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Group Actions, Homeomorphisms, and Matching: A General Framework

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Abstract. This paper constructs metrics on the space of images $\mathcal{I}$ defined as orbits under group actions $\mathcal{G}$. The groups studied include the finite dimensional matrix groups and their products, as well as the infinite dimensional diffeomorphisms examined in Trouvé (1999, *Quaterly of Applied Math.*) and Dupuis et al. (1998). *Quaterly of Applied Math.*). Left-invariant metrics are defined on the product $\mathcal{G} \times \mathcal{I}$ thus allowing the generation of transformations of the background geometry as well as the image values. Examples of the application of such metrics are presented for rigid object matching with and without signature variation, curves and volume matching, and structural generation in which image values are changed supporting notions such as tissue creation in carrying one image to another.

Keywords: image warping, Riemannian metrics, groups of deformations, invariance

1. Introduction

In a seminal series of works (see, in particular, Grenander, 1993), Ulf Grenander and his co-workers have introduced the notion of group action within models of objects. In addition to the essential issue of modeling objects in a way which is tolerant to the action of some parasitic group (rotation, translation, etc.), which has been the subject of a wide range of research in computer vision, the idea was to introduce the group actions in the very nature of the objects themselves, through the notion of *deformable templates*. Roughly speaking, a deformable template simply is an "object" I_{temp} on which a group $\mathcal{G}$ acts and generates, through the orbit $\mathcal{I} = \mathcal{G} \cdot I_{\text{temp}}$ a whole family of new objects.

The interest of this approach is to concentrate most of the modeling effort on the group $\mathcal{G}$, and not on the family of objects $\mathcal{I}$. As a consequence, by studying a sufficiently generic group $\mathcal{G}$, one is able to design a large variety of models of objects, sometimes in very different contexts. In a statistical setting, this implies two modeling phases: i) build a probability model on a group $\mathcal{G}$; ii) for an arbitrary template I_{temp}, deduce, by projection, the corresponding probability model on $\mathcal{I} = \mathcal{G} \cdot I_{\text{temp}}$. In addition, a third phase is also needed to model the object acquisition with an imaging device. Metrics on $\mathcal{I}$ are also deduced from metrics on $\mathcal{G}$ after projection (or "procrustean analysis").

Beside the standard linear actions, more general group actions have been introduced and studied specifically for image analysis. For example, in Grenander and Miller (1994), the action of $SO(2)^n$, the nth power of the group of plane rotations, on closed polygons with n vertices is defined, and a probability model describing the shapes of small organelles is devised from the deformation of a circle. In Grenander and Keenan (1991), an infinite dimensional group action is introduced to model deformations of plane curves, in polar representation (distance to the origin vs. rotating angle). In Younes (1998), a similar group action is introduced, acting on the representation $s \mapsto \tau(s)$ of a plane curve, s being the arc-length and $\tau(s)$ the unit tangent vector, with Bakircioglu et al. (1998) containing transformations of curves in three dimensions. In Trouvé (1999), a theoretical study of a large class of infinite dimensional group actions is studied (including in particular the groups of diffeomorphisms of a given manifold).

In this paper, we introduce a slightly different approach, in which features from both the acting group and the object space are used to build models of deformable objects. This can be seen as generalizing the more conventional deformable template paradigm without loosing the essential features of this theory. In particular it generalizes much of the previous work in which only geometric transformations were modelled in single orbits. We will work on the product space $\mathcal{A} = \mathcal{G} \times \mathcal{I}$, where $\mathcal{G}$ is a group acting on $\mathcal{I}$.[1] Such a generalization will enable us to make object-dependent computations of metrics (in the sense that the deformation effort will depend on the deformed object). The power of the approach will be, for example, to allow us to model variations in the image values themselves, not just in the geometry. Also, our point of view will allow us to formalize a "template-independence" property for probability models of deformable objects.

Figure 1 depicts the overall approach taken. Each curve depicts a particular object under geometric deformation. The metric distance takes into account both the distance between the geometric change as well as the image change, labelled pairs $(h, h \cdot J)$ and $(g, g \cdot I)$.

The paper is organized as follows. We start with an abstract formulation of our object comparison problem, and show how this relies on the design of left-invariant distances on the space of transformations and images. We then describe how such distances can be formally constructed in the case when $\mathcal{A}$ is provided with a differential structure.

The second part of the paper provides several examples of situations which can be plugged into our abstract setting. These examples either come from the deformable template literature, and are revisited according to our new point of view, or are original examples which are illustrated by experiments.

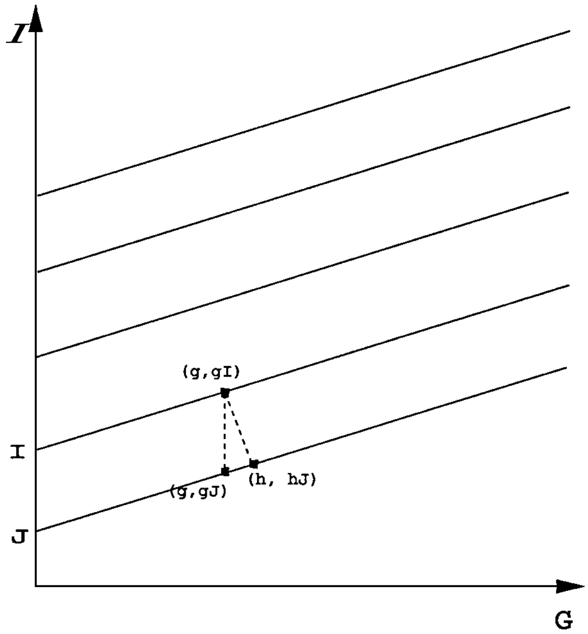


Figure 1. Shown in a comparison of objects I and J by looking for the smallest distance within the set $\mathcal{G} \times \mathcal{I}$ between the set $\mathcal{G} \cdot (\mathrm{id}, I)$ and $\mathcal{G} \cdot (\mathrm{id}, J)$.

2. Distances Between Registered Objects

2.1. *Introduction*

We consider the problem of object comparison in the presence of a group action. The set of object is $\mathcal{I}$ and $\mathcal{G}$ is the acting group. In most applications, an object in $\mathcal{I}$ can be considered as an element of some (often infinite dimensional) space, and can be readily equipped with a distance, d_0.[2] For example, images can be seen, in a continuous setting, as functions defined on a square $[0, 1]^2$, and taking values in $\mathbb{R}$, and they can be compared using any of the functional distances one may want to use, L^p norms, Sobolev norms, ... One of the typical features of this situation is that "small elements" in $\mathcal{G}$ may alter in a very significant way the observable aspect of elements of $\mathcal{I}$, that is, one may find $g \in \mathcal{G}$ and $I \in \mathcal{I}$ such that g is small and $d_0(I, g \cdot I)$ is large. Of course, one must be able to quantify what is meant by a small element in $\mathcal{G}$. This can be done by assuming a functional $g \mapsto \Gamma(g)$ such that $\Gamma(\mathrm{id}) = 0$ and considering that small actions are group elements g for which $\Gamma(g)$ is small.

If one is in a situation in which small elements in $\mathcal{G}$ are not supposed to alter the *essence* of the objects, in the way we want to understand them, the above mentioned distance in the measurement space is a poor candidate for object comparison. Our purpose here is to design distances on $\mathcal{I}$ which will not have this drawback in the sense that they are tolerant to small group actions. We emphasize the fact that we are looking for metrics (i.e. distances) d in $\mathcal{I}$, which must, in particular be symmetrical and satisfy the triangular inequality; in some applications, such a requirement can be unnecessary, but there are many cases in which this is essential, for example for the organisation of object databases.

This first section will essentially remain on an abstract level, and assume minimal structures on the sets which are introduced. We assume that we are given an *object set* $\mathcal{I}$, on which a *group* $\mathcal{G}$ is acting (i.e. we are given an operation $(g, I) \mapsto g \cdot I$, from $\mathcal{G} \times \mathcal{I}$ to $\mathcal{I}$, which satisfies $\mathrm{id} \cdot I = I$ and $(gh) \cdot I = g \cdot ((h \cdot I))$ (id denoting the identity element in $\mathcal{G}$).

2.2. *Least Action Distances*

We want to devise comparison tools between objects in $\mathcal{I}$ which take the action of $\mathcal{G}$ into account. More precisely, we want to be able to decide that two objects I_0 and I_1 in $\mathcal{I}$ are close when there exists an element $g \in \mathcal{G}$ such that $g \simeq \mathrm{id}$ and $g \cdot I_0 \simeq I_1$.

For this purpose, introduce the set of *registered objects*, $\mathcal{A} = \mathcal{G} \times \mathcal{I}$. The group $\mathcal{G}$ acts on $\mathcal{A}$ through the operation

$$h \cdot (g, I) = (hg, h \cdot I). \tag{1}$$

and we define $d(I, I')$ as the set distance between the orbits of (id, I) and (id, I') under this action, that is, we let

$$\begin{aligned} d(I, I') &= D(\mathcal{G} \cdot (\mathrm{id}, I), \mathcal{G} \cdot (\mathrm{id}, I')) \\ &= \inf\{d(a, a') : a = (g, g \cdot I) \\ a' &= (g', g' \cdot I'), g, g' \in \mathcal{G}\}. \end{aligned} \tag{2}$$

This measure obviously satisfies $d(I, I) = 0$ and is symmetric in I, I'. However, since it is a set distance, the triangular inequality cannot, in general, be inherited from D, unless these sets are in some sense "parallel". This property is expressed by the condition that D is left-invariant to the group action.

Definition 1. A distance D on $\mathcal{A}$ is left-invariant if, for all $h, g, g' \in \mathcal{G}$, all $I, I' \in \mathcal{I}$

$$D(h \cdot (g, I), h \cdot (g', I')) = D((g, I), (g', I')). \tag{3}$$

That this is the condition is stated in the next proposition.

Proposition 1. *Let D be a distance on $\mathcal{A}$ which is invariant by the left action of $\mathcal{G}$, then the function d, defined on $\mathcal{I} \times \mathcal{I}$ by*

$$d(I, I') = \inf\{D((g, gI), (g', g'I')) : g, g' \in \mathcal{G}\} \tag{4}$$

is symmetrical, satisfies the triangular inequality and is such that $d(I, I) = 0$ for all I.

If, moreover, the infimum is attained for all I, I' such that $d(I, I') = 0$, then D is a distance.

Note that if D is left invariant, Eq. (4) can also be written

$$d(I, I') = \inf\{D((\mathrm{id}, I), (g', g'I')), g' \in \mathcal{G}\}. \tag{5}$$

Proof: To demonstrate the triangular inequality is satisfied, we must show that, for all $g_1, g_1', g_2', g_2'' \in \mathcal{G}$ and $I, I', I'' \in \mathcal{I}$, there exists $h, h'' \in \mathcal{G}$ such that

$$\begin{aligned} &D((h, h \cdot I), (h'', h'' \cdot I'')) \\ &\quad \le D((g_1, g_1 \cdot I), (g_1', g_1' \cdot I')) \\ &\qquad + D((g_2', g_2' \cdot I'), (g_2'', g'' \cdot I_2)). \end{aligned} \tag{6}$$

But, by left invariance

$$\begin{aligned} &D((g_2', g_2' \cdot I'), (g_2'', g_2'' \cdot I_2)) \\ &\quad = D((g_1', g_1' \cdot I_1'), (h'', h'' \cdot I'')) \end{aligned}$$

with $h'' = g_1' g_2'^{-1} g_2''$, and inequality (6) with $h = g_1$ is the consequence of the triangular inequality for D.

The last statement of Proposition 1 is obvious. □

Remark 1. To relate the previous construction to the classical deformable template point of view in terms of defining distances between objects by minimal group action (as in Grenander (1993), Ch. 12, or Trouvé (1999)), the group H is defined which acts transitively on $\mathcal{I}$. That is for all $I, I' \in \mathcal{I}$, there exists $h \in H$ such that $h \cdot I = I'$. The group H can be interpreted in terms of our setting as follows. Since H is transitive, it describes all possible variations in $\mathcal{I}$. Thus, begin with the starting distance D on H. The objects in $\mathcal{I}$ can be compared in two different ways:

- Template-based comparison: given a reference object $I_0 \in \mathcal{I}$ (the template), one computes, for two objects $I, I' \in \mathcal{I}$, the closest elements $g, g' \in \mathcal{I}$ (for the distance D) which register I and I' to the template, i.e. such that $g^{-1} \cdot I = g'^{-1} \cdot I' = I_0$.
- Homogeneous comparison: one computes, for two objects $I, I' \in \mathcal{I}$, the closest elements $g, g' \in \mathcal{I}$ (for the distance D) which register I and I' to the same object, i.e. such that $g^{-1} \cdot I = g'^{-1} \cdot I'$.

Clearly, the homogenous case corresponds to taking the infimum of the template-based case with respect to all possible choices of the template. The rigorous definition of the corresponding distances are, in the template-based case:

$$d_{\mathrm{temp}}(I, I') = \inf\{D(h, h') : h^{-1} \cdot I = h'^{-1} \cdot I' = I_0\}$$

and in the homogeneous case

$$d_{\mathrm{hom}}(I, I') = \inf\{D(h, h') : h^{-1} \cdot I = h'^{-1} \cdot I'\}$$

The homogeneous case is the one which has been described in Grenander (1993) and Trouvé (1999), among others. The template-based case has been described in Younes (1999).

Sufficient conditions to obtain a distance are most restrictive in the homogeneous case. A similar proof to the one of proposition 1 shows that d_{hom} is a distance as soon as D is left invariant for the action of H on itself, that is, for all $h, g, g' \in H$, $D(hg, hg') = D(g, g')$. A sufficient condition for d_{temp} to be a distance is that D is invariant for the action of the stabilizer $\mathcal{G}$ of I_0 in H, defined by

$$\mathcal{G} = \{h \in H, h \cdot I_0 = I_0\}.$$

This is less restrictive condition than the full invariance required in the homogeneous case.

It is well known that the if group H acts transitively on the set $\mathcal{I}$, and $\mathcal{G}$ is the stabilizer of some $I_0 \in \mathcal{I}$, then the coset space $H/\mathcal{G}$ is in one-to-one correspondence with $\mathcal{I}$. This correspondence can be pushed further to identify H to $\mathcal{A} = \mathcal{G} \times \mathcal{I}$ (see Appendix), and left-invariant distance on H (for the action of $\mathcal{G}$) to left-invariant distances on $\mathcal{A}$. This correspondance permits us to place the template-based setting directly into the framework of Section 2.2. The correspondence between the usual deformable template modeling and our approach is thus done by replacing the "big" group H by the product $\mathcal{G} \times \mathcal{I}$, where $\mathcal{G}$ is the stabilizer of some reference element I_0 in $\mathcal{I}$. We here prefer to directly consider the situation when $\mathcal{G}$ and $\mathcal{I}$ are given, first because this is most of the time more natural in the applications, and leads to simpler computations, and also because this is slightly more general.

