Algebraic Topology
Notes of the Lecture by G. Mislin

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Contents

0 Introduction

0.1 Literature

The book by Allen Hatcher is available for download online!

0.2 Exercises

`www.math.ethz.ch/~mislin` (click on "Algebraic Topology")

0.3 Preliminary Remarks

We will use the language of categories (not the theory, however, so don't worry).
The category $\underline{\text{Top}}$ consists of topological spaces X, Y, etc. (objects) and continuous maps $X \to Y$ (morphisms) between them.
Some "algebraic" categories:

- $\underline{\text{Ab}}$, the category of abelian groups $A, B, \ldots$ with group homomorphisms between them.
- $\underline{\text{Gr}}$, the category of groups.

We will now relate these categories to each other by means of functors:

$$F: \qquad \underline{\text{Top}} \longrightarrow \underline{\text{Gr}}$$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & F(X) \arrow[d, "F(f)"] \\ Y \arrow[r, mapsto] & F(Y) \end{tikzcd}$$

Define $\underline{\text{Top}_*}$ as the category of pointed topological spaces (X with a fixed base-point $x_0 \in X$, with base-point preserving continuous maps).
Then the fundamental group π_1 is an example of a functor:

$$(f : X \to Y) \mapsto (\pi_1 f : \pi_1 X \to \pi_1 Y)$$

Typical problems:

- "$\mathbb{R}^n \cong \mathbb{R}^m \overset{?}{\Rightarrow} n = m$"

  This is interesting because it is actually possible to *continuously* map the unit interval onto the unit square using peano curves!

- "Vector fields on S^2 are singular" where a vector field on S^2 is a continuous map
$$\begin{aligned} v : S^2 &\to \mathbb{R}^3 \\ x &\mapsto v(x) \end{aligned}$$
such that $v(x) \cdot x = 0$ and a singular point is a zero of v. (See chapter on Lefschetz numbers.)

1 Some basic notions concerning topological spaces

Definition 1.1 *Let* $\underline{\text{Top}}$ *be the* category of toplogical spaces*. For* $X, Y \in \underline{\text{Top}}$ *we have the "morphism set"*
$$C(X,Y) = \{f : X \to Y \mid f \textit{ continuous}\}$$
$f : X \to Y$ in $\underline{\text{Top}}$ is a homeomorphism *if there is a $g : Y \to X$ in $\underline{\text{Top}}$ such that $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$.*
We write $X \cong Y$ if $X, Y \in \underline{\text{Top}}$ are homeomorphic.

Definition 1.2 *$X \in \underline{\text{Top}}$ is called* discrete *if all subsets of X are open.*
Note that $f : X \to ?$ continuous for all $f \iff X$ discrete. (Proof: If $f : X \to ?$ is always continuous, choose $A \subset X$, and consider $\chi_A : X \to \{0,1\}$, $\{0,1\}$ with the discrete topology. Since χ_A is continuous, $\chi_A^{-1}(1)$ is open, and this is true for all $A \in X$.)

Definition 1.3 *$X \in \underline{\text{Top}}$ is* indiscrete*, if only $\varnothing \subset X$ and $X \subset X$ are open. ("coarsest topology") Note: X indiscrete $\iff$ every $? \to X$ is continuous.*

Definition 1.4 *$X \in \underline{\text{Top}}$ is called* compact *if X is Hausdorff and every open cover of X admits a finite subcover.*

Definition 1.5 *$X \in \underline{\text{Top}}$ is called* locally compact *if every $x \in X$ has a compact neighbourhood. (Here we do not assume X to be Hausdorff.)*

Definition 1.6 *$X \in \underline{\text{Top}}$ is called* compactly generated *if $A \cap C$ closed in C for every compact $C \subset X$ implies $A \subset X$ closed (in X).*

Example X compact $\Rightarrow$ compactly generated (Take $A \subset X$ with $A \cap C$ closed in C for *all* compact $C \subset X$: so for $X = C$: $A \cap X = A \subset C$ closed : A closed in X).
Also: $\mathbb{R}^n$ compactly generated.

Remark Let X be compactly generated. To prove that $C \subset X \xrightarrow{f} Y$ is continuous, we only need to check that $f|C$ is continuous for all $C \subset X$ compact.

1.1 Quotient spaces

Definition 1.7 *Let $X \in \underline{\text{Top}}$, then $Y \in \underline{\text{Top}}$ is a* quotient space *of X with respect to $\pi : X \to Y$, a surjective map, if $A \subset Y$ closed $\iff$ $\pi^{-1}(A) \subset X$ closed. We then say "Y has the quotient topology".*

Typical situation: $X \in \underline{\text{Top}}$ and "$\sim$" an equivalence relation on X. Then $X/\!\sim\, \in \underline{\text{Top}}$ is the space of equivalence classes, with the topology "$A \subset X/\!\sim$ closed $\iff$ $\pi^{-1}(A) \subset X$ closed" where $\pi : X \to X/\!\sim$ is the projection onto equivalence classes. $X/\!\sim$ is a quotient space of X.

