))) = \Gamma(Z, p_2^*(\mathcal{V})),$$

this gives a natural $\mathcal{U}_\theta$-module morphism of $\Gamma(X, \mathcal{V})$ into $\Gamma(X, I_{s_\alpha}(\mathcal{V}))$. It induces a natural $\mathcal{D}_{s_\alpha\lambda}$-module morphism of $\Delta_{s_\alpha\lambda}(\Gamma(X, \mathcal{V}))$ into $I_{s_\alpha}(\mathcal{V})$, i. e. we have a morphism of the functor $\Delta_{s_\alpha\lambda} \circ \Gamma$ into I_{s_α}. Applying this discussion to the special case $\mathcal{V} = \mathcal{D}_\lambda$, we get, by C.6.1.(i), a natural $\mathcal{D}_{s_\alpha\lambda}$-module morphism of $\mathcal{D}_{s_\alpha}$ into $I_{s_\alpha}(\mathcal{D}_\lambda)$.

For a root $\alpha \in \Pi$, we say that $\lambda \in \mathfrak{h}^*$ is *α-antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer.

Lemma 11. *Let $\alpha \in \Pi$ and $\lambda \in \mathfrak{h}^*$ be α-antidominant. Then the natural morphism of $\mathcal{D}_{s_\alpha\lambda}$ into $I_{s_\alpha}(\mathcal{D}_\lambda)$ is an isomorphism of $\mathcal{D}_{s_\alpha\lambda}$-modules.*

Proof. The assertion is local, so we can apply the previous discussion. It reduces the problem to the corresponding result in the case of $\mathfrak{sl}(2, \mathbb{C})$. In this case there exists only one simple root α and $s_\alpha = -1$; hence we can put $I = I_{s_\alpha}$.

We claim first that the natural morphism from $\mathcal{D}_{-\lambda}$ into $I(\mathcal{D}_\lambda)$ is not zero. To see this we consider the morphism of $\Gamma(X, \mathcal{D}_{-\lambda})$ into $\Gamma(X, I(\mathcal{D}_\lambda))$. It is enough to show that the section $1 \in \mathcal{U}_\theta = \Gamma(X, \mathcal{D}_{-\lambda})$ always maps into a nonzero section of $\Gamma(X, I(\mathcal{D}_\lambda))$. We recall that $I(\mathcal{D}_\lambda)$ is the 0^{th} cohomology of the complex

$$\ldots \longrightarrow 0 \longrightarrow p_{1*}(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) \longrightarrow p_{1*}(p_2^+(\mathcal{D}_\lambda)) \longrightarrow 0 \longrightarrow \ldots,$$

and all other cohomologies of it vanish by 9. Since Z is an affine variety, by the Leray spectral sequence we conclude that this is a left resolution of $I(\mathcal{D}_\lambda)$ by $\Gamma(X, -)$-acyclic $\mathcal{O}_X$-modules. Therefore, the morphism

$$d : \Gamma(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) \longrightarrow \Gamma(Z, p_2^+(\mathcal{D}_\lambda))$$

is injective and $\Gamma(X, I(\mathcal{D}_\lambda))$ is the cokernel of d. The morphism of $\mathcal{U}_\theta = \Gamma(X, \mathcal{D}_{-\lambda})$ into $\Gamma(X, I(\mathcal{D}_\lambda))$ is induced by the natural morphism of $\mathcal{U}_\theta = \Gamma(X, \mathcal{D}_\lambda)$ into $\Gamma(Z, p_2^+(\mathcal{D}_\lambda))$. Therefore, $1 \in \mathcal{U}_\theta$ maps into the image of $1 \in \Gamma(Z, p_2^+(\mathcal{D}_\lambda))$ under the quotient map, and it is enough to show that $1 \in \Gamma(Z, p_2^+(\mathcal{D}_\lambda))$ is not in the image of d. To prove this we use the fact that d is G-equivariant, and analyze the G-action on $\Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))$. If we filter $\mathcal{D}_\lambda$ by degree, we get a filtration by G-homogeneous $\mathcal{O}_X$-modules $\mathrm{F}_p\,\mathcal{D}_\lambda$, $p \in \mathbb{Z}_+$, and $\mathrm{Gr}\,\mathcal{D}_\lambda = S(\mathcal{T}_X)$. Therefore, we have a filtration

$$\mathrm{F}_p(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) = \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^*(\mathrm{F}_p\,\mathcal{D}_\lambda),\ p \in \mathbb{Z}_+,$$

of $\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)$ by G-homogeneous $\mathcal{O}_Z$-modules and

$$\mathrm{Gr}(\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) = \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^*(S(\mathcal{T}_X)) = \mathcal{T} \otimes_{\mathcal{O}_Z} S(p_2^*(\mathcal{T}_X)).$$

By induction in degree we see that higher cohomologies of $\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)$ vanish, hence the filtration of $\mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)$ induces a filtration of its global sections such that

$$\operatorname{Gr} \Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) = \Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^*(S(\mathcal{T}_X))).$$

Because the group G is reductive, the algebraic G-modules $\Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))$ and $\Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} S(p_2^*(\mathcal{T}_X)))$ are isomorphic. On the other hand, $\mathcal{T}_X = \mathcal{O}(-\alpha)$, hence, by 3.(i), $S(p_2^*(\mathcal{T}_X))$ is the direct sum of $p_1^*(\mathcal{O}(k\alpha))$ for $k \in \mathbb{Z}_+$. In addition, $\mathcal{T} = p_1^*(\mathcal{O}(\alpha))$ and finally

$$\Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda)) = \bigoplus_{k=1}^{\infty} \Gamma(Z, p_1^*(\mathcal{O}(k\alpha))).$$

By Frobenius reciprocity, G doesn't act trivially on any submodule of the G-module $\Gamma(Z, \mathcal{T} \otimes_{\mathcal{O}_Z} p_2^+(\mathcal{D}_\lambda))$. This implies that 1 is not in the image of d.

Now we show that the natural morphism of $\mathcal{D}_{-\lambda}$ into $I(\mathcal{D}_\lambda)$ is an isomorphism for α-antidominant λ. First, we remark that both $\mathcal{D}_{-\lambda}$ and $I(\mathcal{D}_\lambda)$ are G-homogeneous $\mathcal{O}_X$-modules and the natural morphism is G-equivariant. Hence it is enough to show that the morphism induces an isomorphism of the geometric fibres at a base point x_0 of X. Right multiplication by elements of $\mathcal{D}_{-\lambda}$ induces on $T_x(\mathcal{D}_{-\lambda})$ a structure of a left $\mathcal{U}(\mathfrak{g})$-module isomorphic to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_{x_0})} \mathbb{C}_{\lambda-\rho}$. Therefore, $T_x(\mathcal{D}_{-\lambda})$ is a module which is the direct sum of one-dimensional weight spaces corresponding to the weights $\{\lambda - \rho - k\alpha \mid k \in \mathbb{Z}_+\}$, and it is irreducible if the α-antidominance condition is satisfied. Moreover, right multiplication by elements of $\Gamma(X, \mathcal{D}_\lambda)$ induces on $T_x(I(\mathcal{D}_\lambda))$ a structure of a left $\mathcal{U}(\mathfrak{g})$-module, and the map of geometric fibres is a morphism of $\mathcal{U}(\mathfrak{g})$-modules. Therefore, if λ is α-antidominant, the morphism of the geometric fibre $T_x(\mathcal{D}_{-\lambda})$ into $T_x(I(\mathcal{D}_\lambda))$ is injective.

The argument from the proof of 9. also implies that

$$T_x(I(\mathcal{D}_\lambda)) = R^0 r_+(\mathcal{D}_F) = \Gamma(F, \mathcal{O}_F)$$

as a vector space. On the other hand, it has the natural structure of $\mathcal{U}(\mathfrak{b}_{x_0})$-module, given by the linear form $-\lambda + \rho$. In addition, the stabilizer B_{x_0} of $x_0 \in X$ acts on this module and induces the natural action on $\Gamma(F, \mathcal{O}_F)$. Therefore, it is the direct sum of one-dimensional weight subspaces corresponding to the weights $\{-k\alpha \mid k \in \mathbb{Z}_+\}$. The action by right multiplication by elements of $\mathcal{U}(\mathfrak{b}_{x_0})$ on $\mathcal{D}_\lambda|F$ induces on $T_x(I(\mathcal{D}_\lambda))$ a left $\mathcal{U}(\mathfrak{b}_{x_0})$-action which is the difference of the second and the first action. Hence, $T_x(I(\mathcal{D}_\lambda))$ is the direct sum of one-dimensional weight subspaces corresponding to the weights $\{\lambda - \rho - k\alpha \mid k \in \mathbb{Z}_+\}$. It follows that the morphism of geometric fibres is an isomorphism. □

For any $S \subset \Sigma^+$, we say that $\lambda \in \mathfrak{h}^*$ is *S-antidominant* if it is α-antidominant for all $\alpha \in S$. Put

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\alpha \in -\Sigma^+\} = \Sigma^+ \cap (-w^{-1}(\Sigma^+))$$

for any $w \in W$.

Lemma 12. *(i)* $\Sigma_{w^{-1}}^+ = -w(\Sigma_w^+)$.

(ii) Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then

$$\Sigma^+_{w'w} = w^{-1}(\Sigma^+_{w'}) \cup \Sigma^+_w.$$

(iii) Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. If $\lambda \in \mathfrak{h}^*$ is $\Sigma^+_{w'w}$-antidominant, then $w\lambda$ is $\Sigma^+_{w'}$-antidominant.

Proof. (i) follows directly from the definition of Σ^+_w.
(ii) follows from ([**LG**], Ch. VI, §1, no. 6, Cor. 2. of Prop. 17).
(iii) follows immediately from (ii). □

Lemma 13. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, for any $w \in W$,*

$$J_w(D(\mathcal{D}_\lambda)) = D(\mathcal{D}_{w\lambda}).$$

Proof. By 11, the statement holds for simple reflections. We prove the general statement by induction in $\ell(w)$. Let $\ell(w) = k+1$. Then there exist $\alpha \in \Pi$ and $w' \in W$ such that $w = s_\alpha w'$ and $\ell(w') = k$. By 12.(iii), $w'\lambda$ is α-antidominant, hence

$$J_w(D(\mathcal{D}_\lambda)) = J_{s_\alpha}(J_{w'}(D(\mathcal{D}_\lambda))) = J_{s_\alpha}(D(\mathcal{D}_{w'\lambda})) = D(\mathcal{D}_{w\lambda}),$$

by 7, 9, 11. and the induction hypothesis. □

Theorem 14. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then:*
(i) For any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$, we have

$$LI_w(\mathcal{V}^\cdot) = J_w(\mathcal{V}^\cdot).$$

(ii) The left cohomological dimension of I_w is $\leq \ell(w)$.

Proof. Clearly, (ii) follows from (i). To prove (i) we have to show that for any projective $\mathcal{P} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, $H^i(J_w(D(\mathcal{P}))) = 0$ for $i \neq 0$. First we observe that, by 5, it is enough to consider the case of regular antidominant λ. In this situation, by 1.3, $\mathcal{P}$ is the localization of a projective $\mathcal{U}_\theta$-module, and therefore a direct summand of $(\mathcal{D}_\lambda)^{(J)}$. It follows that it is enough to treat the case of $\mathcal{P} = \mathcal{D}_\lambda$, and we can apply 13. □

As an immediate consequence, the intertwining functors LI_w, $w \in W$, extend to functors from $D(\mathcal{D}_\lambda)$ into $D(\mathcal{D}_{w\lambda})$. We now prove a preliminary version of the product formula for the intertwining functors.

Lemma 15. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then, for any $\lambda \in \mathfrak{h}^*$, the functors $LI_{w'} \circ LI_w$ and $LI_{w'w}$ from $D^-(\mathcal{D}_\lambda)$ into $D^-(\mathcal{D}_{w'w\lambda})$ are isomorphic.*

Proof. By 8, it is enough to show that, for any projective $\mathcal{P} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, the $\mathcal{D}_{w\lambda}$-module $I_w(\mathcal{P})$ is $I_{w'}$-acyclic. By 5, we can also assume that λ is regular antidominant. In this situation, as in the proof of 14, we can assume that

$\mathcal{P} = \mathcal{D}_\lambda$. Therefore, by 13, $I_w(\mathcal{D}_\lambda) = \mathcal{D}_{w\lambda}$ and the assertion follows by a repeated application of 13. and 14. □

Finally, we have the following result which explains the role of intertwining functors.

Theorem 16. *Let $w \in W$ and let $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant. Then the functors $LI_w \circ L\Delta_\lambda$ and $L\Delta_{w\lambda}$ from $D^-(\mathcal{U}_\theta)$ into $D^-(\mathcal{D}_{w\lambda})$ are isomorphic.*

Proof. We prove this result by induction on $\ell(w)$. Assume first that w is the simple reflection corresponding to $\alpha \in \Pi$. Any $V^\cdot \in D^-(\mathcal{U}_\theta)$ is isomorphic to a complex $F^\cdot$ of free $\mathcal{U}_\theta$-modules. Moreover, $L\Delta_\lambda(V^\cdot) = \Delta_\lambda(F^\cdot)$ and, by 9, 11, 14. and C.6.1.(i), the natural morphism we described before is an isomorphism of

$$L\Delta_{s_\alpha\lambda}(V^\cdot) = \Delta_{s_\alpha\lambda}(F^\cdot) = \Delta_{s_\alpha\lambda}(\Gamma(X, \Delta_\lambda(F^\cdot))),$$

into

$$I_{s_\alpha}(\Delta_\lambda(F^\cdot)) = LI_{s_\alpha}(\Delta_\lambda(F^\cdot)) = LI_{s_\alpha}(L\Delta_\lambda(V^\cdot)).$$

Assume that the statement holds for $w' \in W$ such that $\ell(w') \le k$. Take $w \in W$ such that $\ell(w) = k+1$. Then $w = s_\alpha w'$ for some $\alpha \in \Pi$ and $w' \in W$ with $\ell(w') = k$. By 12.(iii), $w'\lambda$ is α-antidominant. Moreover, by the induction hypothesis and 15, we see that

$$LI_w \circ L\Delta_\lambda = (LI_{s_\alpha} \circ LI_{w'}) \circ L\Delta_\lambda = LI_{s_\alpha} \circ (LI_{w'} \circ L\Delta_\lambda)$$

is isomorphic to $LI_{s_\alpha} \circ L\Delta_{w'\lambda}$. Hence the assertion follows by applying the statement for simple reflections. □

Corollary 17. *Let $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant and let $F \in \mathcal{M}(\mathcal{U}_\theta)$ be a flat $\mathcal{U}_\theta$-module. Then the localization $\Delta_\lambda(F)$ is an I_w-acyclic Δ_λ-module.*

Proof. Since F is a flat module, its higer localizations vanish. Therefore, the assertion follows from the spectral sequence associated to 18. □

The next result is the final form of the product formula for intertwining functors.

Theorem 18. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then, for any $\lambda \in \mathfrak{h}^*$, the functors $LI_{w'} \circ LI_w$ and $LI_{w'w}$ from $D(\mathcal{D}_\lambda)$ into $D(\mathcal{D}_{w'w\lambda})$ are isomorphic.*

Proof. By 5, we can assume that λ is antidominant and regular. In this situation, by 1.4. and 1.12, we know that any complex $\mathcal{V}^\cdot \in D(\mathcal{D}_\lambda)$ is quasi-isomorphic to the localization $\Delta_\lambda(P^\cdot)$ of a complex $P^\cdot \in D(\mathcal{U}_\theta)$ consisting of projective $\mathcal{U}_\theta$-modules. Therefore, by 12, 16. and 17, we have

$$LI_w(\mathcal{V}^\cdot) = LI_w(\Delta_\lambda(P^\cdot)) = \Delta_{w\lambda}(P^\cdot),$$

and $\Delta_{w\lambda}(P^\cdot)$ consists of $I_{w'}$-acyclic $\mathcal{D}_{w\lambda}$-modules. It follows that

$$LI_{w'}(LI_w(\mathcal{V}^\cdot)) = LI_{w'}(\Delta_{w\lambda}(P^\cdot)) = I_{w'}(\Delta_{w\lambda}(P^\cdot))$$
$$= \Delta_{w'w\lambda}(P^\cdot) = LI_{w'w}(\Delta_\lambda(P^\cdot)) = LI_{w'w}(\mathcal{V}^\cdot),$$

and the lemma follows. □

In addition, if we assume that λ is regular, we have:

Theorem 19. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant and regular. Then*

(i) LI_w is an equivalence of the category $D(\mathcal{D}_\lambda)$ with $D(\mathcal{D}_{w\lambda})$ isomorphic to $L\Delta_{w\lambda} \circ R\Gamma$;

(ii) the functors $LI_w \circ L\Delta_\lambda$ and $L\Delta_{w\lambda}$ from $D(\mathcal{U}_\theta)$ into $D(\mathcal{D}_{w\lambda})$ are isomorphic.

Proof. First we prove (ii). Any complex $V^\cdot \in D(\mathcal{U}_\theta)$ is quasi-isomorphic to a complex $P^\cdot$ consisting of projective $\mathcal{U}_\theta$-modules. Therefore, $L\Delta_\lambda(V^\cdot) = \Delta_\lambda(P^\cdot)$ and, by 16. and 17,

$$LI_w(L\Delta_\lambda(V^\cdot)) = I_w(\Delta_\lambda(P^\cdot)) = \Delta_{w\lambda}(P^\cdot) = L\Delta_{w\lambda}(V^\cdot).$$

(i) follows from (ii) and 1.12. □

Theorem 20. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then:*

(i) LI_w is an equivalence of the category $D(\mathcal{D}_\lambda)$ with $D(\mathcal{D}_{w\lambda})$;

(ii) LI_w is an equivalence of the category $D^b(\mathcal{D}_\lambda)$ with $D^b(\mathcal{D}_{w\lambda})$;

(iii) LI_w is an equivalence of the category $D^b_{coh}(\mathcal{D}_\lambda)$ with $D^b_{coh}(\mathcal{D}_{w\lambda})$.

Proof. Assume first that λ is regular antidominant.

In this situation, by 1.12 and 19.(i), we see that the functor LI_w is equivalent to the functor $L\Delta_{w\lambda} \circ R\Gamma$ and the inverse functor is equivalent to $L\Delta_\lambda \circ R\Gamma$. This proves (i). Since the functor Γ has finite right cohomological dimension, and the localization functor Δ_μ has finite left cohomological dimension for regular μ by 1.13, (ii) follows. The last statement follows from 1.16.

The general statement follows from 5. using geometric translation. □

Now we can improve the estimate of left cohomological dimension of intertwining functors.

Theorem 21. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then the left cohomological dimension of $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is $\leq \mathrm{Card}(\Sigma_w^+ \cap \Sigma_\lambda)$.*

Proof. By 5. we can assume that λ is regular and antidominant. The proof is by induction in $\ell(w)$. Assume first that w is the reflection with respect to $\alpha \in \Pi$. Then the left cohomological dimension of I_{s_α} is ≤ 1. Assume in addition that $\alpha\check{}(\lambda) \notin \mathbb{Z}$. Then $s_\alpha(\lambda)$ is also regular and antidominant. By 19.(i), LI_{s_α} is an equivalence of category $D(\mathcal{D}_\lambda)$ with $D(\mathcal{D}_{s_\alpha\lambda})$ isomorphic to $L\Delta_{s_\alpha\lambda} \circ R\Gamma$. Since, by 1.3, functors $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ and $\Delta_{s_\alpha\lambda} : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ are equivalencies of categories, the functor $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ is also an equivalence of categories. It follows that its left cohomological dimension is 0. This ends the proof for simple reflections.

Assume that the statement holds for all $w' \in W$ such that $\ell(w') \leq k$. Let $w \in W$ with $\ell(w) = k+1$. Then there exist $\alpha \in \Pi$ and $w' \in W$ such that $\ell(w') = k$. By 12.(ii),

$$\Sigma^+_{s_\alpha w'} \cap \Sigma_\lambda = (\{\, w'^{-1}(\alpha) \,\} \cup \Sigma^+_{w'}) \cap \Sigma_\lambda,$$

hence

$$\mathrm{Card}(\Sigma^+_w \cap \Sigma_\lambda) = \mathrm{Card}(\Sigma^+_{w'} \cap \Sigma_\lambda) + \mathrm{Card}(\{\alpha\} \cap \Sigma_{w'\lambda}).$$

The lemma follows from 18, the case of simple reflections and the induction hypothesis. □

Corollary 22. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be such that $\Sigma^+_w \cap \Sigma_\lambda = \emptyset$. Then $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is an equivalence of categories and $I_{w^{-1}}$ is its inverse.*

Proof. By 12.(i) and 21, I_w and $I_{w^{-1}}$ are exact. Also, by 5, we can assume that λ is regular and antidominant. This implies that $w\lambda$ is regular and antidominant. By 19.(i), LI_w is an equivalence of category $D(\mathcal{D}_\lambda)$ with $D(\mathcal{D}_{w\lambda})$ isomorphic to $L\Delta_{w\lambda} \circ R\Gamma$. Since, by 1.3, the functors $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ and $\Delta_{w\lambda} : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ are equivalencies of categories, $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is also an equivalence of categories. The same argument applies to $I_{w^{-1}}$. This also implies that their compositions are isomorphic to the identity functor. □

Theorem 23. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ^+_w-antidominant. Then the functors $R\Gamma \circ LI_w$ and $R\Gamma$ from $D(\mathcal{D}_\lambda)$ into $D(\mathcal{U}_\theta)$ are isomorphic.*

Proof. If λ is regular this follows from 1.12. For singular λ, we can find $w' \in W$ and $\nu \in P(\Sigma)$ such that $w'\lambda$ and $-w'\nu$ are antidominant, $w'(\lambda - \nu)$ is antidominant and regular, and $\lambda - \nu$ is Σ^+_w-antidominant. Let $\mathcal{V}^\cdot \in D(\mathcal{D}_\lambda)$. Since the left cohomological dimension of I_w is finite, $\mathcal{V}^\cdot$ is quasi-isomorphic to a complex $\mathcal{C}^\cdot$ consisting of I_w-acyclic $\mathcal{D}_\lambda$-modules. Moreover, by 5, the complex $\mathcal{C}^\cdot(-\nu)$ consists of I_w-acyclic $\mathcal{D}_{\lambda-\nu}$-modules and

$$LI_w(\mathcal{V}^\cdot(-\nu)) = I_w(\mathcal{C}^\cdot(-\nu)) = I_w(\mathcal{C}^\cdot(-\nu)) = I_w(\mathcal{C}^\cdot)(-w\nu) = LI_w(\mathcal{V}^\cdot)(-w\nu).$$

In addition,

$$R\Gamma(LI_w(\mathcal{V}^\cdot)(-w\nu)) = R\Gamma(LI_w(\mathcal{V}^\cdot(-\nu))) = R\Gamma(\mathcal{V}^\cdot(-\nu)),$$

using the statement for regular $\lambda - \nu$. On the other hand, by C.2.1, if we denote by F the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $w'\nu$, we have

$$\begin{aligned} R\Gamma(LI_w(\mathcal{V}^\cdot)) &= R\Gamma((LI_w(\mathcal{V}^\cdot)(-w\nu)\otimes_{\mathcal{O}_X}\mathcal{F})_{[\lambda]}) \\ &= (R\Gamma(LI_w(\mathcal{V}^\cdot)(-w\nu))\otimes_{\mathbb{C}} F)_{[\lambda]} = (R\Gamma(\mathcal{V}^\cdot(-\nu))\otimes_{\mathbb{C}} F)_{[\lambda]} \\ &= R\Gamma((\mathcal{V}^\cdot(-\nu)\otimes_{\mathcal{O}_X}\mathcal{F})_{[\lambda]}) = R\Gamma(\mathcal{V}^\cdot).\square \end{aligned}$$

We finally remark the following fact. It shows that, in general, the estimate of left cohomological dimension of intertwining functors from 21. is the best possible.

Proposition 24. *Let $w \in W$ and $\lambda \in P(\Sigma)$. Then*

$$L^i I_w(\mathcal{O}(\lambda+\rho)) = 0 \text{ for } i \neq -\ell(w)$$

and

$$L^{-\ell(w)} I_w(\mathcal{O}(\lambda+\rho)) = \mathcal{O}(w\lambda+\rho).$$

Proof. By 5. we can assume that λ is antidominant and regular. In this situation the assertion follows immediately from 16. and 1.22. □

L.4 Global Sections of Irreducible $\mathcal{D}_\lambda$-modules

Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then for any quasi-coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$, higher cohomology modules $H^i(X,\mathcal{V})$, $i > 0$, vanish. Therefore, we need to study only the behavior of global sections $\Gamma(X,\mathcal{V})$ of $\mathcal{V}$. We start with the following simple result.

Proposition 1. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ irreducible. Then either*

(i) $\Gamma(X,\mathcal{V}) = 0$, or

(ii) $\mathcal{V}$ is generated by its global sections $\Gamma(X,\mathcal{V})$ and they form an irreducible $\mathcal{U}_\theta$-module.

Proof. As we remarked before, there is a natural morphism of $\Delta_\lambda(\Gamma(X,\mathcal{V}))$ into $\mathcal{V}$, and its image is a coherent $\mathcal{D}_\lambda$-module. Therefore, it is equal to 0 or $\mathcal{V}$. In the first case we have $\Gamma(X,\mathcal{V}) = 0$, and (i) holds. In the second case, $\mathcal{V}$ is generated by its global sections. It remains to prove that $\Gamma(X,\mathcal{V})$ is irreducible. Let $\mathcal{K}$ be the kernel of the epimorphism of $\Delta_\lambda(\Gamma(X,\mathcal{V}))$ onto $\mathcal{V}$. Then we have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta_\lambda(\Gamma(X,\mathcal{V})) \longrightarrow \mathcal{V} \longrightarrow 0$$

and therefore

$$0 \longrightarrow \Gamma(X,\mathcal{K}) \longrightarrow \Gamma(X,\Delta_\lambda(\Gamma(X,\mathcal{V}))) \longrightarrow \Gamma(X,\mathcal{V}) \longrightarrow 0.$$

By 1.1, this implies that $\Gamma(X,\mathcal{K}) = 0$.

Let U be a nonzero submodule of $\Gamma(X,\mathcal{V})$. Then the inclusion i of U into $\Gamma(X,\mathcal{V})$ induces a homomorphism $\Delta_\lambda(i)$ of $\Delta_\lambda(U)$ into $\Delta_\lambda(\Gamma(X,\mathcal{V}))$. Assume that $\operatorname{im}\Delta_\lambda(i)$ is contained in $\mathcal{K}$. By applying Γ we would get that $\Gamma(\Delta_\lambda(i)) = 0$, contradicting 1.1. Therefore, $\operatorname{im}\Delta_\lambda(i)$ is not contained in $\mathcal{K}$. This implies that the natural morphism of $\mathcal{K}\oplus\Delta_\lambda(U)$ into $\Delta_\lambda(\Gamma(X,\mathcal{V}))$ is an epimorphism. By the exactness of Γ it follows that the natural morphism of $\Gamma(X,\mathcal{K}\oplus\Delta_\lambda(U)) = \Gamma(X,\Delta_\lambda(U)) = U$ into $\Gamma(X,\Delta_\lambda(\Gamma(X,\mathcal{V}))) = \Gamma(X,\mathcal{V})$ is surjective; hence, $U = \Gamma(X,\mathcal{V})$. □

The previous result allows the following converse.

Proposition 2. *Let V be an irreducible module from $\mathcal{M}(\mathcal{U}_\theta)$. Let $\lambda\in\mathfrak{h}^*$ be antidominant. Then there exists an irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$ such that $\Gamma(X,\mathcal{V})$ is isomorphic to V. Such $\mathcal{D}_\lambda$-module $\mathcal{V}$ is unique up to an isomorphism.*

We start the proof with a lemma.

Lemma 3. *Let $\lambda\in\mathfrak{h}^*$ be antidominant and $\mathcal{V}\in\mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Then there exists a largest quasi-coherent $\mathcal{D}_\lambda$-submodule $\mathcal{V}'$ of $\mathcal{V}$ with no nonzero global sections.*

Proof. Let $\mathcal{S}$ be the family of all quasi-coherent $\mathcal{D}_\lambda$-modules $\mathcal{U}$ of $\mathcal{V}$ such that $\Gamma(X,\mathcal{U}) = 0$. We assume that $\mathcal{S}$ is ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. By ([**EGA**], Ch. I, 2.2.2) the union W of elements of $\mathcal{C}$ is a quasi-coherent $\mathcal{D}_\lambda$-submodule of $\mathcal{V}$ and $\Gamma(X,\mathcal{W}) = 0$. Hence, by Zorn lemma, there exists a maximal element $\mathcal{V}'$ of $\mathcal{S}$. Let $\mathcal{U}$ be any other element of $\mathcal{S}$. Then $\mathcal{U}+\mathcal{V}$ is a quasi-coherent $\mathcal{D}_\lambda$-submodule of $\mathcal{V}$ and it is a quotient of $\mathcal{D}_\lambda$-module $\mathcal{U}\oplus\mathcal{V}'$. By the exactness of Γ it follows that $\Gamma(X,\mathcal{U}+\mathcal{V}') = 0$, i. e. $\mathcal{U}+\mathcal{V}$ is in $\mathcal{S}$. Therefore, $\mathcal{U}$ is a $\mathcal{D}_\lambda$-submodule of $\mathcal{V}'$. □

Now we can prove 2. The localization $\Delta_\lambda(V)$ of V is a coherent $\mathcal{D}_\lambda$-module generated by its global sections. Let $\mathcal{W}$ be a coherent $\mathcal{D}_\lambda$-submodule of $\Delta_\lambda(V)$ different from $\Delta_\lambda(V)$. Then we have an exact sequence

$$0\longrightarrow\mathcal{W}\longrightarrow\Delta_\lambda(V)\longrightarrow\mathcal{V}\longrightarrow 0$$

of coherent $\mathcal{D}_\lambda$-modules and, since λ is antidominant, we have

$$0\longrightarrow\Gamma(X,\mathcal{W})\longrightarrow\Gamma(X,\Delta_\lambda(V))\longrightarrow\Gamma(X,\mathcal{V})\longrightarrow 0.$$

By 1.1, $\Gamma(X,\Delta_\lambda(V)) = V$; hence our assumption implies that it is irreducible. It follows that $\Gamma(X,\mathcal{W})$ is either 0 or equal to $\Gamma(X,\Delta_\lambda(V))$. In the second case, all global sections of $\Delta_\lambda(V)$ would already be in $\mathcal{W}$. By the definition of $\Delta_\lambda(V)$, it is generated by its global sections as a $\mathcal{D}_\lambda$-module. Therefore, this would imply that $\mathcal{W}$ is equal to $\Delta_\lambda(V)$, contrary to our assumption. It follows that $\Gamma(X,\mathcal{W}) = 0$ and $\Gamma(X,\mathcal{V}) = V$. Therefore, by 3, it follows that $\Delta_\lambda(V)$ has the largest coherent $\mathcal{D}_\lambda$-submodule $\mathcal{W}$ and corresponding $\mathcal{V}$ is irreducible. It is generated by its global sections by 1.

It remains to show the uniqueness. Assume that $\mathcal{U}$ is an irreducible $\mathcal{D}_\lambda$-module such that $\Gamma(X,\mathcal{U}) = V$. Then the image of the natural homomorphism of $\Delta_\lambda(V)$ into $\mathcal{U}$ is either 0 or $\mathcal{U}$. In the first case, $\Gamma(X,\mathcal{U}) = 0$, contrary to our assumption. It follows that this homomorphism is onto, and by the first part of the proof its kernel is $\mathcal{W}$. This ends the proof of 2.

Now we want to study the necessary and sufficient conditions for vanishing of global sections of irreducible $\mathcal{D}_\lambda$-modules for antidominant $\lambda \in \mathfrak{h}^*$. We start with a discussion of the action of intertwining functors, attached to reflections with respect to the roots from Π_λ, on irreducible modules. Let $\alpha \in \Pi_\lambda$. Then

$$\Sigma^+_{s_\alpha} \cap \Sigma_\lambda = \Sigma^+_\lambda \cap (-s_\alpha(\Sigma^+_\lambda)) = \Sigma^+_\lambda \cap (-(\Sigma^+_\lambda - \{\alpha\}) \cup \{\alpha\}) = \{\alpha\},$$

hence, by 3.21, we know that the left cohomological dimension of I_{s_α} is ≤ 1. Assume, in addition, that λ is antidominant. Then $n(s_\alpha\lambda) \leq 1$ and, by C.3.1, we see that the right cohomological dimension of Γ on $\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ is ≤ 1. By 3.23, for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$, we have a spectral sequence with E_2-term $H^p(X, L^qI_{s_\alpha}(V))$ converging to $\Gamma(X,\mathcal{V})$. It follows that this spectral sequence converges at E_2-stage and

(a) $\Gamma(X, L^{-1}I_{s_\alpha}(V)) = H^1(X, I_{s_\alpha}(V)) = 0$,

(b) the $\mathcal{U}_\theta$-module $\Gamma(X,\mathcal{V})$ is an extension of modules $\Gamma(X, I_{s_\alpha}(V))$ and $H^1(X, L^{-1}I_{s_\alpha}(V))$.

Lemma 4. *Let $\alpha \in \Pi_\lambda$ and $\lambda \in \mathfrak{h}^*$ be antidominant and such that α is the only root from Π_λ orthogonal to λ. Assume that $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is such that $\Gamma(X,\mathcal{V}) = 0$. Then*

(i) $I_{s_\alpha}(\mathcal{V}) = 0$,

(ii) $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}$.

Proof. Let $\mu \in P(\Sigma)$ be a regular antidominant weight and F the irreducible finite-dimensional $\mathfrak{g}$-module with lowest weight μ. Denote by $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. Then, as we discussed in C.2, the $\mathcal{U}^\circ$-module $\mathcal{G} = (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\mu]}$ has a natural filtration such that the corresponding graded module is the sum of $\mathcal{V}(\nu)$ for all weights ν of F such that $\lambda + \mu = w(\lambda + \nu)$. This condition implies that $\lambda - w\lambda = w\nu - \mu \in Q(\Sigma)$, hence $w \in W_\lambda$. In addition, since λ is antidominant, the left side of this equation is negative of a sum of roots from Π_λ. On the other hand, since μ is the lowest weight of F, the right side is a sum of roots from Π. This is possible only if both sides are equal to 0. Therefore, $w = 1$ or $w = s_\alpha$. Hence, we have an exact sequence

$$0 \longrightarrow \mathcal{V}(s_\alpha\mu) \longrightarrow \mathcal{G} \longrightarrow \mathcal{V}(\mu) \longrightarrow 0.$$

Also,

$$H^i(X,\mathcal{G}) = H^i(X, \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\mu]} = (H^i(X,\mathcal{V}) \otimes_{\mathbb{C}} F)_{[\lambda+\mu]} = 0,$$

and the long exact sequence of cohomology implies that $\Gamma(X, \mathcal{V}(s_\alpha\mu)) = 0$, $\Gamma(X,\mathcal{V}(\mu)) = H^1(X, \mathcal{V}(s_\alpha\mu))$ and higher cohomology modules of $\mathcal{V}(s_\alpha\mu)$ vanish. This finally implies, by 3.23, that

$$R\Gamma(LI_{s_\alpha}(D(\mathcal{V}(\mu)))) = R\Gamma(D(\mathcal{V}(s_\alpha\mu))[-1]).$$

By the equivalence of derived categories (3.19), we conclude that

$$LI_{s_\alpha}(D(\mathcal{V}(\mu))) = D(\mathcal{V}(s_\alpha \mu))[-1],$$

i. e. that $I_{s_\alpha}(\mathcal{V}(\mu)) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}(\mu)) = \mathcal{V}(s_\alpha \mu)$. The assertions (i) and (ii) follow from 3.5.(ii). □

Lemma 5. *Let $\alpha \in \Pi_\lambda$ and $\lambda \in \mathfrak{h}^*$ such that $-\alpha\check{}(\lambda) = p \in \mathbb{Z}_+$. Let $\mathcal{V}$ be a quasi-coherent $\mathcal{D}_\lambda$-module such that $I_{s_\alpha}(\mathcal{V}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$. Then $\Gamma(X, \mathcal{V}(p\alpha)) = 0$.*

Proof. By 3.23, we have

$$H^i(X, \mathcal{V}) = H^{i+1}(X, L^{-1}I_{s_\alpha}(\mathcal{V})) = H^{i+1}(X, \mathcal{V}(p\alpha))$$

for all $i \in \mathbb{Z}$. □

Lemma 6. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi_\lambda$. Put $-\alpha\check{}(\lambda) = p \in \mathbb{Z}$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then either*

(i) $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$, or

(ii) $I_{s_\alpha}(\mathcal{V}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$.

Proof. By C.3.5.(ii) we can assume that λ is antidominant and regular. Moreover, because of irreducibility of $\mathcal{V}$, $\Gamma(X, \mathcal{V})$ is irreducible $\mathcal{U}_\theta$-module, and either $\Gamma(X, I_{s_\alpha}(\mathcal{V})) = 0$ or $H^1(X, L^{-1}I_{s_\alpha}(\mathcal{V})) = 0$. By C.4.3, we conclude that $I_{s_\alpha}(\mathcal{V}) = 0$ in the first case, and $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$ in the second case. Assume that $I_{s_\alpha}(\mathcal{V}) = 0$. In this case, $L^{-1}I_{s_\alpha}(\mathcal{V}) \neq 0$. By 3.5.(ii) we can now assume that λ is antidominant and that α is the only root from Π_λ orthogonal to it. In addition, from previous discussion and antidominance of $s_\alpha\lambda = \lambda$ we conclude that

$$\Gamma(X, \mathcal{V}) = H^1(X, L^{-1}I_{s_\alpha}(\mathcal{V})) = 0.$$

The assertion follows from 3.5.(ii) and 4.4. □

Lemma 7. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi_\lambda$. Put $-\alpha\check{}(\lambda) = p \in \mathbb{Z}$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module such that $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$. Then $I_{s_\alpha}(\mathcal{V})$ has a largest quasi-coherent submodule $\mathcal{U}$ and we have an exact sequence*

$$0 \longrightarrow \mathcal{U} \longrightarrow I_{s_\alpha}(\mathcal{V}) \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0.$$

Moreover, if $p \geq 0$, $\Gamma(X, \mathcal{U}) = 0$.