2.3. *Relation to Grenander Effort Functionals*

In Grenander (1993) the contruction of the distance d_{hom} described in the previous remark is made using effort functionals. Such a functional is a mapping $\Gamma : \mathcal{G} \to [0, +\infty[$, $\Gamma(g)$ measuring the cost associated to the left action of $g \in \mathcal{G}$ on any element of $\mathcal{I}$. We adapt this construction to our setting, and show how it relates to the above discussion. Given two objects I and I', one should have $d(I, I')$ small as soon as there exists $g \in \mathcal{G}$ such that $\Gamma(g)$ is small and $d_0(I, g^{-1}I')$ is small. An obvious initial guess for d would be to set something like

$$d(I, I') = \inf\{\Gamma(g) + d_0(I, g^{-1} \cdot I'), g \in \mathcal{G}\}$$

Although this provides a valid and robust measure of discrepancy for object comparison, this is not a distance; it is not symmetrical, and would not satisfy the triangular inequality. We shall however look for distances d under the form

$$d(I, I') = \inf\{U(g, I, I'), g \in \mathcal{G}\} \tag{7}$$

for a certain class of functions U which will be admissible; U has to be condidered as a cost function associated to the fact that one compares I to I' via the action of $\mathcal{G}$. We want to provide sufficient conditions on U. One should of course expect that U is non-negative, and $U(\mathrm{id}, I, I) = 0$ for all I. We have

$$d(I', I) = \inf\{U(g, I', I), g \in \mathcal{G}\}.$$

If the infimum in $d(I, I')$ is attained at some g, it is natural to assume that the minimum for $d(I', I)$ is attained at g^{-1}, which yields the symmetry condition:

$$U(g, I, I') = U(g^{-1}, I', I).$$

If one tries to check the triangular inequality, the following condition emerges naturally: for all g, h, I, I', I'',

$$U(hg, I, I'') \le U(g, I, I') + U(h, I', I'').$$

Definition 2. A function $U : \mathcal{G} \times \mathcal{I} \times \mathcal{I} \to [0, +\infty[$ is an effort functional if the following conditions are satisfied for all $g, h \in \mathcal{G}$ and all $I, I', I'' \in \mathcal{I}$:

(i) $U(\mathrm{id}, I, I) = 0$
(ii) $U(g, I, I') = U(g^{-1}, I', I)$
(iii) $U(hg, I, I'') \leq U(g, I, I') + U(h, I', I'')$

The above forementioned conditions are important because then such effort functionals define distances through Eq. (7). This result can be proved directly or simply deduced from Proposition 1 and the next proposition which states that effort functionals and left invariant distances are equivalent.

Proposition 2. *If D is a left-invariant distance on $\mathcal{A}$, then*

$$U(g, I, I') := D[(e, I), (g^{-1}, g \cdot I')]$$

is an effort functional.

Conversely, if U is an effort functional, then one gets a left invariant distance by letting

$$D[(g, I), (g', I')] := U(g'^{-1} \cdot g, g^{-1} \cdot I, g'^{-1} \cdot I')$$

3. Geodesic Distances

3.1. Principles

In general, there is no obvious way for building invariant distances D on $\mathcal{G} \times \mathcal{I}$. Although we shall review some examples for which such a distance can be defined in closed form, the construction requires most of the time using a variational approach in which D would be a geodesic distance.

In cases for which some differential structure exists on $\mathcal{A}$ this generic approach consists in building a left-invariant measure of infinitesimal variations on $\mathcal{A}$, define on this basis the energies of paths on $\mathcal{A}$, and compute the distance by minimizing these energies.[3]

In this section, we give the principle for this construction, without entering into technical details. This discussion is valid for any smooth set $\mathcal{C}$ on which a smooth group $\mathcal{G}$ is acting, and we fix such a set.

Let a differentiable path on $\mathcal{C}$ be a continuous function $\mathbf{a} : [0, 1] \to \mathcal{C}$, for which the time derivative $\frac{d\mathbf{a}}{dt}$ is defined for all $t \in [0, 1]$. We assume that this time derivative can be given a precise meaning, which is, for example, the case when $\mathcal{C}$ is a subset of a Banach space, and will always be the case in the applications. If the derivative exists excepted possibly for a finite number of t, the path is piecewise differentiable.[4]

If $a \in \mathcal{C}$, let $T_a(\mathcal{C})$ be the set which contains all the $\frac{d\mathbf{a}}{dt}_{|t=t_0}$, for all $t_0 \in [0, 1]$ and all differentiable paths on $\mathcal{C}$ such that $\mathbf{a}(t_0) = a$. We want to define a cost function associated to a path (an energy) as an accumulation of infinitesimal efforts. These efforts will be associated to norms $(\|\cdot\|_a, a \in \mathcal{C})$, such that, for all a, $\|\cdot\|_a$ is a norm on $T_a(\mathcal{C})$. For a path $\mathbf{a}$ on $\mathcal{C}$, define

$$E(\mathbf{a}) = \int_0^1 \left\| \frac{d\mathbf{a}}{dt} \right\|^2_{\mathbf{a}(t)} dt. \tag{8}$$

The associated geodesic distance on $\mathcal{A}$ is defined by

$$D(a, a') = \inf\{\sqrt{E(\mathbf{a})}, \mathbf{a}(0) = a, \mathbf{a}(1) = a'\} \tag{9}$$

which can be shown to be a distance.

The left-invariance constraint for D can be ensured by a similar constraint on the norms $\| \cdot \|_a$. For this, we must make an additional simple assumption, which is that, *for any $g \in \mathcal{G}$ and any differentiable path $\mathbf{a}$ on $\mathcal{C}$, the left-translated path $g \cdot \mathbf{a} : t \mapsto g \cdot \mathbf{a}(t)$ also is a differentiable path.* This implies that, for all $g \in \mathcal{G}$ and for all $a \in \mathcal{C}$ we can define an invertible linear mapping, $d_a L_g : T_a(\mathcal{C}) \to T_{ga}(\mathcal{C})$, by the condition that, for all differentiable path $\mathbf{a}$ with $\mathbf{a}(0) = a$, one has

$$\left.\frac{dg\mathbf{a}}{dt}\right|_{t=0} = d_a L_g \cdot \left.\frac{d\mathbf{a}}{dt}\right|_{t=0}. \tag{10}$$

The norms $(\| \cdot \|_a,\ a \in \mathcal{C})$ are left-invariant if for all $a \in \mathcal{C}$, and for all differentiable paths $\mathbf{a}$ with $\mathbf{a}(t_0) = a$, for any $t_0 \in [0, 1]$

$$\left\| \left.\frac{d\mathbf{a}}{dt}\right|_{t=t_0} \right\|_a = \left\| d_a L_g \cdot \left.\frac{d\mathbf{a}}{dt}\right|_{t=t_0} \right\|_{ga}. \tag{11}$$

It is easily shown that, under this condition, the distance defined by (9) is left-invariant.

In the particular case when $\mathcal{C} = \mathcal{A} = \mathcal{G} \times \mathcal{I}$, condition (11) implies that it suffices to define $\|\cdot\|_a$ for elements $a \in \mathcal{A}$ of the kind $a = (\mathrm{id}\ , I)$ for some $I \in \mathcal{I}$. We thus need to define a $\mathcal{I}$-indexed family of norms $(\|\cdot\|_I,\ I \in \mathcal{I})$.

3.2. Functional Objects and Homeomorphisms

3.2.1. Context. In this section, we give a general formulation for the comparison of objects which are defined as mappings (curves, images, etc.). Specific examples will be provided in the next sections.

Fix a bounded domain $\Omega \subset \mathbb{R}^k$. Its closure, $\bar{\Omega}$ will be referred to as the *measurement space.* An *object* is defined as a family of measurements made over Ω. Each measurement provides a value in some set M, *the value space.* Each object therefore is represented as a mapping $I : \Omega \to M$, and the *object space* is some functional space $\mathcal{I}$, containing the *objects* of interest. To simplify, we shall consider that $M = \mathbb{R}^d$.

The group $\mathcal{G}$ generates *homeomorphic actions*: the elements of $\mathcal{G}$ are homeomorphisms $g : \bar{\Omega} \to \bar{\Omega}$ (i.e. invertible continuous mappings, with continuous inverses), subject to some smoothness properties and boundary conditions. A rigorous definition will be given in Section 3.2.4. Define, on $\mathcal{G}$, the product $g \cdot h = h \circ g$, and let the action on objects be $g \cdot I = I \circ g$.

3.2.2. Geodesic Distance on $\mathcal{G}$. Follow the previous discussion of Section 3.1 with $\mathcal{C} = \mathcal{G}$, since $\mathcal{G}$ is obviously acting on itself. Consider that a path[5] $\mathbf{g}$ on $\mathcal{G}$ is differentiable if the partial derivatives $\frac{\partial \mathbf{g}}{\partial t}$ are defined everywhere, and set

$$\left.\frac{d\mathbf{g}}{dt}\right|_{t=t_0} = \frac{\partial \mathbf{g}}{\partial t}(t_0, \cdot).$$

If $h \in \mathcal{G}$, since $h \cdot \mathbf{g}$ is the mapping $(t,\ x) \mapsto \mathbf{g}(t,\ h(x))$ giving

$$\frac{dh \cdot \mathbf{g}}{dt} = \left((t,\ x) \mapsto \frac{\partial \mathbf{g}}{\partial t}(t,\ h(x))\right).$$

This is the expression of the differential of the left-translation in (10). Equation (11) implies that the collection of norms $\|\cdot\|_g$ for $g \in G$ can be deduced from the knowledge of the norm for $g = \text{id}$. Let us assume that $\|\cdot\|_{\text{id}}$ is given, for a path $\mathbf{g}$ such that $\mathbf{g}(t_0) = \text{id}$, by

$$\left\|\frac{d\mathbf{g}}{dt}(t_0)\right\|_{\text{id}} = N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t_0, \cdot)\right)$$

where $N_{\mathcal{G}}$ is a functional norm on the space of functions $v : \Omega \mapsto \mathbb{R}^k$, for example a Sobolev norm, or a norm based on the expansion of the function on some orthonormal basis (Fourier, wavelets, ...).

Now, by left-translation invariance, the norms for any path $\mathbf{g}$ are uniquely specified; if $\mathbf{g}(t_0) = g$,

$$\left\|\frac{d\mathbf{g}}{dt}(t_0)\right\|_{g} = N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t_0, \mathbf{g}^{-1}(t, \cdot))\right)$$

and the energy of the path $\mathbf{g}$ simply is the integral of the above expression with respect to time. For a path $\mathbf{g}$ on $\mathcal{G}$, let $\mathbf{g}^{-1}(t, \cdot) = \mathbf{g}(t, \cdot)^{-1}$ implying

$$E(\mathbf{g}) = \int_0^1 N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{g}^{-1}(t, \cdot))\right)^2 dt$$

Setting

$$\mathbf{v}(t,\ x) = \frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{g}^{-1}(t,\ x)), \tag{12}$$

then

$$E(\mathbf{g}) = \int_0^1 N_{\mathcal{G}}(\mathbf{v}(t, \cdot))^2 dt \tag{13}$$

and the associated geodesic distance:

$$D(g, g') = \inf\{\sqrt{E(\mathbf{g})}, \mathbf{g}(0) = g, \mathbf{g}(1) = g'\}$$

is left-invariant for the action of $\mathcal{G}$ on itself.

3.2.3. Distance on $\mathcal{A}$. We now proceed to the general case for designing distances on $\mathcal{A}$ when the elements of $\mathcal{I}$ are functions $\Omega \to \mathbb{R}^d$. The principles are the same as above, simply the notation become somewhat more complex, since we are dealing with the larger space $\mathcal{A} = \mathcal{G} \times \mathcal{I}$. Consider paths $\mathbf{a} = (g, \mathbf{I})$ on $\mathcal{A}$, and set

$$\frac{d\mathbf{a}}{dt} = \left(\frac{\partial \mathbf{g}}{\partial t}, \frac{\partial \mathbf{I}}{\partial t}\right).$$

If $h \in \mathcal{G}$, this gives

$$\frac{dh \cdot \mathbf{a}}{dt} = \left[(t, x) \mapsto \left(\frac{\partial \mathbf{g}}{\partial t}(t, h(x)), \frac{\partial \mathbf{I}}{\partial t}(t, h(x))\right)\right].$$

To define the collection of norms $\|\cdot\|_g, I$, for $(g, I) \in \mathcal{A}$, because of left invariance it suffices to define them only in the case $g = \text{id}$ (but they still can depend on I). Assume that they are given for a path $\mathbf{a}$ such that $\mathbf{a}(t_0) = (\text{id}\ , I)$ by

$$\left\|\frac{d\mathbf{a}}{dt}(t_0)\right\|^2_{\text{id},\ I} = N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t_0, \cdot); I\right)^2 + N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial t}(t_0, \cdot); I\right)^2.$$

For each fixed I, $N_{\mathcal{G}}$ (resp. $N_{\mathcal{I}}$) is a functional norm on the space of functions $v : \Omega \mapsto \mathbb{R}^k$ (resp. $I : \Omega \mapsto \mathbb{R}^d$). If $N_{\mathcal{G}}$ and $N_{\mathcal{I}}$ do not depend on I, we shall say that we are in a case of *homogeneous deformations*. This exactly corresponds to the homogeneous case presented in Remark 1.

To compute the general expression on the norms, let $\mathbf{a}$ be a path, and set $\mathbf{a}(t_0) = (g, I)$. Then,

$$\begin{aligned}&\left\|\frac{da}{dt}(t_0)\right\|_{g,I}^2\\ &\quad = N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t_0, \mathbf{g}^{-1}(t,\cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t,\cdot))\right)^2\\ &\qquad + N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial t}(t_0, \mathbf{g}^{-1}(t,\cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t,\cdot))\right)^2,\end{aligned}$$

and the associated energy becomes

$$\begin{aligned}E(\mathbf{a}) = &\int_0^1 N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{g}^{-1}(t,\cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t,\cdot))\right)^2 dt\\ &+ \int_0^1 N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial t}(t, \mathbf{g}^{-1}(t,\cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t,\cdot))\right)^2 dt.\end{aligned}$$

To compute the associated distance on $\mathcal{A}$, $D((g_0, I_0), (g_1, I_1))$, the square root of this energy must be minimized over all $\mathbf{a}$ starting at (g_0, I_0) and ending at (g_1, I_1). This energy can be simplified; setting

$$\mathbf{v}(t,x) = \frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{g}^{-1}(t,x)), \tag{14}$$

$$\mathbf{J}(t,x) = \mathbf{I}(t, \mathbf{g}^{-1}(t,x)), \tag{15}$$

and assuming that $\mathbf{J}$ is differentiable in the variable x gives

$$\frac{\partial \mathbf{I}}{\partial t}(t,x) = \frac{\partial \mathbf{J}}{\partial t}(t, g(t,x)) + \frac{\partial \mathbf{J}}{\partial x}(t, g(t,x)) \cdot \frac{\partial \mathbf{g}}{\partial t}(t,x)$$

so that

$$\frac{\partial \mathbf{I}}{\partial t}(t, g^{-1}(t,x)) = \frac{\partial \mathbf{J}}{\partial t}(t,x) + \frac{\partial \mathbf{J}}{\partial x}(t,x) \cdot \mathbf{v}(t,x).$$

This last expression is the Lie derivative of $\mathbf{J}$ in the direction of the vector field $\mathbf{v}$ (sometimes called the *material derivative* of $\mathbf{J}$). Introducing these expressions in the energy E yields one of the main results of the paper.

