Subgroups of Lie groups and separation of variables [a)]

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Separable systems of coordinates for the Helmholtz equation $\Delta_d \Psi = E\Psi$ in pseudo-Riemannian spaces of dimension d have previously been characterized algebraically in terms of sets of commuting second order symmetry operators for the operator Δ_d. They have also been characterized geometrically by the form that the metric $ds^2 = g_{ik}(x)dx^i dx^k$ can take. We complement these characterizations by a group theoretical one in which the second order operators are related to continuous and discrete subgroups of G, the symmetry group of Δ_d. For $d = 3$ we study all separable coordinates that can be characterized in terms of the Lie algebra L of G and show that they are of eight types, seven of which are related to the subgroup structure of G. Our method clearly generalizes to the case $d > 3$. Although each separable system corresponds to a pair of commuting symmetry operators, there do exist pairs of commuting symmetries S_1,S_2 that are not associated with separable coordinates. For subgroup related operators we show in detail just which symmetries S_1,S_2 fail to define separation and why this failure occurs.

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I. INTRODUCTION

The purpose of this article is to investigate the relationship between separation of variables in the Helmholtz equation for a pseudo-Riemannian space and the subgroup structure of the invariance Lie group of the equation. The article thus brings together the results of three different research programs that have been actively pursued during the past few years. These are (i) a systematic algebraic approach to the separation of variables in p.d.e.[1–18]; (ii) the classification of Lie subgroups of Lie groups[1,19–28]; (iii) applications of discrete subgroups of Lie groups.[29–33]

Historically, the approach to separation of variables has been in terms of Riemannian and differential geometry.[34–38] In the algebraic approach[1–18] for a d-dimensional manifold the Helmholtz equation

$$\Delta_d \Psi = E\Psi(x) \tag{1.1}$$

is considered, where $x = (x_1,x_2,...,x_d)$ is a local coordinate system and Δ_d is the Laplace–Beltrami operator on the manifold. It is assumed that Eq. (1.1) has the Lie symmetry group G. Its Lie algebra L consists of first order linear operators X satisfying $[\Delta_d, X] = 0$, and we choose a basis $\{X_1,...X_n\}$ for L. Separable coordinates for Eq. (1.1) are associated with $(d-1)$-tuplets of commuting second-order symmetry operators $\{S_\alpha\}$ for Δ_d. A classification of the sets of operators $\{S_\alpha\}$ into orbits under the action of G provides a classification of separable systems of coordinates. The separable functions

$$\Psi(x) = \prod_{i=1}^{d} f_i(\xi_i) \tag{1.2}$$

are the common eigenfunctions of the operators Δ_d and S_α $(1 \leqslant \alpha \leqslant d-1)$.

There are some puzzling aspects to the algebraic approach. First of all, while there is a mechanical procedure for computing the symmetries $\{S_\alpha\}$ from a separable system of coordinates, the precise relationship between the $\{S_\alpha\}$ and the subgroup structure of G has remained unclear. Furthermore, there exist commuting symmetries $\{S_\alpha\}$ that do not correspond to any separable coordinates at all! The discovery of practical criteria to determine precisely which commuting symmetries lead to variable separation remains one of the most important problems in this theory. Here we show for $d = 3$ the relation between the subgroup structure of G and the coordinate systems yielding separation of variables for the Helmholtz equation on the manifold. (This analysis clearly generalizes to the case $d > 3$.) Furthermore, for subgroup related operators $\{S_\alpha\}$ we show in detail which symmetries fail to define variable separation and why this failure occurs.

Section 2 is devoted to the general theory. We show that separable coordinates fall into different classes, depending on how many of the operators in the set $\{S_\alpha\}$ are squares of the linear operators X (these correspond to Abelian subgroups of G), how many are invariant operators of nonAbelian Lie subgroups, and how many are invariants of discrete subgroups. In Sec. 3 we treat three-dimensional manifolds of constant curvature in some detail.

2. GENERAL THEORY

Let Δ_d be the Laplace–Beltrami operator on a d-dimensional pseudo-Riemannian manifold with metric $ds^2 = \Sigma_{i,j=1}^d g_{ij}dx^i dx^j$, i.e.,

$$\Delta_d \Psi = \sum_{i,j} g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j\psi), \tag{2.1}$$

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where $g = \det(g_{ij})$, $\partial_i = \partial/\partial x^i$, and $\Sigma_j g^{ij} g_{jk} = \delta^i_k$. The Helmholtz equation for this manifold is

$$\Delta_d \Psi = E\Psi, \qquad (2.2)$$

where E is a nonzero constant. In Refs. 17 and 18 the possible coordinate systems that permit separation of variables for the Helmholtz equation have been classified in the cases $d = 2,3,4$. The classification of separable types is closely related to the symmetry algebra of Eq. (2.2). A first order symmetry operator X for Eq. (2.2) is an operator

$$X = \sum_{i=1}^{d} \xi_i(x^j)\partial_i, \qquad (2.3)$$

such that $[X,\Delta_d] = 0$, where $[\cdot,\cdot]$ is the usual commutator of differential operators. (This is equivalent to the assertion that $\{\xi_i\}$ is a Killing vector.) The set of all first order symmetries of Eq. (2.2) forms a Lie algebra L with $\dim L \leqslant d(d+1)/2$. If $\{x^1,...,x^d\}$ is a separable system for Eq. (2.2), we say the variable x^1 is *ignorable* provided $X = \partial_1 \in L$, i.e., provided the tensor g_{ij} in these coordinates is independent of x^1.

In this paper we restrict ourselves to the case $d = 3$. For $d = 3$ each separable system $\{x^1,x^2,x^3\}$ is characterized by a pair of second order differential operators $\{S_1,S_2\}$ such that

$$[S_1,S_2] = 0, \quad [S_j,\Delta_3] = 0, \quad j = 1,2. \qquad (2.4)$$

Here the corresponding separable solutions $\psi = A(x^1)B(x^2)C(x^3)$ of Eq. (2.3) have the characterization

$$S_j\psi = \lambda_j\psi, \quad j = 1,2, \qquad (2.5)$$

where the eigenvalues λ_j are separation constants.

As shown in Ref. 17 the separable systems are of eight distinct types: (I) Three ignorable variables:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \epsilon(dx^3)^2, \epsilon = \pm 1,$$
$$S_1 = \partial_1^2, S_2 = \partial_2^2. \qquad (2.6)$$

Here, L contains a three-dimensional Abelian subalgebra generated by $L_j = \partial_j, j = 1,2,3$, and the manifold is flat. Note that the operator $S_3 = \partial_3^2$ is automatically diagonalized in this case. (II) Two ignorable variables:

$$ds^2 = \sum_{i,j=1}^{3} g_{ij}(x^3)dx^i dx^j, \quad S_1 = \partial_1^2, S_2 = \partial_2^2. \qquad (2.7)$$

Here, L contains a two-dimensional Abelian subalgebra A generated by $L_j = \partial_j, j = 1,2$. The coordinates may be nonorthogonal. The subalgebra A must be maximal Abelian since otherwise the system would be type I. (III) One ignorable variable: This case splits into four subtypes, for each of which L contains the operator $L_1 = \partial_1$, and we have $S_1 = \partial_1^2$:
(III_1) Centralizer coordinates (orthogonal):

$$ds^2 = (dx^1)^2 + (\sigma_2(x^2) + \sigma_3(x^3))[(dx^2)^2 + \epsilon(dx^3)^2],$$
$$S_2 = \frac{1}{\sigma_2 + \sigma_3}[\sigma_3\partial_2^2 - \epsilon\sigma_2\partial_3^2]. \qquad (2.8)$$

(III_2) Centralizer coordinates (nonorthogonal):

$$ds^2 = \sigma_2[\sigma_3(dx^2)^2 + 2dx^1dx^2 + (dx^3)^2],$$
$$S_2 = \partial_3^2 - \sigma_3\partial_1^2. \qquad (2.9)$$