Proof. Again, to prove the first statement we can assume that λ is antidominant and regular. As we remarked before, in this case all higher cohomology modules of $I_{s_\alpha}(\mathcal{V})$ vanish. Let $\mathcal{W}$ be any quasi-coherent $\mathcal{D}_\lambda$-submodule of $I_{s_\alpha}(\mathcal{V})$ different from $I_{s_\alpha}(\mathcal{V})$. Then, by C.3.1. and the long exact sequence of cohomology we conclude that higher cohomology modules of $\mathcal{C} = I_{s_\alpha}(\mathcal{V})/\mathcal{W}$ also vanish. Therefore, by C.4.3, we conclude that $\Gamma(X, \mathcal{C}) \neq 0$. The long exact sequence of cohomology gives

$$0 \longrightarrow \Gamma(X, \mathcal{W}) \longrightarrow \Gamma(X, I_{s_\lambda}(\mathcal{V})) \longrightarrow \Gamma(X, \mathcal{C}) \longrightarrow H^1(X, \mathcal{W}) \longrightarrow 0.$$

We can choose $\nu \in P(\Sigma)$ such that $\lambda + \nu$ is antidominant and α is the only root from Π_λ orthogonal to $\lambda + \nu$. Then, the sequence

$$0 \longrightarrow \mathcal{W}(s_\alpha \nu) \longrightarrow I_{s_\alpha}(\mathcal{V})(s_\alpha \nu) \longrightarrow \mathcal{C}(s_\alpha \nu) \longrightarrow 0$$

is exact and, since $s_\alpha(\lambda + \nu)$ is antidominant, we get the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{W}(s_\alpha \nu)) \longrightarrow \Gamma(X, I_{s_\alpha}(\mathcal{V})(s_\alpha \nu)) \longrightarrow \Gamma(X, \mathcal{C}(s_\alpha \nu)) \longrightarrow 0.$$

Moreover, by 3.5.(ii) and 3.23, $I_{s_\alpha}(\mathcal{V})(s_\alpha \nu) = I_{s_\alpha}(\mathcal{V}(\nu))$ and

$$\Gamma(X, I_{s_\alpha}(\mathcal{V})(s_\alpha \nu)) = \Gamma(X, I_{s_\alpha}(\mathcal{V}(\nu))) = \Gamma(X, \mathcal{V}(\nu)),$$

hence it is either 0 or an irreducible $\mathcal{U}_\theta$-module by 1. We claim now that $\Gamma(X, \mathcal{C}(s_\alpha \nu)) \neq 0$. Assume the contrary, i. e. $\Gamma(X, \mathcal{C}(s_\alpha \nu)) = 0$. Then, by 4, we have

$$I_{s_\alpha}(\mathcal{C}(s_\alpha \nu)) = 0 \text{ and } L^{-1} I_{s_\alpha}(\mathcal{C}(s_\alpha \nu)) = \mathcal{C}(s_\alpha \nu).$$

By 3.5.(ii) this leads to

$$I_{s_\alpha}(\mathcal{C}(-p\alpha)) = 0 \text{ and } L^{-1} I_{s_\alpha}(\mathcal{C}(-p\alpha)) = \mathcal{C}.$$

Therefore, by 5, $\mathcal{C}$ has no global sections, contradicting the preceding discussion. This in turn implies that $\Gamma(X, \mathcal{W}(s_\alpha \nu)) = 0$ and by 4. and 5, we see that $\Gamma(X, \mathcal{W}) = 0$. We proved that any quasi-coherent $\mathcal{D}_\lambda$-submodule $\mathcal{W}$ of $I_{s_\alpha}(\mathcal{V})$ different from $I_{s_\alpha}(\mathcal{V})$ satisfies $\Gamma(X, \mathcal{W}(s_\alpha \nu)) = 0$. Hence, by 3. we conclude that $I_{s_\alpha}(\mathcal{V})$ contains a largest quasi-coherent $\mathcal{D}_\lambda$-submodule $\mathcal{U}$ and that $\Gamma(X, \mathcal{U}) = 0$. Moreover, $(I_{s_\alpha}(\mathcal{V})/\mathcal{U})(s_\alpha \nu)$ is an irreducible $\mathcal{D}_{\lambda+\nu}$-module such that $\Gamma(X, (I_{s_\alpha}(\mathcal{V})/\mathcal{U})(s_\alpha \nu)) = \Gamma(X, \mathcal{V}(\nu))$. By 2. it follows that $(I_{s_\alpha}(\mathcal{V})/\mathcal{U})(s_\alpha \nu) = \mathcal{V}(\nu)$. This implies that $I_{s_\alpha}(\mathcal{V})/\mathcal{U} = \mathcal{V}(p\alpha)$. It remains to show the last statement. We concluded already that $L^{-1} I_{s_\alpha}(\mathcal{U}(-p\alpha)) = \mathcal{U}$ and all other derived intertwining functors vanish on $\mathcal{U}(-p\alpha)$. By 5. we see that $\Gamma(X, \mathcal{U}) = 0$ if $p \geq 0$. □

Corollary 8. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi_\lambda$ be such that $-\alpha\check{}(\lambda) = p \in \mathbb{Z}_+$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module such that $\Gamma(X, \mathcal{V}) \neq 0$ and $L^{-1} I_{s_\alpha}(\mathcal{V}) = 0$. Then $\Gamma(X, \mathcal{V}(p\alpha)) \neq 0$.*

Proof. By 3.23,

$$\Gamma(X, I_{s_\alpha}(\mathcal{V})) = \Gamma(X, \mathcal{V}) \neq 0.$$

Hence, by the left exactness of Γ and 7, $\Gamma(X, \mathcal{V}(p\alpha)) \neq 0$. □

Theorem 9. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and S subset of Π_λ consisting of roots orthogonal to λ. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then the following conditions are equivalent:*

(i) $\Gamma(X, \mathcal{V}) = 0$,

(ii) there exists $\alpha \in S$ such that $I_{s_\alpha}(\mathcal{V}) = 0$.

Proof. (ii)⇒(i) By 4, $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}$. Hence, by 5, $\Gamma(X, \mathcal{V}) = 0$.

(i)⇒(ii) Let $W(\lambda)$ be the stabilizer of λ in W. Then $W(\lambda)$ is generated by reflections with respect to S. Let F be a finite-dimensional representation with regular lowest weight ν, and put $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. Then $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ satisfies

$$\Gamma(X, \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}) = \Gamma(X, \mathcal{V}) \otimes_{\mathbb{C}} F = 0.$$

Moreover $\Gamma(X, (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}) = 0$. On the other hand, if we consider the filtration of F discussed in C.2, it induces a filtration of $(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}$ such that the corresponding graded sheaf is a direct sum of $\mathcal{V}(\mu)$ for all weights μ of F such that $w(\lambda + \nu) = \lambda + \mu$ for some $w \in W$. This implies that $w\lambda - \lambda = \mu - w\nu$, hence $w \in W_\lambda$. Moreover, the left side of the equality $\lambda - w^{-1}\lambda = w^{-1}\mu - \nu$ is a negative of a sum of roots from Π_λ and the right side is a sum of roots from Π. This implies that $w\lambda = \lambda$, i. e. $w \in W(\lambda)$. Let $w_1 \in W(\lambda)$ be such that $\mathcal{V}(w_1\nu)$ is an $\mathcal{O}_X$-submodule of $(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}$, then $\Gamma(X, \mathcal{V}(w_1\nu)) = 0$. Assume now that $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$ for all $\alpha \in S$. We claim that in this case $\Gamma(X, \mathcal{V}(w\nu)) \neq 0$ for any $w \in W(\lambda)$, contradicting our assumption. We prove this by induction in the length of w in $W(\lambda)$ (which is the same as the length in W_λ by ([**LG**], Ch IV, §1, no. 8, Cor. 4. of Prop. 7.)). If $w = 1$, $\lambda + \nu$ is antidominant and regular, and the statement follows from C.4.4. Assume that $\ell_\lambda(w) = k > 0$. Let $w = s_\alpha w'$ with $\alpha \in S$ and $w' \in W(\lambda)$ such that $\ell_\lambda(w') = k-1$. Then, by ([**LG**], Ch. VI, §1, no. 6, Cor. 1 of Prop. 17.), it follows that ${w'}^{-1}\alpha \in \Sigma_\lambda^+$. This implies, by the antidominance of ν, that

$$\alpha\check{}(\lambda + w'\nu) = \alpha\check{}(w'\nu) = ({w'}^{-1}\alpha)\check{}(\nu) \in -\mathbb{N},$$

and $\lambda + w'\nu$ is α-antidominant.

By the induction assumption we have $\Gamma(X, \mathcal{V}(w'\nu)) \neq 0$ and

$$L^{-1}I_{s_\alpha}(\mathcal{V}(w'\nu)) = L^{-1}I_{s_\alpha}(\mathcal{V})(w\nu) = 0$$

by 3.5.(ii). Therefore, the assertion follows from 8. □

L.5 Intertwining Functors and Holonomic Complexes

The category of holonomic modules is a thick subcategory of the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Therefore, we can consider the category $D^b_{hol}(\mathcal{D}_\lambda)$ which is the full subcategory of $D^b(\mathcal{D}_\lambda)$ consisting of complexes with holonomic cohomology. Clearly, the geometric translation functor $\mathcal{V} \to \mathcal{V}(\mu)$, $\mu \in P(\Sigma)$, induces an equivalence of the category $D^b_{hol}(\mathcal{D}_\lambda)$ with the category $D^b_{hol}(\mathcal{D}_{\lambda+\mu})$.

Every holonomic module is of finite length. This results in the following consequence.

Lemma 1. *Irreducible holonomic $\mathcal{D}_\lambda$-modules form a generating class in $D^b_{hol}(\mathcal{D}_\lambda)$.*

Proof. By a standard truncation argument ([**BDM**], I.12.6), all holonomic $\mathcal{D}_\lambda$-modules form a generating class in $D^b_{hol}(\mathcal{D}_\lambda)$. On the other hand, for a holonomic $\mathcal{D}_\lambda$-module $\mathcal{V}$ of length n, let $\mathcal{U}$ be one of its maximal coherent submodules. Then we have the exact sequence

$$0 \to \mathcal{U} \xrightarrow{i} \mathcal{V} \to \mathcal{V}/\mathcal{U} \to 0,$$

where $\mathcal{V}/\mathcal{U}$ is irreducible and $\mathcal{U}$ of length $n-1$. Let

$$D(\mathcal{U}) \xrightarrow{D(i)} D(\mathcal{V}) \to \mathcal{C}_i \to D(\mathcal{V})[1]$$

be the distinguished triangle attached to the morphism i. Then the cone $\mathcal{C}_i$ is isomorphic to $D(\mathcal{V}/\mathcal{U})$. Therefore, by induction in the length, we see that the triangulated subcategory of $D^b_{hol}(\mathcal{D}_\lambda)$ generated by irreducible holonomic $\mathcal{D}_\lambda$-modules contains all complexes of the form $D(\mathcal{V})$ where $\mathcal{V}$ is a holonomic $\mathcal{D}_\lambda$-module. By the first remark, it is equal to $D^b_{hol}(\mathcal{D}_\lambda)$. □

By its definition, for arbitrary $w \in W$, the intertwining functor LI_w maps $D^b_{hol}(\mathcal{D}_\lambda)$ into $D^b_{hol}(\mathcal{D}_{w\lambda})$.

Theorem 2. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then LI_w is an equivalence of the category $D^b_{hol}(\mathcal{D}_\lambda)$ with $D^b_{hol}(\mathcal{D}_{w\lambda})$.*

Proof . By the product formula (3.18), it is enough to show this statement for simple reflections. Using geometric translation and 3.5 we can also assume that λ is antidominant and regular.

There are two possibilities in this case. Either

(a) $\alpha\check{}(\lambda) \notin -\mathbb{N}$, or

(b) $\alpha\check{}(\lambda) \in -\mathbb{N}$.

In case (a), $s_\alpha(\lambda)$ is again regular antidominant. Therefore, by 3.22, I_w is an exact functor and $I_{w^{-1}}$ is its inverse. This immediately implies our assertion.

In case (b), $p = -\alpha\check{}(\lambda) \in \mathbb{N}$. To prove the statement in this case it is enough to show that the full subcategory $\mathcal{A}$ of $D^b_{hol}(\mathcal{D}_{s_\alpha\lambda})$ consisting of objects isomorphic to complexes $LI_{s_\alpha}(\mathcal{C}^\cdot)$, $\mathcal{C}^\cdot \in D^b_{hol}(\mathcal{D}_\lambda)$, is equal to $D^b_{hol}(\mathcal{D}_{s_\alpha\lambda})$. To show this, by 1, it is enough to show that $\mathcal{A}$ contains complexes $D(\mathcal{V}(p\alpha))$ for all irreducible holonomic $\mathcal{D}_\lambda$-modules $\mathcal{V}$.

By 4.6, for an irreducible holonomic $\mathcal{D}_\lambda$-module $\mathcal{V}$, there are two possibilities, either

(i) $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$ and $I_{s_\alpha}(\mathcal{V}) = 0$, or

(ii) $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$.

If (i) holds, we have $LI_{s_\alpha}(D(\mathcal{V})) = D(\mathcal{V}(p\alpha))[1]$. Therefore

$$LI_{s_\alpha}(D(\mathcal{V})[-1]) = D(\mathcal{V}(p\alpha)),$$

and all complexes $D(\mathcal{V}(p\alpha))$, where $\mathcal{V}$ is an irreducible $\mathcal{D}_\lambda$-module of type (i), are in $\mathcal{A}$. This also implies that all complexes $D(\mathcal{U}(p\alpha))$, where $\mathcal{U}$ is a holonomic $\mathcal{D}_\lambda$-module with all composition factors of type (i), are also in $\mathcal{A}$.

If (ii) holds, by 4.7. we have the short exact sequence

$$0 \to \mathcal{U} \to I_{s_\alpha}(\mathcal{V}) \to \mathcal{V}(p\alpha) \to 0.$$

We can choose $\nu \in P(\Sigma)$ such that $\lambda + \nu$ is antidominant and α is the only root in Π_λ orthogonal to $\lambda + \nu$. Then, by 3.5,

$$0 \to \mathcal{U}(s_\alpha\nu) \to I_{s_\alpha}(\mathcal{V}(\nu)) \to \mathcal{V}(\nu) \to 0$$

is exact. Moreover, 4.7 implies that $\Gamma(X, \mathcal{U}(s_\alpha\nu)) = 0$. Since $\lambda + \nu$ is antidominant, Γ is exact by C.3.2, and all composition factors of $\mathcal{U}(s_\alpha\nu)$ have no global sections. By 4.9, this implies that all composition factors of $\mathcal{U}(s_\alpha\nu)$ are of type (i). By 3.5, it follows that all composition factors of $\mathcal{U}$ are of type (i). From the first part of the proof we conclude that $\mathcal{U}$ is in $\mathcal{A}$. Consider now the distinguished triangle associated to the morphism $I_{s_\alpha}(\mathcal{V}) \to \mathcal{V}(p\alpha)$,

$$LI_{s_\alpha}(D(\mathcal{V})) \to D(\mathcal{V}(p\alpha)) \to \mathcal{C}^\cdot \to LI_{s_\alpha}(D(\mathcal{V}))[1].$$

The cone $\mathcal{C}^\cdot$ is isomorphic to $D(\mathcal{U})[-1]$ and, since $LI_{s_\alpha}(D(\mathcal{V}))$ and $D(\mathcal{U})[-1]$ are in $\mathcal{A}$, we see that $D(\mathcal{V}(p\alpha))$ is in $\mathcal{A}$ either. By 1, we see that $\mathcal{A} = D^b_{hol}(\mathcal{D}_{s_\alpha\lambda})$. □

This result has the following consequences.

Theorem 3. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Let $\mathcal{V}$ be a holonomic $\mathcal{D}_\lambda$-module. Then $H^p(X, \mathcal{V})$, $p \in \mathbb{Z}_+$, are $\mathcal{U}_\theta$-modules of finite length.*

Proof. Let $\mu \in \theta$ be antidominant, and $w \in W$ such that $\lambda = w\mu$. By 2, there exists a complex $\mathcal{C}^\cdot$ with holonomic cohomology such that $LI_w(\mathcal{C}^\cdot) = D(\mathcal{V})$. Since Γ is exact functor from $\mathcal{M}_{qc}(\mathcal{D}_\mu)$ into $\mathcal{M}(\mathcal{U}_\theta)$ we see that $R\Gamma(\mathcal{C}^\cdot) = \Gamma(\mathcal{C}^\cdot)$, and $H^p(R\Gamma(\mathcal{C}^\cdot)) = \Gamma(X, H^p(\mathcal{C}^\cdot))$ for arbitrary $p \in \mathbb{Z}$. By 4.1. we also conclude that $H^p(R\Gamma(\mathcal{C}^\cdot))$, $p \in \mathbb{Z}$, are $\mathcal{U}_\theta$-modules of finite length. Therefore, $R\Gamma(\mathcal{C}^\cdot)$ is a complex of $\mathcal{U}_\theta$-modules with cohomology modules of finite length. Finally, by 3.23, we conclude that

$$H^p(X, \mathcal{V}) = H^p(R\Gamma(D(\mathcal{V}))) = H^p(R\Gamma(LI_w(\mathcal{C}^\cdot))) = H^p(R\Gamma(\mathcal{C}^\cdot)).\square$$

Proposition 4. *Let $\lambda \in \mathfrak{h}^*$ be regular antidominant and $\theta = W \cdot \lambda$. Let V be a finitely generated $\mathcal{U}_\theta$-module. Then the following conditions are equivalent:*

(i) $\Delta_\lambda(V)$ is a holonomic $\mathcal{D}_\lambda$-module;

(ii) $L\Delta_{w\lambda}(V)$ is a complex with holonomic cohomology for some $w \in W$;

(iii) $H_p(\mathfrak{n}_x, V)$, $p \in \mathbb{Z}_+$, are finite-dimensional for all $x \in X$;

(iv) there exists $w \in W$ such that $H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)}$, $p \in \mathbb{Z}_+$, are finite-dimensional for all $x \in X$.

Proof. First we remark that, by 1.19, $L\Delta_{w\lambda}(V) \in D^b_{coh}(\mathcal{D}_{w\lambda})$ for any $w \in W$.

The assertions (i) and (ii) are equivalent by 3.19. and 2. Also, (iii) implies (iv).

Assume that (ii) holds. For $x \in X$, denote by i_x the injection of x into X. Then

$$LT_x(L\Delta_{w\lambda}(V)) = Li_x^*(L\Delta_{w\lambda}(V))$$

is a complex with finite-dimensional cohomology. By 2.6, this implies (iv). On the other hand, if (iv) holds, the same result implies that the complex $LT_x(L\Delta_{w\lambda}(V))$ has finite-dimensional cohomology for all $x \in X$. Hence (ii) follows from ([**BDM**], VII.10.7.(ii)). □

Corollary 5. *Let θ be a W-orbit in $\mathfrak{h}^*$ consisting of regular elements. Let V be a finitely generated $\mathcal{U}_\theta$-module. If $H_p(\mathfrak{n}_x, V)$, $p \in \mathbb{Z}_+$, are finite-dimensional for all $x \in X$, the module V is of finite length.*

Proof. Let $\lambda \in \theta$ be antidominant. Then 4. implies that $\Delta_\lambda(V)$ is holonomic, and therefore of finite length. By equivalence of categories this implies that V is of finite length. □

L.6 Tensor Products with Finite-dimensional Modules

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ and $\lambda \in \theta$. Let F be a finite-dimensional representation of $\mathfrak{g}$ and $m = \dim F$. Let $(\mu_i; 1 \le i \le m)$ be the family of all weights of F counted with their multiplicities. Since the weights of F and their multiplicities are invariant under the action of the Weyl group W, the family $S(\theta, F) = (\nu_i = \lambda + \mu_i; 1 \le i \le m)$, $\lambda \in \theta$, depends only on θ and F.

Lemma 1. *Let $V \in \mathcal{M}(\mathcal{U}_\theta)$, and F a finite-dimensional representation of $\mathfrak{g}$. Then*

$$\prod_{\nu \in S(\theta,F)} (\zeta - \chi_\nu(\zeta)), \quad \zeta \in \mathcal{Z}(\mathfrak{g}),$$

annihilates $V \otimes_{\mathbb{C}} F$.

Proof. Let $\lambda \in \theta$ be antidominant. Put $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ and consider its filtration $(\mathcal{F}_i; 1 \le i \le m)$ from the beginning of C.2. Then, it induces a filtration $(\Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}_i; 1 \le i \le m)$ of $\Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}$. The corresponding graded module is the direct sum of $\Delta_\lambda(V)(\mu_i)$, $1 \le i \le m$. It is evident that $\Delta_\lambda(V)(\mu_i)$ is annihilated by $\zeta - \chi_{\nu_i}(\zeta)$ for any $\zeta \in \mathcal{Z}(\mathfrak{g})$. This immediately implies that $\Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}$ is annihilated by $\prod_{i=1}^m (\zeta - \chi_{\nu_i}(\zeta))$ for any $\zeta \in \mathcal{Z}(\mathfrak{g})$. In particular, the module of its global sections is annihilated by these elements. On the other hand,

$$\Gamma(X, \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}) = \Gamma(X, \Delta_\lambda(V) \otimes_{\mathbb{C}} F) = V \otimes_{\mathbb{C}} F$$

by 1.1. □

In particular, $V \otimes_{\mathbb{C}} F$ has a finite increasing filtration by $\mathcal{U}(\mathfrak{g})$-submodules such that all composition factors are modules with infinitesimal character.

Let $\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$ be the full subcategory of $\mathcal{M}(\mathcal{U}(\mathfrak{g}))$ consisting of $\mathcal{U}(\mathfrak{g})$-modules of finite length. Let $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ be the full subcategory of $\mathcal{M}_{fg}(\mathcal{U}(\mathfrak{g}))$ consisting of modules $V \in \mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$ such that $V \otimes_{\mathbb{C}} F \in \mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$ for any finite-dimensional $\mathfrak{g}$-module F.

Remark 2. An example due to T. Stafford shows that there are irreducible $\mathcal{U}_\theta$-modules V such that $V \otimes_{\mathbb{C}} F$ is not artinian. Therefore, $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ is strictly smaller than $\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$.

Lemma 3. *(i) The category $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ is a thick subcategory of the category $\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$.*

(ii) If $V \in \mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ and F is a finite-dimensional $\mathfrak{g}$-module, $V \otimes_{\mathbb{C}} F \in \mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$.

Proof. (i) follows immediately from the exactness of $- \otimes_{\mathbb{C}} F$.

(ii) is evident. □

Lemma 4. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and $\theta = W \cdot \lambda$. Let $V \in \mathcal{M}(\mathcal{U}_\theta)$. Then the assertion:*

(i) $H^p(X, \Delta_\lambda(V)(\nu))$, $p \in \mathbb{Z}_+$, are $\mathcal{U}_\theta$-modules of finite length for any weight $\nu \in P(\Sigma)$,

implies

(ii) $V \otimes_{\mathbb{C}} F$ is an $\mathfrak{g}$-module of finite length for any finite-dimensional $\mathfrak{g}$-module F.

If, in addition λ is regular, (i) and (ii) are equivalent.

Proof. We use the notation from the proof of 1. From the spectral sequence of a filtered object ([**EGA**], III.13.6) we see that there exists a spectral sequence with E_1-term equal to

$$H^{p-q}(X, \mathrm{Gr}_q(\Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F})) = H^{p-q}(X, \Delta_\lambda(V)(\nu_q))$$

which abutts to

$$H^p(X, \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}) = H^p(X, \Delta_\lambda(V)) \otimes_{\mathbb{C}} F$$

and which is equal to $V \otimes_{\mathbb{C}} F$ for $p = 0$ and 0 otherwise. Since the E_1-term consists of $\mathfrak{g}$-modules of finite length and all differentials are morphisms of $\mathfrak{g}$-modules, we conclude that $V \otimes_{\mathbb{C}} F$ is of finite length.

Assume now that λ is regular and that that (ii) holds for $V \in \mathcal{M}(\mathcal{U}_\theta)$.

Let μ is a dominant weight and F the irreducible finite-dimensional module with lowest weight $-\mu$. Then, by C.2.2 and 1.1, we see that

$$\begin{aligned} \Gamma(X, \Delta_\lambda(V)(-\mu)) &= \Gamma(X, \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda-\mu]} \\ &= (\Gamma(X, \Delta_\lambda(V)) \otimes_{\mathbb{C}} F)_{[\lambda-\mu]} = (V \otimes_{\mathbb{C}} F)_{[\lambda-\mu]}. \end{aligned}$$

By 2, $V' = (V \otimes_{\mathbb{C}} F)_{[\lambda-\mu]}$ also has the property (ii) and $\Delta_{\lambda-\mu}(V') = \Delta_\lambda(V)(-\mu)$.

Since an arbitrary weight ν can be written as a difference of two dominant weights μ' and μ, we have

$$\Delta_\lambda(V)(\nu) = \Delta_\lambda(V)(\mu' - \mu) = \Delta_{\lambda-\mu}(V')(\mu').$$

Therefore, it is enough to prove (i) for dominant weights ν.

We complete the proof by induction in p. Assume first that $p = 0$. Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight ν. Then

$$H^p(X, \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}) = H^p(X, \Delta_\lambda(V)) \otimes_{\mathbb{C}} F$$

for all p, $0 \leq i \leq \dim X$; therefore it vanishes for $p > 0$. On the other hand, we have an injection of $\mathcal{F}_1 = \mathcal{O}(\nu)$ into $\mathcal{F}$. It follows that, by tensoring with $\Delta_\lambda(V)$, we get the exact sequence of $\mathcal{U}^\circ$-modules

$$0 \longrightarrow \Delta_\lambda(V)(\nu) \longrightarrow \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Applying Γ to this exact sequence we see that $\Gamma(X, \Delta_\lambda(V)(\nu))$ is a $\mathfrak{g}$-submodule of the tensor product $\Gamma(X, \Delta_\lambda(V)) \otimes_{\mathbb{C}} F = V \otimes_{\mathbb{C}} F$, which is of finite length by our assumption. This proves our assertion for $p = 0$.

Assume now that $p \geq 1$. Then the long exact sequence of cohomology implies that $H^p(X, \Delta_\lambda(V)(\nu))$ is a quotient of $H^{k-1}(X, \mathcal{K})$. On the other hand, from the definition of the filtration of $\mathcal{F}$, it follows that $\mathcal{K}$ has a natural $\mathcal{U}^\circ$-module filtration such that the corresponding graded module $\operatorname{Gr}\mathcal{K}$ is equal to $\oplus\Delta_\lambda(V)(\mu)$, where the sum is taken over all weights μ of F different from ν. By the induction assumption, $H^{k-1}(X, \Delta_\lambda(V)(\mu))$ are $\mathfrak{g}$-modules of finite length. An induction in the length of the filtration of $\mathcal{K}$ implies that $H^{k-1}(X, \mathcal{K})$ is a $\mathfrak{g}$-module of finite length. □

Proposition 5. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and $\theta = W \cdot \lambda$. Let $V \in \mathcal{M}(\mathcal{U}_\theta)$ and F a finite-dimensional representation of $\mathfrak{g}$. If $\Delta_\lambda(V)$ is a holonomic $\mathcal{D}_\lambda$-module, $V \otimes_{\mathbb{C}} F$ is a module of finite length.*

Proof. This follows from 5.3. and 4. □

Therefore, $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ contains all $\mathcal{U}_\theta$-modules with holonomic localizations.

Remark. Is there an irreducible $\mathcal{U}_\theta$-module V in $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$ such that its localization is not holonomic? Let $K(\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g})))$ be the Grothendieck group of $\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$. Denote by ch the natural *character* map from $\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g}))$ into $K(\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g})))$.

Denote by $\mathcal{M}_{cc}(\mathcal{U}_\theta)$ the full subcategory of $\mathcal{M}(\mathcal{U}_\theta)$ consisting of objects which are also in $\mathcal{M}_{cc}(\mathcal{U}(\mathfrak{g}))$.

Theorem 6. *Let $\lambda \in \mathfrak{h}^*$ be regular antidominant and V an irreducible module in $\mathcal{M}_{cc}(\mathcal{U}_\theta)$. Then there exists a unique function Φ from $P(\Sigma)$ into $K(\mathcal{M}_{fl}(\mathcal{U}(\mathfrak{g})))$ such that*

(i) $\Phi(0) = \operatorname{ch}(V)$;

(ii) $\Phi(\nu)$ is a difference of $\mathfrak{g}$-modules with the infinitesimal character $\chi_{\lambda+\nu}$;
(ii) for any finite-dimensional $\mathfrak{g}$-module F

$$\operatorname{ch}(V \otimes_{\mathbb{C}} F) = \sum \Phi(\nu)$$

where the sum is taken over the set of all weights ν of F counted with their multiplicities.

The function Φ is given by the following formula

$$\Phi(\mu) = \sum_{p \in \mathbb{Z}_+} (-1)^p \operatorname{ch}(H^p(X, \Delta_\lambda(V)(\mu)))$$

for $\mu \in P(\Sigma)$.

Proof. First show that the function

$$\Phi(\mu) = \sum_{p \in \mathbb{Z}_+} (-1)^p \operatorname{ch}(H^p(X, \Delta_\lambda(V)(\mu)))$$

has the required properties. The first two properties are evident from the definition. As in 4, from the spectral sequence of a filtered object, we get a spectral sequence with E_1-term equal to

$$H^{p-q}(X, \operatorname{Gr}_q(\Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F})) = H^{p-q}(X, \Delta_\lambda(V)(\nu_q))$$

which abutts to

$$H^p(X, \Delta_\lambda(V) \otimes_{\mathcal{O}_X} \mathcal{F}) = H^p(X, \Delta_\lambda(V)) \otimes_{\mathbb{C}} F$$

and which is equal to $V \otimes_{\mathbb{C}} F$ for $p = 0$ and 0 otherwise. The Euler characteristic of the total complex attached to the E_1-term is

$$\begin{aligned}\sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \operatorname{ch}(H^{p-q}(X, \Delta_\lambda(V)(\nu_q)) \\ &= \sum_{p,q \in \mathbb{Z}} (-1)^{p-q} \operatorname{ch}(H^{p-q}(X, \Delta_\lambda(V)(\nu_q))) \\ &= \sum_{s \in \mathbb{Z}} (-1)^s \operatorname{ch}(H^s(X, \Delta_\lambda(V)(\nu))\end{aligned}$$

where the sum is taken over all weights ν of F counted with their multiplicities. By the Euler principle, this is equal to the Euler characteristics of the total complex of E_∞, i. e. to $\operatorname{ch}(V \otimes_{\mathbb{C}} F)$. This proves the third property.

It remains to show the uniqueness. First, the map $F \longmapsto \operatorname{ch}(V \otimes_{\mathbb{C}} F)$ extends to the Grothendieck group of the category of finite-dimensional $\mathfrak{g}$-modules. Moreover, the notions of weights and their multiplicities transfer directly to this setting. By ([**LG**], Ch. VI, §3, no. 4, Prop. 3), for any dominant weight μ there exists an object in the Grothendieck group of the category of finite-dimensional $\mathfrak{g}$-modules with the set of weights equal to $\{w\mu \mid w \in W\}$ and each weight has multiplicity one. Therefore, $\sum_{w \in W} \Phi(w\mu)$ is uniquely determined by the third property. On the other hand, $\chi_{\lambda+\mu} = \chi_{\lambda+w\mu}$ implies $w'(\lambda+w\mu) = \lambda+\mu$ for some $w' \in W$. This implies that $w'\lambda - \lambda = \mu - w'w\mu \in P(\Sigma)$. Therefore, $w' \in W_\lambda$. Since λ is antidominant, $w'\lambda - \lambda$ is a sum of roots from Σ_λ^+. On the other hand μ is a dominant weight and $\mu - w'w\mu$ is negative of a sum of roots from Σ^+. This implies that $w'\lambda = \lambda$ and $w' = 1$ since λ is regular. But this immediately leads to $w'\mu = \mu$. Hence, by (ii), all summands in $\sum_{w\in W} \Phi(w\mu)$ correspond to different infinitesimal characters. This implies that they are uniquely determined. □

The map Φ is usually called the *coherent family* attached to V.

L.7 Intertwining Functors for Simple Reflections

In this section we study more carefully the action of intertwining functors I_{s_α}, $\alpha \in \Pi$, on irreducible $\mathcal{D}_\lambda$-modules. If $\alpha\check{}(\lambda)$ is not integral, by 3.22, I_{s_α} is an equivalence of the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ with $\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$. A more interesting case which we want to analyze is when $\alpha\check{}(\lambda)$ is an integer. We start with a simple geometric preliminary result.

Lemma 1. *The varieties Z_w, $w \in W$, are affinely imbedded in $X \times X$.*

Proof. The variety $X \times X$ is the flag variety of $\mathfrak{g}\times\mathfrak{g}$. By 3.1.(ii), Z_w, $w \in W$, are the $\mathrm{Int}(\mathfrak{g})$-orbits in $X \times X$ under the diagonal action, hence they are affinely imbedded by H.1.1. □

Let $\alpha \in \Pi$. Denote by X_α the variety of parabolic subalgebras of type α, and by p_α the natural projection of X onto X_α. Let $Y_\alpha = X \times_{X_\alpha} X$ be the fibered product of X with X relative to the morphism p_α. Denote by q_1 and q_2 the corresponding projections of Y_α onto the first and second factor respectively. Then the following diagram

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

is commutative. Moreover, there is a natural imbedding of Y_α into $X \times X$. It identifies Y_α with the closed subvariety of $X \times X$ which is the union of Z_1 and Z_{s_α}. Under this identification, Z_1 is a closed subvariety of Y_α and Z_{s_α} is a dense open subvariety of Y_α. In addition, Z_{s_α} is affinely imbedded into Y_α by 1.

Fix a base point $y \in X_\alpha$ and denote by $P_{\alpha,y}$ the stabilizer of y in G. As we have discussed in (...) the G-homogeneous twisted sheaves of differential operators on X_α are parametrized by $P_{\alpha,y}$-invariant linear forms on the Lie algebra $\mathfrak{q}_{\alpha,y}$ of $P_{\alpha,y}$. Since $P_{\alpha,y}$ is connected, a linear form $\mu \in \mathfrak{p}_{\alpha,y}^*$ is $P_{\alpha,y}$-invariant if and only if it is $\mathfrak{p}_{\alpha,y}$-invariant. Therefore μ is $P_{\alpha,y}$-invariant if and only if it vanishes on the commutator subalgebra $[\mathfrak{p}_{\alpha,y}, \mathfrak{p}_{\alpha,y}]$ of $\mathfrak{p}_{\alpha,y}$. Let $\mathfrak{b}$ be a

Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}_{\alpha,y}$. Then $[\mathfrak{p}_{\alpha,y},\mathfrak{p}_{\alpha,y}]$ contains the nilpotent radical $\mathfrak{n}$ of $\mathfrak{b}$ and we have a canonical map from $\mathfrak{h}$ into $\mathfrak{p}_{\alpha,y}/[\mathfrak{p}_{\alpha,y},\mathfrak{p}_{\alpha,y}]$. This map is surjective and its kernel is spanned by the dual root $\alpha\check{}$ of α. Therefore, G-homogeneous twisted sheaves of differential operators on X_α are parametrized by linear forms $\mu \in \mathfrak{h}^*$ satisfying $\alpha\check{}(\mu) = 0$. In addition, for any $\mu \in \mathfrak{h}^*$ satisfying $\alpha\check{}(\mu) = 0$, the twisted sheaf of differential operators $(\mathcal{D}_{X_\alpha,\mu})^{p_\alpha}$ is a G-homogeneous twisted sheaf of differential operators on X and

$$(\mathcal{D}_{X_\alpha,\mu})^{p_\alpha} = \mathcal{D}_{X,\mu} = \mathcal{D}_{\mu-\rho}.$$

For any $\lambda \in \mathfrak{h}^*$, $(\mathcal{D}_\lambda)^{q_1}$ and $(\mathcal{D}_\lambda)^{q_2}$ are twisted sheaves of differential operators on Y_α. Since $p_\alpha \circ q_1 = p_\alpha \circ q_2$, we see that

$$(\mathcal{D}_{\mu-\rho})^{q_1} = (\mathcal{D}_{\mu-\rho})^{q_2}$$

for any $\mu \in \mathfrak{h}^*$ such that $\alpha\check{}(\mu) = 0$. Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha\check{}(\lambda)$ is an integer. Then we can put $\mu = \lambda + p\rho$. In this case, $\alpha\check{}(\mu) = \alpha\check{}(\lambda) + p\alpha\check{}(\rho) = 0$, and μ satisfies our condition. Therefore, by (...), we get the following result:

$$(\mathcal{D}_\lambda)^{q_1} = ((\mathcal{D}_{\mu-\rho})^{\mathcal{O}((-p+1)\rho)})^{q_1} = ((\mathcal{D}_{\lambda-\rho})^{q_1})^{q_1^*(\mathcal{O}((-p+1)\rho))}$$

and analogously, since $\mu = s_\alpha\lambda + ps_\alpha\rho$,

$$(\mathcal{D}_\lambda)^{q_2} = ((\mathcal{D}_{\mu-\rho})^{\mathcal{O}((-p+1)s_\alpha\rho+\alpha)})^{q_1} = ((\mathcal{D}_{\lambda-\rho})^{q_1})^{q_1^*(\mathcal{O}((-p+1)s_\alpha\rho+\alpha))}$$

Let $\mathcal{L}$ be the invertible $\mathcal{O}_{Y_\alpha}$-module on Y_α given by

$$\mathcal{L} = q_1^*(\mathcal{O}((-p+1)s_\alpha\rho + \alpha)) \otimes_{\mathcal{O}_{Y_\alpha}} q_2^*(\mathcal{O}((-p+1)\rho))^{-1}.$$

Then, by the preceding calculation, we have the following result.