Theorem 1. *Let $(N(\cdot \mid I), I \in \mathcal{I})$ and $(N(\cdot \mid I), I \in \mathcal{I})$ be two collections of norms. Associate to paths $t \mapsto \mathbf{v}(t,\cdot)$ and $t \mapsto \mathbf{J}(t,\cdot)$, where $\mathbf{v}(t,x) \in \mathbb{R}^k$ and $\mathbf{J}(t,x) \in \mathbb{R}^k$, the energy*

$$\begin{aligned}&E(\mathbf{v}, \mathbf{J})\\ &= \int_0^1 N_{\mathcal{G}}(\mathbf{v}(t,\cdot); \mathbf{J}(t,\cdot))^2 dt\\ &\quad + \int_0^1 N_{\mathcal{I}}\left(\frac{\partial \mathbf{J}}{\partial t}(t,\cdot) + \frac{\partial \mathbf{J}}{\partial x}(t,\cdot)\cdot \mathbf{v}(t,\cdot); \mathbf{J}(t,\cdot)\right)^2 dt.\end{aligned} \tag{16}$$

Then, the function

$$\begin{aligned}&d(I, I')\\ &\quad = \inf\{\sqrt{E(\mathbf{v},\mathbf{J})} : \mathbf{v}, \mathbf{J}, \mathbf{J}(0,\cdot) = I, \mathbf{J}(1,\cdot) = I'\}\end{aligned}$$

is a distance on $\mathcal{I}$.

Remark 2. The introduction of this time-dependent vector field in such a context (restricted to the case of homogeneous norms) has been independently proposed in Dupuis et al. (1998), and in Trouvé (1999). In this latter reference, it has been shown that this also provides a way to rigorously define a group $\mathcal{G}$ which shares many of the properties of finite dimensional Lie groups. We give a brief account of these results in Section 3.2.4.

In these references, the following matching problem has been considered: given I_0 and I_1, one minimizes

$$\int_0^1 \|\mathbf{v}(t,\cdot)\|^2\, dt + \int_\Omega |I_1(x) - I_0 \circ \mathbf{g}_{\mathbf{v}}^{-1}(1,x)|^2\, dx \tag{17}$$

(A greedy procedure also has been introduced in Christensen et al. (1996), for efficient minimization in 3 dimensions). The energy in (13) may in fact be used for the same purpose (inexact matching). One of its interests, compared to (17) is to depend only on $\mathbf{v}$ and on $\mathbf{J}$, and thus not to require to compute $\mathbf{g}_{\mathbf{v}}$ for the minimization. On the other hand, it requires the introduction of an extra time-dependent unknown, $\mathbf{J}$.

Remark 3. In Eq. (13) there are two competing components for the matching process. The first one acts on the background space Ω, and is represented by the variations in time, $\mathbf{v}$, of the homeomorphism $\mathbf{g}_{\mathbf{v}}$. The second one acts on the feature space, and can be considered as a variation of the material properties. The path $\mathbf{J}$ represents the evolution of the object, starting from I_0 and arriving at I_1. Notice that the matching path, $\mathbf{g}_{\mathbf{v}}$ does not appear anymore in (13), or in the boundary conditions, which drastically simplifies the design of numerical procedures.

Thus, the first integral in (13) penalizes large variations of the homeomorphism, and the second one penalizes variations in the material which are not due to the deformation, i.e. it penalizes violation of the transport equation. Both penalizations are expressed as norms which depend only on the current object $\mathbf{J}(t, \cdot)$. This a consequence of our left-invariance requirement (Eq. (11)). In (13), the energy does not track the accumulated stress from time 0. For this reason, we classify the matching procedure as *viscous matching* (as first suggested by Rabbitt, Christensen et al., 1996), in opposition to the methods involving elastic matching, Bajcsy et al. (1983), Dengler and Schmidt (1988), Dann et al. (1989), Bajcsy and Kovacic (1989), Miller et al. (1993), Gee et al. (1994), Rabbitt et al. (1995), Davatzikos (1997), Christensen et al. (1999), Christensen et al. (1999). From a point of view analogous to elasticity, it should be harder to make a small deformation of an object J if it is considered to already be a deformation of another object, than to operate the same deformation, but considering that J itself is at rest. Technically, this means that the norms $N_{\mathcal{G}}$ and $N_{\mathcal{I}}$ in (13) would depend, both on the deformed object, and on the current stress associated to $\mathbf{g_v}$, which implies that the left-invariance assumption has to be relaxed. Whether the left-invariance point of view is relevant or not should depend on the application, but we insist on the fact that without this condition, the measure of mismatch given by the minimum energy will not be a distance. Also, when comparing objects using elastic matching, one usually consider that one object is at equilibrium and the other one is deformed, and the result depends on the choice on which object is at equilibrium. This may be relevant in some cases, for example when comparing a face with an expression to a face without expression, but in applications dealing with shapes, silhouettes, or grey-level images of objects which are not generated apriori as deformations of a rest state, an invariant metric is more natural.

3.2.4. Groups of Transformations. We shall be studying the finite dimensional matrix groups as well as infinite dimensional groups of homeomorphisms and diffeomorphisms. The finite dimensional groups are standard. We now give a brief account of the way groups of homeomorphisms have been defined in Dupuis et al. (1998) and Trouvé (1999). Consider a Hilbert space $\mathcal{H}$ of mappings $v : \bar{\Omega} \to \mathbb{R}^k$, with a norm $v \mapsto N(v)$. Elements in $\mathcal{H}$ must satisfy regularity constraints, which can be summarized as follows.

We let $W^{1,\infty}(\Omega, \mathbb{R}^k)$ be the set of mappings $v : \bar{\Omega} \to \mathbb{R}^k$ which have bounded generalized derivative over $\bar{\Omega}$.

(C1) *The canonical inclusion* $i : \mathcal{H} \mapsto W^{1\infty}(\Omega, \mathbb{R}^k)$ *is continuous*, i.e. there exists a constant C_0 such that

$$\|Dv\|_\infty + \|v\|_\infty \leq C_0 \|v\|$$

where Dv is the matrix formeds by the partial derivatives of v.

We let $\mathcal{H}_0$ be the completion in $\mathcal{H}$ of the subset of $\mathcal{H}$ composed with all functions smooth v with compact support included in Ω. Because of (C1), elements of $\mathcal{H}_0$ are such that $v = 0$ and $\frac{dv}{dx} = 0$ on $\partial\Omega$.

We then define the set $\mathcal{K}_0$ of functions $\mathbf{v} : [0, 1] \times \Omega \to \mathbb{R}^k$ such that for all t, $\mathbf{v}(t) \in \mathcal{H}_0$ (where $\mathbf{v}(t)$ is as above the mapping $x \mapsto \mathbf{v}(t, x)$), and such that

$$\|\mathbf{v}\|^2 := \int_0^1 \|\mathbf{v}(t)\|^2 \, dt$$

is finite.

For all $x \in \Omega$, one can define the solution $\mathbf{g_v}(t, x)$ of the differential equation

$$\frac{dy}{dt} = \mathbf{v}(t, y)$$

with initial condition $\mathbf{g_v}(0, x) = x$. The following result is proved in Dupuis et al.(1998) and Trouvé (1999).

Theorem 2 (Dupuis et al., 1998; Trouvé, 1999)**.** *Assume* (C1) *and let* $\mathbf{v} \in \mathcal{K}_0$. *Then, for all* $x \in \Omega$, $\mathbf{g_v}(t, x)$ *exists for all* $t \in [0, 1]$. *Moreover, for all* $t \in [0, 1]$, $x \mapsto \mathbf{g_v}(t, x)$ *is a homeomorphism of* Ω.

Under condition (*C*1) *Trouvé* (1999), $G = A(\mathcal{K}_0)$ *is a group.*

In particular, the mapping $A : \mathbf{v} \mapsto \mathbf{g_v}(1, \cdot)$ associates to each $\mathbf{v} \in \mathcal{K}_0$ a homeomorphism of Ω. One can define $\mathcal{G}$ to be the image of $\mathcal{K}_0$ by this mapping: $\mathcal{G} = A(\mathcal{K}_0)$. The following results follow by standard manipulations on paths (concatenation and time reversal) Obviously, if $\mathbf{v} \in \mathcal{K}_0$, one has $\mathbf{g_v}(t, \cdot) \in \mathcal{G}$ for all $t \in [0, 1]$, so that $\mathbf{g_v}(t, \cdot)$ is a path in $\mathcal{G}$.

Remark 4. The previous approach can be used to address three different problems, listed below:

- Tolerant object comparison; this was the problem we started with: design a distance between objects in $\mathcal{I}$ which accounts for small variations due to the group action.
- Dense homeomorphic matching: this approach provides homeomorphisms which match the various features of the objects. The matching is dense, meaning that each point Ω has an homologous point, and, since it is defined as the solution of an O.D.E, it is also consistent, that is, it provides a homeomorphism in Ω.
- Morphing: the process $\mathbf{J}$ defined in (15) provides a smooth evolution between the initial object and the target.

Estimating the flow, we get the morphing between two objects, the value of the homeomorphism at time 1 providing the dense matching, and the square root of the energy of the optimal path providing the distance. For each of these three problems, there is an unsolved "existence" issue: For the first one, Proposition 1 requires the existence of the optimal matching for object $I = I'$ which are at null distance; the second one needs the existence of an optimal matching between two given objects, and the third one is concerned with the existence of the optimal path. Clearly, this last one is the more general one, and implies the two others. This issue will be addressed in some situations, in Appendix B

3.3. *Landmark Matching*

In this section, we consider the action of groups of diffeomorphisms on finite subsects of Ω. Our set $\mathcal{I}$ is thus composed with N-tuples of *distinct* points of Ω, and we shall write $I = (p_1, \ldots, p_N) \in \mathcal{I}$; $\mathcal{G}$ is a group of diffeomorphisms on Ω. For $g \in G$, and $I = (p_1, \ldots, p_N) \in \mathcal{I}$, we let $g \cdot I$ be given by $g \cdot I = (g^{-1}(p_1), \ldots, g^{-1}(p_N))$, that is, the inverse of the diffeomorphism g applied to each of the p_i. This is a group action, when the product on $\mathcal{G}$ is given, as before, by $g \cdot h = h \circ g$ (one has $(gh) \cdot I = g \cdot (hI)$).

We now follow the general approach developed in Section 3.1, and study paths on $\mathcal{G} \times \mathcal{I}$. Such paths take the form $t \mapsto \mathbf{a}(t) = (\mathbf{g}(t, \cdot)), \mathbf{I}(t)$ with, $\mathbf{I}(t))$, $\mathbf{I}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$. We shall also denote $\mathbf{a}(t) = (\mathbf{g}(t, \cdot), \mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$, relaxing in that way one pair or parentheses. We let the time derivative be

$$\frac{d\mathbf{a}}{dt} = \left(\frac{\partial \mathbf{g}}{\partial t}, \frac{d\mathbf{p}_1}{dt}, \ldots, \frac{d\mathbf{p}_N}{dt}\right)$$

If $h \in G$, we have, since $h \cdot \mathbf{a} = (\mathbf{g}\ (t, h(\cdot)),\ h^{-1}(\mathbf{p}_1), \ldots, h^{-1}(\mathbf{p}_N))$

$$\frac{dh \cdot \mathbf{a}}{dt} = \left(\frac{\partial \mathbf{g}}{\partial t}(t, h(\cdot)), \frac{dh^{-1}}{dx}\frac{d\mathbf{p}_1}{dt}, \ldots, \frac{dh^{-1}}{dx}\frac{d\mathbf{p}_N}{dt}\right)$$

The energy of a path $\mathbf{a}$ should now take the form

$$E(\mathbf{a}) = \int_0^1 \left\| \frac{d\mathbf{g}^{-1}\mathbf{a}}{dt} \right\|_{g^{-1}\cdot \mathbf{I}}$$

Separating, the norm $\|, \cdot\|_I$ as a norm $N_{\mathcal{G}}(\cdot; I)$ for variations in $\mathcal{G}$, and a norm $N_{\mathcal{I}}(\cdot; \mathcal{I})$ for variations in $\mathcal{I}$, and expanding the expression of the of the time derivative yields

$$E(\mathbf{a}) = \int_0^1 N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t} \circ \mathbf{g}^{-1}; \mathbf{g}^{-1} \cdot \mathbf{I}\right)^2 dt + \int_0^1 N_{\mathcal{I}}\left(\frac{\partial \mathbf{g}}{\partial x}\frac{d\mathbf{p}_1}{dt}, \ldots, \frac{\partial \mathbf{g}}{\partial x}\frac{d\mathbf{p}_N}{dt}; \mathbf{g}^{-1} \cdot \mathbf{I}\right)^2 dt$$

Here again, set $\mathbf{v} = \frac{\partial \mathbf{g}}{\partial t} \circ \mathbf{g}^{-1}$, and $\mathbf{J} = \mathbf{g}^{-1} \cdot \mathbf{I}$ so that

$$\begin{aligned}\mathbf{J}(t) :&= (\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t)) \\ &= (\mathbf{g}(t, \mathbf{p}_1(t)), \ldots, \mathbf{g}(t, \mathbf{p}_N(t)))\end{aligned}$$

We have

$$\begin{aligned}\frac{d\mathbf{q}_i}{dt} &= \frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{p}_i(t)) + \frac{\partial \mathbf{g}}{\partial x} \cdot \frac{d\mathbf{p}_i}{dt} \\ &= \mathbf{v}(t, \mathbf{q}_i(t)) + \frac{\partial \mathbf{g}}{\partial x} \cdot \frac{d\mathbf{p}_i}{dt}\end{aligned}$$

so that the energy can be written

$$E(\mathbf{a}) = \int_0^1 N_{\mathcal{G}}(\mathbf{v}; \mathbf{J})^2\, dt + \int_0^1 N_{\mathcal{I}}\left(\frac{d\mathbf{q}_1}{dt} - \mathbf{v}(t, \mathbf{q}_1(t)), \ldots, \frac{d\mathbf{q}_N}{dt} - \mathbf{v}(t, \mathbf{q}_N(t)); \mathbf{J}\right)^2 dt$$

We thus get

Theorem 3. *Let $(N_{\mathcal{G}}(\cdot \mid I), I \in \mathcal{I})$ and $(N_{\mathcal{I}}(\cdot \mid I), I \in \mathcal{I} = \Omega^N)$ be two collections of norms. Associate to paths $t \mapsto \mathbf{v}(t, \cdot)$ and $t \mapsto \mathbf{J}(t)$, where $\mathbf{v}(t, x) \in \mathbb{R}^k$ and $\mathbf{J}(t) \in \Omega^N$, the energy*

$$E(\mathbf{v}, \mathbf{J}) = \int_0^1 N_{\mathcal{G}}(\mathbf{v}; \mathbf{J})^2\, dt + \int_0^1 N_{\mathcal{I}}\left(\frac{d\mathbf{q}_1}{dt} - \mathbf{v}(t, \mathbf{q}_1(t)), \ldots, \frac{d\mathbf{q}_N}{dt} - \mathbf{v}(t, \mathbf{q}_N(t)); \mathbf{J}\right)^2 dt \tag{18}$$

Then, the function

$$d(I, I') = \inf\{\sqrt{E(\mathbf{v}, \mathbf{J})} : \mathbf{v}, \mathbf{J}, \mathbf{J}(0, \cdot) = I, \\ \mathbf{J}(1, \cdot) = I'\}$$

is a distance on $\mathcal{I}$.

