2d curves

A collection of 631 named two−dimensional curves

algebraic curves are a polynomial in x and y
line (1st degree)
conic (2nd)
cubic (3rd)
quartic (4th)
sextic (6th)
octic (8th)
other (otherth)

transcendental curves
discrete
exponential
fractal
gamma & related
isochronous
power
power exponential
spiral
trigonometric

other curves
kid curve
3d curve

derived curves

barycentric
caustic
cissoid
conchoid
curvature
cyclic
derivative
envelope
hyperbolism
inverse
isoptic
parallel
pedal
pursuit
reflecting wall
roulette
strophoid
tractrix

Some programs to draw your own curves:

- GraphEq (2.2 MB) from Pedagoguery Software
- GraphSight (696 kB) from Cradlefields ($19 to register)

For isoptic cubics:

- Isoptikon (866 kB) from the University of Crete

2dcurves in .pdf format (1882 kB)

curve literature

last update: 2003−06−14

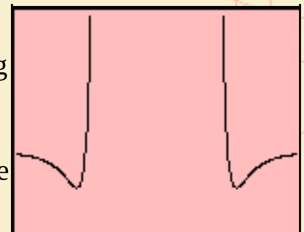
Lennard−Jones potential

higher

last updated: 2002−03−25

$$y = \frac{1}{x^{12}} - \frac{1}{x^{6}}$$

Atoms of an inert gas are in equilibrium with each other by means of an attracting van der Waals force and a repelling forces as result of overlapping electron orbits. These two forces together give the so−called ***Lennard−Jones potential***, which is in fact a curve of thirteenth degree.



Lennard-Jones potential

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backgrounds main

last updated: 2002−11−16

history
I collected curves when I was a young boy. Then, the papers rested in a box for decades. But when I found them, I picked the collection up again, some years spending much work on it, some years less.

questions
I have been thinking a long time about two questions:

1. what is the unity of curve?
   Stated differently as: when is a curve different from another one?
2. which equation belongs to a curve?

1. unity of curve
I decided to aim for simplicity: it does not matter when a curve has been reformatted in a linear way (by ways of translation, rotation or multiplication).
This means that I omit constants in the equations of a curve, as been found by other authors.

Example:
for me the equation of the super ellipse is not

$$\left|\frac{x}{a}\right|^c + \left|\frac{y}{b}\right|^c = 1$$

but:

$$|x|^a + |y|^a = 1$$

Only the parameter 'a' affects the form of the curve.
And all linear transformations of this curve do belong to this same curve 'family'.

2. which formula
I don't want to swim in an ocean of formulae.
Therefore I look for a formula that is as simple as possible, for covering a given curve. Trying to confine myself to Cartesian, polar, bipolar and parametric equations.
Examples:

- Cartesian equation: $y = f(x)$
  Example $y= x^2$ (parabola)
- polar equation: $r = f(\varphi)$
  Example $r = \varphi$ (spiral of Archimedes)
- bipolar equation: $f(r_1, r_2) = a$
  Example $r_1 r_2 = a$ (Cassinian oval)
- parametric equation: $x = f(t)$, $y = g(t)$
  Example: $x = t - a \sin t$ and $y = 1 - a \cos t$ (cycloid)

Sometimes the definition of a curve can not fit in one of these forms:

- textual definition: let there be etc.
  Example: apply the following rule to a grid of black squares: when you get on a black square, make it white and turn to the right; when you get on a white square, make it black and turn to the left (ant of Langton)

Sometimes a much shorter or much more elegant formula can be found, using another way of defining a curve:

- complex equation: $z = f(z_1, z_2)$
  Example: $z = f\ (C_1(t), C_2(t))$ for a curve $C_2(t)$ that rolls over curve $C_1(t)$ (roulette)
- differential equation: $f(x, y, y') = 0$
  Example: $x^2 + y'^2 = 1$ (quadratrix of Abdank−Abakanovicz)

literature

I made up a list of the literature I used.

curve definition

last updated: 2003−05−05

A curve [1] can be defined as a (finite) number of arcs, combined together.

Lawrence defines an ***arc*** as a valid one when its parametric equation (x,y) = (f(t), g(t)) − with t in an open interval I − obeys the following conditions:

- f and g are twice continuously differentiable
- for all t in I, at least one of df/dt and dg/dt is unequal zero
- for all points s and t in I, (f(s), g(s)) = (f(t), g(t)) if and only if s = t

However, not all curves I describe are differentiable: see e.g. the blancmange curve.

A **Jordan curve** is defined as a closed, non−intersecting line, consisting of a finite number of regular segments.
Such a curve with a direction is a **contour**.

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notes

1) In French: courbe.

oval

An oval is a curve resembling a squashed circle, but without a clear mathematical definition. So an oval has not necessarily two axes of symmetry.

We find oval−like curves on the following pages:

- Cartesian oval
- Cassinian oval
- egg curve
- ellipse
- hippopede
- oval of Booth
- spiric section

Sometimes an oval−like curve is constructed as connected arcs, of circles with different radii.

round curve

last updated: 2001−11−17

Is an oval round? Or are only circles **round**?
As for the Latin word 'circum', 'round' has two meanings:

- specific denoting the circle form
- more general, for a closed loopless curve

The second meaning is found in expressions as: a circular road, a circular saw, a circular tour. They all don't match the precise circle form.

Conclusion: there are to meanings of 'round' a strict one (circle) and a more general one.

algebraic curve

last updated: 2003−05−03

A curve is ***algebraic*** when its defining Cartesian equation is *algebraic*, that is a polynomial in x and y. The *order* or *degree* of the curve is the maximum degree of each of its terms $x^i y^j$.
An algebraic curve with degree greater than 2 is called a **higher plane curve**.

An algebraic curve is called a **circular algebraic curve**, when the points (±1, ± i) are on the curve. In that case the highest degree of the Cartesian equation is divisible by $(x^2 + y^2)$. The circle is the only circular conic section.
The curve got its name from the fact that it contains the two imaginary circular points: replace x by x/w and y by y/w, and let the variable w go to zero, we obtain the circular poins.

A **bicircular algebraic curve** passes twice through the points (±1, ± i). In this case the highest degree of the Cartesian equation is divisible by $(x^2 + y^2)^2$.

Every algebraic polynomial is a **Bézier curve**. Given a set of points P_i the Bézier curve of this **control polygon** is the convex envelope of these points.

When a curve is not algebraic, we call the curve (and its function) ***transcendental***.

In the case a function is sufficiently sophisticated it is said to be a ***special function***.

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polynomial main

last updated: 2002−12−29

$$y = \sum_{i=0}^{n} a_i x^i$$

A polynomial is literally[1] an expression of some terms, distinct powers in one ore more variables. On this page we talk about a polynomial in one variable, x. When talking about two variables (x and y), we call it an algebraic curve. A curve for which f(x,y) is a constant, is called a ***equipotential curve***. Other names for the curve are: ***isarithm***, ***isopleth***.

As said, here the polynomial in x, each term is an entire power of x, and the function is also called an ***entire (rational) function***. It is also called an ***algebraic function***.
The highest power in x is the degree (or: order) of the polynomial. An entire function of 2nd degree is called a ***quadratic function***; an entire function of 4th degree is called an ***biquadratic function***.

Some specific polynomials can be found as:

- line
  y is linear in x.
- parabola
  y is quadratic in x.
- cubic
- quartic

The ***root to the nth power*** $\sqrt[n]{\ }$ is the inverse function of the (simple) polynomial with equation $y = x^n$.
A ***broken function*** is a function that is the quotient of two polynomials.Together with the polynomials do they form the group of ***rational functions***.

The following polynomials have a special significance:

- cubic parabola
- generated polynomial
- orthogonal polynomial
  - Hermite polynomial
  - Laguerre polynomial
  - Legendre polynomial
  - Tchebyscheff polynomial
- Pochhammer symbol
- power series
- recurrent polynomial

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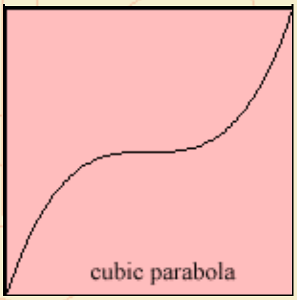
notes

1) polus (Gr.) = many, nomen (Lat.) = name.

cubic parabola

last updated: 2002−03−25

$y = x^3$



A polynomial of the third degree who has its own name is the ***cubic parabola***. Its name has been derived from the parabola, the cubic part refers to the third degree.

The curve has to be distinguished from the cubic parabola that is a divergent parabola.

generated polynomial

last updated: 2003−05−18

Some polynomials are defined inside an expression, e.g. inside an series with infinite coefficients:

Bernoulli polynomial

$$\frac{te^{xt}}{e^t-1}=\sum_{i=0}^{\infty}\frac{y_i(x)}{i!}t^i \quad ,|t|<2\pi$$

The *Bernoulli* polynomials (B_i) are used in some formulae that relate a series and a integral. Some slowly converging series can be written as an (easy to solve) integral.

Euler polynomial

$$\frac{2e^{xt}}{e^t+1}=\sum_{i=0}^{\infty}\frac{y_i(x)}{i!}t^i \quad ,|t|<\pi$$

The *Euler* polynomials (E_i) are used to clarify phenomena of some alternating series [1].

---

notes

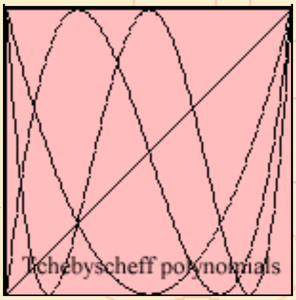
1) Temme 1990 p.13.

orthogonal polynomial

polynomial

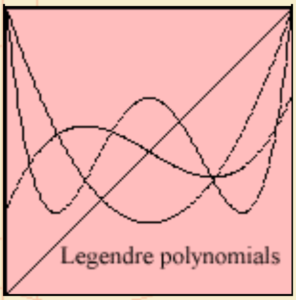
last updated: 2002−11−10

$$\int_a^b w(x)\, f_i(x)\, f_j(x)\, dx = 0$$



According to algebraic theory systems of perpendicular (***orthogonal) polynomial***s f_i can be constructed, that span a so called function space. Orthogonality has the meaning of being zero of the inproducts $\langle f_i, f_j \rangle$ for i j. This function inproduct is defined as above mentioned integral, where w(x) is called the weight function. Always true is that f_i is a polynomial of degree i with the remarkable quality that all points of intersection with the x−axis are real and different, and are in the interval [a, b]. Some orthogonal polynomials have been given special attention, and were named to their explorer:

− polynomial	interval	w(x)	$f_n(x)$	abbreviation
Hermite	$[-\infty, \infty]$	e^{-x^2}	$(-1)^n e^{x^2} D^n(e^{-x^2})$	H_n (x)
Laguerre	$[0, \infty]$	e^{-x}	$\frac{1}{n!} e^{x} D^n(e^{-x} x^n)$	L_n (x)
Legendre	[−1 , 1]	1	$\frac{1}{2^n n!} D^n (x^2-1)^n$	P_n (x)
	[−1 , 1]	$\frac{1}{\sqrt{1-x^2}}$	??	T_n (x)



The polynomials are also solutions of differential equations with the same name (equation of Hermite, Laguerre and so on), with whom I don't want you to torture. The polynomials of Legendre are also called the ***spherical functions of the first kind***. A spherical function is a solution of the equation of Laplace 2), an equation to which a lot of theoretical problems in physics and astronomy can be reduced. The spherical is derived from the solution method, where variables are split in spherical coordinates.
Working out this solution derivatives of polynomials of Legendre, as function of the cosΘ (one of the three spherical coordinates) are formed. These are called the ***associated polynomials of Legendre***.

From the Legendre polyonomial the Legendre trigonometric can be derived.

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notes

1) Or: Tsjebysjev

2) Equation of Laplace: $\Delta f = 0$

Pochhammer polynomial symbol

last updated: 2002−03−25

$$y = \prod_{i=0}^{n-1} (x+i)$$

From the gamma function the ***Pochhammer symbol*** or ***rising factorial*** $(x)_n$ can be derived, in the following way:
y = Γ(x+n) / Γ(x).

The function can be found in series expansions for ***hypergeometric function***s.

Is is evident that on places where the gamma function has infinite values, the Pochammer symbol has definite values.

recurrent polynomial

It's also possible to define a polynomial with a recurrent method, given a starting function. There are an infinite number of polynomials y=f(x), where the nth is to derived from former ones:

Fibonacci polynomial

$$y_n(x) = x\,y_{n-1}(x) + y_{n-2}(x)$$

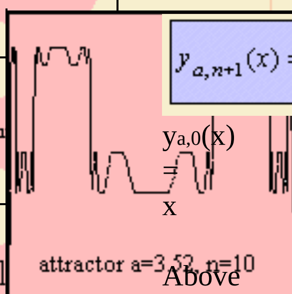
with $y_0(x)=0$, $y_1(x)=1$

Named to *Fibonacci*, famous of his Fibonacci series.

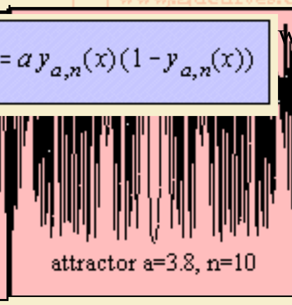
attractor

$$y_{a,n+1}(x) = a\,y_{a,n}(x)(1-y_{a,n}(x))$$

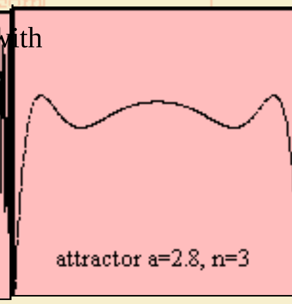
with $y_{a,0}(x) = x$



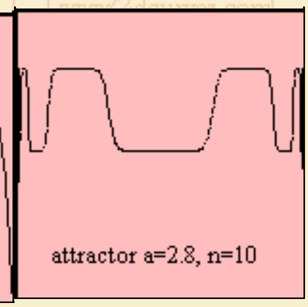
attractor a=3.52, n=10



attractor a=3.8, n=10



attractor a=2.8, n=3



attractor a=2.8, n=10

Above formula states the iteration of a parabolic function, giving a polynomial of degree 2^n. For $0 < a < 4$ convergence to one or more values may happen. Such points of convergence [1] are called attractors, later on this term was also given to the iterated curve.

The character of these patterns depends strongly on the value of the parameter a. For small values for parameter a, one sees just one point of convergence. Enlarging parameter a, at a certain moment a doubling of attractors is to be seen, the point of convergence seems to split, and later on again. Above some critical value [2] no separate attractors are to be distinguished, a chaotic region is entered.

In fact we're looking at polynomials of a rather high degree [3] on a small part of the domain. It appears that sometimes the generated polynomials have regions, where the function values are rather constant. Sometimes the extremities are so close to each other, that it resembles to chaos. This pattern changes with the parameter a.

The relationship can be used in population dynamics as a model (the *Verhulst* model) to describe a retained growth, for insects or other creatures. The parameter a is a measure of the fertility, for small values the species dies out. But there is also a maximum population size, caused by the term 1 − f.

The physicist *Mitchell Feigenbaum* discovered that the quality of the maximum determines the behavior on the long term. It appeared that parabolic behavior is not necessary parabolic, a quadratic maximum satisfies. Efforts are undertaken to bridge the behavior of these curves to the chaos of the physical turbulence (until now, without real success) [4].

These phenomena were quite popular in the beginning of the 20th century. From the 70s on there is a revival, what produced a large stream of publications, especially in relation to chaotic behavior. This renewed attention is partly understood by the easier access to computing facilities, with whom the iterations can be easily calculated.

From these set of curves the fractal parabola bifurcation can be derived.

---

notes

1) The attractor point will always lay on the line y=x.

2) of about 3.56.

3) the 10th iteration leads to a 1000th degree in x.

4) Hofstadter 1988 p. 395.

(generalized) straight line

last updated: 2002−03−25

With ordinary straight lines, many figures can be composed:

- crack function
- cross
- needle function
- polygon
  - triangle
  - quadrilateral
  - pentagon
  - hexagon
  - cross
- star

Also the discrete curves can be seen as built of pieces of straight lines.

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cross

line

last updated: 2002−03−25

Crosses are regular, pointed figures, commonly formed by (often perpendicular) skew lines. They are often used as symbols and signs. A simple cross kan be oblique (as the character x or the multiplying sign x) or perpendicular (as the summation sign +).

In the Netherlands a number of public health organizations have a cross in their emblem: the Red Cross, the Green Cross and the Ivory Cross (and the former organizations: Orange−Green Cross and White−Yellow Cross).
The cross is used as military decoration, and it has religious meaning as the crucifixion of Jesus Christ. And the cross is used symbol for the death.

Painful history has been associated by the ***swastika***, the emblem of the German National−Socialist movement. But for the old Hindu the (reverse oriented) swastika was a sign giving happiness.

Double skew lines are used in music notation as the sharp: #.

The 'cross of Einstein' is a fourfold image of an celestial object, where gravitational fields act as a lens.

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crack function

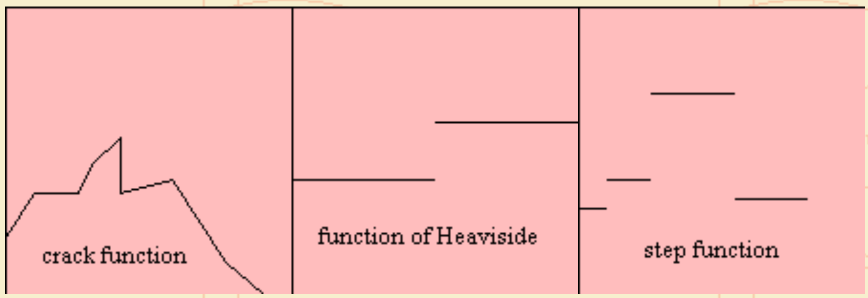
line

last updated: 2003−05−31

A **crack function** is a function built of line segments. In statistics we find her as **frequency polygon**, which curve connects the maximal values of subsequent rectangles. A simple form of the curve is the **modulus** |x|, giving the absolute value of a number.

Using only horizontal line segments [1], the **step function** or **staircase function** is formed. Every discrete function can be seen as such a step function.

Another function that is sometimes called the **step function**, is the **entier function** int(x) or [x]. Now the steps are of equal length: the function gives for each x the first lower natural number. The entier function can be used to form the triangle curve.



The **function of Heaviside** H_a has only one step: until x = a it has a value zero, above the value one. The function describes the velocity of an object as result of a sudden force, during a infinitely small time interval. H is the abbreviated notation for H_0, what is in fact the same as the **unit step function**. At the place of the step the value of the function can be set to ½. Also the **sign function** |x| / x has only one step.

---

notes

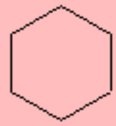
1) By the way, because it's the case of a function, vertical line segments are not admitted.

hexagon

line

last updated: 2002−08−19

The regular hexagon can be divided in three regular triangles, so that the figure is easy to construct with a pair of compasses. Already the Greek group of *Pythagoras* knew that a plane can be filled with regular hexagons.



regular hexagon

At the age of 16 *Pascal* proved a special quality for a hexagon inscribed by a conic section: the cuttings of the opposite sides lie on a straight line: the ***line of Pascal***. Later on, this hexagon had been given the name of **mystical hexagon** [1].

In chemistry the regular hexagon is the formula for the benzene ring. It is made of six carbon atoms, bound in a ring by a hydrogen atom between each neighbor. Also some crystal structure have a regular hexagonal form, for instance zinc sulfide and a special form of diamond.

In real life we see the regular hexagon in the ice crystals by which snow is formed [2].
Probably Johann Kepler was the first to remark this hexagonal symmetry in snow flakes, and he wrote a book about it for his money−lender (in 1611).
The hexagon form is caused by the three parts of a water molecule: two hydrogen and two oxygen atoms.
In construction the hexagonal bolt and the hexagonal screw are used.

In the world of the animals the regular hexagon can be seen in the honey−comb and as the radiate [3].
In 1999 *Thomas Hayles* proved that the hexagonal form constructed by the honey−bees is the best (most efficient) way to divide a plane in small and equal pieces.
This is related to the fact that the hexagon is the most efficient way to lay circles in a plane. This can be understood by realizing that six equal circles fit precisely around a circle with that same radius.

---

notes

1) Known as the 'hexagramma mysticum'.

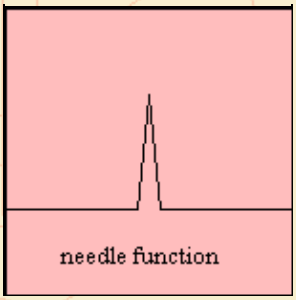
2) All snow flakes have sixfold symmetry, some kinds of snow flakes have the form of a hexagon.

3) An unicellular animal with a silicon skeleton.

needle function

line

last updated: 2002−11−14



The value of the ***needle function*** is only unequal zero in the neighborhood of x = a. Then we make the needle narrower and higher, while keeping the area constant.
In the limit situation that the function is only unequal zero for x = a, we have the **delta function** or ***delta function of Dirac*** δ_a, called after the British physicist *Paul Dirac*.

The simple δ is the abbreviation of δ_0. The function can used for short−during forces. It does not fit in the classical mathematics; modern theories talk about a **distribution** or a **generalized function**.

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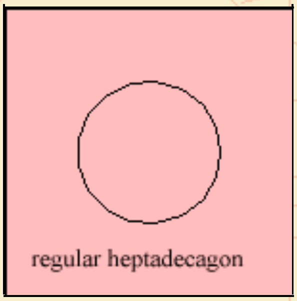
polygon line

last updated: 2002−03−25

By placing together some line segments circle−wise, the ***polygon*** [1] is formed.
In the Netherlands, in the sixties the 'Polygoon' news−reel showed us news from all over the world in the cinema.



An ***equilateral polygon*** has sides of equal length, an **equiangular polygon** has equal angles. Only for a triangle the equilateral and equiangular version gives the same figure. The two qualities together make a ***regular polygon***. The more sides the polygon has, the more it approximates a circle. The genius mathematician *Gauss* showed − in the age of 18 − which regular polygons can be constructed with a pair of compasses and a ruler [2]. For example the ***regular heptadecagon*** has this quality.

The polygons have been given own names, derived from the Greek numerals:

number of sides	***name of polygon***
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon or enneagon
10	decagon
11	undecagon
12	dodecagon
13	tridecagon
14	quadridecagon
15	pentadecagon
16	hexadecagon
17	heptadecagon
18	octadecagon

Some polygons derive their qualities in relation to their position according another curve:

- a ***cyclic polygon*** is a polygon where the sides are chords of a given circle
- for a ***tangent polygon*** the sides are tangent to a given curve (in most cases a circle)
- the ***circumscribed polygon*** and the ***inscribed polygon***: when on each side of a polygon there lies a

vertex of a second polygon, we call the outer polygon a circumscribed polygon, and the inner polygon an inscribed polygon

In the field of Roman ancient history the so called Thijssen polygons are in use: these polygons have been constructed around Roman district capitals, to approximate the districts' boundaries.

---

notes

1) Poly (Gr.) = many, goonia (Gr.) = angle.

2) When n is the number of sides, then for be able to construct by a pair of compasses, n has to obey:

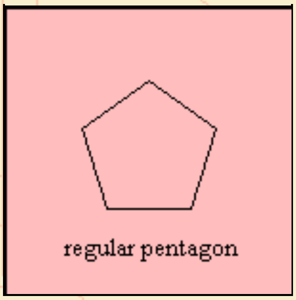
$n = 2^j \prod_{i<>} p_i$ where p_i is the Fermat Number for which

$p_i = 2^{2^i} + 1$

pentagon **line**

last updated: 2002−03−25

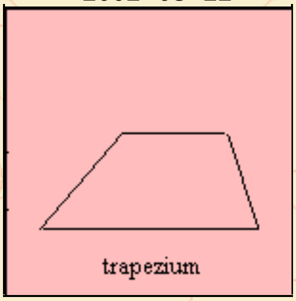
The building of the American Department of War is named the 'Pentagon', while its form is a pentagon.
Some viruses have the form of a regular pentagon.



The pentagonal numbers 1, 5, 12 and so so can be found by surrounding pentagons with larger pentagons.

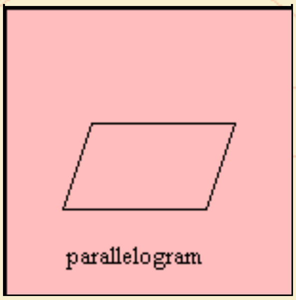
quadrilateral line

last updated: 2002−08−22



There are many quadrilaterals to distinguish. The ***tangent quadrilateral*** and the ***cyclic quadrilateral*** are special cases of the tangent polygon and the cyclic polygon.
For the tangent quadrilateral the opposite sides are of equal length.
A cyclic quadrilateral for which the product of the opposite sides is the same, is called a **harmonic quadrilateral**.

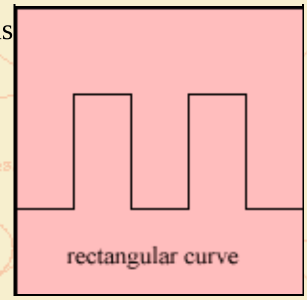
A **trapezium** or ***trapezoid*** is a quadrilateral with only two parallel sides. The trapezoid rule gives a way to find the area under a give curve.
When the non parallel sides are equal in length, the figure is a ***isosceles trapezium***. We see it in the **trapezium curve** − a repetition of trapeziums − and in the so−called trapezoid screw−thread.



The **parallelogram** is a quadrilateral for which the opposite sides are parallel. It can be seen as representing the vector sum of two variables with length and direction (e.g. a force), resulting in the diagonal of the parallelogram.
Anna Isabella Milbanke (who has been married to Lord Byron) had as nickname the 'princess of the parallelograms', for her interest in mathematics.

A crystal lattice in two dimensions is build of a repeating structure in two directions, forming a parallelogram. This shows that a plane can be filled with parallelograms.

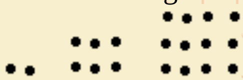
The **rectangle** has only right [1)] angles. It is a very handy form. How many objects have a rectangular form: the pages of a book, this web page, the room I am sitting in, the piece of ground of our house, the window I am looking through, the doors in my house: all rectangles. So while building a house many right angles have to be constructed. We can make this construction easier when we use the Pythagoras theorem, and make windows in a ratio of 3 to 4 for the height related to the length. The right angle is easy to construct by pacing a diagonal of 5 units. In Holland in old farms such windows can be seen.
Also many other object have this rectangular form: writing paper, postcards, football fields, paintings, tables, shelves. Package material as boxes consists often of rectangles, what makes piling up more easy.



In perspective the rectangle seems deformed, for the eye. In such a way that parallel lines − in the direction of the observer − meet each other in the so−called vanishing−point. Our brains can compensate for this deformation, and we will often be able to recognize a rectangle.
In a perspective view the scale is not constant, so that is difficult to measure and and count. An oblique projection, where a rectangle is shown as a parallelogram, is easier for this purpose.

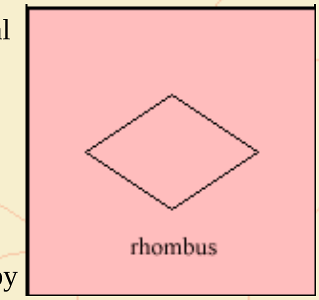
The golden section is a ratio between the sides of a rectangle, that looks quite natural. We define this figure so that when you cut a square from the rectangle, the resulting rectangle is similar to the original rectangle [2] .
We call it the **golden rectangle**.

The rectangular numbers 2, 6, 12, 20 and so on can be defined analogous to the triangular numbers:


and so on.

When the voltage of a signal varies periodically between a minimum and a maximum value, the resulting curve is a block−wave or **rectangular curve**.

A **rhombus** is an oblique square: the four sides are equal in length, but there are no right angles.
In Dutch a rhombus is called a 'ruit', a word which originated from the medieval 'ru(y)te', which also denoted the similar formed plant. A strange phenomenon is that some rectangular objects have the rhombus in their Dutch name: this is the case for the words for: pane, squared paper and tartan. The checker 'ruit' is a real rhombus, as is the 'wyber' (liquorice) and the Renault logo.



When the sides are in pairs equal in length, we see a **kite**.

The **square** is the regular quadrilateral. Of all rectangles with equal circumference the square has the maximum area (what has been proved already by *Euclid*, about 300 BC).

The square numbers 1, 4, 9, 16 and so on, were for the *Pythagoras* group the symbol of justice. The quadratic equation and the quadratic root are also based on the power two.

A magic square is a matrix of numbers, where the sum of rows, columns and diagonals has the same result.

A **pentamino** is a curve formed by joining squares together.

---

notes

1) rectus (Lat.) = right.
The Dutch word 'loodrecht' (lead right, from: lead−line) means: right (perpendicular).

2) The ratio has a value of about 0.62.

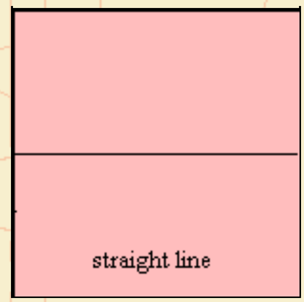
(straight) line <u>line</u>

last updated: 2003−05−18

$y=0$

Let's have a rectangular coordinate system, built of a two straight lines, the horizontal x−axis (the ***abscissa***) and the vertical y−axis (the ***ordinate***). Given two points A and B, a straight line through these points makes the shortest path between A and B. The angle of the line with a coordinate axis is constant, and also the derivative of (a function represented by) a straight line is constant.
The line must be one of the earliest curves studied. *Euclid* studied the straight line in his 'Elements', but he did not consider the line being a curve.
When the center of inversion is not on the line, the inverse of the line is a circle.
And the line is the pedal of the parabola.

The straight line stands for a linear relation, a linear relationship between two variables. In nature there are a lot of phenomena which are related in a linear way. An example is that the relative stretching of a bar grows linearly with the exerted force.



A lot of straight lines have been given special names, because of their relation to other curves or figures (as a circle or a triangle). A straight line that touches a curve is called a ***tangent***. When this occurs in infinity, the curve is called an ***asymptote*** [1]. The line is the only curve for which its tangent coincides with the curve itself.
A ***chord*** is the line between two points on a circle.

In geometry much attention has been given to the triangle. A side of a triangle is called a ***leg***; a special case is the slant side of a right−angled triangle: the ***hypotenuse***. While exploring its qualities, many auxiliary lines has been defined. The ***perpendicular*** is dropped down from a vertex perpendicular to the opposite side. In Dutch, each raised perpendicular is called a 'loodlijn' (plumb−line or lead−line), which use goes back to the Middle Ages, when the plumb−line from a boat was used to gauge the depth of the water.
The ***perpendicular bisector*** is a perpendicular on the middle of a line segment.
The ***median*** directs from a vertex to the opposite side's mid [2]. The ***bisector*** divides the angle in two (equal) parts, the ***trisector*** in three equal parts. A bit far−fetched is the ***symmedian***, the reflection of the median in a bisector. *Euler* proved that − in a triangle − the points of intersection of the perpendicular bisectors, the perpendiculars, and the medians lie on a line: ***Euler's line***.

In a polygon a ***diagonal*** is a line between vertices that are no neighbors.

In a conic section the ***directrix*** is the line for which the distance to the focus is a constant.

Special cases of the straight line are the ***constant function*** (represented by a horizontal line), and the ***identical function*** (line y=x).

The straight line refers to a period, many centuries ago, when measuring (of land) took place with ropes. *Herodotus* describes [3] that after the inundations of the Nile the Egyptian 'rope−stretchers' fixed again the boundaries of the fields. With ropes. They measured too for the pyramids and temples that were being built. That's why the word 'straight' is related to 'to stretch', and the 'line' to 'linen'.

Thousands year ago the Chinese read the future using (uninterrupted) yang lines (saying yes) and (interrupted) yin lines (saying no). This system developed via a set of two lines (earth and heaven) to a set of three lines (trigram). Each trigram has its each own meaning, for example the trigram 'soen' means soft, tenacious, pervasive: ☴ . Later two trigrams were combined to the 'hexagram'.

The straight lines that surround us are mainly constructed by Man. In nature we see them as a result of a force directing in one direction: corn grows in the direction of the sun. A stone falls in a straight line down to earth.
Man−made straight lines we see in our houses as ceilings, walls, window−frames, doors. In this way building is possible faster en more standardized. Think about a home with rooms in the form of ellipses or parabolas and all circular windows. Much more work. Or many curlicues on the ceiling, that is not fit to our age. The modern man goes straight to his goal.

Straight is found in many words and expressions, figurative going in a straight line. A Dutch expression as 'recht door zee' ('straight through the sea': be straightforward) sounds more positive than 'recht in de leer' ('straight in the faith': sound in the faith), where a certain amount of stiffness can be heard.
In situations with moving objects (as cars), we see a straight line when a sharp bend can't be made. You have to do something to gain control in a bend, when you don't do anything, you will follow your impulse in a straight line. Physically a straight line is a very natural line. Physics defines a uniform straight motion, when the forces on an object are in equilibrium. But the magnitude of the velocity is relative, it has only meaning in relation to another object..

---

notes

1) asumptotos (Lat.) = not meeting.

2) In Dutch the median is called the 'zwaartelijn' (gravity line), a good choice because the two parts in a triangle divided by the median are equal in weight.

3) Historiae II, 109.

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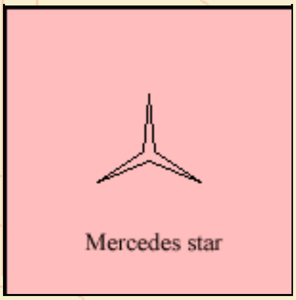
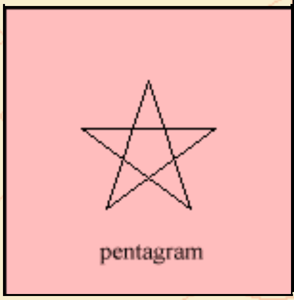
star

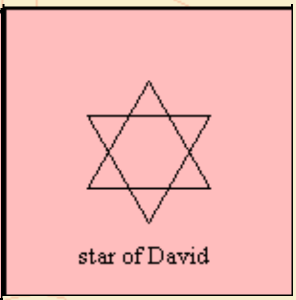
line

Stars are regular formed pointed figures. When the points meet each other at equal angles, the star is a ***regular star***:


Mercedes star

- the regular three−pointed star can be found as the **Mercedes star** of the car of the same name. Their competitor Citroën used two of the three points.
- the regular four−pointed star is to be seen during Christmas time as a ***Christmas star*** (sometimes eight−pointed), referencing to the star occurring in the story of the birth of Jesus in the Bible.
- the regular five−pointed starfish walks over the sea−bottom (but there are sorts with more arms, sometimes up to fifty). And the Dutch football team of MVV carries the sea−star as its emblem.
  By lengthening the sides into the inside of the star we get the **pentagram**. For the Greek Pythagoras it was the symbol of health. Later on the pentagram became the symbol of the medieval master builders.


pentagram


star of David

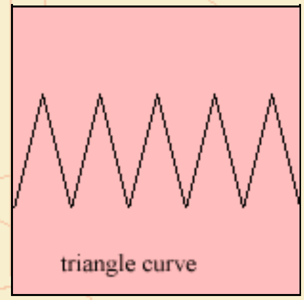
The **star of David** or ***Jews' star***, named to the Jewish king David, is the symbol of the Jews. In fact it is not a star, but a combination of two triangles.

The typographic star or ***asterisk*** is used for foot−notes. It consists of three skew lines: *.

triangle

line

last updated: 2002−08−16



The triangle has been studied in trigonometry by many scholars, already in the Ancient Greek times.

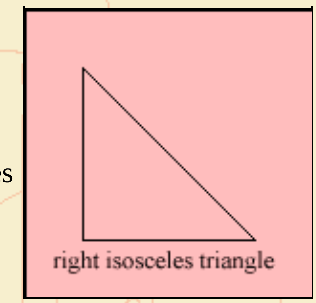
The triangle is a simple figure, but however, many triangles can be distinguished, many of them carrying an own name. For the ***acute−angled triangle*** (or ***acute triangle***) all angles are smaller than 90 degrees. When a triangle has an obtuse angle, it's an ***obtuse−angled triangle*** (or ***obtuse triangle***).

The ***triangle curve*** (or ***zigzag curve***) repeats the **isosceles triangle**, a triangle with two equal legs.
The curve can be defined as giving the distance from x to the natural number that is most close. In formula:

$$\begin{cases} y = x - [x], & \text{for } x - [x] \le 1/2 \\ y = [x+1] - x, & \text{for } x - [x] > 1/2 \end{cases}$$

where [x] is the entier function.
This triangle curve can be used to construct the blancmange curve.

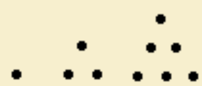
A ***right−angled triangle*** (or ***right triangle***) has one rectangular angle. Then, Pythagoras' theorem states the relation between the oblique side a and the other two sides as $a^2 = b^2 + c^2$.
The triangle inequality relates the three sides in a triangle: $a < b + c$.



The **saw−tooth curve** repeats the right triangle. For equal sides of the right angle we've got of the ***right isosceles triangle***. This triangle is used in mathematical lessons as 'triangle' or 'set square'. Sometimes this set square has the form of a **special right triangle**, a right triangle with angles of 30 and 60 degrees.

The **characteristic triangle** is the triangle with infinitesimal sides dx, dy and ds. This 'triangulum characteristicum' has been used in the 17th century by *Pascal*, *Snellius* and *Leibniz*, while developing infinitesimal mathematics. It seems that is was *Leibniz* who introduced this expression.

In a **equilateral triangle** (or ***regular triangle***) the three sides have equal length. These triangles can fill the plane (also right triangles can, by the way). This quality has already been found by *Pythagoras* (500 BC). The group of scholars around *Pythagoras* had a special liking for numbers, which had for them a magical meaning. They studied the 'figure numbers' (or 'polygon numbers'), constructed from a regular polygon.
In the following way, from a regular triangle:



and so on.

This results in the triangular numbers 1, 3, 6, 10, 15, and so on.
In ***Pascal's triangle*** we find these same numbers als binomial coefficients (n over k) [1)]:

1

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

These coefficients are often used in probability theory and also in formulae for the n^{th} power and the n^{th} derivative. In probability theory the triangle diagram, is used, a way to express the a function of three variables, in a triangle grid.

The Dutch drawer *Escher* was responsible for the 'impossible triangle': an impossible (perspective) triangle that can be drawn, however.
When coping with a project, there is a tension between time, money and quality. You can show this graphically: when you try to diminish the amount of time spent, that cost you money or quality. This is called the devil's triangle.

What triangular objects we see around us? When going to a symphony orchestra we hear the triangle, visiting the sea we can find the triangle mussel the triangle shell and the triangle crab. We have also triangular parts ourselves:

- the 'triangular nerve' (the fifth cerebral nerve)
- the 'triangular bone': a bone in the back row of carpal bones
- the 'triangular muscle': the muscle that goes from the clavicle, shoulder and shoulder−blade to the outer side of the upper arm

A small pair of trunks or pants is called a triangle (in the Netherlands).

It is rather popular to use the triangular form of geographic regions in their names:

- the 'Bermuda triangle' − a sea east of the U.S. that is notorious for his mysterious disappearances, in apparent good weather conditions [2)]
- in the 'gold triangle' is the world's largest resource of opium: the borderland of Thailand, Burma with Laos; the sea−area between Costa Rica, Colombia and Ecuador is also denoted as a 'golden triangle'.
- the 'black triangle' or triangle of death is the extremely polluted borderland of Poland, Czechia with Germany
- the 'iron triangle' is the area northwest of Saigon, where the Vietnam war struck heavy wounds
- the 'brown triangle' is the region of Munich, Nuremberg and Berchtesgarden in Germany, bastion of the brown−clothed nazis
- the 'dark triangle' was a region in Canada, where in 1997 ice−rain made collapse electricity pylons for 3 million people
- 'The Triangle' is North Carolina's version of Silicon Valley
- in Congo a large area of rainforest is called the Goualogo Triangle

Measuring land for mapping by use of trigonometry is called triangulation.

In the 17th and 18th century traders of the Dutch Republic made 'triangular journeys'. Goods were transported by ship in three laps. At first Dutch goods as brandy were transported to western Africa. This was exchanged for slaves who were transported to the Western Indies. There the slaves were exchanged for goods as cacao, cotton and tobacco, to be brought home. These journeys took about 15 months each.
The triangle manifests itself also in social behavior: one speaks of the eternal triangle when there exist intimate relations between three persons. In the Netherlands the

regional cooperation of the Government with the police and the Counsel for the Prosecution is called the 'regional triangle'.

The (mostly regular) triangle is used as a symbol for certain groups or organizations:

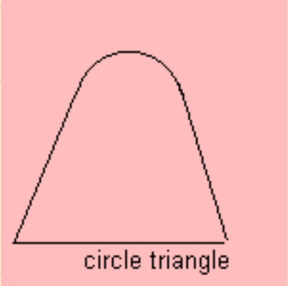
- the pink triangle is the symbol for the homosexual movement; its form has been derived from the German concentration camps, where homosexuals were forced to wear those triangles.
- the red triangle is the symbol of the YMCA
- the blue triangle is the symbol of the YWCA

In traffic a triangular sign has always a warning function (a square sign gives a remark). The warning triangle is also used when your car has a break−down.

At last we know the triangles, for which its quality depends on its position regarding another triangle. The ***cyclic triangle***, the ***inscribed triangle***, the ***circumscribed triangle*** and the ***tangent triangle*** did we already see at the polygon section.
Other special triangles are:

- the ***pedal triangle*** is formed by the dropped altitudes in a triangle
- the ***first Brocard's triangle*** and the ***second Brocard's triangle*** are rather complex triangles, invented by the 19th century French mathematician *Brocard*
- ***perspective triangle***s: two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective in relation to each other when the lines A_1A_2, B_1B_2 en C_1C_2 meet each other in one point. It was *Desargues* who formulated the theory of these phenomena (1648).

A series of triangles can be used to form a spiral of Theodore of Cyrene that approximates the logarithmic spiral.



My 9 year old son Jelle showed me the **circle triangle**, a combination of a part of a circle and a triangle.

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notes

1) In Pascal's 'Triangle aritmetique', published in 1764 (after Pascal's death), the relation between the binomial coefficients and Pascal's triangle is shown.

2) Some investigators doubt the mysterious character: it should be bad weather at time of those accidents, though.

conic section main

last updated: 2002−07−13

$$\frac{1}{r} = \frac{1}{e} + \cos\phi$$

A conic section can be defined as the intersection of a plane and a conic, that's where the name is from.
The relation between the angles of plane and conic determines the kind of conic section:

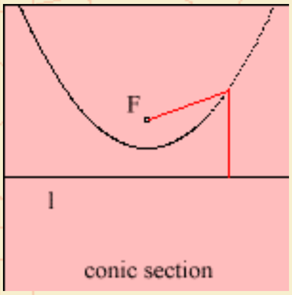
- equal: *parabola*
- smaller: *ellipse*; when the plane is parallel to the ground surface the resulting curve is a *circle*
- larger: *hyperbola*

These qualities were already known by *Apollonius of Perga* (200 BC). He wrote a series of books about the conic sections (Konika [1]), where he used work done by preceding scholars as *Euclid* (who was his teacher).
The first discovering of the conic sections has been made by *Menaichmos* (350 BC), a member of the school of *Plato*. He found the parabola on his attempts to duplicate the cube. But before that period, there was already a certain knowledge of the qualities of the conic sections, for instance in the head of *Pythagoras* (550 BC).

A conic section is an algebraic curve of the 2nd degree (and every 2nd degree equation represents a conic). The kind of conic section can be obtained from the coefficients of the equation in x and y.
The geometric treatment by the old Greek of the conic sections is equivalent with such an equation. But it had a different form, consisting of the theory of the 'adaptation of areas'. This theory can already be read in 'the Elements' of *Euclid*.
Depending the kind adaptation one of the three conic sections is obtained:

- elleipsis: adaptation with defect
- parabole: exact adaptation
- hyperbole: adaptation with excess

Given a point F and a line l.
The conic section can be defined as the collection of points P for which the ratio 'distance to F / distance to l' is constant.
The point F is called the focus of the conic, the line l is called the directrix. The ratio is the *eccentricity*; its value gives the kind of conic section:



eccentricity e	***conic section***
0	circle
$0 < e < 1$	ellipse
1	parabola
> 1	hyperbola

So the value gives the amount of deviation from a circle. When we set the distance of the focus to the line to 1, we can write for a conic section with eccentricity e the polar equation at top of this page. In this equation it is quite clear that the polar

inverse is the limaçon.
The circular conic section is the circle.
The orbit of a comet can in fact take the form of each of the conic sections.

More detailed information about each of the conic sections:

- ellipse
- circle
- parabola
- hyperbola

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notes

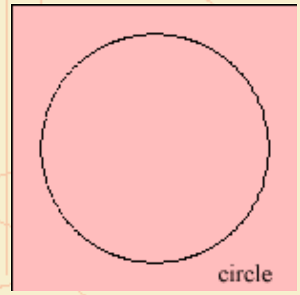
1) koonikos (Gr.) = cone

circle

conic section

last updated: 2003−05−25

$r = 1$

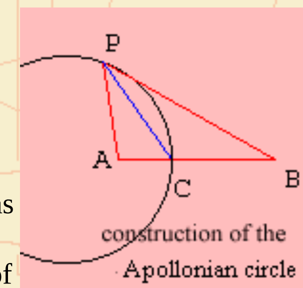


The circle is the ellipse of which the two axes are equal in length.