(III_3) Subgroup coordinates:

$$ds^2 = \sigma_3(dx^3)^2 + \sigma_3\sigma_2[(dx^1)^2 + \epsilon(dx^2)^2],$$
$$S_2 = \frac{1}{\sigma_2}(\partial_1^2 + \epsilon\partial_2^2). \qquad (2.10)$$

(III_4) Generic type III coordinates:

$$ds^2 = (\sigma_2 + \sigma_3)[dx^2)^2 + \epsilon_1(dx^3)^2] + \epsilon_2\sigma_2\sigma_3(dx^1)^2,$$
$$S_2 = \epsilon_2\left(\frac{1}{\sigma_3} - \frac{1}{\sigma_2}\right)\partial_1^2 + \frac{1}{\sigma_2+\sigma_3}(\sigma_3\partial_2^2 - \epsilon_1\sigma_2\partial_3^2) \qquad (2.11)$$
$$+ \frac{1/2}{(\sigma_2+\sigma_3)}\left(\frac{\sigma_3\sigma_2'}{\sigma_2}\partial_2 - \frac{\epsilon_1\sigma_2\sigma_3'}{\sigma_3}\partial_3\right), \quad \epsilon_j = \pm 1.$$

(IV) No ignorable variables: Here there are two subtypes:
(IV_1) We have

$$ds^2 = \sigma_1^2(dx^1)^2 + \sigma_1(\sigma_2 + \sigma_3)[(dx^2)^2 + \epsilon(dx^3)^2],$$
$$S_1 = \frac{1}{\sigma_2+\sigma_3}(\partial_2^2 + \epsilon\partial_3^2), S_2 = \frac{1}{\sigma_2+\sigma_3}(\sigma_3\partial_2^2 - \epsilon\sigma_2\partial_3^2). \qquad (2.12)$$

(IV_2) Generic coordinates:

$$ds^2 = (\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(dx^1)^2 + (\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)(dx^2)^2 + (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)(dx^3)^2, \qquad (2.13)$$
$$S_1 = \frac{\sigma_2+\sigma_3}{(\sigma_1-\sigma_2)(\sigma_3-\sigma_1)}\partial_1^2 + \frac{\sigma_3+\sigma_1}{(\sigma_2-\sigma_3)(\sigma_1-\sigma_2)}\partial_2^2 + \frac{\epsilon(\sigma_1+\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_2-\sigma_3)}\partial_3^2,$$
$$S_2 = \frac{\sigma_2\sigma_3}{(\sigma_1-\sigma_2)(\sigma_3-\sigma_1)}\partial_1^2 + \frac{\sigma_3\sigma_1}{(\sigma_2-\sigma_3)(\sigma_1-\sigma_2)}\partial_2^2 + \frac{\epsilon(\sigma_1\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_2-\sigma_3)}\partial_3^2.$$

In all of the above expressions $\sigma_j = \sigma_j(x^j)$. We refer to systems III_4 and IV_2 as "generic" since all other systems of types III and IV are degenerate cases of these two. It is only for Minkowski space $E_{2,1}$ that all eight separable types actually occur. As shown in Ref. 17, types I, III_1, and III_2 do not appear for space of nonzero constant curvature.

In this paper we are concerned with a purely group theoretic characterization of the various separation types. To successfully characterize a separable system $\{x^j\}$ for Eq. (2.2) in terms of the symmetry algebra L it is necessary that the defining operators S_1,S_2 for the system belong to the enveloping algebra of L. If this is so, we say that the coordinates $\{x^j\}$ are of *class I*; otherwise they are of *class II*. Reference 17 contains a derivation of all class I coordinates for all types except IV_2.

We now describe a general group theoretic procedure for characterizing all class I coordinates associated with the Helmholtz equation on a three-dimensional Riemannian manifold with symmetry algebra L. The validity of this procedure will be demonstrated using the results of Ref. 17 but will also be illustrated by examples in 3. The procedure is as follows:

First we determine if L contains a maximal Abelian subalgebra of dimension 3. This will be the case if and only if the manifold is flat and corresponds to type I (Cartesian) coordinates. Then we find the (conjugacy classes of) maximal Abelian subalgebras of dimension 2. Each such subalgebra determines a type II system.

Next we determine the conjugacy classes of one dimensional subalgebras of L. Let X be a representative from such a class and let cent(X) be the centralizer of X in L. There are four possibilities:

Type III_1: $\mathrm{cent}(X) = \{X\} \oplus (\mathrm{cent}(X)/\{X\})$, cent ($X$) non-Abelian: (2.14)
Let $L_X = (\mathrm{cent}(X)/\{X\})$ and decompose the space of second order elements in the enveloping algebra of L_X into orbits under the action of the normalizer Nor(X) of X in G. Every type III_1 system with $S_1 = X^2$ has the property that S_2 is a representative from one of these orbits. Two representatives from the same orbit correspond to equivalent coordinates.

Type III_2: $\mathrm{cent}(X) \neq \{X\} \oplus (\mathrm{cent}(X)/\{X\})$, cent($X$) non-Abelian: (2.15)
Decompose the space of second order elements in the enveloping algebra of cent(X) into orbits under the action of Nor(X). Every type III_2 system with $S_1 = X^2$ has the property that S_2 is a representative from one of these orbits. Class I coordinates of this type arise only for flat space.

Type III_3: Subgroup type coordinates: (2.16)
Given the one-dimensional subalgebra X, find all subalgebras A of L such that (1) $A \supset X$ (properly), (2) A is non-Abelian, and (3) A has a second order Casimir operator S_2, not equal to Δ_3 or to a linear combination of Δ_3 and the square of an element of L. Every type III_3 system is of the form $S_1 = X^2$, S_2.

Type III_4: Generic type III coordinates: (2.17)
Let X be as above and determine the space S of all second order elements Y in the enveloping algebra of L such that $[X,Y] = 0$. Decompose S into orbits under the adjoint action of Nor(X) and let S_2 be a representative from such an orbit. Every type III_4 system is of the form $S_1 = X^2, S_2$ such that this commuting pair has not already been included under types I–III_3 listed above.

The remaining two types characterize all pairs S_1,S_2 for which neither operator is a perfect square:

Type IV_1: Semisubgroup coordinates: (2.18)
Consider the three-dimensional subalgebras A of L with properties (2) and (3) discussed above in III_3. Take S_1 to be the Casimir operator of such an A and S_2 to be a second order element in the envoloping algebra of A. (Operators S_2 and S'_2 are considered equivalent if they lie on the same orbit under the adjoint action of the maximal group of symmetries whose Lie algebra is A.) Every type IV_1 system is of the form S_1,S_2.

Type IV_2: Generic coordinates: (2.19)
This is the generic case. Here S_1,S_2 are simply a pair of commuting second order symmetries in the enveloping algebra of L, classified into orbits under the action of the symmetry group G, and such that this pair has not already been included under types I–IV_1 above.

For types I, II, and III_3 both operators S_1 and S_2 are invariants of Lie subgroups of G. For III_1, III_2, III_4, and IV_1 only S_1 has this property; for IV_2 neither of the operators is directly related to a Lie subgroup. The group G also contains discrete subgroups and is itself not necessarily connected. We shall see below that those operators S_1 that are not invariants of Lie groups can be characterized by the fact that they occur as invariants of discrete subgroups of G.

Now we demonstrate the validity of our group theoretic classification of defining operators for class I separable coordinates on a three-dimensional Riemannian manifold. First we note that every orbit of two-dimensional vector spaces, each space composed of mutually commuting second order elements in the enveloping algebra of L, belongs to exactly one of the eight classes listed above. Thus, it will be sufficient for us to show that the defining operators S_1,S_2 corresponding to a class I separable system of a given type (2.6)–(2.18) themselves have the group theoretic characterization for the corresponding type listed above. For this we draw on the results of Ref. 17.