The particular case when $N_{\mathcal{G}}(v; J) = \int_\Omega |L \cdot v| dx$ (where L is a differential operator) and

$$N_{\mathcal{I}}(\alpha_1, \ldots, \alpha_N; J) = \alpha_1^2 + \cdots + \alpha_N^2$$

is of interest. It yields

$$E(\mathbf{v}, \mathbf{J}) = \int_0^1 \int_\Omega |L \cdot \mathbf{v}|^2 \, dx \, dt \\ + \int_0^1 \sum_{i=1}^N \left| \frac{d\mathbf{q}_i}{dt} - \mathbf{v}(t, \mathbf{q}_i(t)) \right|^2 dt$$

The integrand can be explicitely minimized in $\mathbf{v}$ for each time t, since it is a quadratic form in $\mathbf{v}(t, \cdot)$. The optimal $\mathbf{v}(t, \cdot)$ is a function of $\mathbf{q}_1, \ldots, \mathbf{q}_N$ and their time derivatives. We eventually obtain an expression which does not contains $\mathbf{v}$ anymore, of the kind

$$Q(\mathbf{J}) = \int_0^1 t \frac{d\mathbf{J}}{dt} A_{\mathbf{J}(t)} \frac{d\mathbf{J}}{dt}$$

where $A_{\mathbf{J}(t)}$ is a symmetric definite positive matrix which depends on the values of $\mathbf{J}$ at time t, and which can be efficiently computed, by finite difference approximation of v, or expansion in an orthonormal basis. The overall analysis should follow that constructed by Joshi (1997).

3.4. Rotation Invariance

We have discussed so far the problem of matching objects which are functions $I : \Omega \mapsto \mathbb{R}^2$, or groups of labelled points in Ω, where Ω is an open subset of $\mathbb{R}^k$, using diffeomorphic actions. We have obtained a distance d on $\mathcal{I}$ which is tolerant to these actions, in the sense which has been discussed in the introduction. In addition to this robustness, one generally wants to incorporate a complete independence with respect to another group action, in the sense that $d(I, r \cdot I) = 0$ for any $i \in \mathcal{I}$ and any element r of this new group. A typical situation is rotation invariant matching, when we want to identify a function I with the function $I \circ r$, for any rotation r in $\mathbb{R}^k$.

From a general point of view, this situation corresponds to having a new group R acting on $\mathcal{I}$, and comparing the orbits of $\mathcal{I}$ under this action. Denoting $[I] = R \cdot I (I \in \mathcal{I})$, we want to devise a distance $\delta([I_0], [I_1])$ between the orbits. A natural definition is

$$\delta([I_0], [I_1]) = \inf\{d(r \cdot I_0, r' \cdot I_1) : r, r' \in R\}$$

A similar discussion to the ones we have made so far would give the fact that δ is a distance as soon as d is invariant by the action of R, that is, for all $I, I' \in \mathcal{I}$, for all $r \in R$,

$$d(r \cdot I, r \cdot I') = d(I, I')$$

As an illustration, we study this property in the particular case of functional objects. Here, R is equal to SE_k, the group of k-dimensional rotations and translations. In this framework, there is a problem related to the fact that if $\Omega \neq \mathbb{R}^k$ and $r \in SE_k$, the function $I \circ r$ will not be defined everywhere in Ω. We will therefore consider that $\Omega = \mathbb{R}^k$, and that all the considered integrals converge. We will also place ourselves in the case in which the energy in Eq. (16) is given by

$$E(\mathbf{v}, \mathbf{J}) = \int_0^1 \int_\Omega |L\mathbf{v}(t, x)|^2 \, dt \, dx + \int_0^1 \int_\Omega \left| \frac{\partial \mathbf{J}}{\partial t}(t, x) \right. \\ \left. + \frac{\partial \mathbf{J}}{\partial x}(t, x) \cdot \mathbf{v}(t, x) \right|^2 dt \, dx$$

where L is a differential operator acting on the x variables.

We have

$$d(I, I') = \inf_{\mathbf{v}, \mathbf{J}} \{\sqrt{E(\mathbf{v}, \mathbf{J})}, \mathbf{J}(0, \cdot) = I, \mathbf{J}(1, \cdot) = I'\}$$

Let us compute $d(I \circ r, I' \circ r)$. Make the change of variables $x = r \cdot y$ in the second term of the energy and set $\mathbf{J}'(t, y) = \mathbf{J}(t, r \cdot y)$. This terms becomes

$$\int_0^1 \int_\Omega \left| \frac{\partial \mathbf{J}'}{\partial t}(t, y) + \frac{\partial \mathbf{J}'}{\partial x}(t, y) \cdot r^{-1} \cdot \mathbf{v}(t, r \cdot y) \right|^2 dt \, dy.$$

Let $\mathbf{v}'(t, y) = r^{-1}\mathbf{v}(t, r \cdot y)$, so that the second integral simply writes

$$\int_0^1 \int_\Omega \left| \frac{\partial \mathbf{J}'}{\partial t}(t, y) + \frac{\partial \mathbf{J}'}{\partial x}(t, y) \cdot \mathbf{v}'(t, y) \right|^2 dt \, dy.$$

If the first integral were also unchanged by the transformation $\mathbf{v} \to \mathbf{v}'$, we would get

$$\begin{aligned} d(I, I') &= \inf_{\mathbf{v},\mathbf{J}} \{\sqrt{E(\mathbf{v}, \mathbf{J})}, \mathbf{J}(0, \cdot) \\ &= r \cdot I, \mathbf{J}(1, \cdot) = r \cdot I'\} \\ &= \inf_{\mathbf{v}',\mathbf{J}'} \{\sqrt{E(\mathbf{v}', \mathbf{J}')}, \mathbf{J}'(0, \cdot) \\ &= I, \mathbf{J}'(1, \cdot) = I'\} \\ &= d(I, I') \end{aligned}$$

by definition of $\mathbf{J}'$. We thus get the proposition

Proposition 3. *Let*

$$\begin{aligned} &E(\mathbf{v}, \mathbf{J}) \\ &\quad = \int_0^1 \int_\Omega |L\mathbf{v}(t, x)|^2 \, dt \, dx \\ &\qquad + \int_0^1 \int_\Omega \left| \frac{\partial \mathbf{J}}{\partial t}(t, x) + \frac{\partial \mathbf{J}}{\partial x}(t, x) \cdot \mathbf{v}(t, x) \right|^2 dt \, dx \end{aligned}$$

and

$$d(I, I') = \inf_{\mathbf{v},\mathbf{J}} \{\sqrt{E(\mathbf{v}, \mathbf{J})}, \mathbf{J}(0, \cdot) = I, \mathbf{J}(1, \cdot) = I'\}$$

Let

$$\delta([I_0], [I_1]) = \inf\{d(I_0 \circ r, I_1 \circ r') : r, r' \in SE_k\}$$

Then δ is a distance between orbits as soon as, for all $r \in SE_k$, for all function $v : \Omega \to \mathbb{R}^k$,

$$\int_\Omega |Lv'|^2 \, dy = \int_\Omega |Lv|^2 \, dx$$

where $v'(y) = r^{-1} v(r \cdot y)$

One can take, for example, L given by the Laplacian (or powers of the Laplacian) applied to each coordinate of v. The analysis in the case of landmark matching is very similar.

4. Minimum Risk Estimators

We present here another context in which left-invariant distances between registered objects come as a natural requirement. This happens when performing minimal risk estimation used in relation to a statistical modeling of object variations (probabilistic deformable templates). We give in this section a formal formulation of the problem; specific examples will be provided in Section 5.

Here again, $\mathcal{G}$ is a group acting on the set $\mathcal{I}$, $\mathcal{A} = \mathcal{G} \times \mathcal{I}$ is the set of registered objects. In addition to this, we consider a set $\mathcal{F}$, the elements of which being *observable* quantities. Using probability distributions modeling the formation of the observation given a true object and a possible element of $\mathcal{G}$ acting on it, the problem is to infer the unknown object from data.

We place ourselves in a Bayesian framework, starting with building a probability model[6] on $\mathcal{A}$. This assumes that some *a priori* knowledge is available to infer this model. Most of the time, it can be assumed that the two components of $\mathcal{A}$ (the group elements and the objects) are independent for this prior probability, which will therefore be written $P(dg, dI) = P_1(dg)P_2(dI)$; moreover, the group element often represents an unknown registration parameter (rotation, translation, change of parametrization, etc) on which no prior modeling can be made, so that the prior distribution on this part will be taken to be uniform on $\mathcal{G}$, in the sense that, for any measurable function f defined on $\mathcal{G}$, and for any $h \in \mathcal{G}$,

$$\int_\mathcal{G} f(g) P_1(dg) = \int_\mathcal{G} f(hg) P_1(dg) \tag{19}$$

Note that this is not always possible for any group $\mathcal{G}$, but can be easily provided for the groups which are typically used in applications (one may need to relax the fact that P_1 is a probability and consider a σ-finite measure instead).

For the construction of P_2, we use Grenander's approach for designing a model by comparing the object I to a reference object (a *template*) which will be denoted I_{temp}; P_2 generally models small variations of the object around the deformable template, and we use the notation $P_2(dI \mid I_{\text{temp}})$ to strengthen the fact that the model is build on the basis of this template. However, the template has to be chosen in a particular position with respect to the action of $\mathcal{G}$, and in fact, any object of the kind $h \cdot I_{\text{temp}}$, for any $h \in \mathcal{G}$ would have made an equally valid template. Given this point, it is then natural to consider the model which should be used if we had chosen $h \cdot I_{\text{temp}}$ as the template. It should be linked to the original one through the equation, which is valid for any function f defined on $\mathcal{I}$

$$\int_I f(I) P_2(dI \mid hI_{\text{temp}}) = \int_I f(h^{-1}I) P_2(dI \mid I_{\text{temp}}) \tag{20}$$

The third part of the model provides a conditional probability for the observation $F \in \mathcal{F}$ given the output of the (unknown) element $(g, I) \in \mathcal{A}$. This conditional probability will be denoted $Q(dF|g^{-1} \cdot I)$.

Assume finally that a distance D is provided on $\mathcal{A}$. A minimal risk estimator is defined as a function $\hat{A} : \mathcal{F} \to \mathcal{A}$, which associates to each observation F an estimated pair $\hat{A}(F) = (\hat{g}(F), \hat{I}(F))$, which is optimal in the sense that it realizes the minimum, over all such functions, of the Bayesian risk

$$R_{I_{\text{temp}}}(\hat{g}(\cdot), \hat{I}(\cdot)) = \int_{\mathcal{A}\times\mathcal{F}} D[(g, I), (\hat{g}(F), \hat{I}(F))] Q(dF|g^{-1} \cdot I) P_1(dg) P_2(dI|I_{\text{temp}})$$

By Bayes rule, this can be rewritten in terms of the posterior probability of g and I given F and I_{temp}, which is denoted $P_{\text{post}}(dg, dI|F, I_{\text{temp}})$, and of the marginal distribution of F (denoted $Q_0(dF)$, as

$$R_{I_{\text{temp}}}(\hat{g}(\cdot), \hat{I}(\cdot)) = \int_{\mathcal{F}} Q_0(dF) \int_{\mathcal{A}} D[(g, I), (\hat{g}(F), \hat{I}(F))] P_{\text{post}}(dg, dI|F, I_{\text{temp}})$$

so that the estimators $\hat{g}$ and $\hat{I}$ must be, for each F, a minimizer (if it exists) of the average posterior distance

$$D_{\text{post}}(\hat{g}, \hat{I}) = \int_{\mathcal{A}} D[(g, I), (\hat{g}, \hat{I})] P_{\text{post}}(dg, dI|F, I_{\text{temp}})$$

In practice, this expression is often minimized by Monte-Carlo sampling of the posterior distribution.

This risk, and the estimators, depends on the particular choice made for the template I_{temp}. Because of the ambiguity on this choice, we must be sure that the procedure behaves consistently when I_{temp} is replaced by $h \cdot I_{\text{temp}}$ for some $h \in G$. This is addressed by the proposition:

Proposition 4. *If D is invariant by the left-action of G, then*

$$R_{h\cdot I_{\text{temp}}}(\hat{g}(\cdot), \hat{I}(\cdot)) = R_{I_{\text{temp}}}(h\hat{g}(\cdot), h\hat{I}(\cdot))$$

so that the minimal risk estimators are modified consistently with a left translation of the template.

The proof of this proposition is a straightforward application of (19), (20) and the left invariance of D, and is left to the reader.

5. Low Dimensional Examples

5.1. 3D-Object Registration

Consider the situation where a family of N objects, $J_1, \ldots, J_N$, is given. We assume that they are represented as *subsets* of $\mathbb{R}^3$. They generate, under the action of $SO(3)$ (the group of 3-D rotations), an object space $\mathcal{I}$ composed with the elements $I = g \cdot J_k$, $g \in SO(3)$ and $k \in \{1, \ldots, N\}$. The objects can be observed on an imaging device, and denoting by $\mathcal{F}$ the set of possible images, the observation is an element $F \in \mathcal{F}$. The problem is to infer from F, the position and the identity of an object present in the scene.