Because of its symmetry the circle is considered as the perfect shape. It is the symbol for the total symmetry of the divine (sic!). The Greek scholar Proclus (500 AC) wrote: "the circle is the first, the simplest and most perfect form". As Christian symbol it represents eternity, and the sleeping eye of God (Genesis 1:2).

And an anonymous poet wrote:

oh, the Circle, she is so divine
her curve is round, unlike the line.



More rational the circle can be described as the ellipse, where the two foci coincide. Or as the collection of points with equal distance to a (center) point. At the top of this page we see the polar equation of a ***unity circle*** [1] with radius 1 and as center the origin.
This definition − which gives the essence of the circle − was already formulated by *Euclid* (300 BC) in book III of his 'Elements'. That's why you can draw the curve with a pair of compasses. The circle's form remains intact while turning, what makes her very useful as lid for closing jars. And also for a watch with turning hands. The diameter of a screw is a circle too. But man got the greatest advantage from the insight that for a wheel of a cart not the square, but the circle is the most suitable form. So the study of the circle goes back beyond recorded history.
The first mathematician to be attributed theorems about circles was the Greek *Thales* (650 BC).

Creating a sand castle, you ought to know that, with a given amount of surface, on the form of a circle the highest sand tower can be constructed. A corresponding quality have circular drum skins: with a given amount of skin they produce the lowest frequency. Another quite practical advantage of the circle is that it has no sharp edges, and distributes the pressure from the inside evenly. That's why the intersection of a hose−pipe has the form of a circle.
You can cut from your circular birthday cake a part, in one straight line (chord). This piece is called a *circle segment*. More common is to divide the cake from the middle in some circle segments. Those wedges can nicely demonstrate the distribution of a variable: a *circle diagram*.

The ***Apollonian circle*** can be constructed for a segment AB, whose points P have a ratio PA/PB = k. When C is the point on AB for which AC/CB = k, it is easy to see that the circle views the segments AC and BC under equal angles [2].
Paris Pamfilos' Isoptikon (796 kB) is a tool for for simply drawing these circles.
Generalization of this circle leads to the Apollonian cubic. The Apollonian circle has been studied by *Apollonius of Perga*.

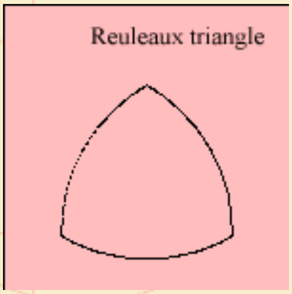
Already centuries before the Christian era mathematicians tried to calculate the area of the circle [3]. The discovery of this area being proportional to the power of the radius, led to the effort to determine this proportional constant (pi) as exactly as possible [4]. This question is related to the problem of the circle's quadrature: to construct a square with the same area as a given circle. The first to deal with this question was *Anaxagoras* (540 BC). As proverb 'squaring a circle' has the meaning of trying something that is completely impossible.

A nice quality of the circle is shown in the story of *Dido*: this Phoenician princess escapes her town Tyrus (in today's Lebanon), after her brother (king Pygmalion) killed her husband. She went to the north coast of Africa, where she aimed to settle a new town. Dido wanted to buy land from king *Jarbas* of Namibia, the emperor at that place. Whether she didn't want to spend a lot of money, or Jarbas didn't want settlements is not clear. But the fact is that they agreed that Dido could get as much land as the skin of an ox could cover. Luckily for her the queen was so clever to cut the skin in a long rope (probably more than a mile in length) and to lay this in the form of a circle, enclosing as much land as possible. This meant the settlement of the city of Cartage (814 BC). The story illustrates the circle being the solution of the *isoperimetric problem* [5]. This problem, one of the oldest problems in variation calculus, was already known by the old Greek. So did Pappus, a Greek scholar from Alexandria (300 BC), already have a good understanding of the problem. In this knowledge, he followed a book of Zenodorus (180 BC) [6].

Some relations of the circle with other curves are the following:

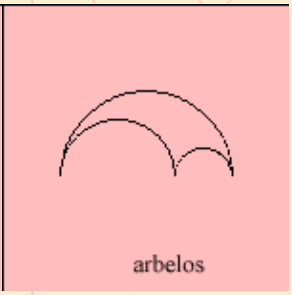
- it is the radial of the cycloid
- it has three well−known catacaustics, depending on the position of the source:
    - cardioid (source on circle)
    - limaçon (source not on circle)
    - nephroid (source at infinity)
- the circle with the pole as center is invariant under inversion and is called the ***circle of inversion***;
  in all situations when the pole is not situated on the circle, the inverse is a circle too
- it is the orthoptic of the deltoid
- the circle is a specimen of the sinusoidal spiral
- the pedal of the circle is the limaçon (pedal point not on the circumference) or the cardioid (pedal point on the circumference)
- the circle is the pedal of the hyperbola
- the hyperbolism of the circle is the witch of Agnesi
- its barycentric is the cochleoid
- the circle is the rose for which c=1

Sometimes the forces of nature cause circular motions.



Drops of fat in the soup have the form of a circle: the circle gives the situation of the least potential energy, for the existing molecular forces.
Object do not move in a circle by their selves, they need a force that is directed to the middle of that circle. Examples of such motions are to be seen in the sport of throwing away a iron shot with a cable, and the movement you make yourself, by standing on the rotating earth. When a point moves with constant velocity in a circular orbit, one speaks of a *harmonic oscillation* [7].
The old Greek had the idea the celestial bodies moving in a constant circular motion, with the earth in the center of those orbits. To fit this model with the experimental facts of the planetary movements, the condition of the earth in the circle's center was omitted. And also combinations of circular motions were proposed: ***epicycles***. These astronomical theories persisted until mid 17th century.
Another celestial phenomenon consisting of circles is the shape of the lighted moon, a combination of two circles. Seen from the earth, one side of that moon is illuminated by the sun.
The brilliant physician Einstein gave his name to the *Einstein circle*s, an optical (relativistic) effect when two solar objects are placed exactly in line behind each other (for an earthly observer). It's in fact an extra solar gravitational lens, and the first observation occurred in 1987.

So the circle is a well−known form in our world, and we can find her back in a lot words and expressions. Sometimes the round form is used: the circle scissors who succeeds to cut round pieces; the round form of the circus ring. The *circle of Willis* is a circular spot on the human body where inner en outer temperature are being compared. And the *Vieth−Muller circle* is a special case of the horopter.
But often the emphasis is on the turning movement. In the case of a circular saw the moment is exactly in a circle, but in words as circular, circuit, circulation only a closed ring remains. In expressions about circles the meaning refers often to that movement: in a circular argument that what is to be proven, is used as given, so that the proof looses it's value. In a circular definition we see two definitions, who point to each other. In a vicious circle one comes once again back in it's starting position.



Till know we spoke about a single circle, but more circles together can be handy too: two circles we find in the English bicycle [8], and the French bicyclette. Three arcs of a circle form together a ***circular triangle***. In the case of three equal arcs beginning in the angles of an equal triangle, it is called the ***Reuleaux triangle***. Already *Archimedes* described the ***arbelos*** or ***shoemaker's knife***, also consisting of three circles.
More concentric circles can be found at a target for shooting or darting.
In the 16th century scientists as *Giordano Bruno* and *John Dee* looked at the spiritual meaning of so−called hermetic forms, which often contained circles. By placing those circles in the right place a resonance with his celestial (pure) counterpart could be reached. An example of these forms is − with four circle arcs − John Dee's ***magic hieroglyphic*** [9].

A combination of five circles we can see in the symbol of the car Audi − circles placed on one line − and in the symbol of the Olympic games − the five continents, the circles now positioned in two lines. Is this similarity the explanation for the sponsorship of Audi of the Dutch Olympic team?
A circle placed around a cross is the symbol of the Ku−klux klan [10)] movement.

During the 19th century it was a popular occupation in geometry to construct all sorts of circles inside a triangle. Relatively easy to imagine are:

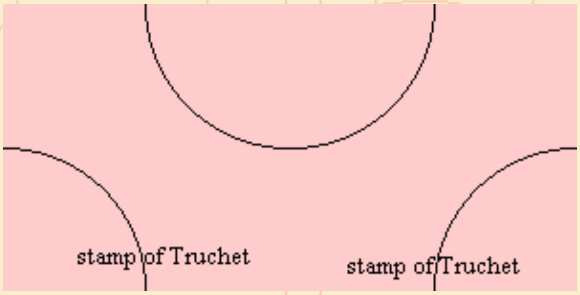
- **outside circle**: circle through the vertices of the triangle
- **inside circle**: largest circle that fits completely inside the triangle
- **touching circle**: circle that touches an outer side of a triangle (two side are being lengthened)

Less common circles are:

- ***Feuerbach 's circle*** (1800−1834) or ***ninepointcircle***
  It appears to be a fact that in each triangle the following points form a circle:

footing points of the perpendiculars; middles of the sides; middles of the lines who connect the high point with the vertices.

- ***first circle of Lemoine*** (1840−1912)
- ***second circle of Lemoine*** or ***cosinecircle***
- ***Brocard's circle***



I save the reader the details of the former three circles.

A more recent invention is an area filling curve in the shape of two circle arcs, the ***stamp of Truchet***. The pattern is formed by 'stamping' the two forms of the Truchet curve next to each other, in a arbitrary sequence.

---

notes

1) When the radius has the value zero, the circle is reduced to a single point, what is called a **point circle**.

2) Line PC bisects the angle APB.

3) The Egypt (1900 BC), the Mesopotamian (1700 BC), the Chinese and the Indian (500 BC). Struik 1977 chapter 2.

4) The value of pi is about 3,14159.

5) Iso (Gr.) = equal, perimetron (Gr.) = perimeter, contour.

6) The ultimate proof was given not earlier than in the second part of the 19th century, by Weierstrass.

7) To be compared with the sine.

8) Bi (Gr.) = two.

9) Davis 1981 p. 100.

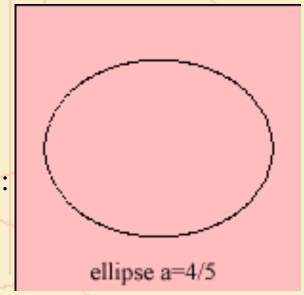
10) Ku−klux is derived from the Greek word for circle: kyklos.

ellipse

conic section

last updated: 2003-05-20

$r_1 + r_2 = 1$



The word ellipse is derived from the Greek word elleipsis (defect). The same defect do we find in the linguistic ellipse: omitting a part of a sentence, in situations that the part can be thought easily. So is the stimulation 'at work!' an linguistic ellipse.
Another fine quality of an ellipse is that for each point the sum of distances to the foci is constant. This gives us an easy way to construct an ellipse: fasten a rope in the two foci, and draw the curve with a pencil tight to the (strained) cord.
This quality leads to above given bipolar equation. In Cartesian coordinates the ellipse is to be written as: $x^2 + a^2 y^2 = 1$
A curve resembling the ellipse is the oval [1]. But the oval is not an ellipse, it has one symmetry axis less.

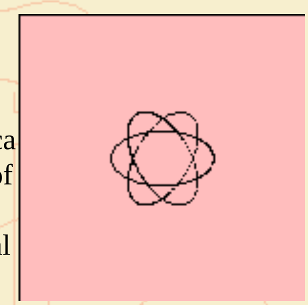
Menaichmos (350 BC) was the first to study the ellipse, *Euclid* wrote about the curve, and its name was given by *Apollonius* (200 BC).

The ellipse can be transformed into a circle by a linear transformation. It is the ellipse with the two axes equal in length. In fact, the ellipse can be seen as the form between the circle (eccentricity 0) and the parabola (eccentricity 1).

Some other qualities of the ellipse are the following:

- when a cycloid rolls over a line, the path of the center of the cycloid is an ellipse
- the ellipse is the pedal of Talbot's curve
- the pedal curve of an ellipse, with its focus as pedal point, is a circle
- the evolute of the ellipse is the astroid
- its isoptic is the circle

Imagine an ellipse as a mirror strip. When you place a lamp at one focus, then there appears a light on the place of the other focus too. In Dutch the focus is designated with the word 'burning point', referring to this phenomenon. By the way, these foci always lay on the longer axis.



The astronomer *Johann Kepler* studied the celestial orbits. While he believed in 1602 the orbit of Mars was oval, later he found that the the form is elliptical, with the sun in the focus [2], what he published in 1609 in his work 'Astronomica Nova'. His conclusion was based on the astronomical data of the astronomer Brahe. Theoretically it's the case of the two body problem [3]. Newton was the first to derive the elliptical orbit out of the forces between planet and sun.
The planetary orbits are close to a circle, the eccentricity of the Mars orbit is about 1/11, the Earth orbit about 1/60.

Comets can move in an elliptical orbit. Halley found in 1705 that the comet, which is now called after him, moved around the sun in an elliptical orbit. Its orbit is close to a parabola, having an eccentricity of about 0.9675.

On a much smaller scale, the atom, one sees electrons orbiting round the atom kernels. A visual representation of this view is used to denote our atom age.

The area of an ellipse has the value πab (where a and b are half of the axes).
The search for the arc length of an ellipse led to the so−called *elliptic functions*: the (incomplete) *elliptic integral* (of the second kind) gives an expression for this length.

The potential equation, a partial differential equation of second order, is also called the *elliptical differential equation*. This is because of the similarity between the equations of ellipse and differential equation [4)]. Normally mathematicians work in an Euclidean geometry, where the fifth Euclidean postulate is valid: given a line l and a point P, not on l; then there exists only one line parallel to l through P. When the assumption is that no such line exists, you enter a geometry that is called the *elliptical geometry*.
In multivariate analysis the Mahalanobis distance weights the differences in variability of the variables. Distances are measured in units of standard deviation. It is easy to see that the lines of constant standard deviation are ellipses.

Brocard pointed out the construction of a specific ellipse, inscribed in a triangle: ***Brocard's ellipse***. The ***Steiner ellipse*** is the ellipse through the three vertices of a triangle and with the centroid of the triangle as its center.
And in our universe we see that some (older) milky ways resemble an ellipse, they are called elliptical star constellations.

---

notes

1) ovum (Lat.) = egg.

2) first Kepler's law.

3) Kuipers 1966 p. 370.

4) substitute the partial derivatives by x en y in:

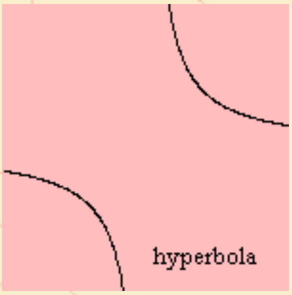
$$\frac{\delta^2\phi}{\delta x^2}+\frac{\delta^2\phi}{\delta y^2}=\rho(x,y)$$

hyperbola

conic section

last updated: 2003−05−21

$xy = 1$ $x^2 - y^2 = 1$



The hyper part in the hyperbola word is derived from the Greek word 'huper', what means to surpass a certain limit [1]. This excess is reached while adapting the areas in the old Greek way, in the case of the hyperbola. The same quality we find in the hyperbole, what has the meaning of a stylistic exaggeration. For example: an ocean of tears. *Apollonius* gave the hyperbola its name.

Nowadays a more common definition of the hyperbola is as being the curve for which the difference between the distances to the foci remains constant. And also the following definition is used:

The hyperbola has two asymptotes, what gives reason to saturation−growth with the curve [2].

We can distinguish three special variants:

- ***orthogonal hyperbola, equilateral hyperbola***.or **rectangular hyperbola**: the two asymptotes are perpendicular.
  This special case has been studied at first by the Greek *Menaechmos*.
- ***degenerate hyperbola***: the hyperbola has been degenerated to his asymptotes
- ***adjungated hyperbola***: given a hyperbola a second one is added, with the same asymptotes, but axes switched. An example of this relationship is formed by a pair of hyperbolas x y = 1 or x y = −1

Some relations with other curves:

- the hyperbola is the isoptic of the parabola
- the hyperbola can be seen as a sinusoidal spiral
- depending on the center of inversion and the form of the hyperbola, as polar inverse the following curves can be obtained:
  - lemniscate (of Bernoulli) (center)
  - (right) strophoid (vertex)
  - trisectrix of MacLaurin (vertex)
- the pedal curve (focus) is the circle
- a pedal (center) of the rectangular hyperbola is the lemniscate
- its isoptic is the circle

The Greek *Euklid* and *Aristaeus* wrote about the general hyperbola, although they confined the curve to one branch. *Apollonius* was the first to see both of the two branches.

Where do we see the hyperbola in our world? Well, take a cistern, filled with water, and place two oscillators, with same frequency and amplitude. Then you see an extinction in the form of hyperbolas, as result of the constant difference in distance

between the two foci (oscillators), on a hyperbola. Above that, we find the hyperbola in various branches of science as variables, that are inversely proportional related.
It appears that the attention for the recent past is inversely proportional to the passed time [3]. The American linguist *George Kingsley Zipf* counted articles' word occurrences. When he arranged the words in order of frequency, the product of number and frequency turned out to be about a constant. In other words: the most used word occurs a 10 ten times more often than the word on the 10th. Zipf studied a lot of texts, differing in languages and subject, but once again he found the amazing relation. Later would have been proven that also texts with random grouped elements (out an alphabet plus the space) obey the rule [4].
George Kingsley got more and more enthusiastic and he discovered similar relations on other subjects, as in the distribution of inhabitants, valid in different countries. The relation is called a *hyperbolic distribution*, and in the case of number and frequency the relations was named the *Zipf's law*. Zipf believed in the validity of his law, but so hard that he justified the annexation of Austria by Germany by stating that his law would fit better! After Zipf's first investigations the relation has been found on many subjects: from the distribution of the reptiles sorts in families to the number of articles handling different topics in a scientific magazine. And recently biologists found that (non coding) junk DNA obeys Zipf's law, but the coding DNA does not.
Sometimes a distribution can not be matched with a hyperbola, the Mandelbrot−Zipf law makes a generalization towards the power function: $y = x^{-a}$ [5].

In another math region we can find the *hyperbolic geometry* (or geometry of Lobachevsky), one of the non−Euclidean geometries, based on the axiom that there exist at least two lines through a point P that are parallel to a line not through P. The hyperbolic functions have not much in common with the hyperbola. Their name is based on the fact that two hyperbolic functions can be drawn geometrically in a hyperbola.
The wave equation, a partial differential equation of second order, is also named the *hyperbolic differential equation*. Reason is the resemblance between the equations [6].

The hyperbolism of a curve f (x,y) = 0 is a curve with equation f (x, xy) = 0.

---

notes

1) Bole is from balloo (Gr.) = to throw

2) In the form: $y = x / (1 + x)$

3) Law of the extinguishing past, based on the frequency of year numbers in newspapers: Pollmann 1996

4) Maybe this is true, but i can't get to this law (for an one or two letter alphabet).

5) For the cities and their inhabitants in many cases a value of the constant a between 0 and 1 has been found. Australia is an exception with a value of a of about 1.2.

6) Substitute the partial derivatives by x en y in:

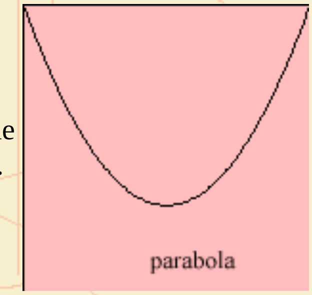
$$\frac{\delta^2 y}{\delta t^2} = \alpha^2 \frac{\delta^2 y}{\delta x^2}$$

parabola

conic section

last updated: 2003−05−21

$y = x^2$



The word 'parabola' refers to the parallelism of the conic section and the tangent of the conic mantle. Also the parable [1)] has been derived from the Greek 'parabole'.
The parabola can be seen as an ellipse with one focus in infinity. This means that a parallel light bundle in a parabolic mirror will come together in one point. It had been told that *Archimedes* did use a parabolic mirror in warfare. It was during the siege of Syracuse (214 − 212 BC) by the Romans, that *Archimedes* constructed reflecting plates in about the form of a parabola. These plates were used to converge the sunlight onto the Roman ships, and put them in fire. Though this event is discussed by some historians, recently the feasibility of Archimedes' plan has been proved [2)].
Other technological parabola shaped objects are the parabolic microphone and the parabolic antenna, used to focus sound and electromagnetic waves, respectively.

Menaichmos (350 BC) found the parabola while trying to duplicate the cube: finding a cube with an area twice that of a given cube [3)]. In fact he tried to solve the equation $x^3 = 2$. *Menaichmos* solved the equation as the intersection of the parabolas $y = x^2$ and $x = ½ y^2$.
Euclid wrote about the parabola, and *Apollonius* (200 BC) gave the curve its name.
Pascal saw the curve as the projection of a circle.

In the ***parabola curve*** the parabola (with its vertex oriented downwards) is being repeated infinitely.

Some properties of the parabola:

- the parabola is the involute of the semi−cubic parabola
- a path of a parabola is formed when the involute of a circle rolls over a line
- the curve is a specimen of the sinusoidal spiral
- it is the pedal of the Tschirnhausen's cubic

Besides, the following curves can be derived from the parabola:

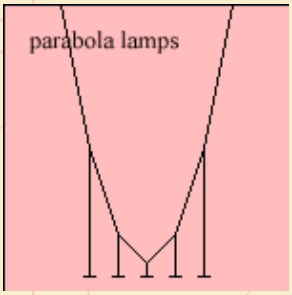
kind of derived curve	***curve***
catacaustic rays perpendicular to the axis	Tschirnhausen's cubic
isoptic	hyperbola
orthoptic	the parabola's directrix
polar inverse focus as the center of inversion	cardioid
polar inverse vertex as the center of inversion	cissoid

roulette a parabola rolls over another (equal) parabola	cissoid: the path of the vertex
roulette a parabola rolls over a line	catenary: the path of the focus

$$y^2 = x^2 \frac{a+x}{1-x}$$

Pedals of the parabola are given by:

a	*pedal point*	*pedal of the parabola*
0	vertex	cissoid (of Diocles)
1	foot of − intersection of axis and − directrix	(right) strophoid
3	reflection of focus in directrix	trisectrix of MacLaurin
−	on directrix	oblique strophoid
−	focus	line



The Italian *Luca Valerio* determined the area of a parabola, in 1606; is was called the quadrature of the parabola. But it was *Archimedes* who first found the value of this area. In his work "Quadrature of a Parabola" he formulated the area as 2/3 from the product of the base and the height of the parabolic sector [3].

At the end of the the Middle Ages the cannon came into use on the battlefield. Therefore it was important to predict the exact location where the projectile would land. Many scientists tried to answer this question, and it was *Galileo Galilei* who found the relation as first.
That the trajectory of a projectile − neglecting effects of friction − has the form of a parabola, can easily be understood as follows: the x coordinate is proportional to the elapsed time, the y coordinate quadratic (as result of the constant gravitation force).
As a matter of fact, this relation was in the 15th century already known as the 'square law'.
When the air friction is taken into account, a braked parabola emerges, what in fact is a kind of exponential curve.

The 'parabolic velocity' is the minimum velocity to escape from (the gravity forces of) a celestial object.
And an arch (e.g. of stone) is called parabolic when its shape has the form of a ....

Now it's time for an experiment: hang on a horizontal chord some lamps, at equal distances. Neglecting the weight of the chord compared to the lamps, the hanging

points of the lamp form a parabola [4]. This form is independent of the weight of the lamps, and also of the distance between the lamps. In the case of a chord without lamps, instead of a parabola, a catenary is formed.
Jungius (1669) proved that *Galileo* was wrong to state this curve being a parabola.

On holidays at the Dutch North Sea beach, you can see dunes in the form of a parabola (parabola dune), formed by the sand, thrown by the wind, but hold by the dune plants.
When experimenting, one will often encounter quadratic relations between variables. So is the kinetic energy of a body proportional with the square velocity. In another branch of mathematics we find the iterated parabola, studied for the first time (in the 60s) by *Myrberg*. In the case of convergence one speaks of an ***attractor***. And the warmth equation, a partial second order differential equation, is also called the *parabolic differential equation*. Reason is the similarity between the relations [5].

---

notes

1) Para = next to, bole (Gr.) = throw, parabole = what is thrown next to (to compare).

2) Mueller 1985 p. 111.

3) one was looking for a cube with contains twice the volume of a given cube: Waerden 1950 p. 88.

4) with thanks to Chris Rorres (Drexel University).

5) This follows from setting to zero of the force momentum, working on the lowest point of the chord.

6) $\frac{\delta y}{\delta t} = \alpha \frac{\delta^2 y}{\delta x^2}$.

cubic

last updated: 2003−05−18

A cubic is an algebraic curve of third degree.

The following curves are − under certain circumstances − cubic:

- Lissajous curve
- polynomial
- polytrope: the cubic hyperbola
- teardrop curve

Newton distinguished four classes of cubics [1], where each class was divided into several species. He used the theorem that each cubic can be obtained from the divergent parabola, by a central projection [2] from a surface on to another surface.

Newton came to 72 species, including the conic sections [3]. This classification was criticized by *Euler*, because of it's lack of generality. And in fact, Newton missed sex species of curves. There are also other classifications, ranging from 57 to 219 types. The one with the 219 classes was given by *Plücker*.

Dick Nickalls describes in the Mathematical Gazette [4] a new approach to solve the cubic. It gives new geometric insights into how and why the standard Cardan algebra works.

I distinguish the following cubic curves:

- circular cubic
    - Apollonian cubic
    - conchoid of de Sluze
- cissoid
- cubic spiral
- divergent parabola
- folium of Descartes
- Newton's diverging parabola (3th species of Newton)
- oblique strophoid
- resonance curve
- serpentine
- strophoid
- trident of Newton (66th species of Newton)
- trisectrix of MacLaurin

---

notes

1) Curves by Sir Isaac Newton in Lexicon Technicum by John Harris, published in London in 1710.

2) Struik 1977 p.138

3) published in 'enumeratio linearum tertii ordinis' (1704)

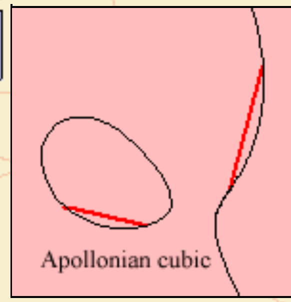
4) RWD Nickalls (1993): a new approach to solving the cubic Cardan's solution revealed. Mathematical Gazette 1993, volume 77 (November), pp 354 359.

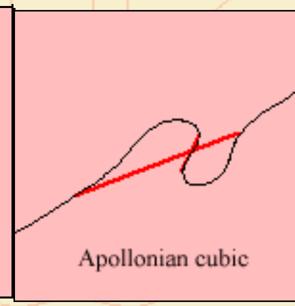
Apollonian cubic

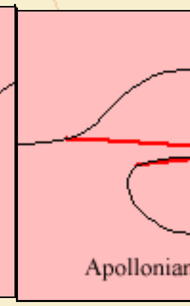
cubic

last updated: 2002−07−13

$$(x-a)(x^2+y^2)+bx+cy=0$$







as follows. Given two line segments AB and CD, the curve is the collection of the points P from which the angles viewing the segments are equal.
The curve is a generalization of the Apollonian circle:

Van Rees found some interesting properties of the curve, in 1829. In 1852 *Steiner* formulated the corresponding problem, unaware of the work of van Rees.
It was *Gomes Teixeira* who remarked this equivalence (in 1915) between the work of *Steiner* and *van Rees*.
These cubic curves got quite popular among mathematicians, e.g. *Brocard*, *Chasles*, *Dandelin*, *Darboux* and *Salmon* studied them.

Nowadays, at the University of Crete, a group around *Paris Pamfilos* is working on these cubics. They gave the curve the name ***Apollonian cubic*** or ***isoptic cubic*** and they constructed a tool, named Isoptikon (796 kB) to draw an Apollonian cubic, given the two segments. An Apollonian cubic can have two parts or just one.
Also the theory about the isoptic cubics has been included in the package, and many characteristics of the curve. It is shown, for instance, that the unifying concept is the Abelian group structure defined on a cubic, as discovered by *Jacobi* (1835).

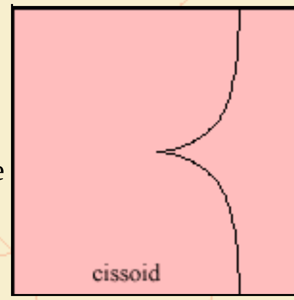
cissoid

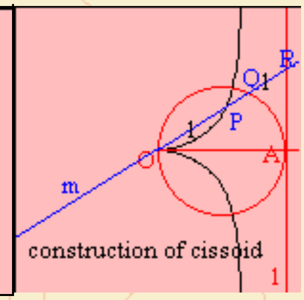
cubic

last updated: 2002−07−13

$r = \sin\phi \tan\phi$

The cissoid can be constructed as follows:
Given a circle C with diameter OA and a tangent l through A. Now, draw lines m through O, cutting circle C in point Q, line l in point R. Then the cissoid is the set of points P for which OP = QR.





This construction has been done by the Greek scholar *Diocles* (about 160 BC) [1]. He used the cissoid to solve the Delian problem, dealing with the duplication of the cube. He did not use the name cissoid. The name of the curve, meaning 'ivy−shaped', is found for the first time in the writings of the Greek *Geminus* (about 50 BC). Because of Diocles' previous work on the curve his name has been added: ***cissoid of Diocles***.
Roberval and *Fermat* constructed the tangent of the cissoid (1634): from a given point there are either one ore three tangents.
In 1658 *Huygens* and *Wallis* showed that the area between the curve and his asymptote is π/4.

The cissoid of Diocles is a special case of the generalized cissoid, where line l and circle C have been substituted by arbitrary curves C1 and C2.
The curve, having one cusp and one asymptote, has as Cartesian coordinates:

$$y^2 = \frac{x^3}{1-x}$$

Some relationships with other curves are:

- the curve is a pedal as well as the polar inverse (with the cusp as center of inversion) of the parabola
- its catacaustic (with the cusp as source) is the cardioid
- when a parabola is rolling over another (equal) parabola, then the path of the vertex is a cissoid
- its pedal (with the focus as pedal point) is the cardioid

In the special case that construction line l passes through the center of the circle, the curve is a strophoid.
And near the cusp the curve approaches the form of a semicubic parabola.

---

notes

1) Waerden 1950 p.297

circular cubic

cubic

last updated: 2002−12−22

$$(x-a)(x^2+y^2)+by^2+cx+dy=e$$

The curve is the cyclic of the parabola.

Some special cases are the following:

- Appolonian cubic (b = e = 0)
- conchoid of de Sluze (b=1+a, c = d = e =0)

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conchoid of de Sluze

cubic

last updated: 2002−03−25

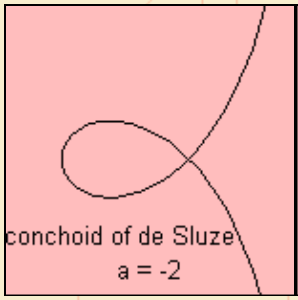
$$r = \frac{1}{\cos\phi} + a\cos\phi$$

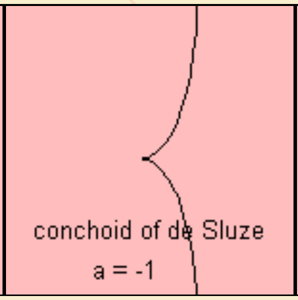
The curve is the work of René François Walter, baron *de Sluze*. This man, who was mathematician as well as important church man, constructed the curve in 1662.

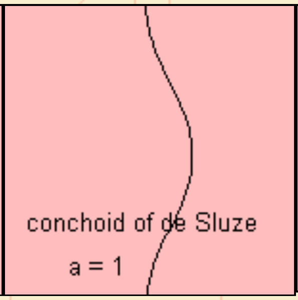
In Cartesian coordinates the curve is written as:

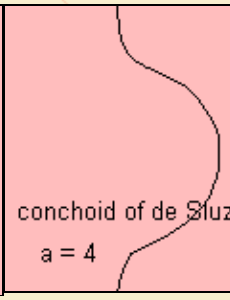
$$(x-1)(x^2+y^2) = ax^2$$

Here we see some specimen of this conchoid curve:









divergent parabola

cubic

last updated: 2002-03-25

$$y^2=x^3+ax^2+bx+c$$

This class of cubic curves has also as name the ***cubic parabola***, to be distinguished from the cubic parabola which is a polynomial in x.

Two special subsets are:

- elliptical curve
- Newton's diverging parabola

elliptic(al)cubic curve

last updated: 2002−03−25

$$y^2 = x^3 + ax + b$$

It is possible to define these divergent parabolas as a type of cubic curve which is topologically equivalent to a torus.
The curves emerged at the studies of the *elliptic integrals*. There is also a recent application in code theory, where they are used to construct error repairing codes.
Quite recently the elliptic curves have been used for cryptography in so called elliptic curve cryptosystems. But in 1997, when a first standard (named P1363) for this method just has been proposed, it appeared that some elliptic curves could be tracked down quite easily because of symmetry reasons (called 'trace one').

egg curve

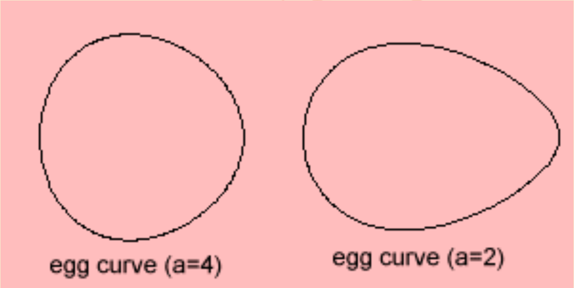
cubic

last updated: 2002-09-29

$$y = (x-a)(1-x^2)$$

Pekka Judin from Helsinki (Finland) sent me this formula, for a cubic that can describe an egg form.

An alternative way to define an egg curve is as a quartic egg curve.



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folium of Descartes

cubic

last updated: 2002−03−25

$$x^3 + y^3 = xy$$

The curve is is symmetric with respect to the line y = x, being a direct result of the polar equation:

$$r = \frac{\sin 2\phi}{\sin^3\phi + \cos^3\phi}$$

Descartes was the first to examine this curve, in 1638. He (incorrectly) believed that the shape was repeated in each quadrant.
Roberval gave the curve the name of the ***fleur de jasmin***, also the result of a incorrect understanding of the shape of the curve. This name was later replaced by the curve's existing name.
In French the curve is sometimes called the '***noeud de ruban***' [1)].

---

notes

1) that means: knot of ribbons. Is this true, isn't here a confusion with the strophoid?

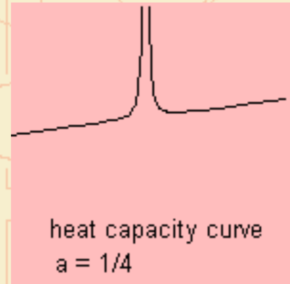
heat capacity curve

cubic

last updated: 2002−03−25

$$y = x + \frac{a}{x^2}$$

This is the formula that follows from the heat capacity model, when linear constants have been omitted [1].



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notes

[1] Normally the formula is written as: y = a + b x + c / x^2

Newton's diverging parabola

cubic

last updated: 2002−03−25

$$y^2 = x(x^2 + ax + b)$$

Newton distinguished four classes of cubics. The diverging parabola is the third class. Newton states: 'in the third Case the Equation was yy = ax^3 + bxx + cx + d; and defines a Parabola whose Legs diverge from one another, and run out infinitely contrary ways.'

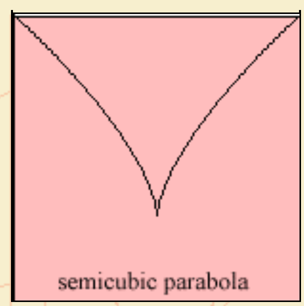
These divergent parabolas can be divided in the following species, depending on the solution on the right hand side of the equation:

- three real roots:
  - all roots equal: semi−cubic(al) parabola
  - three real and unequal roots
  - two equal roots: Tschirnhaus(en)'s cubic
- just one real root

Newton distinguished five species, taking account of the two forms of Tschirnhausen's cubic.

semi−cubic(al) parabola

$$y^2 = x^3$$



As being a kind of a Newton's diverging parabola, it is the situation where the three real roots of the right hand equation are equal.

Its name is derived from the *cubic parabola*. The third (cubic) power is divided by two (semi).

An other name for the curve was derived from *William Neil(e)* (1637−1670) who discovered the curve in 1657: ***Neile's parabola***. This very keen mathematician showed great promise, but the refusal by his father of the marriage with his beloved girl did break him. And he died young.
Newton said already (1710): 'this is the Neilian Parabola, commonly called Semi−Cubical'.

The curve was the first algebraic curve to have its arc length calculated. Congratulations! It was *Wallis* who published this calculation, in 1659; he gave the credits to his pupil, *Neile*.
The Dutch *Van Heuraet* used the curve for a more general construction.

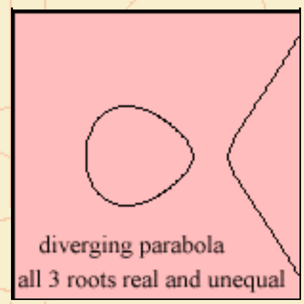
In 1687 *Leibniz* challenged his fellow mathematicians with the question about the isochronous curve: under gravity, the vertical velocity of an object, rolling over this curve, is constant [1]. In 1690 *Jacob Bernoulli* found that the given curve is the semi−cubic parabola (change x and y in above formula). Also *Huygens* found this solution.

You can also find this curve when considering an object as cause of the gravitation of, easy to imagine, the earth. The corresponding equation for the distance s(t) as function of the elapsed time t is $s'' = -g M / s^2$. It is easy to verify that the semicubic parabola t(s) is a solution, isn't it? [2].

The curve is the evolute of the parabola, and the catacaustic of Tschirnhausen's cubic.

three real and unequal roots

$$y^2 = x(x-a)(x-b)$$



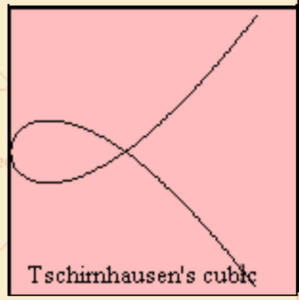
Newton says (1710): 'then the figure is a diverging Parabola of the Form of a Bell, with an Oval at its vertex.'

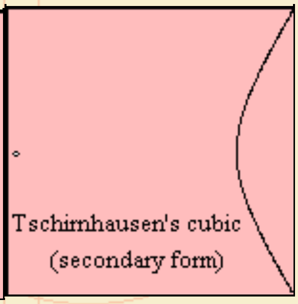
Tschirnhaus(en)'s cubic

$$y^2 = x^2(x+1)$$

or

$$y^2 = x^2(x-1)$$





Two of the three real roots are equal. The curve has two forms, the first is more common in curves' literature. We call the origin in above formulae the pole of the curve.

Newton says (1710) about the curve: 'a Parabola will be formed, either Nodated by toching an Oval, or Punctate, by having the Oval infinitely small'.
Later on, the curve was studied by *Tschirnhaus*, *de L'Hôpital* and *Catalan*. So that for the curve the following names can be found: ***Tschirnhaus(en)'s cubic***, ***(de) L'Hôpital's cubic*** and ***trisectrix of Catalan***.

The curve can also be seen as a sinusoidal spiral.

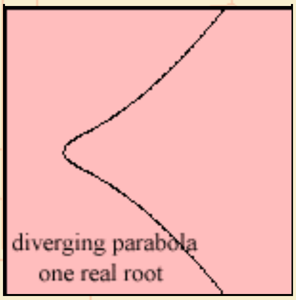
Some interesting properties of the curve are:

- it is the catacaustic of the parabola
- its catacaustic is the semi−cubic parabola
- its pedal is the parabola
- it is the polar inverse of the trisectrix of MacLaurin
- it is the pedal of Talbot's curve

one real root

$$y^2 = x(x^2+a^2)$$

There is just one real root:



Newton said in 1710: 'if two of the roots are impossible, there will be a Pure Parabola of a Bell−like Form'.

---

notes

1) the problem is solved by substituting ds/dt in:

$$ds = \sqrt{dx^2 + dy^2}$$

and dividing numerator and denominator by dx.

2) Walter 1976 p. 4.

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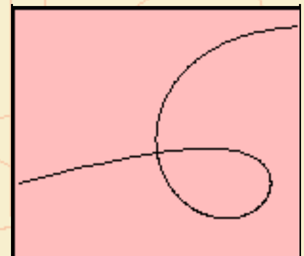
oblique strophoid

cubic

last updated: 2002-03-25

$$r = \frac{\sin(a - 2\phi)}{\sin(a - \phi)}$$



oblique strophoid a=pi/4

This is a more general specimen of the right strophoid.

Now the line OO' through pole O and fixed point O' is no longer perpendicular to the construction line l.
The parameter a = Π/2 for the right strophoid can now also have other values, in Cartesian coordinates:

$$x^3 + xy^2 - x^2 + y^2 = y(x^2 + y^2 - 2x) \cot a$$

The curve is a pedal of the parabola, and a special case of the general strophoid.
Sometimes the oblique strophoid itself is called the 'general strophoid'.

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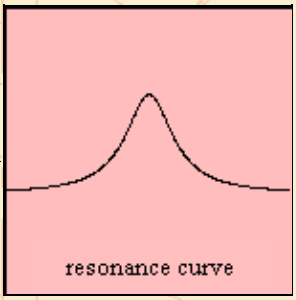
resonance curve

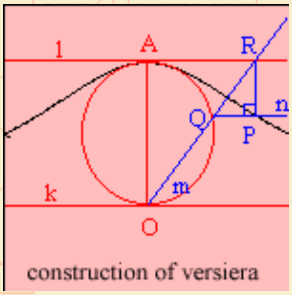
cubic

$$y = \frac{1}{1+x^2}$$


resonance curve

A well−known parameter equation for this curve is the following: $\begin{cases} x = \tan t \\ y = \cos^2 t \end{cases}$

Resonance occurs, when an external oscillation is exerted on a system, with a frequency in the neighborhood of a certain resonance frequency. An example is direct monochromatic light falling onto an atom. The intensity of the radiation, emitted by the atom has the form of a ***resonance curve***, as function of the difference in frequency [1)] (between external and resonance frequency).



construction of versiera

One specimen of this class has been given special attention since the 17th century. We're talking about the ***versiera***, which is constructed as follows.
Given a circle and two parallel lines k and l. Line k is a tangent to this circle in point O, l in point A.
Construct a line bundle m through O. This line m cuts the circle in Q, line l in R. Make now a line n parallel to l and m through Q. Then the point P of the curve is found by letting down a line down on n from point R.
In other words: P has the x−coordinate of point R and the y−coordinate of point Q.
It may be clear that not all resonance curves obey this construction rule.

The curve is:

- a projection of the horopter
- the hyperbolism of the circle
- resembling the transcendent Gauss curve

The curve has been studied by *Fermat* (1666) and *Grandi* (1703).
The Italian female mathematician *Maria Gaetana Agnesi* (1718−1799) wrote about the curve in her book 'Instituzioni analitiche (ad uso della gioventu italiana)' (1748).
She used the Latin word 'versoria' for the curve, meaning 'a rope that turns a sail'. This 'versoria' became the the Italian word 'la versiera', which means 'free to move'. But the translator of the book, the Englishman *John Colson*, translated the word to 'l'aversiera', the Italian word for 'witch'. So nowadays the curve is known as ***versiera*** or ***witch of Agnesi***.

Other names referring to Agnesi are: ***Agnesi's cubic*** and (in French) ***agnésienne***.

---

notes

1) Given is a linear system, i.e. the system is being described by a linear differential equation. Further, the external force fluctuates in the form of a sine. Remember that the intensity of the light is proportional to the square amplitude. Then the given formula follows directly.
Wichmann 1967 p.109

serpentine (curve)

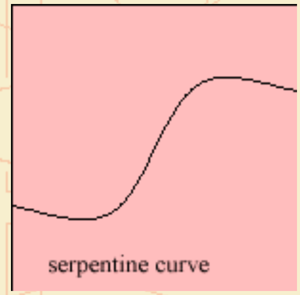
cubic

last updated: 2003−05−05

$$y = \frac{x}{x^2+1}$$

The first normal form in Newton's classification of cubic curves was the hardest case:
$xy^2 + ey = ax^3 + bx^2 + cx + d$.
The ***serpentine*** [1] ***curve***, ***serpentine***, ***serpentine cubic*** or ***anguinea*** [2] is a well known subclass of this form.

The curve can be seen as a projection of the horopter.



The serpentine was studied by *de L'Hôpital* and *Huygens* (in 1692).
But is was *Newton* who gave the curve its serpentine name, when he was working on the curve, in 1701.

---

notes

1) As having the form of a serpent or snake.

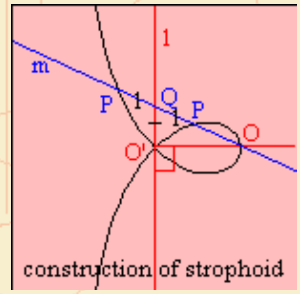
2) Anguis (Lat.) = serpent.
In French: **anguinée**.

(right) strophoid

cubic

last updated: 2003−05−05

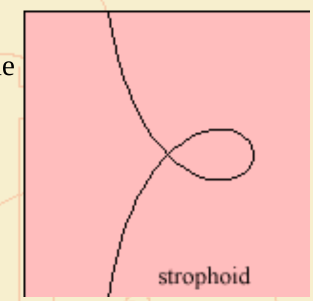
$$r = \frac{\cos 2\phi}{\cos\phi}$$



The strophoid [1] is an extension of the cissoid [2], and in Cartesian coordinates it is written as:

$$y^2 = x^2 \frac{1+x}{1-x}$$

To construct the curve: given a line l, and a point O (called the *pole*), that is perpendicular projected on l in O'. Make now a line bundle m through O. The strophoid is now the collection of the points P on m for which O'Q = PQ. Q is the intersection of l and m.
This construction is closely related to the one of the cissoid, and it can also be seen as a specimen of the general cissoid.
Roberval found the curve as the result of planes cutting a cone: when the plane rotates (about the tangent at its vertex) the collection of foci of the obtained conics gives the strophoid.