The group theoretic characterization of types I and II is obvious.

(III_1) Centralizer coordinates (orthogonal): It follows from the results of Sec. 5 in Ref. 17 that the separable system (2.8) is class I precisely when

$$ds^2 = (dx^1)^2 + d\omega^2(x^2,x^3),$$

where $d\omega^2$ is the metric for a two dimensional Riemannian space of constant curvature [with Lie algebra L' isomorphic to one of e(3), e(2,1), o(4), o(3,1), o(2,2)] and S_2 a second order element in the enveloping algebra of L' which is not a square. Here $L \supseteq \{X\} \oplus L'$, where $X = \partial_1$, so the pair S_1,S_2 is of the form (2.14).

(III_2) Centralizer coordinates (nonorthogonal): According to Ref. 17, coordinates (2.9) are class I only for flat space and the possibilities are listed in Sec. 4 of that paper. One can directly verify that in each case the operators S_1,S_2 are of the form (2.15).

(III_3) Subgroup coordinates: In Ref. 17 it is shown that coordinates (2.10) are class I precisely when

$$ds^2 = \sigma_3(dx^3)^2 + \sigma_3 d\omega^2(x^1,x^2),$$

where $d\omega^2$ is the metric for a two dimensional space of constant curvature, $X = \partial_1$ is a Lie symmetry of $d\omega^2$, and S_2 is the Laplace–Beltrami operator for this two-dimensional space. With $X = \partial_1, S_1 = X^2$ it follows that S_1,S_2 is of the form (2.16).

(III_4) Generic type III coordinates: According to Ref. 17 coordinates (2.11) are class I if and only if the manifold is a space of constant curvature. These coordinates cannot be type III_3 because, as is straightforward to verify for spaces of constant curvature, the subalgebras A in the definition of type III_3 must have Casimir operators that are Laplace–Beltrami operators on two-dimensional manifolds. The operator S_1 [Eq. (2.11)] is clearly not a Laplace–Beltrami operator. The coordinates cannot be type III_2 because among the symmetry algebras for spaces of constant curvature only e(2,1) contains an element X such that $\mathrm{cent}(X) \neq \{X\} \oplus L_X$ and cent(X) is non-Abelian. For this case all corresponding orbits of operators S_2 in the enveloping algebra of cent(X) were computed in Ref. 17 and the coordinates were shown to be of the form (2.9). If the coordinates (2.11) were type III_1, then the manifold would be flat, because among the symmetry algebras for constant curvature spaces, only e(3) and e(2,1) contain elements X such that $\text{cent}(X) = \{X\} \oplus L_X$ with $\text{cent}(X)$ non-Abelian. These cases are classified in the following section and shown to correspond to coordinates (2.8). Thus, class I coordinates (2.11) correspond to operators of the form (2.17).

(IV_1) Semisubgroup coordinates: It is shown in Ref. 17 that coordinates (2.12) are class I provided

$$ds^2 = \sigma_1^2 (dx^1)^2 + \sigma_1 d\omega^2(x^2,x^3) ,$$

where $d\omega^2$ is the metric for a two-dimensional subspace of constant curvature. It is clear from Eq. (2.12) that S_1 is the Laplace–Beltrami operator on this subspace; hence, the Casimir operator for the symmetry algebra L' of the subspace, where $L' \subseteq L$. Since S_2 is defined on the subspace and commutes with S_1, it must be expressable in terms of second order elements in the enveloping algebra of L'. Thus, operators S_1, S_2 are of the form (2.18).

(IV_2) Generic coordinates: Class I coordinates (2.13) cannot be of operator types I–III since we can see by inspection that one cannot construct from a linear combination of S_1 and S_2 an operator which is a perfect square of a Lie symmetry. The operators cannot be of type IV_1 because the only possible choices for the algebra A are e(2), e(1,1), o(3), o(2,1) acting as transitive symmetry algebras on a two-dimensional submanifold. It follows in these cases that the Casimir operator of A is the Laplace–Beltrami operator on the submanifold, and hence that S_1, S_2 can be written in the form (2.12) for appropriate coordinates. Since a set of orthogonal separable coordinates is uniquely determined by its defining operators S_1, S_2 (see Ref. 34), these coordinates must be of the form (2.12), a contradiction. Hence, class I coordinates (2.13) correspond to operators (2.19).

The above results hold for all Riemannian manifolds admitting class I separable coordinates, and there are an infinite number of such manifolds. However, of special interest are the manifolds of constant curvature, since they have the property that all separable coordinates are class I. In the following section we shall study the symmetry algebra L of each of the three-dimensional constant curvature spaces to see in detail how the subalgebra structure of L corresponds to the separable coordinates I–IV_1. We provide a complete orbit analysis for all pairs of commuting operators that correspond to proper subalgebras of L, i.e., for all operator types except IV_2. In a number of cases we will uncover orbits of type III_4 operators that do not correspond to variable separation.

3. THREE-DIMENSIONAL SPACES OF CONSTANT CURVATURE

In this section we illustrate the general theory by considering all spaces of constant curvature.

A. Group E(3)

The algebra e(3) of the group E(3) is generated by the infinitesimal rotations L_i and translations P_i, satisfying the commutation relations

$$[L_i, L_k] = \epsilon_{ikl} L_l, \quad [L_i, P_k] = \epsilon_{ikl} P_l, \quad [P_i, P_k] = 0 . \qquad (3.1)$$

It has two Casimir operators, namely,

$$\Delta = \mathbf{P}^2 = P_1^2 + P_2^2 + P_3^2 \quad \text{and}$$
$$\Delta' = \mathbf{L}\cdot\mathbf{P} = L_1 P_1 + L_2 P_2 + L_3 P_3 . \qquad (3.2)$$

For the representations considered here we have $\Delta' = 0$ (a space of scalar functions in Euclidean space).

The subalgebras of e(3) have been classified into orbits under the action of E(3), e.g., in Ref. 19, where the results are presented in a diagram. Let us use this classification to investigate different types of separable coordinates for the equation $\Delta \Psi = E \Psi$, following Sec. 2.

I. *Three ignorable variables*

The algebra e(3) has precisely one class of maximal Abelian subalgebras (MASA) of dimension 3 represented by $\{P_1, P_2, P_3\}$. This provides *Cartesian coordinates* for which

$$S_i = P_i^2, \quad i = 1,2,3; \quad \Delta = S_1 + S_2 + S_3 . \qquad (3.3)$$

II. *Two ignorable variables*

The algebra e(3) has precisely one class of MASA of dimension 2, represented by $\{L_3, P_3\}$. This provides *cylindrical coordinates*, for which

$$S_1 = L_3^2, \quad S_2 = P_3^2 . \qquad (3.4)$$

III. *One ignorable variable*

To find coordinates of type III_1 and III_2 we must consider separately a representative X of each class of one-dimensional (Abelian nonmaximal) subalgebras and find its centralizer cent$X$ in e(3). We are only interested in non-Abelian centralizers. The only type of element of e(3) having a non-Abelian centralizer can be represented by P_3, where

$$\text{cent}(P_3) = \{P_3\} \oplus \{L_3, P_1, P_2\} , \qquad (3.5)$$

i.e., centP_3 splits into a direct sum of P_3 and $\{\text{cent}(P_3)\}/\{P_3\}$. Hence, no III_2 type coordinates exist in this case. Type III_1 coordinates (orthogonal centralizer type coordinates) are obtained by putting

$$S_1 = P_3^2 , \qquad (3.6)$$

$$S_2 = a L_3^2 + b(L_3 P_1 + P_1 L_3) + c(L_3 P_2 + P_2 L_3)$$
$$+ d(P_1^2 - P_2^2) + 2e(P_1 P_2) + f(P_1^2 + P_2^2) , \qquad (3.7)$$

i.e., S_2 is the most general symmetric second order operator in the enveloping algebra of $\text{e}(2) = \{L_3, P_1, P_2\}$. We must now classify the operators (3.7) into orbits under Nor(P_3), i.e., the normalizer of P_3 in E(3). This is a well-known problem.[1,15] These orbits can be represented by

$$P_1^2, L_3^2, \ L_3^2 + a(P_1^2 - P_2^2), (a > 0), \ \text{and} \ L_3 P_2 + P_2 L_3 . \qquad (3.8)$$

The first two operators should be omitted, since they are squares of generators and lead back to the case I or II. The last two operators provide type III_1 coordinates, namely, *elliptic cylindrical* and *parabolic cylindrical* coordinates, respectively.