Lemma 2. *Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha\check{}(\lambda)$ is an integer. Then*

$$(\mathcal{D}_{s_\alpha\lambda})^{q_1} = ((\mathcal{D}_\lambda)^{q_2})^{\mathcal{L}}.$$

In particular, we have well-defined functors U^j

$$\mathcal{V} \longrightarrow R^j q_{1+}(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})$$

from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$. Since the fibers of q_1 are one-dimensional, $U^j = 0$ for $j \neq -1, 0, 1$. Now we want to analyze the connection between the functors U^j and the intertwining functor I_{s_α}. Denote by i_1 the natural inclusion of Z_1 into Y_α and by i_α the natural inclusion of Z_{s_α} into Y_α. Since i_1 is a closed immersion and i_α is an open affine immersion, we have the distinguished triangle

$$\Gamma_{[Z_1]}(\mathcal{W}^\cdot) \to \mathcal{W}^\cdot \to i_{\alpha*}(\mathcal{W}^\cdot|Z_{s_\alpha}) \to \Gamma_{[Z_1]}(\mathcal{W}^\cdot)[1]$$

in $D^b((\mathcal{D}_{s_\alpha\lambda})^{q_1})$ for any $\mathcal{W}^\cdot \in D^b((\mathcal{D}_{s_\alpha\lambda})^{q_1})$. This leads to the distinguished triangle

$$Rq_{1+}(\Gamma_{[Z_1]}(\mathcal{W}^\cdot)) \to Rq_{1+}(\mathcal{W}^\cdot) \to Rq_{1+}(i_{\alpha*}(\mathcal{W}^\cdot|Z_{s_\alpha})) \to Rq_{1+}(\Gamma_{[Z_1]}(\mathcal{W}^\cdot)[1])$$

in $D^b(\mathcal{D}_{s_\alpha\lambda})$. Moreover, $q_1 \circ i_\alpha = p_1$, hence

$$Rq_{1+}(i_{\alpha*}(\mathcal{W}^\cdot|Z_{s_\alpha})) = Rp_{1+}(\mathcal{W}^\cdot|Z_{s_\alpha}).$$

Assume now that $\mathcal{W}^\cdot = D(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})$ for some $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Then, by 3.3.(i),

$$\mathcal{L}|Z_{s_\alpha} = p_1^*(\mathcal{O}((-p+1)s_\alpha\rho+\alpha)) \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^*(\mathcal{O}((-p+1)\rho))^{-1} = p_1^*(\mathcal{O}(\alpha)) = \mathcal{T}_{s_\alpha}.$$

It follows that, in this case, $\mathcal{W}^\cdot|Z_{s_\alpha} = D(p_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Z_{s_\alpha}}} \mathcal{T}_{s_\alpha})$, and we conclude that

$$Rq_{1+}(i_{\alpha*}(\mathcal{W}^\cdot|Z_{s_\alpha})) = LI_{s_\alpha}(D(\mathcal{V})).$$

By ([**BDM**], 7.12) we know that $R\Gamma_{[Z_1]}(\mathcal{W}^\cdot) = Ri_{1+}(Li_1^+(\mathcal{W}^\cdot)[-1])$, hence

$$Rq_{1+}(R\Gamma_{[Z_1]}(\mathcal{W}^\cdot)) = R(q_1 \circ i_1)_+(Li_1^+(\mathcal{W}^\cdot)[-1]).$$

Here $q_1 \circ i_1$ is the natural isomorphism of the diagonal Z_1 in $X \times X$ with X induced by the projection onto the first factor. If we assume again that $\mathcal{W}^\cdot = D(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})$, we see that

$$Li_1^+(D(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})) = L(q_2 \circ i_1)^+(D(\mathcal{V})) \otimes_{\mathcal{O}_{Z_1}} i_1^*(\mathcal{L}).$$

Here $q_2 \circ i_1$ is again the natural isomorphism of the diagonal Z_1 in $X \times X$ with X induced by the projection onto the second factor. If we use this map to identify Z_1 with X, we see that $i_1^*(\mathcal{L}) = \mathcal{O}(p\alpha)$ and

$$Rq_{1+}(R\Gamma_{[Z_1]}(D(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L}))) = D(\mathcal{V}(p\alpha)[-1]).$$

By applying the long exact sequence of cohomology to the above distinguished triangle this finally leads to the following result.

Theorem 3. *Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha\check{}(\lambda)$ is an integer, and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Then*

(i) $U^{-1}(\mathcal{V}) = L^{-1}I_{s_\alpha}(\mathcal{V})$;

(ii) we have an exact sequence of $\mathcal{D}_{s_\alpha\lambda}$-modules

$$0 \longrightarrow U^0(\mathcal{V}) \longrightarrow I_{s_\alpha}(\mathcal{V}) \longrightarrow \mathcal{V}(p\alpha) \longrightarrow U^1(\mathcal{V}) \longrightarrow 0.$$

We can say more if $\mathcal{V}$ is irreducible.

Theorem 4. *Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha\check{}(\lambda)$ is an integer, and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ an irreducible $\mathcal{D}_\lambda$-module. Then either*

(i) $U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = \mathcal{V}(p\alpha)$ and $U^0(\mathcal{V}) = 0$, and in this case $I_{s_\alpha}(\mathcal{V}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$;

or

(ii) $U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = 0$,

and in this case $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$ *and the sequence*

$$0 \longrightarrow U^0(\mathcal{V}) \longrightarrow I_{s_\alpha}(\mathcal{V}) \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0$$

is exact. The module $U^0(\mathcal{V})$ *is the largest quasi-coherent* $\mathcal{D}_{s_\alpha\lambda}$*-submodule of* $I_{s_\alpha}(\mathcal{V})$ *different from* $I_{s_\alpha}(\mathcal{V})$.

From 4. and 4.7. we immediately see that $U^0(\mathcal{V})$ in (ii) can be characterized as the largest quasi-coherent $\mathcal{D}_{s_\alpha\lambda}$-submodule of $I_{s_\alpha}(\mathcal{V})$.

To prove the remaining assertions, we first we remark that if $U^{-1}(\mathcal{V}) \neq 0$, $L^{-1}I_{s_\alpha}(\mathcal{V}) \neq 0$ by 3. Hence, by 4.6, $I_{s_\alpha}(\mathcal{V}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$. By applying 3. again, we conclude that $U^0(\mathcal{V}) = 0$ and $U^{-1}(\mathcal{V}) = \mathcal{V}(p\alpha)$.

Assume that $U^{-1}(\mathcal{V}) = 0$. Then, by 3, $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$. Hence, by 4.6, we see that $I_{s_\alpha}(\mathcal{V}) \neq 0$. It remains to show that $U^1(\mathcal{V}) = 0$.

Assume that $U^1(\mathcal{V}) \neq 0$. Then, by 3, $U^1(\mathcal{V}) = \mathcal{V}(p\alpha)$. We shall show that this leads to $I_{s_\alpha}(\mathcal{V}) = 0$, what is a contradiction. This argument will also give us some insight in the structure of irreducible $\mathcal{D}_\lambda$-modules with $U^1(\mathcal{V}) \neq 0$.

We start with a preliminary result. Let $\mu \in \mathfrak{h}^*$ be a linear form such that $\alpha\check{}(\mu) = 0$. Let $\mathcal{U}$ be a $\mathcal{D}_{X_\alpha,\mu}$-module on X_α. Then $p_\alpha^+(\mathcal{U})$ is a $\mathcal{D}_{\mu-\rho}$-module on X.

Lemma 5. *Let* $\mathcal{V} = p_\alpha^+(\mathcal{U})$ *for some* $\mathcal{U} \in \mathcal{M}_{qc}(\mathcal{D}_{X_\alpha,\mu})$. *Then* $I_{s_\alpha}(\mathcal{V}) = 0$ *and* $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(\alpha)$.

Proof. As in the discussion preceding the proof of 3.9, the proof reduces to the corresponding statement for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In this case, $\mathcal{V}$ is a direct sum of copies of $\mathcal{O}_X$ and our claim follows from 3.24. □

This result implies that if $\mathcal{W}$ is a translate of a module of the form $p_\alpha^+(\mathcal{U})$, we have $I_{s_\alpha}(\mathcal{W}) = 0$. On the other hand, by applying the base change (...) to the diagram

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

we see immediately that $U^1(\mathcal{V})$ is a translate of such a module. Therefore, $\mathcal{V}$ has also this property, and we conclude that $I_{s_\alpha}(\mathcal{V}) = 0$. This ends the proof of 5.4.

In addition, we proved the following result.

Proposition 6. *Let* $\lambda \in \mathfrak{h}^*$ *be such that* $p = -\alpha\check{}(\lambda)$ *is an integer, and* $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ *an irreducible* $\mathcal{D}_\lambda$*-module. Then the following conditions are equivalent:*

(i) $I_{s_\alpha}(\mathcal{V}) = 0$;

(ii) $\mathcal{V}$ *is a translate of a module of the form* $p_\alpha^+(\mathcal{U})$.

This result, combined with 4.9. implies the following simple criterion for vanishing of global sections of irreducible $\mathcal{D}_\lambda$-modules.

Theorem 7. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Let S be the subset of Π_λ consisting of all roots orthogonal to λ. Assume that S is contained in Π. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then the following conditions are equivalent:*

(i) $\Gamma(X, \mathcal{V}) = 0$;

(ii) there exists $\alpha \in S$ such that $\mathcal{V}$ is a translate of a module of the form $p_\alpha^+(\mathcal{U})$.

L.8 Supports and $\mathfrak{n}$-homology

In this section we prove some results on $\mathfrak{n}$-homology which follow from analysis of the action of intertwining functors.

We start with some geometric preliminaries. Let S be a subset of the flag variety X. For $w \in W$ put

$$E_w(S) = \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } v \text{ with respect to } \mathfrak{b}_y \text{ for some } v \leq w, y \in S\}.$$

Lemma 1. *(i) If S is a subset of X and $w \in W$,*

$$\dim S \leq \dim E_w(S) \leq \dim S + \ell(w).$$

(ii) If S is a subset of X and $w \in W$,

$$E_w(\bar{S}) = \overline{E_w(S)}.$$

(iii) If S is a closed subset of X and $w \in W$, $E_w(S)$ is the closure of the set

$$\{x \in X \mid \mathfrak{b}_x \textit{ is in the relative position } w \textit{ with respect to some } \mathfrak{b}_y,\ y \in S\}.$$

(iv) If S is irreducible, $E_w(S)$ is also irreducible.

(v) Let $w, v \in W$ be such that $\ell(wv) = \ell(w) + \ell(v)$. Then

$$E_{wv}(S) = E_w(E_v(S)).$$

Proof. Let $\alpha \in \Pi$. Denote by X_α the variety of all parabolic subalgebras of type α and by $p_\alpha : X \to X_\alpha$ the natural projection. Then we have

$$\begin{aligned} E_{s_\alpha}(S) &= \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } v \\ &\qquad \text{with respect to } \mathfrak{b}_y \text{ for some } v \leq s_\alpha, y \in S\} \\ &= S \cup \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } s_\alpha \\ &\qquad \text{with respect to } \mathfrak{b}_y \text{ for some } y \in S\} = p_\alpha^{-1}(p_\alpha(S)). \end{aligned}$$

Clearly, since $p_\alpha : X \to X_\alpha$ is a locally trivial fibration with fibre isomorphic to $\mathbb{P}^1$, $E_{s_\alpha}(S)$ is closed (resp. irreducible) if S is closed (resp. irreducible). Moreover, we see that

$$\dim S \le \dim E_{s_\alpha}(S) \le \dim S + 1.$$

Therefore, $E_{s_\alpha}(\bar{S})$ is closed. Hence, $\overline{E_{s_\alpha}(S)} \subset E_{s_\alpha}(\bar{S})$. On the other hand, since $S \subset \overline{E_{s_\alpha}(S)}$ it follows that $\bar{S} \subset \overline{E_{s_\alpha}(S)}$. If $x \in \overline{E_{s_\alpha}(S)}$, the whole fiber $p_\alpha^{-1}(p_\alpha(x))$ is contained in $\overline{E_{s_\alpha}(S)}$. This implies $E_{s_\alpha}(\bar{S}) \subset \overline{E_{s_\alpha}(S)}$. This proves (ii) for simple reflections.

Now we prove (v) by induction in the length of $w \in W$. First we claim that the formula holds if $w = s_\alpha$, $\alpha \in \Pi$. In this case, $E_{s_\alpha}(E_v(S))$ consists of all points $x \in X$ such that either $x \in E_v(S)$ or there exists $y \in E_v(S)$ such that $\mathfrak{b}_x$ is in relative position s_α with respect to $\mathfrak{b}_y$. Hence, it consists of all $x \in X$ such that there exists $y \in S$ and $\mathfrak{b}_x$ is in relative position u with respect to b_y for either $u \le v$ or $u = s_\alpha u'$ with $u' \le v$. In the second case, we have either $\ell(u) = \ell(u') + 1$ and $u \le s_\alpha v$ or $\ell(u) = \ell(u') - 1$ and $u \le u' \le v$. Hence, $E_{s_\alpha}(E_v(S)) \subset E_{s_\alpha v}(S)$. Conversely, if $u \le s_\alpha v$, we have either $u \le v$ or $s_\alpha u \le v$, hence $E_{s_\alpha}(E_v(S)) = E_{s_\alpha v}(S)$.

Assume now that w is arbitrary. Then we can find $\alpha \in \Pi$ and $w' \in W$ such that $\ell(w) = \ell(w') + 1$. Therefore, by the induction assumption,

$$E_w(E_v(S)) = E_{s_\alpha w'}(E_v(S)) = E_{s_\alpha}(E_{w'}(E_v(S))) = E_{s_\alpha}(E_{w'v}(S)),$$

which completes the proof of (v).

Now, for arbitrary $w \in W$, $\alpha \in \Pi$, and $w' \in W$ such that $\ell(w) = \ell(w')+1$, we have $E_w(S) = E_{s_\alpha}(E_{w'}(S))$. Using the first part of the proof and an induction in $\ell(w)$, (i), (ii) and (iv) follow immediately. In addition, we see that $E_w(S)$ is closed, if S is closed.

(iii) Let

$$V = \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } w \text{ with respect to some } \mathfrak{b}_y,\ y \in S\}.$$

Then $V \subset E_w(S)$. Since $E_w(S)$ is closed, $\overline{V} \subset E_w(S)$. Let $y \in S$. Then the closure of the set of all $x \in X$ such that $\mathfrak{b}_x$ is in relative position w with respect to $\mathfrak{b}_y$ is equal to $E_w(\{x\})$. This implies

$$\overline{V} \supset \bigcup_{x \in S} E_w(\{x\}) = E_w(S). \square$$

We say that $w \in W$ is *transversal* to $S \subset X$ if

$$\dim E_w(S) = \dim S + \ell(w).$$

If w is transversal to S, $\ell(w) \le \operatorname{codim} S$.

Lemma 2. *(i) $w \in W$ is transversal to S if and only if it is transversal to $\bar{S}$.*

(ii) Let S be a subset of X and $w, v \in W$ be such that $\ell(wv) = \ell(w) + \ell(v)$. Then the following statements are equivalent:

(a) wv is transversal to S;
(b) v is transversal to S and w is transversal to $E_v(S)$.

Proof. (i) By 1.(ii) we have

$$\dim E_w(S) = \dim \overline{E_w(S)} = \dim E_w(\bar{S}),$$

and the assertion follows from the definition of transversality.

(ii) By 1.(i)

$$\dim E_{wv}(S) \leq \dim S + \ell(wv) = \dim S + \ell(w) + \ell(v),$$

and the equality holds if and only if wv is transversal to S. On the other hand, by 1.(v),

$$\dim E_{wv}(S) = \dim E_w(E_v(S)) \leq \dim E_v(S) + \ell(w) \leq \dim S + \ell(v) + \ell(w).$$

Hence, if (a) holds, the last relation is an equality, i.e.,

$$\dim E_w(E_v(S)) = \dim E_v(S) + \ell(w)$$

and

$$\dim E_v(S) = \dim S + \ell(v).$$

Hence, (b) holds.

Conversely, if (b) holds, we see immediately that wv is transversal to S. □

Lemma 3. *Let S be an irreducible closed subvariety of X and $w \in W$. Then there exists $v \leq w$ such that v is transversal to S and $E_v(S) = E_w(S)$.*

Proof. First we consider the case of $w = s_\alpha$, $\alpha \in \Pi$. In this case $E_{s_\alpha}(S) = p_\alpha^{-1}(p_\alpha(S))$ is irreducible and closed, and we have two possibilities:

a) s_α is transversal to S and $\dim E_{s_\alpha}(S) = \dim S + 1$, or

b) s_α is not transversal to S, $\dim E_{s_\alpha}(S) = \dim S$ and since $S \subset E_{s_\alpha}(S)$, we have $E_{s_\alpha}(S) = S$.

Now we prove the general statement by induction in $\ell(w)$. If $\ell(w) = 0$, $w = 1$ and $E_1(S) = S$, hence the assertion is obvious. Assume that $\ell(w) = k$. Then there exists $w' \in W$ and $\alpha \in \Pi$ such that $w = s_\alpha w'$ and $\ell(w) = \ell(w')+1$. In this case, $E_w(S) = E_{s_\alpha}(E_{w'}(S))$ by 1.(v). By the induction assumption, there exists $v' \in W$, $v' \leq w'$ which is transversal to S and such that $E_{v'}(S) = E_{w'}(S)$.

Now, by the first part of the proof, if s_α is not transversal to $E_{w'}(S)$ we have

$$E_w(S) = E_{s_\alpha}(E_{w'}(S)) = E_{w'}(S) = E_{v'}(S).$$

Since $v' \leq w' \leq w$ the assertion follows. If s_α is transversal to $E_{w'}(S)$, we have

$$\dim E_w(S) = \dim E_{s_\alpha}(E_{w'}(S)) = \dim E_{w'}(S) + 1 = \dim S + \ell(v') + 1.$$

Put $v = s_\alpha v'$. If we have $\ell(v) = \ell(v') - 1$,

$$E_{v'}(S) = E_{s_\alpha}(E_v(S)) = p_\alpha^{-1}(p_\alpha(E_v(S)))$$

by 1.(v) and

$$E_{s_\alpha}(E_{v'}(S)) = p_\alpha^{-1}(p_\alpha(p_\alpha^{-1}(p_\alpha(E_v(S))))) = E_{v'}(S),$$

contrary to transversality of s_α. Therefore, $\ell(v) = \ell(v')+1$, $v \leq w$ and $E_v(S) = E_{s_\alpha}(E_{v'}(S))$. We conclude that $E_w(S) = E_v(S)$,

$$\dim E_v(S) = \dim E_w(S) = \dim S + \ell(v') + 1 = \dim S + \ell(v)$$

and v is transversal to S. □

Fix $\lambda \in \mathfrak{h}^*$. Let $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ be the category of coherent $\mathcal{D}_\lambda$-modules. The support $\operatorname{supp} \mathcal{V}$ of a coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$ is a closed subvariety of X (...). We want to analyze how the action of intertwining functors changes supports of coherent $\mathcal{D}$-modules.

First we remark the following simple fact which is a direct consequence of the definition of the intertwining functors and 1.(iii). If $\mathcal{V}^\cdot$ is a complex in $D^b(\mathcal{D}_\lambda)$, we define the *support* of $\mathcal{V}^\cdot$ as

$$\operatorname{supp} \mathcal{V}^\cdot = \bigcup_{p \in \mathbb{Z}} \operatorname{supp} H^p(\mathcal{V}^\cdot).$$

Clearly, by the above remark, the support of $\mathcal{V}^\cdot \in D^b_{coh}(\mathcal{D}_\lambda)$ is a closed subvariety of X.

Lemma 4. *For any $\mathcal{V} \in D^b_{coh}(\mathcal{D}_\lambda)$ and $w \in W$, we have*

$$\operatorname{supp} LI_w(\mathcal{V}^\cdot) \subset E_w(\operatorname{supp} \mathcal{V}^\cdot).$$

Proof. First we establish this result for simple reflections. If $\alpha \in \Pi$,

$$LI_{s_\alpha}(\mathcal{V}^\cdot) = Rp_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(\mathcal{V}^\cdot))$$

and we have the spectral sequence

$$R^s p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} H^t(p_2^+(\mathcal{V}^\cdot))) \Rightarrow H^{s+t}(LI_{s_\alpha}(\mathcal{V}^\cdot)),$$

hence the support of $LI_{s_\alpha}(\mathcal{V}^\cdot)$ is contained in the closure of the image of the support of $p_2^+(\mathcal{V}^\cdot)$. The support of $p_2^+(\mathcal{V}^\cdot)$ is contained in the closed subset

$$\{(x,x') \in Z_{s_\alpha} \mid x' \in \operatorname{supp} \mathcal{V}^\cdot\} = \{(x,x') \in X \times X \mid \mathfrak{b}_x \text{ is in relative position } s_\alpha \text{ with respect to } \mathfrak{b}_{x'}, x' \in \operatorname{supp} \mathcal{V}^\cdot\}$$

of Z_{s_α}. The projection of this set under p_1 is equal to

$$\{x \in X \mid \mathfrak{b}_x \text{ is in relative position } s_\alpha \text{ with respect to } \mathfrak{b}_{x'}, x' \in \operatorname{supp} \mathcal{V}^\cdot\} \subset E_{s_\alpha}(\operatorname{supp} \mathcal{V}^\cdot).$$

Hence,

$$\operatorname{supp} LI_{s_\alpha}(\mathcal{V}^\cdot) \subset E_{s_\alpha}(\operatorname{supp} \mathcal{V}^\cdot).$$

Now we prove the general statement by induction in $\ell(w)$. Assume that $w = w's_\alpha$, $w' \in W$, $\alpha \in \Pi$ and $\ell(w) = \ell(w') + 1$. Then $LI_w = LI_{w'} \circ LI_{s_\alpha}$ by 3.18. Hence, by the induction assumption and 1.(v),

$$\begin{aligned}\operatorname{supp} LI_w(\mathcal{V}^\cdot) = \operatorname{supp} LI_{w'}(LI_{s_\alpha}(\mathcal{V}^\cdot)) \subset E_{w'}(\operatorname{supp} LI_{s_\alpha}(\mathcal{V}^\cdot)) \\ \subset E_{w'}(E_{s_\alpha}(\operatorname{supp} V^\cdot)) = E_w(\operatorname{supp} \mathcal{V}^\cdot).\square\end{aligned}$$

The next result is more subtle.

Lemma 5. *Let* $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ *and* $\alpha \in \Pi$. *Then*

(i)

$$\dim \operatorname{supp} L^{-1}I_{s_\alpha}(\mathcal{V}) \le \dim \operatorname{supp} \mathcal{V};$$

(ii) if s_α *is transversal to* $\operatorname{supp} \mathcal{V}$,

$$\dim \operatorname{supp} I_{s_\alpha}(\mathcal{V}) = \dim \operatorname{supp} \mathcal{V} + 1.$$

Proof. Let $S = \operatorname{supp} \mathcal{V}$. Then either

(a) s_α is transversal to S, or

(b) s_α is not transversal to S.

Consider first the case (b). Then, $\dim S = \dim E_{s_\alpha}(S)$. Therefore, by 4,

$$\dim \operatorname{supp} L^p I_{s_\alpha}(\mathcal{V}) \le \dim S$$

for $p \in \mathbb{Z}$. In particular, (i) holds in this case.

It remains to study the case (a). We start with some geometric preliminaries. Let S_i , $1 \le i \le n$, be the irreducible components of S. Then

$$E_{s_\alpha}(S) = E_{s_\alpha}\left(\bigcup_{i=1}^n S_i\right) = \bigcup E_{s_\alpha}(S_i).$$

Since $E_{s_\alpha}(S_i)$ are closed and irreducible by 1.(iv), the maximal elements of the family $(E_{s_\alpha}(S_i); 1 \le i \le n)$ are the irreducible components of $E_{s_\alpha}(S)$. Hence,

$$\dim E_{s_\alpha}(S) = \max_{1\le i\le n} \dim E_{s_\alpha}(S_i),$$

and there are irreducible components S_i of S satisfying $\dim E_{s_\alpha}(S_i) = \dim E_{s_\alpha}(S)$. By relabeling the indices, we can assume that this holds for $1 \le i \le m$. Since s_α is transversal to S, we have $\dim E_{s_\alpha}(S) = \dim S + 1$. Therefore, $\dim E_{s_\alpha}(S_i) = \dim S + 1$ for $1 \le i \le m$. On the other hand, $\dim E_{s_\alpha}(S_i) \le \dim S_i + 1$, implies that $\dim S_i = \dim S$ for $1 \le i \le m$. Hence S_i, $1 \le i \le m$, are irreducible components of S of dimension $\dim S$. Since $E_{s_\alpha}(S_i) = p_\alpha^{-1}(p_\alpha(S_i))$ and $p_\alpha : X \to X_\alpha$ is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^1$, we see that

$$\dim p_\alpha(S_i) = \dim p_\alpha^{-1}(p_\alpha(S_i)) - 1 = \dim S_i = \dim S$$

for $1 \le i \le m$. On the other hand, if $m < i \le n$, $\dim E_{s_\alpha}(S_i) \le \dim S$, hence either $\dim S_i < \dim S$ or s_α is not transversal to S_i. In both cases, $\dim p_\alpha(S_i) < \dim S$. Hence, if we denote by S_0 the union of the singular locus of S and $\bigcup_{m<i\le n} S_i$, we see that $\dim p_\alpha(S_0) < \dim S$ and $\dim E_{s_\alpha}(S_0) \le \dim S$. Let $X' = X - S_0$. Then $S \cap X'$ is a smooth closed subvariety of X' and its irreducible components are $S_i' = S_i \cap X'$, $1 \le i \le m$. Therefore, S_i', $1 \le i \le m$, are mutually disjoint smooth subvarieties of dimension $\dim S$, and s_α is transversal to all of them.

If we consider the restrictions $p_\alpha|S_i' : S_i' \to p_\alpha(S_i')$, $1 \le i \le m$, there exist open dense sets U_i in $p_\alpha(S_i)$ such that the fibres $p_\alpha^{-1}(u) \cap S$ are finite for $u \in U = \bigcup_{i=1}^m U$ ([**Mumford**], Ch. I, §8, Theorem 3.). The set $p_\alpha^{-1}(U)$ is open in $E_{s_\alpha}(S)$ of dimension $\dim S + 1$. Since S and $E_{s_\alpha}(S_0)$ are closed subspaces of $E_{s_\alpha}(S)$ of codimension 1, the set

$$V = p_\alpha^{-1}(U) - (S \cup E_{s_\alpha}(S_0))$$

is open in $E_{s_\alpha}(S)$ of dimension $\dim S + 1$ and its complement is a subvariety of dimension $\dim S$.

Let $x \in V$. Consider the projections $p_i : Z_{s_\alpha} \to X$, $i = 1, 2$, induced by projections of $X \times X$ to the first and second factor. Then

$$\begin{aligned} p_1^{-1}(x) \cap p_2^{-1}(S) &= \{(x, x') \in X \times X \mid p_\alpha(x) = p_\alpha(x'),\ x' \in S\} \\ &= \{(x, x') \mid x' \in p_\alpha^{-1}(x) \cap S\} \end{aligned}$$

is a nonempty finite set, and p_2 induces a bijection of this set onto $p_\alpha^{-1}(x) \cap S$.

Now we turn to the analysis of geometric fibres of LI_{s_α}. For any $x \in X$ we denote by i_x the natural injection $\{x\} \to X$. Then we have

$$LT_x(\mathcal{U}^\cdot) = Li_x^+(\mathcal{U}^\cdot) = Ri_x^!(\mathcal{U}^\cdot)[\dim X]$$

for any $\mathcal{U}^\cdot \in D^b(\mathcal{D}_\lambda)$. Therefore, for any $x \in X$, we have

$$\begin{aligned} LT_x(LI_{s_\alpha}(D(\mathcal{V}))) &= Ri_x^!(LI_{s_\alpha}(D(\mathcal{V})))[\dim X] \\ &= Ri_x^!(Rp_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(D(\mathcal{V}))))[\dim X]. \end{aligned}$$

Let $Z_x = p_1^{-1}(x) \subset Z_{s_\alpha}$ be the fibre of p_1 over x. Denote by j_x the immersion of Z_x into Z_{s_α}. Then, by base change ([**BDM**], VI.8.4), we have

$$\begin{aligned} LT_x(LI_{s_\alpha}(D(\mathcal{V}))) &= Ri_x^!(Rp_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(D(\mathcal{V}))))[\dim X] \\ &= R(p_1 \circ j_x)_+(Rj_x^!(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(D(\mathcal{V}))))[\dim X] \\ &= R(p_1 \circ j_x)_+(j_x^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{Z_x}} Rj_x^!(p_2^!(D(\mathcal{V}))[-1]))[\dim X] \\ &= R(p_1 \circ j_x)_+(j_x^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{Z_x}} R(p_2 \circ j_x)^!(D(\mathcal{V})))[\dim X - 1]. \end{aligned}$$

The projection $p_2 \circ j_x$ of Z_x onto its image $F = (p_\alpha(x) - \{x\}) \subset X$ is an ismorphism. Denote by $q_1 : F \to X$ and $q_2 : F \to X$ the compositions of p_1

and p_2 with the inverse of this isomorphism. Then q_2 is the natural inclusion of F into X. Since $\mathcal{T}_{s_\alpha} = p_1^*(\mathcal{O}(\alpha)) = p_2^*(\mathcal{O}(-\alpha))$ by 3.3.(i), we finally see that

$$LT_x(LI_{s_\alpha}(D(\mathcal{V}))) = Rq_{1+}(q_2^*(\mathcal{O}(-\alpha)) \otimes_{\mathcal{O}_F} Rq_2^!(D(\mathcal{V})))[\dim X - 1].$$

Assume now that $x \in V$. Then $F \subset X'$. On the other hand, $S' = S \cap X'$ is a smooth closed subvariety of X'. Let j be the natural immersion of S' into X. Then, by Kashiwara's equivalence of categories ([**BDM**], VI.7.11), we have $\mathcal{V}|X' = j_+(R^0 j^!(\mathcal{V}))|X'$. Hence,

$$Rq_2^!(D(\mathcal{V})) = Rq_2^!(D(j_+(R^0 j^!(\mathcal{V})))) = Rq_2^!(j_+(Rj^!(D(\mathcal{V})))).$$

Let k and h be the natural immersions of $F \cap S$ into F and S' respectively. Then, applying again the base change, we have

$$\begin{aligned} Rq_2^!(D(\mathcal{V})) &= Rq_2^!(j_+(Rj^!(D(\mathcal{V})))) = Rk_+(Rh^!(Rj^!(D(\mathcal{V})))) \\ &= Rk_+(R(j \circ h)^!(D(\mathcal{V}))) = Rk_+(R(q_2 \circ k)^!(D(\mathcal{V}))). \end{aligned}$$

Since the set $F \cap S$ is finite, if we denote by k_y the natural immersion of $\{y\}$ into F, we have

$$\begin{aligned} Rq_2^!(D(\mathcal{V})) &= Rk_+(R(q_2 \circ k)^!(D(\mathcal{V}))) \\ &= \bigoplus_{y \in F \cap S} k_{y+}(Ri_y^!(D(\mathcal{V}))) = \bigoplus_{y \in F \cap S} k_{y+}(LT_y(D(\mathcal{V})))[-\dim X]. \end{aligned}$$

This implies that

$$LT_x(LI_{s_\alpha}(D(\mathcal{V}))) = \bigoplus_{y \in (p_\alpha^{-1}(x) - \{x\}) \cap S} LT_y(D(\mathcal{V}))[-1]$$

as a complex of vector spaces.

By ([**BDM**], VII.9.3), by shrinking U even more, we can assume that $j^!(\mathcal{V})$ is a locally free $\mathcal{O}$-module on $S \cap p_\alpha^{-1}(U)$. This implies that $L^p T_y(j^!(\mathcal{V})) = 0$, for $p \neq 0$ and $y \in S \cap p_\alpha^{-1}(U)$; and

$$E_y = T_y(j^!(\mathcal{V})) \neq 0$$

for $y \in S \cap p_\alpha^{-1}(U)$. Therefore, if we denote by l_y the inclusion of $\{y\}$ into S', we get

$$\begin{aligned} LT_y(D(\mathcal{V})) &= Ri_y^!(D(\mathcal{V}))[\dim X] = Rl_y^!(Rj^!(D(\mathcal{V})))[\dim X] \\ &= LT_y(D(j^!(\mathcal{V})))[\dim X - \dim S] = D(E_y)[\dim X - \dim S] \end{aligned}$$

for all $y \in S \cap p_\alpha^{-1}(U)$. Hence,

$$LT_x(LI_{s_\alpha}(D(\mathcal{V}))) = \bigoplus_{y \in (p_\alpha^{-1}(x) - \{x\}) \cap S} D(E_y)[\dim X - \dim S - 1]$$

for $x \in V$, and $x \in \operatorname{supp} LI_{s_\alpha}(D(\mathcal{V}))$. Therefore, $\operatorname{supp} LI_{s_\alpha}(D(\mathcal{V}))$ contains the closure $\bar{V}$ of V and $\dim \operatorname{supp} LI_{s_\alpha}(D(\mathcal{V})) \geq \dim S + 1$. By 4, we have

$$\dim \operatorname{supp} LI_{s_\alpha}(D(\mathcal{V})) = \dim S + 1.$$

Since

$$\begin{aligned}\dim \sup LI_{s_\alpha}(D(\mathcal{V})) &= \max_{p\in\mathbb{Z}} \dim \operatorname{supp} L^p I_{s_\alpha}(\mathcal{V}) \\ &= \max\left(\dim \operatorname{supp} I_{s_\alpha}(\mathcal{V}), \dim \operatorname{supp} L^{-1} I_{s_\alpha}(\mathcal{V})\right),\end{aligned}$$

to complete the proof we have to show that $\dim \operatorname{supp} L^{-1} I_{s_\alpha}(\mathcal{V}) \leq \dim S$. Let X'' be the complement of the union of the singular locus of $E_{s_\alpha}(S)$ and its irreducible components of dimension $\leq \dim S$. Then $T = E_{s_\alpha}(S) \cap X''$ is a closed smooth subvariety of X'' and all its irreducible components are of dimension $\dim S + 1$. Let $v : T \to X$ be the natural inclusion. By Kashiwara's theorem, $L^p I_{s_\alpha}(\mathcal{V})|X'' = v_+(\mathcal{C}_p)|X''$, for coherent $\mathcal{D}$-modules $\mathcal{C}_p = v^!(L^p I_{s_\alpha}(\mathcal{V}))$ on T. By shrinking X'' if necessary, we can assume that $\mathcal{C}_p$ are locally free $\mathcal{O}$-modules ([**BDM**], VII.9.3). If we denote by h_x the inclusion of $\{x\}$ into T, using again base change, we see that

$$\begin{aligned}LT_x(D(L^p I_{s_\alpha}(\mathcal{V}))) &= LT_x(D(v_+(\mathcal{C}_p))) = Ri_x^!(v_+(D(\mathcal{C}_p)))[\dim X] \\ &= Rh_x^!(D(\mathcal{C}_p))[\dim X] = LT_x(D(\mathcal{C}_p))[\dim X - \dim S - 1] \\ &= D(T_x(\mathcal{C}_p))[\dim X - \dim S - 1]\end{aligned}$$

for $x \in T$ and $p \in \mathbb{Z}$. Hence,

$$L^q T_x(L^p I_{s_\alpha}(\mathcal{V})) = 0$$

for $q \neq \dim S - \dim X + 1$ and

$$L^{\dim S - \dim X + 1} T_x(L^p I_{s_\alpha}(\mathcal{V})) = T_x(\mathcal{C}_p).$$

By this calculation, the spectral sequence

$$L^q T_x(L^p I_{s_\alpha}(\mathcal{V})) \Rightarrow H^{p+q}(LT_x(LI_{s_\alpha}(D(\mathcal{V})))),$$

converges at E_2-stage and

$$H^{\dim S - \dim X + p + 1}(LT_x(LI_{s_\alpha}(D(\mathcal{V})))) = T_x(\mathcal{C}_p)$$

for $x \in T$. Hence,

$$LT_x(LI_{s_\alpha}(D(\mathcal{V}))) = D(T_x(\mathcal{C}_{-1}))[\dim X - \dim S] \oplus D(T_x(\mathcal{C}_0))[\dim X - \dim S - 1]$$

for $x \in T$.

Since the set of all irreducible components of $E_{s_\alpha}(S)$ is equal to the set $(E_{s_\alpha}(S_i); 1 \leq i \leq m)$, we see that $T \cap V$ is dense in T. Comparing the preceding calculations, we see that $T_x(\mathcal{C}_{-1}) = 0$ for $x \in T \cap V$. Since $\mathcal{C}_p$ are locally free, we conclude that $\mathcal{C}_{-1} = 0$. Hence, $\operatorname{supp} L^{-1} I_{s_\alpha}(\mathcal{V}) \subset X - X''$ and $\dim \operatorname{supp} L^{-1} I_{s_\alpha}(\mathcal{V}) \leq \dim S$. This completes the proof of (i) in this case, and implies that (ii) must hold. □

Proposition 6. *Let $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ and $w \in W$. Then*

$$\dim \operatorname{supp} L^p I_w(\mathcal{V}) \leq \dim \operatorname{supp} \mathcal{V} + \ell(w) + p,$$

for $p \in \mathbb{Z}$.

Proof. We prove this result by induction in $\ell(w)$. If w is a simple reflection, this follows from 1.(i), 4. and 5.(i). Let $w = s_\alpha w'$ with $\alpha \in \Pi$ and $w' \in W$ with $\ell(w) = \ell(w') + 1$. Then, by the induction assumption,

$$\begin{aligned}\dim \operatorname{supp} L^p I_{w'}(L^q I_{s_\alpha}(\mathcal{V})) &\leq \dim \operatorname{supp} L^q I_{s_\alpha}(\mathcal{V}) + \ell(w') + p \\ &\leq \dim \operatorname{supp} \mathcal{V} + \ell(w') + 1 + p + q = \dim \operatorname{supp} \mathcal{V} + \ell(w) + p + q,\end{aligned}$$

for $p, q \in \mathbb{Z}$. From the spectral sequence attached to 3.18. we conclude that

$$\dim \operatorname{supp} L^s I_w(\mathcal{V}) \leq \dim \operatorname{supp} \mathcal{V} + \ell(w) + s,$$

for any $s \in \mathbb{Z}$. □

Lemma 7. *Let $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ and $w \in W$ transversal to* $\operatorname{supp} \mathcal{V}$. *Assume that the support S of $\mathcal{V}$ is irreducible. Then*

$$\operatorname{supp} I_w(\mathcal{V}) = E_w(S).$$

Proof. We prove this result by induction in $\ell(w)$. If $\ell(w) = 1$, $w = s_\alpha$ for some $\alpha \in \Pi$. By 4, $\operatorname{supp} I_{s_\alpha}(\mathcal{V}) \subset E_{s_\alpha}(S)$. Also, by 1, both sets are closed and $E_{s_\alpha}(S)$ is irreducible. Since $\dim \operatorname{supp} I_w(\mathcal{V}) = \dim S + 1 = \dim E_{s_\alpha}(S)$ by transversality and 5.(ii), the statement follows.