The polar inverse of the strophoid is the curve itself (with the pole as center of inversion). On the other hand, when the node is taken as center of inversion, its inverse is the hyperbola.
And the curve is a pedal of the parabola.
And the strophoid is formed by the points of contact of parallel tangents to the cochleoid.

In the middle of the 17th century mathematicians like *Torricelli* (1645) and *Barrow* (1670) studies the curve. The name 'strophoid' was proposed by *Montucci* in 1846.

The strophoid is called the ***right strophoid*** to distinguish from the more general oblique strophoid.

---

notes

1) Strophè (Gr.) = turn, swing
In Dutch: **striklijn**.

2) As the strophoid's equation can be rewritten as: r = cosF − sinF tanF, which is equivalent with the term cosF added to the cissoid's equation.

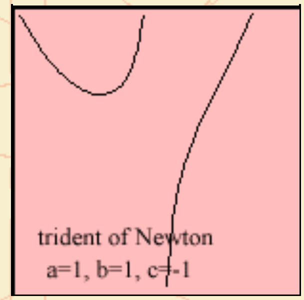
trident of Newton

cubic

last updated: 2003−05−05

$$y = x^2 + ax + b + \frac{c}{x}$$

This was the 66th curve in Newton's classification of cubic curves. The name trident is from the hand of *Newton*. The curve cuts the axis in one or three points.

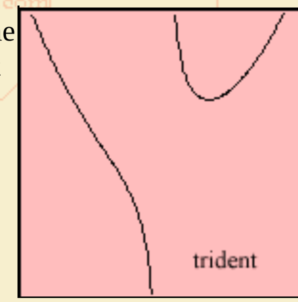


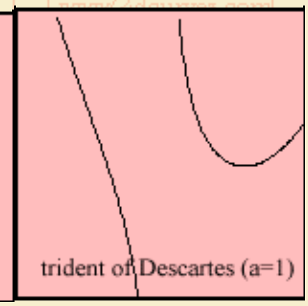
Two subsets of this curve can be mentioned:
Descartes studied the ***trident*** [1] ***of Descartes***,

$$y\,x = (x+a)(x-a)(x-2a)$$

which has also been given the name ***parabola of Descartes*** (although it is not a parabola).

The other subset is the (ordinary) ***trident*** [1]:





$$y = x^2 + \frac{a}{x}$$

---

notes

1) a three−pronged forky form

trisectrix of MacLaurin

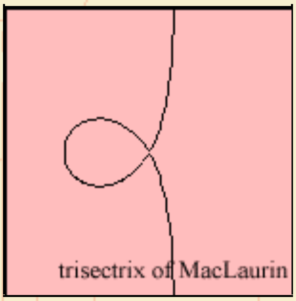
cubic

last updated: 2002−03−25

$$r = \sec\frac{\phi}{3}$$

Colin MacLaurin who was the first to study the curve (in 1742), while looking at the ancient Greek problem of the trisection of an angle: the angle formed by points ABP is three times the angle formed by AOP for points P of the trisectrix.

The area of the loop is equal to $3\sqrt{3}$, and the distance from the origin to the point where the curve cuts the x−axis is equal to 3.



The cubic curve [1] has a second well−known polar form:

$$r = \frac{\sin 3\phi}{\sin 2\phi}$$

Some relationships with other curves are the following:

- the trisectrix is a pedal of the parabola
- it is the polar inverse of Tschirnhausen's cubic (with the focus as center of inversion)
- it is the polar inverse of the hyperbola (with the nod as center of inversion)

It is also an anallagmatic curve.

Freeth (1819−1904) described in a paper published by the London Mathematical Society (1879) the strophoid of the trisectrix.

---

notes

1) With Cartesian coordinates:

$$y^2 = x^2\frac{3+x}{1-x}$$

quartic

last updated: 2003−03−15

A quartic curve is an algebraic curve of the fourth degree.

The following curves do have quartic specimen:

- Lissajous curve
- polytrope
- teardrop curve: the piriform

The following curves are quartic:

- ampersand curve
- besace
  - lemniscate of Gerono
- bicircular quartic
  - plane spiric curve
    - spiric section
      - hippopede
      - Cassini(an) oval
- bicorn
- bifolium
- bullet nose
- Cartesian oval
- conchoid
- cross curve
- deltoid
- devil's curve
- Dürer's conchoid
- egg curve
- falling sand curve
- folium
- kampyle of Eudoxus
- kappa curve
- limaçon
  - cardioid
  - trisectrix
- Mandelbrot oval
- trifolium

ampersand curve

quartic

last updated: 2002−04−06

$$(y^2-x^2)(x-1)(2x-3)=4(x^2+y^2-2x)^2$$

Not much to say about this curve, Cartesian equation is not that pretty [1], and the curve resembles to the ampersand &.

ampersand curve

---

notes

1) Alternative form for the Cartesian equation is the following:

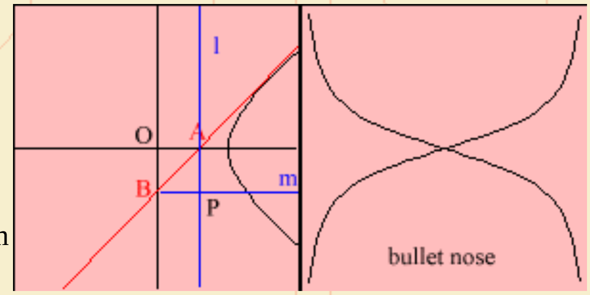
$$6x^4+4y^4+6x^2y^2-13x^3-3xy^2+19x^2-3y^2=0$$

bullet nose

quartic

last updated: 2002−12−30

$$\frac{1}{x^2} - \frac{1}{y^2} = 1$$



The bullet nose can be constructed as follows:
Given a hyperbola H, intersects every tangent with both x− and y−axis, giving points A and B respectively. Draw a line l through A parallel to the y−axis and a line m through B parallel to the x−axis. Then the intersection of lines l and m forms a point of the bullet nose.

The definition is a variation on the definition of the cross curve.

It was *Schoute* (1885) who first described the curve.

bicircular quartic

quartic

last updated: 2003−03−12

$$(x^2+y^2)^2+ax^2+by^2+cx+dy+e=0$$

The ***bicircular quartic*** is a bicircular algebraic curve that is a quartic.
The curve is the cyclic of a conic.

When the conic has the Cartesian equation $x^2 / l + y^2 / m = 1$, the fixed point is (n, o), and the power of inversion is p, then l, m, n, o and p can be written as function of a, b, c, d and e [1].

p=0
When the power of inversion (p) is equal zero, the curve is called rational: a **bicircular rational quartic**.
This curve is the pedal of the circle with respect to one of its points.

We know the following species:

- a=b, d=0 or l=m, o=0: limacon of Pascal
  - with $c^2=2a^3/27$, $e=-a^2/12$ or with $l=n^2$, b=0: cardioid
- b=1, c=d=e=0 or n=o=0: hippopede

p<>0
For p <> 0, the the following curves can be distinguished:

- a=b, d=0 or l=m, o=0: Cartesian oval
  Now the curve is the cyclic of a circle
- a=−b, c=d=0 or $l+m+n^2+o^2-p=0$: Cassinian oval
- a<>b, d=0 or l<>m, o=0: plane spiric curve

The bicircular quartic can also be written in a tripolar equation:

$f r_1 + g r_2 + h r_3 = 0$.

---

notes

1) As follows:
$a = -2(2l+n^2+o^2-p)$
$b = -2(2m+n^2+o^2-p)$
$c = 4ln$
$d = 4mo$
$e = (n^2+o^2-p)^2 - 4(ln^2+mo^2)$

---

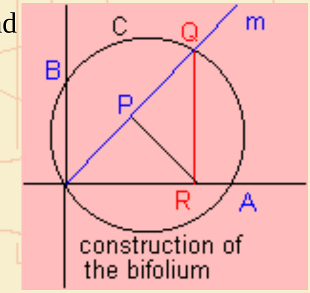
bifolium

quartic

last updated: 2002−08−11

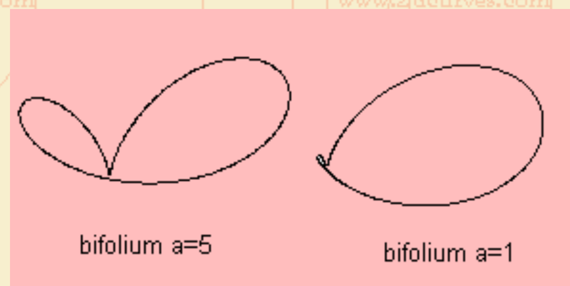
$$(x^2+y^2)^2=(x+ay)x^2$$

The **bifolium** [1] has been studied by *Longchamps* (1886) and *Brocard* (1887), and is also named the **double folium**.



The bifolium can be constructed as follows.
Given a circle C through O, A(0, a) and B(1, 0), and a line m through O that cuts C in Q. Let R be the projection of Q on the x−axis, and P the projection of Q on m.
Then the bifolium is the collection of points P, where m varies in the given circle C.

Special cases are:



- a = 0: a species of the simple folium
- a = ∝: the regular form of the curve: the regular bifolium, a folium

The bifolium for a and for −a are each other's reflection in the x−axis.

---

notes

1) In German: **Zweiblatt**.

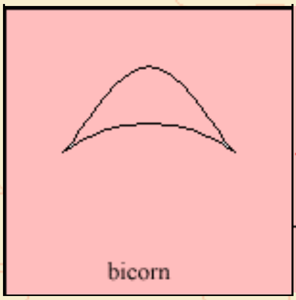
bicorn

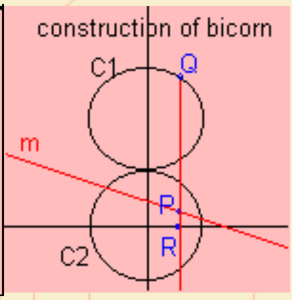
quartic

last updated: 2002−07−13

$$(x^2+2y-1)^2=y^2(1-x^2)$$

The curve has two (bi) horns (corn), that's where its name is from [1].
An alternative name is derived from the **cocked−hat** [2].





The bicorn can be constructed as follows.
Let there be two tangent circles C1 and C2 of equal size.
Let Q be a point on C1, and R its projection on the y−axis.
Let m be the polar inverse (a line) of C1 with respect to Q.
Then the bicorn is formed by the points P which are the intersection of m and QR.

Sylvester (1864) was the first to investigate the curve, followed by *Cayley* in 1867 and *Longchamps* in 1897.

---

notes

1) In German: **Zweihorn**.

2) In Dutch: **steek**.

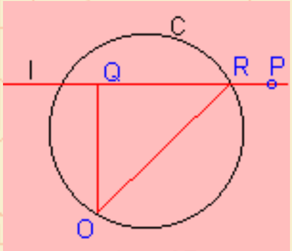
besace

quartic

last updated: 2003−04−18

$$(y+ax^2)^2 = x^2 - x^4$$

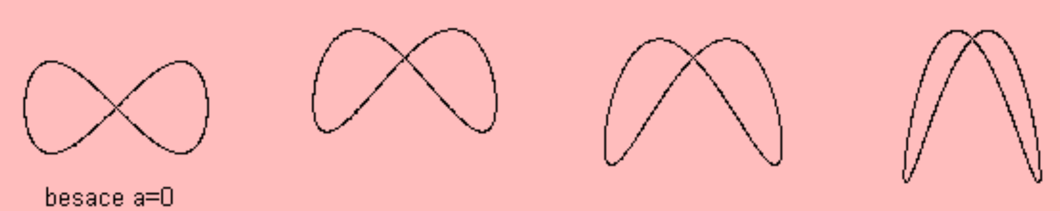
The **besace** is a Lissajous curve. In 1750 *Cramer* gave the curve its name [1].



construction of Besace

The curve can be constructed as follows: given a circle C and a point O draw horizontal lines l which cut the circle in R. Q is the projection of O on l. Then the Besace is the set of points P for which PQ = OR.

The curve is the result of the projection of the pancake curve on a plane through the central z−axis of the curve.


besace a=0 (lemniscate of Gerono) | besace a=1/2 | besace a=1 | besace a=2

For large a, the curve approximates the parabola.

For a = 0 the lemniscate of Gerono results.

$$y^2 = x^2 - x^4$$

The form of the curve is that of ribbon [2], as the other lemniscate, the more famous lemniscate of Bernoulli. Because of the resemblance with the digit 8, the curve is also called the ***(figure) eight curve***.
But I think the figure eight curve that is formed by the hippopede makes a better eight.

Its Cartesian equation resembles the equation of the kampyle of Eudoxus.

---

notes

1) In German: **Quersackkurve**.

2) lemniskos (Gr.) = ribbon

Cassini(an) oval

quartic

last updated: 2003−05−18

$$r_1 r_2 = a$$

Given two foci $F_1(-1,0)$ and $F_2(1,0)$, you can distinguish two polar coordinates, with respect to each of the foci. The curve for which the product of this two polar radii is a constant, is the **Cassinian oval**.
The curve can be generalized to the Cassinian curve.

The French astronomer *Giovanni Dominico Cassini* (1748−1845) found the curves in 1680, while attempting to describe the movement of the earth relative to the sun. He believed the orbit of the earth was a cassinoid, with the sun in one focus.
Malfatti studied the curve in 1781.

For the curve the product of distances to the two focal points is a constant. This definition resembles the definition of the ellipse, with a product instead of an addition. That's why the curve has also been given the name of the ***Cassini(an) ellipse***. The curve is also named a ***cassinoid***.

The name Cassini has been given to the pilotless spaceship that is right now on his way to the planet Saturn.

The curve is a bicircular quartic [1], and an anallagmatic curve
In polar coordinates the curve is written as:

$$r^2 = 2\cos 2\phi + \frac{a-1}{r^2}$$

The value of the variable named a determines the form of the oval: for a > 1, we see one curve, for a < 1 two egg−shaped forms.
For a < 4, the oval is squeezed in the middle, for a > 4, the curve goes towards a circle.

Cassinian oval a=.9 | Cassinian oval, a=1 (lemniscate) | Cassinian oval a=1.5 | Cassinian oval a=4

The curve is a spiric section, for which the distance from the plane that forms the curve to the axis of the torus is equal to its inner radius.

$$r^2 = \cos 2\phi$$

When the variable a is equal to one, the last term in the polar formula vanishes:

with $(0, \pm 1/\sqrt{2})$ as foci.

The resulting curve is the ***lemniscate*** or ***lemniscate of Bernoulli***. In 1694 *Jacob Bernoulli* (1654−1705) wrote an article in Acta Eruditorum about the curve, and he gave it the name of 'lemniscus' [2]. He didn't know that 14 years earlier Cassini had already described the more general case of the Cassini ovals.
More properties of the lemniscate were found by *Fagnano* in 1750. Research of Gauss and Euler on the length of the arc of the curve led to the ***elliptic functions***. In determining this arc the *Lemniscate constant* was established, a constant which has a value of 2 or 4 times the constant L. This constant L has a value of about 5.2, more exactly a form with a gamma function included: $\frac{1}{2}\Gamma^2(\frac{1}{4}) / (2\pi)^{\frac{1}{2}}$.

Because the curve is the inverse of an equilateral hyperbola, it is also called the ***hyperbolic lemniscate***. In fact, the curve is also the pedal of this hyperbola variant.

The lemniscate can be seen as a special case of the:

- hippopede
- sinusoidal spiral
- generalized cissoid
  The lemniscate is the cissoid of two equal circles, where the distance from the center to the center of the circle is √2 times the circle's radius.
- Watt's curve: the length of the rod and the distance between the circle's centers are equal, and the length of the rod is √2 times the radius of the circle)

And the curve is the inverse of the rectangular hyperbola.

Maybe the infinity sign ∞ has been derived from the lemniscate, as this curve is also going round infinitely. Lemniscate is also the name of the longest composition of Western music, made by Simeon ten Holt (30 hours).
A lemniscate is a nice logo, e.g. used by the Dutch publisher 'Lemniscaat' (well−known by his child books).

A three dimensional variation on the lemniscate is the *Möbius strip*.

---

notes

1) with Cartesian equation: $((x+1)^2+y^2)((x-1)^2+y^2)=a^2$ which leads to:

$(x^2+y^2)^2 + 2x^2 - 2y^2 = b$, with $b=a^2 -1$

2) Lemniskos (Gr.) = ribbon

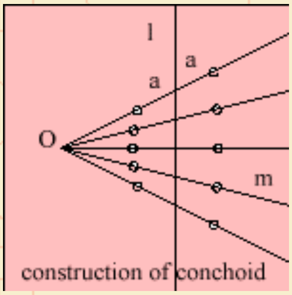
conchoid

quartic

last updated: 2003−05−05

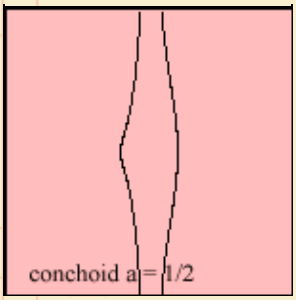
$$r = \frac{1}{\cos\phi} \pm a$$

Because the conchoid's resemblance to a shell [1] it is also called ***shell curve***. Another name for this curve is ***cochloid***.



construction of conchoid

Given a line l and a line bundle m through focus O, the curve is constructed by pacing at both sides of l a distance a on all lines m. Setting the distance from l to O to 1, above shown polar equation follows. It's easy to convert the polar equation to a Cartesian fourth order equation [2].
When the variable a is greater than 1, a curl becomes visible.

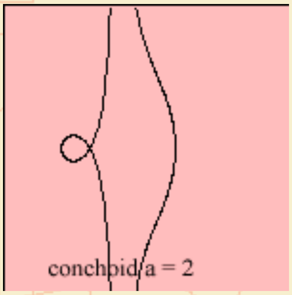

conchoid a = 1/2

The area between either branch and the asymptote is infinite.

The first who described the curve was the Greek *Nicomedes* (about 200 BC). He used the curve working at the problem of the trisection of an angle, and also when duplicating the cube. *Nicomedes* was also the one to describe a tool to construct the curve.
The conchoid was a favorite of the 17th century mathematicians.

In the science of mechanics one use a *conchoidal movement*. A surface makes a movement in such a way, that:
− a line of the moving surface makes his way through a fixed point
− a point at the surface keeps a tight right track
This is just the way the above described instrument operates, and every point on the moving surface has a path in the form of a conchoid.

Further, the conchoid has been used in the construction of buildings, as the shape of the vertical section of columns.


conchoid a = 2

The conchoid has been generalized by substituting the line l by a arbitrary curve C: the general conchoid. The simple conchoid of this chapter then, can be distinguished with the name ***conchoid of Nicomedes***. The conchoid can also be seen as a specimen of a generalized cissoid.

---

notes

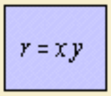
1) Concha (Lat.) = shell, mussel

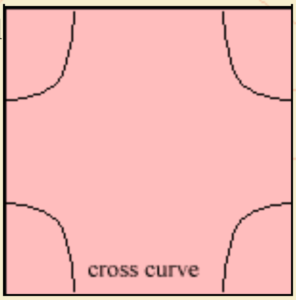
2) $(x^2 + y^2)(x - 1) = a^2 x^2$

cross curve

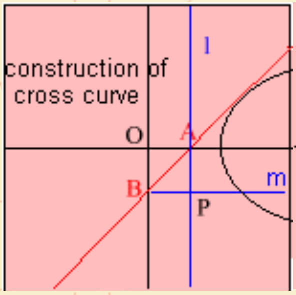
quartic

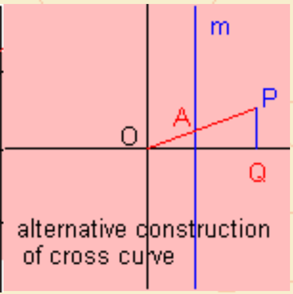
last updated: 2002−07−14





The cross curve can be constructed in the following way.
Given a ellipse H, construct for every tangent the intersection with both x−axis and y−axis, giving points A and B respectively. Then draw a line l through A parallel to the y−axis, and a line m through B parallel to the x−axis. Then the intersection of line l and line m lies on the cross curve.





This definition is a variation of the definition of the bullet nose. As is also to been seen at the curve's formula: $\frac{1}{x^2} + \frac{1}{y^2} = 1$

The curve can be constructed in an alternative way as follows.
Given a vertical line m. The cross curve is formed by the points P for which the distance to the x−axis is equal to the distance from the origin to the crossing of OP and m (PQ = OA in picture).

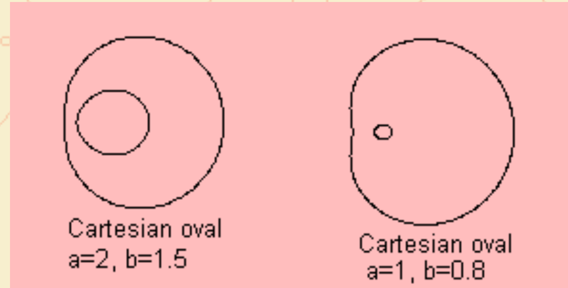
Other names for the curve are: **(equilateral) cruciform (curve)** and **stauroid**.

Cartesian oval

quartic

last updated: 2003−03−09

$a r_1 + b r_2 = 1$



with foci F_1 (−1, 0) and F_2 (1,0).
This bipolar equation defines the Cartesian oval as the collection of points for which the distances to F_1 and F_2 are related linearly.
The curve is also called the **oval of Descartes**, or the **Cartesian curve** [1].
One can imagine that

The curve is a quartic, in fact a bicircular quartic and a cyclic of a circle.
When working out the bipolar equation into the Cartesian form [2], a second oval appears. In fact the bipolar equation extends to $a r_1 \pm b r_2 = \pm 1$

When the inside and outside of an Cartesian oval have refraction indices n_1 and n_2, respectively, with $n_1/n_2 = b/a$, then the refracted rays sent from one focal point, seem to be issued from the other focal point. This led to the name of **aplanatic curve** [3].

Some special cases of the Cartesian oval are:

- a = b: ellipse
- a = −b: hyperbola
- a = 1: limaçon

It was *Descartes* (1637) who was the first to describe the curve; *Newton* studied the curve while classifying his cubic curves.

---

notes

1) in French: **cartésienne**.

2) Cartesian equation of the curve is, with $c = a^2 - b^2$ and $d = a^2 + b^2$:
$(c(x^2 + y^2 + 1) - 2dx))^2 = 2d(x^2 + y^2 + 1) - 4cx - 1$

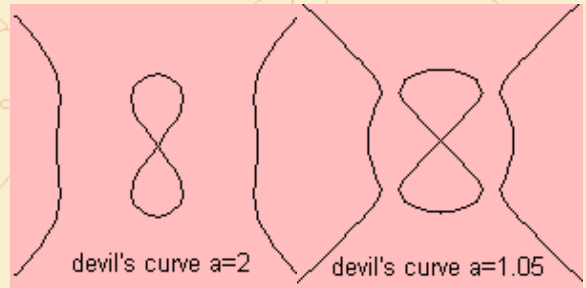
This is said to be equivalent with the following:
$(x^2 + y^2)^2 + k(x^2 + y^2) + lx + m = 0$

3) in French: **courbe aplanétique**

devil's curve quartic

last updated: 2002−03−25

$$y^4 - y^2 = x^4 - ax^2$$



This **devil's curve** is also known as the ***devil on two sticks***.

Gabriel Cramer (1704 − 1752) was the first to investigate the curve, in 1750. Cramer was a Swiss mathematician, he is best known for his work on determinants.
Lacroix studied the curve in 1810.

The *Nouvelles Annales* in 1858 wrote about the curve.

Many authors place an extra constant before the y^2 term, but this is only a linear distortion of the curve.

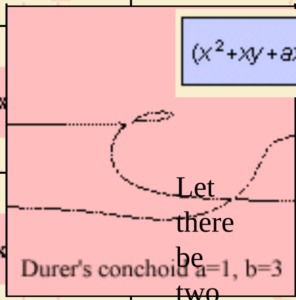
For a=25/24 the curve is called the **electric motor curve**.

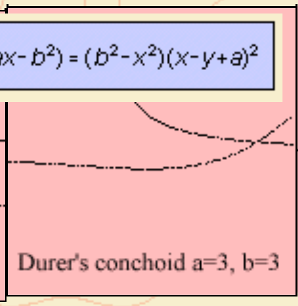
Dürer's conchoid

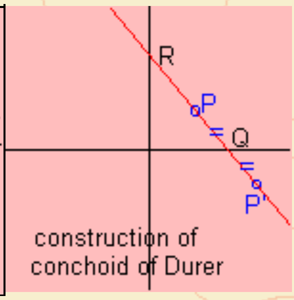
quartic

last updated: 2002−07−14

$$(x^2+xy+ax-b^2)=(b^2-x^2)(x-y+a)^2$$







Let there be two points Q(q,0) and R(0,r) where the sum of q and r is constant (a). Now construct points P on line l through RQ, so that PQ is equal to a constant (b). Then the locus of P defines Dürer's conchoid.
The definition is a variation on the (normal) conchoid.
Sometimes the curve is called **Dürer's shell curve**.

The curve appeared in a work of *Albrecht Dürer* (1471 − 1528), named 'Instruction in measurement with compasses and straight edge' (1525). He only found one of the two branches of the curve. Albrecht called the curve 'ein muschellini' [1]. Dürer looked at the situation that a=13, for points P and P' where PRQ and RQP' both have a length of 16 units.

It can be shown that the envelope of the line P'QRP is a parabola.

---

notes

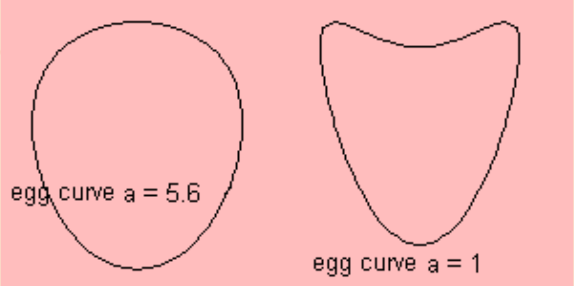
1) Muschellini = conchoid = shell.

egg curve

quartic

last updated: 2002−09−09

$$(y-x^2)^2 = 1 - ax^2$$



An often raised question is that about the oval curve that describes an egg.
I did not find a real good definition in terms of x and y, so I propose one myself.
Starting with an ellipse, with proper measures, make one end smaller than the other one.
This leads [1)] to a quartic function.
When taking linear transformation into account, parameters b (roundness) and c (difference between the ends) vanish into one parameter a (b/c) in above equation.

A good chicken egg has a value for parameter a between 5 and 6.

There is also a cubic egg curve that shows like an egg.

---

notes

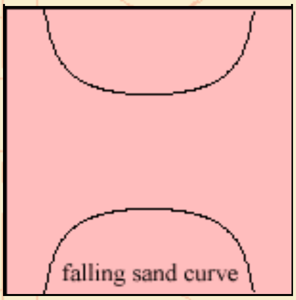
1) Via $y = \pm \sqrt{(1 - bx^2)} + c\,x^2$

falling sand curve

quartic

last updated: 2002-03-25

$$y^2 = \frac{1}{1-x^2}$$



Imagine a basket of sand, oscillating harmonically on a string. When there is a hole in the basket, so that the sand pours down on the ground, a dune of sand emerges. We admit that the amount of falling sand is constant with time, and we neglect the horizontal velocity of the sand [1]. Then the profile of the sand dune has a curve as shown in the drawn curve.
Until I find a more common name, I call the curve the ***falling sand curve***.

---

notes

1) It may be more proper to do the experiment with light instead, so that we can neglect the horizontal movement of the falling property.

folium [1] quartic

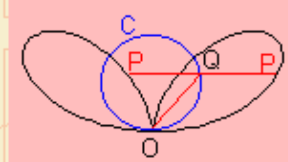
last updated: 2003−05−05

$r = (\sin^2\phi - a)\cos\phi$

The folium [2] has three forms:

- a >= 1: ***single folium*** or ***simple folium***
  Some authors confine the simple folium to a = 1.

- a = 0: **regular bifolium** (or **regular double folium)**
  The curve is sometimes called the **bifolium**, but I see the curve as a special case of this bifolium.
  Alternative names for the curve are: **right bifolium**, **right double folium** or **rabbit−ear** [3].
  The curve can be generalized to the generalized regular bifolium.

  construction of the regular bifolium

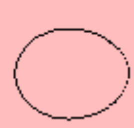
  The curve can be constructed with a given circle C through O as follows.
  Draw for each point Q on C points P, so that PQ = OQ. Then the collection of the points P forms the regular bifolium.

  The Cartesian equation of the curve can be written as $y = \pm x \pm \sqrt{x(1-x)}$, the regular bifolium can also be constructed as the mediane curve of a parabola and an ellipse.

- 0 < a < 1: ***trifolium***
  The special case that a=1/4 gives the ***regular trifolium***, which is in fact a rosette.

Each of the three folia is a (different) pedal of the deltoid.

It was *Johann Kepler* (1609) who was the first to describe the curve.


folium a=2


folium a=1/2


folium a=0

---

notes

1) In Cartesian coordinates: $(x^2 + y^2)^2 + a x^3 = (1-a) x y^2$

2) Folium (Lat.) = leaf.

3) In French: **oreilles de lapin**.

hippopede

quartic

last updated: 2002−12−26

$$r^2 = 1 - a\sin^2\phi$$

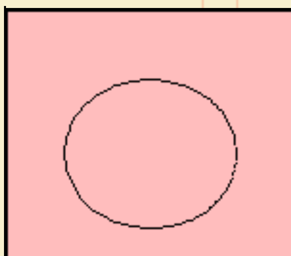
Hippopede means literally 'foot of a horse'; the curve is also known as the ***horse fetter***.
Because of the work done by of *J. Booth* (1810−1878), the curve is also named the **curve of Booth**.

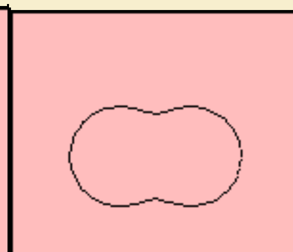
The curve has three forms:

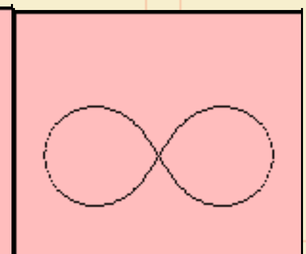
- a < 1/2: an oval
- 1/2 < a < 1: an indentated oval
- a > 1: a **(figure) eight curve**

For a < 1, the curve is the **oval of Booth**.
For a > 1, the curve is the **lemniscate of Booth**.
When the parameter a has the value 2, the curve is the lemniscate of Bernoulli.

Proclus (75 BC) and *Eudoxus* were the first to investigate the curve.
After Proclus the curve has been named **hippopede of Proclus**.


hippopede a=1/2


hippopede a=3/4


hippopede a=4/3

In Cartesian coordinates the curve has the form: $(x^2 + y^2)^2 = b\,x^2 + y^2$ with b = 1 − a.
And we see that the curve is a bicircular (rational) quartic.

The hippopede can be seen as:

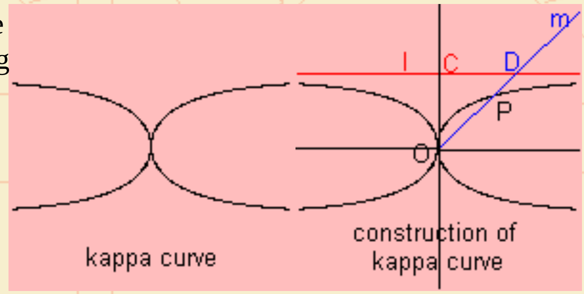
- the cissoid of two circles of equal size
- a Watt's curve for which the length of the rod and the distance between the circles are equal
- a special case of the spiric section for which the plane is tangent to the interior of the torus

kappa curve

quartic

last updated: 2003−03−02

$r = \tan\phi$



This curve resembling the Greek character kappa (k) (in fact its form is more a character 'x') can be defined as follows: let the line l be parallel to the x−axis, intersecting the y−axis in point C. Consider now lines m through O that meet line l in point D. Then the kappa curve is defined as the collection of points P, for which OP = CD.

Another name for the curve is ***Gutschoven's curve***, named after the first one who studied the curve, being *G. van Gutschoven* (in 1662).
The curve has also been studied by *Newton* and, later, by *Johann Bernoulli* and *de Sluze* (1862).

The curve is the radial of the tractrix.

In Cartesian coordinates the curve can be written as:

$(x^2 + y^2)\, y^2 = x^2$

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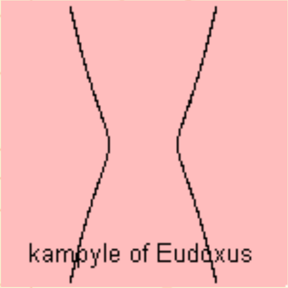
kampyle of Eudoxus

quartic

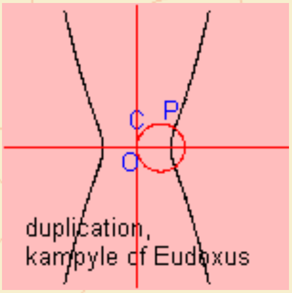
last updated: 2003−02−02

$$y^2 = x^4 - x^2$$



This quartic curve has been studied by *Eudoxus of Cnide* (406−355 BC), an astronomer, philosopher and mathematician who was a pupil of *Plato.*

Eudoxus worked on the curve in relation to the classical problem of the duplication of the cube.
Let there be a circle C through O with radius $\sqrt{1/2}$, which cuts the kampyle in P.
Then OP has as length $\sqrt[3]{2}$.



In polar coordinates the curve can be written as $r = \cos^{-2} \varphi$.

The kampyle [1)] is also named **Clairaut's curve**.

And the kampyle is:

- the radial of the catenary
- the inverse of the double egg

Its Cartesian equation resembles the equation of the lemniscate of Gerono.

---

notes

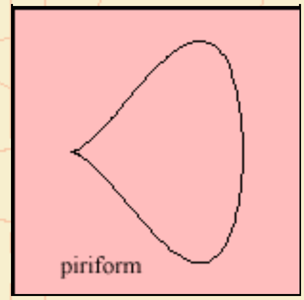
1) Kampyle (Gr.) = curved

piriform

quartic

last updated: 2002−05−19

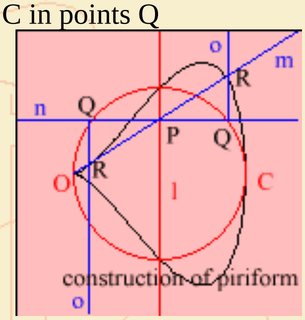
$$y^2 = x^3(1-x)$$



This curve has a form resembling a pear [1], why it is also called ***pear−shaped quartic*** or ***pear−shaped curve***.
The first to study the curve was the French mathematician *G. de Longchamps* (1886).

The curve is defined in a rather complex way, in the following steps:

- let there be a circle C with a point O on it
- let there be a line l, perpendicular to the diameter of the circle through O
- now, draw an arbitrary line m through O, crossing l in P
- draw a line n perpendicular to l through P, crossing C in points Q
- then, draw a line o parallel to l through Q



- now the curve is formed by the points R that are intersection of the lines o and the arbitrary line m

Could you follow me? If not, take a look at the picture.

In fact, the given formula is a (linear) generalization of the curve. The construction is only valid for the curve when the constant 1 is replaced by the 2.

The piriform has as parameter equation: $\begin{cases} x = 1 + \sin t \\ y = (1 + \sin t)\cos t \end{cases}$

The curve is also called the ***pegtop***, it is easy to see why.

In your throat there is a place where pieces of food can be stuck, e.g. a fish bone. This pit is called the piriform fossa [2].

The curve can be generalized to the teardrop curve.

---

notes

1. Pirum (Lat.) = pear

2. Fossa (Lat.) = moat, pit.

spiric section

quartic

last updated: 2003−03−15

$$(x^2+y^2+1-a^2+b^2)^2 = 4(x^2+b^2)$$


spiric section a=1/3, b=0.6


spiric section a=1/3, b=0.7

When the plane that intersects the torus is parallel to the axis of the torus, the plane spiric curve turns into the **spiric section** [1].
It was the Greek scholar *Perseus* (150 BC) who investigated the curve, what lead to the name of the **spiric of Perseus**.

Given a circle of radius 'a' and an axis of revolution in the same plane. The torus or spiric surface can be defined as the surface generated by rotating the circle along a circle of radius r. Then the spiric section is formed by the intersection of the spiric surface with a plane, which is parallel to the axis of revolution.
Let the value of b give the distance of the cutting plane from the center of the torus, and setting r to 1, above formula represents the curve.

For b = 0 the curve consists of two circles.

According to *Geminus, Perseus* wrote an epigram on his discovery of the curve: "Three curves upon five sections finding, Perseus made offering to the gods...". There have been given different explications of this sentence. *Bulmer−Thomas* states suggests that *Perseus* found five sections, but only three of them gave new curves.

When the plane is tangent to the interior of the torus, we get the hippopede.
When b is equal to the inner radius of the torus (c=1 [1]), we get the Cassinian oval.

---

notes

1) The Cartesian equation can also be written as:

$$(x^2+y^2) + 2x^2 - 2cy^2 = d$$

plane spiric curve

quartic

last updated: 2003−03−15

$$(x^2+y^2)^2+ax^2+by^2+cx+d=0$$

The conic section can be constructed as the intersection of a conic with a plane.
Replacing the conic by a torus, we get the **plane spiric curve** [1].
The curve is a bicircular quartic.

When the plane is parallel to the axis of the torus (a=2, c=0), a spiric section is formed.

---

notes

1) speira (Gr.) = torus

sextic

last updated: 2003−01−27

A sextic curve is an algebraic curve of the sixth degree.
Sextic curves are also to be seen as:

- astroid
- Lissajous curve
- nephroid
- polytrope
- quadrifolium
- teardrop curve

Other sixth order curves are:

- arcs of Samothrace
- atriphtothalassic curve
- butterfly curve
- Cayley's sextic
- Ceva's trisectrix
- dumbbell curve
- kiss curve
- Watt's curve

atriphtothalassic sextic curve

last updated: 2002−12−25

This **atriphtothalassic curve** has been studied by *Haughton* and *Longchamps*.

The only information I have found is that the curve is a circular sextic.

---

notes

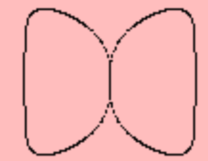
1) atriptos (Gr) = without friction, thalassa (Gr) = sea.

butterfly curve

sextic

last updated: 2002−03−25

$$y^6 = x^2 - x^6$$



butterfly curve

As you might expect, the curve has the form of a butterfly.
It has to be distinguished from the exponential butterfly curve.

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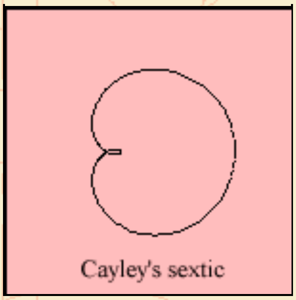
Cayley's sextic

sextic

last updated: 2002−03−25

$$r = \cos^3 \frac{\phi}{3}$$

As we know, the curve has been studied for the first time by *MacLaurin*. Later *Cayley* took a closer look to the curve. *Archibald* named the curve after Cayley in a paper classifying some curves, a paper that was published in 1900 in Strasbourg.



$$4(x^2+y^2-x)^3 = 27(x^2+y^2)^2$$

This sextic curve is also a sinusoidal spiral.

The Cartesian equation is as follows:

Some relationships with other curves:

- it is a roulette, formed by a cardioid rolling over another cardioid with the same size
- the curve is the pedal of he cardioid
- it is the involute of the nephroid

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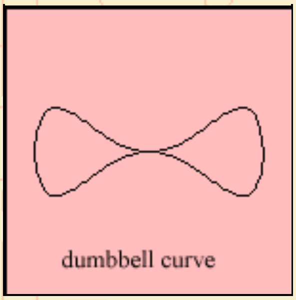
dumbbell curve

sextic

last updated: 2002−05−02

$$y^2 = x^4 - x^6$$



As a regular guest of our local fitness center, the curve reminds me to the dumb−bells I'm exercising with.

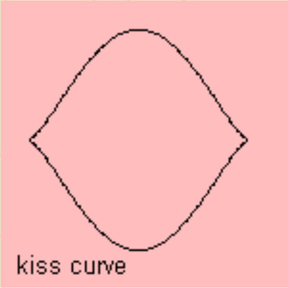
kiss curve

sextic

last updated: 2003−01−01

$$y^2 = (1 - x^2)^3$$

I found the **kiss curve** at *Robert Ferreol*'s site.

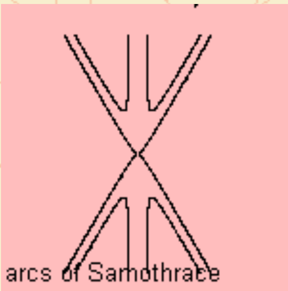


arcs of Samothrace

sextic

last updated: 2002−12−10

$$r = \frac{\tan\phi}{1 + 2\cos 2\phi}$$



It's a rather rare curve to find in math.
Its form is a bit alike a hyperbola, but it is not one.

And it is a sextic curve, with Cartesian equation:

$$x^2 (3x^2 - y^2)^2 = y^2 (x^2 + y^2)$$

The curve can be extended to the generalized arcs of Samothrace.

(extended) Ceva's trisectrix

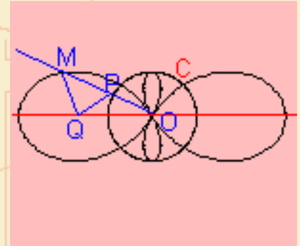
sextic

last updated: 2003−01−28

$r = 1 + b \sin 2\phi$

This sextic [1] is a botanic curve.

Giovanni C (1648−1734), an Italian mathematician and engineer, studied the curve for b=2 [2].



construction of trisection

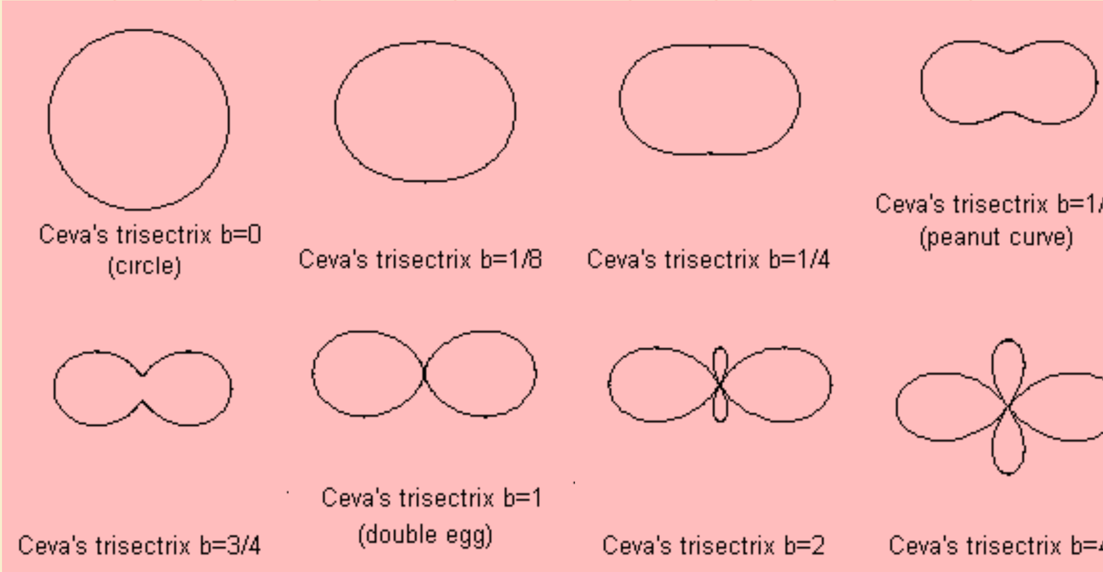
The curve can be used for the trisection of an angle, as follows (see picture to the right).
Let there be a circle C with center O. Draw a line through O which cuts C in P.
Construct a point Q on the x−axis so that OP = PQ.
Then the Ceva's trisectrix is the collection of points M for which:

- M lies on the line through OP
- MP = PQ

Now the angle OQM is triple the angle QOM.

For b = 1/2, the curve is called the **peanut curve**.
For b = 1, the curve is called the **double egg**.

Some examples:



For large values of parameter b, the curve approximates the quadrifolium (rhodonea c=2).

notes

1) Cartesian equation: $(x^2 + y^2)^3 = ((b+1)x^2 - (b-1)y^2)^2$

2) This curve can also be written as: $r = \sin 3\phi / \sin\phi$

octic curve

last updated: 2002−12−27

An octic curve is an algebraic curve of the eight degree.

They can be seen at the:

- Lissajous curve
- polytrope
- teardrop curve

Specific eight degree curves are the following:

- illumination curve
- pear curve

The octic curve is also called the **biquartic**.

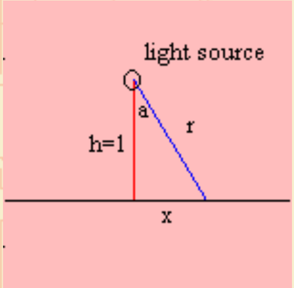
illumination curve

octic

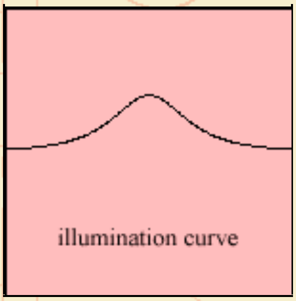
last updated: 2002−04−27

$$y^2 = \frac{1}{(x^2+1)^3}$$

A lamppost will give the most light directly under the lamp. The illumination will decrease when you move away from the light source. This relation is shown by a curve, which I give the name ***illumination curve***.