Type III_3 coordinates (subgroup type) are obtained by taking a representative X of each orbit of generators of e(3) and finding all proper subalgebras of e(3) that properly contain X, are non-Abelian, and have a second order Casimir operator, not equal to $\Delta = \mathbf{P}^2$ or to a linear combination of Δ

and the square of a generator. The only such chain of subalgebras is

$$e(3) \supset o(3) \supset o(2),$$

and we have

$$S_1 = L_3^2, \ S_2 = \mathbf{L}^2, \tag{3.9}$$

i.e., S_2 is the Casimir operator of o(3), providing *spherical* coordinates.

Type III_4 coordinates are obtained by running through all representative generators X, and for each X finding the most general second order operator S_2 in the enveloping algebra of e(3) satisfying $[X,S_2] = 0$. We find a representative of each orbit and eliminate representatives already encountered, i.e., corresponding to squares of generators, members of the enveloping algebra of cent(X), or Casimir operators of subalgebras. Let us examine each case separately.

(i) $X = L$, nor $L_3 = \{L_3,P_3\}$, and

$$S_2 = a\mathbf{L}^2 + b(L_1P_2 + P_2L_1 - L_2P_1 - P_1L_2) + c(P_1^2 + P_2^2) + dL_3P_3 \tag{3.10}$$

[we have dropped the Casimir operator of e(3) from Eq. (3.10)]. Separable coordinates (u,v,Φ) of this type satisfy

$$L_3 \equiv \frac{\partial}{\partial\Phi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial x}{\partial\Phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\Phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\Phi}\frac{\partial}{\partial z}. \tag{3.11}$$

Hence, we have

$$\frac{\partial x}{\partial\Phi} = -y, \quad \frac{\partial y}{\partial\Phi} = x, \quad \frac{\partial z}{\partial\Phi} = 0. \tag{3.12}$$

The relations (3.12) imply

$$x = f(u,v)\cos\Phi, y = f(u,v)\sin\Phi, z = h(u,v). \tag{3.13}$$

The operators

$$S_1 = L_3^2 \text{ and } S_2$$

in Eq. (2.11) are invariant under the reflection $\Phi \to -\Phi$ (i.e., $y \to -y$). Since L_3P_3 does not have this invariance, property, we must put $d = 0$ in Eq. (3.10), i.e., operator (3.10) with $d \neq 0$ does not correspond to variable separation. We can now use the translation $\exp\alpha P_3$, belonging to the normalizer of L_3 in E(3), to simplify S_2. For $a \neq 0$ we can reduce Eq. (3.10) to

$$S_2 = \mathbf{L}^2 + c(P_1^2 + P_2^2), \tag{3.14}$$

For $c > 0$ and $c < 0$ this corresponds to oblate and prolate spheriodal coordinates, respectively. If $a = 0$, $b \neq 0$, we can reduce S_2 to

$$S_2 = L_1P_2 + P_2L_1 - L_2P_1 - P_1L_2, \tag{3.15}$$

corresponding to *parabolic* coordinates.

If $a = b = 0$ we return to type II coordinates.

(ii) $X = P_3$; nor $P_3 = \{L_3,P_1,P_2,P_3\}$: We have

$$S_2 = aL_3^2 + b(L_3P_1 + P_1L_3) + c(L_3P_2 + P_2L_3) + dL_3P_3 + c_{ik}P_iP_k. \tag{3.16}$$

The coordinates (u,v,x_3) satisfy

$$P_3 \equiv \frac{\partial}{\partial x_3} = \frac{\partial x}{\partial x_3}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x_3}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x_3}\frac{\partial}{\partial z}.$$

Hence,

$$x = x(u,v), \quad y = y(u,v), \quad z = x_3,$$

and L_3P_3 changes sign under the reflection $z \to -z$. Hence, $d = 0$ in Eq. (3.16) in order to yield variable separation. Similarly, $c_{i3} = c_{3i} = 0$.

If $a \neq 0$, we use the normalizer of P_3 to reduce S_2 to $S_2 = L_3^2 + c(P_1^2 - P_2^2)$, corresponding to II or III_1 type coordinates. If $a = 0$, $b^2 + c^2 \neq 0$, we obtain $S_2 = L_3P_1 + P_1L_3$, corresponding to the type III_1. If $a = b = c = 0$, we obtain type I coordinates.

(iii) $X = L_3 + aP_3$; nor $(L_3 + aP_3) = \{L_3,P_3\}$: A straightforward computation shows that in this case S_2 satisfying $[X,S_2] = 0$ can be reduced to

$$S_2 = P_3(bL_3 + cP_3). \tag{3.17}$$

Since L_3 and P_3 commute, a diagonalization of P_3 and L_3 separately is equivalent to a diagonalization of any polynomials in L_3 and P_3. We thus reobtain case II.

(IV) No ignorable variables: Neither of the operators S_1 or S_2 is the square of a generator of e(3).

Type (IV_1): We return to the non-Abelian subalgebras of e(3) discussed above in III_3. We take S_1 to be the Casimir operator of such a subalgebra and S_2 some second order element of the enveloping algebra of the corresponding subalgebra. These operators S_2 must be classified into orbits under the group A whose Lie algebra is A. Only one such case occurs for e(3), namely

$$S_1 = \mathbf{L}^2, \ S_2 = L_1^2 + rL_2^2, \quad 0 < r < 1, \tag{3.18}$$

corresponding to *spheroconical* coordinates (S_2 is not allowed to be the square of a generator).

Type (IV_2): Here S_1 and S_2 are simply commuting second order operators in the enveloping algebra of e(3). Neither of them is the square of a generator nor a Casimir operator of any Lie algebra. This is the generic case with the lowest symmetry. The remaining coordinates *ellipsoidal* and *paraboloidal* are of this type.

This completes the list of all 11 types of separable coordinates in Euclidean 3-space.

Finally, let us discuss the question of discrete symmetries that further characterize some of the coordinate systems. Indeed, for coordinates of the type III_1, III_4, and IV_1 only one of the diagonal operators is characterized by the fact that it is an invariant operator of a one or higher dimensional Lie algebra. For coordinates of the type IV_2 neither S_1 nor S_2 has this property. These operators will, in general, be invariants of certain discrete subgroups of E(3). No operator of the type

$$S = a_{ik}L_iL_k + b_{ik}P_iP_k + c_{ik}(L_iP_k + P_kL_i) \tag{3.19}$$

is left invariant by discrete translations (unless $a_{ik} = c_{ik} = 0$ and we have continuous translational invariance). We can hence restrict ourselves to point groups and indeed to groups of reflections in planes through the origin. Let us use X, Y, and Z to denote a reflection of the coordinate x, y, and z, respectively, and $I_2n\,(A_1,\dots,A_n)$ to denote the Abelian group of order 2^n generated by $A_1,\dots,A_n$. By inspection we see that the operators S_i not related to Lie subgroups have the following invariance groups:

$$
\begin{aligned}
&L_3^2 + a(P_1^2 - P_2^2) && : I_4(X,Y)\,, \\
&L_3P_2 + P_2L_3 && : I_2(Y)\,, \\
&L_1P_2 + P_2L_1 - L_2P_1 - P_1L_2 && : I_4(X,Y)\,, \\
&\mathbf{L}^2 \pm a(P_1^2 + P_2^2) && : I_4(Z,X\ Y)\,, \\
&\left.\begin{array}{l} L_2^2 + aL_1^2 + bP_3^2 \\ \mathbf{L}^2 + bP_1^2 + aP_2^2 + (a+b)P_3^2 \end{array}\right\} && : I_8(X,Y,Z)\,, \\
&\left.\begin{array}{l} L_3^2 - c^2P_3^2 + c(L_2P_1 + P_1L_2 + L_1P_2 + P_2L_1) \\ L_2P_1 + P_1L_2 - L_1P_2 - P_2L_1 + c(P_2^2 - P_1^2) \end{array}\right\} && : I_4(X,Y)\,.
\end{aligned}
\tag{3.20}
$$