Let $w \in W$ with $\ell(w) = k > 1$. Then $w = s_\alpha w'$ with $\alpha \in \Pi$ and $\ell(w') = k - 1$. Since w is transversal to S, w' is transversal to S and s_α is transversal to $E_{w'}(S)$ by 2. By the induction assumption, $\operatorname{supp} I_{w'}(\mathcal{V}) = E_{w'}(S)$. Hence, by 3.8 and 1.(iv), we have

$$\operatorname{supp} I_w(\mathcal{V}) = \operatorname{supp} I_{s_\alpha}(I_{w'}(\mathcal{V})) = E_{s_\alpha}(E_{w'}(S)) = E_w(S).\square$$

To any coherent $\mathcal{D}_\lambda$-module we attach two subsets of the Weyl group W:

$$S(\mathcal{V}) = \{w \in W \mid \operatorname{supp} I_w(\mathcal{V}) = X\}$$

and

$$\mathcal{E}(\mathcal{V}) = \text{ the set of minimal elements in } S(\mathcal{V}).$$

Proposition 8. *Suppose $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ has irreducible support. Then*
(i) the set $S(\mathcal{V})$ is nonempty;
(ii)

$$\mathcal{E}(\mathcal{V}) = \{w \in W \mid w \text{ is transversal to } \operatorname{supp} V \text{ and } \ell(w) = \operatorname{codim} \operatorname{supp} \mathcal{V}\},$$

i. e., $\mathcal{E}(\mathcal{V})$ consists of all $w \in W$ transversal to $\operatorname{supp} \mathcal{V}$ *with the maximal possible length.*

Proof. Assume that $w \in W$ is transversal to $\operatorname{supp}\mathcal{V}$ and $\ell(w) = \operatorname{codim}\operatorname{supp}\mathcal{V}$. Then, by 7, we conclude that $w \in S(\mathcal{V})$. If $v < w$, $\ell(v) < \operatorname{codim}\operatorname{supp}\mathcal{V}$, and $\dim\operatorname{supp} I_v(\mathcal{V}) < \dim X$ by 4. Hence, $v \notin S(\mathcal{V})$, i. e. $w \in \mathcal{E}(\mathcal{V})$.

Conversely, assume that $w \in \mathcal{E}(\mathcal{V})$. Then, by 4, we have $E_w(\operatorname{supp}\mathcal{V}) = X$. Since the support of $\mathcal{V}$ is irreducible, by 3. we can find $v \le w$ such that v is transversal to $\operatorname{supp}\mathcal{V}$ and $E_v(\operatorname{supp}\mathcal{V}) = X$. By 7. this implies $v \in S(\mathcal{V})$. Since w is a minimal element in $S(\mathcal{V})$ we must have $w = v$, and w is transversal to $\operatorname{supp}\mathcal{V}$. This proves (ii).

To show (i) it is enough to show that $\mathcal{E}(\mathcal{V})$ is nonempty. Clearly, if w_0 is the longest element in W, $E_{w_0}(S) = X$. By 3, there exists w transversal to S such that $E_w(S) = X$, hence the assertion follows from (ii). □

To formulate the main result of this section we need another notion. Let V be a finitely generated $\mathcal{U}_\theta$-module. We say that $\lambda \in \theta$ is an *exponent* of V if the set

$$\{x \in X \mid H_0(\mathfrak{n}_x, V)_{(\lambda+\rho)} \ne 0\}$$

contains an open dense subset of X.

We say that $\lambda \in \mathfrak{h}^*$ is *strongly antidominant* if $\operatorname{Re}\alpha\check{}(\lambda) \le 0$ for any $\alpha \in \Sigma^+$. Clearly, a strongly antidominant λ is antidominant.

We also define a partial ordering on $\mathfrak{h}^*$ by: $\lambda \preccurlyeq \mu$ if $\mu - \lambda$ is a linear combination of simple roots in Π with coefficients with non-negative real parts. This order relation is related to the ordering on the Weyl group W by the following observation.

Lemma 9. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Then for any $v, w \in W$, $v \le w$ implies $v\lambda \preccurlyeq w\lambda$.*

Proof. Clearly, it is enough to show that for any $w \in W$ and $\alpha \in \Pi$ such that $\ell(s_\alpha w) = \ell(w) + 1$, we have $w\lambda \preccurlyeq s_\alpha w\lambda$. But $s_\alpha w\lambda = w\lambda - \alpha\check{}(w\lambda)\alpha$, hence

$$s_\alpha w\lambda - w\lambda = (w^{-1}\alpha)\check{}(\lambda)\alpha,$$

and it is enough to prove that $\operatorname{Re}(w^{-1}\alpha)\check{}(\lambda) \ge 0$. Since $w^{-1}\alpha$ is in Σ^+ ([**LG**], Ch. VI, §1, no. 6, Cor. 2 of Prop. 17), this follows immediately from strong antidominance of λ. □

Theorem 10. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ be such that $S = \operatorname{supp}\mathcal{V}$ is irreducible. Put $V = \Gamma(X, \mathcal{V})$.*

(i) If ω is an exponent of V, there exists $w \in W$ transversal to S with $\ell(w) = \operatorname{codim} S$ such that $w\lambda \preccurlyeq \omega$.

(ii) Assume that $\mathcal{V}$ is irreducible and $V \ne 0$. If $w \in W$ is transversal to S and $\ell(w) = \operatorname{codim} S$, then $w\lambda$ is an exponent of V.

Proof. (i) Let μ be a regular dominant weight and F the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight μ. Let $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. Then $\lambda - \mu$ is regular and strongly antidominant. Let $U = \Gamma(X, \mathcal{V}(-\mu))$. Then, by C.2.1,

$$\mathcal{V} = (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

This implies

$$\begin{aligned} V = \Gamma(X, \mathcal{V}) &= \Gamma(X, (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}) \\ &= \Gamma(X, \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} = (\Gamma(X, \mathcal{V}(-\mu)) \otimes_{\mathbb{C}} F)_{[\lambda]} = (U \otimes_{\mathbb{C}} F)_{[\lambda]}. \end{aligned}$$

Let ω be an exponent of V, i.e., $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} \neq 0$ for all x in some open dense subset of X. Then

$$H_0(\mathfrak{n}_x, V) = H_0(\mathfrak{n}_x, (U \otimes_{\mathbb{C}} F)_{[\lambda]})$$

is the direct sum of generalized $\mathcal{U}(\mathfrak{h})$-eigenspaces of $H_0(\mathfrak{n}_x, U \otimes F)$ corresponding to weights $v\lambda + \rho$, $v \in W$. Hence,

$$H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} = H_0(\mathfrak{n}_x, U \otimes_{\mathbb{C}} F)_{(\omega+\rho)}.$$

Let $(F_p; 1 \leq p \leq n)$ be an increasing $\mathfrak{b}_x$-invariant maximal flag in F. It induces a filtration $(U \otimes_{\mathbb{C}} F_p; 1 \leq p \leq n)$ of the $\mathfrak{b}_x$-module $U \otimes_{\mathbb{C}} F$. The corresponding graded module is the sum of modules of the form $U \otimes_{\mathbb{C}} \mathbb{C}_\nu$, where ν goes over the set of weights of F. Clearly, the semisimplification of $H_0(\mathfrak{n}_x, U \otimes_{\mathbb{C}} F)$ is a submodule of the direct sum of modules $H_0(\mathfrak{n}_x, U) \otimes_{\mathbb{C}} \mathbb{C}_\nu$. Since the infinitesimal character of U is regular, $H_0(\mathfrak{n}_x, U)$ is a semisimple $\mathfrak{h}$-module by L.2.4. This implies that $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)}$ is a submodule of the direct sum of modules $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \otimes_{\mathbb{C}} \mathbb{C}_\nu$. In particular, if $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} \neq 0$, $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \neq 0$ for some weight ν of F. Since the set of weights is finite, we can assume that $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \neq 0$ for all x in an open dense subset of X. On the other hand, $\omega - \nu = v(\lambda - \mu)$ for some uniquely determined $v \in W$. This implies that $v^{-1}(\omega - \nu) = \lambda - \mu$. Since $\omega = u\lambda$ for some $u \in W$, we see that

$$v^{-1}u\lambda - \lambda = -(\mu - v^{-1}\nu).$$

Since μ is the highest weight of F, the right side is the negative of a sum of positive roots. Hence $v^{-1}u \in W_\lambda$ and since λ is antidominant, we see that the left side is a sum of positive roots. It follows that both sides must be zero, $v^{-1}u$ is in the stabilizer of λ and $\omega = u\lambda = v\lambda$. Since $\lambda - \mu$ is regular, $\mathcal{V}(-\mu) = \Delta_{\lambda-\mu}(U)$. Moreover, from 2.6. we conclude that $\operatorname{supp} \Delta_{v(\lambda-\mu)}(U) = X$. Since $I_v(\mathcal{V}(-\mu)) = I_v(\Delta_{\lambda-\mu}(U)) = \Delta_{v(\lambda-\mu)}(U)$ by 3.16, we see that $v \in S(\mathcal{V}(-\mu)) = S(\mathcal{V})$. Hence, by 6. there exists $w \leq v$ such that w is transversal to S and $\ell(w) = \operatorname{codim} S$. But, by 7, this implies that $w\lambda \preccurlyeq v\lambda = \omega$.

(ii) If $\mathcal{V}$ is irreducible, $\mathcal{V}(-\mu)$ is also irreducible and their support S is irreducible. Hence, U is irreducible by the equivalence of categories. Since w is transversal to S and $\ell(w) = \operatorname{codim} S$, by 7. we see that $\operatorname{supp} \Delta_{w(\lambda-\mu)}(U) = X$. Put $\mathcal{U} = \Delta_{w(\lambda-\mu)}(U)$. Since U is irreducible, by applying 1.16. with $p = 0$, we get $U \subset \Gamma(X, \mathcal{U})$.

Assume that $s \in U$ is a global section of $\mathcal{U}$ which vanishes on the open dense subset in X. Then it generates a submodule of global sections supported in the complement of this open set. This submodule must be either equal to U or to zero. The first possibility would imply that the localization $\Delta_{w(\lambda-\mu)}(U)$

is also supported in the complement of this open set, contradicting our assumption. Therefore this submodule is equal to zero, i.e., $s = 0$. This implies that the support of any nonzero global section in U is equal to X. Let F be the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight μ. Then, as before, by C.2.1,

$$\mathcal{U}(w\mu) = (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

Hence, we see

$$\begin{aligned} \Gamma(X, \mathcal{U}(w\mu)) &= \Gamma(X, (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}) \\ &= \Gamma(X, \mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} = (\Gamma(X, \mathcal{U}) \otimes_{\mathbb{C}} F)_{[\lambda]} \supset (U \otimes_{\mathbb{C}} F)_{[\lambda]} = V. \end{aligned}$$

Moreover, the support of any nonzero global section of $\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{U} \otimes_{\mathbb{C}} F$ which comes from $U \otimes_{\mathbb{C}} F$ is equal to X, and the support of any nonzero global section of its subsheaf $\mathcal{U}(w\mu)$ which belongs to $(U \otimes_{\mathbb{C}} F)_{[\lambda]} = V$ is also equal to X. Since $\mathcal{U}(w\mu)$ is coherent, there exists an open dense subset O in X such that $\mathcal{U}(w\mu)|O$ is a locally free $\mathcal{O}_O$-module ([**BDM**], VII.9.3). Therefore, on this set, a section vanishes if and only if its values (i.e. its images in geometric fibres) vanish everywhere. Hence, there exists an open dense subset O' of O, such that for $x \in O'$, some sections from V do not vanish at x. On the other hand, for any $x \in O'$, the global sections in $\mathfrak{n}_x V$ vanish at that point. Therefore, for $x \in O'$, the geometric fibre map $\mathcal{U}(w\mu) \longmapsto T_x(\mathcal{U}(w\mu))$ induces a nonzero map of V into $T_x(\mathcal{U}(w\mu))$, which factors through $H_0(\mathfrak{n}_x, V)$, and this factor map is a morphism of $\mathfrak{b}_x$-modules. It follows that $H_0(\mathfrak{n}_x, V)_{(w\lambda+\rho)} \neq 0$ for $x \in O'$, i.e., $w\lambda$ is an exponent of V. □

The next result is a direct consequence of 10.

Theorem 11. *Let $V \neq 0$ be a finitely generated $\mathcal{U}_\theta$-module. Then the set of exponents of V is nonempty. In particular, there exists an open dense subset U of X such that $H_0(\mathfrak{n}_x, V) \neq 0$ for $x \in U$.*

Proof. Since V is nonzero, it has an irreducible quotient $U \neq 0$. Let $\lambda \in \theta$ be strongly antidominant. Then $U = \Gamma(X, \mathcal{U})$ for some irreducible $\mathcal{D}_\lambda$-module $\mathcal{U}$ by 4.2. By 8.(ii), there exists $\omega \in \theta$ which is an exponent of U. Since $H_0(\mathfrak{n}_x, U)$ is a quotient of $H_0(\mathfrak{n}_x, V)$, it follows that ω is an exponent of U. □

H. Harish-Chandra Modules

H.1 Group Actions on Flag Varieties

Let $\mathfrak{g}$ be a semisimple Lie algebra and $G = \text{Int}(\mathfrak{g})$. The following result will play an important role later.

Proposition 1. *Let K be a subgroup of G. Then K-orbits in the flag variety X are affinely imbedded.*

The proof is based on the following observations.

Lemma 2. *Let S be a solvable algebraic group and S' its closed subgroup. Then S/S' is an affine variety.*

Proof. Assume first that S is unipotent. Let $\mathfrak{s}$ be the Lie algebra of S and $\mathfrak{s}'$ the Lie algebra of S'. If $S' \neq S$, there exists a Lie subalgebra $\mathfrak{r}$ of $\mathfrak{s}$ of codimension one which contains $\mathfrak{s}'$. Since the exponential map is an isomorphism of $\mathfrak{s}$ onto S, the variety S is isomorphic to the product of an affine line with the closed subgroup R determined by $\mathfrak{r}$. Moreover, S/S' is isomorphic to the product of the affine line with R/S'. By induction in codimension of S' in S, it follows that S/S' is an affine space.

Assume now that S is arbitrary and S' is unipotent. Then S' is a closed subgroup of the unipotent radical N of S ([**Bo**], III.10.6). By the Levi decomposition, in this case S/S' is isomorphic to the product of a maximal torus T of S and N/S'. This reduces the proof to the first case.

Consider now arbitrary S'. Let N' be its unipotent radical and T' a maximal torus in S'. By Levi decomposition, S' is the semidirect product of N' with T'. By the first part of the proof, S/N' is an affine variety. The group T' acts on the variety S/N' and the quotient is S/S'. Since T is reductive this quotient is an affine variety. □

Now, let Y be a homogeneous space for G. We define an action of G on $Y \times X$ by

$$g(y, x) = (gy, gx)$$

for $g \in G$, $x \in X$, $y \in Y$.

Lemma 3. *The G-orbits in $Y \times X$ are affinely imbedded.*

Proof. Fix a point $v \in X$. Let B be the Borel subgroup corresponding to v. Every G-orbit in $Y \times X$ intersects $Y \times \{v\}$. Let $u \in Y$. Then the intersection of the G-orbit Q of (u, v) with $Y \times \{v\}$ is equal to $Bu \times \{v\}$. Let $\bar{N}$ be the unipotent radical of a Borel subgroup opposite to B. Then $\bar{N}v$ is an open neighborhood of v in X, and the map $\bar{n} \longrightarrow \bar{n}v$ is an isomorphism of $\bar{N}$ onto this neighborhood. The intersection of Q with $Y \times \bar{N}v$ is equal to the image of the variety $Bu \times \bar{N}$ under the map $(y, \bar{n}) \longrightarrow \bar{n}(y, v)$, which is obviously an immersion. Since B is solvable, its orbit Bu is an affine variety by 2. Then Bu and it is an affine variety. Clearly, this implies that $Bu \times \bar{N}$ is an affine variety. It follows that the intersection of Q with $Y \times \bar{N}v$ is affine. Therefore we can construct an open cover of $Y \times X$ such that the intersection of Q with any element of the cover is an affine variety. Since the affinity of a morphism is a local property with respect to the target variety, this ends the proof. □

Now we can prove 1. Let $Y = G/K$ and $u \in Y$ the identity coset. Then the image of the immersion $i_u : X \longrightarrow Y \times X$ given by $i_u(x) = (u, x)$ is a closed subvariety of $Y \times X$ isomorphic to X. Let $v \in X$. Denote by Q' the K-orbit of v. Then the intersection of the image of i_u with the G-orbit Q of (u, v) in $Y \times X$ is equal to

$$i_u(X) \cap Q = (\{u\} \times X) \cap Q = \{u\} \times Q'.$$

Let U be an open affine subset in $Y \times X$. Then $U \cap Q$ is open affine subset of Q by 2. Moreover, since $i_u(X)$ is closed in $Y \times X$, $U \cap (i_u(X) \cap Q)$ is open affine subset in $i_u(X) \cap Q$. This implies that $U \cap (\{u\} \times Q')$ is an open affine subset of $\{u\} \times Q'$. Furthermore, since i_u is a closed immersion, $V = i_u^{-1}(U)$ is an open affine subset of X and $V \cap Q'$ is an open affine subset of Q'. Clearly, this implies that Q' is affinely imbedded into X and completes the proof of 1.

H.2 Harish-Chandra Pairs

Let K be an algebraic group and $\varphi : K \to \mathrm{Int}(\mathfrak{g})$ a morphism of algebraic groups such that the differential of φ is injective. In this case we can identify the Lie algebra of K with a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Clearly, the group K acts naturally on X.

We say that the pair $(\mathfrak{g}, K)$ is a *Harish-Chandra pair* if the K-action on X has finitely many orbits.

If $(\mathfrak{g}, K)$ is a Harish-Chandra pair, K has an open orbit in X. Actually, these two properties are equivalent [**Brion**].

Theorem 1. *Let K be a closed subgroup of* $\mathrm{Int}(\mathfrak{g})$. *Then the following conditions are equivalent:*

(i) K has an open orbit in X;

(ii) K has finitely many orbits in X.

An example of a Harish-Chandra pair is the pair $(\mathfrak{g}, B)$ where B is a Borel subgroup of $\mathrm{Int}(\mathfrak{g})$. The finiteness of B-orbits in X is the Bruhat lemma ([**Bo**], 14.11).

Another important class of examples arises in the following way. Let σ be an involution of $\mathfrak{g}$ and $\mathfrak{k}$ the Lie subalgebra of all vectors in $\mathfrak{g}$ fixed by σ. We say that $\mathfrak{k}$ is an *involutive* subalgebra of $\mathfrak{g}$.

Proposition 2. *Let K be a closed subgroup of* $\mathrm{Int}(\mathfrak{g})$ *such that its Lie algebra $\mathfrak{k}$ is an involutive subalgebra of $\mathfrak{g}$. Then K acts with finitely many orbits on X.*

We denote the involutive automorphism of $G = \mathrm{Int}(\mathfrak{g})$ with differential σ by the same letter. The key step in the proof is the following lemma. First, define an action of G on $X \times X$ by

$$g(x, y) = (gx, \sigma(g)y)$$

for any $g \in G$, $x, y \in X$.

Lemma 3. *The group G acts on $X \times X$ with finitely many orbits.*

Proof. We fix a point $v \in X$. Let B_v be the Borel subgroup of G corresponding to v, and put $B = \sigma(B_v)$. Every G-orbit in $X \times X$ intersects $X \times \{v\}$. Let $u \in X$. Then the intersection of the G-orbit Q through (u, v) with $X \times \{v\}$ is equal to $Bu \times \{v\}$. Because of the Bruhat decomposition ([**Bo**], IV.14.11), this implies the finiteness of the number of G-orbits in $X \times X$. □

Now we show that 2. is a consequence of 3. First we can assume that K is connected. Let Δ be the diagonal in $X \times X$. By 3, that the orbit stratification of $X \times X$ induces a stratification of Δ by finitely many irreducible, affinely imbedded subvarieties which are the irreducible components of the intersections of the G-orbits with Δ. These strata are K-invariant, and therefore unions of K-orbits. Let V be one of these subvarieties, $(x, x) \in V$ and Q the K-orbit of (x, x). If we let $\mathfrak{b}_x$ denote the Borel subalgebra of $\mathfrak{g}$ corresponding to x, the tangent space $T_x(X)$ of X at x can be identified with $\mathfrak{g}/\mathfrak{b}_x$. Let p_x be the projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{b}_x$. The tangent space $T_{(x,x)}(X \times X)$ to $X \times X$ at (x, x) can be identified with $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. If the orbit map $f : G \to X \times X$ is defined by $f(g) = g(x, x)$, its differential at the identity in G is the linear map $\xi \to (p_x(\xi), p_x(\sigma(\xi)))$ of $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. Then the tangent space to V at (x, x) is contained in the intersection of the image of this differential with the diagonal in the tangent space $T_{(x,x)}(X \times X)$, i.e.

$$\begin{aligned} T_{(x,x)}(V) &\subset \{(p_x(\xi), p_x(\xi)) | \xi \in \mathfrak{g} \text{ such that } p_x(\xi) = p_x(\sigma(\xi))\} \\ &= \{(p_x(\xi), p_x(\xi)) | \xi \in \mathfrak{k}\} = T_{(x,x)}(Q). \end{aligned}$$

Consequently the tangent space to V at (x, x) agrees with the tangent space to Q, and Q is open in V. By the irreducibility of V, this implies that V is a K-orbit, and therefore our stratification of the diagonal Δ is the stratification induced via the diagonal map by the K-orbit stratification of X. Hence, 2. follows.

The following result is just a reformulation of 2.

Theorem 4. *Let K be an algebraic group and $\varphi : K \to \mathrm{Int}(\mathfrak{g})$ a morphism of algebraic groups with injective differential. Assume that the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is involutive. Then $(\mathfrak{g}, K)$ is a Harish-Chandra pair.*

Such Harish-Chandra pair is called an *involutive Harish-Chandra pair.*

3. Harish-Chandra Modules and Harish-Chandra Sheaves

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. A *Harish-Chandra module* V is a vector space which is

(i) a finitely generated $\mathcal{U}(\mathfrak{g})$-module, which is locally finite as a $\mathcal{Z}(\mathfrak{g})$-module;

(ii) an algebraic K-module;

(iii) the actions of $\mathfrak{g}$ and K are compatible, i.e., the action of $\mathfrak{k}$ given by the differential of the K-action is the same as the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ and

$$(\varphi(k)\xi) \cdot v = k \cdot \xi \cdot k^{-1} \cdot v$$

for $k \in K$, $\xi \in \mathfrak{g}$ and $v \in V$.

A morphism of Harish-Chandra modules is a linear map which is a morphism of $\mathcal{U}(\mathfrak{g})$-modules and K-modules. We denote by $\mathcal{M}_{fg}(\mathcal{U}(\mathfrak{g}), K)$ the category of Harish-Chandra modules. For $\lambda \in \mathfrak{h}^*$, $\theta = W \cdot \lambda$, we denote by $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ the full subcategory of $\mathcal{M}_{fg}(\mathcal{U}(\mathfrak{g}), K)$ consisting of modules with infinitesimal character χ_λ.

The objects of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ are called *Harish-Chandra sheaves.*

Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. By ..., it is evident that for any object of $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$, the localization $\Delta_\lambda(V)$ is an object of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. Moreover, by L.1.21, the cohomology modules $H^i(X, \mathcal{V})$, $0 \leq i \leq \dim X$, of a Harish-Chandra sheaf $\mathcal{V}$ in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ are finitely generated as $\mathcal{U}_\theta$-modules. Since they are algebraic K-modules by ..., it follows that they are in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$.

Lemma 1. *Any Harish-Chandra sheaf $\mathcal{V}$ has a good filtration $\mathrm{F}\,\mathcal{V}$ consisting of K-homogeneous coherent $\mathcal{O}_X$-modules.*

Proof. By tensoring with $\mathcal{O}(\mu)$ for sufficiently negative $\mu \in P(\Sigma)$ we can assume that λ is antidominant and regular. In this case, by L.1.3, $\mathcal{V} = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$, where $V = \Gamma(X, \mathcal{V})$. Since V is an algebraic K-module and finitely generated $\mathcal{U}_\theta$-module, there is a finite-dimensional K-invariant subspace U which generates V as a $\mathcal{U}_\theta$-module. Then $F_p\mathcal{D}_\lambda \otimes_{\mathbb{C}} U$, $p \in \mathbb{Z}_+$, are K-homogeneous coherent

$\mathcal{O}_X$-modules. Since the natural map of $F_p\mathcal{D}_\lambda \otimes_{\mathbb{C}} U$ into $\mathcal{V}$ is K-equivariant, the image $\mathrm{F}_p\,\mathcal{V}$ is a K-homogeneous coherent $\mathcal{O}_X$-submodule of $\mathcal{V}$ for arbitrary $p \in \mathbb{Z}_+$.

We claim that $\mathrm{F}\,\mathcal{V}$ is a good filtration of the $\mathcal{D}_\lambda$-module $\mathcal{V}$. Clearly, this is a $\mathcal{D}_\lambda$-module filtration of $\mathcal{V}$ by K-homogeneous coherent $\mathcal{O}_X$-modules. Since $\mathcal{V}$ is generated by its global sections, to show that it is exhaustive it is enough to show that any global section v of $\mathcal{V}$ lies in $\mathrm{F}_p\,\mathcal{V}$ for sufficiently large p. Since V is generated by U as an $\mathcal{U}_\theta$-module, there are $T_i \in \mathcal{U}_\theta$, $u_i \in U$, $1 \le i \le m$, such that $v = \sum_{i=1}^m T_i u_i$. On the other hand, there exists $p \in \mathbb{Z}_+$ such that T_i, $1 \le i \le m$, are global sections of $\mathrm{F}_p\,\mathcal{D}_\lambda$. This implies that $v \in \mathrm{F}_p\,\mathcal{V}$. Finally, by the construction of $\mathrm{F}\,\mathcal{V}$, it is evident that $\mathrm{F}_p\,\mathcal{D}_\lambda\,\mathrm{F}_q\,\mathcal{V} = \mathrm{F}_{p+q}\,\mathcal{V}$ for all $p, q \in \mathbb{Z}_+$, i.e., $\mathrm{F}\,\mathcal{V}$ is a good filtration. □

The critical result on Harish-Chandra sheaves is the following remark.

Theorem 2. *Harish-Chandra sheaves are holonomic $\mathcal{D}_\lambda$-modules. In particular, they are of finite length.*

We shall actually prove a stronger result. First we need some notation. Let Y be a smooth algebraic variety of pure dimension and Z a smooth subvariety of Y. Then we define a smooth subvariety $N_Z(Y)$ of $T^*(Y)$ as the variety of all points $(z, \omega) \in T^*(Y)$ where $z \in Z$ and $\omega \in T_z(Y)^*$ is a linear form vanishing on $T_z(Z) \subset T_z(Y)$. We call $N_Z(Y)$ the *conormal variety* of Z in Y.

Lemma 3. *The dimension of the conormal variety $N_Z(Y)$ of Z in Y is equal to* $\dim Y$.

Proof. The dimension of the space of all linear forms in $T_z(Y)^*$ vanishing on $T_z(Z)$ is equal to $\dim T_z(Y) - \dim T_z(Z) = \dim Y - \dim_z Z$. Hence, $\dim_z N_Z(Y) = \dim Y$. □

Let $\lambda \in \mathfrak{h}^*$. Then, by ..., $\mathrm{Gr}\,\mathcal{D}_\lambda = \pi_*(\mathcal{O}_{T^*(X)})$, where $\pi : T^*(X) \to X$ is the natural projection. Let $\xi \in \mathfrak{g}$. Then ξ determines a global section of $\mathcal{D}_\lambda$ of order ≤ 1, i.e. a global section of $\mathrm{F}_1\,\mathcal{D}_\lambda$. Therefore, the symbol of this section is a global section of $\pi_*(\mathcal{O}_{T^*(X)})$ independent of λ. Let $x \in X$. Then the differential at $1 \in G$ of the orbit map $f_x : G \to X$, given by $f_x(g) = gx$, maps the Lie algebra $\mathfrak{g}$ onto the tangent space $T_x(X)$ at x. The kernel of this map is $\mathfrak{b}_x$, i.e. the differential $T_1(f_x)$ of f_x at 1 identifies $\mathfrak{g}/\mathfrak{b}_x$ with $T_x(X)$. The symbol of the section determined by ξ is given by the function $(x, \omega) \longmapsto \omega(T_1(f_x)(\xi))$ for $x \in X$ and $\omega \in T_x(X)^*$.

Let K be a closed subgroup of $\mathrm{Int}(\mathfrak{g})$ and $\mathfrak{k}$ its Lie algebra. Denote by $\mathcal{I}_K$ the ideal in the $\mathcal{O}_X$-module $\pi_*(\mathcal{O}_{T^*(X)})$ generated by the symbols of sections attached to elements of $\mathfrak{k}$. Let $\mathcal{N}_K$ the set of zeros of this ideal in $T^*(X)$.

Lemma 4. *The variety $\mathcal{N}_K$ is the union of the conormal varieties $N_Q(X)$ to all K-orbits Q in X.*

Proof. Let $x \in X$ and denote by Q the K-orbit through x. Then,

$$\begin{aligned}\mathcal{N}_K \cap T_x(X)^* &= \{\omega \in T_x(X)^* \mid \omega \text{ vanishes on } T_1(f_x)(\mathfrak{k})\,\} \\ &= \{\omega \in T_x(X)^* \mid \omega \text{ vanishes on } T_x(Q)\,\} = N_Q(X) \cap T_x(X)^*,\end{aligned}$$

i.e. $\mathcal{N}_K$ is the union of all $N_Q(X)$. □

Corollary 5. *Assume that K acts on X with finitely many orbits. Then:*

(i) $\dim \mathcal{N}_K = \dim X$.

(ii) If K is connected, the irreducible components of $\mathcal{N}_K$ are the closures $\overline{N_Q(X)}$ of the conormal varieties $N_Q(X)$ of K-orbits Q in X.

Proof. For any K-orbit Q in X, its conormal variety $N_Q(X)$ has dimension equal to $\dim X$ by 3. Since the number of K-orbits in X is finite, by 3. and 4, $\mathcal{N}_K$ is a finite union of subvarieties of dimension $\dim X$. This implies (i).

Moreover,

$$\mathcal{N}_K = \bigcup_Q \overline{N_Q(X)}.$$

If K is connected, its orbits in X are also connected. Hence, their conormal varieties $N_Q(X)$ are connected too. Since they are smooth this immediately implies that they are irreducible. Hence their closures $\overline{N_Q(X)}$ are irreducible closed subvarieties of $\mathcal{N}$ of dimension $\dim X$. Therefore, they are the irreducible components of $\mathcal{N}_K$. This proves (ii). □

Therefore, 2. is an immediate consequence of the following result.

Proposition 6. *Let $\mathcal{V}$ be a Harish-Chandra sheaf. Then the characteristic variety* $\operatorname{Char}(\mathcal{V})$ *of $\mathcal{V}$ is a closed subvariety of $\mathcal{N}_K$.*

Proof. By 1, $\mathcal{V}$ has a good filtration $\mathrm{F}\,\mathcal{V}$ consisting of K-homogeneous coherent $\mathcal{O}_X$-modules. Therefore, the global sections of $\mathcal{D}_\lambda$ corresponding to $\mathfrak{k}$ map $F_p\mathcal{V}$ into itself for $p \in \mathbb{Z}$. Hence, their symbols annihilate $\operatorname{Gr}\mathcal{V}$ and $\mathcal{I}_K$ is contained in the annihilator of $\operatorname{Gr}\mathcal{V}$ in $\pi_*(\mathcal{O}_{T^*(X)})$. This implies that the characteristic variety $\operatorname{Char}(\mathcal{V})$ is a closed subvariety of $\mathcal{N}_K$. □

The following result is an immediate consequence of 2.

Theorem 7. *Every Harish-Chandra module is of finite length.*

Proof. Let V be a Harish-Chandra module. Since it is finitely generated as a $\mathcal{U}(\mathfrak{g})$-module and locally finite as a $\mathcal{Z}(\mathfrak{g})$-module, there exists a finite-dimensional $\mathcal{Z}(\mathfrak{g})$-submodule U of V which generates V. Therefore, there exist a finite set $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathfrak{h}^*$ and $n \in \mathbb{N}$ such that

$$P(\xi) = \prod_{i=1}^{k} (\xi - \chi_{\lambda_i}(\xi))^n,$$

$\xi \in \mathcal{Z}(\mathfrak{g})$, annihilates U. Since U generates V as a $\mathcal{U}(\mathfrak{g})$-module, it follows that $P(\xi)$ annihilates V for each $\xi \in \mathcal{Z}(\mathfrak{g})$. Therefore, V is a direct sum of submodules

$$V_i = \{v \in V | (\xi - \chi_{\lambda_i}(\xi))^n v = 0 \text{ for all } \xi \in \mathcal{Z}(\mathfrak{g}) \text{ and sufficiently large } n \in \mathbb{N}\}$$

for $1 \leq i \leq k$.

Therefore, we can assume that V is annihilated by $(\xi - \chi_\lambda(\xi))^n$, $\xi \in \mathcal{Z}(\mathfrak{g})$. We claim that such V has a finite filtration $\mathrm{F}V$ by Harish-Chandra submodules such that the corresponding graded module is has infinitesimal character χ_λ. Te proof is by induction in $\dim U$. If $\dim U = 1$ the assertion is evident. In general, U contains an one-dimensional eigenspace U_1 for $\mathcal{Z}(\mathfrak{g})$. Clearly, U_1 generates a Harish-Chandra submodule V_1 of V with infinitesimal character χ_λ. The quotient $V_2 = V/V_1$ is generated by the image U_2 of U and $\dim U_2 \leq \dim U - 1$.

Therefore, we can assume that V has an infinitesimal character, i.e., V is in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$. In this case we can choose $\lambda \in \theta$ which is antidominant. The localization $\Delta_\lambda(V)$ is a Harish-Chandra sheaf, and therefore a of finite length by 2. By L1.1, $V = \Gamma(X, \Delta_\lambda(V))$. Hence, the exactness of Γ and L.4.1 imply that V has finite length. □

4. $\mathfrak{n}$-homology of Harish-Chandra modules

Let V be a Harish-Chandra module. For any $x \in X$, the $\mathfrak{n}_x$-homology of V can be calculated from the standard complex $C^\cdot(\mathfrak{n}_x, V)$ given by

$$C^p(\mathfrak{n}_x, V) = \wedge^{-p}\mathfrak{n}_x \otimes_{\mathbb{C}} V,\ p \in \mathbb{Z};$$

with the differential

$$d(\xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_p \otimes v) = \sum_{i=1}^{p} (-1)^{i+1} \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_p \otimes \xi_i v$$
$$+ \sum_{i<j} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_p \otimes v$$

for $\xi_1, \ldots \xi_p \in \mathfrak{n}_x$ and $v \in V$. This immediately implies that the $\mathfrak{h}$-modules $H_p(\mathfrak{n}_x, V)$ for various points x in the same K-orbit Q in X are canonically isomorphic.

Theorem 1. *Let V be a Harish-Chandra module and $x \in X$. Then all $\mathfrak{n}_x$-homology modules $H_p(\mathfrak{n}_x, V)$, $p \in \mathbb{Z}$, are finite-dimensional.*

Proof. In the proof of 3.7 we constructed a finite filtration of V by Harish-Chandra submodules, such that its composition factors are Harish-Chandra modules with infinitesimal character. Therefore, using the spectral sequence of a filtered object (...), we see immediately that it enough to prove the statement for $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta, K)$.

If θ is a regular orbit of W, this result follows from L.5.4 and 3.2. Assume now that θ is an arbitrary Weyl group orbit. Fix an antidominant $\lambda \in \theta$. Let

F be a finite-dimensional irreducible $\mathfrak{g}$-module F with regular highest weight $\mu \in P(\Sigma)$. Then $\lambda - \mu$ is regular antidominant, and

$$\Delta_\lambda(V) = (\Delta_\lambda(V)(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}$$

by C.2.1. Therefore,

$$V = \Gamma(X, \Delta_\lambda(V)) = (\Gamma(X, \Delta_\lambda(V)(-\mu)) \otimes_{\mathbb{C}} F)_{[\lambda]}.$$

Clearly, $U = \Gamma(X, \Delta_\lambda(V)(-\mu))$ is a Harish-Chandra module with regular infinitesimal character $\chi_{\lambda-\mu}$, and $H_p(\mathfrak{n}_x, U)$, $p \in \mathbb{Z}$, are finite dimensional. Let $(F_i; 1 \leq i \leq m)$ be an increasing Jordan-Hölder filtration of F as an $\mathfrak{n}_x$-module. Then the corresponding graded module is a trivial $\mathfrak{n}_x$-module. Therefore, $(U \otimes_{\mathbb{C}} F_i; 1 \leq i \leq m)$ is a $\mathfrak{n}_x$-module filtration of $U \otimes_{\mathbb{C}} F$ such that the corresponding graded module is a direct sum of m copies of U. From the spectral sequence of the filtered object (...) it follows that $H_p(\mathfrak{n}_x, U \otimes F)$, $p \in \mathbb{Z}$, are finite-dimensional. The assertion follows from $V = (U \otimes_{\mathbb{C}} F)_{[\lambda]}$. □

The next result is considerably deeper it follows from the main result of L.8. Let Q_o be the unique K-open orbit in X.

Theorem 2. *Let V be a Harish-Chandra module. If V is a nonzero module, $H_0(\mathfrak{n}_x, V) \neq 0$ for all $x \in Q_o$.*

Proof. Since Q_o is open and dense in X and $H_0(\mathfrak{n}_x, V)$ are canonically isomorphic for all $x \in Q_o$, we see that $\lambda \in \mathfrak{h}$ is an exponent of a Harish-Chandra module V if and only if $H_0(\mathfrak{n}_x, V)_{(\lambda+\rho)} \neq 0$ for $x \in Q_o$. Hence, the result follows from L.8.11. □

5. Irreducible Harish-Chandra Sheaves

Now we want to describe all irreducible Harish-Chandra sheaves for a Harish-Chandra pair $(\mathfrak{g}, K)$. For simplicity we assume that K is connected. We start with the following remark.

Lemma 1. *Let $\mathcal{V}$ ba an irreducible Harish-Chandra sheaf. Then its support* $\operatorname{supp}(\mathcal{V})$ *is the closure of a K-orbit Q in X.*

Proof. Since K is connected, the Harish-Chandra sheaf $\mathcal{V}$ is irreducible if and only if it is irreducible as a $\mathcal{D}_\lambda$-module. To see this we may assume, by twisting with $\mathcal{O}(\mu)$ for sufficiently negative μ, that λ is antidominant and regular. In this case the statement follows from the equivalence of categories and the analogous statement for Harish-Chandra modules (which is evident).