For the construction of a Bayesian model for image formation, we refer to Grenander et al. (in which a more general formulation is presented, the presence of an unknown number of objects in the scene). We here focus on the construction of a distance D on $\mathcal{A}$.

Let us make the following hypotheses: the set $\mathcal{G} \cdot I_k$ and $\mathcal{G} \cdot I_l$ are disjoint as soon as $k \neq l$, and for any k, one can have $g \cdot I_k = I_k$ only if $g = \text{id}$ (which implies that there is enough structure in the object to make the registration identifiable).

Because of our first hypotheses, the set $\mathcal{J} = \{J_1, \ldots, J_N\}$ can be identified to the coset space $\mathcal{I}/G$. We therefore set $\mathcal{A} = SO(3) \times \mathcal{J}$. As a consequence, the distance D on $\mathcal{A}$ can be set to be any distance of the kind

$$D((g, I), (h, I')) = d_0(g, h) + \Delta\big(J_{k(I)}, J_{k(I')}\big)$$

where d_0 is a left invariant distance on $SO(3)$ (see next section for a construction), $k(I)$ is the index of the coset to which I belongs and Δ is any distance on the set $\mathcal{J}$.

5.2. Invariant Distances on $SO(k)$.

We here review some standard facts on invariant distances on $SO(k)$. We use this setting to illustrate, in a simple case, the construction of Section 3.1, which here leads to closed form formulas.

The Hilbert-Schmidt norm of a $k \times k$ matrix A is defined by

$$\|A\|^2 = \text{trace} A \cdot A^t.$$

It is easily shown that this norm is left (and right)-invariant by the action of $SO(k)$, so that, the distance

$$d_1(g, g') = \|g - g'\|$$

is a possible left-invariant distance on $SO(k)$.

Another distance may be defined, as in Section 3.1, by using the geodesic distance on $SO(k)$. We now compute it, as an illustration. For this, we must start with a norm on the tangent space to $SO(k)$ at the identity, which will be denoted $\|\cdot\|_{\mathrm{Id}}$, and extend it to the tangent spaces at any $g \in SO(k)$ by left translation:

$$\|X\|_g = \|g^{-1} \cdot X\|_{\mathrm{Id}}$$

(the tangent space to $SO(k)$ at the identity is composed with $k \times k$ anti-symmetric matrices, X, i.e. such that $X + X^t = 0$).

If we let $\|X\|_{\mathrm{Id}}$ be the Hilbert-Schmidt norm, then, by left-invariance of this norm, one also has $\|X\|_g = \sqrt{\mathrm{trace} X X^t}$. The geodesic distance between g and g' satisfies

$$\begin{aligned} d_2(g, g')^2 &= \inf_g \int_0^1 \left\| \frac{\partial}{\partial t} \mathbf{g}(t) \right\|_{\mathbf{g}(t)}^2 dt \\ &= \inf_g \int_0^1 \mathrm{trace}\left[\frac{\partial}{\partial t} \mathbf{g}(t) \frac{\partial}{\partial t} \mathbf{g}(t)^t \right] dt \end{aligned}$$

The infimum being taken over all paths $\mathbf{g}$ on $SO(k)$ such that $\mathbf{g}(0) = g$ and $\mathbf{g}(1) = g'$. Because the Hilbert-Schmidt norm is both left and right invariant, it can be shown that the minimizing paths are the group exponential, i.e. that they take the form $\mathbf{g}(t) = e^{Xt}$, for an anti-symmetric matrix X with $e^X \cdot g = g'$. Admitting this general result (see, for example, Do Carmo (1992), Ch. 3, Exercise 3), we thus see that

$$d_2(g, g') = \sqrt{\mathrm{trace} X \cdot X^t}.$$

If we define the $\frac{(k-1)}{2} k$ entries of the skew symmetric matrix to be $x_1, x_2, \ldots$, then

$$d_2(g, g')^2 = 2 \sum_{i=1}^{\frac{(k-1)k}{2}} x_i^2.$$

Let us study more precisely the expression, and the fundamental differences between d_1 and d_2. As described in Grenander et al., for $SO(k)$ the Hilbert Schmidt distance becomes

$$d_1(g, g') = 2k - 2\ \mathrm{trace}[g g'^t].$$

Since d_1 and d_2 are both left and right invariant, one has, for any g, g', and for $i = 1, 2$,

$$d_i(g, g') = d_i(\mathrm{Id}, g^t g')$$

and

$$d_i(\mathrm{Id}, g') = d_i(\mathrm{Id}, g^t g' g).$$

Any element $g \in SO(k)$ can be written $g = r^t h r$ where $r, h \in SO(k)$ and h is a block-diagonal matrix $h = \mathrm{diag}(\mathrm{Id}_{k-2p}, R(\theta_1), \ldots, R(\theta_p))$ where Id_{k-2p} is the identity $(k - 2p) \times (k - 2p)$ matrix, and $R(\theta_i)$ is the 2×2 block given by

$$R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}.$$

This h is given by e^X where X is the block diagnoal matrix equal to diag $(0_{k-2p}, r(\theta_1), \ldots, r(\theta_p))$, 0_{k-2p} being the null matrix of size $k - 2p$, and

$$r(\theta_i) = \begin{bmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{bmatrix}.$$

One has

$$d_1(\mathrm{Id}, h)^2 = 2p - 2 \sum_{i=1}^{p} \cos\theta_i$$

and

$$d_2(\mathrm{Id}, h)^2 = 2 \sum_{i=1}^{p} \theta_i^2$$

(assuming that the θ_i are taken in$]-\pi, \pi]$).

The distances between g and g' can therefore be computed after putting $O = g^t g'$ under a canonical form. This can be done explicitly in 2 or 3 dimensions: in $SO(2)$, a rotation is always given under the form $R(\theta)$ for some θ; for $SO(3)$, there is also only one angle θ and $1 + 2\ \cos\theta = \mathrm{trace}(g)$.

5.3. *Image Matching via the Orthogonal Group*

Let us now provide a distance, defined along the lines of Section 3.1, in a tolerant way under the action of $SO(3)$. This situation assumes that the object are roughly aligned with respect to the rotations, so that comparison will implicitely assume that the rotation is small.

Define the object space to be composed with mappings $I : \mathbb{R}^3 \to \mathbb{R}$, and let the orthogonal group $G := SO(3)$ act on it with group action $g \in \mathcal{G} : I \mapsto gI$, with $[g \cdot I](x) = I(g^{-1} \cdot x)$. Using the notation of Section 3.2.3, we define the norms at identity, for paths such that $\mathbf{g}(t) = \mathrm{id}$, $N_{\mathcal{G}}(\frac{\partial}{\partial t}\mathbf{g}(t))^2 = \mathrm{trace}\,[\dot{\mathbf{g}}(t)\dot{\mathbf{g}}(t)^t]$ and

$$N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial \mathbf{t}}(t, \cdot)\right)^2 = \int_{\mathbb{R}^3} \left| \frac{\partial \mathbf{I}}{\partial t}(t, x) \right|^2 dt.$$

Then, the associated distance D on $\mathcal{A} = \mathcal{G} \times \mathcal{I}$ becomes

$$\begin{aligned} d((g, I), (g', I')) &= \inf\left(\int_0^1 N_{\mathcal{G}}(\mathbf{g}(t)^{-1}\frac{\partial}{\partial t}\mathbf{g}(t))^2\, dt \right. \\ &\quad \left. + \int_0^1 \int_{\mathbb{R}^3} \left\| \frac{\partial}{\partial t}\mathbf{I}(t, \mathbf{g}x) \right\|^2 dt\, dx \right) \\ &= \inf\left(\int_0^1 \|\dot{\mathbf{g}}(t)\mathbf{g}(t)^t\|^2 dt \right. \\ &\quad \left. + \int_0^1 \int_{\mathbb{R}^3} \left\| \frac{\partial \mathbf{I}}{\partial t}(t, y) \right\|^2 dt\, dy \right) \\ &= d_2(g, g') + \int_{\mathbb{R}^3} |I(x) - I'(x)|^2\, dx. \end{aligned}$$

Distance between images is then given by

$$\begin{aligned} d(I_0, I_1) &= \min\{d((g_0, I), (g_1, I'))\ g_0 I_0 = I, g_1 I_1 = I'\} \\ &= \min\big\{d((id, g_0^{-1} I), (g_0^{-1}, g_1, g_0^{-1} I')), \\ &\quad g_0^{-1} g_1 I_1 = g_0^{-1} I'\big\} \\ &= \min\{d((id, I_0), (\tilde{g}, \tilde{I}))\tilde{g} I_1 = \tilde{I}\} \\ &= \min_{\{g \in SO(3)\}} \{d((id, I_0), (g, gI_1))\} \\ &= \min_{X \text{antisymmetric}} \{d((id, I_0), (e^X, e^X I_1))\} \\ &= \min_X \|X\|^2 + \int_{\mathbb{R}^3} |I_0(x) - I_1(e^X x)|^2\, dx. \end{aligned}$$

In Cooper et al. (1996), the following variant has been proposed. The variations of what corresponds to the intensity[7] have been expressed, for a template, as a linear combination of given functions $\phi^1(\cdot), \ldots, \phi^K(\cdot)$ defined on $\Omega = \mathbb{R}^3$, such that $(\phi^1, \ldots, \phi^K)$ form an orthonormal system of functions. The observed model is given by

$$I(x) = I_{\text{temp}}(g^{-1} \cdot x) + \sum_{i=1}^{k} \lambda_i \phi^i(g^{-1} \cdot x) \qquad (21)$$

for $x \in \Omega$ and $g \in SO(n)$. When the pair (g, I) is given, this equation uniquely defines $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. This implies that one can identify the set of registered objects $\mathcal{A} = \mathcal{G} \times \mathcal{I}$ with the set $\mathcal{G} \times \mathbb{R}^K$. To be more explicit, to (g, I), one associates (g, λ) with

$$\lambda_i = \int_\Omega \langle I(x) - I_{\text{temp}}(g^{-1} \cdot x), \phi^i(g^{-1} \cdot x)\rangle\, dx.$$

The action $h \cdot (g, I) = (h \cdot g, h \cdot I)$ must be transduced to an action $h \cdot (g, \lambda) = (h \cdot g, \lambda')$, with

$$\begin{aligned} \lambda'_i = \int_\Omega &\langle I(h^{-1}x) \\ &- I_{\text{temp}}(g^{-1} \cdot h^{-1}x), \phi^i(g^{-1} \cdot h^{-1}x)\rangle\, dx. \end{aligned}$$

But, since h has determinant 1, we see that $\lambda'_i = \lambda_i$ for all i. Thus, the action of $\mathcal{G}$ on $\mathcal{A}$ simply becomes $h \cdot (g, \lambda) = (hg, \lambda)$. This implies that registered objects can be compared by the very simple left-invariant distance

$$D((g, \lambda), (g', \lambda'))^2 = d_0(g, g') + \|\lambda - \lambda'\|^2$$

where d_0 is a left-invariant distance on $SO(n)$ computed in the previous section.

6. High Dimensional Examples

6.1. Plane Curves

6.1.1. Matching Trajectories Curves are the simplest cases of functional objects (Section 3.2.3), for which the set Ω is an interval in $\mathbb{R}$. Any differentiable curve can be represented by a mapping $m : [0, 1] \mapsto \mathbb{R}^k$. Let us apply the construction of Section 3.1 to this particular context. The paths $\mathbf{v}$ are functions $\mathbf{v}(t, x)$ defined on $[0, 1] \times [0, 1]$, with null partial derivatives up to some order at $x = 0$ and $x = 1$, for all t. Comparison can be held by defining norms for infinitesimal deformations by

$$N_{\mathcal{G}}(v; m) = N_{\mathcal{G}}(v) = \int_0^1 |L \cdot v|^2\, dx$$

and

$$N_{\mathcal{I}}(z; m) = N_{\mathcal{I}} = \int_0^1 |z|^2\, dx$$

where L is some differential operator (these norms are homogeneous).

Which leads to defining the energy of a deformation path $(\mathbf{v}, \mathbf{m})$ by

$$\begin{aligned} &\int_0^1 \int_0^1 |L\mathbf{v}(t, x)|^2\, dx\, dt \\ &\quad + \int_0^1 \int_0^1 \left| \frac{\partial \mathbf{m}}{\partial t}(t, x) + \frac{\partial \mathbf{m}}{\partial x}(t, x)\mathbf{v}(t, x) \right|^2 dt\, dx. \end{aligned}$$

If m_0 and m_1 are given and the above energy is minimized subject to the contraint $m_0(\mathbf{g_v}(1, x)) = m_1(x)$, this approach does provide a distance for matching parametrized curves, and can be used in situations for which the parametrization x is significant (for example, to perform on-line signature identification).

6.1.2. Matching signatures. The previous setting depends on the curve parametrization, in the sense that, if m_0 and m_1 are replaced by $m_0 \circ \psi_0$ and $m_1 \circ \psi_1$, the optimal matching will not vary consistently, that is, if s_0 and s_1 were initially matched, $\psi_0(s_0)$ and $\psi_1(s_1)$ need not be matched when comparing $m_0 \circ \psi_0$ and $m_1 \circ \psi_1$. In other terms, the comparison is not valid from a geometric point of view.

To obtain a geometric comparison, fix a well defined parameterization, the simplest one being the (Euclidean) arc-length, which corresponds to the constraint that the derivative $\frac{dm}{dx}$ has a constant norm along the curve (the value of this constant being the length of the curve). We consider curves $x \mapsto m(x)$ defined for $x \in [0, 1]$, such that

$$L(m) := \left|\frac{dm}{dx}\right|$$

is constant along the curve; $L(m)$ is the length of the curve. We devise our comparison norms by making a small deformation analysis.