When the lamp gives light homogenous in all directions, the light stream for each unit of area is inversely proportional with r^2. For a sloping bundle of light the light stream is proportional with the cosine of the angle α.
Now setting the light stream at the source to 1, as well as the height from the lamp to the ground, gives the equation above.



higher degree curve

last updated: 2003−05−04

Algebraic curves of degree higher than eight can be found at the following curves:

- Cassinian curve
- circle tangent
- epicycloid
- generalized arcs of Samothrace
- hypocycloid
- Lennard−Jones potential
- Lissajous curve
- Nickalls−Pulfrich curve
- Plateau curve
- polynomial (in x)
- polytrope
- Saint−Hilaire skating rink curve
- Talbot's curve
- teardrop curve

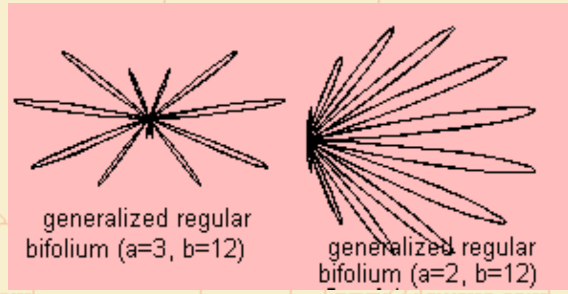
generalized regular bifolium

higher

last updated: 2002−12−04

$r = \sin^a b\phi \cos\phi$

The curve is a generalization of the regular bifolium (a = 2, b = 1).

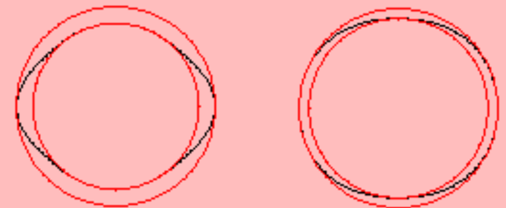


circle tangent

last updated: 2003−04−19

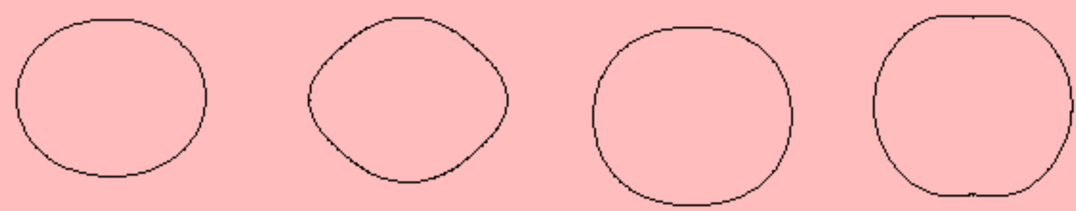
$$r = 1 + a \sin^b \phi$$

Given two concentric circles, with radius r_0 and r_1 respectively, a curve tangent to both of them can be imagined. I named the curve the **circle tangent**.
The constant a is equal to $r_1/r_0 - 1$.



circle tangent a=1/5, b=8 (construction) circle tangent a=-1/10, b=4 (construction)

The degree of the corresponding Cartesian equation is equal to 2b + 2.
For large b, the curve has a long tangent to one of the circles, the inner for positive values for parameter a, the outer for negative values for the parameter a.



circle tangent a=1/5, b=2 circle tangent a=1/5, b=8 circle tangent =-1/10, b=4 circle tangent a=-1/ b=16

It is possible that the Saint−Hilaire skating rink curve belongs to this family.

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Cassinian curve

higher

last updated: 2003−05−04

$$\prod_{i=1}^{n} r_i = a$$

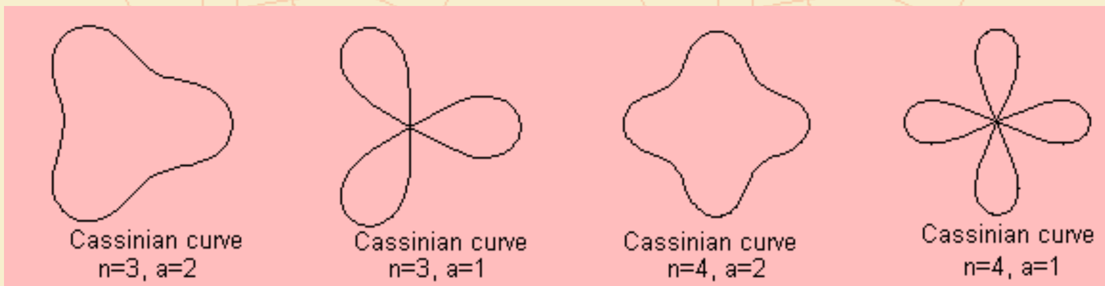
It is the curve for which the product of multiple polar radii is constant.
The curve is an algebraic curve of degree 2n.

In the case of a regular polygon the curve can be written in polar coordinates as

$$r^n = 2\cos n\phi + \frac{a-1}{r^n}$$ 1)

For a < 1, the curve consists of n pieces.
For a > 1, the curve is just one closed curve.
For a = 1, the curve is a sinusoidal spiral.

For n = 2 we see the Cassinian oval.



Think about a magnetic field created by n parallel threads with the same current. The magnetic field lines in a plane orthogonal to the threads are Cassinian curves for which the foci are the intersections of the threads with the plane.

The curve has been studied by *Serret* (in 1843), and the curve is named after *Cassini*.

---

notes

1) In other words:

$$r = \sqrt[n]{\cos n\phi \pm \sqrt{a - \sin^2 n\phi}}$$

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Lissajous curve

higher

last updated: 2002−12−29

$$\begin{cases} x = \sin t \\ y = \sin(at+b) \end{cases}$$

where $0 \le b \le \pi/2$.

When the constant a is rational, the curve is algebraic and closed.
If a is irrational, the curve fills the area [−1,1] x [−1,1].
It is easy to see that the curves for 1/a and a are equal in form. This means that we can confine ourselves to the case a ≥1.

Jules−Antoine Lissajous (1822−1880) discovered these elegant curves (in 1857) while doing his sound experiments.
But it is said that the American *Nathaniel Bowditch* (1773−1838) found the curves already in 1815. After him the curve bears the name of ***Bowditch curve***.
Another name that I found is the **play curve of Alice**.

The curves are constructed as a combination of two perpendicular harmonic oscillations. Patterns occur as a result of differences in frequency ratio (a) and phase (b).
At high school we used the oscilloscope to make the curve visible (nowadays a computer would do), by connecting different harmonic signals to the x− and y−axis entrance.

The curves have applications in physics, astronomy and other sciences.

Each Lissajous curve can be described with an algebraic equation.

Write a as the smallest integers m, n for which a = m/n.
Then the degree of the equation obeys the following rules:

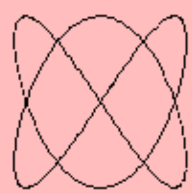
- degree = m
  for b = 0 and m is odd
  for b = π/2 and m is even
- degree = 2m
  for b ∈]0, π/2] and m is odd
  for b ∈ [0, π/2[and m is even

Some examples of Lissajous curves are the following:

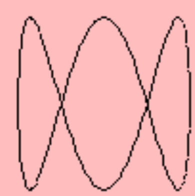
a	b	name of the curve
1	0	straight line

1	$\pi/2$	circle
1	<> 0, $\pi/2$	ellipse
2	$\pi/2$	parabola
2	<> $\pi/2$	besace: its parameter a is equal to the tangent of the Lissajous curve's parameter b.
3	0	the cubic $y = 2x^3 - x$ (see the picture below)
3	$\pi/2$	the sextic $y^2 = (1 - x^2)(1-4x^2)$ (see the picture below)


Lissajous curve a=3/2,b=1/3


Lissajous curve a=3/2, b=0


Lissajous curve a=3, b=0


Lissajous curve a=3, b=pi/2

In the situation that the parameter a is an integer, three forms of the curve can be distinguished:

1. the curve is **symmetric** with respect to the x− and y−axis
   When a is even: b = 0. When a is odd: b = $\pi/2$.
2. the curve is **asymmetric**
   For this form (and the one before, as a matter of fact) the number of compartments is equal to the value of a.
3. the curve has **no compartments**, the two parts of the curve come together
   When a is even: b = $\pi/2$. When a is odd: b = 0.

There is a relation with the Tchebyscheff polynomial: the Lissajous curve (a, b) corresponds with Tchebyscheff function T_a for a integer and b=$\pi/2$ (for even a) or b=0 (for odd a).

The Lissajous curve can be extended to the generalized Lissajous curve.

---

notes

1) In French: **joue courbe d'Alice**.

Mandelbrot lemniscate

last updated: 2002−03−25

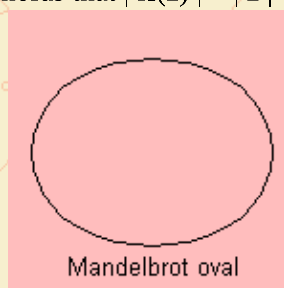
$$|f_n(x+iy)| = 1$$

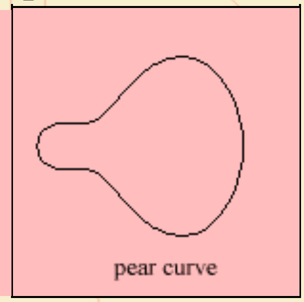
where f_n is defined by the quadratic recurrence relation $f_{n+1}(z) = f_n^2(z) + z$ with initial $f_0(z) = 0$.

These curves are called ***equipotential curve**s* by *Peitgen* and *Saupe* (1988).
In general, the curve is a polynomial in x and y of degree 2^n.

The first three Mandelbrot lemniscates are:

- n = 1: circle
  because for the unity circle holds that $| f_1(z) | = | z | = 1$
- n = 2: **Mandelbrot oval**
  Some authors tell us that this quartic oval curve is a Cassinian oval.
  But I am not sure about this, the relation $| z^2 + z | = 1$ leads to a quartic Cartesian equation [1] that does not seem equivalent with the Cassinian oval's equation. That's why I call the curve the Mandelbrot oval, for the time being.





- n = 3: ***pear curve***
  Although the resembles more a bulb, the curve has been given the name of the pear curve.

When n goes to infinity, the curve is the Mandelbrot set.

---

notes

1) Oval: $(x^2 - y^2 + x)^2 + y^2 (2x+1)^2 = 1$.

Nickalls–Pulfrich higher curve

last updated: 2002–03–25

$$\begin{cases} x = (1/2u)(2b\cos^2(t-a) + 2c(b\cos a - c\sin a)\cos(t-a) + (b\cos 2a - c\sin 2a - b)) \\ y = (1/u)(b\sin(t-a)\cos(t-a) + (bc\cos a + b^2 \sin a)\sin(t-a)) \\ u = (b\cos a + c\sin a)\cos(t-a) + \sin a\cos a + bc \end{cases}$$

The Pulfrich effect is a remarkable visual illusion, seen when a moving object is viewed binocularly with a neutral density filter in front of one eye. It is generally accepted that this phenomenon is due to a unilateral increase in visual latency, resulting from the decrease in retinal image intensity due to the filter.

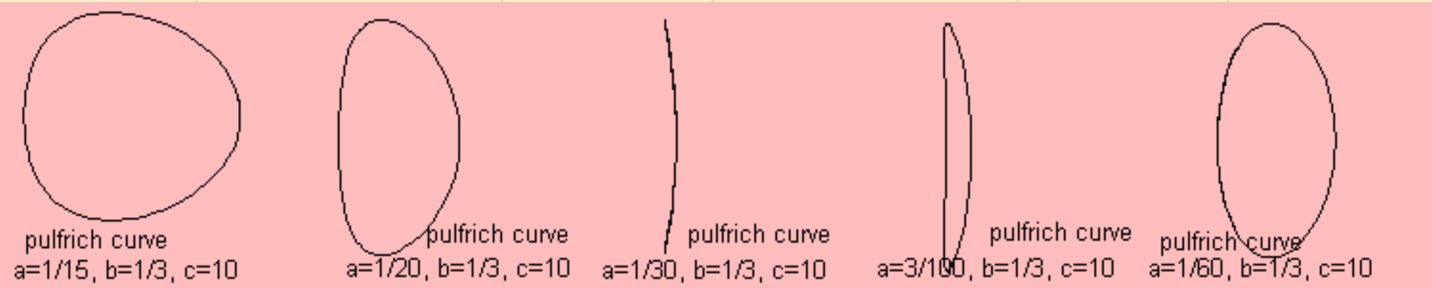
Now imagine that a vertical rod, moving in a horizontal circle on a turntable, is viewed horizontally from the side with a filter in front of one eye. It was *Pulfrich* (1922) who described some of the illusions that are to be seen while the turntable is slowing down. This slowing down makes studying of the phenomenon very difficult, that's why *Dick Nickalls* chose to observe a situation with constant angular velocity and a view from various distances.
He found the theoretical apparent curves of the rod, and verified them experimentally [1)], as function of:

- the latency as result of the filter
- the distance from the rod to the center of the turntable
- the distance between the two eyes
- the distance form the observer to the center of the turntable

These parameters are related as follows to the constants in the given formula:

- 2a is the latency angle (latency difference in seconds * angular velocity)
- 2b is the distance between the eyes / distance from the rod to the center of the turntable
- c is the distance between the observer and the center of the turntable / distance from the rod to the center of the turntable

When varying the latency, and keeping the distances the same, the rod appears to have the same (nearby) or the opposite (far away) rotation direction of the turntable. In the transition situation the curve is an arc of a circle. Beneath five situations are given with latency angle growing from right to left.



notes

[1] RWD Nickalls: the rotating Pulfrich effect, and a new method of determining visual latency differences, Vision Research, Vol 26, pp 367−372, 1986.
While working on the Pulfrich experiments a generalized conic theorem was found, to be read in the Mathematical Gazette (2000), vol. 84 (July), pp 232−241.

polytrope

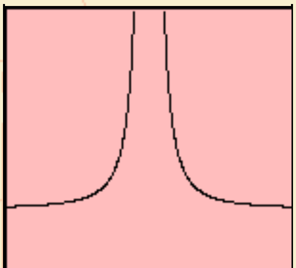
higher

last updated: 2002−03−25

$y x^n = 1$

A polytrope [1] has two asymptotes, and is either symmetric round the y−axis (even n), either symmetric round zero (odd n).
Some special cases:

- n = 1 delivers a conic section: the hyperbola
- n = 2 delivers a cubic: the ***cubic hyperbola***


cubic hyperbola

Polytropic equations are used by nature for a variety of forces: the force is inversely proportional to an integer power of the distance from the source. Each force is related to a so−called potential, described by a polytrope with a degree of one more [2].

Best known cases are *gravitation* and the *electrostatic force*. The force is inversely proportional to the square distance. This relation reminds us of phenomena as the decreasing of light and sound with distance from the source. The propagating oscillation fills a whole area's sphere, and that area is proportional to the square of the radius.

There are other also polytropic forces that fall down faster with the distance. Those are forces that work on an atomic scale.

When we consider an ideal (inert) gas, we see an attraction between the atoms, as result of the dipole nature of the atoms. It can be derived that this so called van der Waals interaction [3] has a potential that is inversely proportional to the sixth power of the distance.

Equilibrium in this situation is settled by another, yet a repulsive power. This power works as result of the overlap of electron tracks. Experimental has been settled that this potential is inversely proportional to the twelfth power of the distance.

The softer sciences encounter polytropes too. There has been an investigation how the year's numbers in a newspaper are distributed [4]. It occurred that there was a difference between events before and after 1910. Before this year (about the length of a man's life ago) the events extinguish inversely to the square of the passed time, after that year inversely to the passed time. What do you think, is this science, or is this an invention of a relation that is not there?

---

notes

1) Poly = strong, tropè = turn.

2) As we know for the relation between potential and force:

$$\bar{F} = -\nabla V(r) = V'(r)\frac{\bar{r}}{r}$$

3) Consider the atoms as harmonic vibrating dipoles, Kittel 1976 p.78

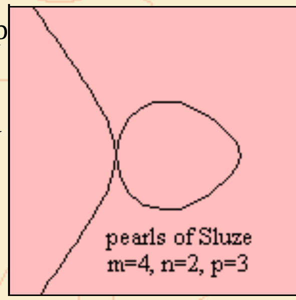
4) Pollmann 1996

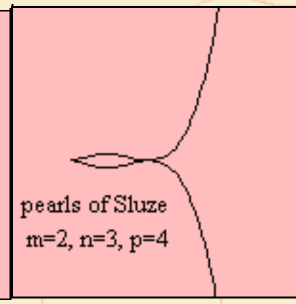
pearls of Sluze

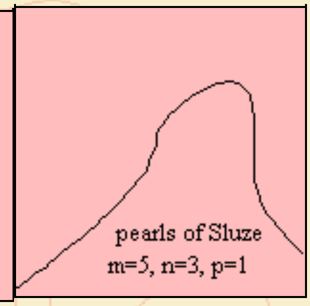
last updated: 2002−03−25

$$y^m = x^n(1-x)^p$$

The curve is a generalization of the teardrop curve: set in above equation: m = 2 and p = 1 and you get the tears...







This group of curves was studied by *de Sluze* between 1657 and 1698. It was *Blaise Pascal* who named the curve to *de Sluze*.

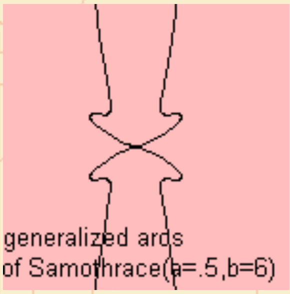
generalized arcs of Samothrace

higher

last updated: 2002−12−10

$$r = \frac{\tan\phi}{1 + a\cos b\phi}$$

The curve is a generalization of the arcs of Samothrace (a=2, b=2).

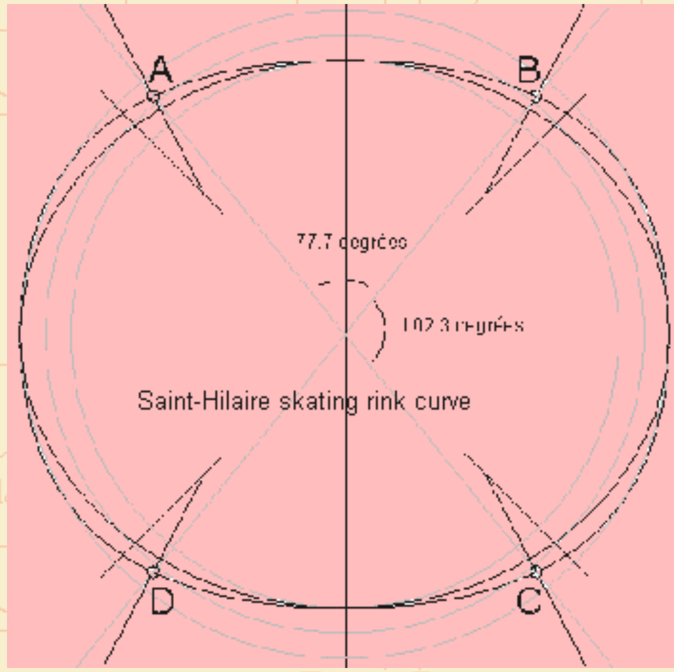


Saint−Hilaire skating rink curve

higher

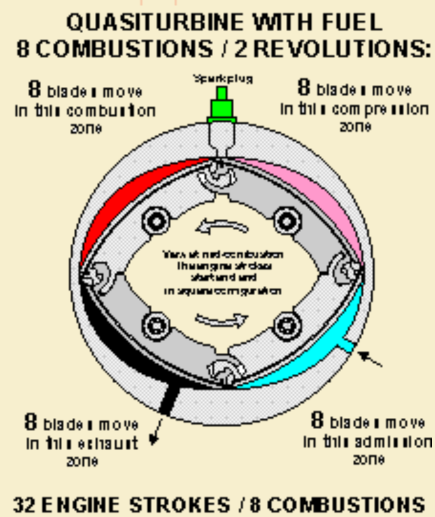
last updated: 2003−04−19

The curve is an oval curve fitting between two circles of different radius.
The curve is tangent to the two concentric circles of different radius. Coming from the smaller circle, the curve rapidly reaches the maximum diameter, it follows the larger circle for about half its length. The curve is in fact an oval curve fitting between two circles of given radius
It seems that there is no formula available for the curve. A computer program is needed to calculate its form.



The physicist *Saint−Hilaire* was the first to calculate the profile of this curve.

An application of the profile has been found in engine design. The Quasiturbine (or Qurbine) is a rotating engine with a rotor, trapped inside an internal housing contour, which does not require a central shaft or support. This concept brings the engine dead time to zero.
The method of operation allows the engine to be extremely flexible. The developer claims that it is possible to use it for pneumatics, hydraulics, steam, petrol, natural gas and hydrogen.



It is possible that the curve is a subspecies of the circle tangent.

teardrop curve

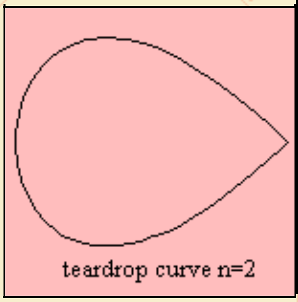
higher

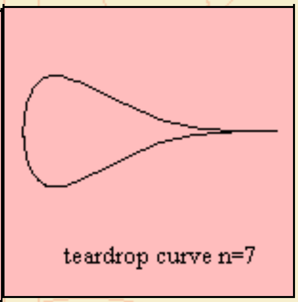
last updated: 2002−03−25

$$y^2 = x^n(1-x)$$

Tears, tears...

The curve is a generalization of the quartic piriform: this is the curve that results when the parameter n has the value 3.





For n =1 a circle results, for n = 2 a cubic, and so on.

Sometimes the teardrop curve is given in a parametric equation as:

$$\begin{cases} x = \cos t \\ y = \sin t \sin^{n-1}\left(\frac{t}{2}\right) \end{cases}$$

A further generalization of the curve is given by the pearls of Sluze.

algebraic curve

last updated: 2003−05−03

A curve is ***algebraic*** when its defining Cartesian equation is *algebraic*, that is a polynomial in x and y. The *order* or *degree* of the curve is the maximum degree of each of its terms $x^i y^j$.
An algebraic curve with degree greater than 2 is called a **higher plane curve**.

An algebraic curve is called a **circular algebraic curve**, when the points (±1, ± i) are on the curve. In that case the highest degree of the Cartesian equation is divisible by $(x^2 + y^2)$. The circle is the only circular conic section.
The curve got its name from the fact that it contains the two imaginary circular points: replace x by x/w and y by y/w, and let the variable w go to zero, we obtain the circular poins.

A **bicircular algebraic curve** passes twice through the points (±1, ± i). In this case the highest degree of the Cartesian equation is divisible by $(x^2 + y^2)^2$.

Every algebraic polynomial is a **Bézier curve**. Given a set of points P_i the Bézier curve of this **control polygon** is the convex envelope of these points.

When a curve is not algebraic, we call the curve (and its function) ***transcendental***.

In the case a function is sufficiently sophisticated it is said to be a ***special function***.

discrete curve

last updated: 2001−12−27

A discrete function has values only for the natural numbers (n). Giving the interval [n − 1/2, n + 1/2] the value of f(n), we get a bar formed curve, a **histogram**.
When the discrete function gives a percentage, and the function values have been ordered from high to low, the curve is called a **Pareto diagram**.

We shall see that some discrete curves for large n can be approached by (often exponential) continuous functions.

The following discrete curves can be distinguished:

- factorial
- **prime** related function
  - Euler's indicatrix
  - inverse prime summation
  - Möbius function
  - pi function
  - prime function
- (discrete) probability distribution
  The function gives the probability that a value n occurs. Simple functions are the ***univalent*** distribution (just one value) and the ***alternative*** distribution (two different values).
  Others **distributions** are:
  - binomial distribution
  - Fermi−Dirac distribution
  - hypergeometric distribution
  - Poisson distribution

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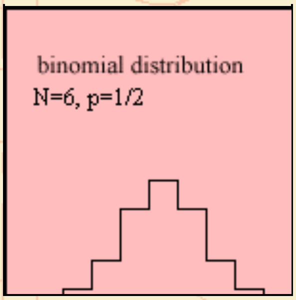
binomial distribution

discrete curve

last updated: 2001−12−29

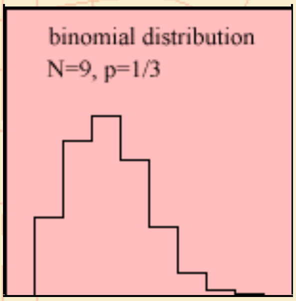
$$y = \binom{N}{n} p^{n} (1-p)^{N-n}$$



where $\binom{N}{n}$ is called the binomial coefficient ' N over n',

what stands for the number of combinations when taking n objects out a collection of N objects.
The formula is an abbreviation of:

$$\frac{N!}{n!(N-n)!}$$



Suppose that an event has a probability p to occur.
Examples of such events are 'heads or tails', and the chance on a certain number on a dice. The binomial distribution gives the probability of n successes for a number of N events.

When p=1/2, we find the ***binomial***:

$$y = \binom{N}{n}$$

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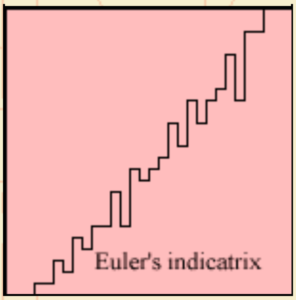
Euler's indicatrix

discrete curve

last updated: 2001-12-27

$$y = \phi(n)$$

This function φ, also known as ***Euler's function***, is defined as the number of natural numbers less than n which have no common divisor with n.



Examples: φ(6) = 2, φ(p) = p − 2, where p is a prime.

factorial

discrete curve

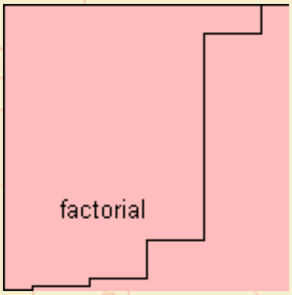
last updated: 2002-01-14

$y = n!$

The exclamation mark is a symbol for the multiplication

$$n! = \prod_{k=1}^{n} k$$

The function gives the number of possible arrangement of n objects.



The Stirling curve gives a continuous approximation of the function.

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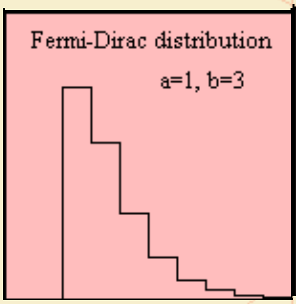
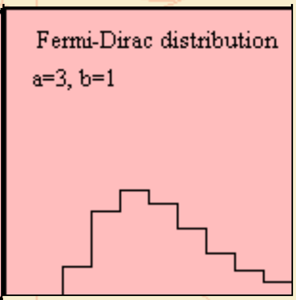
Fermi−Dirac discrete distribution

last updated: 2001−11−05

$$y = \frac{n^a}{e^{n-b} + 1}$$

The ***Fermi−Dirac distribution*** is used in statistical mechanics.





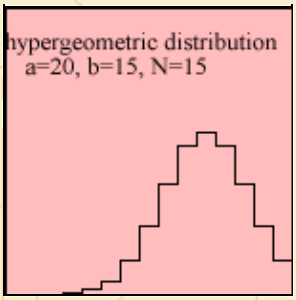
The sum for n = 0 into the infinite is a combination of a gamma function and a Lerch transcendent: $e^b\,\Gamma(a+1)\,\Phi(-e^b, a+1, 1)$.

hypergeometric distribution

last updated: 2001−11−05

$$y = \frac{\binom{a}{n}\binom{b}{N-a}}{\binom{a+b}{N}}$$

Given the situation of a basket filled with balls, which are white either black. There are number of a white balls and b black ones. Now take a random selection of N balls. The probability that n white balls have been taken, is given by above formula.



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inverse prime summation

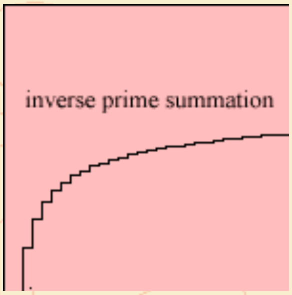
discrete curve

last updated: 2002−01−14

$$y = \sum_{p<n} \frac{1}{p}$$

where p is a prime.

This curve is approximated by the logarithmic logarithm.



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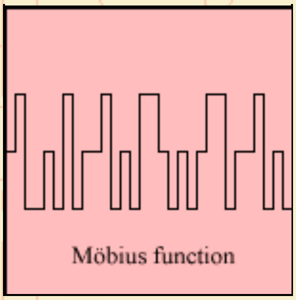
Möbius function

discrete curve

last updated: 2001−12−27

$$y = \mu(n)$$


Möbius function

where the Möbius function μ is defined as follows:
let a natural number n to be written as a multiplication of primes:

$$n = \prod_{i=1}^{r} p_i^{k_i}$$

Then $\mu(n) = (-1)^r$ where $k_1 = \ldots = k_r = 1$; $\mu(1) = 1$; $\mu(n) = 0$ otherwise.

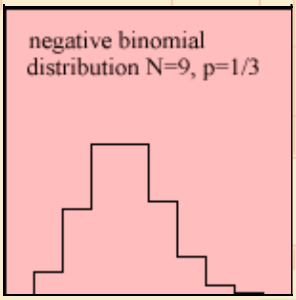
It follows that $\mu = -1$ for each prime. This extraordinary function is used in the Prime Number Theorem.

negative binomial distribution

discrete curve

last updated: 2001−12−29

$$y = \binom{N-1}{n-1} p^n (1-p)^{N-n}$$



The **negative binomial distribution** is a variation on the binomial distribution. Events with probability p are being continued until the N−th success. The distribution gives the probability that n experiments are needed to reach N successes.

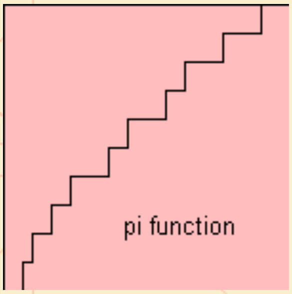
In formula, the negative distribution is related with a factor n/N to the normal binomial distribution.

prime counting function

discrete curve

last updated: 2003−05−31

$y = \pi(n)$



The **prime counting function** π(n) gives the number of primes equal or less than n. According the Prime Number Theorem [1], for large n the function approximates the prime counting function approximation.

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notes

1) Ash 1971 p. 196

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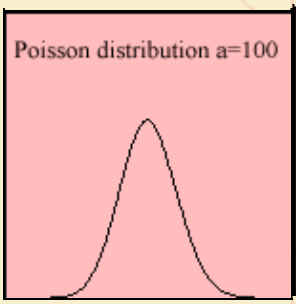
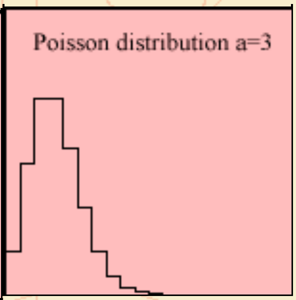
Poisson distribution

discrete curve

last updated: 2001−12−27

$$y = \frac{a^n}{n!}$$





The Poisson distribution (Siméon Denis *Poisson*, 1781−1840) can be obtained from the binomial distribution: let the number of events N go to infinity, where the most probable number of successes remains constant [1)].
The function is used to be normalized with a factor e^{-a}.
It is not symmetric, as the Gaussian distribution.

We can use the Poisson distribution in the case of a discrete probability process, with a constant frequency. Examples can be found in nuclear physics while counting elementary particles.

---

notes

1) product of n and p, denoted as: a.

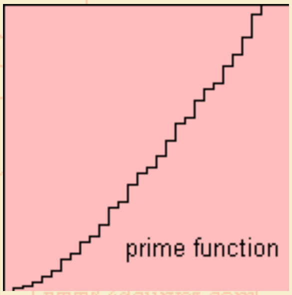
prime function

discrete curve

last updated: 2003−05−31

$y = p_n$

where p_n is the nth prime.
A prime is a natural number > 0 that is only divisible by itself and by the number one.



For large n the prime function is approached by the prime function approximation.

exponential curve

last updated: 2002−07−03

In this chapter all curves that have to do with exponential functions:

- **exponential**
  - exponential exponential
  - Gauss curve
  - generalized logarithm
  - linear exponential
  - pursuit curve
  - reciprocal exponential
  - Weibull curve
- gudermannian
- **Hoerl curve**
  - vapor pressure curve
- ***hyperbolic function***:
  - hyperbolic
    - − cosecant
    - − cosine
    - − cotangent
    - − secant
    - − sine
    - − tangent
  - Planck distribution
  - Planck radiation curve
  - tractrix
- **integral**
  - error function
  - exponential − (1)
  - exponential − (2)
  - hyperbolic sine −
  - logarithmic −
- **oscillation**
  - damped −
    - forced −
- pi function approximation
- **sigmoidal curves**
  - logistic growth
  - Richards curve

alysoid

exponentional

last updated: 2002−03−28

$$\begin{cases} x = \int_0^t \frac{\cos u}{\cos^2 au} du \\ y = \int_0^t \frac{\sin u}{\cos^2 au} du \end{cases}$$

with $-\pi/2a < t < \pi/2a$

The **alysoid** [1)] is a generalization of the catenary (a=1).
However, sometimes the alysoid is used as an alternative name for this catenary.

The alysoid can be defined as the curve for which the center of curvature describes the path of a parabola, when rolling over a straight line (with the parabola perpendicular to the straight line).

It was *Césaro* who started to study the curves, in 1886.

The intrinsic Whewell equation of the alysoid has a rather simple form: $s = \tan a\phi$, s being the arc length.

---

notes

1) Alusion (Gr.) = little chain.

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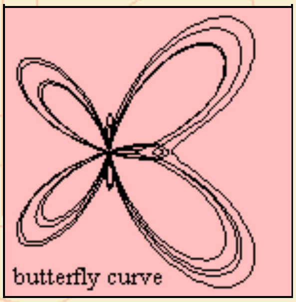
butterfly curve

exponential

last updated: 2002−03−25

$$r = e^{\cos\phi} - 2\cos 4\phi + \sin^5 \frac{\phi}{12}$$

T. Fay published in 1989 in American Mathematics Monthly a rather complex curve, which was named to the butterfly. It is not the same curve as the sextic butterfly curve.



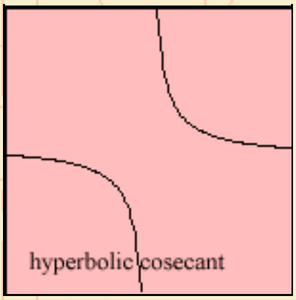
hyperbolic cosecant

exponential

last updated: 2001−11−05

$y = 1/\sinh x$

The hyperbolic cosecant csch(x) is the divider of the hyperbolic sine, analogous to the trigonometric functions.



Its inverse is the ***arc hyperbolic cosecant*** arccsch(x).

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damped harmonic oscillation

exponential

last updated: 2001−12−23

Suppose a harmonic oscillation that is damped by a force, which is magnitude proportional to the velocity [1]. This is called a free damped harmonic oscillation.
There are three possible situations, which are treated on this page.
The following curves give the amplitude as function of time.

strong damping

$$y = e^{-ax} \sinh x$$

The function is the product of an exponential function and a hyperbolic sine.

weak damping

$$y = e^{-ax} \sin x$$

The function is the product of an exponential function and a sine.

semi weak damping

$$y = x e^{-x}$$

The function is the product of an exponential function and a line.

notes

1) Differential equation: y'' + a y' + b y = 0 with y(0) = 0.

Kuipers 1966 p.345.

exponential function

exponential

last updated: 2003−05−18

$y = e^x$

The variable x in the equation is called the 'exponent', the function is called the ***exponential function***. The oldest exponential series is: 1, 2, 4, 8 and so on. There is an old legend about a scholar who could choose his own reward for some job. He asked to place grains of corn on a chess board, starting with one, on each field the double amount of the former...
With the Euler number e[1] as base of the power, the function is the ***natural exponential function*** exp(x), sometimes abbreviated as exponential function. For other values of the base of the power the function is a ***general exponential function***.

The inverse of the exponential function is the ***logarithmic function*** or ***logarithm***. This ***general logarithm*** is written as y = $^{a}\log(x)$ or y = $\log_a(x)$.
For e as the base of the logarithm the curve is called the ***natural logarithm*** ln(x) [2].
Her derivative is the hyperbola y=1/x. The natural logarithm is sometimes called the ***Napierian logarithm***, but these two are slightly different [3]. *John Napier* (1550−1617), a Scottish baron from Merchiston, was the first to relate the geometric series and the arithmetic series. For this purpose he uses the logarithm function.
For 10 as the base of the logarithm we've got the ***common logarithm*** log(x), also called ***Briggs' logarithm*** or ***Briggian logarithm***. *Henry Briggs* (1561−1631) was an English mathematician, an admirer of Napier, who wrote the natural logarithm in its modern form (1624). It is common to measure the volume of sound in decibels, which has been defined as the $^{10}\log$ of the proportion of the measured sound pressure to the atmospheric pressure.
For 2 as the base of the logarithm the ***binary logarithm*** lb(x) is the case.
In relation to the logarithm, the exponential function can be denoted as ***inverse logarithm*** or ***antilogarithm***.

The logarithm can be generalized as the polylogarithm.

In ancient times, when calculators and computers were uncommon (or not existent at all), many calculations were done with the help of logarithmic derivations. Operations as multiplying, dividing or raising to a power can be simplified to the adding and subtracting of logarithms by use or relations as log ab = log a + log b and log a^k = k log a.
You can use a slide−rule with a logarithmic scale, on two parts, one movable. By pulling out the movable part a logarithmic addition or subtraction can be made. The result can be seen at once. When more precise resulted are necessary, you can use logarithmic tables, with the desired accuracy. Ciphering is now looking up various logarithmic values, subtracting or adding, and looking up the final result in the logarithmic table.

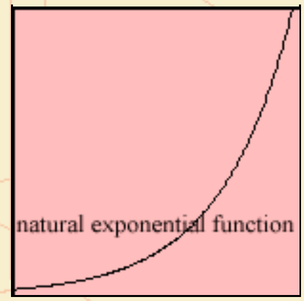
In the time of the logarithmic tables the **cologarithm** was used to prevent negative results. The function has been defined as: colog(x) = $^{10}\log(1/x)$.

When experimentally testing an exponential relation between variables, it makes sense to display the results on logarithmic paper. This is squared paper where one of the two axes has a logarithmic (semi logarithmic) scale. When a straight line

emerges, the exponential relation has been confirmed. From the angle with the x−axis the exponent can be derived. Because $^{b}\log z = {}^{c}\log z / {}^{c}\log b$, different bases of power are related with a constant factor. So one logarithmic choice satisfies.
When using two logarithmic (double logarithmic) axes, a power relation between two variables − like $y = x^{a}$ − can be tested.

Consider that something grows, proportional with time as well with quantity. In such a simple model growth takes place exponentially as function of the time. This can also be formulated as dy = a y dt, so y' = a y and $y(t) = y(0)\, e^{at}$.
That's why the curve is also called ***growth curve***. The exponential function grows faster than each polynomial (in x), and we can prove that :

$$\lim_{x \to \infty} \left(\frac{x^{a}}{e^{x}} \right) = 0$$



Many physical processes pass away as a (natural) exponential function of the time. In many cases there is a negative exponent, a phenomenon gradually approaches some equilibrium.
For radioactive decay the constant in the exponent is determined by the kind of isotope. For the process of absorption of light the constant is called the absorption coefficient. For a closed electrical circuit with voltage V, resistance R and capacity (of capacitor) C the charge on the capacitor approaches asymptotically to C V [4].
We go on: for a turbulent flow you can describe the distance between points in the flow as increasing exponentially with time.
And in ionic crystals there exists, except for an attractive Coulomb force, also a repellent force. A force for which the potential is an exponential function of the distance (with negative exponent).
For the high water for the Dutch coast you can graph how many times a certain height has been reached. This curve is also a (negative) exponential curve, according wait time theory.

The law of Euler gives the relationship between the exponential and the trigonometric functions: $e^{ix} = \cos x + i \sin x$, so that the following wonderful relationship holds: $e^{i} + 1 = 0$.

---

notes

1) The Euler number e is about 2.71828. She can be defined in different ways, for instance as:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}$$ or $$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Because of his many contributions to mathematics it is quite rightly that the number e has been named tot Euler.

2) Sometimes the natural logarithm is written as log(x).

3) For the Napierian logarithm y = Neplog(x) the relation between x and y is as follows:

$x = 10^7 e^{-10^{-7} y}$, Struik 1977 p. 113.

4) Solution of the differential equation R Q'(t) + Q(t) / C = V is:

$$Q(t) = C\,V(1 - e^{-\frac{t}{RC}})$$

exponential exponential

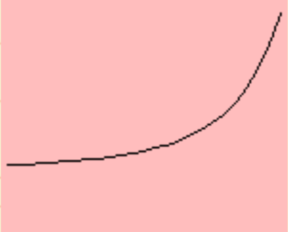
exponential

last updated: 2002−01−20

$$y = e^{e^x}$$

The curve is the exponential of an exponential.

Its inverse is the **logarithmic logarithm**, e.g. y = ln (ln x). This logarithmic logarithm is an approximation of the inverse prime summation.



exponential exponential

The **Gompertz curve** [1] is a linear form of the exponential exponential.

---

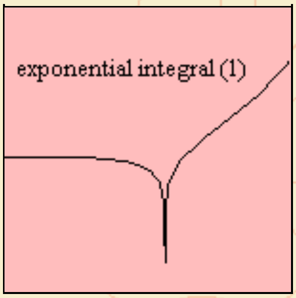
1) the Gompertz relation has the form of:

$$y = e^{-e^{a-bx}}$$

exponential exponential integral (1)

last updated: 2001-12-23

$$y = \int_{-\infty}^{x} \frac{e^{t}}{t}\, dt$$



Often mathematical authors give some extra minus signs for this exponential integral Ei(x).

There is another exponential integral, which is somewhat different and has a supplementary parameter.

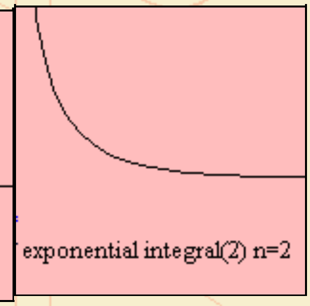
exponential integral (2)

exponential

last updated: 2001−12−26

$$y_n = \int_1^\infty \frac{e^{-xt}}{t^n}\,dt$$

This exponential integral $E_n(x)$ has a exponential integral sister, which is somewhat different and has no parameter.



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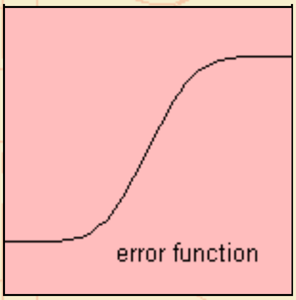
error function

exponential

last updated: 2002−02−26

$$y = \int_0^x e^{-t^2} dt$$



The ***error function*** erf(x), usually normalized to one with a factor 2 / π, is the integral of the Gauss curve. It is also called the **integrated Gauss function** or the ***integrated distribution***. The function gives the probability that a measurement under the influence of accidental errors has a distance less than x from the average value at the center.
The ***complementary error function*** erfc(x) is the same integral, but now from x to infinity.
For the ***generalized error function*** the upper as well as the lower integral boundary is a variable.
The error function is widely used in statistical calculations.

The ***inverse error function*** is used in computing confidence intervals in statistics and in some algorithms for generating Gaussian random numbers.

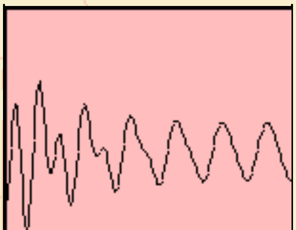
forced damped harmonic oscillation

exponential

last updated: 2001−06−24

$$y = \sin x + a e^{-bx} \sin c x$$

This is the situation of a damped harmonic oscillation where a supplementary force (also harmonic) is exerted.



forced damped harmonic oscillation a=1, b=0.1, c=2

Solution of the differential equation [1] leads to a stationary part (first part) and a part that extinguishes in time (second part).

---

notes

1) Differential equation: y'' + a y' + b y = c sin dx with y(0) = 0.

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Gauss curve

exponential

last updated: 2002−02−16

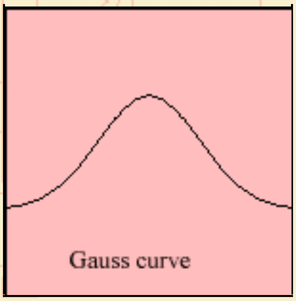
$$y = e^{-x^2}$$

The **Gauss curve** is one of the most common *distribution functions*, especially in error theory, following to names as:

- ***normal curve***
- **normal curve of error**
- **frequency curve**

Mathematicians who worked on the curve were the English−French mathematician *de Moivre* (in 1733), and later *Laplace* and *Gauss*.
For the distribution corresponding to the curve we find as names:

- ***de Moivre distribution***
- **Gaussian distribution**
- ***Gauss Laplace distribution***



The Gaussian distribution can be proved in the situation that each measurement is the result of a large amount of small, independent error sources. These errors have to be of the same magnitude, and as often positive as negative.
When measuring a physical variable one tries to eliminate systematic errors, so that only accidental errors have to be taken into account. In that case the measured values will spread around the average value, as a Gauss curve. It can be proved that in the case when the average value of a measured value is the 'best value', a Gaussian distribution holds. The 'best value' is here defined as that value, for which the chance on subsequent measurements is maximal [1)].
Because in general an estimation of errors is rather rough, the distribution to be used has not to define the error very precise. More important is that the distribution is easy to work with. And the Gaussian distribution has that quality in many situations.