Therefore, by ..., we know that $\operatorname{supp}(\mathcal{V})$ is an irreducible closed subvariety of X. Since it must also be K-invariant, it is a union of K-orbits. The finiteness of K-orbits implies that there exists an orbit Q in $\operatorname{supp}(\mathcal{V})$ such that

$\dim Q = \dim \operatorname{supp}(\mathcal{V})$. Therefore, $\bar{Q}$ is a closed irreducible subset of $\operatorname{supp}(\mathcal{V})$ and $\dim \bar{Q} = \dim \operatorname{supp}(\mathcal{V})$. This implies that $\bar{Q} = \operatorname{supp}(\mathcal{V})$. □

Let $\mathcal{V}$ be an irreducible Harish-Chandra sheaf and Q the K-orbit in X such that $\operatorname{supp}(\mathcal{V}) = \bar{Q}$. Let $X' = X - \partial Q$. Then X' is an open subvariety of X and Q is a closed subvariety of X'. By ..., the restriction $\mathcal{V}|X'$ of $\mathcal{V}$ to X' is irreducible. Let $i : Q \to X$, $i' : Q \to X'$ and $j : X' \to X$ be the natural immersions. Hence, $i = j \circ i'$. Therefore, $Ri^! = R(i')^! \circ j^!$, where $j^! = j^+$ is just the ordinary restriction to the open subvariety X' of X. It follows that $R^p i^!(\mathcal{V}) = R^p(i')^!(\mathcal{V}|X')$ for $p \in \mathbb{Z}$. Since $i' : Q \to X'$ is an immersion of a closed smooth subvariety and $\operatorname{supp}(\mathcal{V}|X') = Q$, by Kashiwara's equivalence of categories, we see that $R^p i^!(\mathcal{V}) = 0$ for $p \neq 0$ and $\tau = i^!(\mathcal{V})$ is an irreducible $(\mathcal{D}_\lambda^i, K)$-module on Q. Moreover, $i'_+(\tau) = \mathcal{V}|X'$. Since $\mathcal{V}$ is holonomic by 3.2, τ is a holonomic module. This implies, by ..., that there exists an open dense subset U in Q such that $\tau|U$ is a connection. Since K acts transitively on Q, τ must be a K-homogeneous connection.

Therefore, to each irreducible Harish-Chandra sheaf we attach a pair (Q, τ) consisting of a K-orbit Q and an irreducible K-homogeneous connection τ on Q such that:

(i) $\operatorname{supp}(\mathcal{V}) = Q$;

(ii) $i^!(\mathcal{V}) = \tau$.

We call the pair (Q, τ) *the standard data* attached to $\mathcal{V}$.

Let Q be a K-orbit in X and $\tau \in \mathcal{M}(\mathcal{D}_\lambda^i, K)$ an irreducible K-homogeneous connection on Q. Then, by ..., $\mathcal{I}(Q, \tau) = i_+(\tau)$ is a $(\mathcal{D}_\lambda, K)$-module. Moreover, by ..., it is holonomic, i.e., $\mathcal{I}(Q, \tau)$ is a Harish-Chandra sheaf. We call it *the standard Harish-Chandra sheaf* attached to (Q, τ).

Lemma 2. *Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q. Then the standard Harish-Chandra sheaf $\mathcal{I}(Q, \tau)$ contains a unique irreducible Harish-Chandra subsheaf.*

Proof. Clearly,

$$\mathcal{I}(Q, \tau) = i_+(\tau) = j_+(i'_+(\tau)).$$

Therefore, $\mathcal{I}(Q, \tau)$ contains no sections supported in ∂Q. Hence, any nonzero $\mathcal{D}_\lambda$-submodule $\mathcal{U}$ of $\mathcal{I}(Q, \tau)$ has a nonzero restriction to X'. By Kashiwara's equivalence of categories, $i'_+(\tau)$ is an irreducible $(\mathcal{D}_\lambda|X')$-module. Hence, $\mathcal{U}|X' = \mathcal{I}(Q, \tau)|X'$. Therefore, for any two nonzero $\mathcal{D}_\lambda$-submodules $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{I}(Q, \tau)$, $\mathcal{U} \cap \mathcal{U}' \neq 0$. Since $\mathcal{I}(Q, \tau)$ is of finite length, it has a minimal $\mathcal{D}_\lambda$-submodule and by the preceding remark this module is unique. By its uniqueness it must be K-equivariant, therefore it is a Harish-Chandra sheaf. □

We denote by $\mathcal{L}(Q, \tau)$ the unique irreducible Harish-Chandra sheaf of $\mathcal{I}(Q, \tau)$. The following result gives a classification of irreducible Harish-Chandra sheaves.

Theorem 3. *(i) An irreducible Harish-Chandra sheaf $\mathcal{V}$ with the standard data (Q, τ) is isomorphic to $\mathcal{L}(Q, \tau)$.*

(ii) Let Q and Q' be K-orbits in X, τ and τ' irreducible K-homogeneous connections on Q and Q' respectively. Then $\mathcal{L}(Q,\tau) \cong \mathcal{L}(Q',\tau')$ if and only if $Q = Q'$ and $\tau \cong \tau'$.

Proof. (i) Let $\mathcal{V}$ be an irreducible Harish-Chandra sheaf and (Q,τ) the corresponding standard data. Then, as we remarked above, $\mathcal{V}|X' = (i')_+(\tau)$. By the universal property of j_+, there exists a nontrivial morphism of $\mathcal{V}$ into $\mathcal{I}(Q,\tau) = j_+((i')_+(\tau))$ which extends this isomorphism. Since $\mathcal{V}$ is irreducible its kernel must be zero and its image must be $\mathcal{L}(Q,\tau)$ by 2.

(ii) Since $Q = \operatorname{supp} \mathcal{L}(Q,\tau)$, it is evident that $\mathcal{L}(Q,\tau) \cong \mathcal{L}(Q',\tau')$ implies $Q = Q'$. The rest follows from the formula $\tau = i^!(\mathcal{L}(Q,\tau))$. □

V. Verma Modules

V.1 Category of Highest Weight Modules

Fix a Borel subalgebra $\mathfrak{b}_0$ in $\mathfrak{g}$ and $\mathfrak{n}_0 = [\mathfrak{b}_0, \mathfrak{b}_0]$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_0$. The root system Σ specializes to the root system R_0 in $\mathfrak{h}_0^*$ and the root subspaces corresponding to positive roots from R_0^+ span $\mathfrak{n}_0$. To simplify the notation in the following, when it doesn't cause confusion, we shall identify the Cartan triple $(\mathfrak{h}, \Sigma, \Sigma^+)$ with $(\mathfrak{h}_0, R_0, R_0^+)$ via this specialization. Denote by $\bar{\mathfrak{n}}_0$ the nilpotent subalgebra spanned by root subspaces corresponding to the negative roots in R_0. A $\mathfrak{g}$-module V is called a *highest weight module* (with respect to $\mathfrak{b}_0$) if

(i) V is finitely generated,

(ii) V is $\mathcal{U}(\mathfrak{b}_0)$-finite, i. e. for any $v \in V$, $\mathcal{U}(\mathfrak{b}_0)v$ is finite-dimensional.

We call the full subcategory of the category $\mathcal{M}_{fg}(\mathcal{U}(\mathfrak{g}))$ consisting of highest weight modules the *category of highest weight modules.*

Let V be a highest weight module. For $\lambda \in \mathfrak{h}_0^*$ we put

$$V^\lambda = \{v \in V | (\xi - \lambda(\xi))^k v = 0,\ \xi \in \mathfrak{h}_0,\ \text{for some } k \in \mathbb{N}\}.$$

Then V^λ is a $\mathfrak{h}_0$-submodule of V and V is the direct sum of V^λ, $\lambda \in \mathfrak{h}_0^*$. If $V^\lambda \neq 0$ we say that λ is a *weight* of V.

Lemma 1. *Let V be a finitely generated $\mathfrak{g}$-module. Then the following conditions are equivalent:*

(i) V is a highest weight module,

(ii) V satisfies:

(a) $V = \oplus V^\lambda$ and V^λ, $\lambda \in \mathfrak{h}_0^*$, are finite-dimensional.

(b) There exists a finite set of weights S_0 of V such that for any weight ν of V there exists $\mu \in S_0$ such that $\mu - \nu$ is a sum of roots from R_0^+.

Proof. Assume that V is a highest weight module. By definition, V is generated as a $\mathfrak{g}$-module by a finite-dimensional $\mathfrak{b}_0$-invariant subspace U. Hence, by the Poincaré-Birkhoff-Witt theorem, the natural map of $\mathcal{U}(\bar{\mathfrak{n}}_0) \otimes_{\mathbb{C}} U$ into V is a surjective morphism of $\mathfrak{h}_0$-modules. This clearly implies (a) and (b).

Assume now that V satisfies (a) and (b). Let $v \in V^\lambda$. Then $\mathcal{U}(\mathfrak{b}_0)v$ is contained in the direct sum of V^ν for weights ν such that $\nu - \lambda$ is a sum of positive roots. The number of such weights is finite by (b). Therefore, (a) implies that $\mathcal{U}(\mathfrak{b}_0)v$ is finite-dimensional. □

Lemma 2. *Let*

$$0 \longrightarrow V \longrightarrow V' \longrightarrow V'' \longrightarrow 0$$

be an exact sequence of $\mathfrak{g}$*-modules. Then* V' *is a highest weight module if and only if* V *and* V'' *are highest weight modules.*

Proof. It is clear that if V' is a highest weight module V and V'' are highest weight modules either. Assume that V and V'' are highest weight modules. Then V' is clearly finitely generated. Also, it satisfies the conditions in 1.(ii). Hence, V' is a highest weight module. □

We say that $v \in V$ is $\mathcal{Z}(\mathfrak{g})$-finite if $\mathcal{Z}(\mathfrak{g})v$ is finite-dimensional. Clearly, V^λ are $\mathcal{Z}(\mathfrak{g})$-invariant, and consist of $\mathcal{Z}(\mathfrak{g})$-finite vectors by 1.(ii). This implies that all vectors in V are $\mathcal{Z}(\mathfrak{g})$-finite, i. e. V is a $\mathcal{Z}(\mathfrak{g})$-finite module. Finally, since V is finitely generated, we have the following result.

Lemma 3. *Let* V *be a highest weight module. Then the annihilator of* V *in* $\mathcal{Z}(\mathfrak{g})$ *is of finite codimension.*

Also, we have the following converse.

Proposition 4. *Let* V *be a* $\mathfrak{g}$*-module satisfying the following conditions:*
(i) V *is finitely generated,*
(ii) for any $v \in V$*, there exists* $k \in \mathbb{N}$ *such that* $\mathfrak{n}_0^k \cdot v = 0$*,*
(iii) the annihilator of V *in* $\mathcal{Z}(\mathfrak{g})$ *is of finite codimension.*
Then V *is a highest weight module.*

Proof. Let U be a finite-dimensional $\mathfrak{n}_0$-invariant subspace which generates V. We shall prove that V is a highest weight module by induction in $\dim U$. If $\dim U = 1$, U is annihilated by $\mathcal{U}(\mathfrak{g})\mathfrak{n}_0$. On the other hand, from the properties of the Harish-Chandra homomorphism we know that the projection of $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{h}_0) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}_0$ into $\mathcal{U}(\mathfrak{h}_0)$ is an algebra homomorphism and that $\mathcal{U}(\mathfrak{h}_0)$ is finitely generated over its image. This clearly implies that $\mathcal{U}(\mathfrak{h}_0)U = \mathcal{U}(\mathfrak{b}_0)U$ is a finite-dimensional subspace in V. One checks easily that the linear subspace V' consisting of all vectors $u \in V$ such that $\mathcal{U}(\mathfrak{b}_0)u$ is finite-dimensional is a $\mathfrak{g}$-submodule of V. It contains U by the preceding discussion, what in turn implies that it is equal to V, i. e. V is a highest weight module. Assume now that $\dim U > 1$. Then by Engel's theorem U has an one-dimensional subspace U_0 such that $\mathfrak{n}_0 U_0 = 0$. Let V_0 be the $\mathfrak{g}$-submodule of V generated by U_0. Then V_0 is a highest weight module by the first part of the proof. Let $V_1 = V/V_0$. Then V_1 is generated by $U_1 = U/(U \cap V_0)$ and $\dim U_1 \leq \dim U - 1$. Therefore, V_1 is a highest weight module by the induction assumption. By 2. we see that V is a highest weight module. □

Let N_0 be the unipotent subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{n}_0$. Then, by 4, one can exponentiate the action of $\mathfrak{n}_0$ to an algebraic action of N_0 and view highest weight modules as elements in $\mathcal{M}_{fg}(\mathfrak{g}, N_0)$. Actually, in this way one can identify the category of highest weight modules with the full subcategory of $\mathcal{M}_{fg}(\mathfrak{g}, N_0)$ consisting of modules annihilated by ideals in $\mathcal{Z}(\mathfrak{g})$ of finite codimension.

Now we want to describe irreducible objects in $\mathcal{M}_{fg}(\mathfrak{g}, N_0)$, i. e. irreducible highest weight modules. First we construct some closely related modules. Let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}_0$-module defined by $\lambda \in \mathfrak{h}_0^*$. Then, if we consider $\mathcal{U}(\mathfrak{g})$ as a right $\mathcal{U}(\mathfrak{b}_0)$-module via right multiplication, the tensor product $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_\lambda$ has a natural structure of a left $\mathcal{U}(\mathfrak{g})$-module given by left multiplication at the first factor. It is clearly a highest weight module; and we put

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{\lambda-\rho}.$$

The highest weight module $M(\lambda)$ is called the *Verma module* determined by λ.

Lemma 5. *Let $\lambda \in \mathfrak{h}_0^*$. Then*

(i) all weights of $M(\lambda)$ are of the form $\lambda - \rho - \nu$ where ν is a sum of positive roots,

(ii) $\dim M(\lambda)^{\lambda-\rho} = 1$*,*

(iii) $M(\lambda)$ has a unique maximal $\mathfrak{g}$-submodule $N(\lambda)$,

(iv) $N(\lambda)^{\lambda-\rho} = 0$.

Proof. By the Poincaré-Birkhoff-Witt theorem, we see that $M(\lambda)$, considered as a $\mathfrak{h}_0$-module, is isomorphic to $\mathcal{U}(\bar{\mathfrak{n}}_0) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$. This immediately implies (i) and (ii). Clearly, by definition of $M(\lambda)$, the one-dimensional subspace $M(\lambda)^{\lambda-\rho}$ generates $M(\lambda)$ as a $\mathfrak{g}$-module. Therefore, any $\mathfrak{g}$-submodule different from $M(\lambda)$ cannot contain $M(\lambda)^{\lambda-\rho}$. Let M be a maximal $\mathfrak{g}$-submodule of $M(\lambda)$ and N any $\mathfrak{g}$-submodule different from $M(\lambda)$. Then, either $N \subset M$ or $M + N = M(\lambda)$. In the second case we would have

$$M(\lambda)^{\lambda-\rho} = (M+N)^{\lambda-\rho} = M^{\lambda-\rho} + N^{\lambda-\rho} = 0,$$

what is clearly impossible. Therefore, M is the unique maximal $\mathfrak{g}$-submodule. □

This implies that $M(\lambda)$ has the unique irreducible quotient $\mathfrak{g}$-module $L(\lambda)$. Also, $L(\lambda)^{\lambda-\rho}$ is one-dimensional. We say that $\lambda - \rho$ is the *highest weight* of $L(\lambda)$.

Proposition 6. *(i) Any irreducible highest weight module is isomorphic to some $L(\lambda)$.*

(ii) $L(\lambda)$ is isomorphic to $L(\mu)$ if and only if $\lambda = \mu$.

Proof. (i) Let V be an irreducible highest weight module. Let S be the set of all weights of V. Then, by 1., we can find a weight $\lambda \in S$ such that $\lambda + \alpha$ is not in S for any $\alpha \in \Sigma^+$. This implies that V^λ is annihilated by $\mathfrak{n}_0$. Therefore, if we take any nonzero $v \in V^\lambda$, the homomorphism $\xi \longrightarrow \xi \cdot v$ from $\mathcal{U}(\mathfrak{g})$ into V is surjective and factors through $M(\lambda + \rho)$. This implies that V is isomorphic to $L(\lambda + \rho)$.

(ii) This follows from 5.(i) and (ii). □

Lemma 7. *The center $\mathcal{Z}(\mathfrak{g})$ acts on $M(\lambda)$ via χ_λ.*

Proof. This follows from the definition of the Harish-Chandra homomorphism. □

Proposition 8. *Highest weight modules have finite length.*

Proof. Let V be a highest weight module. If V is not of finite length, we can construct a decreasing $\mathfrak{g}$-module filtration $(V_i; i \in \mathbb{Z}_+)$ of V such that V_i/V_{i-1}, $i \in \mathbb{N}$, are irreducible. Therefore, by 6.(i), $L(\lambda_i) = V_i/V_{i-1}$ for some $\lambda_i \in \mathfrak{h}_0^*$. By 3. and 7, it follows that the set of possible λ_i is finite. Therefore, by 6.(ii), the set of possible $L(\lambda_i)$ is finite, what contradicts the finite-dimensionality of weight subspaces of V. □

By ([**LG**], Ch. VIII, §5, Prop. 2), there exists an involutive automorphism ι of $\mathfrak{g}$ with the property that $\iota|\mathfrak{h}_0 = -1$. Then, $\iota(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for any $\alpha \in R_0^+$. Let τ be the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ which is the product of the principal antiautomorphism of $\mathcal{U}(\mathfrak{g})$ and the automorphism which extends ι. Then τ is the identity on $\mathfrak{h}_0$ and it maps $\mathfrak{n}_0$ into $\bar{\mathfrak{n}}_0$.

Lemma 9. *The antiautomorphism τ acts as identity on $\mathcal{Z}(\mathfrak{g})$.*

Proof. Let γ be the Harish-Chandra homomorphism, i. e. the projection of $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{h}_0) \oplus \mathfrak{n}_0\mathcal{U}(\mathfrak{g})$ into $\mathcal{U}(\mathfrak{h}_0)$ along $\mathfrak{n}_0\mathcal{U}(\mathfrak{g})$. By definition, the antiautomorphism τ acts as identity on $\mathcal{U}(\mathfrak{h}_0)$ and maps $\mathfrak{n}_0\mathcal{U}(\mathfrak{g})$ into $\mathcal{U}(\mathfrak{g})\bar{\mathfrak{n}}_0$. On the other hand, the intersections of $\mathfrak{n}_0\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})\bar{\mathfrak{n}}_0$ with the centralizer of $\mathfrak{h}_0$ in $\mathcal{U}(\mathfrak{g})$ are equal, what implies immediately that γ and $\gamma \circ \tau$ agree on $\mathcal{Z}(\mathfrak{g})$. The injectivity of γ implies that $\tau|\mathcal{Z}(\mathfrak{g})$ is the identity. □

For any highest weight module V, let V^* be its linear dual. We define the action of $\mathcal{U}(\mathfrak{g})$ on V^* by

$$(\xi f)(v) = f(\tau(\xi)v),\ \xi \in \mathcal{U}(\mathfrak{g}),\ f \in V^*,\ v \in V.$$

In this way V^* becomes a $\mathfrak{g}$-module. Let $V\tilde{}$ be the subspace consisting of $f \in V^*$ such that $\mathcal{U}(\mathfrak{h}_0)f$ is finite-dimensional. It can be easily checked that $V\tilde{}$ is a $\mathfrak{g}$-module.

Lemma 10. *(i) $V\tilde{}$ is a highest weight module.*
(ii) $(V\tilde{})\tilde{} = V$.
(iii) $(V\tilde{})^\lambda = (V^\lambda)^*$ for any $\lambda \in \mathfrak{h}_0^*$.

Proof. Clearly $V\tilde{} = \oplus (V^\lambda)^*$ as an $\mathfrak{h}_0$-module and $(V\tilde{})^\lambda = (V^\lambda)^*$. Hence, the set of weights of $V\tilde{}$ is the same as the set of weights of V. The canonical map of V into $(V\tilde{})\tilde{}$ is injective, a $\mathfrak{g}$-module morphism and

$$\dim((V\tilde{})\tilde{})^\lambda = \dim(V\tilde{})^\lambda = \dim V^\lambda,$$

for any $\lambda \in \mathfrak{h}_0^*$, what implies that it is an isomorphism. Let U be an $\mathfrak{g}$-submodule of $V\tilde{}$ and $U^\perp$ be the subspace of $(V\tilde{})\tilde{} = V$ orthogonal to U. Then $U^\perp$ is a $\mathfrak{g}$-submodule of V. Also, $(U^\perp)^\perp = U$. This implies that every $\mathfrak{g}$-submodule of $V\tilde{}$ is the orthogonal of some $\mathfrak{g}$-submodule of V. By 8, it follows that $V\tilde{}$ is of finite length. In particular, $V\tilde{}$ is finitely generated and a highest weight module by 1. □

Therefore, $V \longrightarrow V\tilde{}$ is an exact contravariant functor from $\mathcal{M}_{fg}(\mathfrak{g}, N_0)$ into itself. We call $V\tilde{}$ the *dual* of V. Also, for any orbit θ of the Weyl group W in $\mathfrak{h}^*$, we conclude from 9. that $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta, N_0)$ implies that $V\tilde{} \in \mathcal{M}_{fg}(\mathcal{U}_\theta, N_0)$, i. e. $V \longrightarrow V\tilde{}$ is an antiequivalence of the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, N_0)$ with itself.

Lemma 11. *For any $\lambda \in \mathfrak{h}_0^*$, $L(\lambda)\tilde{} = L(\lambda)$.*

Proof. It follows from 10. that $L(\lambda)\tilde{}$ is an irreducible highest weight module with the highest weight λ. By 6, $L(\lambda)\tilde{}$ is isomorphic to $L(\lambda)$. □

We put $I(\lambda) = M(\lambda)\tilde{}$. Then, by 5. and 11, $I(\lambda)$ has a unique irreducible $\mathfrak{g}$-submodule $L(\lambda)$. The modules $I(\lambda)$ have the following universal property.

Lemma 12. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$.*

(i) Let V be a highest weight module such that $\lambda - \rho$ is a weight of V and $\lambda - \rho + \alpha$ is not a weight of V for any positive root $\alpha \in \Sigma^+$. Then there exists a nonzero morphism of V into $I(\lambda)$.

(ii) Let V be a highest weight module satisfying following conditions:

(a) V contains a unique irreducible submodule isomorphic to $L(\lambda)$;

(b) $\dim V^\mu = \dim I(\lambda)^\mu$ for any $\mu \in \mathfrak{h}^*$.

Then V is isomorphic to $I(\lambda)$.

Proof. (i) By 10. $V\tilde{}$ is a highest weight module such that $\lambda - \rho$ is a weight of $V\tilde{}$ and $\lambda - \rho + \alpha$ is not a weight of $V\tilde{}$ for any positive root $\alpha \in \Sigma^+$. Hence, if $v \in (V\tilde{})^{\lambda-\rho}$, $v \neq 0$, v is annihilated by $\mathfrak{n}_0$. Therefore, the homomorphism $\xi \longrightarrow \xi \cdot v$ from $\mathcal{U}(\mathfrak{g})$ into V factors through $M(\lambda)$, and we constructed a nonzero morphism ϕ of $M(\lambda)$ into $V\tilde{}$. By duality, $\phi\tilde{}$ is a nonzero morphism of V into $I(\lambda)$.

(ii) By (i) there exists a nonzero morphism ϕ of V into $I(\lambda)$. Since the image of ϕ is nontrivial, it must contain the unique irreducible submodule $L(\lambda)$ of $I(\lambda)$. Denote by K the kernel of ϕ. Since $\dim V^{\lambda-\rho} = \dim I(\lambda)^{\lambda-\rho} = 1$ by (b), and $\dim L(\lambda)^{\lambda-\rho} = 1$, we conclude that $K^{\lambda-\rho} = 0$. By (a), this implies that $K = 0$, and ϕ is injective. From (b) we finally conclude that ϕ is an isomorphism. □

Now we can relate our results to the general geometric scheme for classification of irreducible objects in $\mathcal{M}_{fg}(\mathcal{U}_\theta, N_0)$. First, by the Bruhat lemma ([**Bo**], 14.11), N_0 has finitely many orbits in the flag variety X. Therefore, we have the following remark which enables us to apply the results of ... .

Proposition 13 $(\mathfrak{g}, N_0)$ *is a Harish-Chandra pair.*

The N_0-orbits in X are the Bruhat cells $C(w)$, $w \in W$. They are affine subvarieties of X ([**Bo**], 14.11) and

$$\dim C(w) = \ell(w),\ w \in W.$$

Therefore, if $C(s)$ is in the boundary $\partial C(w) = \overline{C(w)} - C(w)$ of $C(w)$, we have $\ell(s) < \ell(w)$.

Let $\lambda \in \mathfrak{h}^*$. Let $C(w)$ be a Bruhat cell and $i_w : C(w) \longrightarrow X$ the canonical immersion. Then, from ... we see that the only irreducible N_0-homogeneous $(\mathcal{D}_\lambda)^{i_w}$-connection on $C(w)$ is $\mathcal{O}_{C(w)}$. The standard $\mathcal{D}_\lambda$-module corresponding to data $(C(w), \mathcal{O}_{C(w)})$ we denote by $\mathcal{I}(w, \lambda)$, and its unique irreducible $\mathcal{D}_\lambda$-submodule by $\mathcal{L}(w, \lambda)$. The key connection between the geometric classification of irreducible objects and 6. is given by the following result.

Theorem 14. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then*

$$\Gamma(X, \mathcal{I}(w, \lambda)) = I(w\lambda),\ w \in W.$$

To prove 14. we need some preparation. We start with a very special case of 14.

Lemma 15. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then*

$$\Gamma(X, \mathcal{I}(1, \lambda)) = M(\lambda) = L(\lambda) = I(\lambda).$$

Proof. Clearly, $\mathcal{I}(1, \lambda)$ is an irreducible $\mathcal{D}_\lambda$-module. Hence, by L.4.1, the $\mathcal{U}_\theta$-module $\Gamma(X, \mathcal{I}(1, \lambda))$ is either an irreducible highest weight module or zero. Since $\mathcal{I}(1, \lambda)$ is supported at the point $C(1)$ the second possibility is automatically eliminated. Therefore, to prove the statement it is enough to establish the first equality.

Now we need to describe the structure of the direct image module $\mathcal{I}(1, \lambda) = i_{1+}(\mathbb{C}_{\lambda+\rho})$. We can view it as a right $\mathcal{D}_{-\lambda}$-module. Then it is equal to $i_1^*(\mathcal{D}_{-\lambda, C(1)\to X})$ as a right $\mathcal{D}_{-\lambda}$-module in the natural way. On the other hand, as in the proof of L.2.4, we conclude that as a right $\mathcal{U}(\mathfrak{g})$-module

$$\begin{aligned} \mathcal{D}_{-\lambda, C(1)\to X} &= T_{x_0}(\mathcal{D}_{-\lambda}) \\ &= (\mathcal{U}(\mathfrak{g})/\mathfrak{n}_0\mathcal{U}(\mathfrak{g}))/(I_{-\lambda+\rho}(\mathcal{U}(\mathfrak{g})/\mathfrak{n}_0\mathcal{U}(\mathfrak{g}))) = \mathbb{C}_{-\lambda+\rho} \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathcal{U}(\mathfrak{g}), \end{aligned}$$

where we denoted by x_0 the point in $C(1)$. This implies that, as a left $\mathcal{U}(\mathfrak{g})$-module, $\mathcal{I}(1, \lambda)$ is equal to $M(\lambda)$. □

Now we need some results about the action of the intertwining functors on the standard modules.

Lemma 16. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then*

$$LI_w(D(\mathcal{I}(1, \lambda))) = D(\mathcal{I}(w^{-1}, w\lambda)).$$

Proof. We use the notation from the L.3. Let $Z_w \subset X \times X$ be the variety of ordered pairs of Borel subalgebras in relative position $w \in W$. Denote by p_i, $i = 1, 2$, the projections to the i^{th} factor in $X \times X$. Then

$$\begin{aligned} p_2^{-1}(C(1)) = \{(x, x') \in X \times X \mid & \\ & \mathfrak{b}_{x'} = \mathfrak{b}_0,\ \mathfrak{b}_x \text{ in relative position } w^{-1} \text{ with respect to } \mathfrak{b}_0\} \\ & = C(w^{-1}) \times C(1), \end{aligned}$$

i. e. we have the following commutative diagram

$$\begin{array}{ccc} C(w^{-1}) \times C(1) & \xrightarrow{j} & Z_w \\ {\scriptstyle pr_2}\downarrow & & {\scriptstyle p_2}\downarrow \\ C(1) & \xrightarrow{i_1} & X \end{array}$$

and by base change ([**BDM**], VI.8.4), since pr_2 and p_2 are submersions and i_1 and j affine immersions, we have

$$p_2^+(\mathcal{I}(1, \lambda)) = p_2^+(i_{1+}(\mathcal{O}_{C(1)})) = j_+(pr_2^+(\mathcal{O}_{C(1)})) = j_+(\mathcal{O}_{C(w^{-1}) \times C(1)}).$$

The projection p_1 induces an immersion of $p_2^{-1}(C(1))$ into X and its image is equal to $C(w^{-1})$, i. e. we have the following commutative diagram

$$\begin{array}{ccc} C(w^{-1}) \times C(1) & \xrightarrow{j} & Z_w \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle p_1}\downarrow \\ C(w^{-1}) & \xrightarrow{i_{w^{-1}}} & X \end{array}$$

and we get, after checking the appropriate twists, that

$$\begin{aligned} LI_w(D(\mathcal{I}(1, \lambda))) &= Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{I}(1, \lambda))) \\ &= Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} j_+(\mathcal{O}_{C(w^{-1}) \times C(1)})) \\ &= D(i_{w^{-1}+}(\mathcal{O}_{C(w^{-1})}))) = D(\mathcal{I}(w^{-1}, w\lambda)).\square \end{aligned}$$

Corollary 17. *Let $w, w' \in W$ be such that $\ell(ww') = \ell(w) + \ell(w')$. Then*

$$LI_w(D(\mathcal{I}({w'}^{-1}, \lambda))) = D(\mathcal{I}({w'}^{-1} w^{-1}, w\lambda)).$$

Proof. Clearly, by 16.

$$LI_{ww'}(D(\mathcal{I}(1, {w'}^{-1}\lambda))) = D(\mathcal{I}({w'}^{-1} w^{-1}, w\lambda)).$$

On the other hand, by L.3.18 we have

$$\begin{aligned} LI_{ww'}(D(\mathcal{I}(1, {w'}^{-1}\lambda))) &= LI_w(LI_{w'}(D(\mathcal{I}(1, {w'}^{-1}\lambda))) \\ &= LI_w(D(\mathcal{I}({w'}^{-1}, \lambda))).\square \end{aligned}$$

In particular, if w_0 is the longest element in W, and w an arbitrary element of W, the element $w' = w_0 w$ satisfies $ww'^{-1} = w_0^{-1} = w_0$ and $\ell(w') = \ell(w_0) - \ell(w)$ ([**LG**], Ch. VI, §1, no. 6, Cor. 3. of Prop. 17.). It follows that $LI_{w'}(D(\mathcal{I}(w,\lambda))) = D(\mathcal{I}(w_0, w'\lambda))$.

The next result is critical for the proof of 14.

Lemma 18. *Let $\lambda \in \mathfrak{h}^*$. Then*
(i) $H^p(X, \mathcal{I}(w_0,\lambda)) = 0$ for $p > 0$;
(ii) $\Gamma(X, \mathcal{I}(w_0,\lambda)) = I(w_0\lambda)$.

Proof. Since $C(w_0)$ is an affine open subvariety of X,

$$H^p(X, \mathcal{I}(w_0,\lambda)) = H^p(C(w_0), \mathcal{O}_{C(w_0)}) = 0$$

for $p > 0$. This proves (i).

Now we can prove (ii). Assume that $\lambda \in \mathfrak{h}^*$ is antidominant. Then, by 15, we have $\Gamma(X, \mathcal{I}(1,\lambda)) = M(\lambda)$ and it is an irreducible $\mathfrak{g}$-module. If we take a nonzero $v \in \Gamma(X, \mathcal{I}(1,\lambda))$, it generates a finite-dimensional $\mathfrak{b}_0$-invariant subspace U. By Engel's theorem, there exists a vector $v' \in U$ which spans a $\mathfrak{b}_0$-invariant subspace. Therefore, v' is a weight vector of $M(\lambda)$ for some weight $\mu \in \mathfrak{h}^*$ and it is annihilated by $\mathfrak{n}_0$, hence there exists a natural morphism of $M(\mu+\rho)$ into $M(\lambda)$. Since $M(\lambda)$ is irreducible, we have $L(\mu+\rho) = M(\lambda) = L(\lambda)$. By 6. we finally conclude that $\lambda = \mu + \rho$. Hence, we have proved that every $\mathfrak{b}_0$-invariant subspace U of $M(\lambda)$ contains the highest weight subspace $M(\lambda)^{\lambda-\rho}$. By 16, we also conclude that for antidominant $\lambda \in \mathfrak{h}^*$,

$$\Gamma(X, \mathcal{I}(w_0, w_0\lambda)) = M(\lambda)$$

and every $\mathfrak{b}_0$-invariant subspace U of it contains the highest weight subspace. As we remarked before, for arbitrary $\mu \in \mathfrak{h}^*$,

$$\Gamma(X, \mathcal{I}(w_0,\mu)) = \Gamma(C(w_0), \mathcal{O}_{C(w_0)}).$$

Therefore, the constant function 1 on $C(w_0)$ is a global section of $\mathcal{I}(w_0,\mu)$. It is clearly N_0-invariant. Denote by $x_{w_0} \in C(w_0)$ the point corresponding to the Borel subalgebra which contains $\mathfrak{h}_0$ and is opposite to $\mathfrak{b}_0$. The section $n \cdot x_{w_0} \longrightarrow \mathrm{Ad}(n)\xi$, $\xi \in \mathfrak{h}_0$, of $\mathcal{U}^\circ|C(w_0)$ maps into the constant section $(w_0\mu - \rho)(\xi)$ in $\mathcal{D}_{w_0\mu}|C(w_0)$, hence it acts on the section 1 as multiplication by $(w_0\mu - \rho)(\xi)$. This in turn implies that ξ acts on this section as multiplication by $(w_0\mu-\rho)(\xi)$, i. e. $1 \in \Gamma(X, \mathcal{I}(w_0,\mu))^{w_0\mu-\rho}$. In particular, for antidominant $\lambda \in \mathfrak{h}^*$, any $\mathfrak{b}_0$-invariant subspace U of $\Gamma(X, \mathcal{I}(w_0, w_0\lambda))$ contains the highest weight subspace $\Gamma(X, \mathcal{I}(w_0, w_0\lambda))^{\lambda-\rho}$ consisting of constant functions on $C(w_0)$. Since the geometric translation of $\mathcal{I}(w_0,\mu)$ is $\mathcal{I}(w_0,\mu) \otimes_{\mathcal{O}_X} \mathcal{O}(\nu) = \mathcal{I}(w_0, \mu+\nu)$, and $\mathcal{O}(\nu)|C(w_0) = \mathcal{O}_{C(w_0)}$ as an $\mathcal{O}_{C(w_0)}$-module, we have a natural isomorphism of the $\mathfrak{b}_0$-module $\Gamma(X, \mathcal{I}(w_0, \mu+\nu))$ with $\Gamma(X, \mathcal{I}(w_0,\mu)) \otimes_{\mathbb{C}} \mathbb{C}_\nu$. Hence, we see that for arbitrary $\mu \in \mathfrak{h}^*$:

(a) $\dim \Gamma(X, \mathcal{I}(w_0,\mu))^\omega = \dim M(w_0\mu)^\omega = \dim I(w_0\mu)^\omega$ for any weight $\omega \in \mathfrak{h}^*$;

(b) any $\mathfrak{b}_0$-invariant subspace U of $\Gamma(X, \mathcal{I}(w_0, \mu))$ contains the highest weight subspace $\Gamma(X, \mathcal{I}(w_0, \mu))^{w_0\mu-\rho}$ consisting of constant functions on $C(w_0)$.

In particular, from (b) we conclude that any $\mathfrak{g}$-submodule of $\Gamma(X, \mathcal{I}(w_0, \mu))$ contains the constants. This implies that $\Gamma(X, \mathcal{I}(w_0, \mu))$ has a unique irreducible $\mathfrak{g}$-submodule L.

Then (a) implies that $w_0\mu - \rho$ is the highest weight of L, i. e. $L = L(w_0\mu)$. Finally, by 12.(ii), we conclude that $\Gamma(X, \mathcal{I}(w_0, \mu)) = I(w_0\mu)$. □

Now we can prove 14. If $\lambda \in \mathfrak{h}^*$ is antidominant, w an arbitrary element of W and $ww'^{-1} = w_0$, we have

$$R\Gamma(D(\mathcal{I}(w, \lambda))) = R\Gamma(LI_{w'}(D(\mathcal{I}(w, \lambda)))) = R\Gamma(D(\mathcal{I}(w_0, w'\lambda)))$$

by the preceding discussion and L.3.23. This implies that

$$\Gamma(X, \mathcal{I}(w, \lambda)) = \Gamma(X, \mathcal{I}(w_0, w'\lambda)) = I(w_0w'\lambda) = I(w\lambda).$$

and proves 14.

Now it is quite straightforward to determine the global sections of irreducible modules.

Theorem 19. *Let $\lambda \in \mathfrak{h}^*$ be regular antidominant. Then, for any $w \in W$, we have*

$$\Gamma(X, \mathcal{L}(w, \lambda)) = L(w\lambda).$$

Proof. Since λ is regular and antidominant, the functor $\Gamma(X, -)$ is an equivalence of categories. Therefore, by 14, the global sections of the unique irreducible submodule $\mathcal{L}(w, \lambda)$ of $\mathcal{I}(w, \lambda)$ are isomorphic to the unique irreducible submodule $L(w\lambda)$ of $I(w\lambda)$. □

It remains to discuss the behavior of $\Gamma(X, \mathcal{L}(w, \lambda))$ for singular antidominant $\lambda \in \mathfrak{h}^*$. Let $W(\lambda)$ be the stabilizer of λ. First, by L.4.1, we know that $\Gamma(X, \mathcal{L}(w, \lambda))$ is an irreducible $\mathfrak{g}$-module or zero. By exactness of Γ and 14, if it is nonzero, it must be the unique irreducible submodule $L(w\lambda)$ of $I(w\lambda) = \Gamma(X, \mathcal{I}(w, \lambda))$.