We consider a curve m and a small perturbation along m, $x \mapsto \Delta(x) \in \mathbb{R}^2$. The following analysis will be made assuming that Δ and all its derivatives are infinitesimally small, and keeping only first order terms. We obtain a new curve $x \mapsto m(x) + \Delta(x)$, which can be parametrized by arc-length. Denote by $\tilde{m}$ the obtained curve, and by g the diffeomorphim providing the arc-length parametrization of $m + \Delta$, in the sense that, for all $x \in [0, 1]$

$$\tilde{m} \circ g(x) = m(x) + \Delta(x)$$

Set $\xi(x) = g(x) - 1$. Denote by τ, ν and k the unit tangent, normal, and curvature of m. We have

$$\begin{aligned} \frac{dm}{dx} &= L(m)\tau, \\ \frac{d^2m}{dx^2} &= L(m)\frac{d\tau}{dx} = L(m)^2\kappa\nu \end{aligned}$$

Similarly, denote by $\tilde{\tau}$, $\tilde{\nu}$, $\tilde{\kappa}$ the tangent, normal and curvature of $\tilde{m}$; let also $L = L(m)$ and $\tilde{L} = L(\tilde{m})$. The small deformation energy will be defined in terms of the derivatives of Δ and some characteristics of the curve m. We have, neglecting quantities of order larger than 2 (quantities of order 1 are $\tilde{L} - L, \xi,\ \tilde{\tau} \circ \mathbf{g} - \tau$, etc).

$$\begin{aligned} \frac{d\Delta}{dx} &= \left(1 + \frac{d\xi}{dx}\right)\frac{d\tilde{m}}{dx} \circ g - \frac{dm}{dx} \\ &= \tilde{L}\frac{d\xi}{dx}\tilde{\tau} \circ g + \tilde{L}\tilde{\tau} \circ g - L \cdot \tau \\ &\simeq L\frac{d\xi}{dx}\tau + (\tilde{L} - L)\tau + L(\tilde{\tau} \circ g - \tau) \end{aligned}$$

A first definition of the energy of small deformations can be

$$E = \int_0^L \left|\frac{dV}{dx}\right|^2 = \frac{1}{L}\int_0^1 \left|\frac{d\Delta}{dx}\right|^2 dx$$

with $V(x) = \Delta(x/L)$. Noticing that, at first order, $\tilde{\tau} \circ g - \tau$ is perpendicular to τ, we have

$$E \simeq \frac{(\tilde{L} - L)^2}{L} + L\int_0^1 \left|\frac{d\xi}{dx}\right|^2 dx + L\int_0^1 |\tilde{\tau} \circ g - \tau|^2\, dx$$

This expression leads to define a group $\mathcal{G}$ of diffeomorphisms of [0,1], and an object space $\mathcal{I}$ composed with elements of the kind $I = (L, \tau)$, where L is a positive number and $\tau : [0, 1] \mapsto \mathbb{R}^2$ is such that $|\tau(x)| = 1$ for all x. If L is the length and τ the unit tangent, this characterizes a curve up to translations. Define norms $N_{\mathcal{G}}$ and $N_{\mathcal{I}}$ by

$$N_{\mathcal{G}}(\xi; L, \tau) = L\int_0^1 \left|\frac{d\xi}{dx}\right|^2 dx,$$

and

$$N_{\mathcal{I}}(\delta_L, \delta_\tau; L, \tau) = \frac{\delta_L^2}{L} + L\int_0^1 |\delta_\tau|^2\, dx$$

Now, if $\mathbf{a} = (\mathbf{g}, \mathbf{L}, \mathbf{T}')$ is a path on $\mathcal{A} = G \times \mathcal{I}$, we must define, according to Section 3.1, the total energy

$$\begin{aligned} E(\mathbf{a}) = &\int_0^1 \frac{1}{\mathbf{L}}\frac{d\mathbf{L}^2}{dt}\,dt + \int_0^1\int_0^1 \mathbf{L}\left|\frac{d\mathbf{v}}{dx}\right|^2 dx\,dt \\ &+ \int_0^1\int_0^1 \mathbf{L}\left|\frac{\partial \mathbf{T}}{\partial t} + \frac{\partial \mathbf{T}}{\partial x}\mathbf{v}\right|^2 dt\,dx. \qquad (22) \end{aligned}$$

where we have made the usual change of variables $\mathbf{v} = \frac{d\mathbf{g}}{dt} \circ \mathbf{g}^-1$ and $\mathbf{T} = \mathbf{T}' \circ \mathbf{g}^{-1}$.

For this particular context, the distance D on $\mathcal{A}$ can be explicitely computed (c.f. Younes (1998)). Letting

$a_0 = (g_0, L_0, T_0)$ and $a_1 = (g_1, L_1, T_1)$ be two registered objects, we have

$$\begin{aligned} D(a_0, a_1) &= L_0 + L_1 - \sqrt{2} L_0 L_1 \\ &\quad \times \int_0^1 \sqrt{\dot{g}_0(x)\dot{g}_1(x)(1 + \langle T_0(x), T_1(x)\rangle)}\, dx. \end{aligned}$$

The associated optimal deformation paths being also explicitely known.

6.1.3. The distance we have obtained compares tangent angles, which are rotation dependent. To compare curves modulo rotations, we pursue the small deformation analysis and compute the second derivative of Δ. This yields

$$\begin{aligned} \frac{d^2\Delta}{dx^2} &\simeq L^2 \frac{d\xi}{dx}\kappa\nu + L\frac{d^2\xi}{dx^2}\tau + L(\tilde{L} - L)\kappa\nu \\ &\quad + L\tilde{L}\left(1 + \frac{d\xi}{x}\right)\tilde{\kappa}\circ g\tilde{\nu}\circ g - L^2\kappa\nu \\ &\simeq L\frac{d^2\xi}{dx^2}\tau + 2L^2\frac{d\xi}{dx}\kappa\nu + 2L(\tilde{L} - L)\kappa\nu \\ &\quad + L^2(\tilde{\kappa}\circ g - \kappa)\nu + L^2\kappa(\tilde{\nu}\circ g - \nu) \end{aligned}$$

At order 1, $L(\tilde{\nu}\circ g - \nu)$ is perpendicular to ν, and is obtained by a rotation of $\pi/2$ from $L(\tilde{\tau}\circ g - \tau)$, which is the normal part of the first derivative of Δ. Thus, we have $L^2(\tilde{\nu}\circ g - \nu) \simeq -L\langle \frac{d\Delta}{dx}, \nu\rangle\tau$. We finally get

$$\begin{aligned} \frac{d^2\Delta}{dx^2} &+ L\kappa\left\langle \frac{d\Delta}{dx}, \nu\right\rangle\tau \simeq L\frac{d^2\xi}{dx^2}\tau + 2L^2\frac{d\xi}{dx}\kappa\nu \\ &+ 2L(\tilde{L} - L)\kappa\nu + L^2(\tilde{\kappa}\circ g - \kappa)\nu \end{aligned}$$

We define the small deformation energy

$$E = \frac{1}{L^3}\int_0^1 \left| \frac{d^2\Delta}{dx^2} + L\kappa\left\langle \frac{d\Delta}{dx}, \nu\right\rangle\tau \right|^2 dx,$$

the factor L^{-3} stemming from a "real size analysis" (letting as before $V(x) = \Delta(x/L)$). This energy can be written as

$$\begin{aligned} E &= \frac{1}{L}\int_0^1 \left|\frac{d^2\xi}{dx^2}\right|^2 dx \\ &\quad + L\int_0^1 \left| 2\frac{d\xi}{dx}\kappa + 2\frac{\tilde{L} - L}{L}\kappa + \tilde{\kappa}\circ g - \kappa \right|^2 dx \end{aligned}$$

We shall here work with objects of the kind $I = (L, \kappa)$, where L is the length and κ is the curvature, and deduce from this small deformation analysis the definition of the length of a path, introducing again the variables $\mathbf{v}$, and letting $(\mathbf{L}, \mathbf{K})$ be the evolving object at time t, by

$$\begin{aligned} &\int_0^1\int_0^1 \frac{1}{\mathbf{L}}\left|\frac{d^2\mathbf{v}}{dx^2}\right|^2 dx\, dt + \int_0^1\int_0^1 \mathbf{L}\left| 2\frac{d\mathbf{v}}{dx}\mathbf{K} \right. \\ &\quad \left. + 2\frac{1}{\mathbf{L}}\mathbf{K}\frac{d\mathbf{L}}{dt} + \frac{\partial\mathbf{K}}{\partial t} + \mathbf{v}\frac{\partial\mathbf{K}}{\partial x}\right|^2 dx\, dt \end{aligned}$$

6.2. *Image Matching*

6.2.1. Energy. Let an image be a mapping defined on $\bar{\Omega} = [0, 1]^2 \subset \mathbb{R}^2$, with values in $\mathbb{R}$, and let G be a group of homeomorphisms acting on the images via the action considered in Section 3.2.3.

In the homogeneous case, an image deformation functional can be defined from the norms

$$N_{\mathcal{G}}\left(\frac{\partial\mathbf{g}}{\partial t}(t_0, \cdot); \mathbf{I}(t_0)\right)^2 = \int_0^1 \left| \frac{\partial^2}{\partial x^2}\frac{\partial\mathbf{g}}{\partial t}(t_0, x)\right|^2 dx$$

and

$$N_{\mathcal{I}}\left(\frac{\partial\mathbf{I}}{\partial t}(t_0, \cdot); \mathbf{I}(t_0)\right)^2 = \alpha\int_0^1 \left|\frac{\partial\mathbf{I}}{\partial x^2}(t, x)\right|^2 dx$$

Where α is a positive parameter. The associated functional is

$$\begin{aligned} &\int_0^1\int_\Omega \left\| \frac{\partial^2\mathbf{v}}{\partial x^2}(t, x)\right\|^2 dt\, dx \\ &\quad + \alpha\int_0^1\int_\Omega \left\| \frac{\partial\mathbf{J}}{\partial t}(t, x) + \langle\nabla\mathbf{J}, \mathbf{v}\rangle\right\|^2 dt\, dx. \end{aligned}$$

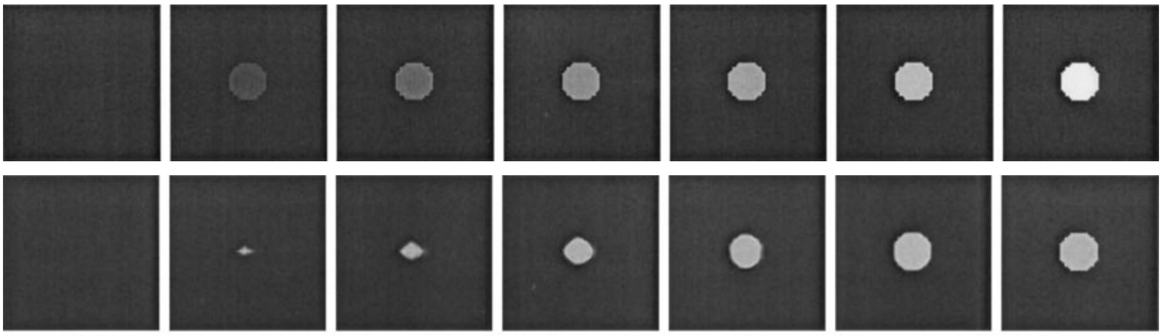
Comparison between two images I_0 and I_1 can be performed by minimizing this functional over all paths which satisfy $\mathbf{v}(t, x) = \frac{\partial\mathbf{v}}{\partial x} = 0$ for all $t \in [0, 1]$ and all $x \in \partial\Omega$, and $\mathbf{J}(0, x) = I_0(x)$, $\mathbf{J}(1, x) = I_1(x)$. Examples of such computations are provided in the next section.

6.2.2. Experiments. The pictures in the following figures provide the estimated morphing process $\mathbf{J}(t, \cdot)$ between two images I_0 and I_1.

The first experiments show how the interpolation process deals with the need for creating pixel intensity: a totally black image is matched to an image containing a white disc in its center. Depending on the choice of the parameter α, the process will allow for the creation of a large quantity of pixel intensity, yielding a morphing which looks like fading, with almost no deformation at all (α small), or will prefer introducing a small white zone in the center of the disc, and deform it into a disc, yielding a process which resembles an explosion (α large).

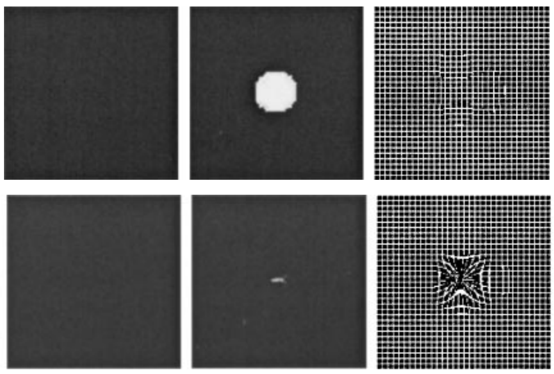
The second experiment presents the transformation of a square into a disc. The remaining ones deal with anatomical gray-level images (brain scans and macaque brain slices).

6.2.3. Explosion vs. Fading. The first set of experiments illustrates how luminance can be created during the optimal morphing process. Depending on the value of the parameter α (which penalizes non-conservation of luminance) the results are quite different: for small α, the disk is appearing without any deformation, simply the grey levels vary. For large α, a white spot first appears at the center of the disk and the deformation generates the expansion. The first and last picture in each row are the data provided to the algorithm, and the other ones are synthesized intermediate steps.



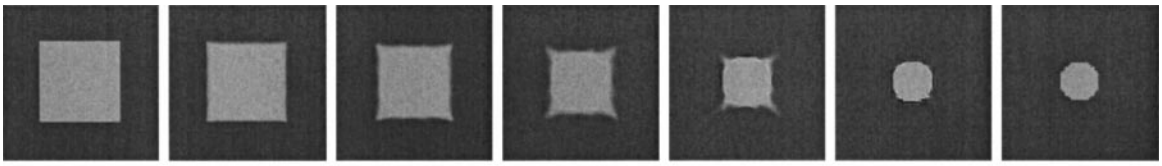
First row: small α: the white disc fades in the picture; second row: small α: a white spot appears and expands.

Here is the inverse deformation applied to the last image: we first give the initial image for reference, and then the final one after application of the inverse deformation. The differences show the places at which luminance has been created. The third picture gives the reversed deformation itself applied to a regular grid.

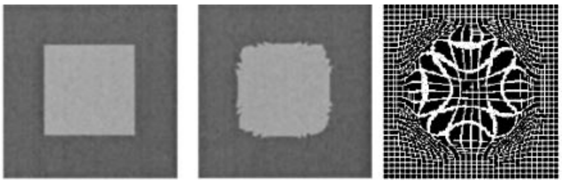


First row: reference image, registered target and deformation for small α; second row: the sames for large α

6.2.4. Squared Disc. The next experiment shows the matching starting with a square and ending with a smaller disc. A large α is chosen to minimize variations of luminance. Some numerical instabilities can be noticed at the boundaries.

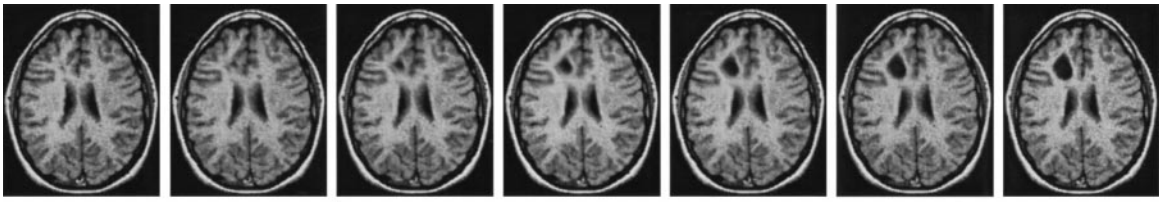


A square deformed into a disc.