Some real life examples of the Gauss distribution:

- distribution of the length of persons (given the sex)
- distribution of the weight of machine packed washing powder
- distribution of the diameter of machine made axes

Taking the definition of the standard deviation [2)] it can be seen that σ is the standard deviation in the Gauss distribution of the form:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

The points of inflection are situated at x = ± σ. For this distribution about two of the three measurements has a distance less than σ from the maximum value. And about one of the twenty measurements has a distance of more than 2σ.
Another interesting quality of the Gauss curve is that it is the only function which remains unchanged for a Fourier transform.

Because its form the curve is also called the ***bell curve***.

notes

[1] Squires 1972 p. 40.

2) The standard deviation of a distribution function f(x) is defined as:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

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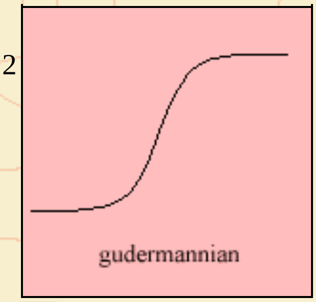
gudermannian exponential

last updated: 2001−12−26

$\tan y = e^x$

The gudermannian is a combination of a tangent and the exponential function. We see often a supplementary factor 2 and constant $-\pi/2$.



The inverse of the gudermannian function can be written as a combination of the natural logarithm, the secant and the tangent:

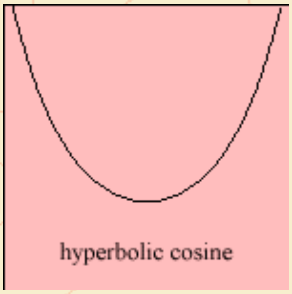
$$gd^{-1}(x) = \ln(\sec(x) + \tan(x))$$

hyperbolic cosine

exponential

last updated: 2003−05−05

$$y = e^{x} + e^{-x}$$



It is common practice to add a factor 1/2 forming the hyperbolic cosine cosh(x) (or ch(x)). Then the curve is constructed as the mid−points of the vertical line segments between e^{x} and e^{-x}.
The curve is generally known as the ***catenary*** and also the (French) name ***chainette*** is used.
The curve is the result of a (perfect, uniform) flexible and inextensible chain between two supports, as result of gravity. *Galileo* was wrong supposing this curve is the parabola.
In 1690 *Jakob Bernoulli* challenged his colleagues to find this 'chain−curve'. *Christiaan Huygens* as well as *Johann Bernoulli* and *Leibniz* succeeded independently to describe the phenomenon mathematically (1691).
It was *Huygens* who used the word 'catenary' for the first time, in a letter to *Leibniz* (1690).
There was also a man named David Gregory who wrote a treatise on the curve (1690).

There are several ways to show that the chain has the form of the hyperbolic cosine. One way is to look at the vertical and horizontal component of the forces in a small piece of the chain. Solving the resulting differential equation [1)] gives the catenary. The form is independent of the length and other qualities of the chain.
An alternate way is to calculate the minimal potential energy, or − which is equivalent − have the center of gravity as low as possible. By way of calculus of variations the minimum of the found integral can be derived [2)], and the catenary results.

A catenary, upside down, forms a state with maximal potential energy. A bridge, constructed as a catenary with exactly sawn wooden blocks, moves in the same manner as the chain, without breaking down. This construction is of course much less stable than the hanging chain.

Some relationships with other curves:

- its evolute is the tractrix
- its radial is the kampyle of Eudoxus.
- its inverse is the ***arc hyperbolic cosine*** arccosh(x)
- when a parabola is rolling over a line, then the path of the focus is a catenary

The catenary can be generalized to the alysoid.

---

notes

1) Differential equation:

$$y'' = \frac{m}{c}\sqrt{1 + y'^2}$$

with m the mass density and c the stress in the horizontal direction in the chain. Blij 1975 p. 301.

2) Minimizing differential equation:

$$\int_a^b y\sqrt{1+y'^2}\,dx$$

with boundary conditions y(a) = y(b) leads to equation

$y = c\sqrt{a+y'^2}$, Arthurs 1975 p.14.

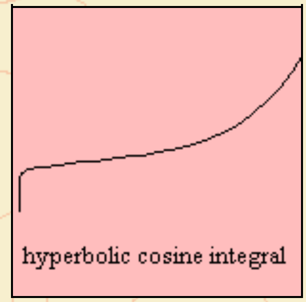
hyperbolic cosine integral

exponential

last updated: 2002−01−12

$$y = \int_0^x \frac{\cosh(t)}{t}\,dt$$

This is the integral of the (damped) hyperbolic cosine: the hyperbolic cosine integral Chi(x).
To avoid singularities you can put under the integral a form like (cosh(t) − 1) / t.



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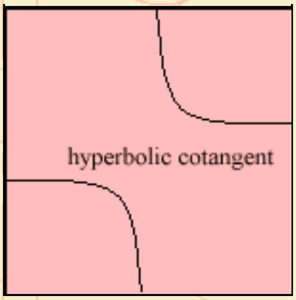
hyperbolic cotangent

$$y = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$



The hyperbolic cotangent cotanh(x) or coth(x) is defined − not surprisingly − as the hyperbolic cosine divided by the hyperbolic sine.

The inverse of the curve is the ***arc hyperbolic cotangent*** arccoth(x).

Hoerl function

exponentional

last updated: 2001−12−26

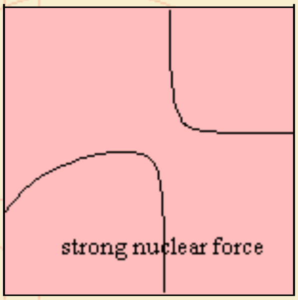
$$y = x^a e^x$$

The **Hoerl curve** is used for curve fitting purposes.
The reciprocal of the curve is found as the vapor pressure curve.

For a = −1 and −x we find a formula for the strong nuclear force:

$$y = \frac{e^{-x}}{x}$$

The elementary particles of the atom's nucleus, are being held together by the strong nuclear force. The potential of this force, as function of the distance between the particles, is derived by the given formula.
In the **strong nuclear force curve** it's the part on the right side.

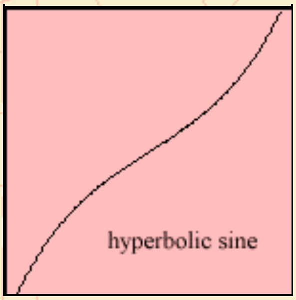


hyperbolic sine

exponential

last updated: 2002−01−12

$$y = e^{x} - e^{-x}$$



In general a factor 1/2 is added to the formula for the hyperbolic sine sinh(x) or sh(x).

The inverse of the curve is the ***arc hyperbolic sine*** arcsinh(x).

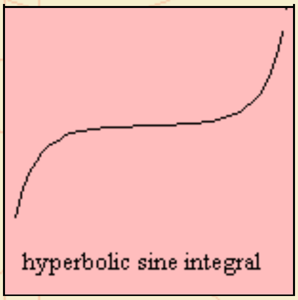
hyperbolic sine integral

exponential

last updated: 2002−01−14

$$y = \int_0^x \frac{\sinh(t)}{t}\,dt$$

This is the integral of the (damped) hyperbolic sine: the hyperbolic sine integral Shi(x).



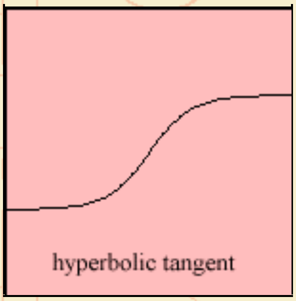
hyperbolic tangent

exponential

last updated: 2002−01−14

$$y = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

The hyperbolic tangent tanh(x) or th(x) is defined − not surprisingly − as the hyperbolic sine divided by the hyperbolic cosine. It has a typical sigmoid form.



The inverse of the curve is the ***arc hyperbolic tangent*** arctanh(x).

logarithmic integral

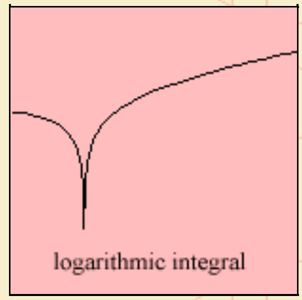
exponential

last updated: 2002−01−14

$$y = \int_0^x \frac{1}{\ln t}\,dt$$

The logarithmic integral gives the integral over a function with the logarithm in the denominator.

Some authors use the value 2 as starting point of the integral.



The logarithmic integral li(x) or Li(x) gives an approximation of the pi function.

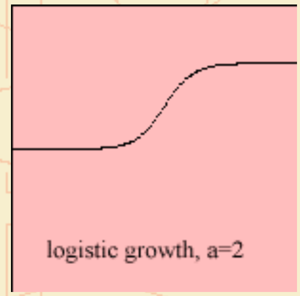
logistic growth

exponential

last updated: 2003−05−19

$$y = \frac{1}{1 + ae^{x}}$$

Especially saturation processes can be described fairly well with this curve [1].



This curve is called the **S−curve**.

---

notes

1) normally you find −x instead of x.

linear exponential

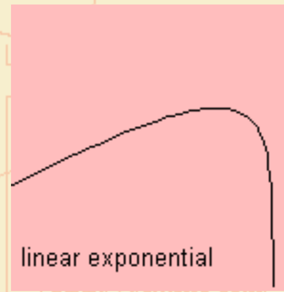
exponential

last updated: 2002−07−04

$$y = e^{x+ay}$$

The curve has a linear form of y and x in the exponent, also to be written as:

y = ln x + a x



The curve emerges for a ball thrown away when taken account of air friction, proportional to the speed [1]. Without that friction, the curve is a parabola, for which reason Robert Ferreol calls the curve a **braked parabola**.

---

notes

1) This leads to the following differential equation:

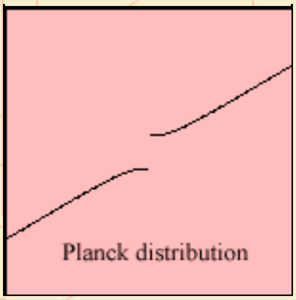
$$\frac{d^2(x,y)}{dt^2} = -a\frac{d(x,y)}{dt} - b(0,y)$$

Planck distribution

exponential

last updated: 2001−12−23

$$y = \coth \frac{1}{x}$$



The Planck distribution is directly related to the hyperbolic cotangent.

For an amount of oscillators under temperature equilibrium, the curve describes the average quantum state as function of the temperature [1].
It was the physicist *Planck* who found the formula.

An alternate way to formulate the curve is:

$$y = \frac{1}{e^{\frac{1}{x}} - 1}$$

---

notes

1) Kittel 1976 p. 128.

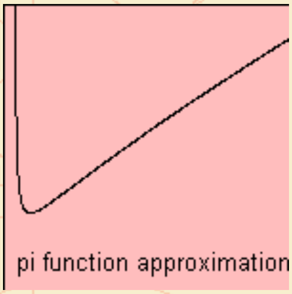
pi function approximation

exponential

last updated: 2002-01-14

$$y = \frac{x}{\ln x}$$

The function approximates the pi function.

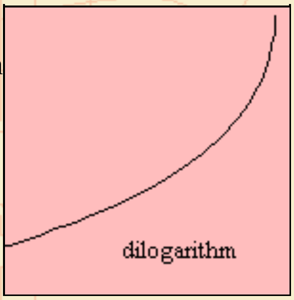


polylogarithm exponential

last updated: 2001−12−26

$$y = \sum_{i=1}^{\infty} \frac{x^i}{i^n}$$

The ***polylogarithm function*** $Li_n(x)$ is a generalization of the logarithm: the **generalized logarithm**. It has also been given the name of ***Jonquière's function***.



Special cases:

- n = 1: the function equivalent to the logarithm.
- n = 2: the ***dilogarithm***.
  $Li_2(1 - x)$ is also known as ***Spence's integral*** [1].
- n<= 0: we see a polynomial in x divided by $(1-x)^i$

The function is found in the study of elementary particle physics in Feynman diagram integrals.

The function can be generalized as the Lerch transcendent.

---

notes

1) As

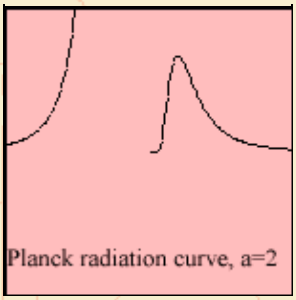
$$Li_2(1-x) = \int_{1-x}^{\infty} \frac{\ln(1-t)}{t}\, dt$$

Planck radiation curve

exponential

last updated: 2001−12−23

$$y = \frac{\coth \frac{1}{ax}}{x^5}$$



The physicist Planck stated as a result of theoretical and experimental considerations a formula for the temperature radiation energy of a black body, as function of wavelength and temperature [1]. This theory was one of the foundations of quantum physics. Crucial was Planck's assumption of packets of energy. Characteristic for the relationship is that for higher temperatures the wavelength with maximum radiation energy gets smaller. This fact could not be rightly understood with the classical theories.
In the curve the right side gives the radiation energy (y) as function of the wavelength (x). The parameter a is the temperature, apart from a constant (k / hc [2]).

The hyperbolic cotangent part in the formula is the Planck distribution.

---

notes

1) Kittel 1976 p. 128.

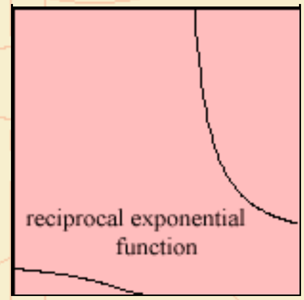
2) k: Boltzmann constant; h: Heisenberg constant; c: light velocity.

reciprocal exponential function

exponential

last updated: 2001-12-23

$$y = e^{1/x}$$



The curve has asymptotes for x = 1 and y = 0.
It has two parts, the left side in the picture is to be seen in quantum mechanics. Given a set of harmonic oscillators, the proportion of the amount in quantum state n+1 to the amount in state n − as function of the temperature − is given by the curve [1].

The right side can be found (as inverse), for the density of primes [2].

The function is also called the **modified exponential function**.

Its inverse is the **reciprocal logarithm**.

---

notes

1) Kittel 1976 p.128.

2) y = 1 / ln(n), Blij 1990 p.14.

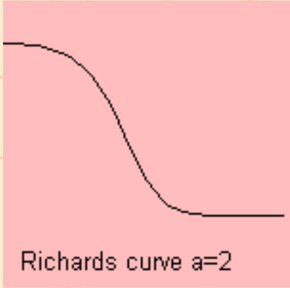
Richards curve

exponential

last updated: 2002−01−20

$$y = \frac{1}{(1+e^{x})^{a}}$$

The curve is used in curve modeling [1].



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notes

1) you often find x replaced by a linear form with −x.

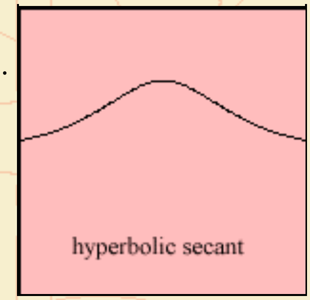
hyperbolic secant

exponential

last updated: 2001−12−23

$$y = 1/\cosh x$$

The hyperbolic secant sech(x) is the divider of the hyperbolic cosine, analogous to the trigonometric functions.



Its inverse is the ***arc hyperbolic secant*** arcsech(x).

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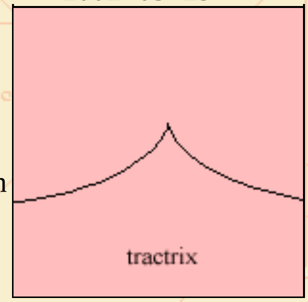
tractrix

exponential

last updated: 2002−03−29

$$\begin{cases} x = \dfrac{1}{\cosh t} \\ y = t - \tanh t \end{cases}$$



The study of this curve started with a problem stated by *Huygens* (1692): given a inextensible string, moving in a straight line, dragging an object. What then, is the path of that object, asked Huygens himself. And he gave the curve the name of the ***tractrix***. Also *Leibniz* and *Johann Bernoulli* studied the curve.

Because the dragging quality the curve has been given the names ***drag curve*** [1] and ***tractory curve***. Also the name of **donkey curve** [2] can be heard.
For the tractrix the length of a tangent from the tangent point to the asymptote is constant, leading to its alternative name of ***equitangential curve***.

The curve must not be confused with the pursuit curve, while there is no fixed length between dragging and dragged object.

Relationships with other curves are the following:

- its evolute is the catenary
- its radial is the kappa curve
- when a hyperbolic spiral is rolling over a line, then the path of the pole is a tractrix

Beltrami (1868) used the tractrix to rotate around its asymptote, in his study of non−Euclidean geometry. The surface of the resulting pseudosphere has a constant negative curvature.

---

notes

1) trahere (Lat.) = to drag
In German: Schleppkurve.

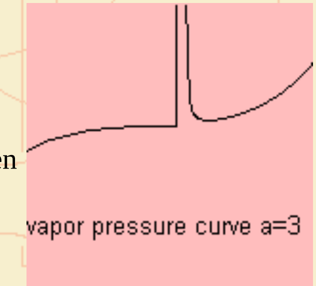
2) In French: courbe de l'âne

vapor pressure curve

exponential

last updated: 2001−12−26

$$y = x^a e^{1/x}$$



The curve is used in vapor pressure modeling. It can be seen as a **reciprocal Hoerl curve**.

Weibull curve

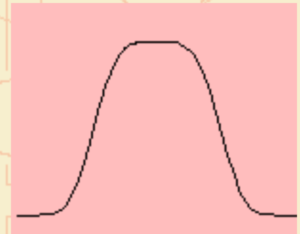
exponential

last updated: 2002−01−20

$$y = e^{-x^a}$$

The Weibull curve is used in curve modeling.

We have seen some curves already for some values for the parameter a:
a = 1: exponential curve
a = 2: Gauss curve


Weibull curve a=4

fractal

last updated: 2003−05−03

Since everyone has a computer, the fractal is a popular curve to play with.
But what is a fractal? The name fractal is derived from the idea that the dimension of such a curve is a fraction of an integer. Since some interesting fractal−like curves do have an integer dimension, there is no ground left for this relationship. However, the name fractal persisted.
You could say that a fractal is a complex curve, for which a certain amount of similarity can be seen, when magnifying parts of the curve.
The curves were discovered in the late 17th century, and they shocked the mathematicians of that era.

A special case of the fractals are the **space−filling curve**s, curves which completely fill an area.

The following fractals can be distinguished:

- ant of Langton
- bifurcation
- blancmange curve
- Mandelbrot set
- Sierpinski square

In 3d:

- Menger sponge

ant of Langton

fractal

last updated: 2002−03−25

Apply the following rule to a grid of black squares: when you get on a black square, make it white and turn to the right; when you get on a white square, make it black and turn to the left.

Surprisingly, after about 1000 steps, a regular path occurs.

You can download a program (302 kB) to make the path visual.
Then you see some kind of ant, wandering around, and at last marching in a stable direction.

blancmange curve

fractal

last updated: 2002−12−29

$$y = \sum_{k=0}^{\infty} \frac{f(2^k x)}{2^k}$$

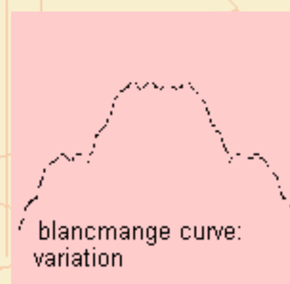
where f is the triangle curve.

The curve has been introduced by *Tagaki* (1903) to give an example of a curve that is nowhere derivable.

As you see the curve has the form of a pudding that has been turned on. That's why *John Mills* gave the curve the name of the **blancmange curve** (or **blanc−mange curve**). In fact, the 'blanc−manger' is a French kind of pudding made of almond milk and cream.

Other names of the curve are **Tagaki curve**, named after his introducer, and **van der Waerden curve**, after another (Dutch) mathematician (1930) who studied the curve.

A variation on the curve can be found for even k:



When you extend the curve to a surface, the **mount Tagaki** is formed.

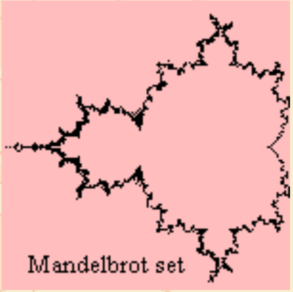
Mandelbrot set

fractal

last updated: 2002−08−16

$$y = \lim_{n}^{\infty} f_{x+iy,n}$$

where f is a Mandelbrot lemniscate



The Mandelbrot set is one of the most popular fractals, on many places on the internet pictures (often colored) of the curve are to be seen.
The curve is a generalization of the bifurcation to the complex numbers.
The boundary of the set is very complex, and while enlarging and enlarging this part of the picture, again and again different beautiful structures are conveyed. In some parts miniature Mandelbrot set formed *Mandelbrot baby*'s are to be seen.

Before Mandelbrot already the Hungarian mathematician *Riesz* (1952) posed questions related tot the Mandelbrot set. In 1978 *Brooks* and *Matelski* made the first computer graphics of the curve. *Adrien Douady* and *John Hubbard* studied the curve, they proved that the set is *connected*, and named it to Mandelbrot.
In 1994 it was *Shishikura* who proved the (Hausdorff) dimension of the curve being 2.
It is not yet known whether the curve is *path−connected*.

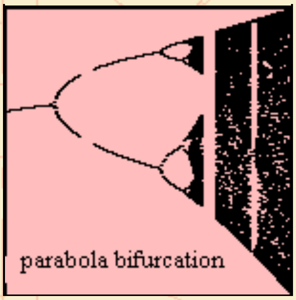
The main part of the curve, on the right, has the form of a cardioid; the part on the right has the form of a circle.

(parabola) bifurcation

fractal

last updated: 2002−08−16

$y = \lim_{n \to \infty} f_{x,n}$ where $f_{x,n}$ is an attractor



For an infinite recurrent parabola y = ax (1−x) we can compute the possible function values, as function of the parameter a. The resulting pattern is a ***bifurcation*** [1], a forklike ramification. The shown bifurcation belongs to the above mentioned parabola. For other polynomials sometimes symmetry around the x−axis be reached.
Mind the vertical white bands and the black arcs in the chaotic part at the right side of the picture. These are regions where some values are impossible (white) or very popular (black).

Special situations on the curve are:

- a=0 and a=1 are on the curve
- at a=3 the curve is split in two branches
- at a= $1 + \sqrt{6}$ the curve is split from two to four branches

The generalization for the bifurcation to the complex numbers is the Mandelbrot set.

---

notes

1) furca (Lat.) = fork, bis (Lat.) = twice

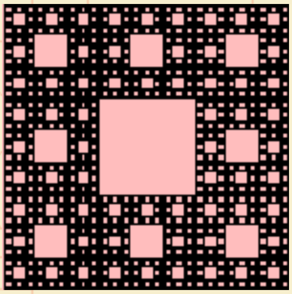
Sierpinski square

fractal

last updated: 2002−12−01

The curve is also known as the **Sierpinski universal plane curve**.

This curve can be built from a square: divide it into nine equal squares and omit the interior of the center one, then perform this operation on each of the remaining 8 squares, then on the remaining 64 squares, and so on.



The curve is the only plane locally connected one−dimensional continuum *S* such that the boundary of each complementary domain of *S* is a simple closed curve and no two of these complementary domain boundaries intersect.

Some kind of shells (conus textilus, conus gloriatnatis) have patterns that ressemble the Sierpinksi square.

gamma and related

last updated: 2003−06−15

In this part several functions can be seen, which can be derived from the gamma function:

- Airy function
- Bessel function
  - modified −
  - spherical −
- beta function
  - Catalan −
  - incomplete −
- gamma function
  - incomplete −
  - di−
  - logarithmic −
  - poly−
- Lerch transcendent
- Pochhammer symbol
- theta function
  - Riemann−Siegel −
- zeta function
  - generalized −
  - Riemann−Siegel −

Airy function **gamma**

last updated: 2001−11−05

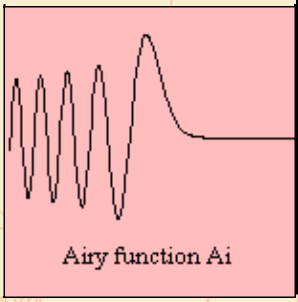
The ***Airy function*** is the solution of the differential equation:

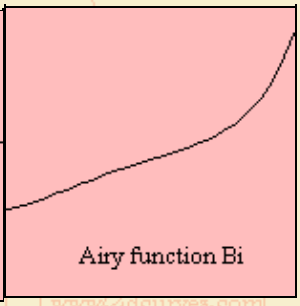
y " − x y = 0

There are two independent solutions:

- Ai(x) approaches to zero for large x
- Bi(x) goes to infinity for large x

The Airy function is found as solution to boundary value problems in quantum mechanics and electromagnetic theory.





Given a potential that increases linearly from left to right. Then the Airy function Ai(x) gives for a particle the quantum−mechanical amplitude. The region at the right side is classically inaccessible, quantum mechanically exponentially damped.

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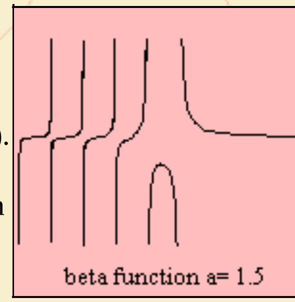
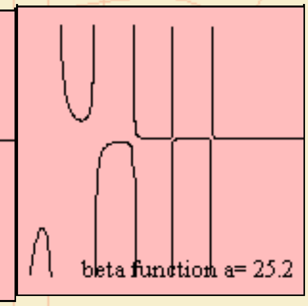
beta function gamma

last updated: 2003−05−18

$$y = \frac{\Gamma(x)}{\Gamma(x+a)}$$

Given the gamma function, a function of two variables can be derived: the ***beta function***: B(a,b) to be defined as: Γ(a) Γ(b) / Γ(a+b). As the gamma function is defined as an integral, the beta function can also be written in an integral form [1].





The name of the curve is given by *Binet* (1839). But it has been studied before, by *Euler* (1771) and *Legendre* (1809). So that the curve is also called the ***Euler beta function*** or the ***first Euler function***.

For positive integer values for the parameter a the beta function reduces to a polynomial (in x and y) [2].

The beta function comes into picture when calculating the total density by performing convolutions of classical state densities which have a power law form. An example is the situation of two subsystems composed of different sets of harmonic oscillators s1 and s2. Then the density is a function of B(s1, s2).

---

notes

1)

$$B_a(x) = \int_0^1 t^{a-1}(1-t)^{x-1}\,dt$$

2)

$$y = \prod_{i=0}^{a-1} \frac{1}{x+i}$$

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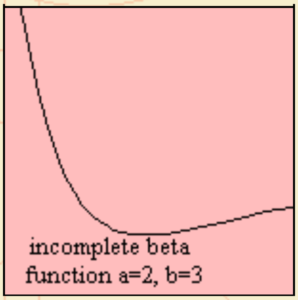
incomplete beta function

gamma

last updated: 2000-01-17

$$y = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

The ***incomplete beta function*** $B_x(a,b)$ is a variation on the beta function.



Sometimes the function is normalized to the ***regularized incomplete beta function*** I(x,a,b), defined as: $B_x(a,b)$ / B(a,b).

The incomplete beta function and its inverse are used in statistics.

Bessel function

gamma

last updated: 2001−06−24

The Bessel function is the solution of the Bessel differential equation:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

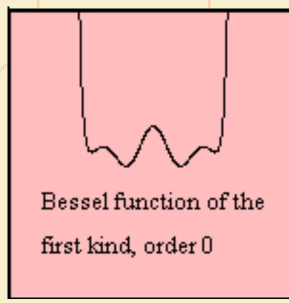
The functions are found for systems with cylindrical symmetry.
Bessel was a mathematician who lived from 1784 to 1846.

There are two linear independent solutions:

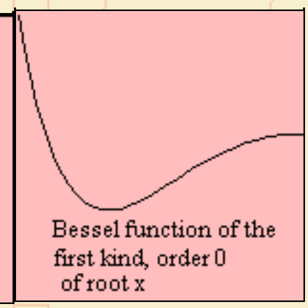
- Bessel function of the first kind
- Bessel function of the second kind

Bessel function of the first kind

When speaking of the ***Bessel function***, normally the ***Bessel function of the first kind*** order n, $J_n(x)$, is meant.



Bessel function of the first kind, order 0



Bessel function of the first kind, order 0 of root x

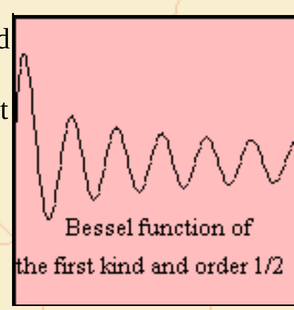
It can be written as an infinite polynomial with terms derived from the gamma function:

$$J_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(n+i+1)} \left(\frac{x}{2}\right)^{2i+n}$$

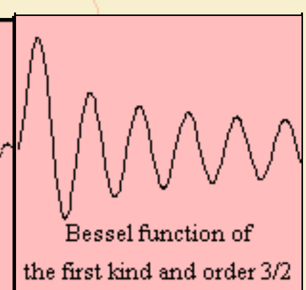
It is found when solving wave equations. For instance, in the case of a wave equation on a membrane [1], the solution is a Bessel function of integer order (a). For a circular membrane the solution can be expressed as a Bessel function of order 0: $J_0(x)$.

The Bessel function of the first kind and order 0 can occur when you wiggle an (idealized) chain, fixed at one side:

$$J_0(\sqrt{x})$$



Bessel function of the first kind and order 1/2



Bessel function of the first kind and order 3/2

The Bessel functions $J_{n+½}(x)$ are found in the definition of spherical Bessel functions.
These functions can be expressed in sine and cosine terms.
So can be found that:

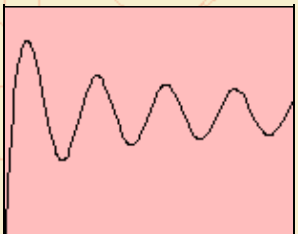
$$J_{1/2}(x) = \frac{\sin x}{\sqrt{x}} \quad \text{and} \quad J_{3/2}(x) = \frac{1}{\sqrt{x}}\left(\frac{\sin x}{x} - \cos x\right)$$

From the Bessel function of the first kind two ***Kelvin function***s $ber_n(x)$ and $bei_n(x)$ can be derived, in the following way:

$$ber_n(x) + i\, bei_n(x) = e^{n\pi i} J_n(x\, e^{-\pi i/4})$$

Bessel function of the second kind

The ***Bessel function of the second kind*** of order n, $Y_n(x)$, is also called the ***Weber function*** or the ***Neumann function*** $N_n(x)$.


Neumann function n=0

it can be written as an infinite polynomial with terms derived from the gamma function:

$$N_n(x) = \lim_{i \to n} \frac{J_i(x)\cos i\pi - J_{-i}(x)}{\sin i\pi}$$

The ***Bessel function of the third kind*** or ***Hankel function*** $H_n(x)$ is a (complex) combination of the two solutions: the real part is the Bessel function of the first kind, the complex part the Bessel function of the second kind.

---

notes

1)

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

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modified Bessel function

gamma

last updated: 2000−02−15

The ***modified Bessel function*** is the solution of the differential equation:

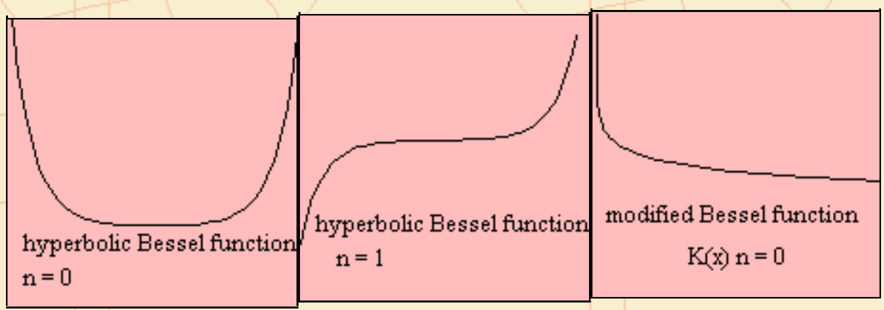
$x^2 y'' + x y' - (x^2 + n^2) y = 0$

which is slightly different from the Bessel differential equation that defines the Bessel function.

There are two solutions:

- ***hyperbolic Bessel function*** $I_n(x)$
- modified Bessel function $K_n(x)$
  From the modified Bessel function two ***modified Kelvin functions*** $\mathrm{ker}_n(x)$ and $\mathrm{kei}_n(x)$ can be derived, in the following way:

  $\mathrm{ker}_n(x) + i\,\mathrm{kei}_n(x) = e^{-n\pi i/2} K_n(x\, e^{\pi i/4})$



spherical Bessel function

gamma

last updated: 2000−02−13

$$y = \frac{1}{\sqrt{x}} Bessel_{n+1/2}(x)$$

The ***spherical Bessel function*** is found in the case of systems with spherical symmetry.
For instance in the situation of a threedimensional wave, which obeys the standard wave equation [1].

spherical Bessel function of the first kind, order 3/2

The function can easily expressed as a Bessel function, as we can see in the formula on top (omitting constants).

For the Bessel function of the first kind and the order (n) is equal to 0, the function is equivalent to the damped sine.

---

notes

1)

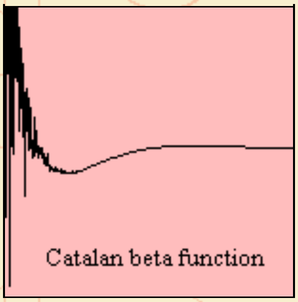
$$\Delta u(r) = \frac{1}{a^2} u_{tt}$$

Catalan beta function

gamma

last updated: 2003−06−15

$$y = \sum_{i=0}^{\infty} (-1)^i (2i+1)^{-x}$$



The ***Catalan beta function*** β(x) can be expressed in the Lerch transcendent as:

$$\beta(x) = 2^{-x}\, \Phi(-1, x, \tfrac{1}{2})$$

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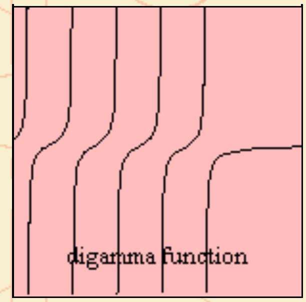
digamma function

gamma

last updated: 2000−01−18

$$y = \frac{\Gamma'(x)}{\Gamma(x)}$$



The derivative of the logarithmic gamma function, the ***digamma function*** $\psi(x)$ can be used to clarify the sum of some rational series.

The following relation can be used:

$$\psi(x) = -\gamma + \sum_{i=0}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+x}\right)$$

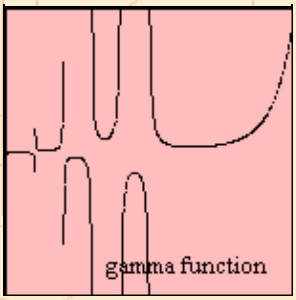
The digamma function can be generalized to the nth derivative as the polygamma function.

gamma function

gamma

last updated: 2003−05−18

$$y = \int_0^\infty e^{-t} t^{x-1} dt$$



The gamma function Γ(x) can be seen as a generalization of the factorial: for a positive integer the function value of the gamma function is equal to the factorial.

Alternative definitions of the curve are defined by *Bourbaki* and *Weierstrass* (with infinite products).

The gamma function has some remarkable qualities: it has an infinite number of maxima and minima. And it can be proven that it cannot be the solution of a differential equation with algebraic coefficients.
The function plays a major role in difference equations, as the exponential functions has its role for differential equations. The curve can be found in a lot of mathematical problems.

The name and the notation of the function has been set by *Legendre* (1809). Because of the work of *Euler* on the curve the function is also called the ***Euler gamma function*** or the ***second Euler function***.

On the internet approximations of gamma functions can be found, for instance on the site of Victor T. Toth.

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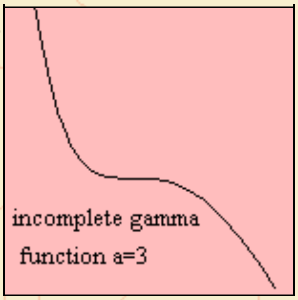
incomplete gamma function

gamma

last updated: 2000-01-17

$$y = \int_x^\infty e^{-t} t^{a-1}\, dt$$

The ***incomplete gamma function*** Γ(a,x) is a variation on the gamma function.



An alternative incomplete gamma function can be defined as the integral from 0 to x. It may be clear that this is the same function, apart from a constant.

The function can be normalized to the ***regularized incomplete gamma function*** Q(a,x) defined by: Γ(a,x) / Γ(a).

The incomplete gamma function and its inverse are used in statistics.

logarithmic gamma function

gamma

last updated: 2000−01−18

$$y = \ln(\Gamma(x))$$

The natural logarithm of the gamma function is used in some computations in number theory, with the help of the derivative of this function: the digamma function.

logarithmic gamma function

The function is only defined for positive values.

Lerch transcendent

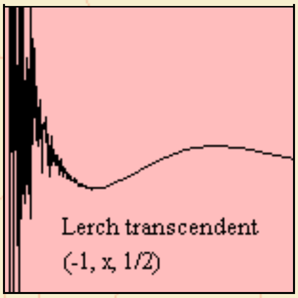
gamma

last updated: 2003−06−15

$$y = \sum_{i=0}^{\infty} \frac{x_1^{\,i}}{(i+a)^{x_2}}$$

where the terms with i + a = 0 are excluded.

The ***Lerch transcendent*** $\Phi(x_1, x_2, a)$ is a generalization of the polylogarithm (regarding x_1) and the (generalized) zeta function (regarding x_2).



Many infinite sums (of reciprocal powers) can expressed as a Lerch transcendent, for instance the Catalan beta function.
The function is also related to integrals of the Fermi−Dirac distribution. And it is used to evaluate some series in number theory (Dirichlet L−series).

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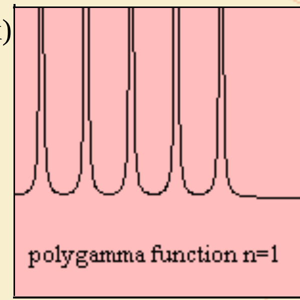
polygamma function

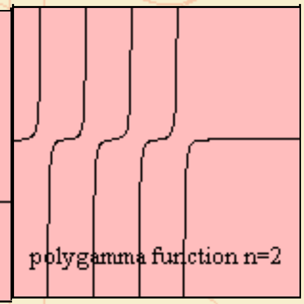
gamma

last updated: 2000−01−19

$$y = \frac{d^n\psi(x)}{dx^n}$$

The ***polygamma functions*** $\psi^{(n)}(x)$ are the derivatives of the digamma function.





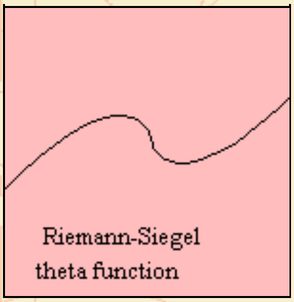
For n = 0 we see the digamma function.

Riemann−Siegel gamma theta function

last updated: 2000−01−22

$$y = \mathrm{Im}\,\ln\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right) - \frac{x}{2}\ln(\pi)$$

While studying the zeta function for arguments ½ + i t, the ***Riemann−Siegel theta function*** θ(x) was composed.



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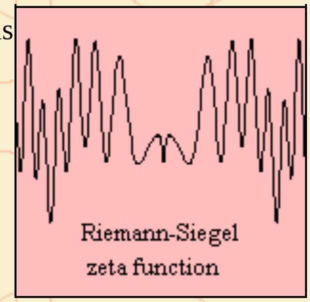
Riemann−Siegel gamma zeta function

last updated: 2000−01−22

$$y = e^{i\theta(x)}\zeta\left(\frac{1}{2}+ix\right)$$

where θ(x) is the Riemann−Siegel theta function, and ζ(x) is the zeta function.



While studying the zeta function for arguments ½ + i x, the ***Riemann−Siegel zeta function*** Z(x) was composed.

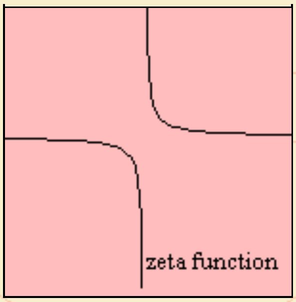
zeta function

gamma

last updated: 2003−05−18

$$y = \sum_{i=1}^{\infty} i^{-x}$$

The ***zeta function*** ζ (x) also named the ***Riemann zeta function*** or the ***Euler zeta function***.



For integer values of the argument the zeta function can be found in some sums and integrals.

The zeta function that has been generalized for complex arguments is used to describe the distribution of primes. With its help the Prime Number Theorem can be derived.

The function can be generalized as the generalized zeta function.

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generalized zeta function

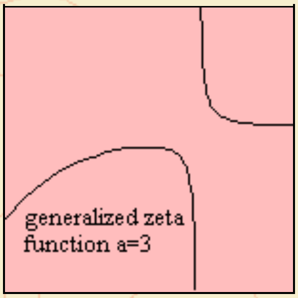
gamma

last updated: 2003−06−15

$$y = \sum_{i=1}^{\infty} (i+a)^{-x}$$

where terms with i+a=0 are excluded.

The ***generalized (Riemann) zeta function*** or ***Hurwitz zeta function*** ζ (x, a) is a generalization of the (Riemann) zeta function.



isochronous curve

main

last updated: 2001−11−17

These are curves where some aspect of time (chronos) is equal (iso).

Two of those curves are:

- semi−cubic(al) parabola: the vertical velocity of an object, rolling over this curve, is constant
- cycloid: a bead rolling along the curve, needs the same time to get down, independent from the starting point.

power curve

last updated: 2002−11−05

On this page I describe some functions in a power of x:

- straight pursuit curve
- power function
- power series
- special power curve
- super ellipse

power function

power

last updated: 2002−03−25

$$y = x^a$$

When the value of the constant a is a rational number, the curve reduces to an algebraic curve. When a is an integer, the curve reduces to a simple sort of polynomial.

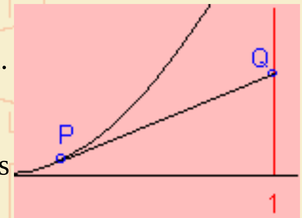
straight pursuit curve

power

last updated: 2002−11−16

$$y = \frac{x^{1+a}}{1+a} - \frac{x^{1-a}}{1-a}$$

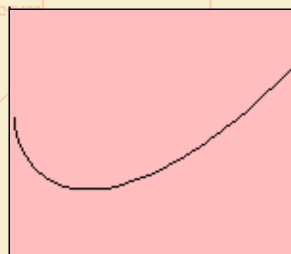
Given a dog (P) chasing a pedestrian (Q). The anxious human creature runs in a straight line, with constant speed u. The dog runs also with a constant speed (v), always in the direction of its prey. The path the dog runs has been given the name ***pursuit curve***. The ratio u/v of the two velocities is just the value of the parameter 'a' in the formula [1].
When the movement doesn't take place in a straight line, we can call the curve a generalized pursuit curve, to be distinguished from the **straight pursuit curve** on this page.



construction of the straight pursuit curve

Leonardo da Vinci was the first to consider this problem.

In some cases the curve degenerates to an algebraic curve, for instance when the parameter 'a' has the value 1/2.


pursuit curve a=1/2


pursuit curve a=1

Some authors confine the pursuit curve to the case that a = 1, when the velocities are the same. This leads to the following formula [2]:

$$y = x^2 - \ln x^2$$

It was *Arthur Bernhart* who paid special attention to the curve.

It is good to state that this pursuit curve for a=1 is not the tractrix: equal distance between pursued and pursuer is not equivalent with equal speed between them.

---

notes

1) the tangent to the curve of pursuit can be expressed as dy/dx = (y − ut) / (x −1) (i)
The arc length of the curve of pursuit can be written as vt = $\int\sqrt{(1+ (dy/dx)^2)}dx$ (ii)
Eliminate t form (i) and (ii), then set dy/dx = p(x) and differentiate (Kuipers 1966 p.360)

2) To understand that holds:

$$\lim_{a\to 1} \frac{x^{1-a}}{1-a} = \ln x$$

realize that

$$\lim_{x\to 0} \frac{a^x}{x} = f'(0)$$

with $f(x) = a^x$

power series power

last updated: 2002−12−27

$$y = \sum_{i=1}^{\infty} \frac{f^{i}(0)}{i!} x^{i}$$

The theorem of *Taylor* predicts that a function that behaves sufficient neatly can be written as a ***power series*** of polynomials.
This series can be used to define a function (with curve y = f(x)). In that way trigonometric functions can be built, without using geometrical insights.

In the case of a finite number of polynomials the function is an algebraic function. In the infinite case, the function is transcendental.

super ellipse

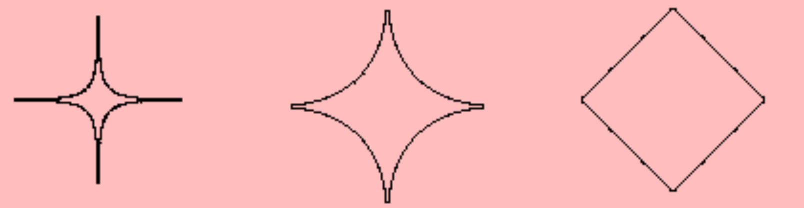
power

last updated: 2003−04−18

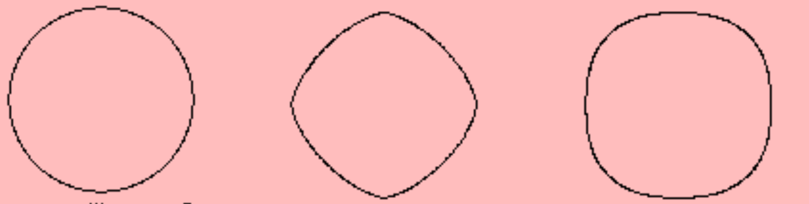
$$|x|^a + |y|^a = 1$$

It was *Lamé* who was the first to study this class of curves (in 1818), leading to the name of **Lamé curve**.
The curve can be seen as a kind of (distorted) ellipse, a **super ellipse**.