Thus, the operators S_1,S_2 for each of the 11 separable coordinate systems can be viewed as corresponding to a certain subgroup reduction of E(3) and both Lie subgroups and discrete subgroups figure in the reductions. The subgroups will determine the symmetry properties of the separated solutions of the Helmholtz equations. In particular, the discrete subgroups are often important in physical applications, especially in the context of "symmetry adapted basis functions" in molecular physics and general many body theories.[5,29-33]

The results of this section are summarized in Table I. We do not spell out the explicit form of the coordinates. The ones used are listed, for example, in Ref. 38.

B. The group O(4)

Separable systems of coordinates in s_3, the unit sphere, were first obtained by Eisenhart[34] and studied from the algebraic point of view in Ref. 13. Let us now classify them from the subgroup point of view. The continuous subgroups of O(4) are listed, for example, in Ref. 20 (they were first obtained by Goursat[39]).

Using the isomorphism $o(4)\sim o(3)\oplus o(3)$ we write the algebra o(4) as $\{A_i,B_i,i=1,2,3,\}$, satisfying

$$[A_i,A_k] = \epsilon_{ikl}A_l,\ \ [B_i,B_k] = \epsilon_{ikl}B_l,\ \ [A_i,B_k] = 0\,. \tag{3.21}$$

The algebra o(4) has precisely one MASA [up to conjugacy under O(4)], namely, $\{A_3,B_3\}$. Hence, no class I systems exist and just one class II system. The one-dimensional subalgebras are A_3,$A_3 + xB_3(0<x<1)$, and $A_3 + B_3$. No type III_1 or III_2 coordinates exist on s_3; III_2 is excluded because cent(A_3) is a direct sum and III_1 is not realized because the operators $(A_3^2, B_1^2 + k^2B_2^2)$ would correspond to separation on $s_2\otimes s_2$ rather than s_3 (the III_1 and III_2 type coordinates only exist on flat three-dimensional manifolds). The only non-Abelian subalgebra of o(4) with a second order Casimir operator that is not a Casimir operator of o(4) is $\{A_1+B_1, A_2+B_2, A_3+B_3\}$. This provides III_3 coordinates for

$$
\begin{aligned}
S_1 &= (A_1+B_1)^2 + (A_2+B_2)^2 + (A_3+B_3)^2, \\
S_3 &= (A_3+B_3)^2\,,
\end{aligned}
\tag{3.22}
$$

and type IV_1 coordinates for

$$
\begin{aligned}
S_1 &= (A_1+B_1)^2 + (A_2+B_2)^2 + (A_3+B_3)^2\,, \\
S_3 &= (A_1+B_1)^2 + k^2(A_2+B_2)^2 \quad (0<k^2<1)\,.
\end{aligned}
\tag{3.23}
$$

Type III_4 coordinates are obtained from A_3+B_3 only. The operator S_2 commuting with A_3+B_3 can be reduced to

$$S_2 = A_1B_1 + A_2B_2 + \alpha A_3B_3, \quad \alpha>0, \alpha\neq 1$$

and we distinguish between $0<\alpha<1$ and $1<\alpha\quad\infty$.

Type IV_1 coordinates were discussed above and type IV_2 also occurs.[13]

Notice that only pairs of operators S_1,S_2 that are invariant under parity Π, i.e.,

$$\Pi:(x_1,x_2,x_3,x_4)\to(-x_1,-x_2,-x_3,x_4)\,, \tag{3.24}$$

lead to separable coordinates on s_3, as was shown in Ref. 13. Here,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\,.$$

TABLE I. Separable coordinate systems for E(3).

Type	Coordinates	Diagonal operators		Subgroup chain
I	Cartesian	P_1,P_2,P_3		T(3)
II	Cylindrical	L_3,P_3		$O(2)\otimes T(1)$
III_1	Elliptic cylindrical	$P_3,L_3^2 + a(P_1^2 - P_2^2)$	$a>0$	$E(2)\otimes T(1)\supset I_4(X,Y)\otimes T(1)$
	Parabolic cylindrical	$P_3,L_3P_2 + P_2L_3$		$E(2)\otimes T(1)\supset I_2(Y)\otimes T(1)$
III_3	Spherical	$L_3,L_1^2 + L_2^2 + L_3^2$		$O(3)\supset O(2)$
III_4	Parabolic	$L_3,L_1P_2 + P_2L_1 - L_2P_1 - P_1L_2$		$O(2)\otimes I_4(X,Y)$
	Oblate spheroidal	$L_3,L_1^2 + L_2^2 + L_3^2 + a(P_1^2 + P_2^2)$	$a>0$	$O(2)\otimes I_4(Z,XY)$
	Prolate spheroidal	$L_3,L_1^2 + L_2^2 + L_3^2 - a(P_1^2 + P_2^2)$	$a>0$	$O(2)\otimes I_4(Z,XY)$
IV_1	Spheroconical	$L_1^2 + L_2^2 + L_3^2, L_1^2 + rL_2^2$	$0<r<1$	$O(3)\supset I_8(X,Y,Z)$
IV_2	Ellipsoidal	$L_2^2 + aL_1^2 + bP_3^2$,	$a>b>0$	
		$L_1^2 + L_2^2 + L_3^2 + bP_1^2 + aP_2^2 + (b+a)P_3^2$		$I_8(X,Y,Z)$
	Paraboloidal	$L_3^2 - c^2P_3^2 + c(L_2P_1 + P_1L_2 + L_1P_2 + P_2L_1)$,		
		$L_2P_1 + P_1L_2 - L_1P_2 - P_2L_1 + c(P_2^2 - P_1^2)$	$c>0$	$I_4(X,Y)$

TABLE II. Separable coordinate systems on s_3.

Type	Coordinates	Operators	Subgroup reduction
II	Cylindrical	L_3^2, J_3^2	$O(2)\times O(2)$
III_3	Spherical	$L_1^2+L_2^2+L_3^2, L_3^2$	$O(3)\supset O(2)$
III_4	Elliptic cylindrical I and II	$L_1^2+L_2^2+L_3^2+a(L_3^2-J_3^2), L_3^2 \quad (a\neq 0)$	$O(2)\times I_8(X_1,X_2,X_3,X_4)$
IV_1	Spheroelliptic	$L_1^2+L_2^2+L_3^2, L_1^2+rL_2^2 \quad (0<r<1)$	$O(3)\supset D_2$
IV_2	Ellipsoidal	$L_1^2-J_1^2+\frac{1-a+b}{a+b-1}(L_2^2-J_2^2)+\frac{1+a-b}{a+b-1}(L_3^2-J_3^2)$, $L_1^2+J_1^2+\frac{b-a-1}{a+b-1}(L_2^2-J_2^2)+\frac{a-b}{b(a-1)}(L_3^2+J_3^2)+\frac{a(b-1)(a-b-1)}{b(a-1)(a+b-1)}(L_3^2-J_3^2) \quad (1<b<a)$	D_2

The results of this paragraph are summarized in Table II, together with the discrete subgroup properties of each system. Again $I_{2n}(A_1,\ldots,A_n)$ will be a group of reflections in hyperplanes through the origin with, for example, X reflecting the Cartesian coordinate x only. We write the invariant operators S_1 and S_2 in terms of $L_i = A_i + B_i$ and $J_i = A_i - B_i$, rather than A_i and B_i directly (the J_i do not constitute a subalgebra).