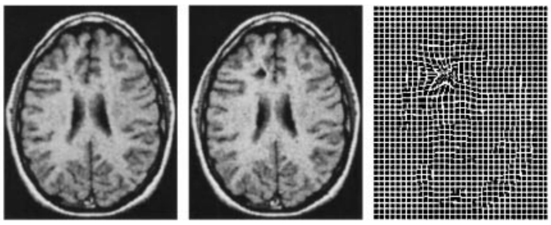


Matching a square and a disc: Reference image, registered target and deformation

6.2.5. Tumor. In this next experiment, a brain without a tumor is matched to the same one with a tumor. The tumor appears progressively during the morphing process. One notices large deformations around the tumor, and almost no deformation in other places.

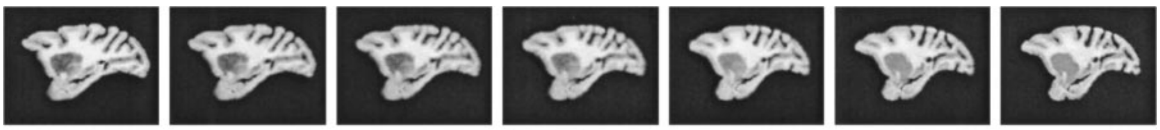


A tumor progressively appearing on a brain

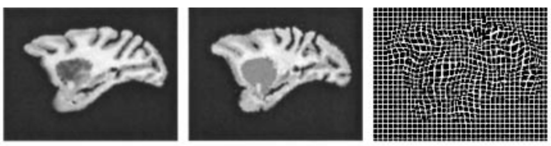


Tumor: Reference image, registered target and deformation

6.2.6. Macaque Brain Slices. Two slices of macaque brains are compared. The morphing, registered image and deformation are provided.



Macaque brain slices comparison



Macaque brain slices: Reference image, registered target and deformation

6.3. Numerical Facts

The minimization of

$$E(\mathbf{v},\mathbf{J}) = \int_0^1 \int_\Omega \left\| \frac{\partial^2 \mathbf{v}}{\partial x^2}(t,x) \right\|^2 dt\, dx + \alpha \int_0^1 \int_\Omega \left\| \frac{\partial \mathbf{J}}{\partial t}(t,x) + \langle \nabla \mathbf{J}, \mathbf{v} \rangle \right\|^2 dt\, dx \tag{23}$$

is performed after time and space discretization, the derivatives being estimated by finite differences. If N^2 is the dimension of the space grid and T the number of time steps, there are, because of the boundary conditions, a little less than $3N^2T$ variables to estimate ($\mathbf{v}$ has two coordinates and $\mathbf{J}$ has one).

For fixed $\mathbf{v}$ (resp. fixed $\mathbf{J}$), the energy (23) is quadratic in $\mathbf{J}$ (resp. $\mathbf{v}$). For this reason, we used two step relaxation procedure, which alternates a conjugate gradient descent for $\mathbf{J}$ with a gradient descent for $\mathbf{v}$. A complication comes from the fact that the discretization of the intensity conservation term

$$\frac{\partial \mathbf{J}}{\partial t}(t,x) + \langle \nabla \mathbf{J}, \mathbf{v} \rangle$$

has to be done with care. A direct linear discretization with finite difference leads to very unstable numerical procedures, and one has to use a non-linear, but more stable approximation for the last term, of the kind

$$\left\langle \frac{\partial \mathbf{J}}{\partial x}, \mathbf{v} \right\rangle (t,x) = \sum_{k=1}^{2} (\mathbf{J}(t, s+e_k) - \mathbf{J}(t,s))\mathbf{v}^+(k,t,s)/h - (\mathbf{J}(t,s) - \mathbf{J}(t, s-e_k))\mathbf{v}^-(k,t,s)/h$$

h being the space step, $(e_1,\ e_2)$ the canonical basis of $\mathbb{R}^2$ and $\mathbf{v}^+$ (resp. $\mathbf{v}^-$) the positive (resp. negative) part of $\mathbf{v}$.

We also use a hierarchical, multigrid, procedure: the deformations are first estimated on a coarse grid (small N) and then iteratively refined until the finer grid is reached.

The choice of the number of time steps, T, is quite interesting. For $T = 2$ ($t = 0$ or 1), (23) reduces to the regularized intensity-conservation cost function which is of standard use for optical flow estimation. It is noticeable that such an energy fails to identify large deformations, as illustrated by the next very simple example

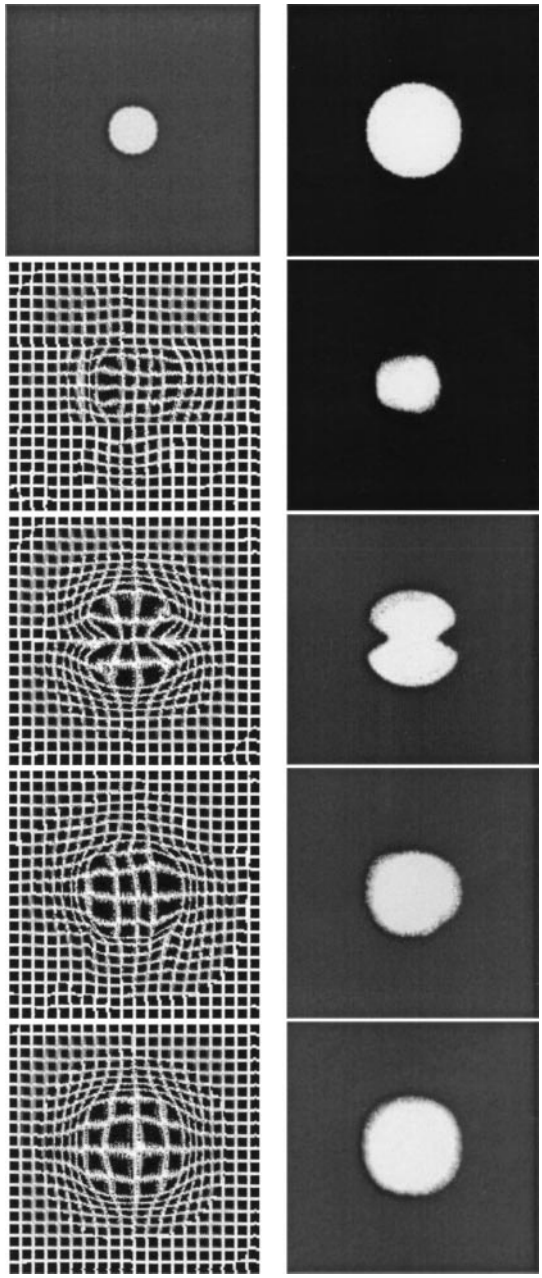


Figure 2. First line: a small disc and a large disc, the latter being the target; lines 2 to 5: left estimated deformation applied on a regular grid, and applid to the small disc (to be compared with the large disc), for $T = 2, 3, 4, 6$.

in which a small disc is mapped to a larger one with identical center. We show, in Fig. 2, several matchings which has been obtained by the same algorithm, with the same value of α, and various choices of T. The increased power for generating large deformation obtained by introducing the extra time dimension into the initially static problem is clearky exhibited by this example.

7. Conclusion

In this paper we have developed a general framework for examining object comparison problems in image analysis which naturally leads to object registration in a low dimensional setting and object matching in high dimensions. The reasoning was essentially in two parts: starting from the problem of designing metrics on the object set $\mathcal{I}$ which are tolerant to the action of a group $\mathcal{G}$, the analysis led to the need of defining left-invariant distances on $\mathcal{G} \times \mathcal{I}$. We then argued that, in the absence of any other hint for defining such distances, we could use a a generic construction of left-invariant Riemannian distances on $\mathcal{G} \times \mathcal{I}$, and were able to generate explicit and feasible comparison functionals in several practical frameworks.

When performing registration with low dimensional matrix groups, our formalism naturally led to standard left invariant distances on these groups, which can be computed, and used in close form. Dealing with groups of homeomorphisms, the formalism directly led to the introduction of the velocity field, which was used in Christensen et al. (1996) by analogy with mechanics, and because it can generate smooth large deformation diffeomorphisms. However, working on the product space $\mathcal{G} \times \mathcal{I}$ provided a setting in which the optimal deformation path is able to cross orbits of the action of $\mathcal{G}$ on $\mathcal{I}$, by generating simultaneous evolutions in $\mathcal{G}$ and in $\mathcal{I}$. The introduction of the Lie derivative (material derivative) of the image evolution introduces explicitly the notion of non-homogeneous evolutions with respect to the object part of the model. This has allowed us, for example, to rigorously design curvature based curve-matching procedures. In particular, it quantify the balance of energy associated with the evolution in image matching associated with geometric variation, and object image variation.

Appendix A. Transitive Group Action

As announced in Remark 1, we show how placing oneself on the set of registered objects can be seen as a generalization of the usual deformable template approach. The latter considers that a group, H, acts transitively on the object space $\mathcal{I}$. Given a distance D on $\mathcal{A}$, the least action distance can be defined as

$$d(I, I') = \inf\{D(\mathrm{id}, a), a \in H, a \cdot I = I'\}. \quad (24)$$

It can be shown that, if D is left-invariant by the action of $\mathcal{A}$, then d is symmetric and satisfies the triangular inequality (in Grenander (1993), $D(\mathrm{id}\,, a)$ is called an effort functional; similar concepts are introduced in Hagedoorn and Veltkamp (1999); Pennec and Ayache (1998)).

A more general definition has been introduced in Younes (1999) in the framework of transitive action. This definition relies on fixing a *reference object*, I_{ref}, which can be an arbitrary element of I, and let

$$d(I, I') = \inf\{D(a, a'), a, a' \in H, \\ I_{\mathrm{ref}} = a \cdot I = a' \cdot I'\}. \quad (25)$$

Letting $\mathcal{G}$ be the stabilizer of the reference object, i.e.

$$\mathcal{G} = \{a, a \in H, a \cdot I_{\mathrm{ref}} = I_{\mathrm{ref}}\},$$

d satisfies the triangular inequality as soon as D is left invariant by the action of $\mathcal{G}$ on H, i.e. $D(g \cdot a, g \cdot a') \equiv D(a, a')$ for all $g \in \mathcal{G}$. It can also easily be checked that, if D is left-invariant by the action of H on itself, then (25) boils down to (24).

We finally show that this construction can be seen as a particular case of the construction of Section 2.2. As indicated by the notation, H will be identified to the set of referenced objects *under the action of* $\mathcal{G}$, $\mathcal{A} = \mathcal{G} \times \mathcal{I}$, in a way such that the group multiplication $(g,\ a) \mapsto g \cdot a$. is identified to the left action of $\mathcal{G}$ on $\mathcal{G} \times \mathcal{I}$, as defined in (1). For this, we associate to each $I \in \mathcal{I}$ an element $\rho(I) \in H$ such that $\rho(I) \cdot I = I_{\mathrm{ref}}$ (we know that such an element exists, because the action of H on $\mathcal{I}$ is transitive). We then define

$$\begin{aligned} \Psi : \mathcal{G} \times \mathcal{I} &\to H \\ (g, I) &\mapsto g\rho(g^{-1} \cdot I). \end{aligned}$$

One can show that Ψ is bijective: $\Psi(g, I) = a$ is equivalent to $g = a \cdot \rho(a^{-1} I_{\mathrm{ref}})^{-1}$ and $I = ga^{-1} \cdot I_{\mathrm{ref}}$. Moreover, we have, for all $h \in \mathcal{G}$, $\Psi(h \cdot g, h \cdot I) = h \cdot \Psi(g, I)$ so the left-action of $\mathcal{G}$ on $\mathcal{G} \times \mathcal{I}$ is, as required, mapped to the left-action of $\mathcal{G}$ on H.

Appendix B. Appendix: Existence of Optimal Matchings

B.1. Introduction

In this section, we study the existence of optimal matchings in the framework of Section 3.2.3. We thus assume that we are given two norms, $N_{\mathcal{G}}(\cdot; I)$ and $N_{\mathcal{I}}(\cdot; I)$, and that the matching functional is

$$E(\mathbf{g}, \mathbf{I}) = \int_0^1 N_{\mathcal{G}}\left(\frac{\partial \mathbf{g}}{\partial t}(t, \mathbf{g}^{-1}(t, \cdot)); \mathbf{I}((t, \mathbf{g}^{-1}(t, \cdot))\right)^2 dt + \int_0^1 N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial t}(t, \mathbf{g}^{-1}(t, \cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t, \cdot))\right)^2 dt$$

As remarked in Section 3.2.4, diffeomorphisms are more easily and more efficiently defined as the solutions of an ordinary differential equation associated to a time-dependen vector field $\mathbf{v}(t, \cdot)$ by

$$\begin{cases} \mathbf{g}(0, x) = g_0(x) \\ \dfrac{\partial \mathbf{g}}{\partial t}(t, x) = \mathbf{v}(t, \mathbf{g}(t, x)) \end{cases}$$

In terms of $\mathbf{v}$, the functional can be written

$$U(\mathbf{v}, \mathbf{I}) = \int_0^1 N_{\mathcal{G}}(\mathbf{v}(t, \cdot); \mathbf{I}(t, \mathbf{g}^{-1}(t, \cdot)))^2 \, dt + \int_0^1 N_{\mathcal{I}}\left(\frac{\partial \mathbf{I}}{\partial t}(t, \mathbf{g}^{-1}(t, \cdot)); \mathbf{I}(t, \mathbf{g}^{-1}(t, \cdot))\right)^2 dt \tag{26}$$

For $p > 0$, we let $L^2(\Omega, \mathbb{R}^p)$ be the set of measurable functions f defined on $\bar{\Omega}$ and taking values in $\mathbb{R}^p$ such that

$$\| f \|_2 = \sqrt{\int_{\bar{\Omega}} | f(x)|^2 \, dx} < \infty$$

Assume that the norms $N_{\mathcal{G}}(\cdot; I)$ and $N_{\mathcal{I}}(\cdot; I)$ are such that, for some constant $\gamma > 0$, and for all $f \in L^2(\Omega, \mathbb{R}^k)$ (resp. $f \in L^2(\Omega, \mathbb{R}^d)$), one has $\gamma \| f \|_2 \leq N_{\mathcal{G}}(f, I)$ (resp. $\gamma \| f \|_2 \leq N_{\mathcal{I}}(f, I)$). Thus, for fixed I, the set of f such that $N_{\mathcal{G}}(f; I) < \infty$ can be seen as a Hilbert subspace of $L^2(\Omega; \mathbb{R}^k)$ which will be denoted by $\mathcal{H}_I$. To indicate that there is a relation of the kind $\gamma \| f \|_2 \leq N_{\mathcal{G}}(f, I)$, we will use the notation $\mathcal{H}_I \hookrightarrow L^2(\Omega, \mathbb{R}^k)$. Similarly, we denote by $\mathcal{J}_I$ the set of all functions f such that $N_{\mathcal{I}}(f, I) < \infty$, and we have $\mathcal{J}_I \hookrightarrow L^2(\Omega, \mathbb{R}^d)$.