For a < 2, the curve is also named a **hypoellipse**.
A circle goes via a square (a = 1) in the direction of a cross (a = 0).
A special case is the astroid, for a = 2/3. Some authors call the general case the astroid.



super ellipse a=1/4 — super ellipse a=1/2 — super ellipse a=1 (line)



super ellipse a=2 (circle/ellipse) — super ellipse (a=3/2) — super ellipse a=5/2 (Piet Hein's ellipse)

For a > 2, the curve is also named a **hyperellipse**.
A special case is ***Piet Hein's ellipse***, for a = 5/2. This Danish poet and architect used the curve for architecture objects as motorway bridges.

The curve can be extended to the **generalized super ellipse**:

$$|x|^a + |y|^b = 1$$

The three−dimensional version of the hyper ellipse has been called the **super egg**, by *Piet Hein*.

power exponential curves

last updated: 2002−01−14

On this page I describe some functions with terms x^x:

- power exponential curve
- prime function approximation
- reciprocal power exponential curve

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m@ail me

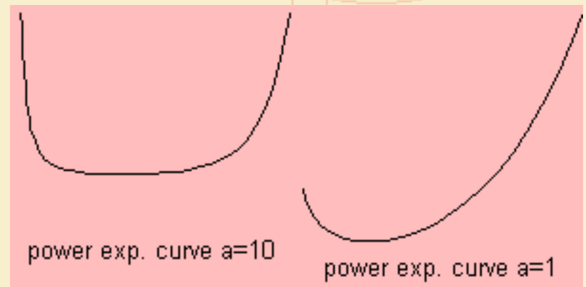
power exponential curve

power exponential

last updated: 2002−01−14

$$y = x^{ax}$$

In curve fitting the **power exponential curve** (you know a better name?) is used for the so−called 'geometric fit'.

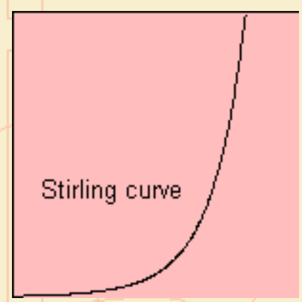


The logarithm of the curve is the prime function approximation.

The curve can be generalized to:

$$y = x^{ax+b}$$

A = e and b = 1/2 gives the **Stirling curve** which is an approximation for the factorial [1].
Stirling was the one to discover this relation.



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notes

[1] Often an expression is given as:

$$y = 2\pi (x/e)^x \sqrt{x}$$

prime function approximation

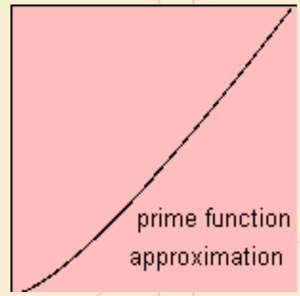
power exponential

last updated: 2002−01−14

$y = x \ln x$

The curve is an approximation of the prime function.

It is the logarithm of the power exponential curve.



Convergence does not take place rapidly: after 1000 primes the approximation is more than 10 % less than the discrete function.

reciprocal power exponential curve

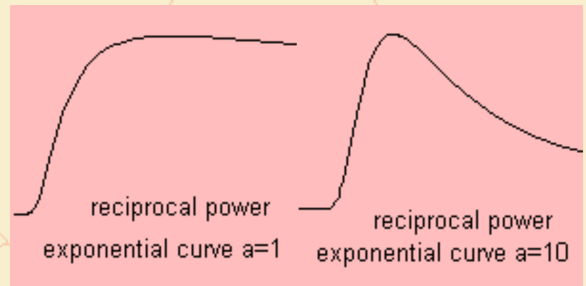
power exponential

last updated: 2002−01−14

$y = x^{a/x}$

The curve is the reciprocal of the power exponential curve.

In curve fitting the curve is used for the so−called 'modified geometric fit'.



spiral

last updated: 2003−04−21

A spiral is a curve that winds itself round a certain point [1]. While not being a circle, the radius will vary along the angle.
Not all spiral−named curves have this winding quality, see e.g. the epi spiral.

I am not primarily concerned with 3D spirals [2].

In 2D we see the spiral in nature in the snail−shell, the cochlea (in your ear), the composites' flower−head, the shell, spiral star constellations. The spiral is also used in architecture, it's a very old ornament. According to *Proclus* the Greek *Perseus* was the first to describe the spiral curve.

In the universe some of the star systems have a spiral form. And the spiral theory is a model of our solar system, which has been constructed by the medieval *Alpetragius*. This theory is a variant on the system of *Aristotle*. The naming to a spiral is not that precise, because concentric spheres instead of spirals are used in the theory.

The following spirals can be distinguished:

- Archimedean spiral
- Archimedes' spiral
- atom−spiral
- Atzema spiral
- cochleoid
- cubic spiral
- epi spiral
- Euler's spiral
- involute of a circle
- logarithmic spiral
- Poinsot's spiral
- sinusoidal spiral

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notes

[1] Spira (Lat.) = twisting (of a snake)

[2] 3D spirals are often composed of a circular and a linear movement (in different directions). E.g.:

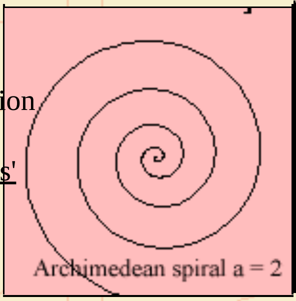
- the spiral staircase
- the movement of a particle in a magnetic field
- the contraceptive spiral
- the spiral grain in some kinds of trees (clockwise)

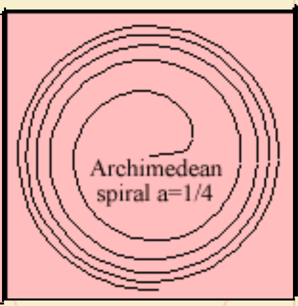
A three−dimensional spiral movement with water or air material is called a **vortex**.

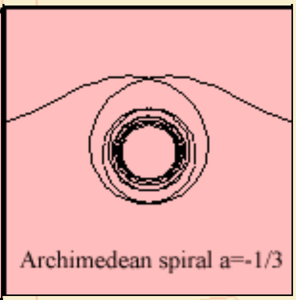
Archimedean spiral spiral

last updated: 2002−03−25

$$r = \phi^a$$







This spiral is a generalization of Archimedes' spiral (a=1), named to the Greek *Archimedes* (225 BC).
The inverse of the spiral with a constant a is an Archimedean spiral with a constant −a.

There are some special situations:

- a = −1: hyperbolic spiral
- a = −1/2: lituus
- a = 1/2: Fermat's spiral
- a = 1: Archimedes' spiral

An Archimedean spiral with parameter a has as polar inverse an Archimedean spiral with parameter −a: so the lituus and Fermat's spiral are inversely related, and also the hyperbolic and the Archimedes' spiral.

The Cesaró equation writes a curve in terms of a *radius of curvature* ρ and an *arc length* s. For the Archimedean spiral, the two are equal: ρ = s.

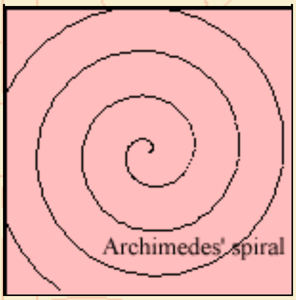
It was *Sacchi* (1854) who distinguished this group of spirals, for the first time.

Archimedes' spiral

spiral

last updated: 2002−08−19

$r = \phi$



The Archimedes' spiral (or ***spiral of Archimedes***) is a kind of Archimedean spiral.
Some authors define this spiral as the combination of the curves r = Φ and r = − Φ.

Because there is a linear relation between radius and the angle, the distance between the windings is constant.

There are two classical mathematical questions for which this spiral gives a simple solution:
The quadrature of the circle was the quest to construct a square with the same area as a given circle. A first step would be to construct a line with a length π times the the length of a given line. When the Archimedes' spiral is available, this is easy: for an angle of magnitude π the radius is equal to this π.
With the trisection was meant the quest to divide an angle in three equal parts. This is easy with the Archimedes' spiral, too. While the angle is linear with the radius, a division of the radius in n equal parts gives n equal angles in the spiral.
The Greek − who found these construction methods − were not satisfied, however: they preferred to use no other construction tools than a ruler and a pair of compasses.

The curve can be used to convert uniform angular motion into uniform linear motion: build a cam consisting of two branches of the spiral curve. Rotating this device with constant speed about its center results in a uniform linear motion at the crossing with a straight line through the center. This quality has been used in old sewing−machines (see picture).

In nature the curve can be seen for the pine−cone (see picture).
Some ammonites have the form of a spiral of Archimedes, while most the form of a logarithmic spiral.
The Archimedes' spiral describes a growth that just adds, in contrary to the logarithmic spiral, that grows related to its size.

Three−dimensionally the curve is the orthogonal projection (on a plane perpendicular to the axis) of the spiral cone of *Pappus*.

Some relations with other curves are the following:

- the curve is the pedal of the involute of a circle
- its inverse is the hyperbolic spiral
- the conchoid of the curve is also an Archimedes' spiral.

It was of course *Archimedes* to explore this curve for the first time (225 BC). He wrote about it in his work 'on spirals'. It was his friend *Conon* who already considered the curve.
Later it was *Sacchi* who studied the curve (in 1854).

This spiral can be seen on the desktop at startup of your computer, when your friend has been affected by the win32.hybris virus. This email worm is able to extend its functionality by using plugins. One of these plugins is the one that displays an Archimedes' spiral, which is next to impossible to close.

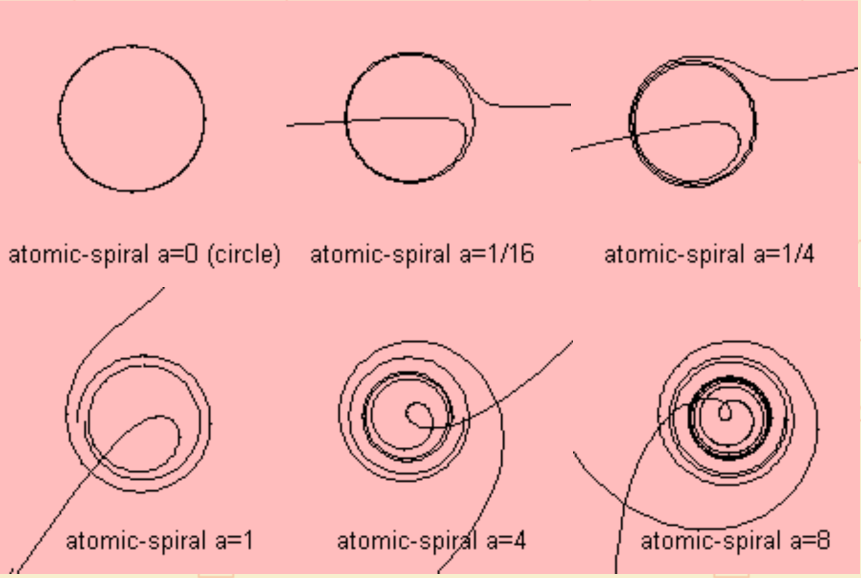
atom−spiral

spiral

last updated: 2003−02−03

$$r = \frac{\phi}{\phi - a}$$

The atom−spiral can be used to design an e with an appealing curl like in the @ symbol.
The name has been given by the Belgian *Annie van Maldeghem*, named after a symposium called Matomium, held in 2002 in Brussels.



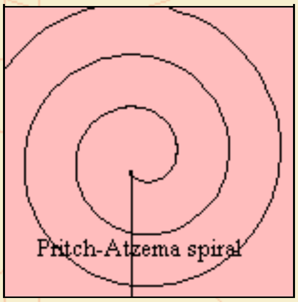
The spiral has two asymptotes, the unity circle and a line (ϕ = a).
The smaller a, the faster the curve approximates the circle.

Atzema spiral spiral

last updated: 2002−03−25

$$\begin{cases} x = \dfrac{\sin t}{t} - 2\cos t - t\sin t \\ y = -\dfrac{\cos t}{t} - 2\sin t + t\cos t \end{cases}$$

The ***Atzema spiral*** is the curve whose catacaustic is a circle.
Eisso Atzema (with *Yael Pritch*) proved this recently (2000).



For large t, the curve approximates the Archimedes' spiral.
However, the spiral has an asymptote for x = 0.

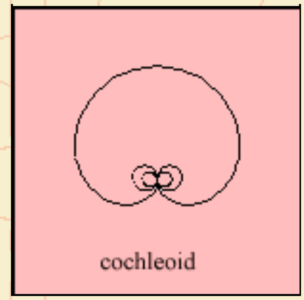
cochleoid

spiral

last updated: 2002−04−21

$$r = \frac{\sin\phi}{\phi}$$

This curve has been named to the snail [1] by *Benthan* and *Falkenburg* (in 1884).
It can be constructed as follows: given a point O and the y−axis. For all circles through O (so tangent to the y−axis), pace a constant distance on the circle. The collection of those points is the ***cochleoid*** [2].


cochleoid

Some relations with other curves are the following:

- it is the polar inverse of the quadratrix
- the points of contact of parallel tangents to the curve are lying on a strophoid
- it is the barycentric of the circle

The first mathematicians to study the curve were *Peck* (1700) and *Bernoulli* (1726).

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notes

1) Kochlias (Gr.) = snail.

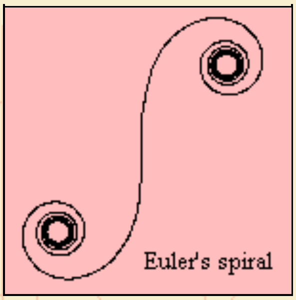
2) This can be proven as follows: let the circle have a radius t, and center C(t, 0).
Now we know that the length of the arc has to be equal to a constant, say a, what is the same as Φ t. With $\sin \Phi = y / t$, this leads to $t \arcsin(y/t) = a$ (1).
Now a point of the curve is on the circle, so that $(x-t)^2 + y^2 = t^2$ and $y = 2 r^2/2x$ (2).
Combination of (1) and (2) leads to the desired formula.

Euler's spiral spiral

last updated: 2003−05−18

$$\begin{cases} x = \pm \int_0^t \frac{\sin u}{\sqrt{u}} du \\ y = \pm \int_0^t \frac{\cos u}{\sqrt{u}} du \end{cases}$$

The formulas under the integral can also be written as the Bessel functions $J_{1/2}$ (for y) en $J_{-1/2}$ (for x).



For the spiral, which is also called ***clothoid*** or ***spiral of Cornu***, the x and y component are both a Fresnel integral of a square root.
Its curvature is linearly related to its arc length [1].
When the path of the curve is followed with an uniform velocity, the speed of rotation is linear (in time). That's why the curve with the reverse relation is called the anti−clothoid.

It was the famous *Leonhard Euler* (1744) who started investigating the curve.

In the early days of railways it was perfectly adequate to form the railways with series of straight lines and flat circular curves. When speeds increased the need developed for a more gradual increase in radius of curvature R concomitant with an elevation of the outer rail, so that the transition to the circular curve became smooth. The clothoid makes a perfect **transition spiral**, as its curvature increases linearly with the distance along the spiral.
A first order approximation of this spiral is the cubic spiral.

For the same reason the spiral is used in ship design, specifying the curvature distribution of an arc of a plane curve while drawing a ship.

The curve for which the curvature is *cubically* related to the arc length, would be even more useful. This curve is called the **generalized Cornu spiral** or the **polynomial spiral.**

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notes

[1] $d\phi/ds = 2a^2 s$ leads to $\phi(s) = (as)^2$, so that
$dx/ds = \cos(as)^2$ and $dy/ds = \sin(as)^2$

which leads to the given integrals.

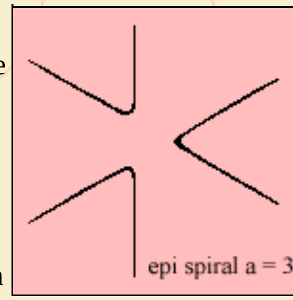
epi spiral

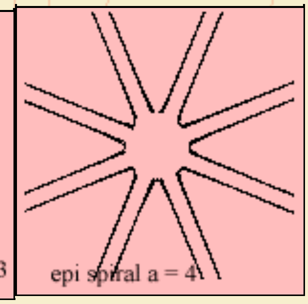
spiral

last updated: 2002−03−25

$$r \cos a\phi = 1$$

This curve doesn't have the characteristic spiral form, it is more the form of a radial compound hyperbola.
It is the polar inverse of the rhodonea.





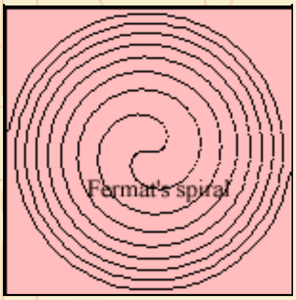
The number of sections depends on the value of parameter a: for odd a there are a number of a sections, for even a: 2*a sections.

Fermat's spiral

spiral

last updated: 2003-06-14

$$r^2 = \phi$$



The spiral of Fermat is a kind of Archimedean spiral.
Because of its parabolic formula the curve is also called the **parabolic spiral**.

It was the great mathematician *Fermat* (1636) who started investigating the curve, so that the curve has been given his name.
Sometimes the curve is called the **dual Fermat's spiral** when both both negative and positive values are accepted.

The people who practice omphaloskepsis adopted the curve as their symbol.
Omphaloskepsis is a way of life, around the act of contemplating one's navel as a method of achieving higher meditation.

The inverse of the curve is the lituus.

hyperbolic spiral

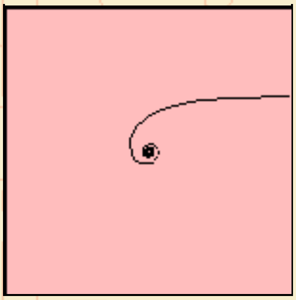
spiral

last updated: 2002-03-25

$r\phi = 1$

This spiral is a kind of Archimedean spiral.

The hyperbolic name reflects the relation between radius and angle being the same as the x,y relation for the hyperbola. This relationship leads also the name of the ***reciprocal spiral***.



When a hyperbolic spiral is rolling over a line, then the path of the pole forms a tractrix, being a roulette of the spiral.

Its inverse is the Archimides' spiral.

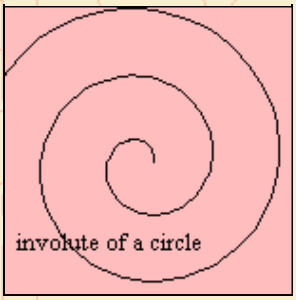
It was *Pierre Varignon* who studied the curve as first, in 1704. Later *Johann Bernoulli* (1710 − 1713) and *Cotes* (1722) worked on the curve.

involute of a circle

spiral

last updated: 2002−04−08

$$r^2 = \phi^2 + 1$$



The curve is the involute of a circle [1].
Some authors add the points (−y, x) to the curve as shown on the page.

For large angles (>>1), the curve approximates the Archimedes' spiral, this curve also being the pedal of the involute of a circle.
When the curve is rolling over a line, then the path of the center is a parabola. And its pedal is an Archimedes' spiral.

The Cesaró equation writes a curve in terms of a radius of curvature ρ and an arc length s. For the involute of a circle, the equation has the elegant form: $\rho^2 = s$.

When the path of the curve is followed in a linear way (in time), the rotation speed is constant. While this is the reverse definition as for the clothoid, the involute of a circle is also called the **anti−clothoid.**

Huygens used the curve in his experiments to have a cycloid swing in his pendulum clocks.
In mechanical engineering the curve is used for the profile of a gear−wheel, in the situation of non−parallel axes.

notes

1) In French: développante du cercle; in German: Kreisevolvente; in Dutch: cirkelevolvente.

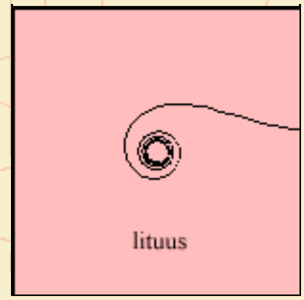
lituus

spiral

last updated: 2003−05−05

$$r^2\phi = 1$$

The lituus is a species of the Archimedean spiral.
Some authors add (−r, φ) forming a double shape.



The curve is formed by the loci of a point P moving in such a way that the area of a circular sector remains constant.

Roger Cotès (1682−1716) was the first to study the curve. Appointed professor at Cambridge at the age of 24 his work was published only after his death.

Maclaurin gave the curve its name in his book 'Harmonia Mensurarum' (1722) using the resemblance of the curve to a crosier [1)].

The inverse of the curve is Fermat's spiral.

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notes

1) lituus (Lat.) = bend, crook.
In the old Roman empire the lituus was a crosier.

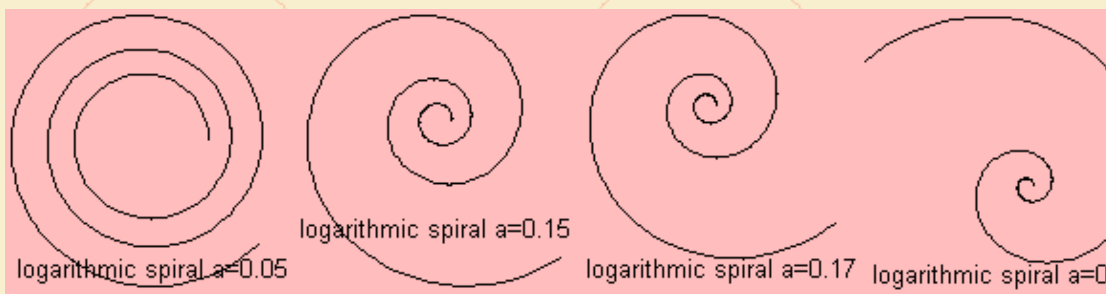
logarithmic spiral

last updated: 2003−02−22

$$r = e^{a\phi}$$

This is the spiral for which the radius grows exponentially with the angle. The logarithmic relation between radius and angle leads to the name of ***logarithmic spiral*** or ***logistique*** (in French).
The distances where a radius from the origin meets the curve are in geometric progression.

The curve was the favorite of *Jakob (I) Bernoulli* (1654−1705). On his request his tombstone, in the Munster church in Basel, was decorated with a logarithmic spiral. The curve, which looks by the way more like an Archimedes' spiral, has the following Latin text accompanied: *eadem mutata resurgo*. In a free translation: 'although changed, still remaining the same'. This refers to the various operations for which the curve remains intact (see below).
Therefore the curve is also called the **Bernoulli spiral**.



However, *Rene Descartes* (1638) was the first to study the curve. *Torricelli* worked on the curve independently, and found the curve's length.
The curve is also named to *Fibonacci* as the **Fibonacci spiral**.

What are those remarkable qualities of this spiral?
The curve is identical to its own:

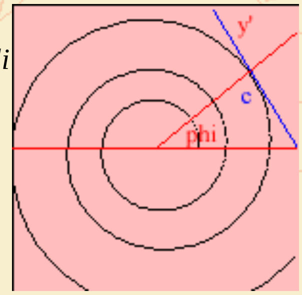
- caustic
- evolute
- inverse
- involute
- orthoptic
- pedal
- radial

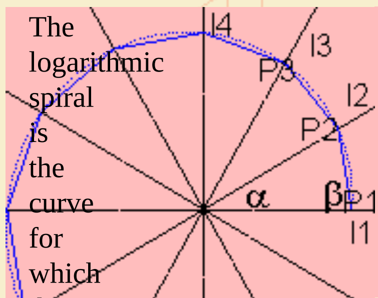
Other qualities of the spiral are the following:

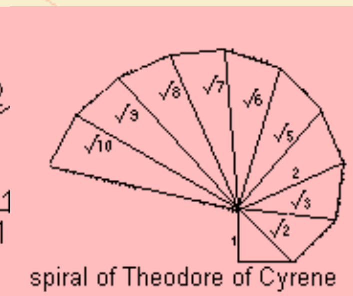
- the radius of curvature is equal to the arc length

- when a logarithmic spiral is rolling over a line, then the path of each point of the spiral is a line
- the multiplication of the logarithmic spiral is equivalent with a rotation
- the length from the origin O to a point $P(r_0, \varphi_0)$ of the spiral is equal to $r_0 \sec(b)$, where $a = \cot(b)$ [1]

Remarkable! This spiral is a real ***spira mirabilis***, as *Jakob Bernoulli* called the curve in 1692.







spiral of Theodore of Cyrene

The logarithmic spiral is the curve for which the angle between the tangent and the radius (the polar tangent) is a constant.
Suppose that an insect flies in such a way that its orbit makes a constant angle b with the direction to a lamp. Then the poor creature travels in the form of a logarithmic spiral, eventually reaching the lamp [2]. It seems that a night−moth flies under a constant angle with respect to the moon. But its orbit resembles much that of a right line, while the orbit of the moth is much smaller that the distance to the moon. This property gives the spiral the name of ***equiangular spiral***.
For angle $b = \pi/2$, the result is a circle.

The logarithmic spiral can be approximated by a series of straight lines as follows: construct a line bundle l_i through O with slope $i\alpha / 2\pi$.
Starting with a given point P_1 on l_1, construct point P_2 on l_2 so that the angle between P_1P_2 and OP_2 is β. Then the points P_i approximate a logarithmic spiral (for small α) with $a = \cot \beta$. For $\beta = \pi / 2$, the line series is called the **spiral of Theodore of Cyrene**.

We see the curve in nature, for organisms where growth is proportional to their size. An example is the Nautilus shell, where a kind of octopus hides (showed on this page).
For that proportionality the curve bears the name of the ***growth spiral***: a growth that is proportional to its size. A growth that just adds, is shown by the Archimedes' spiral.
D'Arcy Thompson explains the curve in his book 'Growth and Form'.

Two physical properties related to the spiral are:

- the force that makes a point move in a logarithmic spiral orbit is proportional to $1/r^3$ [3)].
- a particle moving in a uniform magnetic field, perpendicular to that field, forms a logarithmic spiral.

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notes

1) Remember that the arc length s can be described in polar coordinates as (ds)2 = (dr)2 +r2 (dj)2

2) It can be proven that the desired curve is the logarithmic spiral: the curve can be found as the solution of the differential equation, which results out of the relation y' = tan(b + F):

$$y' = \frac{\tan b + \frac{y}{x}}{1 - \frac{y}{x}\tan b}$$

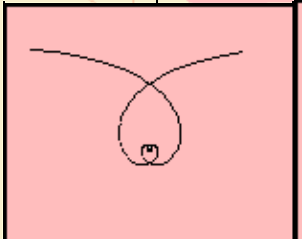
Substituting y = x z and rewriting in polar coordinates gives the spiral's equation. See, for example, van der Blij 1975, p. 204.
Then it follows that formula's constant a is equal to the cotangent of angle b.

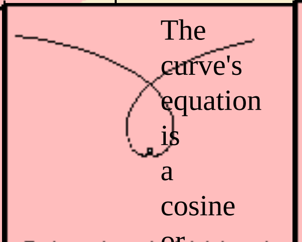
3) To be proven with help of the formula of Binet.

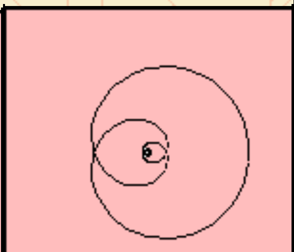
Poinsot's spiral

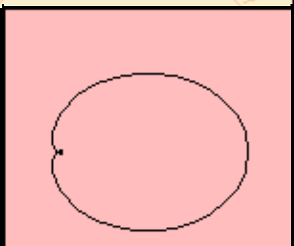
last updated: 2002-03-25

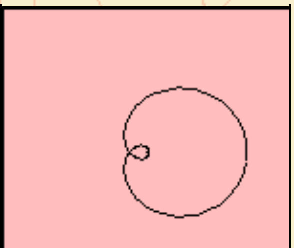
$r \sinh a\phi = 1$ or $r \cosh a\phi = 1$


Poinsot's spiral (sin) a=1/2


Poinsot's spiral (sin) a=1


Poinsot's spiral (cos) a=1/2


Poinsot's spiral (cos) a=1.8


Poinsot's spiral (cos) a=1

The curve's equation is a cosine or sinus hyperbolic function in the radius and the angle.

sinusoidal spiral

last updated: 2003−05−05

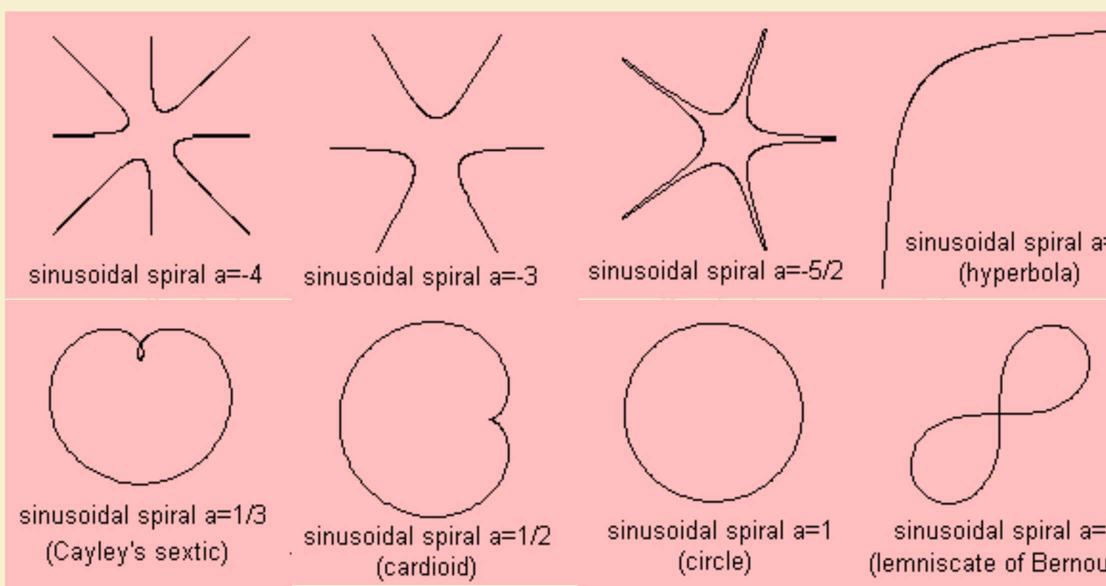
$r^a = \sin a\phi$

The relation between the power of the radius and the angle Φ has the form of a sine. Some authors confine the parameter a to a rational number.

For many values of a the spiral coincides with another curve:

a	*curve*
− 2	hyperbola
− 1	line
− 1/2	parabola
− 1/3	Tschirnhausen's cubic
0	logarithmic spiral
1/3	Cayley's sextic
1/2	cardioid
1	circle
2	lemniscate of Bernoulli

The smaller parameter a the more the curve has a spiral form.



There are several operations that keep the sinusoidal spiral intact:

- the isoptic of a sinusoidal spiral is again a sinusoidal spiral
- it is easy to see that the polar inverse of a sinusoidal spiral with parameter a is the sinusoidal spiral with parameter −a
- the pedal of a sinusoidal spiral (with the center as pedal point) is again a sinusoidal spiral

It was the Scotsman *Colin MacLaurin* (1718) who was the first to study these group of curves.

The curve can be generalized to the Cassinian curve.

trigonometric curve

last updated: 2003−05−18

Literally trigonometry does mean the measurement of angles [1]. *Euler* was the first who defined (1749) ***the trigonometric function***s as proportions of lines. He was also the one who introduced the notation for these functions.
But long for the starting of the era scholars in the old Greece and in India where already studying the trigonometric functions. In the Middle Ages (800 AD) the functions have been studied intensively by Arab speaking mathematicians. All used the functions as cords, it took until the 18th century that the current view was adopted.

The basic trigonometric functions are:

- cyclometric function
- secant
- sine
- tangent

Some more advanced trigonometric functions are :

- catastrophic sine
- cosine integral
- damped sine
- Fresnel integral
- handwriting curve
- Legendre trigonometric
- meander curve
- quadratrix
- quadratrix of Abdank−Abakanovicz
- sine integral
- sine summation

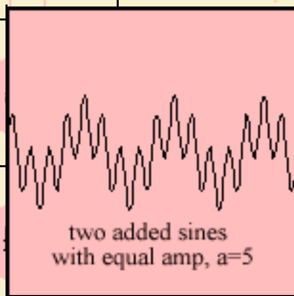
---

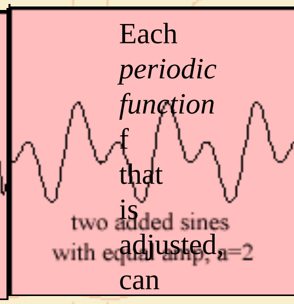
notes

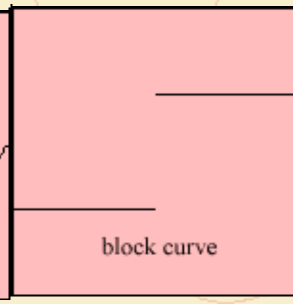
1) goonia (Gr.) = angle, metreoo (Gr.) = to measure

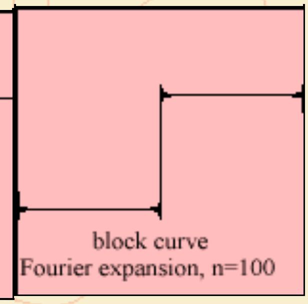
sine summation

last updated: 2002−03−25









Each *periodic function* f that is adjusted, can be written as a infinite sum of sine and cosine terms. It was *Fourier* (1768−1830) who was the first to realize this, so that this infinite sum is called a Fourier series [1]. This vanished the difference between *function* and *curve*: each function has a curve, and for each curve there is a function (its Fourier expansion).
For odd f, the expansion contains only sine terms (sine series), for even f only cosine terms (cosine series).
The function f need not to be periodic. In that case we take into account the interval [a, b], to be seen as the period of the curve.

While a function has convergence in each point, differences are possible between the graph of a function and its Fourier expansion. According to the *effect of Gibbs*, at a discontinuity the amplitude is too large, for the Fourier expansion. This is illustrated in a picture for a block curve and its Fourier expansion after 100 terms.

An easy example of adding sines is when there are two sines with same amplitude. This is the situation of two harmonic oscillations with the same power. The patterns of the sum are periodic when the ratio of the two wave lengths (a) is rational.
A musical tone is a combination of a fundamental tone (with frequency f) with its overtones (with frequencies n*f). The proportion of the amplitudes makes the 'timbre' of the tone.

---

notes

1) The Fourier expansion for a function f with period 2 Π:

$$y = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

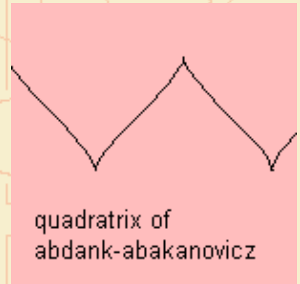
According to Fourier theory the terms a and b can be expressed in an integral in f and cos(nx) and sin(nx), respectively.

quadratrix of Abdank−Abakanovicz

last updated: 2002−11−02

$x^2 + y'^2 = 1$

The curve has been studied by the Russian engineer *Abdank−Abakanovicz* (1852 − 1907), in 1889.



The curve gives the area under a circle, as function of x, so that the curve is also called the **circular integral**.

The curve can be constructed by the 'intergraph of Abdank − Abakanovicz', an instrument used for the construction of curves.

cyclometric function

trigonometric

last updated: 2002−03−25

The cyclometric functions[1] are the inverse functions of some basic trigonometric functions. As a trigonometric function gives the length of a arc as function of an angle, the inverse cyclometric function gives the angle as function of an arc. That's why the curves are arc functions:

- arc cosecant
- arc cosine
- arc cotangent
- arc secant
- arc sine
- arc tangent

---

notes

1) Cyclometric function: a description used in the Netherlands

catastrophic sine

trigonometric

last updated: 2002−03−25

$$y = (x + a)\sin x$$

When on a system an oscillation is exerted, and there is no extinction, the amplitude of the oscillation gets larger and larger. The oscillation is amplified more and more. Until... the catastrophe, so that we call the curve a ***catastrophic sine***.

The differential equation of the phenomenon [1)] leads to above equation.

---

notes

1) Kuipers 1966 p. 357. Differential equation:

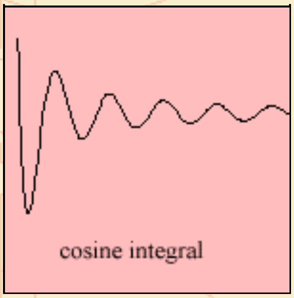
$$y'' + \alpha^2 y = \beta \cos(\alpha t)$$

cosine integral

last updated: 2002−03−25

$$y = \int_x^\infty \frac{\cos t}{t} dt$$

The ***cosine integral*** Ci(x) is the integral of cos(x) / x. Often a minus−sign is added to the formula.



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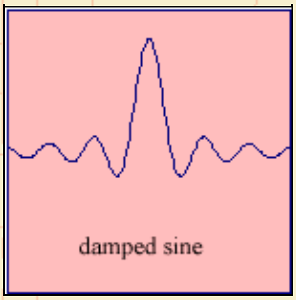
damped sine **trigonometric**

last updated: 2002−03−25

$$y = \frac{\sin x}{x}$$

A harmonic oscillation for which the amplitude diminishes proportionally with time is called a ***damped sine***.
The function is not periodic.

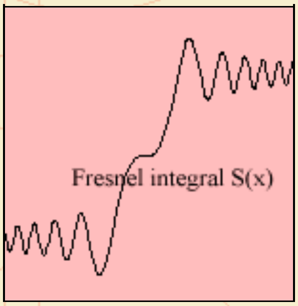


Fresnel integral

last updated: 2003−04−20

$$y = \int_0^x \sin(t^2)\,dt$$

The Fresnel integral plays a role in diffraction theory. Two kinds of the integral are distinguished: the Fresnel integral S(x) with a sine in the integral, and the Fresnel integral C(x) with the cosine in the integral.



Most authors have a term $\pi t^2 / 2$ instead of t^2 in the integral.

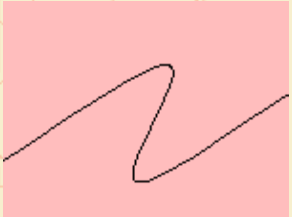
The form under the integral is the Bessel function $J_{1/2}$.

handwriting curve

last updated: 2002−11−03

$$\begin{cases} x = t - a \sin t \\ y = 1 - a \sin t \end{cases}$$

I found the curve as a variation on the cycloid. For the time being, I call the curve the **handwriting curve**.



handwriting curve a=2

Joukowski curve

trigonometric

last updated: 2002−12−14

$$z = \zeta + a/\zeta$$

where z and z are in complex notation and z is on a circle, with given radius and center.

Let (c, d) be the center of the circle with radius b, then the curve can be written in parametric notation as [1]:

$$\begin{cases} x = c + b\cos(t) + \dfrac{a^2(c + b\cos(t))}{(c + b\cos(t))^2 + (d + b\sin(t))^2} \\ y = d + b\sin(t) + \dfrac{a^2(d + b\sin(t))}{(c + b\cos(t))^2 + (d + b\sin(t))^2} \end{cases}$$

The curve is used for studying airfoil profiles. *James* and *James* (1992) state that then the point z=1 has to be inside the circle.

Some typical Joukowski forms are the following:

Joukowski curve
a=.3, b=.5, c=1.4, d=1.2

Joukowski curve
a=.2, b=.2, c=1.2, d=1

---

notes

1) let z = (x, y) = (c + b cos(t), d + b sin(t)), then work out the expression for z.

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generalized Lissajous curve

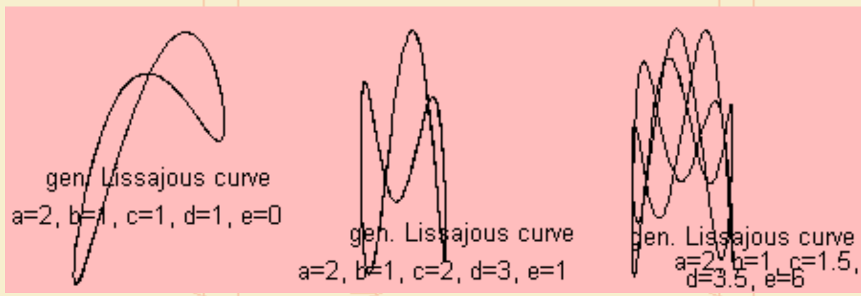
trigonometric

last updated: 2002−12−28

$$\begin{cases} x = \sin t \\ y = \sin(at+b) + c\sin(dt+e) \end{cases}$$

The curve is a generalization of the Lissajous curve.

As you can understand, a wide variety of curves are possible. On this page I show some of them.



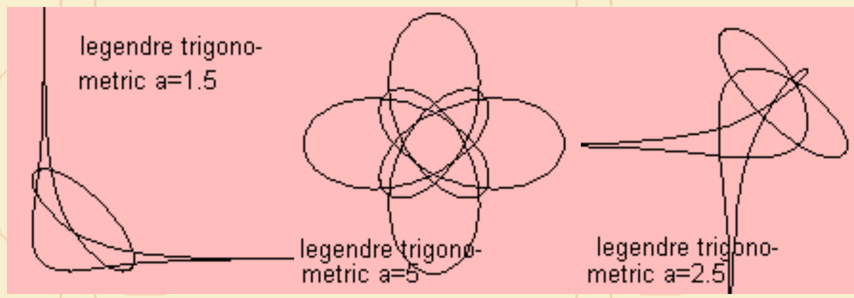
Legendre trigonometric

$$\begin{cases} x = P_a(\cos t) \\ y = P_a(\sin t) \end{cases}$$

where P_a is the Legendre polynomial of degree a.

For time being, I call the curve the Legendre trigonometric, any other suggestions are welcome.

Some examples pictures:



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meander curve

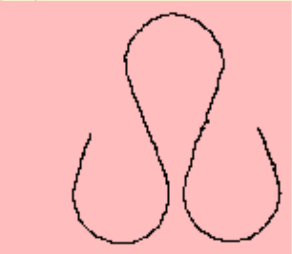
trigonometric

last updated: 2002−04−20

$$y' = a \sin s$$

where s is the arc length [1].

Luna Leopold and *W.B. Langbein* used the curve to approximate the sinuosity of river channels [2]. In other words: they found that the angular direction of the channel (with respect to the mean down valley direction) is a sine function of the distance measured along the channel.
For this reason they called the curve a **sine−generated curve**.
The characteristic repeating pattern of a rive is called a 'meander', what leads to the name of **meander curve**.



meander curve a=2

It appears that this meander curve has the smallest variation of changes of direction, in comparison to other curves with a meander−like form.

When you hold a thin strip of spring steel firmly at two points, and allow the length between those points to assume an unconstrained shape, the same curve form results.

---

notes

1) where

$$s = \int \sqrt{1 + y'^2}\,dx$$

2) Scientific American of June 1966

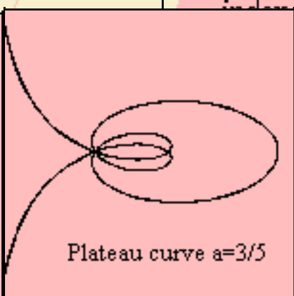
Plateau curve

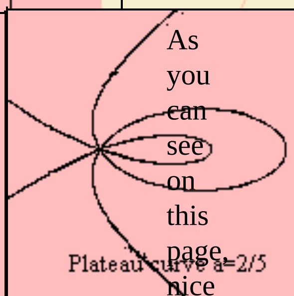
trigonometric

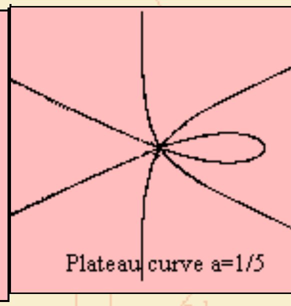
last updated: 2002−03−25

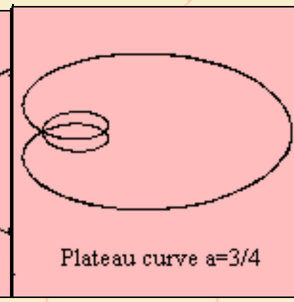
$$\begin{cases} x = \dfrac{\sin(1+a)t}{\sin(1-a)t} \\ y = \dfrac{\sin t \sin at}{\sin(1-a)t} \end{cases}$$

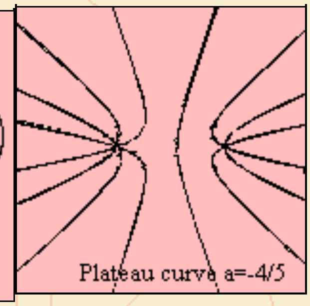
These curves were studied by the Belgium *Joseph Plateau*.









As you can see on this page nice pictures appear for different values for a.
For a = 1 the Plateau curve has no meaning.
The Plateau curve with parameter 1/a is equal to the Plateau curve with parameter a.
So we can confine ourselves to values for the parameter a between −1 and 1.
For a = 1/2, the Plateau curve degenerates to an ellipse, for a = − 1/2 to a hyperbola.
For a = 0 a straight line results.

When the parameter 'a' is a positive rational number, the curve consists of a number of ellips−like forms. Give the parameter 'a' its simplest form, then the number of elliptic forms is equal to the numerator of 'a'.
The denominator of 'a' minus one gives the number of elliptic and asymptotic forms.