C. The group O(3,1)

The subalgebras of o(3,1) have been classified[1] under the action of O(3,1) and the results are reproduced in, for example, Ref. 20.

The algebra o(3,1) is generated by the rotations L_i and boosts K_i, satisfying

$$[L_i,L_j]=\epsilon_{ijk}L_k,\ [L_i,K_j]=\epsilon_{ijk}K_k,\ [K_i,K_j]=-\epsilon_{ijk}L_k\,. \tag{3.25}$$

The Casimir operators are $\Delta = \mathbf{L}^2 - \mathbf{K}^2$ and $\Delta' = \mathbf{L}\cdot\mathbf{K}$ (we have $\Delta' = 0$). All separable coordinates for O(3,1) hyperboloids were obtained by Olevskiĭ[36]; the pairs of commuting operators S_1 and S_2 corresponding to these 34 coordinate systems are also known.[14]

The algebra o(3,1) has two MASA. Both are two-dimensional, namely,

$$\{L_3,K_3\} \text{ and } \{L_2+K_1, L_1-K_2\}\,.$$

We hence have no type I coordinates and two type II coordinate systems.

The one-dimensional subalgebras are $\{L_3\}$, $\{K_3\}$, $\{L_2+K_1\}$, and $\{L_3+aK_3; a>0\}$. None of these have non-Abelian centralizers, so we obtain no III_1 or III_2 type coordinate systems. Subgroup type coordinates III_3 are obtained from the subgroups O(3), O(2,1), and E(2). The corresponding pairs of operators are

$$(\mathbf{L}^2,L_3^2),(K_1^2+K_2^2-L_3^2,L_3^2),(K_1^2+K_2^2-L_3^2,K_1^2)\,, \tag{3.26}$$

$$(K_1^2+K_2^2-L_3^2,(K_1+L_3)^2) \text{ and } ((K_1+L_2)^2+(L_1-K_2)^2,L_3^2)\,.$$

Now let us consider III_4 type coordinates:

(i) $S_1 = L_3^2$, $\mathrm{nor}(L_3)=\{L_3,K_3\}$: The most general second order operator S_2 commuting with L_3 can, after linear combinations with Δ,Δ', and L_3^2 have been accounted for, be written as

$$S_2 = a(K_1^2+K_2^2+L_1^2+L_2^2)+b(K_2L_1+L_1K_2 - K_1L_2-L_2K_1)+cK_3^2+dL_3K_3\,. \tag{3.27}$$

The transformation $\exp\alpha K_3$ induces a hyperbolic rotation between the first two terms. Hence, if $|a|<|b|$, we can transform a into zero; if $|a|>|b|$, we can transform b into zero; and if $|a|=|b|$, the first two terms reduce to $(L_1+K_2)^2+(L_2-K_1)^2$. In these coordinates we have $L_3 = \partial/\partial\phi$ and the term L_3K_3 will be odd under the transformation $\phi\to-\phi$ which should leave S_2 invariant. Hence, $d=0$. If $|a|=|b|$, K_3 can be used to scale the value of a (and b) with respect to c. Using $\exp\alpha K_3$, parity, and linear combinations with Δ we can finally reduce S_2 to one of the forms:

$$K_1^2+K_2^2+aK_3^2, L_1K_2+K_2L_1-L_2K_1-K_1L_2+aK_3^2 \quad (a\geqslant 0) \tag{3.28}$$

$$(L_1+K_2)^2+(L_2-K_1)^2+\epsilon K_3^2 \quad (\epsilon=\pm 1)\,.$$

In the first case we distinguish between the regions $0<a<1$, $1<a<\infty$, and $-\infty<a<0$.

(ii) $S_1 = K_3^2$, $\mathrm{nor}(K_3)=\{L_3,K_3\}$: Imposing $[K_3^2,S_2]=0$ and using linear combinations with Δ,Δ', and K_3^2 we have

$$S_2 = a(L_1^2-K_2^2-L_2^2+K_1^2)+b(L_1L_2+L_2L_1+K_1K_2 + K_2K_1)+c(L_1^2-K_2^2+L_2^2-K_1^2)+dL_3K_3\,. \tag{3.29}$$

In these coordinates we have $K_3 = \partial/\partial\beta$ and L_3K_3 changes sign for $\beta\to-\beta$. Hence, $d=0$. The operator $\exp\alpha L_3$ will rotate between the first two terms. Hence, we can always rotate b in zero (the case $a^2+b^2=0$ would lead back to type II coordinates). We thus obtain

$$S_2 = K_1^2 - L_2^2 + a(L_1^2-K_2^2),\quad 0<|a|<1 \tag{3.30}$$

and we distinguish between $0<a<1$ and $-1<a<0$.

(iii) $S_1=(K_1+L_2)^2$,

$$\mathrm{nor}(K_1+L_2)=\{K_3,K_1+L_2,K_2-L_1\}: \tag{3.31}$$

The operator S_2 satisfying $[K_1+L_2,S_2]=0$ can be written as

$$S_2 = a(K_1^2+K_3^2-L_2^2)+b[(K_1+L_2)L_3+L_3(K_1+L_2) + K_3(K_2-L_1)+(K_2-L_1)K_3] + c[(K_2-L_1)^2+(K_1+L_2)^2] + d(K_1+L_2)(K_2-L_1)\,. \tag{3.32}$$

The separable coordinates (u,v,t) will be such that $K_1 + L_2 = \partial/\partial_t$. The term $(K_1 + L_2)(K_2 - L_1)$ will be odd under the reflection $t \to -t$; hence, $d = 0$. If $a \neq 0$, we put $a = 1$ and use $\exp\alpha(K_2 - L_1)$ to transform $b \to 0$. Further $\exp\beta K_3$ will scale c with respect to a. We obtain

$$S_2 = K_1^2 + K_3^2 - L_2^2 + \epsilon[(K_2 - L_1)^2 + (K_1 + L_2)^2], \quad \epsilon = \pm 1. \qquad (3.33)$$

($c = 0$ is excluded, since it would lead to type III_3.) If $a = 0$, $b \neq 0$, we put $b = 1$ and use $\exp\alpha(K_2 - L_1)$ to transform $c \to 0$. Finally $a = b = 0$ is excluded, since it would lead to type II.

(iv) $S_1 = (L_3 + aK_3)^2$, $\text{nor}(L_3 + aK_3) = \{L_3, K_3\}$:

$$(3.34)$$

The most general second-order operator commuting with $L_3 + aK_3$ $(a \neq 0)$ can be reduced to, for example, $K_3(bL_3 + cK_3)$ and hence leads back to type II coordinates.

Type IV_1 coordinates are obtained similarly as type III_3 ones. Indeed, we consider the subgroups O(3), O(2,1), and E(2) of O(3,1) and take S_1 as the corresponding Casimir operator. The operator S_2 will then be a second-order operator in the enveloping algebra of o(3), o(2,1), or e(2), respectively. These operators must be classified into orbits under O(3), O(2,1), or E(2), as the case may be, and orbits corresponding to squares of generators must be excluded. For O(3), O(2,1), and E(2) we obtain one, six, and two orbits, respectively.[15]

Finally, we are left with the generic case IV_2. The operators S_1 and S_2 are such that neither of them is the square of a generator or a Casimir operator of a subgroup of O(3,1) [nor is it conjugate under O(3,1) to such operators].