We also denote by $\mathcal{H}_I^0$ the Hilbert space generated by functions $v \in \mathcal{H}_I$, with compact support included in Ω. The boundary conditions associated to the minimisation of (26) are the following:

- $\mathbf{v}(t, \cdot) \in \mathcal{H}^0_{1(t, \cdot)}$ for all t
- $\mathbf{I}(0, \cdot) = I_0(\cdot)$, $\mathbf{I}(1, \cdot) = I_1(\mathbf{g_v}(1, \cdot))$

The existence result will be stated under conditions of embeddings of the Hilbert spaces $\mathcal{H}_I$ and $\mathcal{J}_I$ into some more restrictive spaces than the L^2 spaces.

B.2. Existence Result

We will assume that the norms $N_{\mathcal{G}}(, ; I)$ and $N_{\mathcal{I}}(\cdot; I)$ can be controlled by homogeneous norms $\bar{N}_{\mathcal{G}}(\cdot)$ and $\bar{N}_{\mathcal{I}}(\cdot)$ which do not depend on I. We will in fact make the assumptions: (H1) There exists a positive constant κ such that, for all $I \in \mathcal{I}$, for all $f \in \mathcal{H}_I$,

$$\kappa \bar{N}_{\mathcal{G}}(f) \leq N_{\mathcal{G}}(f; I) \leq \frac{1}{\kappa} \bar{N}_{\mathcal{G}}(f)$$

and for all $f \in \mathcal{J}_{\mathcal{I}}$,

$$\kappa \bar{N}_{\mathcal{I}}(f) \leq N_{\mathcal{I}}(f; I) \leq \frac{1}{\kappa} \bar{N}_{\mathcal{I}}(f)$$

This implies in particular that the Hilbert spaces $\mathcal{H}_I$ and $\mathcal{J}_I$ do not depend on I, so that we can drop the subscript I when refering to them.

We will also ask for some regularity of the norms with respect to I:
(H2) There exists a positive constant κ such that, for all I, I', for all $f \in \mathcal{H}$

$$|N_{\mathcal{G}}(f; I) - N_{\mathcal{G}}(f; I')| \leq \kappa \bar{N}_{\mathcal{G}}(f) \|I - I'\|_\infty$$

and, for all $f \in \mathcal{J}$,

$$|N_{\mathcal{I}}(f; I) - N_{\mathcal{I}}(f; I')| \leq \kappa \bar{N}_{\mathcal{I}}(f) \|I - I'\|_\infty$$

We let $\mathcal{K}$ be the Hilbert space consisting of time dependent functions $\mathbf{v} : [0, 1] \times \bar{\Omega} \to \mathbb{R}^k$ and $\mathbf{Z} : [0, 1] \times \bar{\Omega} \to \mathbb{R}^d$ such that, for all t, $\mathbf{v}(t, \cdot) \in \mathcal{H}_0$ and $\mathbf{Z}(t, \cdot) \in \mathcal{J}$ and

$$\|(\mathbf{v}, \mathbf{Z})\|_\kappa^2 := \int_0^1 \bar{N}_{\mathcal{G}}(\mathbf{v}(t, \cdot))^2 \, dt + \int_0^1 \bar{N}_{\mathcal{I}}(\mathbf{Z}(t, \cdot))^2 \, dt \leq \infty$$

As in Trouvé (1999), we reformulate the matching problem in terms of these "time-dependent vector fields". For $(\mathbf{v}, \mathbf{Z}) \in \mathcal{K}$, we let the mappings $\mathbf{g}_\mathbf{v} : \bar{\Omega} \to \bar{\Omega}$ and $\mathbf{I}_{\mathbf{v},\mathbf{Z}} : \bar{\Omega} \to \mathbb{R}^d$, be the solutions (when they exist) of the differential system

$$\begin{cases} \dfrac{\partial \mathbf{g}_\mathbf{v}}{\partial t}(t, x) = \mathbf{v}(t, \mathbf{g}_\mathbf{v}(t, x)) \\ \dfrac{\partial \mathbf{I}_{\mathbf{v},\mathbf{Z}}}{\partial t}(t, x) = \mathbf{Z}(t, \mathbf{g}_\mathbf{v}(t, x)) \end{cases}$$

with initial conditions $\mathbf{g}_\mathbf{v}(0, x) = x$ and $\mathbf{Z}(0, x) = 0$.

When the initial point I_0 and the target I_1 are given, the matching problem can be expressed as minimizing

$$U(\mathbf{v}, \mathbf{Z}) = \int_0^1 N_\mathcal{G}(\mathbf{v}(t, \cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{Z}}(t, \cdot))^2\, dt + \int_0^1 N_\mathcal{I}(\mathbf{Z}(t, \cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{Z}}(t, \cdot))^2\, dt$$

with the constraint that $\mathbf{I}_{\mathbf{v},\mathbf{Z}}(1, \cdot) = I_1 - I_0$.

Let $\mathcal{V}(I)$ be the subset of $\mathcal{K}$ composed with those $(\mathbf{v}, \mathbf{Z})$ such that $\mathbf{I}_{\mathbf{v},\mathbf{Z}}(1, \cdot) = I$. One can prove that minimizers of U exist by first showing that U is weakly lower-semi-continuous on $\mathcal{K}$, and that the intersection of $\mathcal{V}(I)$ with a closed ball in $\mathcal{K}$ is weakly compact. Part of these results have already be addressed in Trouvé (1999), in which the homogeneous case has been handled. The essential condition, which is also required in this more general case, is that the norms $\bar{N}_\mathcal{G}$ and $\bar{N}_I$ are admissible, in the sense of the following condition:

(H3) there exists a constant $\kappa > 0$ such that, for all $f \in \mathcal{H}$,

$$\|f\|_\infty + \left\|\frac{df}{dx}\right\|_\infty \le \kappa \bar{N}_\mathcal{G}(f)$$

and, for all $z \in \mathcal{J}$,

$$\|z\|_\infty + \left\|\frac{dz}{dx}\right\|_\infty \le \kappa \bar{N}_\mathcal{I}(z)$$

Note that, if (H3) is true, the results quoted in Section 3.2.4 state that $\mathbf{g}_\mathbf{v}$ is well defined for all $(\mathbf{v}, \mathbf{Z}) \in \mathcal{K}$.

The function $\mathbf{I}_{\mathbf{v},\mathbf{Z}}$ is defined by

$$\mathbf{I}(t, x) = \int_0^t \mathbf{Z}(s, \mathbf{g}_\mathbf{v}(s, x))\, ds$$

which is finite since, by (H3), for all t and x, $\mathbf{Z}(t, x)$ is bounded by $\bar{N}(\mathbf{Z}(t, \cdot))$ which is integrable in the t variable.

Moreover, it is proved in Dupuis et al. (1998) and Trouvé (1999) that, still under condition (H3),

Lemma 1. *There exists a constant K depending only on Ω such that, for all $\mathbf{v}$ such that $\int_0^1 \bar{N}(\mathbf{v}(t, \cdot))^2\, dt < \infty$:*

for all $t, s \in [0, 1]$, for all $x, y \in \Omega$, one has

$$|\mathbf{g}_\mathbf{v}(t, x) - \mathbf{g}_\mathbf{v}(s, x)| \le K\|\mathbf{v}\|\sqrt{|t - s|}$$

and

$$|\mathbf{g}_\mathbf{v}(t, x) - \mathbf{g}_\mathbf{v}(t, y)| \le K\|\mathbf{v}\|x - y\|$$

The next lemma is essential for the results we aim at. We skip its proof, since it is essentially in Dupuis et al. (1998).

Lemma 2. *If $(\mathbf{v}_n, \mathbf{Z}_n)$ is a bounded sequence in $\mathcal{K}$ which weakly converges to $(\mathbf{v}, \mathbf{Z}) \in \mathcal{K}$ then $\mathbf{g}_{\mathbf{v}_n}$ and I_{v_n}, $\mathbf{Z}_n$ both converge uniformly over $[0, 1] \times \bar{\Omega}$ to $\mathbf{g}_\mathbf{v}$ and $\mathbf{I}_{\mathbf{v},\mathbf{Z}}$.*

Finally, we quote this last result from Trouvé (1999), which is an almost direct consequence of the previous lemmas. For $R > 0$, we let $\mathcal{B}_\mathcal{K}(R)$ be the closed ball in $\mathcal{K}$ with radius R.

Theorem 4. *If condition (H3) is true, then, for all $I \in \mathcal{I}$ and all $R > 0$, the set $\mathcal{V}(I) \cap \mathcal{B}_\mathcal{K}(R)$ is weakly compact.*

The next result deals with the lower semi-continuity of the matching functional:

Theorem 5. *Assume conditions (H1) to (H3). Then, the functional U is weakly lower-semi-continuous on $\mathcal{K}$*

We must prove that, if a sequence $(\mathbf{v}_n, \mathbf{Z}_n)$ weakly converges to $(\mathbf{v}, \mathbf{Z}) \in \mathcal{K}$, then $U(\mathbf{v}, \mathbf{Z}) \le \liminf U(\mathbf{v}_n, \mathbf{Z}_n)$.

Our first step in this proof is the following lemma. We let $U_{\mathbf{v},\mathbf{Z}}$ be the functional defined on $\mathcal{K}$ by

$$U_{\mathbf{v},\mathbf{Z}}(\mathbf{w}, \mathbf{Y}) = \int_0^1 N_\mathcal{G}(\mathbf{w}(t, \cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{Z}}(t, \cdot))^2\, dt + \int_0^1 N_\mathcal{I}(\mathbf{Y}(t, \cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{Z}}(t, \cdot))^2\, dt$$

Lemma 3. *If* $(\mathbf{v}_n, \mathbf{Z}_n)$ *is a bounded sequence which weakly converges in* $\mathcal{K}$ *to* $(\mathbf{v}, \mathbf{Z})$, *then*

$$\lim_{n\to\infty} |U(\mathbf{v}_n, \mathbf{Z}_n) - U_{\mathbf{v},\mathbf{Z}}(\mathbf{v}_n, \mathbf{Z}_n)| = 0$$

By assumption, there exists $R > 0$ such that $(\mathbf{v}_n, \mathbf{Z}_n) \in \mathcal{B}_{\mathcal{K}}(R)$ for all n. We have, for all t,

$$\begin{aligned}
&|N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v}_n,\mathbf{z}_n}(t,\cdot))^2 - N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{z}}(t,\cdot))^2| \\
&\quad \leq \sqrt{2}\sqrt{N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v_n},\mathbf{z_n}}(t,\cdot))^2 + N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{z}}(t,\cdot))^2} \\
&\qquad \times |N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v_n},\mathbf{z_n}}(t,\cdot)) - N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{z}}(t,\cdot))| \\
&\quad \leq 2\kappa^2 \bar{N}_{\mathcal{G}}(\mathbf{v}_n(t,\cdot))\|\mathbf{I}_{\mathbf{v_n},\mathbf{z_n}}(t,\cdot) - \mathbf{I}_{\mathbf{v},\mathbf{z}}(t,\cdot)\|_\infty
\end{aligned}$$

by conditions (H1) and (H2). This implies

$$\begin{aligned}
&\left|\int_0^1 N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v}_n,\mathbf{Z}_n}(t,\cdot))^2\,dt \right.\\
&\quad \left. - \int_0^1 N_{\mathcal{G}}(\mathbf{v}_n(t,\cdot); I_0 + \mathbf{I}_{\mathbf{v},\mathbf{Z}}(t,\cdot))^2\,dt\right| \\
&\quad \leq\; 2\kappa^2 R\|\mathbf{I}_{\mathbf{v}_n,\mathbf{Z}_n} - \mathbf{I}_{\mathbf{v},\mathbf{Z}}\|_\infty
\end{aligned}$$

which tends to 0 by Lemma 2. The same argument holds for the second integral in U.

We then have

Lemma 4. *The functional* $U_{\mathbf{v},\mathbf{Z}}$ *is weakly lower-semi-continuous.*

By a standard theorem of functional analysis a strongly continuous convex functional is weakly lower-semi-continuous. Since $U_{\mathbf{v},\mathbf{Z}}$ is obviously convex, we must only show that it is strongly continous. However, this property trivially derives from the fact that, for all $f_1,\ f_2$ and for all I,

$$\begin{aligned}
&|(N_{\mathcal{G}}(f_1; I)^2 - N_{\mathcal{G}}(f_2; I)^2| \\
&\quad \leq\; |(N_{\mathcal{G}}(f_1; I) + N_{\mathcal{G}}(f_2; I))N_{\mathcal{G}}(f_1 - f_2; I) \\
&\quad \leq\; \kappa(\bar{N}_{\mathcal{G}}(f_1) + \bar{N}_{\mathcal{G}}(f_2))\bar{N}_{\mathcal{G}}(f_1 - f_2)
\end{aligned}$$

and the similar inequality for $N_{\mathcal{I}}$.

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Notes

1. In fact, as will be seen later, the acting groups $\mathcal{G}$ in the case of deformable templates, and the group $\mathcal{G}$ in this formulation do not have exactly the same meaning; the correct point of view implies that the latter is a *subgroup* of the former, see Remark 1.
2. If E is a set, a mapping $d : E \times E \to [0, +\infty[$ is a distance (or a metric) if and only if the three following properties are satisfied:

    - $d(x, x') = 0 \Leftrightarrow x = x'$
    - $d(x, x') = d(x', x)$ (symmetry)
    - $d(x, x') + d(x', x'') \leq d(x, x'')$ (triangular inequality).

    .
3. To keep this paper essentially self-contained, we shall avoid refering to too many concepts of differential geometry, although our discussion is obviously based on this theory.
4. In the following, we use boldface letters **a**, **g** to refer to paths on $\mathcal{C}$, $\mathcal{G}$, or any other space, and leave roman letters for denoting individual elements of these sets.
5. Notice that the function **g** and depends on both time variable $t \in [0, 1]$ and space variable $x \in \Omega$.
6. To be rigorous, we must assume that a structure of *measurable space* is placed on $\mathcal{A}$ and $\mathcal{F}$, that is, that these sets are equipped with σ-algebras, and that probability distributions are defined with respect to these σ-algebras. We do not formally introduce these quantities, in order to limit the notational burden of this section. The concerned reader will easily complete these gaps.
7. In Cooper et al. (1996), the intensity is a temperature field measured over the surface of the template.

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