On the other hand, by L.4.2, there exists the unique irreducible $\mathcal{D}_\lambda$-module $\mathcal{L}(s, \lambda)$, $s \in W$, such that $\Gamma(X, \mathcal{L}(s, \lambda)) = L(w\lambda)$. Then $L(w\lambda)$ is isomorphic to an irreducible $\mathfrak{g}$-submodule of $\Gamma(X, \mathcal{I}(s, \lambda))$. By 14, $\Gamma(X, \mathcal{I}(s, \lambda)) = I(s\lambda)$ and $L(s\lambda)$ is isomorphic to $L(w\lambda)$. By 6.(ii), it follows that $s\lambda = w\lambda$, i. e. $s \in wW(\lambda)$. Let $u \in wW(\lambda)$, such that $\Gamma(X, \mathcal{L}(u, \lambda)) = 0$. Then,

$$\Gamma(X, \mathcal{I}(u, \lambda)/\mathcal{L}(u, \lambda)) = I(w\lambda),$$

hence it contains $L(w\lambda)$ as its composition factor. Moreover, it follows that $\mathcal{L}(s, \lambda)$ is a composition factor of $\mathcal{I}(u, \lambda)/\mathcal{L}(u, \lambda)$. Hence, $C(s) \subset \partial C(u)$ and $\ell(s) < \ell(u)$. Therefore, ℓ attains at s its minimum on the coset $wW(\lambda)$. Moreover, s is uniquely determined by this property.

The preceding discussion has the following consequence.

Theorem 20. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Then the annihilator of $M(\lambda)$ in $\mathcal{U}_\theta$ is $\{0\}$.*

Proof. It is enough to show that the annihilator of $I(\lambda)$ is trivial for any $\lambda \in \mathfrak{h}^*$. But, by 1.18.(ii), this is equivalent to showing that no nontrivial element of $\Gamma(X, \mathcal{D}_\lambda)$ annihilates $\Gamma(X, \mathcal{I}(w_0, \lambda)) = \Gamma(C(w_0), \mathcal{O}_{C(w_0)})$ which is evident. □

Finally, we want to discuss the necessary and sufficient conditions for the irreducibility of standard modules $\mathcal{I}(w, \lambda)$ and Verma modules. First we analyze a critical special situation.

Lemma 21. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi$ be such that $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$. Let $w = w's_\alpha$ with $\ell(w) = \ell(w') + 1$. Then:*

(i) we have an exact sequence

$$0 \longrightarrow U^0(\mathcal{I}(w', s_\alpha\lambda)) \longrightarrow \mathcal{I}(w, \lambda) \longrightarrow \mathcal{I}(w', \lambda) \longrightarrow 0;$$

(ii) $U^0(\mathcal{I}(w', s_\alpha\lambda)) \neq 0$ and it is a translate of a module of form $p_\alpha^+(\mathcal{V})$.

Proof. By 16. and L.3.18, we have

$$\mathcal{I}(w, \lambda) = I_{w^{-1}}(\mathcal{I}(1, w\lambda)) = I_{s_\alpha}(I_{w'^{-1}}(\mathcal{I}(1, w\lambda))) = I_{s_\alpha}(\mathcal{I}(w', s_\alpha\lambda)).$$

Hence, by L.5.3.(ii), we have an exact sequence

$$\begin{array}{ccccccccc} 0 \to & U^0(\mathcal{I}(w', s_\alpha\lambda)) & \to & \mathcal{I}(w, \lambda) & \to & \mathcal{I}(w', s_\alpha\lambda)(-p\alpha) & \to & U^1(\mathcal{I}(w', s_\alpha\lambda)) & \to 0 \\ & & & & & \Vert & & & \\ & & & & & \mathcal{I}(w', \lambda) & & & \end{array}.$$

Let $p_\alpha : X \longrightarrow X_\alpha$ be the natural projection of the flag variety X onto the variety of all parabolic subalgebras of type α. Then, using the notation from L.5. we have the commutative diagram

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

and by base change, using the fact that the composition of $p_\alpha \circ i_{w'}$ is an immersion of the affine variety $C(w')$ into X_α, we conclude that $U^1(\mathcal{I}(w', s_\alpha\lambda)) = 0$, $U^0(\mathcal{I}(w', s_\alpha\lambda)) \neq 0$ and it is a translate of a module of form $p_\alpha^+(\mathcal{V})$. This implies both assertions. □

Theorem 22. *Let $\lambda \in \mathfrak{h}^*$ and $w \in W$. Then the following conditions are equivalent:*

(i) $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$;

(ii) $\mathcal{I}(w, \lambda)$ is irreducible $\mathcal{D}_\lambda$-module.

Proof. (i)⇒(ii) If $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$, by L.3.22, the intertwining functor $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is an equivalence of categories and $I_{w^{-1}}$ its inverse. By 16, we have

$$I_{w^{-1}}(\mathcal{I}(1, w\lambda)) = \mathcal{I}(w, \lambda).$$

Since $\mathcal{I}(1, w\lambda)$ is evidently irreducible, $\mathcal{I}(w, \lambda)$ is an irreducible $\mathcal{D}_\lambda$-module.

Now we shall prove, by induction in $\ell(w)$, that $\Sigma_w^+ \cap \Sigma_\lambda \neq \emptyset$ implies that $\mathcal{I}(w, \lambda)$ is a reducible $\mathcal{D}_\lambda$-module. If $\ell(w) = 0$, $w = 1$ and the assertion is obvious. Therefore, we can assume that the statement holds for $w' \in W$ with $\ell(w') < k$. Let $\ell(w) = k$. Then $w = w's_\alpha$ for some $\alpha \in \Pi$ and $w' \in W$ with $\ell(w') = k - 1$. As in the preceding proof, from 16. and L.3.18 we deduce that

$$\mathcal{I}(w, \lambda) = I_{s_\alpha}(\mathcal{I}(w', s_\alpha\lambda)).$$

Moreover, by L.3.12.(ii),

$$\Sigma_w^+ \cap \Sigma_\lambda = s_\alpha(\Sigma_{w'}^+ \cap \Sigma_{s_\alpha\lambda}) \cup (\{\alpha\} \cap \Sigma_\lambda).$$

If $\alpha \notin \Sigma_\lambda$, $\mathrm{Card}(\Sigma_w^+ \cap \Sigma_\lambda) = \mathrm{Card}(\Sigma_{w'}^+ \cap \Sigma_{s_\alpha\lambda})$, and by induction assumption $\mathcal{I}(w', s_\alpha\lambda)$ is a reducible $\mathcal{D}_{s_\alpha\lambda}$-module. Since, by L.3.22, in this case $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda}) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories, $\mathcal{I}(w, \lambda)$ is a reducible $\mathcal{D}_\lambda$-module.

If $\alpha \in \Sigma_\lambda$, $\mathcal{I}(w, \lambda)$ is reducible by 21. □

Now we deduce a necessary and sufficient condition for irreducibility of Verma modules.

Theorem 23. *Let $\lambda \in \mathfrak{h}^*$. Then the following conditions are equivalent:*

(i) λ is antidominant;

(ii) $M(\lambda)$ is irreducible.

Proof. (i)⇒(ii) If λ is antidominant, $I(\lambda) = \Gamma(X, \mathcal{I}(1, \lambda))$ by 14. Moreover, $\mathcal{I}(1, \lambda)$ is clearly irreducible. By L.4.1, $I(\lambda)$ is an irreducible $\mathfrak{g}$-module, and $M(\lambda) = I(\lambda)\check{}$ is also irreducible.

(ii)⇒(i) Take λ which is not antidominant. Assume that $M(\lambda)$ is irreducible. Let $w \in W$ be a shortest element of W such that $w^{-1}\lambda$ is antidominant. Then, by 14, we have

$$\Gamma(X, \mathcal{I}(w, w^{-1}\lambda)) = I(\lambda) = M(\lambda)\check{} = L(\lambda)\check{} = L(\lambda) = M(\lambda).$$

Let $\alpha \in \Pi$ such that $w = w's_\alpha$ with $\ell(w) = \ell(w') + 1$. Then $w'^{-1}\lambda$ is not antidominant. We claim that $p = -\alpha\check{}(w^{-1}\lambda) \in \mathbb{N}$. Since $w^{-1}\lambda$ is antidominant, $\beta\check{}(w^{-1}\lambda) \notin \mathbb{N}$ for any $\beta \in \Sigma^+$. In addition, s_α permutes the roots of $\Sigma^+ - \{\alpha\}$, hence $(s_\alpha\beta)\check{}(w^{-1}\lambda) = \beta\check{}(w'^{-1}\lambda) \notin \mathbb{N}$ for any $\beta \in \Sigma^+ - \{\alpha\}$, and since $w'^{-1}\lambda$ is not antidominant, $\alpha\check{}(w'^{-1}\lambda) \in \mathbb{N}$. From 21.(i) we get the exact sequence

$$0 \longrightarrow U^0(\mathcal{I}(w', w'^{-1}\lambda)) \longrightarrow \mathcal{I}(w, w^{-1}\lambda) \longrightarrow \mathcal{I}(w', w^{-1}\lambda) \longrightarrow 0$$

of $\mathcal{D}_{w^{-1}\lambda}$-modules. Since $w^{-1}\lambda$ is antidominant, by C.3.2, we get the exact sequence

$$0 \to \Gamma(X, U^0(\mathcal{I}(w', w'^{-1}\lambda))) \to \Gamma(X, \mathcal{I}(w, w^{-1}\lambda)) \to \Gamma(X, \mathcal{I}(w', w^{-1}\lambda)) \to 0,$$

and $\Gamma(X, \mathcal{I}(w, w^{-1}\lambda)) = I(\lambda) = M(\lambda)\tilde{}$ is irreducible. By 14, we have

$$\Gamma(X, \mathcal{I}(w', w^{-1}\lambda)) = I(s_{w\alpha}\lambda) \neq 0.$$

This implies that $\Gamma(X, U^0(\mathcal{I}(w', w'^{-1}\lambda))) = 0$ and $I(\lambda) = I(s_{w\alpha}\lambda)$. By dualizing we conclude that $M(\lambda) = M(s_{w\alpha}\lambda)$, what is possible only if $\lambda = s_{w\alpha}\lambda$. This in turn implies that s_α stabilizes $w^{-1}\lambda$, what contradicts $\alpha\check{}(w^{-1}\lambda) \in \mathbb{N}$. Therefore, $M(\lambda)$ must be reducible. □

Finally we want to discuss the action of the intertwining functors for simple reflections on irreducible modules $\mathcal{L}(w, \lambda)$.

Proposition 24. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi$. Then:*

(i) If $\alpha\check{}(\lambda) \notin \mathbb{Z}$, $I_{s_\alpha}(\mathcal{L}(w, \lambda)) = \mathcal{L}(ws_\alpha, s_\alpha\lambda)$.

(ii) If $\alpha\check{}(\lambda) \in \mathbb{Z}$, there are two possibilities:

(a) if $\ell(ws_\alpha) = \ell(w) + 1$, $I_{s_\alpha}(\mathcal{L}(w, \lambda)) \neq 0$ and $L^{-1}I_{s_\alpha}(\mathcal{L}(w, \lambda)) = 0$;

(b) if $\ell(ws_\alpha) = \ell(w) - 1$, $I_{s_\alpha}(\mathcal{L}(w, \lambda)) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{L}(w, \lambda)) = \mathcal{L}(w, s_\alpha\lambda)$.

Proof. (i) In this case, I_{s_α} is an equivalence of categories by L.3.22, hence it is an exact functor. By L.3.5. we can assume in addition that λ is antidominant and regular. This implies that $s_\alpha\lambda$ is also antidominant and regular. Therefore, the statement follows from L.1.16, L.3.23. and 19.

(ii) Let P_α be the parabolic subgroup of type α containing a Borel subgroup B. Then $P_\alpha = B \cup Bs_\alpha B$. This implies that $p_\alpha^{-1}(p_\alpha(C(w))) = C(w) \cup C(ws_\alpha)$. Since p_α is a locally trivial projection with fibres isomorphic to $\mathbb{P}^1$, it follows that $p_\alpha^{-1}(p_\alpha(\overline{C(w)})) = \overline{C(w)} \cup \overline{C(ws_\alpha)}$. If $\ell(ws_\alpha) = \ell(w) + 1$, $C(w) \subset \overline{C(ws_\alpha)}$ and $p_\alpha^{-1}(p_\alpha(\overline{C(w)})) = \overline{C(ws_\alpha)} \neq \overline{C(w)}$; if $\ell(ws_\alpha) = \ell(w) - 1$, $C(ws_\alpha) \subset \overline{C(w)}$ and $p_\alpha^{-1}(p_\alpha(\overline{C(w)})) = \overline{C(w)}$. Since $\operatorname{supp} \mathcal{L}(w, \lambda) = \overline{C(w)}$, we see that $\mathcal{L}(w, \lambda)$ can be a translate of a module of the form $p_\alpha^+(\mathcal{V})$ only if $\ell(ws_\alpha) = \ell(w) - 1$. Hence, by L.5.6, $I_{s_\alpha}(\mathcal{L}(w, \lambda)) = 0$ implies that $\ell(ws_\alpha) = \ell(w) - 1$.

It remains to prove the converse. Let $\ell(ws_\alpha) = \ell(w) - 1$. In this case, by 21, we have the exact sequence

$$0 \longrightarrow U^0(\mathcal{I}(ws_\alpha, s_\alpha\lambda)) \longrightarrow \mathcal{I}(w, \lambda) \longrightarrow \mathcal{I}(ws_\alpha, \lambda) \longrightarrow 0$$

and $U^0(\mathcal{I}(ws_\alpha, s_\alpha\lambda))$ is a non-zero translate of a module of the form $p_\alpha^+(\mathcal{V})$. Hence, $U^0(\mathcal{I}(ws_\alpha, s_\alpha\lambda))$ contains $\mathcal{L}(w, \lambda)$ as its unique irreducible submodule and, in particular, $\operatorname{supp} U^0(\mathcal{I}(ws_\alpha, s_\alpha\lambda)) = \overline{C(w)}$. Moreover by L.5.5, $L^{-1}I_{s_\alpha}(U^0(\mathcal{I}(ws_\alpha, s_\alpha\lambda))) = U^0(\mathcal{I}(ws_\alpha, \lambda))$. Since, by 17, $L^{-1}I_{s_\alpha}(\mathcal{I}(ws_\alpha, \lambda))$ vanishes, from the long exact sequence of derived functors of I_{s_α} applied to the preceding short exact sequence, we conclude that

$$L^{-1}I_{s_\alpha}(\mathcal{I}(w, \lambda)) = U^0(\mathcal{I}(ws_\alpha, \lambda)) \neq 0.$$

Also, since $L^{-1}I_{s_\alpha}$ is left exact, we conclude that $L^{-1}I_{s_\alpha}(\mathcal{L}(ws_\alpha,\lambda)) = 0$. Consider now the short exact sequence

$$0 \longrightarrow \mathcal{L}(w,\lambda) \longrightarrow \mathcal{I}(w,\lambda) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\operatorname{supp}\mathcal{Q} \subset \partial C(w) = \overline{C(w)} - C(w)$. By the preceding discussion, the conditions $C(v) \subset \partial C(w)$ and $p_\alpha^{-1}(p_\alpha(\overline{C(v)})) = \overline{C(w)}$ imply that $v = ws_\alpha$. Therefore, if $\mathcal{L}(v,\lambda)$ would be an irreducible constituent of $\mathcal{Q}$ with $\operatorname{supp} L^{-1}I_{s_\alpha}(\mathcal{L}(v,\lambda)) = \overline{C(w)}$, we would have $v = ws_\alpha$ what is impossible by the preceding remark. Hence, by induction in the length of $\mathcal{Q}$, we conclude that $\operatorname{supp} L^{-1}I_{s_\alpha}(\mathcal{Q}) \neq \overline{C(w)}$. Therefore, the part of the corresponding long exact sequence of derived intertwining functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & L^{-1}I_{s_\alpha}(\mathcal{L}(w,\lambda)) & \longrightarrow & L^{-1}I_{s_\alpha}(\mathcal{I}(w,\lambda)) & \longrightarrow & L^{-1}I_{s_\alpha}(\mathcal{Q}) \\ & & & & \| & & \\ & & & & U^0(\mathcal{I}(ws_\alpha,\lambda)) & & \end{array}$$

implies that the second horizontal arrow is nonzero, hence $L^{-1}I_{s_\alpha}(\mathcal{L}(w,\lambda)) \neq 0$. This implies by L.5.4. that $I_{s_\alpha}(\mathcal{L}(w,\lambda)) = 0$.

Therefore, (ii) is a consequence of L.5.4. □

V.2 Kazhdan-Lusztig Algorithm

In this section we want to develop an algorithm for calculating the multiplicities in the composition series of Verma modules. We start with a critical combinatorial result.

Let W be the Weyl group of a reduced root system Σ and S the set of simple reflections attached to a set of simple roots Π. Denote by $\ell : W \longrightarrow \mathbb{Z}_+$ the length function on (W,S). Let $\mathbb{Z}[q,q^{-1}]$ be the localization of $\mathbb{Z}[q]$ at (q), i. e. the ring of finite Laurent series in q. Denote by $\mathcal{H}$ the $\mathbb{Z}[q,q^{-1}]$-module with basis δ_w, $w \in W$. Let $\alpha \in \Pi$. Then, for any $w \in W$, either $\ell(ws_\alpha) = \ell(w)+1$ or $\ell(ws_\alpha) = \ell(w)-1$. We define a $\mathbb{Z}[q,q^{-1}]$-module endomorphism T_α of $\mathcal{H}$ by

$$T_\alpha(\delta_w) = \begin{cases} q\delta_w + \delta_{ws_\alpha} & \text{if } \ell(ws_\alpha) = \ell(w)+1; \\ q^{-1}\delta_w + \delta_{ws_\alpha} & \text{if } \ell(ws_\alpha) = \ell(w)-1. \end{cases}$$

The mentioned combinatorial result is the following theorem.

Theorem 1. *There exists a unique function $\varphi : W \longrightarrow \mathcal{H}$, such that the following properties are satisfied:*

(i) for $w \in W$ we have

$$\varphi(w) = \delta_w + \sum_{v<w} P_{wv}\delta_v,$$

where $P_{wv} \in q\mathbb{Z}[q]$.

(ii) for $\alpha \in \Pi$ *and* $w \in W$ *such that* $\ell(ws_\alpha) = \ell(w) - 1$, *there exist* $c_v \in \mathbb{Z}$, *which depend on* α *and* w, *such that*

$$T_\alpha(\varphi(ws_\alpha)) = \sum_{v \leq w} c_v \varphi(v).$$

The function $\varphi : W \longrightarrow \mathcal{H}$ determines an unique family $\{P_{wv} \mid w, v \in W, v \leq w\}$ of polynomials in $\mathbb{Z}[q]$ such that $\varphi(w) = \sum_{v \leq w} P_{wv} \delta_v$ for $w \in W$. These polynomials are called the *Kazhdan-Lusztig polynomials* for (W, S).

Remark 2. Our Kazhdan-Lusztig polynomials differ in normalization from the ones defined originally [**KL**]. We shall discuss the connection of the two normalizations later ... .

First we shall prove the uniqueness part of 1. To prove the existence, we need it in a slightly stronger form. For $k \in \mathbb{Z}_+$, denote by $W_{\leq k}$ the set of elements $w \in W$ such that $\ell(w) \leq k$.

Lemma 3. *Let* $k \in \mathbb{N}$. *Then there exists at most one function* $\varphi : W_{\leq k} \longrightarrow \mathcal{H}$, *such that the following properties are satisfied:*

(i) for $w \in W_{\leq k}$ *we have*

$$\varphi(w) = \delta_w + \sum_{v < w} P_{wv} \delta_v,$$

where $P_{wv} \in q\mathbb{Z}[q]$.

(ii) for $\alpha \in \Pi$ *and* $w \in W_{\leq k}$ *such that* $\ell(ws_\alpha) = \ell(w) - 1$, *there exist* $c_v \in \mathbb{Z}$, *which depend on* α *and* w, *such that*

$$T_\alpha(\varphi(ws_\alpha)) = \sum_{v \leq w} c_v \varphi(v).$$

Proof. The proof is by induction in k. Let $k = 0$. Then $W_{\leq k} = \{1\}$. Clearly, (i) implies that $\varphi(1) = \delta_1$ and (ii) is void in this case.

Assume that $k > 1$. By the induction assumption, $\varphi | W_{\leq k-1}$ is unique. Then, for $w \in W_{\leq k}$ such that $\ell(w) = k$ we can find a simple root α such that $\ell(ws_\alpha) = \ell(w) - 1 = k - 1$. By (ii) we know that

$$T_\alpha(\varphi(ws_\alpha)) = \sum_{v \leq w} c_v \varphi(v),$$

and, by evaluating at $q = 0$ and using (i),

$$T_\alpha(\varphi(ws_\alpha))(0) = \sum_{v \leq w} c_v \delta_v.$$

By the induction assumption, the left side is uniquely determined. This implies that c_v are uniquely determined. On the other hand, if we put $y = ws_\alpha$, we have

$$T_\alpha(\varphi(ws_\alpha)) = T_\alpha(\delta_y + \sum_{v<y} P_{yv}\delta_v) = T_\alpha(\delta_y) + \sum_{v<y} P_{yv}T_\alpha(\delta_v)$$
$$= q\delta_y + \delta_w + \sum_{v<y} P_{yv}T_\alpha(\delta_v).$$

By the construction, $\ell(v) < \ell(y) = k-1$. Hence, terms in the expansion of $T_\alpha(\delta_y)$ can involve only δ_u with $\ell(u) \le k-1$. In particular, they cannot involve δ_w. This implies that $c_w = 1$. But this yields to

$$\varphi(w) = T_\alpha(\varphi(ws_\alpha)) - \sum_{v<w} c_v\varphi(v).\square$$

The uniqueness part of 1. follows immediately from 3. The difficult part of the proof of 1. is the existence. We shall prove the existence by relating the Kazhdan-Lusztig polynomials with the structure of the category $\mathcal{M}_{coh}(\mathcal{D}_X, N_0)$. As a byproduct of this analysis we shall get a connection between the Kazhdan-Lusztig polynomials and the multiplicities of irreducible $\mathfrak{g}$-modules in Verma modules.

First we want to establish a "parity" property of solutions of 3. Define additive involutions i on $\mathbb{Z}[q, q^{-1}]$ and ι on $\mathcal{H}$ by

$$i(q^m) = (-1)^m q^m \quad \text{for} \quad m \in \mathbb{Z},$$

$$\iota(q^m\delta_w) = (-1)^{m+\ell(w)} q^m \delta_w \quad \text{for} \quad m \in \mathbb{Z} \quad \text{and} \quad w \in W.$$

Then $\iota T_\alpha \iota$ is $\mathbb{Z}[q, q^{-1}]$-linear endomorphism of $\mathcal{H}$, and we have

$$(\iota T_\alpha \iota)(\delta_w) = (-1)^{\ell(w)}\iota(T_\alpha(\delta_w)) = (-1)^{\ell(w)}\iota(q\delta_w + \delta_{ws_\alpha})$$
$$= -(q\delta_w + \delta_{ws_\alpha}) = -T_\alpha(\delta_w),$$

if $\ell(ws_\alpha) = \ell(w) + 1$, and

$$(\iota T_\alpha \iota)(\delta_w) = (-1)^{\ell(w)}\iota(T_\alpha(\delta_w)) = (-1)^{\ell(w)}\iota(q^{-1}\delta_w + \delta_{ws_\alpha})$$
$$= -(q^{-1}\delta_w + \delta_{ws_\alpha}) = -T_\alpha(\delta_w),$$

if $\ell(ws_\alpha) = \ell(w) - 1$. Therefore,

$$\iota T_\alpha \iota = -T_\alpha.$$

Lemma 4. *Let $k \in \mathbb{N}$. Let $\varphi : W_{\le k} \longrightarrow \mathcal{H}$, be a function satisfying the properties 3.(i) and 3.(ii). Then*

$$P_{wv} = q^{\ell(w)-\ell(v)} Q_{wv}$$

where $Q_{wv} \in \mathbb{Z}[q^2, q^{-2}]$.

Proof. Define $\psi(w) = (-1)^{\ell(w)}\iota(\varphi(w))$. Then $\psi : W_{\leq k} \longrightarrow \mathcal{H}$, and

$$\psi(w) = (-1)^{\ell(w)}\iota(\delta_w + \sum_{v<w} P_{wv}\delta_v) = \delta_w + \sum_{v<w}(-1)^{\ell(w)-\ell(v)}i(P_{wv})\delta_v,$$

hence ψ satisfies 3.(i). By the previous remark, for $\alpha \in \Pi$ and $w \in W_{\leq k}$ such that $\ell(ws_\alpha) = \ell(w) - 1$, we have

$$\begin{aligned} T_\alpha(\psi(ws_\alpha)) &= -(-1)^{\ell(w)}T_\alpha(\iota(\varphi(ws_\alpha))) = (-1)^{\ell(w)}\iota(T_\alpha(\varphi(ws_\alpha))) \\ &= (-1)^{\ell(w)}\iota(\sum_{v\leq w} c_v\varphi(v)) = (-1)^{\ell(w)}\sum_{v\leq w} c_v\iota(\varphi(v)) \\ &= \sum_{v\leq w}(-1)^{\ell(w)-\ell(v)}c_v\psi(v), \end{aligned}$$

hence ψ satisfies also 3.(ii). Therefore, by 3, we conclude that $\psi = \varphi$. □

Let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, N_0)$. For $w \in W$ we denote by i_w the canonical immersion of the Bruhat cell $C(w)$ into X. Clearly, for any $k \in \mathbb{Z}$, $L^{-k}i_w^+(\mathcal{F})$ is N_0-equivariant connection on $C(w)$, i. e. it is isomorphic to a sum of copies of $\mathcal{O}_{C(w)}$. On the other hand, $\dim C(w) = \ell(w)$, hence $R^{n-\ell(w)-k}i_w^!(\mathcal{F}) = L^{-k}i_w^+(\mathcal{F})$ for any $k \in \mathbb{Z}$. We put

$$\nu(\mathcal{F}) = \sum_{w\in W}\sum_{m\in\mathbb{Z}} \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}))q^m\delta_w.$$

Therefore, ν is a map from $\mathcal{M}_{coh}(\mathcal{D}_X, N_0)$ into $\mathcal{H}$.

For any $w \in W$, we put

$$\mathcal{I}_w = \mathcal{I}(w, -\rho) \quad \text{and} \quad \mathcal{L}_w = \mathcal{L}(w, -\rho).$$

The existence part of 1. follows from the next result.

Proposition 5. *Let $\varphi(w) = \nu(\mathcal{L}_w)$. Then φ satisfies 1.(i) and 1.(ii).*

Checking that φ satisfies 1.(i) is quite straightforward.

Lemma 6. *Let $\varphi(w) = \nu(\mathcal{L}_w)$. Then*

$$\varphi(w) = \delta_w + \sum_{v<w} P_{wv}\delta_v$$

where $P_{wv} \in q\mathbb{Z}[q]$.

Proof. Clearly, $\operatorname{supp}\mathcal{L}_w = \overline{C(w)}$. By definition of the Bruhat order, $v \leq w$ is equivalent with $C(v) \subset \overline{C(w)}$. Therefore, we see that $R^m i_v^!(\mathcal{L}_w) = 0$, for all $m \in \mathbb{Z}$, if v is not less than or equal to w. By Kashiwara's theorem, we conclude that

$$R^0 i_w^!(\mathcal{L}_w) = R^0 i_w^!(\mathcal{I}_w) = R^0 i_w^!(R^0 i_{w+}(\mathcal{O}_{C(w)})) = \mathcal{O}_{C(w)},$$

and, for $m \neq 0$,

$$R^m i_w^!(\mathcal{L}_w) = R^m i_w^!(\mathcal{I}_w) = R^m i_w^!(R^0 i_{w+}(\mathcal{O}_{C(w)})) = 0.$$

Finally, if $v < w$, denote by X' the complement in X of the boundary $\partial C(v) = \overline{C(v)} - C(v)$ of $C(v)$. Let j_v be the natural inclusion of $C(v)$ into X'. Then j_v is a closed immersion. Clearly, $\mathcal{L}_w|X'$ is again irreducible, and by Kashiwara's theorem, $R^0 j_{v+}(R^0 i_v^!(\mathcal{L}_w))$ is the $\mathcal{D}_{X'}$-submodule of $\mathcal{L}_w|X'$ consisting of sections supported on $C(v)$. Since the restriction of an irreducible $\mathcal{D}_X$-module to a nonempty open set is again irreducible, $\mathcal{L}_w|X'$ is an irreducible $\mathcal{D}_{X'}$-module, which implies that $R^0 j_{v+}(R^0 i_v^!(\mathcal{L}_w)) = 0$. Hence, $R^m i_v^!(\mathcal{L}_w) \neq 0$ is possible only if $m > 0$. □

The main part of the proof is to establish that $\varphi(w) = \nu(\mathcal{L}_w)$ satisfies 1.(ii). First we need an auxiliary result.

Let $\alpha \in \Pi$ and X_α the corresponding flag variety of parabolic subalgebras of type α. Denote by $p_\alpha : X \longrightarrow X_\alpha$ the natural projection map. Let $C(v)$ be a Bruhat cell in X for $v \in W$. Since it is isomorphic to $\mathbb{C}^{\ell(v)}$, the natural imbedding $i_v : C(v) \longrightarrow X$ is an affine morphism. The projection $p_\alpha(C(v))$ of $C(v)$ to X_α is also an affine space, and therefore affinely imbedded into X_α. Since the fibration $p_\alpha : X \longrightarrow X_\alpha$ is locally trivial, we conclude that $p_\alpha^{-1}(p_\alpha(C(v)))$ is a smooth affinely imbedded subvariety of X. If P_α is the standard parabolic subgroup of type α containing the Borel subgroup B, we have $P_\alpha = B \cup B s_\alpha B$. This implies that $p_\alpha^{-1}(p_\alpha(C(v))) = C(v) \cup C(v s_\alpha)$. One of these Bruhat cells is open and dense in $p_\alpha^{-1}(p_\alpha(C(v)))$, the other one is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$. We have either $\ell(v s_\alpha) = \ell(v)+1$ or $\ell(v s_\alpha) = \ell(v)-1$. In the first case, $\dim p_\alpha^{-1}(p_\alpha(C(v))) = \ell(v)+1$, $C(v s_\alpha)$ is open and $C(v)$ closed in $p_\alpha^{-1}(p_\alpha(C(v)))$. In the second case, $\dim p_\alpha^{-1}(p_\alpha(C(v))) = \ell(v)$, $C(v)$ is open and $C(v s_\alpha)$ closed in it. Moreover, in the first case $p_\alpha : C(v) \longrightarrow p_\alpha(C(v))$ is an isomorphism, while in the second case it is a fibration with fibres isomorphic to an affine line. We define the functors

$$U_\alpha^q(\mathcal{F}) = p_\alpha^+(R^q p_{\alpha+}(\mathcal{F})),$$

from $\mathcal{M}_{qc}(\mathcal{D}_X)$ into itself, for any $q \in \mathbb{Z}$. Since the fibres of the projection map $p_\alpha : X \longrightarrow X_\alpha$ are one-dimensional, U_α^q can be nonzero only for $q \in \{-1, 0, 1\}$. These functors are closely related to the functors we discussed in L.5. In particular, we have the following lemma.

Lemma 7. *Let $w \in W$ and $\alpha \in \Pi$ be such that $\ell(w s_\alpha) = \ell(w) - 1$. Then:*
(i) $U_\alpha^q(\mathcal{L}_{w s_\alpha}) = 0$ for all $q \neq 0$;
(ii) $U_\alpha^0(\mathcal{L}_{w s_\alpha})$ is a direct sum of $\mathcal{L}_v$ for $v \leq w$.

Proof. First, by the construction, $U_\alpha^q(\mathcal{L}_{w s_\alpha})$ are holonomic $(\mathcal{D}_X, N_0)$-modules supported inside the closure of $p_\alpha^{-1}(p_\alpha(C(w)))$, which is equal to the closure of $C(w)$ by the preceding discussion. This implies that $U_\alpha^q(\mathcal{L}_{w s_\alpha})$ are of finite length and their composition factors could be only $\mathcal{L}_v$ for $v \leq w$. Since p_α is a locally trivial fibration with fibres isomorphic to $\mathbb{P}^1$ and $\mathcal{L}_{w s_\alpha}$ is the direct image of $\mathcal{O}_{C(ws_\alpha)}$ and therefore of geometric origin (...), by the decomposition theorem (...) $R^q p_{\alpha+}(\mathcal{L}_{ws_\alpha})$ are semisimple. This implies, using again the local triviality of p_α, that $U^q_\alpha(\mathcal{L}_{ws_\alpha})$ are semisimple, and completes the proof of (ii).

To prove (i) we establish the connection with the results in L.5. Let $Y_\alpha = X \times_{X_\alpha} X$ denote again the fibered product of X with X relative to the morphism p_α. Denote by q_1 and q_2 the corresponding projections of Y_α onto the first and second factor respectively. Then the following diagram

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\Big\downarrow & & \Big\downarrow{\scriptstyle P_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

is commutative. By base change,

$$U^q_\alpha(\mathcal{L}_{ws_\alpha}) = p^+_\alpha(R^q p_{\alpha+}(\mathcal{L}_{ws_\alpha})) = R^q q_{1+}(q_2^+(\mathcal{L}_{ws_\alpha})).$$

Since $\mathcal{D}_X = \mathcal{D}_{-\rho}$, we easily check that $U^q_\alpha(\mathcal{L}_{ws_\alpha})(\alpha) = U^q(\mathcal{L}_{ws_\alpha})$. Hence, (i) follows immediately from L.5.4. if we show that $I_{s_\alpha}(\mathcal{L}_{ws_\alpha}) \neq 0$. On the other hand, this follows immediately from 1.21. □

Now we want to calculate $\nu(U^0_\alpha(\mathcal{L}_{ws_\alpha}))$ for $w \in W$ and $\alpha \in \Pi$ such that $\ell(ws_\alpha) = \ell(w) - 1$. First, let $v \in W$ be such that $v \leq w$. Then $C(v)$ is in the closure of $C(w)$. Since, by our assumption, the closure of $C(w)$ is also the closure of $p_\alpha^{-1}(p_\alpha(C(w)))$, we conclude that $p_\alpha^{-1}(p_\alpha(C(v))) = C(v) \cup C(vs_\alpha)$ is also contained in the closure of $C(w)$, i. e. $vs_\alpha \leq w$. Therefore, without any loss of generality, we can assume that $\ell(v) = \ell(vs_\alpha) + 1$, i. e. $C(v)$ is open in $Z_\alpha = p_\alpha^{-1}(p_\alpha(C(v)))$. Let $j : Z_\alpha \longrightarrow X$ and $j_v : p_\alpha(C(v)) \longrightarrow X_\alpha$ be the natural inclusions. Then we have the following commutative diagram

$$\begin{array}{ccc} Z_\alpha & \xrightarrow{j} & X \\ {\scriptstyle q_\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle p_\alpha} \\ p_\alpha(C(v)) & \xrightarrow{j_v} & X_\alpha \end{array}$$

and by base change and 7. we get

$$\begin{aligned} R^k j^!(U^0_\alpha(\mathcal{L}_{ws_\alpha})) &= H^k\left(Rj^!(p^+_\alpha(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha}))))\right) \\ &= H^{k-1}\left(Rj^!(Rp^!_\alpha(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha}))))\right) \\ &= H^{k-1}\left(R(p_\alpha \circ j)^!(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha})))\right) \\ &= H^{k-1}\left(R(j_v \circ q_\alpha)^!(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha})))\right) \\ &= H^{k-1}(Rq^!_\alpha(Rj^!_v(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha}))))) \\ &= q^+_\alpha\left(H^k\left(Rj^!_v(Rp_{\alpha+}(D(\mathcal{L}_{ws_\alpha})))\right)\right) \\ &= q^+_\alpha\left(H^k\left(Rq_{\alpha+}(Rj^!(D(\mathcal{L}_{ws_\alpha})))\right)\right). \end{aligned}$$

Now we analyze in more details the structure of the complex $Rj^!(D(\mathcal{L}_{ws_\alpha}))$. As we remarked before, $Z_\alpha = C(v) \cup C(vs_\alpha)$, $C(v)$ is open in Z_α and $C(vs_\alpha)$

is closed in it. If we denote by $i : C(v) \longrightarrow Z_\alpha$ and $i' : C(vs_\alpha) \longrightarrow Z_\alpha$ the canonical affine immersions, we have the following distinguished triangle

$$i'_+(Ri'^!(\mathcal{F}^\cdot)) \longrightarrow \mathcal{F} \longrightarrow i_+(\mathcal{F}^\cdot|C(v))$$

in the category $D^b(\mathcal{D}_{Z_\alpha})$, for any object $\mathcal{F}^\cdot$. Therefore, in particular we have the following distinguished triangle

$$i'_+(Ri'^!(Rj^!(D(\mathcal{L}_{ws_\alpha})))) \longrightarrow Rj^!(D(\mathcal{L}_{ws_\alpha})) \longrightarrow i_+(Rj^!(D(\mathcal{L}_{ws_\alpha}))|C(v))$$

and

$$i'_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha}))) \longrightarrow Rj^!(D(\mathcal{L}_{ws_\alpha})) \longrightarrow i_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha}))).$$

By applying the functor $Rq_{\alpha+}$ we get the distinguished triangle

$$Rq_{\alpha+}(i'_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))) \to Rq_{\alpha+}(Rj^!(D(\mathcal{L}_{ws_\alpha}))) \to Rq_{\alpha+}(i_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha}))))$$

in $D^b(\mathcal{D}_{p_\alpha(C(v))})$. Since $p_\alpha(C(v))$ is a N_0-orbit in X_α, and all $\mathcal{D}$-modules involved in the preceding arguments are N_0-equivariant, the cohomologies of the complexes in this triangle are sums of copies of $\mathcal{O}_{p_\alpha(C(v))}$. In addition,

$$Rq_{\alpha+}(i'_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))) = R(q_\alpha \circ i')_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))$$

and $q_\alpha \circ i' : C(vs_\alpha) \longrightarrow p_\alpha(C(v))$ is an isomorphism. Therefore,

$$\dim_{\mathcal{O}} H^k\left(R(q_\alpha \circ i')_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))\right) = \dim_{\mathcal{O}} R^k i^!_{vs_\alpha}(\mathcal{L}_{ws_\alpha})$$

for any $k \in \mathbb{Z}$. On the other hand,

$$Rq_{\alpha+}(i_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha})))) = R(q_\alpha \circ i)_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha})))$$

and $q_\alpha \circ i : C(v) \longrightarrow p_\alpha(C(v))$ is a a locally trivial projection with fibres isomorphic to an affine line. Therefore, since cohomologies of $Ri^!_v(D(\mathcal{L}_{ws_\alpha}))$ are sums of copies of $\mathcal{O}_{C(v)}$,

$$\dim_{\mathcal{O}} H^k\left(R(q_\alpha \circ i)_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha})))\right) = \dim_{\mathcal{O}} R^{k+1} i^!_v(\mathcal{L}_{ws_\alpha})$$

for any $k \in \mathbb{Z}$. This also leads to the long exact sequence

$$\begin{aligned} \cdots \to H^k\left(R(q_\alpha \circ i')_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))\right) \to H^k\left(Rq_{\alpha+}(Rj^!(D(\mathcal{L}_{ws_\alpha})))\right) \\ \to H^k\left(R(q_\alpha \circ i)_+(Ri^!_v(D(\mathcal{L}_{ws_\alpha})))\right) \to H^{k+1}\left(R(q_\alpha \circ i')_+(Ri^!_{vs_\alpha}(D(\mathcal{L}_{ws_\alpha})))\right) \to \cdots \end{aligned}$$

consisting of $\mathcal{D}_{p_\alpha(C(v))}$-modules which are are sums of copies of $\mathcal{O}_{p_\alpha(C(v))}$.