A further subclassification is obtained by considering discrete subgroups of O(3,1) leaving the individual pairs of operators invariant we omit all details here but summarize the results in Table III, where we give the invariant operators, the subgroup reductions, and identify the coordinate system by the number it carries in Refs. 14 and 36.

D. The group O(2,2)

We shall consider this case in somewhat less detail than the previous ones. Separable systems of coordinates on the hyperboloid $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$ were discussed in Ref. 16. The subalgebras of o(2,2) were classified in Ref. 20 and a

TABLE III. Diagonal operators and corresponding subgroup chains for separable coordinate systems on the O(3,1) hyperboloid.

Type	Diagonal operators		Subgroup chain
II	L_3^2, K_3^2		$O(2)\otimes O(1,1)$
	$(L_1+K_2)^2, (L_2-K_1)^2$		T(2)
III_3	$L_1^2+L_2^2+L_3^2, L_3^2$		$O(3)\supset O(2)$
	$K_1^2+K_2^2-L_3^2, L_3^2$		$O(2,1)\supset O(1,1)$
	$K_1^2+K_2^2-L_3^2, K_1^2$		$O(2,1)\supset O(1,1)$
	$K_1^2+K_2^2-L_3^2, (K_1+L_3)^2$		$O(2,1)\supset T(1)$
	$(L_1-K_2)^2+(L_2+K_1)^2, L_3^2$		$E(2)\supset O(2)$
III_4	$L_3^2, K_1^2+K_2^2+aK_3^2$	$0<a<1, 1<a<\infty$ or $-\infty<a<0$	$O(2)\times I_4(Z,T)$
	$L_3^2, L_1K_2+K_2L_1-L_2K_1-K_1L_2+aK_3^2$	$a\geqslant 0$	$O(2)\times I_4(X,Y)$
	$L_3^2, (L_1+K_2)^2+(L_2-K_1)^2+\epsilon K_3^2$	$\epsilon=\pm 1$	$O(2)\times I_4(X,Y)$
	$K_3^2, K_1^2-L_2^2+a(L_1^2-K_2^2)$	$0<a<1$ or $-1<a<0$	$O(1,1)\times I_{16}(X,Y,Z,T)$
	$(K_1+L_2)^2, K_1^2+K_3^2-L_2^2+\epsilon[(K_2-L_1)^2+(K_1+L_2)^2]$	$\epsilon=\pm 1$	$T(1)\times I_4(X,Y)$
	$(K_1+L_2)^2, (K_1+L_2)L_3+L_3(K_1+L_2)+K_3(K_2-L_1)+(K_2-L_1)K_3$		$T(1)\times I_2(X)$
IV_1	$L_1^2+L_2^2+L_3^2, L_1^2+aL_2^2$	$0<a<1$	$O(3)\supset I_{16}(X,Y,Z,T)$
	$K_1^2+K_2^2-L_3^2, L_3^2-aK_2^2$	$a<-1$ or $0<a$	$O(2,1)\supset I_{16}(X,Y,Z,T)$
	$K_1^2+K_2^2-L_3^2, K_1^2+a(K_2L_3+L_3K_2)$	$0<a$	$O(2,1)\supset I_4(Y,Z)$
	$K_1^2+K_2^2-L_3^2, L_3^2+(L_3K_2+K_2L_3)$		$O(2,1)\supset I_4(Y,Z)$
	$K_1^2+K_2^2-L_3^2, K_2^2+(L_3K_2+K_2L_3)$		$O(2,1)\supset I_4(Y,Z)$
	$K_1^2+K_2^2-L_3^2, K_1K_2+K_2K_1+K_2L_3+L_3K_2$		$O(2,1)\supset I_2(Z)$
	$(K_1+L_2)^2(K_2-L_1)^2, L_3^2+(K_1+L_2)^2$		$E(2)\supset I_4(X,Y)$
	$(K_1+L_2)^2+(K_2-L_1)^2, L_3(K_1+L_2)+(L_2+K_1)L_3$		$E(2)\supset I_2(X)$
IV_2	$M_1^2+bM_2^2+aM_3^2-(a+b)K_1^2-(a+1)K_2^2-(b+1)K_3^2, abK_1^2+aK_2^2+bK_3^2,$	$1<b<a$	$I_{16}(X,Y,Z,T)$
	$M_1^2-aK_2^2-bK_3^2-(a+b)K_1^2+(a+1)M_3^2+(b+1)M_2^2,$		$I_{16}(X,Y,Z,T)$
	$abK_1^2-aM_3^2-bM_2^2,$	$1<b<a$	
	$2aM_1^2-(a+1)(K_3^2-M_2^2)-a(K_2^2-M_3^2)-b(K_2M_3+M_3K_2-M_2K_3-K_3M_2),$		$I_4(Y,Z)$
	$(a^2+b^2)M_1^2-a(K_3^2-M_2^2)+b(K_3M_2+M_2K_3),$	$a,b\in\mathbb{R}$	
	$(K_2+M_3)^2+(K_3+M_2)^2+(a+1)K_1^2+K_3^2-M_2^2+a(M_3^2-K_2^2),$		$I_4(Y,Z)$
	$(K_3+M_2)^2-a(K_2+M_3)^2+aK_1^2,$	$1<a$	
	$(K_2+M_3)^2+(K_3+M_2)^2-(a+1)K_1^2-M_2^2+K_3^2-a(K_2^2-M_3^2),$		$I_4(Y,Z)$
	$(K_3+M_2)^2-a(K_2+M_3)^2-aK_1^2$	$1<a$	
	$(K_2+M_3)^2-(K_3+M_2)^2-(a-1)K_1^2-M_3^2+K_2^2-a(M_2^2-K_3^2),$		$I_4(Y,Z)$
	$(K_2+M_3)^2-a(K_3+M_2)^2-aK_1^2$	$0<a$	
	$M_2^2-K_3^2-M_1^2-(M_2-K_3)^2-M_1(M_2-K_3)-(M_2-K_3)M_1,$		$I_4(Y,Z)$
	$(M_2-K_3)^2-K_1(K_2-M_3)-(K_2-M_3)K_1$		

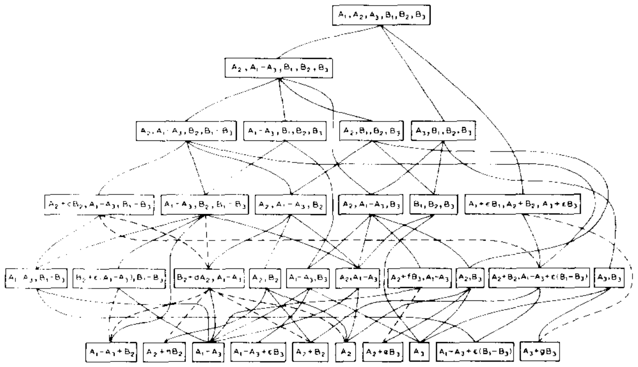

FIG. 1. The O(2,2) conjugacy classes of subalgebras of $o(2,2)$. The parameters take the following real values: $\epsilon = \pm 1, 0 < |c| \leqslant 1, d \geqslant 0, e > 0, f \neq 0, 0 < |g| \leqslant 1, 0 < h < 1$. A line connects each algebra with its maximal subalgebras. A solid (broken) line indicates an inclusion for all (some) values of the parameters involved.

diagram of them is given in Fig. 1. We use the isomorphism $o(2,2) \sim o(2,1) \oplus o(2,1)$ and write the algebra o(2,2) in the form $\{A_i, B_i\}$:

$$[A_1,A_2] = -A_3,\ [A_3,A_1] = A_2,\ [A_2,A_3] = A_1\,,$$
$$[B_1,B_2] = -B_3,\ [B_3,B_1] = B_2,\ [B_2,B_3] = B_1\,, \quad (3.35)$$
$$[A_i,B_k] = 0,\quad i,k = 1,2,3$$

(A_3 and B_3 are the compact elements).