When the parameter 'a' is a negative rational number, the Plateau curve has the form of a number of hyperbolas. How many hyperbolas? Give the parameter 'a' its simplest form, then the number of hyperbolas is equal to the sum of the numerator and the denominator minus one. Remarkable.

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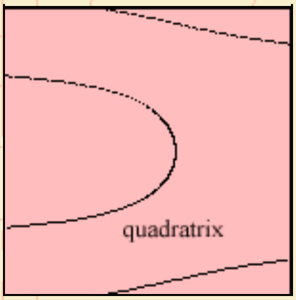
quadratrix

trigonometric

last updated: 2002−03−25

$$y = \frac{x}{\tan x}$$



The ***quadratrix*** can be defined as the intersection of two moving lines. The first line rotates (for example counter clockwise), the second line moves along (for example in the direction of the positive y−axis). Both movements are with constant velocity. Can you imagine?

It was the Greek *Hippias of Ellias* (or *Elis*), who used the curve (about 420 BC) for the trisection of an angle. So that the curve is also called the ***quadratrix of Hippias***.
This *Hippias* was a traveller, being a statesman and a philosopher; *Plato* describes him as a vain and boastful person.

Deinostratos (about 350 BC) and *Nicomedes* used the curve [1)] while working on the problem of squaring the circle. That where the curve's name is from.

Funny is that the curve's equation can be expressed in the mixed Cartesian−polar form: y = Φ.

It is the polar inverse of the cochleoid.

---

notes

1) according to *Pappos*.

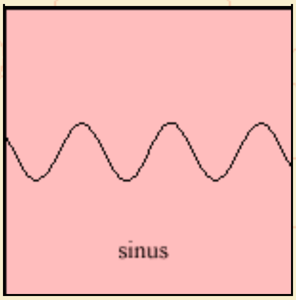
2) See at the Archimedes' spiral for a short exposure on these Classical problems.

sine

trigonometric last updated: 2003−05−18

$y = \sin x$

In the antiquity the sine was used as the length of a chord. Given a circle with a certain radius, for a large amount of values of the angles the lengths of chords were written down [1]. Such cord tables can be found in Ptolemy's' Almagest (150 AC) and in the Indian Surya Siddanta, and were used especially by astronomers.



In the late Middle Ages there was a short period that one used to write the tangent as sixtieths of the radius, and sixtieths of sixtieths. It was the German astronomer *Regiomontanus* (*Johannes Muller*, c. 1450*)*, who prevented this practice. He wrote the trigonometric functions as lengths of chords for a circle with a radius of 10^n (n = 5 − 15), so that the results could be written as whole numbers.

It was not earlier than the 19th century that the sine got his actual meaning as the proportion of line segments. Geometrically the sine can be defined, in a rectangular triangle (Euler) as the proportion of the opposite side to the hypotenuse. Another way to define the curve is as a power series [2].
A sine curve is also called a ***sinusoid***.

A point moves with constant speed on the circumference of a circle. Take as center of the circle the origin, then the coordinates of the moving point as function of the rolled angle a are: (cos(a), sin(a)). So the component along an arbitrary diameter is a sine function of time. This is the definition of a *harmonic oscillation*. Necessary for this movement is a force, which is proportional to the movement: $F = -k x$.
The sine obeys this rule, because for its second derivative applies: $y'' = -y$.

In fact, the harmonic oscillation is the form in which we meet the sine in our physical world.
A moving harmonic oscillation gives as result a sinusoid wave. The periodicity of the sine is called the *wavelength*, the amplitude is the factor of sin x.
Imagine a triangle, the *sine formula* states that the ratio of the sine of an angle and the opposite side is equal for all three angles.
An ***equalized sine*** is the result when the negative values have been turned: $y = |\sin x|$.

Sine means bend, curve, bosom. It is a literal translation of the Arab *gaib*, what has been derived from *gib*, a way to spell the Indian *jya* (cord). So there is a relation with the old habit of using the sine as the length of a cord. In natural life we see the sine form in the bosom of a wife.
In screwing activities we make a three dimensional ***helical line***, when projecting the line on a surface through the screw's axis the result is again a sine.
In the bicycle−loving country of Holland, research led to the conclusion that a street threshold (to limit the speed of the cars) in the form of a sine is the most comfortable form for bicycling.

The ***cosine*** is defined − in the same rectangular triangle − as the proportion of the adjacent side to the hypotenuse.
The cosine is a translated sine: $\cos x = \sin (x + \Pi/2)$.
The cosine formula is an extension of the theorem of Pythagoras. For a triangle without a rectangular angle, it states that $a^2 = b^2 + c^2 - 2bc \cos \Phi$ [3].
Directly derived from the sine are the ***versine*** and the ***haversine***. The versine is defined as $\text{vers}(x) = 1 - \cos(x)$; the haversine is defined as $\text{hav}(x) = 1/2 \text{ vers}(x)$.

The inverse functions of the sine and the cosine are called the ***arc sine*** arcsin(x) and the ***arc cosine*** arccos(x), respectively. These are cyclometric functions.

Sine series and *cosine series* are obtained while adding sines or cosines, in the sine summation.

---

notes

1) The length of a cord can be written in our current notation as:
$\text{cord}(a) = 2 R \sin (a/2)$

2) The sine as a power series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

3) The angle Φ is the angle opposite line a.

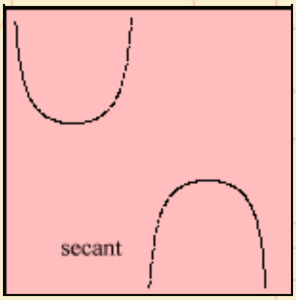
secant

trigonometric

last updated: 2002−03−25

$$y = \frac{1}{\sin x}$$

As the secant sec(x) is defined as the divider of the cosine, the ***cosecant*** csc(x) is that of the sine. These curves are equal under translation.



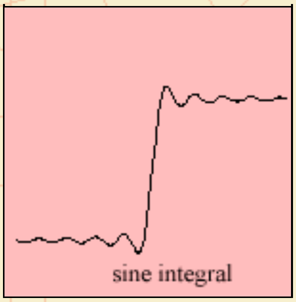
The inverse functions of the secant and the cosecant are called the ***arc secant*** arcsec(x) and the ***arc cosecant*** arccsc(x), respectively. These are cyclometric functions.

sine integral trigonometric

last updated: 2002−03−25

$$y = \int_0^x \frac{\sin t}{t}\,dt$$

The ***sine integral*** Si(x) is the integral of the damped sine.



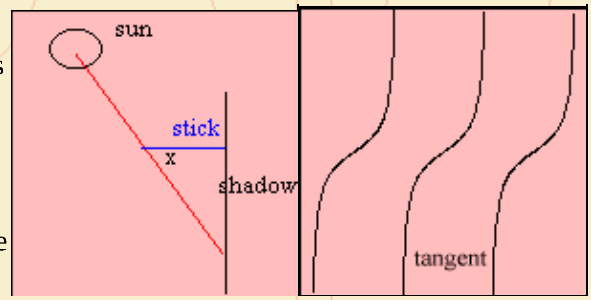
tangent

trigonometric

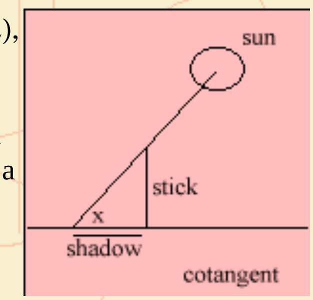
last updated: 2002−03−25

$$y = \frac{\sin x}{\cos x}$$

The tangent can geometrically be defined, in a rectangular triangle, as the proportion of the opposite and the adjacent side. So that tan(x) = sin(x) / cos(x). It has vertical asymptotes for cos(x) = 0.
The old Arabs scholars could derive the height (angle) x of the sun as function of the shadow of a horizontal stick on a vertical plane: they called the shadow, which length is a factor tan(x) of the stick: *umbra versa* [1].



The tangent rule states a relationship between tan((α + β)/2), tan ((α – β)/2) and the sides a and b.



The ***cotangent*** is defined as cot(x) = 1 / tan(x). It is evident that the curve is not more than a tangent that is mirrored in a vertical [2].
The Arab astronomer *Al−Battani* (900 AD) discovered the cotangent while studying shadows [3]. He put a stick in the sand, vertically. He took the stick's shadow as a measure for the height of the sun. The shadow was called the *umbra recta* or the *umbra extensa* [4]. Its length is a factor cot(x) of the length of the stick, where x is the angle the sun is to be seen.

The cotangent rule gives cot(α/2) as function of the sides a, b and c.

The inverse functions of the tangent and the cotangent are called the ***arc tangent*** arctan(x) and the ***arc cotangent*** arccot(x), respectively. In the Netherlands these functions are called the cyclometric functions.

---

notes

1) Umbra (Lat.) = shadow, versa (Lat.) = turned.

2) It is easy to see that cot(x) = − tan(x + Π / 2)

3) Dijksterhuis 1977, p. 302.

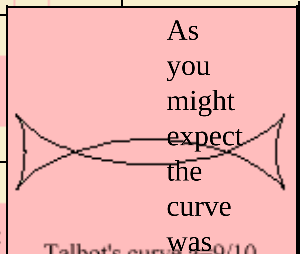
4) Rectus (Lat.) = right, extensa (Lat.) = extended.

Talbot's curve

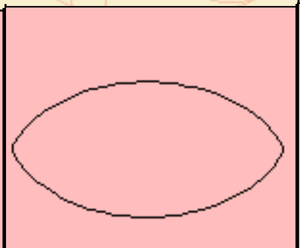
trigonometric

last updated: 2002−03−25

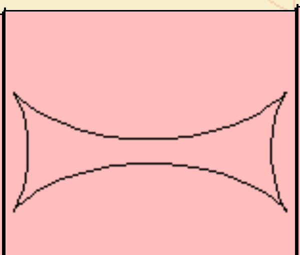
$$\begin{cases} x = (1 + a \sin^2 t) \cos t \\ y = (1 - a - a \cos^2 t) \sin t \end{cases}$$



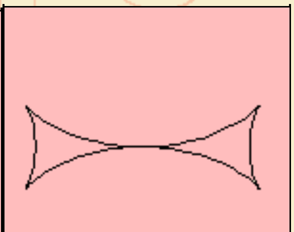
Talbot's curve a=9/10



Talbot's curve a=1/2



Talbot's curve a=11/10



Talbot's curve a=1

As you might expect the curve was investigated as first by *Talbot*.

The pedal of the curve is the ellipse.

The traditional form of the curve is the first shown. Then the parameter a lies between ½ and 1.

barycentric

last updated: 2002−05−20

$$\begin{cases} x = \int_C x\, ds / \int_C ds \\ y = \int_C y\, ds / \int_C ds \end{cases}$$

where the integral over ds is the path along a curve C.

Given an (homogenous) arc of a curve C, its **barycentric** curve gives the center of gravity of that arc.

curve C	**barycentric of C**
circle	cochleoid
line	line with half the slope of C

caustic

last updated: 2003−02−22

Given a curve C and a point source S emitting light rays. The envelope of the rays, after reflection at C, is called a ***catacaustic***.

In the case of refraction the envelope is a ***diacaustic***, also named ***orthotomic*** or ***secondary caustic***.

The word **caustic** can denote the **catacaustic** as well as both the **diacaustic** and the **catacaustic**.

The **catacaustic** of the logarithmic spiral is the curve itself. For a cycloid this is half true: for a cycloid arch, rays perpendicular to the x−axis result in two cycloid arches.

Some other catacaustic curves are:

curve	***source***	***catacaustic***
cardioid	cusp	nephroid
circle	on circumference	cardioid
	not on circumference	limaçon
	at infinity	nephroid
cissoid of Diocles	focus	cardioid
deltoid	infinity	astroid
parabola	infinity, rays perpendicular to the axis	Tschirnhausen's cubic
Pritch−Atzema spiral	center	circle
quadrifolium	center	astroid
Tschirnhausen's cubic	pole	semi−cubical parabola

The caustic can be generalized for different refraction indices (n_1, n_2) at the two sides (same and opposite side of S, in relation to the tangent). This curve is called the **anticaustic**.
For n (n_1/n_2) = 1 the **orthotomic** results.
It is said that the orthotomic can be identified with a cyclic.

Some anticaustic curves are the following:

curve	***source***	***anticaustic***
circle	on circle	Cartesian oval

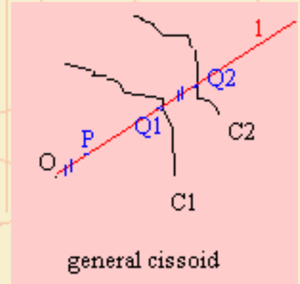
straight line	symetric on line	ellipse (n < 1)
	symetric on line	hyperbola (n > 1)

The anticaustic curves has been studied by *Mannheim*.

(generalized) cissoid

last updated: 2002-12-25

Given curves C1 and C2 and a fixed point O. Draw lines l through O that intersect C1 in Q1, and C2 in Q2. Then the ***cissoid*** is defined as the collection of points P (on l) for which OP is equal to the distance between the intersections (Q1Q2).



Some special cases of the cissoid:

curve C1	***curve C2***	***pole O***	***cissoid***
line	line parallel to C1	any point	line
	circle	center of the circle	conchoid (of Nicomedes)
circle	concentric circle	center	circle
	line tangent to the circle	on circle, opposite the tangent	cissoid (of Diocles)
	line tangent to the circle	on circle	***oblique cissoid***
	radial line	on circle	(right) strophoid
	circle of equal size	any point	hippopede

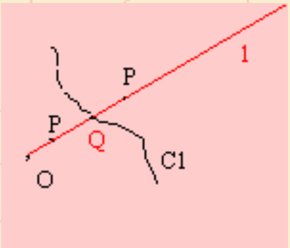
The first cissoid to be discovered was the cissoid of Diocles.

(generalized) conchoid

main

last updated: 2003−01−03

Given a curve C1 and a fixed point O. Draw lines l through O that intersect C1 in Q. Then the ***conchoid*** is defined as the collection of points P (on l) for which PQ is equal to a constant a.



general conchoid

This general conchoid is obtained from the 1st curve that has been named the conchoid, the conchoid that is derived from the straight line.

In the following table some conchoids have been collected:

C1	**O**	**conchoid of C1**
Archimedes' spiral	center	Archimedes' spiral
circle	on C1	limaçon
	on C1 (constant 1 equal to diameter of C1)	cardioid
curve with polar equation r = f(ϕ)	the origin	curve with polar equation r = f(ϕ) + a
rose	center	botanic curve
straight line		(simple) conchoid

The conchoid of Dürer does not belong to this family, it is another variation on the conchoid.

curvature curve

last updated: 2002−03−25

There are four derived curves that are related to curvature:

- evolute
- involute
- radial
- the curvature itself

Given a curve C1, the ***evolute*** is the curve C2 defined by the loci of the centers of curvature of C1. In other words: construct in each point P of curve C1 a circle that is a tangent to C1 in P; then the center of the circle belongs to C2.

When C1 is given by (x, y) = (f(t), g(t)), then C2 has the form:

$$\begin{cases} x = \dfrac{f(t)\, g'^2(t) - g(t)\, f'(t)\, g'(t)}{f'(t)g''(t) - f''(t)g'(t)} \\ y = g(t) + \dfrac{(f'^2(t) + g'^2(t))\, f'(t)}{f'(t)g''(t) - f''(t)g'(t)} \end{cases}$$

If a line l rolls without slipping as a tangent along a curve C1, then the path of a point P on l forms a new curve C2, the ***involute*** of C1. Involution is the reverse operation of evolution: if C2 is the involute of C1, then C1 is the *evolute* of C2. You might ask yourself whether there exists a curve whose involute is exactly the same curve. Well, there are two curves with this property:

- cycloid
- logarithmic spiral

Besides, there are some curves whose involute is the same curve, but not equal in position or magnitude:

- epicycloid
- hypocycloid

Some other involute−evolute couples are:

involute	***evolute***
ellipse	astroid
epicycloid	epicycloid (similar, but smaller)
involute of a circle	circle
tractrix	catenary
roulette: when a line is rolling over curve C, the path	any curve C

of any point on the line.	

The ***radial*** is a variation on the *evolute*: draw, from a fixed point, lines parallel to the radii of curvature, with the same length as the radii. The set of end points is the radial. The logarithmic spiral is the curve whose radial is the curve itself.

Radials of some other curves are:

base curve	***radial***
astroid	quadrifolium
catenary	kampyle of Eudoxus
Cayley's sextic	nephroid
cycloid	circle
deltoid	trifolium
epicycloid	rhodonea
parabola	semi cubical parabola
tractrix	kappa curve

Given a curve, the **curvature** κ is defined as the inclination per arc length: dφ/ds.

For a curve y = f (x) the curvature is given by: $$\kappa = \frac{f''(x)}{(1+(f'(x))^2)^{3/2}}$$

For Euler's spiral the curvature κ is linearly related to the arc length s.

The **radius of curvature** ρ is the reciprocal of the absolute value of the curvature κ. The Cesaró equation writes a curve in terms of a radius of curvature ρ and an arc length s.

For the logarithmic spiral the radius of curvature ρ is equal to the arc length s.

cyclic

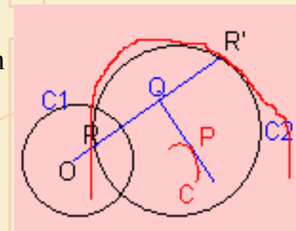
last updated: 2003−03−11

$$r^2 - 2rf(\phi) + a = 0$$

where r=f(ϕ) is the polar equation of the pedal of the curve to be derived.

Given a curve C and a fixed point O.
Let O be the pedal point, then select a point P of curve C and draw its pedal Q.
Then construct points R and R' on OQ for which OR*OR' = a, and OR+OR'=2OQ.
The collection of the points R and R' form the **cyclic** of C.

Graphically you can see that the construction can be seen as in inversion: define the inversion circle C1 as the circle with center O and radius √a. Then construct a circle C2 with center P, which is orthogonal to the inversion circle C1.
Then the envelope of the circles C2 is the cyclic of curve C.
The parameter a is called the power (of inversion).



construction of cyclic

Parameter 'a' being unequal zero, the cyclic is an anallagmatic curve.
For parameter 'a' being zero, the cyclic is said to be equal to an orthotomic. Then it is called rational.

Some cyclic relations are the following:

curve C1	center of inversion	cyclic of C1
conic	center	bicircular quartic
hyperbola	center	ovale of Cassini
parabola	focus	circular cubic

Generalizing the notion to a surface you get the **cyclid**.

derivative and integral

last updated: 2001−11−05

Given a curve C1, the ***derivative*** is the curve C2 defined by the derivation of the y component to the x component. So the y component of C2 gives the slope of C1.
When the curve is not smooth, or not a function, special effects can occur.

Suppose C1 is given by (x, y) = (f(t), g(t)) then C2 is given by y(x) = g '(x) / f '(x).

When the curve is formulated in polar coordinates, a polar pendant of the derivative is: dr/dΦ.This is then the slope in r as function of Φ.

The reverse of differentiation is the operation of integration:
Given a curve C1, the ***integral*** is the curve C2 defined by the integration of the y component to the x component. So the y component of C2 gives the cumulated area between C1 and the x−as.

When C1 is given by (x, y) = (f(t), g(t)), then C2 has the form:

$$y(x) = \int_{x_0}^{x} g(t) f'(t)\, dt$$

In polar coordinates, integration is over a sector between angles Φ0 and Φ.

envelope

last updated: 2003−05−20

An envelope of a set of curves is a curve that is tangent to these curves.

Examples are the following:

- an astroid is the envelope of straight lines
- an astroid is also the envelope of co−axial ellipses for which the sum of the major and minor axe is constant

generalized tractrix

last updated: 2002−03−29

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x-x_0}{y-y_0} \\ (x-x_0)^2 + (y-y_0)^2 = 1 \\ C_1(x_0, y_0) = 0 \end{cases}$$

Given a point P which is fixed to a point Q with an inextensible stick. When P follows a path according curve C1, then Q will follows as path a curve a C2, defined as the **tractrix** of C1.

Examples in reality are the back wheels of a car (or bike). Then the path of the back wheels is a tractrix of the path of the front wheels.

Some authors call the curve of the dragged object the **tractory** of curve C, reserving the name **tractrix** for C1 being a straight line.
The generalized tractrix on this page is a generalization of this straight line tractrix.

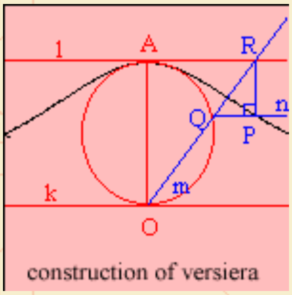
hyperbolism

last updated: 2002−03−31

Given a curve C1, a point O, and a line l, then the ***hyperbolism*** of C1 is the curve C2 formed by the points P as follows:

- let Q be a point on C1, then line m through OQ intersects l in R
- let n be a line parallel to line l through Q
- then P is the projection from R on m

Look at the witch of Agnesi to illustrate the construction:



We can normalize the hyperbolism to a curve C1 with respect to a point O and a line x = 1.
In this case holds that when C1 has equation f (x,y) = 0, the hyperbolism of C1 has as equation: f (x, xy) = 0.
The term xy makes the link to the hyperbola.

The inverse of the hyperbolism is the **antihyperbolism**: when C2 is the hyperbolism of C1, then C1 is the antihyperbolism of C2.

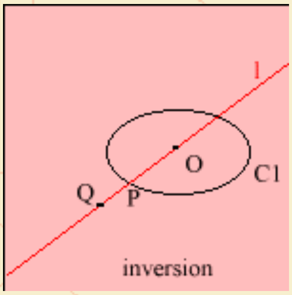
Some examples are the following:

C1	O wrt C1	l wrt C1	l wrt C2	O wrt C2	C2
circle	on circle	parallel to diameter through O	−	center	serpentine
circle	on circle	tangent on opposite side of O	parallel to asymptote	center	witch of Agnesi

inverse

last updated: 2003−03−04

When a curve C1 is given by a Cartesian function y = f(x), the inverse function is defined as the function g for which g(f(x)) = x. The corresponding ***inverse*** curve C2 has the same form as C1, only the x− and y−axis are interchanged.



A more interesting case is polar inversion, where each point is inverted along the line through the center of inversion. Given a curve C1, draw a line l through O. Line l intersects C1 in a point P. Now construct points Q of C2 so that OP * OQ = 1.

When curve C1 has been defined as r = f(Φ), then the ***polar inverse*** − with O as the center of inversion − is a curve C2, which has as polar equation: r = 1 / f(Φ). When C2 is the inverse of C1, then C1 is the inverse of C2.
Some curves have several curves inversely related to them. Each inverse then has a different center of inversion.

The first mathematician who discussed the curves was *Steiner* (1824).

A curve which is invariant under inversion − given a certain point of inversion − is called an ***anallagmatic curve*** [1]. Examples include:

- cardioid
- Cartesian oval
- Cassini oval
- circle (center of inversion not on circle)
- logarithmic spiral (pole)
- limaçon
- sinusoidal spiral (pole)
- strophoid (pole)
- trisectrix of MacLaurin

Moutard introduced the notion, in 1860.

An anallagmatic curve can be identified with a cyclic.

Other interesting inverse relations are the following.

curve 1	*center of inversion (curve 1)*	*center of inversion (curve 2)*	*curve 2*
Archimedean spiral (parameter a)	pole	pole	Archimedean spiral (parameter −a)
cissoid (MacTutor: cardioid)	cusp	vertex	parabola
cochleoid	pole	−	quadratrix
ellipse	focus	pole or node	ordinary limaçon
epi spiral	pole	pole	rhodonea
	pole	pole	lituus

Fermat's spiral			
hyperbola	focus	pole or node	limaçon (with a noose)
line	not on line	on circle	circle
parabola	focus	cusp	cardioid
rectangular –	center	center	lemniscate
rectangular –	vertex	node	(right) strophoid
asymptote angle: π/3	vertex	node	trisectrix of MacLaurin
sinusoidal spiral (parameter a)	pole	pole	sinusoidal spiral (parameter −a)
trisectrix of MacLaurin	focus	–	Tschirnhausen's cubic

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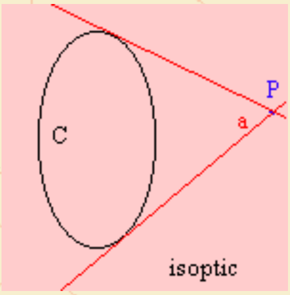
notes

1) Without change, from allagma (Gr.) = change.

isoptic

last updated: 2003−05−20

Let's have a curve C. Vary two tangents to the curve in such a way that the angle between the two lines is a constant angle. Then the intersections of the tangents form the ***isoptic*** of C.
The sinusoidal spiral is the curve for which the isoptic is also a sinusoidal spiral.



Some isoptic curves are the following:

curve	*isoptic*
cycloid	trochoid
epicycloid	epitrochoid
hypocycloid	hypotrochoid
parabola	hyperbola

When the constant angle between the tangents has the value Π / 2, the curve is called an ***orthoptic***.
For the logarithmic spiral its orthoptic is equal to itself.

Some other orthoptic curves are:

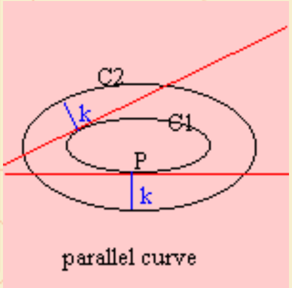
curve	*orthoptic*
astroid	quadrifolium
cardioid	limaçon
deltoid	circle
ellipse	
hyperbola	
parabola	directrix

La Hire (1704) was the first to mention the curve.

Another curve with isoptic properties is the isoptic cubic.

parallel curve

last updated: 2001−11−05



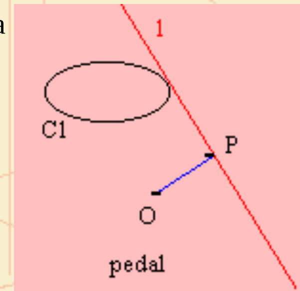
Construct for each point P on a given curve C1 two points Q1 and Q2, at a distance k from the point P. The points Q1 and Q2 form ***parallel curve***s of C1.

Leibniz was the first author to mention the curves (1692).

pedal curve **main**

last updated: 2002−07−13

$$\begin{cases} x = \dfrac{f(t)\, g'^2(t) - g(t)\, f'(t)\, g'(t)}{f'^2(t) + g'^2(t)} \\ y = \dfrac{g(t)\, f'^2(t) - f(t)\, f'(t)\, g'(t)}{f'^2(t) + g'^2(t)} \end{cases}$$



Given a curve C1 and a (pedal) point O, construct for each tangent l of C1 a point P, for which OP is perpendicular to the tangent. The collection of points P forms a curve C2, the ***(positive) pedal*** of C1 (with respect to the pedal point).
When C1 is given by (x, y) = (f(t), g(t)), and we translate C1 in such a way that the pedal point is the origin, then C2 has the form:

Two curves are invariant for making a pedal:

- the sinusoidal spiral (pedal is unequal)
- the pedal of a logarithmic spiral is an equal logarithmic spiral

$$y^2 = x^2 \frac{a + x}{1 - x}$$

The pedal of the parabola is the curve given by the equation:

a	*pedal point*	*pedal of the parabola*
0	vertex	cissoid (of Diocles)
1	foot of − intersection of axis and − directrix	(right) strophoid
3	reflection of focus in directrix	trisectrix of MacLaurin
−	on directrix	oblique strophoid
−	focus	line

Some other pedals are:

curve	*pedal point*	*pedal*
astroid	center	quadrifolium
cardioid	cusp	Cayley's

		sextic
circle	any point	limaçon
	on circle	cardioid
cissoid (of Diocles)	focus	cardioid
deltoid	cusp	simple folium
	vertex	bifolium
	center	trifolium
ellipse	focus	circle
epicycloid	center	rhodonea
hyperbola	focus	circle
rectangular hyperbola	center	lemniscate
hypocycloid	center	rhodonea
involute of a circle	center	Archimedes' spiral
line	any point	point
parabola	focus	line
Talbot's curve	center	ellipse
Tschirnhausen's cubic	focus	parabola

The reverse operation of making a pedal is to construct from each point P of C2 a line l that is perpendicular to OP. The lines l together form an envelope of the curve C1. Now we call C1 the ***negative pedal*** [1)] of C2. When C1 is a pedal of C2, then C2 is the negative pedal of C1.
Because of this definition, the curve is in fact also an **orthocaustic**: the orthocaustic of a curve C1 (with respect to a point O) is the envelope of the perpendiculars of P on OP (P on C1).

Instead of tangents to a curve we can consider normals to that curve. This pedal curve is called the ***normal pedal curve***.

MacLaurin was the first author to investigate pedal curves (1718).

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notes

1) In French: antipodaire. In German: Gegenfusspunktskurve.

pursuit curve general

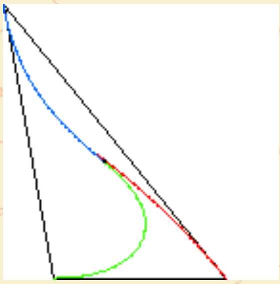
last updated: 2001−10−28

Given a point P that moves (with constant velocity) along a curve C; then the **(generalized) *pursuit curve*** is described by a point Q that is always directed towards P.

It was *Pierre Bouguer* [1] who was the first to study the curve (in 1732).

When above mentioned curve is a straight line, we call the curve a (straight) pursuit curve.

Martin Gardner stated the classic problem of four beetles, chasing each other, initially located at the vertices of a square. This gives four logarithmic spirals to the center.
Darryl Nester worked on the problem for three beetles in a triangle, the resulting curves can be seen at the Bluffton College site as animated gif:



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notes

1) He was the French scientist who tried to measure the earth's density using the deflection of a plumb line by (the attraction of) a mountain.

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reflecting wall curve

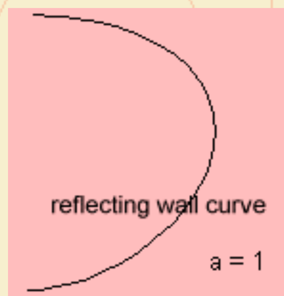
general

last updated: 2001−05−21

$$\begin{cases} x = f_a(t) \\ y = t\, f_a(t) \end{cases}$$

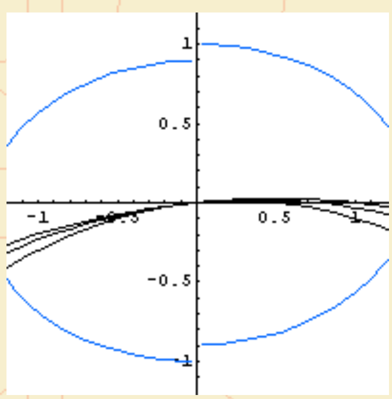
with the following $f_a(t)$:

$$\begin{cases} a^2 < 8: & \mathrm{Exp}\left[\dfrac{a}{\sqrt{8-a^2}} \arctan \dfrac{(4t-a)}{\sqrt{8-a^2}}\right] / \sqrt{1-at+2t^2} \\ a^2 = 8: & \dfrac{\mathrm{Exp}\left[\dfrac{a}{a-4t}\right]}{a-4t} \\ a^2 > 8: & \dfrac{\mathrm{Abs}\left[\dfrac{4t-a-\sqrt{a^2-8}}{4t-a+\sqrt{a^2-8}}\right]^{\frac{a}{2\sqrt{a^2-8}}}}{\sqrt{|1-at+2t^2|}} \end{cases}$$



Suppose throwing a ball from a fixed point, say the origin, at a given angle ϕ, with varying initial velocity. Then an orthogonal trajectory of the parabolic paths can be seen as the wall that would bounce (reflect) the ball back to the origin. We call this trajectory the *reflecting wall curve*. The tangent of the angle ϕ corresponds with the value a in the formulas above.

Darryl Nester worked on this curve, he constructed an animated gif to show the curve as function of the angle.



roulette

last updated: 2003−05−20

$$z(t) = C_1(t) - \frac{C_1'(t)}{C_2'(t)}(C_2(t) - z(0))$$

Given two curves C_1 and C_2 and a point P attached to curve C_2. Now let curve C_2 roll along curve C_1, without slipping. Then P describes a ***roulette*** [1].
When P is on curve C2, the curve is called a **point−roulette**. When P is on a line attached to C2, the curve is called a **line−roulette**.

When working in the complex plane and both curves C_1 and C_2 are expressed as function of the arc length, the resulting curve z can be written as a function of C_1 and C_2 [2].

The first to describe these curves where *Besant* (1869) and *Bernat* (1869), respectively.

The following roulettes can be distinguished, for a curve C_2 that is rolling over a curve C_1:

fixed curve C_1	***rolling curve C_2***	***fixed point on curve C_2***	***roulette***
any curve C	line	on line	involute of curve C
catenary	line	center of catenary	line
circle	circle (on outside)	on circle	epicycloid
	circle (on inside)	on circle	hypocycloid
line	circle	on circle	cycloid
	circle	outside circle	prolate cycloid
	circle	inside circle	curtate cycloid
	cycloid	center	ellipse
	ellipse	focus	elliptic catenary
	hyperbola	focus	hyperbolic catenary
	hyperbolic spiral	pole	tractrix
	involute of a circle	center	parabola
	involute of a circle	any point	circle
	logarithmic spiral	any point	line
	parabola	focus	catenary
parabola	(equal) parabola	vertex	cissoid

The point−roulettes for which a circle rolls on a line or on another circle, are known as **cycloidal curves**.

The same curves can be defined as a ***glissette*** [3]: as the locus of a point or a envelope of a line which slides between two given curves C_1 and C_2.
An wide−known example of a glissette is the astroid.

---

notes

1) rouler (Fr.) = to roll.
In Dutch: **rolkromme**.

2) The formula follows from the isometric insight that $C_1(t) - z(t) / C_2(t) - z(0) = C_1'(t) / C_2'(t)$.

Example for the cycloid: C1(t) = t and C2(t) = i − i e it where t is the arc length.

3) glisser (Fr.) = to glide.

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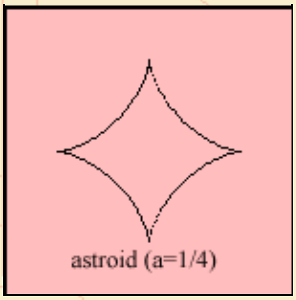
astroid

roulette

last updated: 2003−05−20

$$x^{2/3} + y^{2/3} = 1$$



It's the hypocycloid for which the rolled circle is four times as large as the rolling circle.

The curve can also be constructed as the envelope of the lines through the two points (cos t, 0) and (0, sin t). A mechanical device composed from a fixed bar with endings sliding on two perpendicular tracks is called a *trammel of Archimedes*.

The length of the this unit astroid curve is 6, and its area is 3π/8.

This sextic curve [1] is also called the ***regular star curve*** [2]. 'Astroid' is an old word for 'asteroid', a celestial object in an orbit around the sun, intermediate in size between a meteoroid and a planet.
The curve acquired its astroid name from a book from *Littrow*, published in 1836 in Vienna, replacing existing names as **cubocycloid**, **paracycle** and **four−cusp−curve**.
Because of its four cusps it is also called the **tetracuspid**.
Abbreviation for a hypocycloid with four cusps (1=1/4) led to the name of **H4**.

Some relations with other curves:

- its radial, pedal and orthoptic are all the same curve: the quadrifolium
- the curve is the catacaustic of the quadrifolium
- it is the deltoid's catacaustic
- it is the evolute of the ellipse

The first to investigate the curve was *Roemer* (1674). Also *Johann Bernoulli* (1691) worked on the curve. *Leibniz* corresponded about the curve in 1715, and in 1748 *d'Alembert* did some work on the curve.

The curve can be generalized into the super ellipse or Lamé curve.
Some authors call this generalization the astroid.

---

notes

1) In Cartesian coordinates:
$$(x^2+y^2-1)^3+27x^2y^2=0$$

2) Astrum (Lat.) = star

botanic curve

roulette

last updated: 2003−01−29

$r = 1 + d \sin c\phi$

The **botanic curve** is the conchoid of the rhodonea.
Its name is derived from her petal−like form.

Other names for the curve are:

- **geometric petal**
- **rose of Troy**

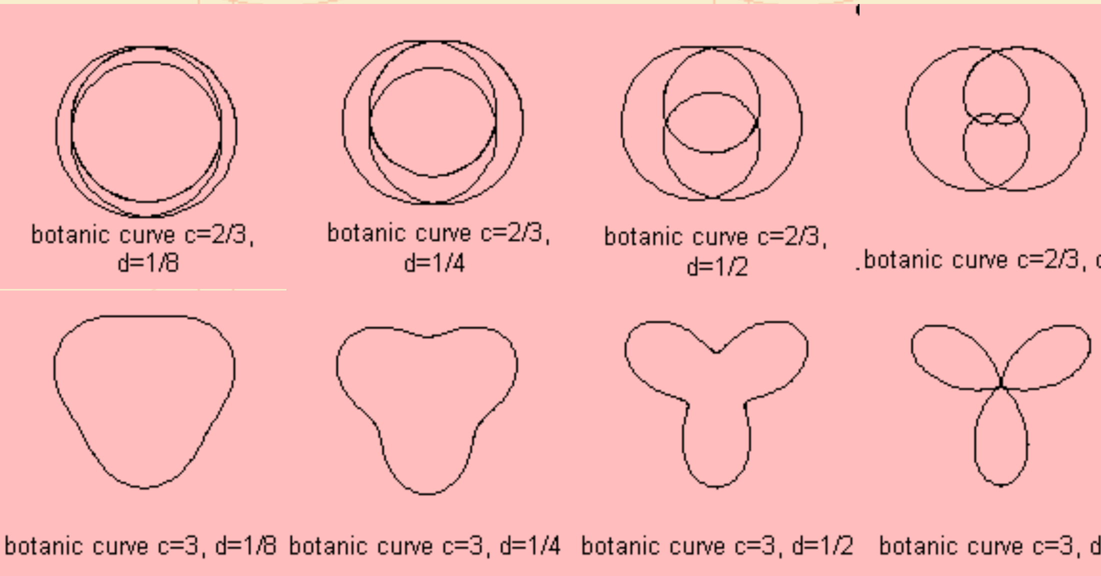
The following qualities of the rhodonea hold also for the botanic curve:

- the number of petals is the denominator of 1/2 − 1/(2c)
- for integer values for c the number of petals is c (c odd), or 2c (c even)
- when c is irrational the curve does not close, and the number of petals is infinite
- when the parameter c is rational, the resulting curve is algebraic

The following botanic curves have been given a special name:

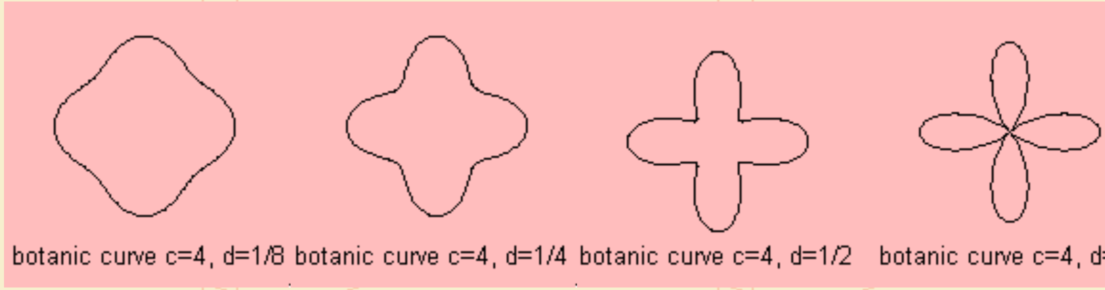
c	curve
1/2	Freeth's nephroid
1	limaçon of Pascal
2	Ceva's trisectrix

Some examples for other values of c:



botanic curve c=2/3, d=1/8 | botanic curve c=2/3, d=1/4 | botanic curve c=2/3, d=1/2 | botanic curve c=2/3, d

botanic curve c=3, d=1/8 | botanic curve c=3, d=1/4 | botanic curve c=3, d=1/2 | botanic curve c=3, d



You can observe that the value of d defines the form of the petals: for d < 1 the petal is open, it closes for d=1 and it forms an extra small petal for d>1.

cycloid

roulette

last updated: 2003−05−31

$$\begin{cases} x = t - a\sin t \\ y = 1 - a\cos t \end{cases}$$

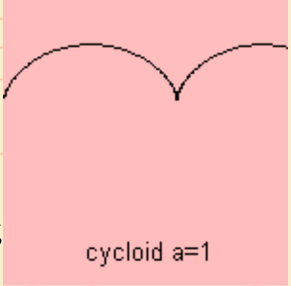
The curve is formed by the locus of a point, attached to a circle (cycle −> cycloid), that rolls along a straight line [1]. In other words: the combination of a linear (term t) and a circular motion (terms sin t and cos t).
The old Greek were already acquainted with this curve..

The value of the parameter 'a' determines the starting point in relation to the circle:

- (ordinary) cycloid
  The starting point is situated on the circle (a = 1).
  When the starting point is not on the circle, the curve is called a trochoid:
- prolate cycloid (Fr. cycloïde allongée)
  The starting point is situated outside the circle (a > 1).
- curtate cycloid (Fr. cycloïde raccourcie)
  The starting point is situated inside the circle (a < 1).

When you have a steady hand, you can make your own cycloid on a blackboard, combining a linear and a circular motion.

(ordinary) cycloid



In Holland we use for this curve also the name "wheel line" [2], being to the track followed by a point on a cycling wheel.

The curve has two remarkable qualities:

The first quality is that the cycloid is the ***brachistochrone*** [3], that is the curve between two points in a vertical plane, along which a bead needs the shortest time to travel [4]. *Galilei* (who gave the curve its name in 1699) stated in 1638 (falsely) that the brachistochrone has to be the arc of a circle. And in June 1696 *Johann Bernoulli* challenged his brother *Jakob Bernoulli* − the both were rivals − to solve the problem. December 1696 Johann [5] repeated his challenge in the 'Acta eruditorum', asking to send solutions before Easter 1697. Besides *Johann* and *Jakob* also *Leibniz*, *Newton*, and *de L'Hôpital* solved the problem.
This is one of the first variational problems, to be studied.
As a matter of fact, this curve is the opposite (mirroring in the x−axis) of the shown curve. Without any formula, it can be understood that along this curve the path is faster than along a straight line. For a cycloid the resulting component of the gravity is larger, and so the acceleration and speed, immediately after the start. It is also easy to verify experimentally that the way along a straight line takes more time.
This knowledge can used while skiing: it is faster choosing a way down so that you gain speed, than to avoid the slopes.

The second quality is that the cycloid is the ***tautochrone*** (sometimes called: ***isochrone***) [6]. This means that a bead along the curve, needs the same time to get down, independent from the starting point. Wonderful! It was *Christiaan Huygens* who discovered this fact, in 1659. In his treatise 'Horologium Oscilatorium' (1673) he designs a clock with a pendulum with variable length. The pendulum moves between

two cheeks, both having the form of a cycloid. Swaying to the outer side, the pendulum shortens. *Huygens* used the insight that the involute of a cycloid is the same cycloid (of course also valid for the evolute).
By this construction the irregularity in a normal pendulum was compensated. For a normal pendulum then, the time of oscillation is only in a first approximation independent of the position aside [7]. *Huygens* was quite impressed, he wrote above the prove: 'magna nec ingenijs investigata priorum', to be translated as: 'this is something great, and never before investigated by a genius'.
In his experiments *Huygens* used also the involute of a circle in his pendulum clock to approximate the cycloid path.

However, use of the tautochrone principle in designing pendulum clocks posed too many mechanical problems, to make it common.

Some interesting properties of the cycloid:

- its radial is the circle
- when a cycloid rolls over a line, the path of the center of the cycloid is an ellipse
- the isoptic of the cycloid is a trochoid

Cusa was the first to study the curve in modern times, when trying to find the circle's area. *Mersenne* (1599) gave the first proper definition of the cycloid, he tried to find the area under the curve but failed. He posed the question to *Roberval*, who solved it in 1634. Later, *Torricelli* found the curve's area, independently.
Descartes found how to draw a tangent to the cycloid, he challenged *Roberval* to find the solution, Roberval failed, but *Fermat* succeeded. Also *Viviani* found the tangent.
Blaise Pascal wrote in 1658 the 'traité général de la Tourette', dedicated to the cycloid. It is said that for him studying the curve was a good diversion from his severe toothache.
Pascal published, under the name of *Dettonville* a challenge, offering two prizes, for solving the area and the center of gravity of a segment of the cycloid. *Wallis* and

Lalouére entered, both were not successful. *Sluze*, *Ricci*, *Huygens*, *Wren* and *Fermat* did not enter the competition, but all wrote their solution to Pascal. Pascal published his own solutions, together with an extension of *Wren*'s result.

Desargues proposed teeth for gear wheels, in the form of a cycloid (c 1635).

Jakob and *Johann Bernoulli* showed (1692) that the cycloid is the catacaustic of a circle, where light rays come from the circumference.
A cycloid arch, with rays perpendicular to the x−axis, results in two cycloid arches.

So the cycloid was very popular among 17th century mathematicians. That's why later the curve has been given the name ***quarrel curve*** [8].