Now we want to prove that $\varphi(w) = \nu(\mathcal{L}_w)$ satisfies 1.(ii) by induction in the length of $w \in W$. If $\ell(w) = 0$, $w = 1$ and 1.(ii) is void in this case. Therefore, we can assume that $\varphi(w) = \nu(\mathcal{L}_w)$ satisfies 1.(ii) on $W_{\leq k}$ for some $k \in \mathbb{N}$. By 4, it satisfies the parity condition on $W_{\leq k}$, i. e. for any $u \in W_{\leq k}$, we have $R^k i^!_v(\mathcal{L}_u) = 0$ for all $v \in W$ and $k \in \mathbb{Z}_+$ such that $k \equiv \ell(v) - \ell(u) - 1 \pmod 2$. Let $w \in W$ be such that $\ell(w) = k+1$. Then there exists $\alpha \in \Pi$ such that $\ell(ws_\alpha) = k$, i. e. $ws_\alpha \in W_{\leq k}$. Then for any $v \in W$ in the preceding calculation, we either have $k \equiv \ell(v) - \ell(ws_\alpha) \pmod 2$ or $k \equiv \ell(v) - \ell(ws_\alpha) - 1 \pmod 2$. In the first case, we have $R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) = 0$ and $R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha}) = 0$, what in turn implies that $H^k(Rq_{\alpha+}(Rj^!(D(\mathcal{L}_{ws_\alpha}))) = 0$. In the second case, we see that

$$\dim_{\mathcal{O}} H^k(Rq_{\alpha+}(Rj^!(D(\mathcal{L}_{ws_\alpha}))) = \dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha}).$$

This implies that $R^k j^!(U_\alpha^0(\mathcal{L}_{ws_\alpha})) = 0$ if $k \equiv \ell(v) - \ell(ws_\alpha) \pmod 2$, and

$$\dim_{\mathcal{O}} R^k j^!(U_\alpha^0(\mathcal{L}_{ws_\alpha})) = \dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})$$

for $k \equiv \ell(v) - \ell(ws_\alpha) - 1 \pmod 2$.

By restricting further to $C(v)$ and $C(vs_\alpha)$ we finally get, for all $k \in \mathbb{Z}_+$, that

$$\dim_{\mathcal{O}} R^k i_v^!(U_\alpha^0(\mathcal{L}_{ws_\alpha})) = \dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})$$

and

$$\dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(U_\alpha^0(\mathcal{L}_{ws_\alpha})) = \dim_{\mathcal{O}} R^k i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^{k-1} i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha}),$$

what leads to

$$\begin{aligned}
\nu(U_\alpha^0(\mathcal{L}_{ws_\alpha})) &= \sum_{v \in W} \sum_{k \in \mathbb{Z}} \dim_{\mathcal{O}} R^k i_v^!(U_\alpha^0(\mathcal{L}_{ws_\alpha}))\ q^k \delta_v \\
&= \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \dim_{\mathcal{O}} R^k i_v^!(U_\alpha^0(\mathcal{L}_{ws_\alpha}))\ q^k \delta_v \\
&+ \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(U_\alpha^0(\mathcal{L}_{ws_\alpha}))\ q^k \delta_{vs_\alpha} \\
&= \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \big(\dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})\big)\ q^k \delta_v \\
&+ \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \big(\dim_{\mathcal{O}} R^k i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^{k-1} i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})\big)\ q^k \delta_{vs_\alpha} \\
&= \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \big(\dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha}) + \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})\big)\ q^k (\delta_v + q\delta_{vs_\alpha}) \\
&= \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \dim_{\mathcal{O}} R^{k+1} i_v^!(\mathcal{L}_{ws_\alpha})\ q^{k+1} (q^{-1}\delta_v + \delta_{vs_\alpha}) \\
&+ \sum_{vs_\alpha < v} \sum_{k \in \mathbb{Z}} \dim_{\mathcal{O}} R^k i_{vs_\alpha}^!(\mathcal{L}_{ws_\alpha})\ q^k (\delta_v + q\delta_{vs_\alpha}) \\
&= T_\alpha(\nu(\mathcal{L}_{ws_\alpha})) = T_\alpha(\varphi(ws_\alpha)).
\end{aligned}$$

In combination with 7. we get

$$T_\alpha(\varphi(ws_\alpha)) = \nu(U_\alpha^0(\mathcal{L}_{ws_\alpha})) = \sum_{v \le w} c_v \nu(\mathcal{L}_v) = \sum_{v \le w} c_v \varphi(v),$$

i. e. 1.(ii) holds for φ on $W_{\le k+1}$. By induction we see that φ satisfies 1.(ii), and this ends the proof of 5. This also completes the proof of 1.

Now we want to establish the connection between the Kazhdan-Lusztig polynomials and the multiplicities of irreducible $\mathfrak{g}$-modules in Verma modules. We start with the following observation.

Lemma 8. *The evaluation of the map ν at -1 factors through the Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N_0))$ of $\mathcal{M}_{coh}(\mathcal{D}_X, N_0)$.*

Proof. Evidently

$$\nu(\mathcal{F})(-1) = \sum_{w\in W}\sum_{m\in\mathbb{Z}} \left((-1)^m \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}))\right)\delta_w.$$

On the other hand, if

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is an exact sequence in $\mathcal{M}_{coh}(\mathcal{D}_X, N_0)$, we get a long exact sequence

$$\dots \longrightarrow R^m i_w^!(\mathcal{F}_1) \longrightarrow R^m i_w^!(\mathcal{F}_2) \longrightarrow R^m i_w^!(\mathcal{F}_3) \longrightarrow R^{m+1} i_w^!(\mathcal{F}_1) \longrightarrow \dots$$

of N_0-homogeneous connections on $C(w)$. By the Euler principle,

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}_2))\\ &= \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}_1)) + \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}_3)).\square\end{aligned}$$

Also we need the following simple fact.

Lemma 9. $\nu(\mathcal{I}_w) = \delta_w$.

Proof. By definition, $\mathcal{I}_w = R^0 i_{w+}(\mathcal{O}_{C(w)})$. Therefore, by Kashiwara's theorem

$$R^0 i_w^!(\mathcal{I}_w) = R^0 i_w^!(R^0 i_{w+}(\mathcal{O}_{C(w)})) = \mathcal{O}_{C(w)},$$

and, for $m \neq 0$,

$$R^m i_w^!(\mathcal{I}_w) = R^m i_w^!(R^0 i_{w+}(\mathcal{O}_{C(w)})) = 0.$$

Moreover, by the base change, for any $y \in W$, $y \neq w$, we have

$$R^m i_y^!(\mathcal{I}_w) = R^m i_y^!(R^0 i_{w+}(\mathcal{O}_{C(w)})) = 0.\square$$

Let $\chi : \mathcal{M}_{coh}(\mathcal{D}_X, N_0) \longrightarrow K(\mathcal{M}_{coh}(\mathcal{D}_X, N_0))$ denote the natural map of the category $\mathcal{M}_{coh}(\mathcal{D}_X, N_0)$ into its Grothendieck group.

Theorem 10. *Let P_{wv}, $w, v \in W$, be the Kazhdan-Lusztig polynomials of (W, S). Then*

$$\chi(\mathcal{L}_w) = \chi(\mathcal{I}_w) + \sum_{v<w} P_{wv}(-1)\chi(\mathcal{I}_v).$$

Proof. Since $\mathcal{I}_w$ contains $\mathcal{L}_w$ as the unique irreducible submodule, and all other composition factors are $\mathcal{L}_v$ for $v < w$, we see that $\chi(\mathcal{I}_w)$, $w \in W$, form a basis of $K(\mathcal{M}_{coh}(\mathcal{D}_X, N_0))$. Hence

$$\chi(\mathcal{L}_w) = \sum_{v \leq w} \lambda_{wv}\, \chi(\mathcal{I}_v)$$

with $\lambda_{wv} \in \mathbb{Z}$. By 8, $\nu(-1)$ factors through $K(\mathcal{M}_{coh}(\mathcal{D}_X, N_0))$ and by 9, $\nu(\mathcal{I}_v)(-1) = \delta_v$ for $v \in W$, what leads to

$$\nu(\mathcal{L}_w)(-1) = \sum_{v \leq w} \lambda_{wv} \nu(\mathcal{I}_v)(-1) = \sum_{v \leq w} \lambda_{wv} \delta_v.$$

Hence, from definition of P_{wv} it follows that $\lambda_{ww} = 1$ and $P_{wv}(-1) = \lambda_{wv}$. □

This gives an effective algorithm to calculate the multiplicities of irreducible modules in Verma modules for infinitesimal character χ_ρ. We can order the elements of W by an order relation compatible with the Bruhat order. Then the matrix $(\lambda_{wv}; w, v \in W)$ is lower triangular with 1 on the diagonal. If $(\mu_{wv}; w, v \in W)$ are the coefficients of its inverse matrix, we see from 10. that

$$\begin{aligned}\chi(\mathcal{I}_w) &= \sum_{u \in W} \sum_{v \in W} \mu_{wv} \lambda_{vu}\ \chi(\mathcal{I}_u) = \sum_{v \in W} \mu_{wv} \left(\sum_{u \in W} \lambda_{vu}\ \chi(I_u) \right) \\ &= \sum_{v \in W} \mu_{wv}\ \chi(\mathcal{L}_v) = \sum_{v \leq w} \mu_{wv}\ \chi(\mathcal{L}_v)\end{aligned}$$

and $\mu_{ww} = 1$ for any $w \in W$. By 1.11, 1.14. and 1.19. we finally get the following result.

Corollary 11. *The multiplicity of irreducible module $L(-v\rho)$ in the Verma module $M(-w\rho)$ is equal to μ_{wv}.*

Clearly, by twisting by a homogeneous invertible $\mathcal{O}_X$-module we get the results analogous to 10. for standard modules in $\mathcal{M}_{coh}(\mathcal{D}_\mu, N_0)$ for arbitrary weight $\mu \in P(\Sigma)$. This immediately leads to an analogue of 11. for Verma modules with infinitesimal character χ_μ for regular weights $\mu \in P(\Sigma)$. In the next section we shall discuss the analogous problem for Verma modules with arbitrary regular infinitesimal character.

At the end, we list a few simple properties of P_{wv}.

Corollary 12. *The coefficients of the Kazhdan-Lusztig polynomials P_{wv} are non-negative integers.*

Proof. This follows immediately from 5. and the definition of the map ν. □

Lemma 13. *Let $w \in W$ with $\ell(ws_\alpha) = \ell(w) - 1$. Then, for any $v \in W$, $vs_\alpha \leq w$ is equivalent to $v \leq w$. If $v \leq w$ with $\ell(vs_\alpha) = \ell(v) - 1$, we have $qP_{wv} = P_{w\, vs_\alpha}$.*

Proof. In the proof of 1.21, we have shown that $\ell(ws_\alpha) = \ell(w) - 1$ is equivalent to $\overline{C(w)} = p_\alpha^{-1}(p_\alpha(\overline{C(w)}))$. Therefore, $C(v) \subset \overline{C}(w)$ implies that

$$C(v) \cup C(vs_\alpha) = p_\alpha^{-1}(p_\alpha(C(v))) \subset \overline{C(w)},$$

i. e. $vs_\alpha \leq w$. This proves the first assertion.

Moreover, by 1.24. and L.5.6, $\mathcal{L}_w$ is of the form $p_\alpha^+(\mathcal{V})$. Therefore, using the notation from the proof of 5, we have

$$\begin{aligned} \dim_{\mathcal{O}} R^p i^!_{vs_\alpha}(\mathcal{L}_w) &= \dim_{\mathcal{O}} L^{p-n+\ell(v)-1} i^+_{vs_\alpha}(\mathcal{L}_w) = \dim_{\mathcal{O}} L^{p-n+\ell(v)-1} j^+_v(\mathcal{V}) \\ &= \dim_{\mathcal{O}} L^{p-n+\ell(v)-1} i^+_v(\mathcal{L}_w) = \dim_{\mathcal{O}} R^{p-1} i^!_v(\mathcal{L}_w), \end{aligned}$$

for arbitrary $v \leq w$ such that $\ell(vs_\alpha) = \ell(v) - 1$. □

G. Generalized Verma Modules

G.1 Cosets in Weyl Groups

Let Σ be a reduced root system and Σ^+ a set of positive roots. Denote by Π the corresponding set of simple roots. As before, we put

$$\Sigma_w^+ = \Sigma^+ \cap \{-w^{-1}(\Sigma^+)\} = \{\alpha \in \Sigma^+ \mid w\alpha \in -\Sigma^+\}.$$

Lemma 1. *Let $w \in W$ and $\alpha \in \Pi$. Then the following statements are equivalent:*

(i) $\ell(ws_\alpha) = \ell(w) + 1$,

(ii) $\alpha \notin \Sigma_w^+$.

Proof. Let $\ell(ws_\alpha) = \ell(w) + 1$. Then by L.3.12.(ii) we have

$$\Sigma_{ws_\alpha}^+ = s_\alpha(\Sigma_w^+) \cup \{\alpha\},$$

i. e. $\alpha \in \Sigma_{ws_\alpha}^+$. This implies that

$$-w\alpha = ws_\alpha\alpha \in -\Sigma^+,$$

i. e. $\alpha \notin \Sigma_w^+$.

If $\ell(ws_\alpha) = \ell(w) - 1$, $\ell(w's_\alpha) = \ell(w') + 1$ for $w' = ws_\alpha$ and $\alpha \notin \Sigma_{w'}^+$. This in turn implies that

$$w\alpha = w's_\alpha\alpha = -w'\alpha \in -\Sigma^+,$$

and $w \in \Sigma_w^+$. □

Let $\Theta \subset \Pi$. Denote by Σ_Θ the root subsystem of Σ generated by Θ, and by W_Θ the subgroup of W generated by simple reflections $S_\Theta = \{s_\alpha \mid \alpha \in \Theta\}$. Clearly, the length function of (W_Θ, S_Θ) is the restriction of ℓ to W_Θ. Also, define the set

$$W^\Theta = \{w \in W \mid \Sigma_w^+ \cap \Theta = \emptyset\} = \{w \in W \mid \Theta \subset w^{-1}(\Sigma^+)\}.$$

Theorem 2. *Every element $w \in W$ has a unique decomposition in the form $w = w't$, $w' \in W^\Theta$, $t \in W_\Theta$. In addition, $\ell(w) = \ell(w') + \ell(t)$.*

Proof. By 1, for any $w \in W$, the following conditions are equivalent:

(i) $w \in W^{\Theta}$;

(ii) $\ell(ws_\alpha) = \ell(w) + 1$ for any $\alpha \in \Theta$.

First we claim that a shortest element in a left W_Θ-coset must be in W^Θ. Assume that w is a shortest element in a left W_Θ-coset and that $w \notin W^\Theta$. Then there would exist an $\alpha \in \Theta$ with $\ell(ws_\alpha) = \ell(w) - 1$. Therefore, there would exist an element in the same left W_Θ-coset of shorter length, contradicting our assumption.

Now we prove, by induction in length, that every element w in a left W_Θ-coset has a decomposition of the form $w = w't$ with $w' \in W^\Theta$, $t \in W_\Theta$ and $\ell(w) = \ell(w') + \ell(t)$. We already proved this for elements of minimal length. Take an arbitrary element w of a left W_Θ-coset. If it is in W^Θ we are done. If it is not in W^Θ, by the preceding remark we can find an $\alpha \in \Theta$ with $\ell(ws_\alpha) = \ell(w) - 1$. By the induction assumption $ws_\alpha = w't'$ for some $w' \in W^\Theta$ and $t' \in W_\Theta$ with $\ell(ws_\alpha) = \ell(w') + \ell(t')$. This implies that $w = w't's_\alpha$. Put $t = ts_\alpha$. Then $w = w't$, $t \in W_\Theta$ and $\ell(t) \leq \ell(t') + 1$. Moreover, we have

$$\ell(w) \leq \ell(w') + \ell(t) \leq \ell(w') + \ell(t') + 1 = \ell(ws_\alpha) + 1 = \ell(w),$$

which implies that we have the equality $\ell(w) = \ell(w') + \ell(t)$. This completes the proof of the existence of the decomposition.

To prove that this decomposition is unique it is enough to show that there is at most one element of W^Θ in each left W_Θ-coset. Assume that $w, w' \in W^\Theta$ and $w = w't$ with $t \in W_\Theta$. Then

$$\Theta \subset w^{-1}(\Sigma^+) = t^{-1}{w'}^{-1}(\Sigma^+).$$

The set $\Sigma_\Theta^+ = \Sigma_\Theta \cap \Sigma^+$ is the set of positive roots in Σ_Θ determined by Θ. Then

$$\Sigma_\Theta^+ \subset t^{-1}{w'}^{-1}(\Sigma^+),$$

and

$$t(\Sigma_\Theta^+) \subset {w'}^{-1}(\Sigma^+).$$

Analogously,

$$\Sigma_\Theta^+ \subset {w'}^{-1}(\Sigma^+).$$

Since W_Θ is isomorphic to the Weyl group of Σ_Θ, if $t \neq 1$, there would exist a root $\beta \in \Sigma_\Theta^+$ such that $-\beta \in t(\Sigma_\Theta^+)$. This would imply that $\beta, -\beta \in {w'}^{-1}(\Sigma^+)$, which is impossible. Therefore, $t = 1$ and $w = w'$. □

Let w_Θ be the longest element in W_Θ. This element is characterized by the following property.

Lemma 3. *The element w_Θ is the unique element in W with the following properties:*

(i) $w_\Theta(\Theta) = -\Theta$;

(ii) w_Θ permutes positive roots outside Σ_Θ^+.

Proof. To prove that w_Θ satisfies (i) it is enough to remark that w_Θ maps positive roots in Σ_Θ into negative roots.

Let $\beta \in \Theta$. Then the reflection s_β permutes elements in $\Sigma^+ - \{\beta\}$ and also roots in Σ_Θ. This implies that it also permutes positive roots outside Σ_Θ^+. By induction in length we conclude that any element of W_Θ permutes positive roots outside Σ_Θ^+. This shows that w_Θ satisfies (ii).

On the other hand, if w satisfies the conditions (i) and (ii),

$$w(\Sigma^+) = \left(-\Sigma_\Theta^+\right) \cup \left(\Sigma^+ - \Sigma_\Theta^+\right),$$

and, since W acts simply transitively on all sets of positive roots in Σ, there is only one element of W with this property. □

Theorem 4. *(i) Each left W_Θ-coset in W has a unique shortest element. It lies in W^Θ.*

(ii) If w is the shortest element in a left W_Θ-coset C, ww_Θ is the unique longest element in this coset.

(iii) Each right W_Θ-coset in W has a unique shortest element.

(iv) If w is the shortest element in a right W_Θ-coset C, $w_\Theta w$ is the unique longest element in this coset.

Proof. The statements (i) and (ii) follow immediately from 2. Since the anti-automorphism $w \longmapsto w^{-1}$ of W preserves W_Θ, w_Θ and the length function $\ell : W \to \mathbb{Z}_+$, and maps left W_Θ-cosets into right W_Θ-cosets, (i) and (ii) imply also (iii) and (iv). □

Therefore, the set W^Θ is a section of the left W_Θ-cosets in W consisting of the shortest elements of each coset. Hence, the shortest elements of right W_Θ-cosets in W are the elements of the set

$$\{w \in W \mid w^{-1} \in W^\Theta\} = \{w \in W \mid \Theta \subset w(\Sigma^+)\}.$$

This implies the following result.

Lemma 5. *The set*

$${}^\Theta W = \{w \in W \mid \Theta \subset -w(\Sigma^+)\}$$

is the section of the set of right W_Θ-cosets in W consisting of the longest elements of each coset.

Proof. Let w be the shortest element of a right W_Θ-coset. Then, by the preceding discussion, $\Theta \subset w(\Sigma^+)$. Therefore,

$$-\Theta = w_\Theta(\Theta) \subset w_\Theta w(\Sigma^+),$$

and the longest element of this right coset $w_\Theta w$ is in ${}^\Theta W$.

On the other hand, if $w \in {}^\Theta W$, $\Theta \subset -w(\Sigma^+)$ and

$$-\Theta = w_\Theta(\Theta) \subset -w_\Theta w(\Sigma^+),$$

what implies that $w_\Theta w$ is the shortest element in its right W_Θ-coset. Therefore, w is the longest element in this coset. □

For a right W_Θ-coset C in W we denote by w^C the corresponding element in ${}^\Theta W$. We define an order relation on the set $W_\Theta \backslash W$ of all all right W_Θ-cosets by transfering the order relation on ${}^\Theta W$ induced by the Bruhat order on W.

Proposition 6. *Let C be a right W_Θ-coset in W and $\alpha \in \Pi$. Then we have the following three possibilities:*

(i) $Cs_\alpha = C$;

(ii) $Cs_\alpha > C$, and in this case $w^{Cs_\alpha} = w^C s_\alpha$ and $\ell(ws_\alpha) = \ell(w) + 1$ for any $w \in C$;

(iii) $Cs_\alpha < C$, and in this case $w^{Cs_\alpha} = w^C s_\alpha$ and $\ell(ws_\alpha) = \ell(w) - 1$ for any $w \in C$.

Proof. Assume that $Cs_\alpha \neq C$. Let w be the shortest element in C. Then there are two possibilities, either $\ell(ws_\alpha) = \ell(w) + 1$ or $\ell(ws_\alpha) = \ell(w) - 1$.

Assume first that $\ell(ws_\alpha) = \ell(w) + 1$. Let $t \in W_\Theta$. Suppose that $\ell(tws_\alpha) = \ell(tw) - 1$. Since $\ell(ws_\alpha) = \ell(w) + 1$, by the exchange condition we conclude that there exists $t' \in W_\Theta$ such that $ws_\alpha = t'w$. This implies that $Cs_\alpha = C$, contrary to our assumption. Therefore, $\ell(tws_\alpha) = \ell(tw) + 1$. This implies that $w^C s_\alpha$ is the longest element in Cs_α and (ii) holds.

Assume now that $\ell(ws_\alpha) = \ell(w) - 1$. Let $t \in W_\Theta$. Then

$$\ell(tws_\alpha) \leq \ell(t) + \ell(ws_\alpha) = \ell(t) + \ell(w) - 1 = \ell(tw) - 1,$$

which implies $\ell(tws_\alpha) = \ell(tw) - 1$. Therefore, $w^C s_\alpha$ is the longest element in Cs_α and (iii) holds. □

Let C be a right W_Θ-coset and w its shortest element. Then $w^C = w_\Theta w$ and $\ell(w^C) = \ell(w) + \ell(w_\Theta)$. This implies that $w_\Theta \leq w_\Theta w = w^C$ in the Bruhat order, i. e. $W_\Theta \leq C$ in the ordering on $W_\Theta \backslash W$. Hence, W_Θ is the smallest element in $W_\Theta \backslash W$. Later we shall need the following characterization of this element.

Lemma 7. *Let $C \in W_\Theta \backslash W$. Assume that for any $\alpha \in \Pi$ we have either $Cs_\alpha = C$ or $Cs_\alpha > C$. Then $C = W_\Theta$.*

Proof. Let w be the shortest element in C. Our assumption implies that $\ell(ws_\alpha) = \ell(w) + 1$ for any $\alpha \in \Pi$. But this is possible only if $w = 1$ and $C = W_\Theta$. □

Finally, we remark the following fact.

Lemma 8. *If $w \in {}^\Theta W$ and $t \in W_\Theta$, we have*

$$\ell(tw) = \ell(w) - \ell(t).$$

Proof. Let $w \in {}^\Theta W$ and $t \in W_\Theta$. Then $w_\Theta w$ is the shortest element in the right W_Θ-coset of w. Moreover, by 2,

$$\begin{aligned}\ell(tw) &= \ell((tw_\Theta)(w_\Theta w)) = \ell(tw_\Theta) + \ell(w_\Theta w)\\ &= \ell(w_\Theta) - \ell(t) + \ell(w_\Theta w) = \ell(w) - \ell(t).\square\end{aligned}$$

Let B be a Borel subgroup of G and P_Θ the standard parabolic subgroup of G of type Θ containing B. Then the P_Θ-orbits in X are B-invariant, and therefore unions of Bruhat cells in X. More precisely, we have the following result.

Lemma 9. *Let O be a P_Θ-orbit in X and $C(w) \subset O$. Then*

$$O = \bigcup_{t \in W_\Theta} C(tw).$$

Proof. This follows from ([**LG**], Ch. IV, §2, no. 5, Prop. 2). □

Therefore, we have a bijection between $W_\Theta \backslash W$ and the set of P_Θ-orbits in X.

Let C be a right W_Θ-coset in W and O the corresponding P_Θ-orbit. Then

$$\dim O = \max_{t \in W_\Theta} \dim C(tw^C) = \max_{t \in W_\Theta} \ell(tw^C) = \ell(w^C),$$

by 8. Therefore, $C(w^C)$ is the open Bruhat cell in O. This implies the following result.

Proposition 10. *The map attaching to P_Θ-orbit O in the flag variety X the unique Bruhat cell $C(w)$ open in O is a bijection between the set of all P_Θ-orbits in X and the set of Bruhat cells $C(w)$ with $w \in {}^\Theta W$.*

Finally we want to give a geometric interpretation of the order relation on $W_\Theta \backslash W$.

Proposition 11. *Let $C \in W_\Theta \backslash W$. Let O be the P_Θ-orbit in X corresponding to C. Then the closure of O consists of all P_Θ-orbits in X corresponding to $D \leq C$.*

Proof. The Bruhat cell $C(w^C)$ is open in O. Since O is irreducible, $C(w^C)$ is dense in O and $\bar{O} = \overline{C(w^C)}$. Therefore, $\bar{O} = \cup_{v \leq w^C} C(v)$. On the other hand, $\bar{O}$ is a union of P_Θ-orbits. Let D correspond to a P_Θ-orbit in $\bar{O}$. Then $C(w^D)$ is in $\bar{O}$ and $w^D \leq w^C$, i. e. $D \leq C$. If $D \leq C$, we have $w^D \leq w^C$ and $C(w^D) \subset \overline{C(w^C)} = \bar{O}$. If O' is the orbit corresponding to D we get

$$O' \subset \overline{O'} = \overline{C(w^D)} \subset \overline{C(w^C)} = \bar{O}.\square$$

G.2 Generalized Verma Modules

Let $\mathfrak{b}_0$ be a fixed Borel subalgebra of $\mathfrak{g}$ and denote by B_0 the corresponding Borel subgroup of the simply connected covering group G of $\mathrm{Int}(\mathfrak{g})$. To each subset Θ of Π we associate a standard parabolic subalgebra $\mathfrak{p}_\Theta$ containing $\mathfrak{b}_0$ and denote by P_Θ the corresponding parabolic subgroup of G. Let $\mathfrak{q}_\Theta = [\mathfrak{p}_\Theta, \mathfrak{p}_\Theta]$ and Q_Θ the commutant of P_Θ.

Lemma 1. *(i) The unipotent radical N_Θ of P_Θ is the unipotent radical of Q_Θ.*

(ii) Let L_Θ be a Levi factor of P_Θ. Then the commutator subgroup S_Θ of L_Θ is a Levi factor of Q_Θ.

(iii) The P_Θ-orbits in X are also Q_Θ-orbits.

(iv) The stabilizer in P_Θ of $x \in X$ is connected.

(v) The stabilizer in Q_Θ of $x \in X$ is connected.

Proof. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_0$ and R the root system in $\mathfrak{c}^*$. We identify Θ with a subset of the set of simple roots in R which corresponds to Π under the specialization defined by $\mathfrak{b}_0$. Then $\mathfrak{p}_\Theta$ is spanned by $\mathfrak{c}$, and the root subspaces $\mathfrak{g}_\alpha$ corresponding to positive roots in R and rots of the form $-\beta$ where β is a sum of roots in Θ. This implies that the Lie algebra $\mathfrak{n}_\Theta$ of N_Θ is spanned by the root subspaces $\mathfrak{g}_\alpha$ corresponding to positive roots which are not sums of roots in Θ. Also, the Lie algebra $\mathfrak{l}_\Theta$ of a Levi factor L_Θ is spanned by $\mathfrak{c}$ and the root subspaces $\mathfrak{g}_\alpha$ which are sums of roots in Θ or their negatives. Put $\mathfrak{s}_\Theta = [\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$. Then, since $[\mathfrak{c}, \mathfrak{n}_\Theta] = \mathfrak{n}_\Theta$ we have

$$\mathfrak{q}_\Theta = [\mathfrak{p}_\Theta, \mathfrak{p}_\Theta] = [\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] + [\mathfrak{l}_\Theta, \mathfrak{n}_\Theta] + [\mathfrak{n}_\Theta, \mathfrak{n}_\Theta] = \mathfrak{s}_\Theta + \mathfrak{n}_\Theta.$$

By the conjugacy of Levi decompositions these results are independent of the choice of $\mathfrak{c}$. Since P_Θ is connected, the decompositions for groups follow immediately from the results for their Lie algebras. This completes the proof of (i) and (ii).

(iii) We see that Q_Θ and B_0 generate P_Θ. Let O be a P_Θ-orbit in X and $x \in O$. Then $B_x \cap B_0$ contain a common Cartan subgroup C of G, and N_0 and C generate B_0. This implies that Q_Θ and C generate P_Θ, and the Q_Θ-orbit of x agrees with O.

The stabilizer $P_\Theta \cap B_x$ is the semidirect product of the Cartan subgroup C with the unipotent radical $P_\Theta \cap N_x$ of the stabilizer. Analogously, the stabilizer $Q_\Theta \cap B_x$ is the semidirect product of a Cartan subgroup in S_Θ with the unipotent radical $Q_\Theta \cap N_x$ of the stabilizer. This immediately implies the statements (iv) and (v). □

Clearly, $(\mathfrak{g}, Q_\Theta)$ is a Harish-Chandra pair for any $\Theta \subset \Pi$. We want to analyze the categories $\mathcal{M}_{coh}(\mathcal{D}_\lambda, Q_\Theta)$. First we consider standard Harish-Chandra sheaves.

Let $\lambda \in \mathfrak{h}^*$. Fix a P_Θ-orbit O in X. Let $i_0 : O \to X$ be the natural immersion of O into X. The homogeneous twisted sheaf of differential operators $\mathcal{D}_\lambda$

on X defines a P_Θ-homogeneous twisted sheaf of differential operators $(\mathcal{D}_\lambda)^{i_O}$ on O. Then the stabilizer of x in Q_Θ is connected by 1.(v). Therefore, there exists at most one irreducible Q_Θ-homogeneous $(\mathcal{D}_\lambda)^{i_O}$-connection on O.

Put

$$P(\Sigma_\Theta) = \{\lambda \in \mathfrak{h}^* \mid \alpha\check{}(\lambda) \in \mathbb{Z} \text{ for } \alpha \in \Theta\}.$$

Lemma 2. *Let $\lambda \in \mathfrak{h}^*$. Let O be a P_Θ-orbit and $w \in {}^\Theta W$ such that the Bruhat cell $C(w)$ is open in O. Then the following conditions are equivalent:*

(i) $w\lambda \in P(\Sigma_\Theta)$;

(ii) there exist an irreducible Q_Θ-homogeneous $(\mathcal{D}_\lambda)^{i_O}$-connection on O.

If such connection exists, it is unique.

Proof. Let $x \in C(w)$. Then $\mathfrak{b}_x$ is a Borel subalgebra in $\mathfrak{g}$ in relative position w with respect to $\mathfrak{b}_0$. Let $\mathfrak{c}$ be a Cartan subalgebra contained in $\mathfrak{b}_x \cap \mathfrak{b}_0$. Then, with respect to specialization s defined by $\mathfrak{b}_0$, $\mathfrak{n}_x$ is spanned by root subspaces $\mathfrak{g}_\alpha$ corresponding to $\alpha \in w(\Sigma^+)$. By 1.(v), the stabilizer of x in Q_Θ is the connected subgroup with the Lie algebra $\mathfrak{q}_\Theta \cap \mathfrak{b}_x$. Therefore, with respect to our specialization, the Lie algebra of the stabilizer is spanned by $\mathfrak{q}_\Theta \cap \mathfrak{c}$ and $\mathfrak{g}_\alpha$ with $\alpha \in \Sigma^+ \cap w(\Sigma^+)$ and $\alpha \in \Sigma_\Theta \cap w(\Sigma^+)$. Since $w \in {}^\Theta W$, we see that the second set is equal to $-\Sigma_\Theta^+$. Therefore, the stabilizer of x is the semidirect product of a Borel subgroup of S_Θ opposite to $B_0 \cap S_\Theta$ with the normal subgroup $N_0 \cap N_x$. This implies that the differential of the representation of the stabilizer which determines an irreducible Q_Θ-homogeneous $(\mathcal{D}_\lambda)^{i_O}$-connection on O must be given by the restriction of the specialization of an element in $P(\Sigma_\Theta)$. The specialization s' defined by $\mathfrak{b}_x$ is equal to $s \circ w$. Hence, the linear form which determines the connection is $s'(\lambda + \rho) = s(w\lambda + w\rho)$ and this implies that $w\lambda$ must be in $P(\Sigma_\Theta)$. □

This implies that for any $\lambda \in \mathfrak{h}^*$ there exists at most one standard Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, Q_\Theta)$ attached to the orbit O. We denote it by $\mathcal{I}(O, \lambda)$, and its unique irreducible Harish-Chandra subsheaf we denote by $\mathcal{L}(O, \lambda)$. First, we observe that the irreducible Harish-Chandra sheaves $\mathcal{L}(O, \lambda)$ are actually isomorphic to the irreducible modules we encountered before. More precisely, we have the following result.

Proposition 3. *Let O be a P_Θ-orbit in X and $C(w)$ the Bruhat cell open in O. Let $w\lambda \in P(\Sigma_\Theta)$. Then $\mathcal{L}(O, \lambda) = \mathcal{L}(w, \lambda)$.*

Proof. Let $j : C(w) \to O$ be the natural immersion. Denote by τ the unique irreducible Q_Θ-homogeneous $(\mathcal{D}_\lambda)^{i_O}$-connection on O. Then its restriction to $C(w)$ is an irreducible N_0-connection and therefore isomorphic to $\mathcal{O}_{C(w)}$. This implies that $\tau \subset R^0 j_+(\mathcal{O}_{C(w)})$. Hence,

$$\mathcal{I}(O, \lambda) = R^0 i_{O+}(\tau) \subset R^0 i_{O+}(R^0 j_+(\mathcal{O}_{C(w)})) = R^0 i_{w+}(\mathcal{O}_{C(w)}) = \mathcal{I}(w, \lambda).$$

Therefore, $\mathcal{L}(O, \lambda) \subset \mathcal{I}(w, \lambda)$. Since $\mathcal{L}(O, \lambda)$ is irreducible, it must be equal to $\mathcal{L}(w, \lambda)$. □

It remains to analyze $\mathcal{I}(O, \lambda)$. We use the method from V.2.

Lemma 4. *Let O be a P_Θ-orbit in X and $C(w)$ the Bruhat cell open in O. Then $R^p i_v^!(\mathcal{I}(O,\lambda)) = 0$ for all $v \in W$ such that $C(v) \not\subseteq O$ and $p \in \mathbb{Z}$. If $C(v) \subset O$, $R^p i_v^!(\mathcal{I}(O,\lambda)) = 0$ if $p \neq \ell(w) - \ell(v)$, and*

$$R^{\ell(w)-\ell(v)} i_v^!(\mathcal{I}(O,\lambda)) = \mathcal{O}_{C(v)}.$$

Proof. We use the notation from the preceding proof. Since O is affinely imbedded in X and $\mathcal{I}_O = R^0 i_{O+}(\tau)$, the first assertion follows from the base change.

If we denote by j_v the immersion of $C(v)$ into O, by the base change we conclude that

$$R^p i_v^!(\mathcal{I}(O,\lambda)) = R^p i_v^!(R^0 i_{O+}(\tau)) = R^p j_v^!(\tau)$$

for any $p \in \mathbb{Z}$. On the other hand,

$$R^p j_v^!(\tau) = L^{p-\dim O+\ell(v)} j_v^+(\tau),$$

what is nonzero only if $p - \dim O + \ell(v) = 0$, i. e. if $p = \dim O - \ell(v) = \ell(w) - \ell(v)$, since τ is a connection. Moreover, since τ is locally isomorphic to $\mathcal{O}_O$, $L^0 j_v^+(\tau)$ is an N_0-homogeneous connection on $C(V)$ locally isomorphic to $\mathcal{O}_{C(v)}$, i. e. $L^0 j_v^+(\tau) = \mathcal{O}_{C(v)}$. □

By 1.8, we have $\ell(vw) = \ell(w) - \ell(v)$ for $v \in W_\Theta$, hence we get

$$\nu(\mathcal{I}(O,\lambda)) = \sum_{v\in W}\sum_{m\in\mathbb{Z}} \dim_{\mathcal{O}}(R^m i_v^!(\mathcal{I}(O,\lambda)))\, q^m \delta_v = \sum_{v\in W_\Theta} q^{\ell(v)} \delta_{vw}.$$

As in the proof of V.2.10 this leads to the following result.

Proposition 5. *Let O be a P_Θ-orbit in X and $C(w)$ the Bruhat cell open in O. Then*

$$\chi(\mathcal{I}(O,\lambda)) = \sum_{v\in W_\Theta} (-1)^{\ell(v)} \chi(\mathcal{I}(vw,\lambda)).$$

In particular, if $\Theta = \Pi$, $P_\Theta = G$ acts transitively on X and the big cell is the Bruhat cell attached to the G-orbit X. Therefore, we have the following consequence.