Let us discuss the individual classes of coordinates.

Type I: The algebra o(2,2) has no MASA of dimension 3, and hence this class does not occur.

Type II: There exist six different MASA of dimension 2, each corresponding to a different system of coordinates. Systems I1, I2, and I3 of Ref. 16 are orthogonal and correspond to the subalgebras $\{A_2,B_3\}$, $\{A_2,B_2\}$, and $\{A_1 - A_3, B_1 - B_3\}$, respectively. Systems I4, I5, and I6 are nonorthogonal and correspond to the subalgebras $\{A_3, B_1 - B_3\}$, $\{A_2, B_1 - B_3\}$, and $\{A_3, B_2\}$, respectively.

Type III_1 and III_2: These do not occur since the centralizers of all one-dimensional subalgebras are either Abelian, or reductive of the type

$$\{A_3\} \oplus \{B_1,B_2,B_3\},\ \{A_2\} \oplus \{B_1,B_2,B_3\}\,,$$
$$\text{or } \{A_1 - A_3\} \oplus \{B_1,B_2,B_3\}\,.$$

These do not lead to separable coordinate systems on the considered hyperboloid [they would on the direct product of two O(2,1) hyperboloids].

Type III_3: The algebra o(2,2) has three non-Abelian subalgebras with second order Casimir operators distinct from the Casimir operators of o(2,2). These are

(i) $e(1,1)$: $\{A_2 - B_2, A_1 - A_3, B_1 - B_3\}$

[here $S_1 = (A_2 - B_2)^2, S_2 = (A_1 - A_3)(B_1 - B_3)$ leads to one coordinate system],

(ii) $o(2,1)$: $\{A_1 + B_1, A_2 + B_2, A_3 + B_3\}$,

(iii) $o(2,1)$: $\{A_1 - B_1, A_2 + B_2, A_3 - B_3\}$.

Each of the o(2,1) subalgebras leads to three different subgroup type coordinate systems.

Type III_4: The one-dimensional subalgebras providing III_4 type coordinates are $\{A_3 + B_3\}$, $\{A_3 - B_3\}$, $\{A_2 + B_2\}$, $\{A_1 - A_3 + B_1 - B_3\}$, and $\{A_1 - A_3 - B_1 + B_3\}$, leading to two, two, nine, three, and three systems, respectively.

Type IV_1: The same subalgebras e(1,1) and o(2,1) as in case III_3 lead to these "semisubgroup" type coordinates, of which there exist 8 +6 +6 = 20.

Type IV_2: The remaining generic case leads to 22 more coordinate systems.[16]

Thus, altogether 74 separable coordinate systems exist. Of these exactly six are nonorthogonal. We shall not discuss their discrete symmetries here.

E. The group E(2,1)

Separation of variables in three-dimensional Minkowski space has not been investigated with the same amount of detail as in the other three-dimensional spaces of constant curvature. The coordinate systems can however be extracted from Refs. 17 and 14. The subgroup structure of E(2,1) on the other hand is known.[21] We write the algebra e(2,1) in the form $\{K_1,K_2,L_3,P_0,P_1P_2\}$:

$$[K_1,K_2] = -L_3, [L_3,K_1] = K_2, [L_3,K_2] = -K_1\,,$$
$$[K_i,P_0] = P_i, [K_i,P_k] = \delta_{ik}P_0, [L_3,P_0] = 0\,, \quad (3.36)$$
$$[L_3,P_1] = P_2, [L_3,P_2] = -P_1, [P_\mu,P_\nu] = 0\,,$$
$$(i,k = 1,2; \mu,\nu = 0,1,2)\,.$$

Type I: There is one three-dimensional MASA: $\{P_0,P_1,P_2\}$ corresponding to Cartesian coordinates.

Type II: There are four different MASA of dimension 2. Two of them $\{K_1,P_2\}$ and $\{L_3,P_0\}$ correspond to orthogonal coordinates, and two others correspond to nonorthogonal ones. These are

$$S_1 = (P_0 - P_2)^2,\ S_2 = (L_3 + K_1)^2:$$
$$x = x_2x_3,\ y = x_1 - \tfrac{1}{2}x_2^2x_3,\ t = -x_1 + x_3 + \tfrac{1}{2}x_2^2x_3\,,$$

and

$$S_1 = (P_0 - P_2)^2,\ S_2 = (L_3 + K_1 + P_0 + P_2)^2:$$
$$x = x_2(x_2 + x_3),\ y = x_1 + x_2 - x_2^2\left(\frac{x_2}{3} + \frac{x_3}{2}\right),$$
$$t = -x_1 + x_2 + x_3 + x_2^2\left(\frac{x_2}{3} + \frac{x_3}{2}\right).$$

Type III_1: Among the nine types of one-dimensional subalgebras of e(2,1) precisely three algebras have non-Abelian centralizers, two of which are direct sums. These are as follows: (i) $\{P_1\}$ with cent$(P_1) = P_1 \oplus \{K_2,P_0,P_2\}$: Hence, $S_1 = P_1^2$ and S_2 is an element of the enveloping algebra of e(1,1), not equal to the Casimir operator, nor to the square of a generator. This leads to eight orthogonal coordinate systems. (ii) $\{P_0\}$ with cent$(P_0) = P_0 \oplus \{L_3,P_1,P_2\}$: Hence, $S_1 = P_0^2$ and S_2 is either $L_3P_1 + P_1L_3$ or $L_3^2 + a(P_1^2 - P_2^2)$ with $a > 0$ (two orthogonal systems).

Type III_2: The only element of e(2,1) that has a nonseparable centralizer is $(P_0 - P_2)$ with cent$(P_0 - P_2) = \{L_3 - K_1, P_0 + P_2, P_1, P_0 - P_2\}$ (this is a nilpotent algebra). In this case we have nor $(P_0 - P_2) = \{K_2, L_3 - K_1, P_0 + P_2, P_1, P_0 - P_2\}$. The choice $S_1 = (P_0 - P_2)^2$ and S_2 a member of the enveloping algebra of cent$(P_0 - P_2)$ (not equal to a square of a generator, nor to a Casimir operator) leads to three nonorthogonal coordinate systems. These are as follows:

(i) $S_2 = P_1(L_3 - K_1) + (L_3 - K_1)P_1$,

$$x = x_3\sqrt{x_2},\ y - t = x_1 - \tfrac{1}{4}x_3^2,\quad y + t = 2x_2;$$

(ii) $S_2 = (L_3 - K_1)^2 + 4P_1^2$,

$$x = x_3\sqrt{1 + x_2^2},\ y - t = x_1 - \tfrac{1}{2}x_3^2 x_2,\quad y + t = 2x_2;$$

(iii) $S_2 = (L_3 - K_1)^2 + 8aP_1(P_1 - P_2),\quad a > 0$,

$$x = x_2x_3 + \frac{a}{x_2}, y - t = x_1 - \tfrac{1}{2}x_2x_3^2 + a\frac{x_3}{x_2} + \frac{a^2}{6(x_2)^3},$$

$$y + t = 2x_2.$$

Type III_3: Subgroup type coordinates in this case only originate from the O(2,1) subgroup. We obtain three coordinate systems, corresponding to $S_1 = L_3^2, K_2^2$, or $(L_3 - K_1)^2$ and $S_2 = K_1^2 + K_2^2 - L_3^2$.

Type III_4: Taking $S_1 = L_3^2, K_2^2$, or $(L_3 - K_2)^2$ we obtain 10 orthogonal coordinate systems.

Type IV_1: Semisubgroup type coordinates again originate from O(2,1) only and six types of them exist.

Type IV_2: The generic class here consists of 22 types of coordinates.

The total is 54 orthogonal coordinate systems, and five nonorthogonal ones. We shall not go into the problem of discrete symmetries here.

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