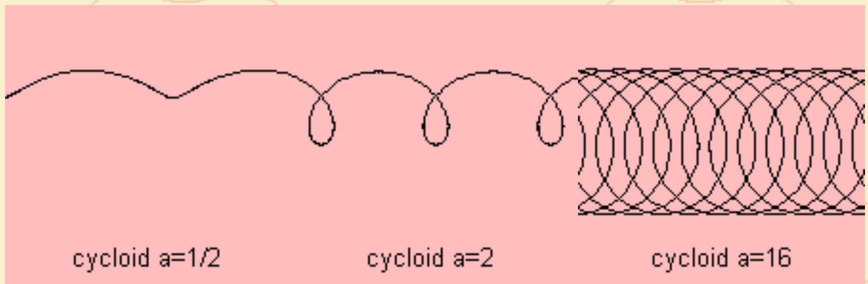
In the Piano Museum in Hopkinton[9] one finds a piano, whose back edge has the form of a cycloid. The maker, Henry Lindeman, named the instrument 'the Cycloid Grand', in the late 1800s.
But seen from the upside you see that its form differs from a real cycloid:

Gear wheels have a cycloid form, they can be approximated by a series of circular arcs. Also numeric tables can be used, as *George Grant*'s Odontograph, which is also the name for an instrument for laying off the outlines of the teeth of the gear wheels.

The curve is a cycloidal curve.

trochoid[10)]

Now the point being followed is not lying on the circle. When the point lays outside the circle, the curve is called a ***prolate cycloid*** (or ***extended cycloid***). When the point lays inside the rolling circle, the curve is called a ***curtate cycloid*** (or ***contracted cycloid***). The latter curve is followed by the valve of a bike. That's where the name ***valve curve*** for the cycloid is from.



The first to study the curve were *Dürer* (1525) and *Römer* (1674).

---

notes

1) Let there be a circle with center (0,R) and a point (p, 0) as starting point to roll. Then the coordinates of the cycloid, as function of the rolled angle t are

$$\begin{cases} \frac{x}{R} = t - \frac{p}{R}\sin t \\ \frac{y}{R} = 1 - \frac{p}{R}\cos t \end{cases}$$

2) In Dutch: radlijn

3) Brakhisto (Gr.) or brachus (Lat.) = short, chronos (Gr.) = time

4) At height y the bead becomes a velocity √gy, so that minimizing the travel time means minimizing the integral

$$\int \frac{(\sqrt{1+y'^2}}{\sqrt{y}}dx \ .$$

Solving this equation leads via differential equation
$y(1+y'^2)=c$ to the cycloid.

5) In that period professor in mathematics in Groningen, Holland

6) Tauto = equal, chronos = time: the curve to be followed in equal time.

7) The correct relation is given by a complete elliptical integral of the first kind.

8) In Dutch: kibbelkromme.

9) Hopkinton, Mass., about 1/2 hr. west of Boston.

10) Trochus (Lat.) = hoop.
Sometimes the meaning of cycloid and trochoid is interchanged: trochoid for the general case, cycloid only for the situation that the starting point is lying on the circle.

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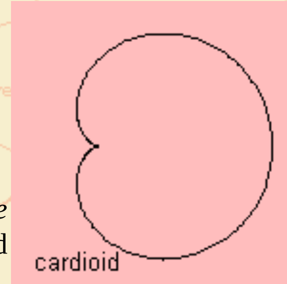
cardioid roulette

last updated: 2003-03-07

$r = 1 + \sin\phi$

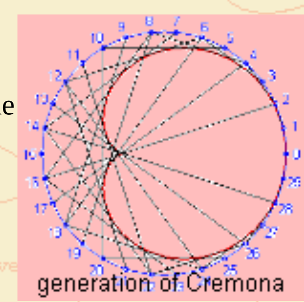


This quartic [1] curve has the form of a heart [2], so that it is also called the ***heart curve*** [3]. *De Castillon* used the name cardioid for the first time, in a paper in the Philosophical Transactions of the Royal Society (1741).
The curve is the epicycloid for which the rolling circle and the rolled circle have the same radius. Besides, the curve can be seen as a special case of the limaçon, and it is also a sinusoidal spiral.

When light rays fall on a concave mirror with a large angle, no focus point but a focus line is to be seen. This line is a cardioid. The same phenomenon can occur nearby a lamp, or as a result of light falling in a cup of tea.
This seems to me the property of the cardioid that it is the catacaustic of the circle (with the source on the circumference). The cardioid is also the pedal of the circle (pedal point not on circle).

Other properties of the curve are:

- the evolute of a cardioid is an(other) cardioid
- its catacaustic (with the cusp as source) is the nephroid
- it is the catacaustic as well as a pedal of the cissoid of Diocles
- it is the (polar) inverse of the parabola
- it is the conchoid of a circle
- its pedal is Cayley's sextic
- its orthoptic is the limaçon
- it can be found in the middle of the Mandelbrot set



The cardioid is the envelope of the chords of a circle, between points P and Q, which follow the circle in the same direction, where one point has the double speed of the other. This construction is called the generation of Cremona.
This means in the figure that the points 10 and 20, 11 and 22, and so on, have been connected.

Given a circle C through the origin. Then the cardioid is the envelope of the circles with diameter the line through the origin and a point on C.

cardioid, envelope

The first to study the curve was *Römer* (1674), followed by *Vaumesle* (1678) and *Koërsma* (1689). *La Hire* found its length (4) in 1708.

---

notes

1) equation: $(x^2 + y^2 - y)^2 = x^2 + y^2$

2) kardia (Gr.) = heart

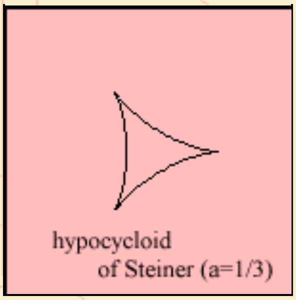
3) In German: Herzkurve

deltoid

roulette

last updated: 2003−05−18

$$\begin{cases} x = 2\cos t + \cos 2t \\ y = 2\sin t - \sin 2t \end{cases}$$



This curve, with the form of the Greek capital delta, has been given also as names:

- ***Steiner's hypocycloid***: the Swiss mathematician *Jakob Steiner* studied the curve in 1856
- ***tricuspoid***: because of its three cusps

This quartic curve [1] is the hypocycloid for which the rolled circle is three times as large as the rolling circle.
Given a tangent l to the curve. It cuts the curve in points P and Q. Then PQ has a length of 4, and the tangents to the curve in P and in Q make a right angle.
The length of the deltoid is 16, and the area it encloses: 2 π.

Some other properties of the curve:

- the radial of the deltoid is the trifolium
- its catacaustic (with the source at infinity) is the astroid
- its orthoptic is the circle

Three pedals of the curve are:

pedal point	*pedal curve*
cusp	simple folium
vertex	double folium
center	trifolium

The curve has been investigated as first by *Leonhard Euler* (1745), while studying an optical problem.

notes

1) In Cartesian coordinates:
$(x^2+y^2)^2 - 8x(x^2-3y^2) + 18(x^2+y^2) = 27$

epicycloid

roulette

last updated: 2003−05−18

$$\begin{cases} x = (1+a)\cos at - ab\cos(1+a)t \\ y = (1+a)\sin at - ab\sin(1+a)t \end{cases}$$

The curve is formed by the locus of a point, attached to a circle, that rolls on [1] the outside of another circle [2]. In the curve's equation the first part denotes the relative position between the two circles, the second part denotes the rotation of the rolling circle.

The value of the constant b determines the starting point in relation to the circle:

- (ordinary) epicycloid
  The starting point is situated on the rolling circle (b = 1).
  When the starting point is not on the circle, the curve is called an epitrochoid:
- prolate epicycloid
  The starting point is situated outside the rolling circle (b > 1).
- curtate epicycloid
  The starting point is situated inside rolling the circle (b < 1).

The isoptic of the ordinary epicycloid is an epitrochoid.

The curve has a closed form when the ratio of the rolling circle and the other circle (a) is equal to a rational number. When giving this ratio its simplest form, the numerator is the number of revolutions around the resting circle, before the curve closes. The denominator is the number of rotations of the rolling circle before this happens.
In this rational case the curve is algebraic, otherwise transcendental.

(ordinary) epicycloid

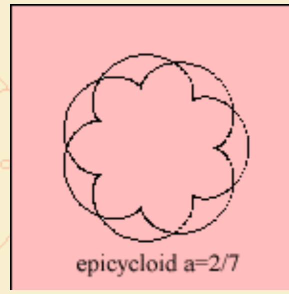
The epicycloid curves have been studied by a lot of mathematicians around the 17th century: *Dürer* (1515), *Desargues* (1640), *Huygens* (1679), *Leibniz*, *Newton* (1686), *de L'Hôpital* (1690), *Jakob Bernoulli* (1690), *la Hire* (1694), *Johann Bernoulli* (1695), *Danilel Bernoulli* (1725) and *Euler* (1745, 1781).

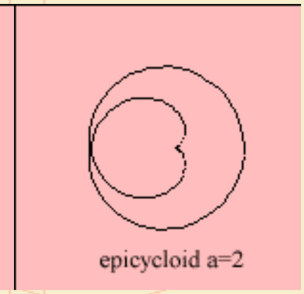
Apollonius of Perga (about 200 BC), had the idea to describe the celestial movements as combinations of circular movements. It was *Hipparchos of Nicaea* (about 150 BC), the greatest astronomer in Greek antiquity, who worked out this theory in detail. The results did become famous by the books of *Ptolemy* (about 150 AD). The earth is thought as standing in (or nearby) a celestial center, around which the other celestial bodies rotate. The combination of the rotation of the earth and the planet's rotation around her makes an epicycloid. This geocentric theory should be the accepted theory for almost 2000 years. The heliocentric theory (as constructed by Copernicus), was also discussed by the Greek, but refused for emotional reasons.

Some relations with other curves:

- the radial and the pedal (with the center as pedal point) of the curve is the rhodonea
- the evolute of an epicycloid is a similar epicycloid, but smaller in size

There are some epicycloids that have been given an own name:



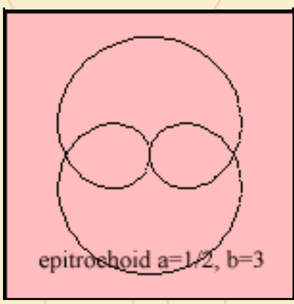


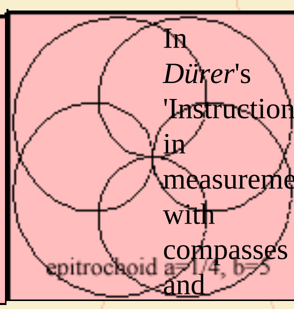
- a=1/2: ***nephroid***
- a=1: ***cardioid***

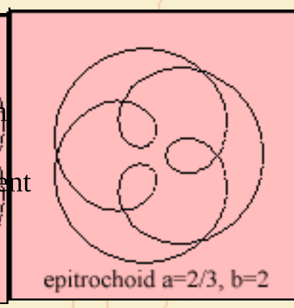
The curve is a cycloidal curve.

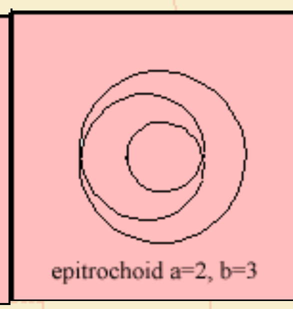
epitrochoid [3)]

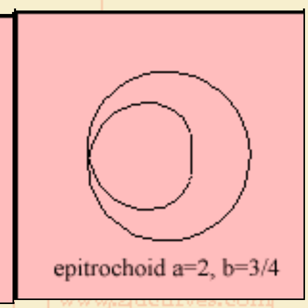
Now the point being followed is not lying on the rolling circle. When the point lays outside the circle (b>1), the curve is called a ***prolate epicycloid***. When the point lays inside the rolling circle (b<1), the curve is called a ***curtate epicycloid***.
With a spirograph set many epitrochoids can be drawn.











In *Dürer*'s 'Instruction in measurement with compasses and straight edge' (1525) occurs an example of an epitrochoid. He called them **spider lines** because of the form of the construction lines he used.
Other mathematicians who studied the curves were: *la Hire*, *Desargues* and *Newton*.

And now some examples of epitrochoids:

When the two circles have equal radius (a=1), the epitrochoid is a ***limaçon***.

notes

1) Epi = on

2) Let a circle with radius r roll on the outside of a circle with radius R. Take as center of the coordinate system the center of the rolled circle. Now let the starting point be on a distance b r from the center of the rolling circle. Then the coordinates of the epicycloid as a function of the rolled angle t are:

$$\begin{cases} x = (R+r)\cos\frac{r}{R}t - b\,r\cos(1+\frac{r}{R})t \\ y = (R+r)\sin\frac{r}{R}t - b\,r\sin(1+\frac{r}{R})t \end{cases}$$

3) Trochus (Lat.) = hoop.
Sometimes the meaning of epicycloid and epitrochoid is interchanged: epitrochoid for the general case, epicycloid only for the situation that the starting point is lying on the circle.

hypocycloid

roulette

last updated: 2003−05−21

$$\begin{cases} x = (1-a)\cos at + ab\cos(1-a)t \\ y = (1-a)\sin at - ab\sin(1-a)t \end{cases}$$

The curve is formed by the locus of a point, attached to a circle, that rolls on the inside [1] of another circle [2]. In the curve's equation the first part denotes the relative position between the two circles, the second part denotes the rotation of the rolling circle.

The value of the constant b determines the starting point in relation to the circle:

- (ordinary) hypocycloid
  The starting point is situated on the rolling circle (b = 1).
  When the starting point is not on the circle, the curve is called a hypotrochoid:
- prolate hypocycloid
  The starting point is situated outside the rolling circle (b > 1).
- curtate hypocycloid
  The starting point is situated inside rolling the circle (b < 1).

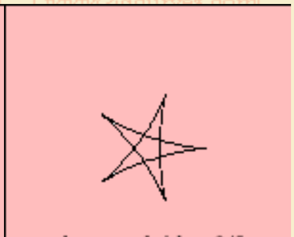
The curve has a closed form when the ratio of the rolling circle and the other circle (a) is equal to a rational number. When giving this ratio its simplest form, the numerator is the number of revolutions inside the resting circle, before the curve closes. The denominator is the number of rotations of the rolling circle before this happens.
In this rational case the curve is algebraic, otherwise transcendental.

We let the rolling circle be smaller than the rolled circle [3], so that a < 1. And because the curve for a* = 1 − a is the same as for a, we can state that 0 < a <= 1/2.

(ordinary) hypocycloid

The resulting curve is a kind of star with a number of cusps. This amount is equal to the numerator of the constant a.


hypocycloid a=2/7


hypocycloid a=2/5

Some special hypocycloids are the following:

- 1 = 1/2: ellipse
- a = 1/3: deltoid
- a = 1/4: astroid

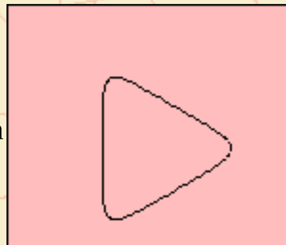
Related curves:

- the pedal of the hypocycloid (center as pedal point) is the rhodonea
- the isoptic of the hypocycloid is a hypotrochoid
- the evolute of the curve is the same hypocycloid, but smaller in size

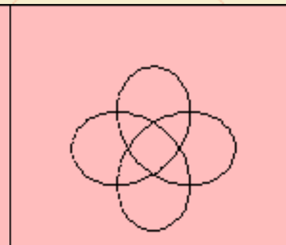
The curve is a cycloidal curve.

hypotrochoid [4)]

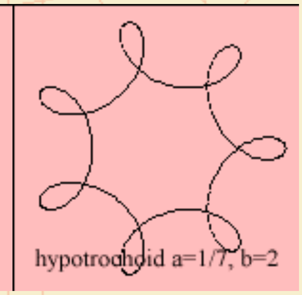
Now the point being followed is not lying on the rolling circle. When the point lays outside the circle (b>1), the curve is called a ***prolate hypocycloid***. When the point lays inside the rolling circle (curtate hypocycloid.



hypotrochoid a=1/3, b=1/2



hypotrochoid a=1/4, b=5



hypotrochoid a=1/7, b=2

With a spirograph set many hypotrochoids can be drawn.

When the factors for the two terms in the parameter equation for x are equal, the resulting curve is a ***rhodonea***.
In this case holds b = 1/a − 1.

---

notes

1) Hypo (Gr.) = under

2) Let a circle with radius r roll on the inside of a circle with radius R. Take as origin the center of the rolled circle. Now let the starting point be on a distance b r from the center of the rolling circle. Then the coordinates of the hypocycloid as a function of the rolled angle t are:

$$\begin{cases} x = (R-r)\cos\frac{r}{R}t + br\cos(\frac{r}{R}-1)t \\ y = (R-r)\sin\frac{r}{R}t + br\sin(\frac{r}{R}-1)t \end{cases}$$

3) If not, the rolling circle and the rolled circle can be interchanged.

4) Trochus (Lat.) = hoop.
Sometimes the meaning of hypocycloid and hypotrochoid is interchanged: hypotrochoid for the general case, hypocycloid only for the situation that the starting point is lying on the circle.

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limaçon

roulette

last updated: 2003-05-05

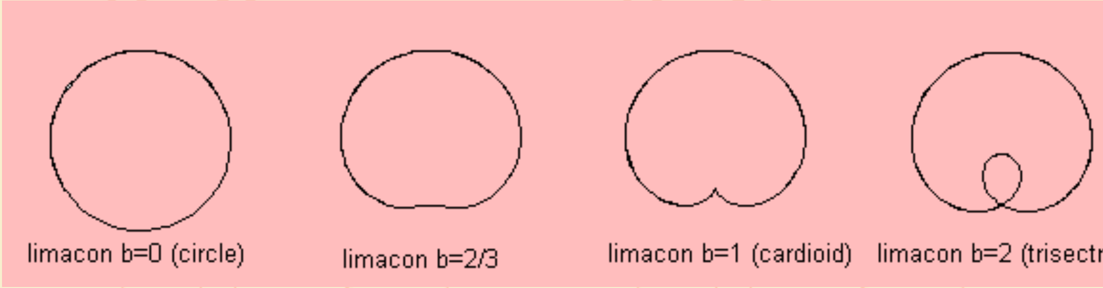
$r = 1 + b \sin\phi$

The curve has the form of a snail [1], so that it is also called the ***snail curve***. It's the epitrochoid for which the rolling circle and the rolled circle have the same radius.
The curve can also be defined as a conchoid.

The curve is called the ***limaçon of Pascal*** (or ***snail of Pascal***). It is not named to the famous mathematician Blaise Pascal, but to his father, *Etienne Pascal*. He was a correspondent for *Mersenne*, a mathematician who made a large effort to mediate new knowledge (in writing) between the great mathematicians of that era.
The name of the curve was given by *Roberval* when he used the curve drawing tangents to for differentiation.

But before *Pascal*, *Dürer* had already discovered the curve, since he gave a method for drawing the limaçon, in 'Underweysung der Messung' (1525).

Sometimes the limaçon is confined to values b < 1. We might call this curve an ***ordinary limaçon***. It is a transitional form between the circle (b=0) and the cardioid (b=1).
When we extend the curve to values for b > 1, a noose appears.



For b = 2, the curve is called the ***trisectrix***.

An alternative name for the curve is **arachnid** [2] or **spider curve**.

Some fine properties of the curve are:

- the limaçon is the catacaustic and also the pedal of the circle
  The catacaustic quality was shown by *Thomas de St Laurent* in 1826
- the ordinary limaçon is the inverse of the ellipse
- the limaçon with a noose is the inverse of the hyperbola
  In fact, the constant b is the same as the eccentricity for a conic section
- the limaçon is the orthoptic of the cardioid
- the curve is a special kind of botanic curve

The limaçon is an anallagmatic curve.

For b unequal zero, the curve is a quartic, in Cartesian coordinates it can be written as a fourth degree equation [3].

---

notes

1) limax (Lat.) = snail

2) from arakne (Gr.) = spider.
In French: **arachnée**

3) equation: $(x^2 + y^2 - by)^2 = x^2 + y^2$

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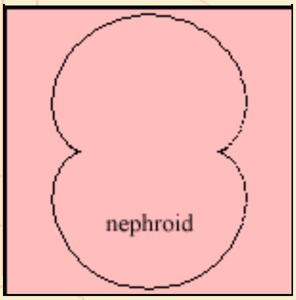
nephroid

roulette

last updated: 2002−03−25

$$\begin{cases} x = 3\cos t - \cos 3t \\ y = 3\sin t - \sin 3t \end{cases}$$



This kidney formed curve [1] is the epicycloid with two cusps. The rolling circle is half the size as the rolled circle. The English mathematician *R.A. Proctor* (1837−1888) gave the curve its name, in 1878 in his book *The geometry of cycloids*.

Some other properties of the curve are:

- the curve is the catacaustic of the cardioid, with the cusp as source
- the nephroid is the evolute of Cayley's sextic
- the curve is the catacaustic of the circle, with the source at infinity *Huygens* found this in 1678, and he published this findings in 1690 in his essay 'Traité de la lumière'. One had to wait for 1838 that *Airy* proved the fact theoretically.

The nephroid has a length of 24, and its area is 12 π.
In Cartesian coordinates the curve can be written as a sixth degree equation [2].
The curve is said to be the perfect multi−seater dining−table.

The first to distinguish the nephroid was *Christiaan Huygens* (1678).

---

notes

1) Nephros = kidney

2) Equation:

$$(x^2+y^2-4)^3 = 108y^2$$

rhodonea

roulette

last updated: 2003−05−21

$r = \sin c\phi$

Because of its flower−like form, the ***rhodonea*** (***curve)*** is also called the ***rose*** [1], or the ***rosette***.
It is a hypocycloid for which the amplitudes of the two terms of the parameter equation are equal [2].

The number of petals is the denominator of: 1/2 − 1/(2c).
For irrational c the curve does not close, and the number of petals is infinite. For rational c, the resulting curve is algebraic.
Example: for c=5/2 the number of petals is 10 (see below).

Some relationships with other curves:

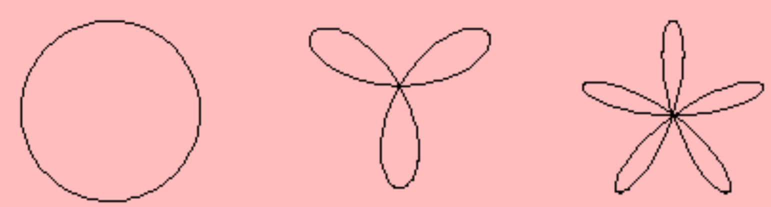
- the rhodonea is the radial and the pedal of the epicycloid
- it is also the pedal of the hypocycloid
- it is the polar inverse of the epi spiral
- the conchoid of the curve is the botanic curve

Some authors confine the constant c to integer values. In that case the number of petals is:

- c (for odd c)
- 2c (for even c)

The degree of the Cartesian equation of the curve is then:

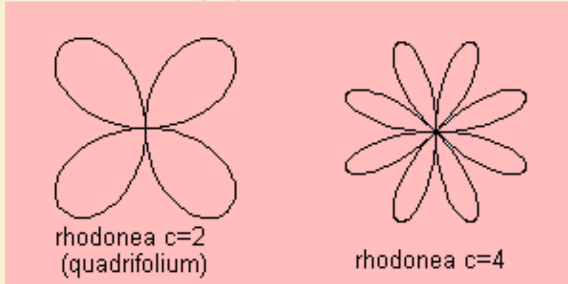
- c + 1 (for odd c)
- 2 (c + 1) (for even c)



rhodonea c=1 (circle) rhodonea c=3 (trifolium) rhodonea c=5

Some integer rhodonea curves are the following:

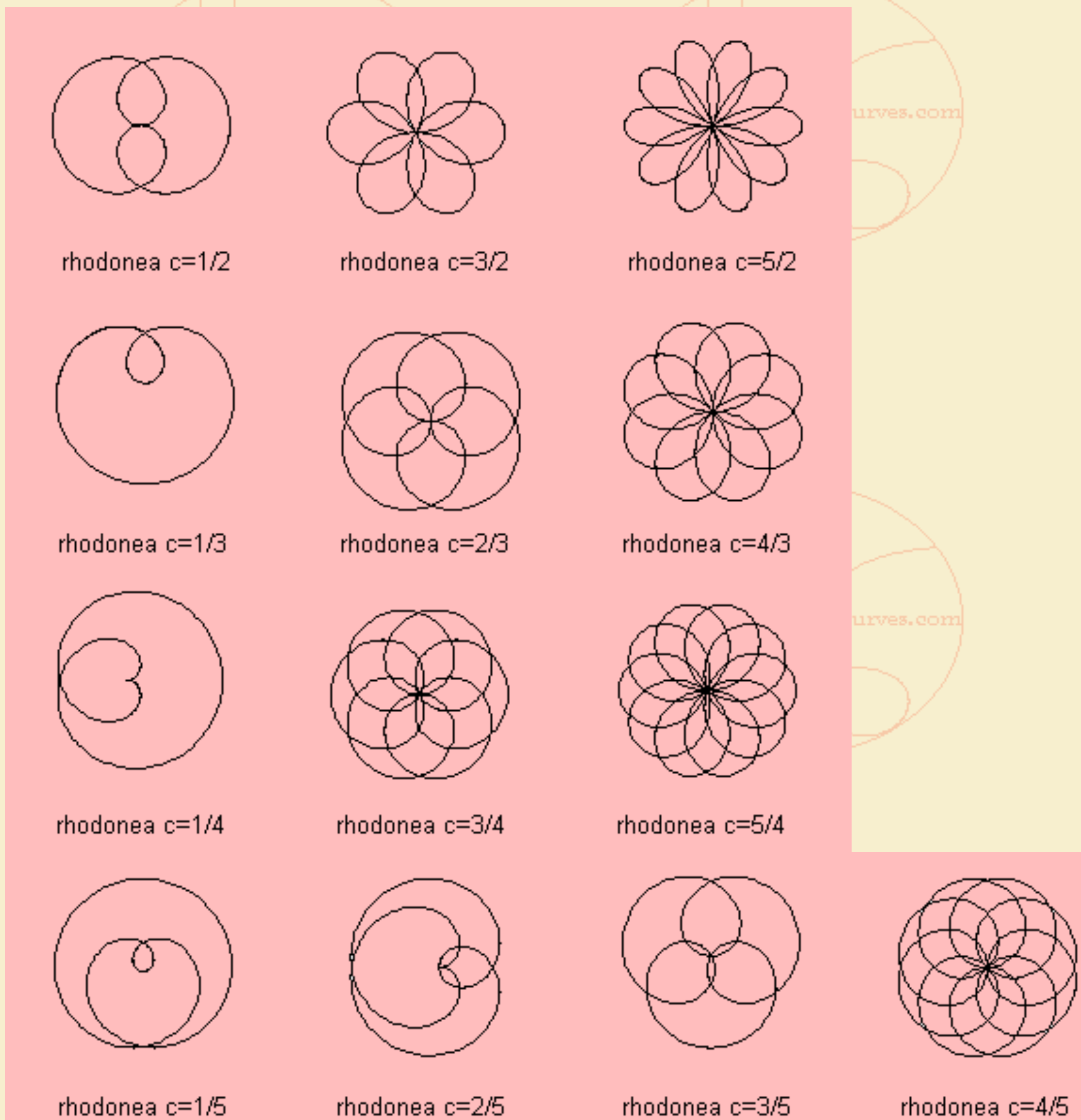
The circle is the rose with c=1.
The ***(regular) trifolium*** [5], **the three−leaved rose**, a quartic curve [6], is is the radial of the deltoid.
I add the regular prefix in the name to distinguish the curve from the more general trifolium.



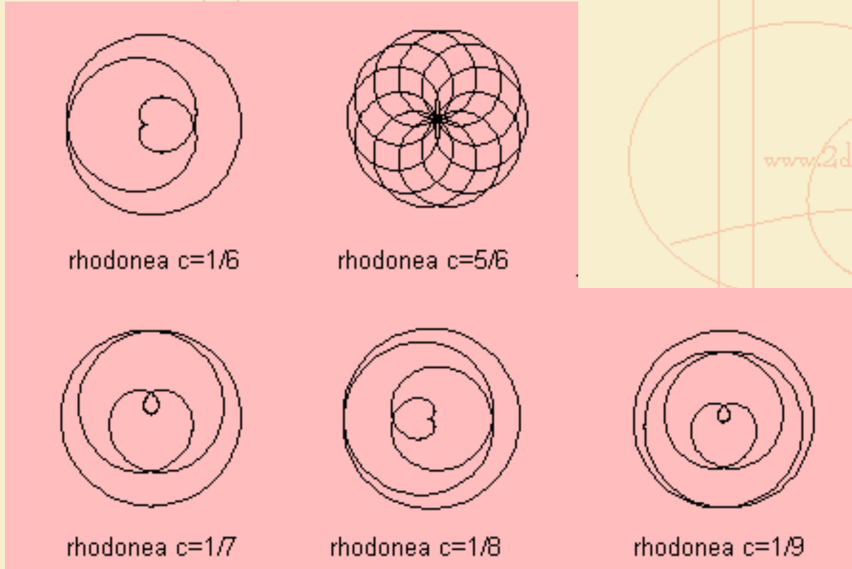
rhodonea c=2 (quadrifolium) rhodonea c=4

The ***quadrifolium***[3], the **four−leaved rose**, a sextic curve[4], has a remarkable relationship with the astroid: the curve is the radial, the pedal and the orthoptic of the astroid
And its catacaustic (with the source in the center) is also an astroid.

Next step is to investigate the curves with non−integer values for parameter c:



rhodonea c=1/2 rhodonea c=3/2 rhodonea c=5/2
rhodonea c=1/3 rhodonea c=2/3 rhodonea c=4/3
rhodonea c=1/4 rhodonea c=3/4 rhodonea c=5/4
rhodonea c=1/5 rhodonea c=2/5 rhodonea c=3/5 rhodonea c=4/5



The first to investigate the curve was *Grandi* (1723), an Italian priest, member of the order of the Camaldolites. He was professor in Mathematics at the University of Pisa.

---

notes

1) Rhodon = rose.

2) The result is a formula in which the radius is a cosine function of the rolled angle (which is proportional to the polar angle). The relation between the rhodonea's parameter c and the parameters a and b of the hypotrochoid is: $1/c = 1 - 2a$, where $b = 1/a - 1$.

3) Quattuor (Lat.) = four, folium (Lat.) = leaf.

4) Its Cartesian equation is the following: $(x^2+y^2)^3 = x^2y^2$

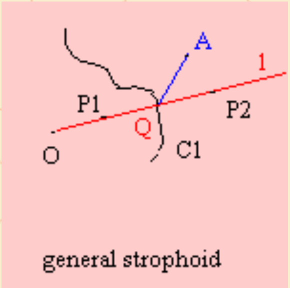
5) Tres (Lat.) = three, folium (Lat.) = leaf.

6) In Cartesian coordinates: $(x^2 + y^2)^2 = x(x^2 - 3y^2)$

(generalized) strophoid

last updated: 2002−02−16

Given a curve C1 and fixed points O (pole) and A (another point). Draw lines l through O that intersect C1 in Q. Then the ***strophoid*** is defined as the collection of points P (on l) for which PQ is equal to a AQ.



Some special cases:

curve C1	*pole O*	*fixed point A*	*strophoid*
line	not on C1	on C1	oblique strophoid
line	not on C1	projection of O on C1	(right) strophoid
circle	center of C1	on C1	Freeth's nephroid
trisectrix	...	...	**Freeth's strophoid**

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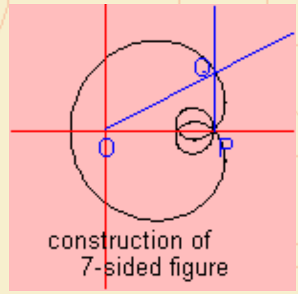
(extended) strophoid Freeth's nephroid

last updated: 2003−01−27

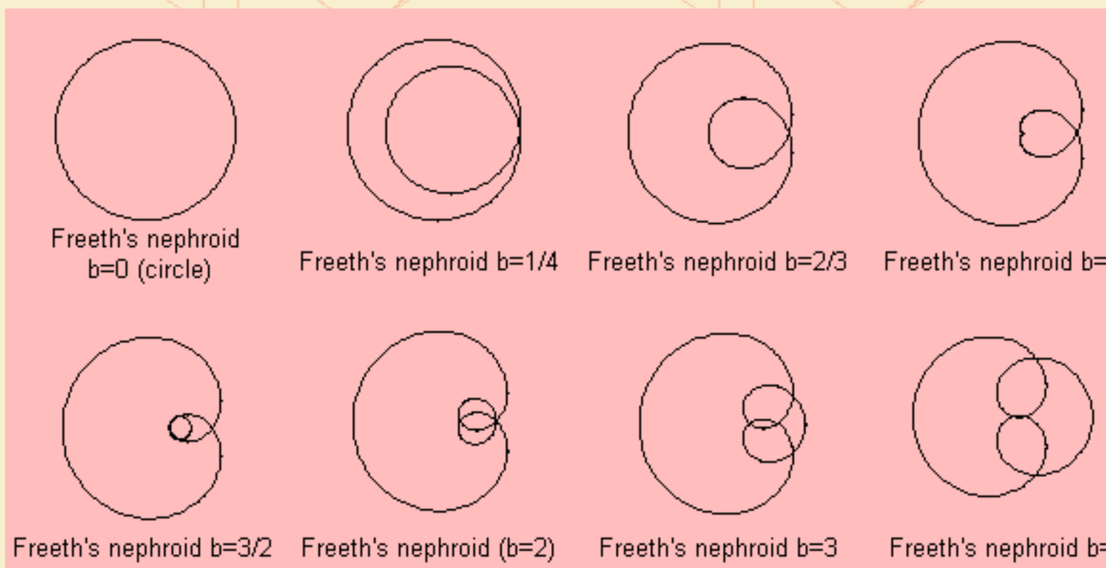
$$r = 1 + b \sin\frac{\phi}{2}$$



construction of 7-sided figure

The curve, a botanic curve, is an extension of **Freeth's nephroid** (b = 2). This specimen is the strophoid constructed for a circle, named after the English mathematician *T.J. Freeth* (1819−1904).

The Freeth's nephroid (b=2) can be used to construct a regular 7−sided figure, as follows:
Let P be the node where the curve crosses itself three times.
Line l parallel the y−axis through P cuts the nephroid in Q.
Then the angle POQ is 3π/7.



For large b the curve approximates the rhodonea (c=1/2).

Don't mistake this curve with another nephroid, the (ordinary) nephroid [1].

---

notes

1) Nephros = kidney

3d curve

last updated: 2002−12−29

My www.2dcurves.com site is meant to be a collection of two−dimensional curves. I chose to place the 6 three−dimensional curves and the surfaces I encountered on this page, not mixing them with their 2d brothers:

- cyclid
- horopter
- Menger sponge
- pancake curve
- super egg
- mount Tagaki

horopter

3d curve

last updated: 2003−05−25

$$\begin{cases} z = xy \\ r = a \end{cases}$$

Given a binocular system, to be seen as a model of the human eyes. The ***horopter*** [1] then, is a 3D curve that can be defined as the set of points for which the light falls on corresponding areas in the two retinas [2]. This definition could be equivalent to the intersection of a *cylinder* and a *hyperbolic paraboloid* (exercise is to be done).

The curve is a kind of one−turn helix around a cylinder. In the special situation that he horopter is in the horizontal plane through the eyes which contains the center of the retinas [3], the horopter takes the form of a circle, named the *Vieth−Muller circle*, plus a line along the cylinder: the ***vertical horopter***.

The first to mention the curve was *Aquilonius* (1613).

---

notes

1) From the Greek words horos (= boundary) and opter (observer).

2) This is called the condition of zero retinal disparity: one object seems to lay in the same direction for both eyes.
There can be confusion about the definition of this term. When when you define 'zero retinal disparity' as that objects appear to lie in the same direction, or at the same distance, you get a similar, but different curve.

3) The center region of the retina, where the light falls on when you look straight to an object, is called the fovea.

With thanks to *Edgar Erwin* for his remarks about the curve.

Menger sponge

3d curve

last updated: 2003−01−09

$$\begin{cases} (x,y,z) \in I^3 \\ (x,y) \in S \\ (x,z) \in S \\ (y,z) \in S \end{cases}$$

where I3 is the unit cube, and S the Sierpinski square.
In fact, the sponge can be seen as the three−dimensional analog of the Sierpinski square. It is a fractal.

The cube can be constructed as follows: take a cube, divide it into 3 x 3 x 3 smaller cubes of equal size. Then remove the cube in the center, and also the six cubes that share sides with it.
Then, repeat the process on each of the remaining twenty cubes. And again, and again, infinitely.

The first picture shows the square after three iterations, the second and third picture show the square after six iterations.
The 2nd picture has been constructed with Open Inventor at the Interdisciplinary Center for Scientific Computing (IWR) in Heidelberg/Germany by *C. Dartu* and *D. Volz* in 1997.
The 3rd picture has also been made in Heidelberg.

The curve is the most popular creation of the mathematician *Karl Menger*, while working on dimension theory.
The curve has as alternative names: the **Menger universal curve**, or the **Sierpinski sponge**.

pancake curve

3d curve

last updated: 2002−12−27

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = \sin 2t \end{cases}$$

The curve is formed by a circle in the x, y plane sine wise transformed in the z−direction. Because her resemblance to the circumference of a pancake thrown ahead to turn, *Robert Ferréol* gave the curve its name.

When projecting the curve on plane through the central z−axis of the curve, a Besace is formed.

kids' curves

last updated: 2001−11−05

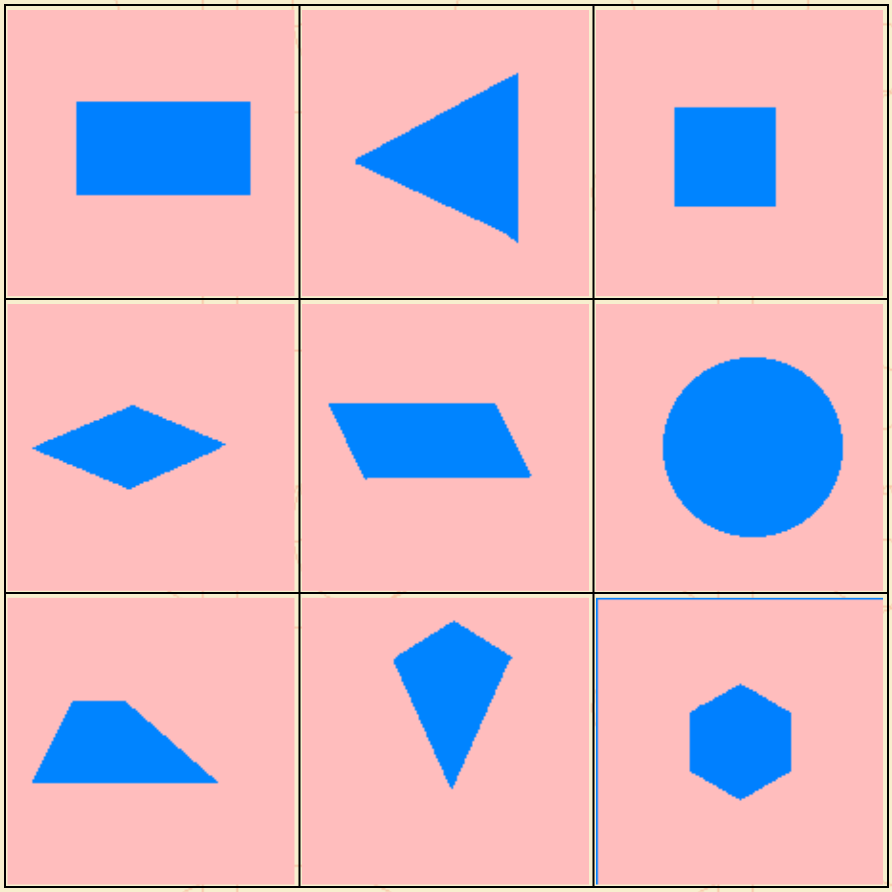
The simplest curves, especially for our kids, placed on one page, for each language:

- in Dutch
- in English

kids page in Dutch

kids' page

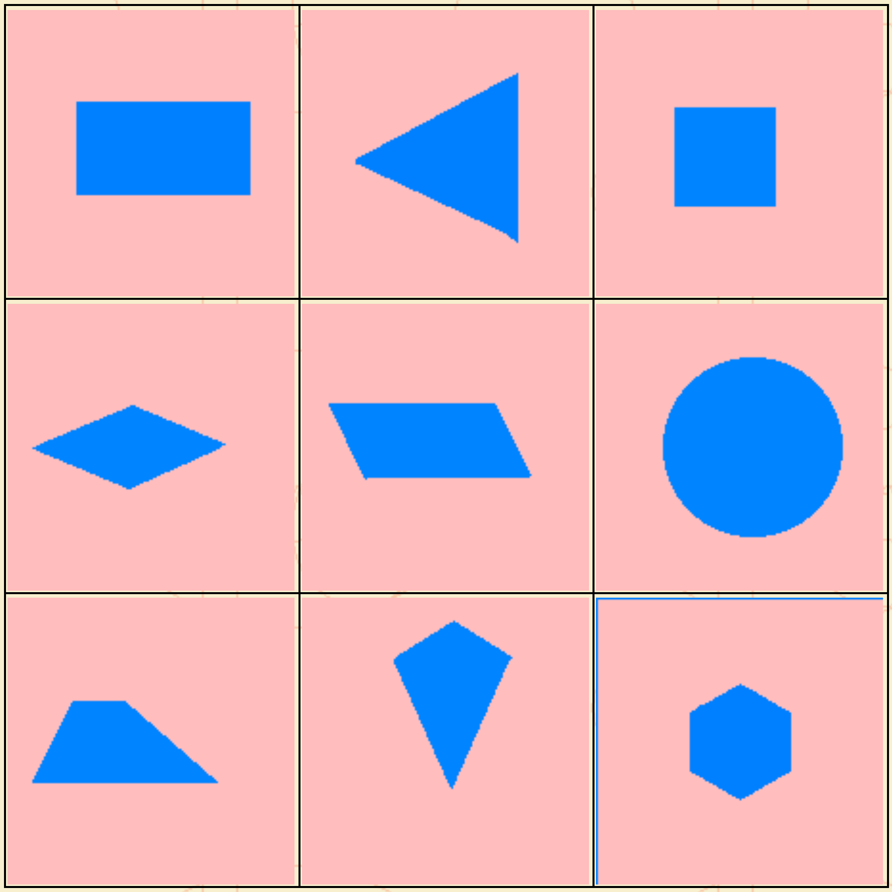
last updated: 2001−10−14



kids page in English

kids' page

last updated: 2001−10−14



curve literature

last updated: 2002−08−19

- Arthurs, A.M., *calculus of variations*, Routledge & Kegan Paul, London and Boston, 1975.
- Ash, Robert B., *complex variables*, Academic Press, New York and London 1971.
- Blij, F. van der, and J. van Tiel, *infinitesimaalrekening*, Spectrum, Utrecht/Antwerpen, 1975.
- Blij, F. van der e.a., *kaleidoscoop van de wiskunde I: van priemgetal tot populatiegenetica*, Epsilon uitgaven, Utrecht, 1990.
- Brinkman, P.L., *handboek voor de communicatie in de wiskunde*, Agon Elsevier, Amsterdam/Brussel, 1974.
- Davis, Philip J., and Reuben Hersh, *the mathematical experience*, Houghton Mifflin Company, Boston, 1981.
- Dijksterhuis, E.J., *de mechanisering van het wereldbeeld*, Meulenhoff, Amsterdam, 1977.
- Kervel, E. van, *prisma van de wiskunde*, Spectrum, Utrecht, 1990.
- Kittel, C., *introduction to solid state physics*, John Wiley & Sons Inc., New York, 1976.
- Kuipers, Dr.L., en Dr.R. Timman (red), *handboek der wiskunde 1*, Scheltema & Holkema N.V., Amsterdam, 1966.
- Lauwerier, Hans, *Een wereld van fractals*, Aramith Uitgevers, Bloemendaal, 1990.
- Lauwerier, Hans, *(spelen met) graphics en fractals*, Academic Service, Schoonhoven, 1994.
- Lawrence, J. Dennis, *a catalog of special plane curves*, Dover Publications, Inc., New York, 1972.
- Wolfram research Inc., *Mathematica 3.0*, 1996.
- Mueller, George (red), *fenomena*, Vrij Geestesleven, Zeist, 1985.
- Pollmann, Thijs: *de wet van het uitdovend verleden*, in: NRC Handelsblad of 11/01/1996.
- Struik, D.J., *geschiedenis van de wiskunde*, SUA, Amsterdam 1977.
- Stewart, Ian, *what shape is a snowflake*, Ivy Press, Lewes, 2001.
- Teller, Otto, *vademecum van de wiskunde*, Spectrum, Utrecht/Antwerpen, 1965.
- Temme, N.M., *speciale functies in de mathematische fysica*, Epsilon uitgaven (deel 15), Utrecht, 1990.
- Waerden, B.L. van der, *ontwakende wetenschap: Egyptische, Babylonische en Griekse wiskunde*, P.Noordhoff N.V., Groningen, 1950.
- Walter, Wolfgang: *Gewöhnliche Differentialgleichungen*, Springer, Berlin, 1976.
- Wichmann, Eyvind H., *quantum physics*, Berkeley physics course vol.4, McGraw−Hill, New York, 1967.

2d curves links

Some interesting sites on the internet about two−dimensional curves:

- *eric weisstein* moved his 'treasure trove of mathematics' to wolfram's mathworld
- *mike notte* made an Italian version of the mac tutor curves
- panagiotis stefanides from Athens, Greece
- *robert ferréol* shows the curves in French
- *university of st. andrews* in Scotland wrote the 'mac tutor history of mathematics':
  - biographies
  - curves
- *xah lee* made a visual dictionary of special plane curves

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