Corollary 6.

$$\chi(\mathcal{O}(\lambda+\rho)) = \sum_{w\in W} (-1)^{\ell(w)} \chi(\mathcal{I}(ww_0,\lambda)).$$

Now we want to describe the highest weight modules which correspond to standard Harish-Chandra sheaves $\mathcal{I}(O,\lambda)$ under the equivalence of categories for regular antidominant λ. The first step is to construct some objects in the category of highest weight modules. Let

$$P_{++}(\Sigma_\Theta) = \{\nu \in P(\Sigma_\Theta) \mid \alpha\check{}(\lambda) \in \mathbb{Z}_+\}.$$

For any $\nu \in P_{++}(\Sigma_\Theta)$, if we use the specialization defined by $\mathfrak{b}_0$, there exists a unique irreducible finite-dimensional $\mathfrak{l}_\Theta$-module V^ν with highest weight ν. The action of $\mathfrak{s}_\Theta$ on this module is clearly the differential of a unique algebraic S_Θ-module action. Therefore, if we extend the actions to $\mathfrak{p}_\Theta$ and Q_Θ by assuming that they are trivial on $\mathfrak{n}_\Theta$ and N_Θ respectively, we can view V^ν as $(\mathfrak{p}_\theta, S_\Theta)$-module. For $\mu \in \mathfrak{h}^*$ such that $\mu - \rho \in P_{++}(\Sigma_\Theta)$, we define the *generalized Verma module*

$$M_\Theta(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\Theta)} V^{\mu-\rho},$$

here the $\mathfrak{g}$-action is given by left multiplication in the first factor and Q_Θ action is given as the tensor product of the adjoint action on the first factor and the natural action on the second factor. Clearly, $M_\Theta(\mu)$ is in $\mathcal{M}_{fg}(\mathcal{U}(\mathfrak{g}), Q_\Theta)$.

Lemma 7. *Let $\mu \in \mathfrak{h}^*$ such that $\mu - \rho \in P_{++}(\Sigma_\Theta)$. The module $M_\Theta(\mu)$ is the largest quotient of the Verma module $M(\mu)$ which is $\mathfrak{p}_\Theta$-finite.*

In particular, $M_\Theta(\mu)$ is a highest weight module with infinitesimal character χ_μ.

Proof. The $\mathfrak{p}_\Theta$-module $V^{\mu-\rho}$ is a quotient of the $\mathfrak{p}_\Theta$-module $\mathcal{U}(\mathfrak{p}_\Theta) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{\mu-\rho}$. Therefore, $M_\Theta(\mu)$ is a quotient of

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\Theta)} (\mathcal{U}(\mathfrak{p}_\Theta) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{\mu-\rho}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{\mu-\rho} = M(\mu).$$

Let N be a quotient of $M(\mu)$ which is $\mathfrak{p}_\Theta$-finite. Then it contains a vector v which is the image of the generator $1 \otimes 1 \in M(\mu)$. This vector is a weight vector of weight $\mu - \rho$ and it spans the one-dimensional weight subspace in N. Let N' be the finite-dimensional $\mathfrak{l}_\Theta$-submodule generated by v. Then N' is a direct sum of irreducible $\mathfrak{l}_\Theta$-submodules, and only one of these submodules can contain the weight subspace corresponding to the weight $\mu - \rho$. This implies that N' is actually irreducible and isomorphic to $V^{\mu-\rho}$. Therefore, the projection of $M(\mu)$ onto N factors through $M_\Theta(\mu)$. □

This implies that $M_\Theta(\mu)$ is in $\mathcal{M}_{fg}(\mathcal{U}_\theta, Q_\Theta)$ for $\theta = W \cdot \mu$.

We know that $\chi(M(\lambda))$, $\lambda \in \theta$, is a basis of the Grothendieck group $K(\mathcal{M}(\mathcal{U}_\theta, N_0))$. Therefore, we should be able to express $\chi(M_\Theta(\mu))$ in terms of $\chi(M(\lambda))$, $\lambda \in \theta$. By Poincaré-Birkhoff-Witt theorem, the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a free right $\mathcal{U}(\mathfrak{p}_\Theta)$-module for right multiplication. This implies that the *induction* functor $V \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\Theta)} V$ from the category of $\mathcal{U}(\mathfrak{p}_\Theta)$-modules into the category $\mathcal{U}(\mathfrak{g})$-modules is exact. If we consider the category of highest weight modules for $\mathfrak{s}_\Theta$ with respect to its Borel subalgebra $\mathfrak{b}_0 \cap \mathfrak{s}_\Theta$, we can define the natural functor to the category of $\mathcal{U}(\mathfrak{p}_\Theta)$-modules by extending the action to the center of $\mathfrak{l}_\Theta$ by a linear form and trivially to $\mathfrak{n}_\Theta$. The composition of this functor with the induction functor defines an exact functor from the category of highest weight modules for $\mathfrak{s}_\Theta$ into the category of highest weight modules for $\mathfrak{g}$. Therefore, it defines a morphism of the corresponding Grothendieck groups.

Next we need the following simple observation.

Lemma 8. *Let $\nu \in P(\Sigma)$ be a dominant weight and F^ν the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight ν. Then, in the Grothendieck category of highest weight modules for $\mathfrak{g}$ we have*

$$\mathrm{ch}(F^\nu) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}(M(w(\nu + \rho))).$$

Proof. The lowest weight of F^ν is $w_0\nu$. By 6. we have

$$\chi(\mathcal{O}(w_0\nu)) = \sum_{w \in W} (-1)^{\ell(w)} \chi(\mathcal{I}(ww_0, w_0\nu - \rho)).$$

By the equivalence of categories and V.1.14 this implies that

$$\mathrm{ch}(F^\nu) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}(I(w\nu - ww_0\rho)) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}(M(w(\nu + \rho))).\square$$

Proposition 9. *Let $\mu \in \mathfrak{h}^*$ such that $\mu - \rho \in P_{++}(\Sigma_\Theta)$. Then*

$$\mathrm{ch}(M_\Theta(\mu)) = \sum_{v \in W_\Theta} (-1)^{\ell(v)} \mathrm{ch}(M(v\mu)).$$

Proof. The length function of (W_Θ, S_Θ) is the restriction of the length function ℓ of (W, S) to W_Θ. Therefore, the statement follows from 6. applied to the representation $V^{\mu-\rho}$ of $\mathfrak{l}_\Theta$ and the preceding observation about the morphism of Grothendieck groups defined by the induction functor. □

Put $I_\Theta(\mu) = M_\Theta(\mu)\check{}$.

Theorem 10. *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Let O be a P_Θ-orbit in X and $w \in W$ be such that $C(w)$ is open in O. Assume that $w\lambda \in P(\Sigma_\Theta)$, so that $\mathcal{I}(O, \lambda)$ exists. Then:*

(i) $\alpha\check{}(w\lambda) \in \mathbb{Z}_+$ for $\alpha \in \Theta$;

(ii) if $\alpha\check{}(w\lambda) = 0$ for some $\alpha \in \Theta$, we have $\Gamma(X, \mathcal{I}(O, \lambda)) = 0$;

(iii) if $\alpha\check{}(w\lambda) \neq 0$ for $\alpha \in \Theta$, $w\lambda - \rho \in P_{++}(\Sigma_\Theta)$ and we have

$$\Gamma(X, \mathcal{I}(O, \lambda)) \cong I_\Theta(w\lambda).$$

All modules $I_\Theta(\mu)$, with $\mu \in W \cdot \lambda$ and $\mu - \rho \in P_{++}(\Sigma_\Theta)$, are obtained in this way.

Proof. Since $w \in {}^\Theta W$, (i) holds. Since the functor Γ is exact for antidominant λ, we have by 5. and V.1.14,

$$\begin{aligned}\operatorname{ch}(\Gamma(X,\mathcal{I}(O,\lambda))) &= \sum_{v\in W_\Theta}(-1)^{\ell(v)}\chi(\Gamma(X,\mathcal{I}(vw,\lambda)))\\ &= \sum_{v\in W_\Theta}(-1)^{\ell(v)}\operatorname{ch}(I(vw\lambda)).\end{aligned}$$

Hence, if $\alpha\check{}(w\lambda)=0$ for some $\alpha\in\Theta$, $s_\alpha(w\lambda)=w\lambda$ and $\operatorname{ch}(\Gamma(X,\mathcal{I}(Q,\lambda)))=0$. This clearly implies (ii).

If the assumption in (ii) doesn't hold, $\alpha\check{}(w\lambda-\rho)=\alpha\check{}(w\lambda)-1\in\mathbb{Z}_+$, hence $w\lambda-\rho\in P_{++}(\Sigma_\Theta)$. The restriction of the irreducible Q_Θ-homogeneous $(\mathcal{D}_\lambda)^{i_O}$-connection τ to $C(w)$ is equal to $\mathcal{O}_{C(w)}$, so we can identify $\mathcal{I}(O,\lambda)$ with a submodule of $\mathcal{I}(w,\lambda)$. Since Γ is exact for antidominant λ, this implies that $\Gamma(X,\mathcal{I}(O,\lambda))$ is a $\mathcal{U}_\theta$-submodule of $\Gamma(X,\mathcal{I}(w,\lambda))$. By V.1.14, it follows that $\Gamma(X,\mathcal{I}(O,\lambda))$ is a $\mathfrak{p}_\Theta$-finite submodule of $I(w\lambda)$. By dualizing the statement of 7, we see that $\Gamma(X,\mathcal{I}(O,\lambda))$ is a submodule of $I_\Theta(w\lambda)$. Finally, by the preceding calculation and 9, we conclude that

$$\operatorname{ch}(\Gamma(X,\mathcal{I}(O,\lambda)))=\operatorname{ch}(I_\Theta(w\lambda)),$$

and therefore $\Gamma(X,\mathcal{I}(O,\lambda))\cong I_\Theta(w\lambda)$.

Let $\mu\in W\cdot\lambda$, $\mu-\rho\in P_{++}(\Sigma_\Theta)$. This implies that $\alpha\check{}(\mu)\in\mathbb{N}$ for $\alpha\in\Theta$. Let $w\in W$ be such that $-\beta\check{}(\mu)\in\mathbb{Z}_+$ for all $\beta\in w(\Sigma_+)$. Then $\lambda=w^{-1}\mu$ is antidominant, and $\Theta\subset -w(\Sigma^+)$, i. e, $w\in {}^\Theta W$. □

Now we want to discuss the irreducibility of standard Harish-Chandra sheaves $\mathcal{I}(O,\lambda)$. First we need a result about the action of the intertwining functors.

Lemma 11. *Let O be a P_Θ-orbit in X and $C(w)$, $w\in {}^\Theta W$, the Bruhat cell open in O. Let $\alpha\in\Pi$ be such that $ws_\alpha\in {}^\Theta W$, $\ell(ws_\alpha)=\ell(w)+1$, and O' the corresponding P_Θ-orbit. Let $w\lambda\in P(\Sigma_\Theta)$. Then*

$$I_{s_\alpha}(\mathcal{I}(O,\lambda))=\mathcal{I}(O',s_\alpha\lambda)\ \textit{and}\ L^{-1}I_{s_\alpha}(\mathcal{I}(O,\lambda))=0.$$

Proof. First we remark that $(ws_\alpha)s_\alpha\lambda=w\lambda\in P(\Sigma_\Theta)$, hence the standard module $\mathcal{I}(O',s_\alpha\lambda)$ exists by 2.

Let p_α be the projection of X onto the generalized flag variety X_α of parabolic subalgebras of type α. Let Q be a P_Θ-orbit in X corresponding to $C\in W_\Theta\backslash W$. Then $p_\alpha(Q)$ is a P_Θ-orbit in X_α corresponding to the double coset $W_\Theta\backslash W/W_{s_\alpha}$ in W. Therefore, $p_\alpha^{-1}(p_\alpha(Q))$ consists of either:

(a) one P_Θ-orbit if $Cs_\alpha=C$;

(b) two P_Θ-orbits if $Cs_\alpha\neq C$.

By our assumption, $p_\alpha^{-1}(p_\alpha(O))=O\cup O'$ and $\dim O'=\dim O+1$. Hence,

$$\dim p_\alpha(O)=\dim p_\alpha^{-1}(p_\alpha(O))-1=\dim O.$$

Let $x'\in O$. Let $P_{p_\alpha(x')}$ be the stabilizer of $p_\alpha(x')$. Then the stabilizers $P_\Theta\cap B'_x$ and $P_\Theta\cap P_{p_\alpha(x')}$ have the same dimension, and since they are both connected, they must be equal. This implies that $p_\alpha^{-1}(p_\alpha(x')) \cap O$ consists only of x'. Therefore,

$$p_\alpha^{-1}(p_\alpha(x')) = \{x'\} \cup (p_\alpha^{-1}(p_\alpha(x')) \cap O'),$$

and $P_\Theta \cap P_{p_\alpha(x')}$ acts transitively on $p_\alpha^{-1}(p_\alpha(x')) \cap O'$. Therefore, we finally conclude that $P_\Theta \cap B_{x'}$ acts transitively on $p_\alpha^{-1}(p_\alpha(x')) \cap O'$, i. e. on the set of all Borel subalgebras in the relative position s_α with respect to $\mathfrak{b}_{x'}$.

Now we use the notation from L.3. Let $Z_{s_\alpha} \subset X \times X$ be the variety of ordered pairs of Borel subalgebras in relative position s_α. Since

$$\begin{aligned} &p_2^{-1}(O) \\ &=\{(x,x') \in X \times X \mid x' \in O,\ \mathfrak{b}_x \text{ in relative position } s_\alpha \text{ with respect to } \mathfrak{b}_{x'}\}, \end{aligned}$$

by the preceding remark we see that $p_2^{-1}(O)$ is a P_Θ-orbit of dimension $\dim O + 1$. This implies that $p_1(p_2^{-1}(O))$ is also a P_Θ-orbit, hence $p_1(p_2^{-1}(O)) = O'$. The stabilizer $P_\Theta \cap B_x$ of the point $x \in O'$ in P_Θ contains the stabilizer $P_\Theta \cap B_x \cap B_{x'}$ of $(x,x') \in p_2^{-1}(O)$. Since the dimensions of orbits are the same and the stabilizers connected, we conclude that p_1 induces an isomorphism of $p_2^{-1}(O)$ onto O'.

By L.3.2 and 2.3, we see that $p_2^{-1}(O)$ is affinely imbedded. Therefore, if we denote by j the affine immersion of $p_2^{-1}(O)$ into Z_{s_α} and by q_2 the morphism of $p_2^{-1}(O)$ into O induced by p_2, we get

$$p_2^+(\mathcal{I}(O,\lambda)) = p_2^+(R^0 i_{O+}(\tau)) = R^0 j_+(q^+(\tau))$$

by base change. This implies that

$$LI_{s_\alpha}(D(\mathcal{I}(O,\lambda))) = Rp_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} D(R^0 j_+(q^+(\tau)))) = R(p_1 \circ j)_+(\tau'),$$

where τ' is an irreducible Q_Θ-equivariant connection on $p_2^{-1}(O)$. The image of $p_1 \circ j$ is equal to O', and the map is an immersion. Therefore, $LI_{s_\alpha}(D(\mathcal{I}(O,\lambda)))$ is equal to $D(\mathcal{V})$, where $\mathcal{V}$ is a standard $\mathcal{D}_{s_\alpha\lambda}$-module attached to O'. By 2, this standard module is $\mathcal{I}(O', s_\alpha\lambda)$. □

Let $w \in {}^\Theta W$. Then $\Sigma_\Theta^+ \subset -w(\Sigma^+)$ and therefore $-w^{-1}(\Sigma_\Theta^+) \subset \Sigma^+$. If $w\lambda \in P(\Sigma_\Theta)$, we have $\Sigma_\Theta \subset \Sigma_{w\lambda}$ and $w^{-1}(\Sigma_\Theta) \subset \Sigma_\lambda$. This implies that

$$\Sigma_w^+ \cap \Sigma_\lambda \supset -w^{-1}(\Sigma_\Theta^+).$$

Theorem 12. *Let O be a P_Θ-orbit in X and $C(w)$ the Bruhat cell open in O. Let $w\lambda \in P(\Sigma_\Theta)$. Then the following conditions are equivalent:*

(i) the standard module $\mathcal{I}(O,\lambda)$ is irreducible;

(ii) $\Sigma_w^+ \cap \Sigma_\lambda = -w^{-1}(\Sigma_\Theta^+)$.

Proof. We prove this by induction in $\ell(w)$. If $\ell(w)$ is minimal, $w = w_0$ and O is the closed P_Θ-orbit in X. In this case $\mathcal{I}(O,\lambda)$ is allways irreducible. On the other hand, $w_0\lambda \in P(\Sigma_\Theta)$ is equivalent to $\lambda \in P(\Sigma_\Theta)$ and

$$\Sigma_{w_0}^+ \cap \Sigma_\lambda = \Sigma_\Theta^+ \cap \Sigma_\lambda = \Sigma_\Theta^+ = -w_0^{-1}(\Sigma_\Theta^+).$$

Assume now that the statement holds for $\ell(w) < k$. Let $\ell(w) = k$. Then, by 1.7, we can find $\alpha \in \Pi$ such that $w' = ws_\alpha \in {}^\Theta W$ and $\ell(w') = k - 1$. Let O' be the P_Θ-orbit corresponding to w'. Since $w's_\alpha\lambda = w\lambda \in P(\Sigma_\theta)$, the standard module $\mathcal{I}(O', s_\alpha\lambda)$ exists. Now, 11 implies that

$$I_{s_\alpha}(\mathcal{I}(O', s_\alpha\lambda)) = \mathcal{I}(O, \lambda),$$

and $L^{-1}I_{s_\alpha}(\mathcal{I}(O', s_\alpha\lambda)) = 0$.

Assume that (ii) holds. Since $\Sigma_w^+ = s_\alpha(\Sigma_{w'}^+) \cup \{\alpha\}$, we see that

$$\Sigma_{w'}^+ \cap \Sigma_{s_\alpha\lambda} = s_\alpha((\Sigma_w^+ - \{\alpha\}) \cap \Sigma_\lambda) \subset -s_\alpha w^{-1}(\Sigma_\Theta^+) = -{w'}^{-1}(\Sigma_\Theta^+).$$

By the discussion preceding the theorem, this implies that the above inclusion is an equality. Therefore, $\mathcal{I}(O', s_\alpha\lambda)$ is irreducible by the induction assumption, and $\alpha \notin \Sigma_\lambda$. This implies that $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda}) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories and therefore $\mathcal{I}(O, \lambda)$ is also irreducible.

Assume that $\mathcal{I}(O, \lambda)$ is irreducible. If $\alpha\check{}(\lambda)$ is not an integer, $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda}) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is an equivalence of categories and $\mathcal{I}(O', s_\alpha\lambda)$ must also be irreducible. By the induction assumption, we have $\Sigma_{w'}^+ \cap \Sigma_{s_\alpha\lambda} = -{w'}^{-1}(\Sigma_\Theta^+)$. Therefore,

$$\Sigma_w^+ \cap \Sigma_\lambda = s_\alpha(\Sigma_{w'}^+) \cap \Sigma_\lambda = s_\alpha(\Sigma_{w'}^+ \cap \Sigma_{s_\alpha\lambda}) = -s_\alpha {w'}^{-1}(\Sigma_\Theta^+) = -w^{-1}(\Sigma_\Theta).$$

If $\alpha\check{}(\lambda)$ is an integer, by L.7.3 and 11 we have the exact sequence

$$0 \to U^0 \to \mathcal{I}(O, \lambda) \to \mathcal{I}(O', \lambda) \to U^1 \to 0,$$

where the middle arrow is nontrivial. Since the support of $\mathcal{I}(O, \lambda)$ is larger than the support of $\mathcal{I}(O', \lambda)$, we have a contradiction with the assumption that $\mathcal{I}(O, \lambda)$ is irreducible. □

This result has the following consequence.

Theorem 13. *Let $\lambda \in P_{++}(\Sigma_\Theta)$ be regular. Then the following conditions are equivalent:*

(i) $M_\Theta(\lambda)$ is irreducible;

(ii) $\Sigma_\Theta^+ = \{\alpha \in \Sigma^+ \mid \alpha\check{}(\lambda) \in \mathbb{N}\}$.

Proof. Clearly, instead of $M_\Theta(\lambda)$ we can consider $I_\Theta(\lambda)$. Let $w \in {}^\Theta W$ be such that $w^{-1}\lambda$ is antidominant (such w exists by the proof of 10). Let O be the P_Θ-orbit attached to the left W_Θ-coset of w. Then $\Gamma(X, \mathcal{I}(O, w^{-1}\lambda)) = I_\Theta(\lambda)$, and by the equivalence of categories and 12, this module is irreducible if and only if $\Sigma_w^+ \cap \Sigma_{w^{-1}\lambda} = -w^{-1}(\Sigma_\Theta^+)$. This is equivalent with

$$\Sigma_{w^{-1}}^+ \cap \Sigma_\lambda = \Sigma_\Theta^+.$$

Let $\beta \in \Sigma_{w^{-1}}^+ \cap \Sigma_\lambda$. Then $\beta\check{}(\lambda) \in \mathbb{Z}$ and $-\beta \in w(\Sigma^+)$, hence $-w^{-1}\beta \in \Sigma^+$ and

$$-\beta\check{}(\lambda) = -(w^{-1}\beta)\check{}(w^{-1}\lambda) \in -\mathbb{N}.$$

Therefore, $\beta\check{}(\lambda) \in \mathbb{N}$. If $\beta\check{}(\lambda) \in \mathbb{N}$ for some $\beta \in \Sigma^+$, then $-w^{-1}\beta\check{}(w^{-1}\lambda) \in -\mathbb{N}$, and since $w^{-1}\lambda$ is antidominant, $-w^{-1}\beta \in \Sigma^+$. This implies that $\beta \in -w(\Sigma^+)$ and finally $\beta \in \Sigma^+_{w^{-1}} \cap \Sigma_\lambda$. Therefore,

$$\Sigma^+_{w^{-1}} \cap \Sigma_\lambda = \{\beta \in \Sigma^+ \mid \beta\check{}(\lambda) \in \mathbb{N}\}.\square$$

G.3 Kazhdan-Lusztig Algorithm for Generalized Verma Modules

Let $\Theta \subset \Pi$. In this section we assume that $\lambda = -\rho$ and denote

$$\mathcal{I}_O = \mathcal{I}(O, -\rho) \quad \text{and} \quad \mathcal{L}_O = \mathcal{L}(O, -\rho).$$

Consider the $\mathbb{Z}[q, q^{-1}]$-modules $\mathcal{H}$ introduced in V.2. For each right W_Θ-coset C in W we denote by w^c the longest element in this coset and by δ_C the element of $\mathcal{H}$ given by

$$\delta_C = \sum_{w \in W_\Theta} q^{\ell(v)} \delta_{vw^C}.$$

Let $\mathcal{H}_\Theta$ be the $\mathbb{Z}[q, q^{-1}]$-submodule of $\mathcal{H}$ spanned by δ_C, $C \in W_\Theta \backslash W$.

Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$. Then, by 1.6, we have the following three possibilities: $Cs_\alpha = C$; $Cs_\alpha > C$ and $Cs_\alpha < C$. We want now to calculate the action of T_α on δ_C in these cases.

Lemma 1. *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$. Then:*

(i) if $Cs_\alpha = C$, we have

$$T_\alpha(\delta_C) = \left(q + q^{-1}\right)\,\delta_C;$$

(ii) $Cs_\alpha > C$, we have

$$T_\alpha(\delta_C) = q\,\delta_C + \delta_{Cs_\alpha};$$

(iii) $Cs_\alpha < C$, we have

$$T_\alpha(\delta_C) = q^{-1}\,\delta_C + \delta_{Cs_\alpha}.$$

Proof. Consider first the case (i). In this case the left multiplication by s_α permutes the elements of C. Let

$$C_+ = \{w \in C \mid \ell(ws_\alpha) = \ell(w) + 1\}$$

and

$$C_- = \{w \in C \mid \ell(ws_\alpha) = \ell(w) - 1\}.$$

Then $C_+ s_\alpha = C_-$ and $C_- s_\alpha = C_+$. Therefore, if we denote by w^C the longest element in C, we have

$$\begin{aligned} T_\alpha(\delta_C) &= \sum_{v\in W_\Theta} q^{\ell(v)} T_\alpha(\delta_{vw^C}) \\ &= \sum_{vw^C\in C_+} q^{\ell(v)} T_\alpha(\delta_{vw^C}) + \sum_{vw^C\in C_-} q^{\ell(v)} T_\alpha(\delta_{vw^C}) \\ &= \sum_{vw^C\in C_+} q^{\ell(v)} (q\,\delta_{vw^C} + \delta_{vw^C s_\alpha}) + \sum_{vw^C\in C_-} q^{\ell(v)} (q^{-1}\,\delta_{vw^C} + \delta_{vw^C s_\alpha}). \end{aligned}$$

As we remarked before, if $vw^C \in C_+$, $vw^C s_\alpha \in C_-$, and therefore $vw^C s_\alpha = v'w^C$ with

$$\ell(v') = \ell(w^C) - \ell(v'w^C) = \ell(w^C) - \ell(vw^C s_\alpha) = \ell(w^C) - \ell(vw^C) - 1 = \ell(v) - 1.$$

Analogously, if $vw^C \in C_-$, $vw^C s_\alpha \in C_+$ and $vw^C s_\alpha = v'w^C$ with

$$\ell(v') = \ell(v) + 1.$$

Therefore,

$$\begin{aligned} T_\alpha(\delta_C) &= \sum_{vw^C\in C_+} q^{\ell(v)+1}\,\delta_{vw^C} + \sum_{v'w^C\in C_-} q^{\ell(v')+1} \delta_{v'w^C} \\ &+ \sum_{vw^C\in C_-} q^{\ell(v)-1} \delta_{vw^C} + \sum_{v'w^C\in C_+} q^{\ell(v')-1} \delta_{v'w^C} \\ &= \sum_{v\in W_\Theta} (q+q^{-1}) q^{\ell(v)}\,\delta_{vw^C} = \left(q+q^{-1}\right)\,\delta_C. \end{aligned}$$

In case (ii), by 1.6, we have $w^C s_\alpha = w^{Cs_\alpha}$. Therefore,

$$T_\alpha(\delta_C) = \sum_{v\in W_\Theta} q^{\ell(v)} T_\alpha(\delta_{vw^C}) = \sum_{v\in W_\Theta} q^{\ell(v)} (q\,\delta_{vw^C} + \delta_{vw^C s_\alpha}) = q\,\delta_C + \delta_{Cs_\alpha}.$$

In case (iii), by 1.6, we have $w^c s_\alpha = w^{Cs_\alpha}$. Therefore,

$$\begin{aligned} T_\alpha(\delta_C) &= \sum_{v\in W_\Theta} q^{\ell(v)} T_\alpha(\delta_{vw^C}) \\ &= \sum_{v\in W_\Theta} q^{\ell(v)} (q^{-1}\,\delta_{vw^C} + \delta_{vw^C s_\alpha}) = q^{-1}\,\delta_C + \delta_{Cs_\alpha}. \square \end{aligned}$$

Corollary 2. *$\mathcal{H}_\Theta$ is invariant under T_α, $\alpha \in \Pi$.*

Lemma 3. *Let $C \in W_\Theta \backslash W$. Then $\varphi(w^C) \in \mathcal{H}_\Theta$ and*

$$\varphi(w^C) = \delta_C + \sum_{D<C} P_{w^C\,w^D} \delta_D.$$

Proof. Let O be the P_Θ-orbit corresponding to w^C. Then, by 2.3, we know that $\mathcal{L}_{w^C} = \mathcal{L}_O$. Let O' be a P_Θ-orbit in $\bar{O}$. Denote by $i_{O'}$ the natural inclusion of

O' into X. Then $R^p i^!_{O'}(\mathcal{L}_O)$ is a P_Θ-equivariant connection on O', i. e. a sum of copies of $\mathcal{O}_{O'}$. Assume that $D \in W_\Theta \backslash W$ corresponds to O'. Then

$$O' = \bigcup_{v \in W_\Theta} C(vw^D).$$

Moreover, for any $v \in W_\Theta$, $C(vw^D)$ is a smooth subvariety of O of codimension $\ell(v)$. Therefore, if we denote by j the immersion of $C(vw^D)$ into O, we see that $R^p j^!(\mathcal{O}_O) = 0$ for $p \neq \ell(v)$, and $R^{\ell(v)} j^!(\mathcal{O}_O) = \mathcal{O}_{C(vw^D)}$. This implies that, for any $p \in \mathbb{Z}$,

$$R^p i^!_{vw^D}(\mathcal{L}_O) = R^{\ell(v)} j^!(R^{p-\ell(v)} i^!_{O'}(\mathcal{L}_O)).$$

Therefore,

$$\dim_{\mathcal{O}} R^p i^!_{w^D}(\mathcal{L}_O) = \dim_{\mathcal{O}} R^p i^!_{O'}(\mathcal{L}_O)$$

and

$$\dim_{\mathcal{O}} R^p i^!_{vw^D}(\mathcal{L}_O) = \dim_{\mathcal{O}} R^{p-\ell(v)} i^!_{O'}(\mathcal{L}_O) = \dim_{\mathcal{O}} R^{p-\ell(v)} i^!_{w^D}(\mathcal{L}_O),$$

for any $p \in \mathbb{Z}$ and $v \in W_\Theta$. This implies that

$$\begin{aligned}
\varphi(w^C) = \nu(\mathcal{L}^C_w) = \nu(\mathcal{L}_O) &= \sum_{w \in W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_w(\mathcal{L}_O))\, q^m\, \delta_w \\
&= \sum_{v \in W_\Theta} \sum_{D \in W_\Theta \backslash W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^{m-\ell(v)} i^!_{w^D}(\mathcal{L}_O))\, q^m\, \delta_{vw^D} \\
&= \sum_{v \in W_\Theta} \sum_{D \in W_\Theta \backslash W} \sum_{p \in \mathbb{Z}} \dim_{\mathcal{O}}(R^p i^!_{w^D}(\mathcal{L}_O))\, q^{p+\ell(v)}\, \delta_{vw^D} \\
&= \sum_{D \in W_\Theta \backslash W} \sum_{p \in \mathbb{Z}} \dim_{\mathcal{O}} R^p i^!_{w^D}(\mathcal{L}_O)\, q^p\, \delta_D = \sum_{D \in W_\Theta \backslash W} P_{w^C\, w^D}\, \delta_D. \square
\end{aligned}$$

In the following, we put

$$\varphi(C) = \varphi(w^C)$$

for $C \in W_\Theta \backslash W$.

Finally, we remark the following fact.

Lemma 4. *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$. Then*

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D)$$

for some $c_v \in \mathbb{Z}$.

Proof. By 2. and 3. we know that $T_\alpha(\varphi(Cs_\alpha)) \in \mathcal{H}_\Theta$, i. e.

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \in W_\Theta \backslash W} Q_D \delta_D = \sum_{v \in W_\Theta} \sum_{D \in W_\Theta \backslash W} Q_D q^{\ell(v)} \delta_{vw^D}$$

with $Q_D \in \mathbb{Z}[q,q^{-1}]$. On the other hand,by 1.6 we know that $w^{Cs_\alpha} = w^C s_\alpha$ and $\ell(w^C s_\alpha) = \ell(w^C) - 1$, hence, by V.2.1, we have

$$T_\alpha(\varphi(Cs_\alpha)) = T_\alpha(\varphi(w^C s_\alpha)) = \sum_{v \leq w^C} c_v \varphi(v).$$

This implies that Q_D are in $\mathbb{Z}[q]$. By evaluating these expressions at 0 we get

$$\sum_{D \in W_\Theta \backslash W} Q_D(0)\delta_{w^D} = T_\alpha(\varphi(Cs_\alpha))(0) = \sum_{v \leq w^C} c_v \delta_v.$$

This shows that $c_v \neq 0$ implies that $v = w^D$ for some $D \in W_\Theta \backslash W$. Therefore, $\varphi(v) = \varphi(D)$ and $w^D \leq w^C$, i. e. $D \leq C$. □

This leads to the following result generalization of V.2.1.

Theorem 5. *There exists a unique function $\varphi : W_\Theta \backslash W \longrightarrow \mathcal{H}_\Theta$, such that the following properties are satisfied:*

(i) for $C \in W_\Theta \backslash W$ we have

$$\varphi(C) = \delta_C + \sum_{D < C} P_{CD}\delta_D.$$

where $P_{CD} \in q\mathbb{Z}[q]$;

(ii) for $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$ such that $Cs_\alpha \neq C$ and $\ell(w^C s_\alpha) = \ell(w^C) - 1$, there exist $c_D \in \mathbb{Z}$, which depend on α and C, such that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D).$$

The polynomials P_{CD} are given by the Kazhdan-Lusztig polynomials for (W,S) by

$$P_{CD} = P_{w^C\, w^D}$$

for $C, D \in W_\Theta \backslash W$, $D \leq C$.

Proof. We already established the existence. It remains to prove the uniqueness. This part of the argument is analogous to the proof of uniqueness in V.2.1. First we can assume that $\Theta \neq \Pi$, since in the case $\Theta = \Pi$ we have $W_\Theta = W$ and the proof is trivial.

The proof is by induction in $\ell(w^C)$. The function $C \longmapsto \ell(w^C)$ attains its minimal value on w_Θ and in this case $C = W_\Theta$. Clearly, (i) implies that $\varphi(W_\Theta) = \delta_{W_\Theta}$ and (ii) is void in this case.

Take $C \in W_\Theta \backslash W$ such that $\ell(w^C) > \ell(w_\Theta)$. By the induction assumption, φ is uniquely determined on $D \in W_\Theta \backslash W$ with $\ell(w^D) < \ell(w^C)$. Then, by 1.7, we can find a simple root α such that $Cs_\alpha < C$. By 1.6, we have then $\ell(w^C s_\alpha) = \ell(w^C) - 1$.

By (ii) we know that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \le C} c_D \varphi(D),$$

and, by evaluating at $q = 0$ and using (i),

$$T_\alpha(\varphi(Cs_\alpha))(0) = \sum_{D \le C} c_D \delta_D.$$

By the induction assumption, the left side is uniquely determined. This implies that c_D are uniquely determined. On the other hand, if we put $C' = Cs_\alpha$, we have

$$\begin{aligned} T_\alpha(\varphi(Cs_\alpha)) &= T_\alpha(\delta_{C'} + \sum_{D < C'} P_{C'D}\delta_D) \\ &= T_\alpha(\delta_{C'}) + \sum_{D<C'} P_{C'D} T_\alpha(\delta_D) = q\delta_{C'} + \delta_C + \sum_{D<C'} P_{C'D} T_\alpha(\delta_D). \end{aligned}$$

By the construction, $\ell(w^D) < \ell(w^{C'}) = \ell(w^C) - 1$. Hence, terms in the expansion of $T_\alpha(\delta_D)$ can involve only $\delta_{D'}$ with $\ell(w^{D'}) \le \ell(w^C) - 1$. In particular, they cannot involve δ_C. This implies that $c_C = 1$. But this yields to

$$\varphi(C) = T_\alpha(\varphi(Cs_\alpha)) - \sum_{D<C} c_D \varphi(D),$$

which proves the uniqueness of $\varphi(C)$. □

Theorem 6. *Let P_{CD}, $C, D \in W_\Theta \backslash W$, be the polynomials of from 5. For $C \in W_\Theta / W$, denote by O_C the corresponding P_Θ-orbit. Then*

$$\chi(\mathcal{L}_{O_C}) = \chi(\mathcal{I}_{O_C}) + \sum_{D<C} P_{CD}(-1)\chi(\mathcal{I}_{O_D}).$$

Proof. Since $\mathcal{I}_{O_C}$ contains $\mathcal{L}_{O_C}$ as the unique irreducible submodule, and all other composition factors are $\mathcal{L}_{O_D}$ for $D < C$, we see that $\chi(\mathcal{I}_{O_C})$, $C \in W_\Theta \backslash W$, form a basis of $K(\mathcal{M}_{coh}(\mathcal{D}_X, P_\Theta))$. Hence

$$\chi(\mathcal{L}_{O_C}) = \sum_{D \le C} \lambda_{CD}\, \chi(\mathcal{I}_{O_D})$$

with $\lambda_{CD} \in \mathbb{Z}$. Since $\nu(-1)$ factors through $K(\mathcal{M}_{coh}(\mathcal{D}_X, N_0))$ and by 2.6, $\nu(\mathcal{I}_D)(-1) = \delta_D(-1)$ for $D \in W_\Theta \backslash W$, what leads to

$$\nu(\mathcal{L}_{O_C})(-1) = \sum_{D \le C} \lambda_{CD} \nu(\mathcal{I}_{O_D})(-1) = \sum_{D \le C} \lambda_{CD} \delta_D(-1).$$

Hence, from definition of P_{CD} it follows that $\lambda_{CC} = 1$ and $P_{CD}(-1) = \lambda_{CD}$. □

This gives an effective algorithm to calculate the multiplicities of irreducible modules in generalized Verma modules for infinitesimal character χ_ρ.

We can order the elements of $W_\Theta\backslash W$ by an order relation compatible with the Bruhat order. Then the matrix $(\lambda_{CD}; C, D \in W_\Theta\backslash W)$ is lower triangular with 1 on the diagonal. If $(\mu_{CD}; C, D \in W_\Theta\backslash W)$ are the coefficients of its inverse matrix, we see from 6. that

$$\begin{aligned}
\chi(\mathcal{I}_{O_C}) &= \sum_{E\in W_\Theta\backslash W} \sum_{D\in W_\Theta\backslash W} \mu_{CD}\lambda_{DE}\ \chi(\mathcal{I}_{O_E}) \\
&= \sum_{D\in W_\Theta\backslash W} \mu_{CD} \left(\sum_{E\in W_\Theta\backslash W} \lambda_{DE}\ \chi(I_{O_E}) \right) \\
&= \sum_{D\in W_\Theta\backslash W} \mu_{CD}\ \chi(\mathcal{L}_{O_D}) = \sum_{D\leq C} \mu_{CD}\ \chi(\mathcal{L}_{O_D})
\end{aligned}$$

and $\mu_{CC} = 1$ for any $C \in W_\Theta\backslash W$. Hence, from 2.12, 2.5 and V.1.19, we finally get the following result.

Corollary 7. *The multiplicity of irreducible module $L(-v\rho)$, $v \in {}^\Theta W$, in the generalized Verma module $M_\Theta(-w\rho)$, $w \in {}^\Theta W$, is equal to μ_{CD} where $C, D \in W_\Theta\backslash W$ are the cosets of w and v respectively.*