Note If $Y \in \underline{\text{Top}}$ is a quotient space of X with respect to $f : X \to Y$ (a surjective map) then $Y \cong X/\!\sim$ where "$\sim$" is defined by $x_1 \sim x_2 \iff f(x_1) = f(x_2)$, $x_1, x_2 \in X$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\!\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

f is constant on equivalence classes, $\bar{f}$ is continuous ($A \subset Y$ closed $\Rightarrow \bar{f}^{-1}(A)$ closed because $\pi^{-1}\bar{f}^{-1}(A) = f^{-1}(A)$ is closed.) and $\bar{f}$ is a bijection of closed subsets $\Rightarrow \bar{f}$ a homeomorphism.

Definition 1.8 *Let $A \subset X \in \underline{\text{Top}}$, $A \neq \varnothing$, then:*

$$X/A := X/\!\sim$$

where $x_1 \sim x_2 \iff x_1 = x_2$ or $x_1, x_2 \in A$

Example $[0, 1]/\{0, 1\} \cong S^1$

Theorem 1.9 *$\varnothing \neq A \subset X$ in $\underline{\text{Top}}$: X/A has the following* universal property*:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "can"'] & Y \\ X/A \arrow[ur, dashed, "\exists!(f/A)"'] & \end{tikzcd}$$

for every f constant on A.

Example $A \subset B$ in $\underline{\mathrm{Gr}}$ (i.e. a subgroup):

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d] & C \\ \text{“}B/A\text{”} \arrow[ur, dashed] & \end{tikzcd}$$

with f constant on A ($f|A = 0$). Take “B/A” to be $B/N(A)$, where $N(A)$ is the smallest normal subgroup containing A.

Definition 1.10 *Let $X \in \underline{\mathrm{Top}}$. The subsets $A \subset X$, such that $A \cap C$ closed in C for all compact $C \subset X$, form the closed subsets of a topology on X, called the* compactly generated topology *of X. We write X_K for X with this topology.*

Note $\mathrm{id} : X_K \to X$ is continuous. X itself is called compactly generated, if $\mathrm{id} : X \to X_K$ is continuous as well.

1.2 Products and Coproducts in Top

Definition 1.11 *Let $\underline{\mathrm{C}}$ be a category, and $A, B \in \underline{\mathrm{C}}$. Then $A \amalg B \in \underline{\mathrm{C}}$ together with $p_A : A \amalg B \to A$, $p_B : A \amalg B \to B$ is called a* product *of A and B, if it has the following universal property:*

$$\begin{tikzcd} & & A \\ C \arrow[r, dashed, "\exists!"', "\{f,g\}"'] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \amalg B \arrow[ur, "p_A"] \arrow[dr, "p_B"'] & \\ & & B \end{tikzcd}$$

From the topology course of last semester, we know that “$\underline{\mathrm{Top}}$ has products”: $X \times Y$ with the *product topology* and p_X, p_Y the canonical projections.

Definition 1.12 *Let $\underline{\mathrm{C}}$ be a category, and $A, B \in \underline{\mathrm{C}}$. Then $A \amalg B \in \underline{\mathrm{C}}$ together with $i_A : A \to A \amalg B$, $i_B : B \to A \amalg B$ is called a* coproduct *of A and B, if it has the following universal property:*

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, bend left, "f"] & & \\ & A \amalg B \arrow[r, dashed, "\exists!", "\langle f,g\rangle"'] & C \\ B \arrow[ur, "i_B"'] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Theorem 1.13 $\underline{\text{Top}}$ *has coproducts:* $X, Y \in \underline{\text{Top}}$. *We write* $X \amalg Y \in \underline{\text{Top}}$ *for the disjoint union of* X *and* Y *with the topology coming from the open subsets in* X *and* Y, *and* i_X, i_Y *the canonical inclusions.*

Definition 1.14 $X \in \underline{\text{Top}}$ *is called connected, if for any two open, disjoint* $A, B \subset X$ *such that* $A \cup B = X$, *it follows that* $A = \varnothing$ *or* $B = \varnothing$. *(Equivalently: every map* $X \to \{0,1\}$, *where* $\{0,1\}$ *has the discrete topology, is constant.)*

Fact $X, Y \in \underline{\text{Top}}$ connected $\iff$ $X \times Y$ connected.

Corollary 1.15 $\mathbb{R} \ncong \mathbb{R}^2$.

Proof If $\phi : \mathbb{R} \xrightarrow{\cong} \mathbb{R}^2$, then

$$\phi|(\mathbb{R} \setminus \{0\}) : \underbrace{\mathbb{R} \setminus \{0\}}_{\text{not conn.}} \xrightarrow{\cong} \underbrace{\mathbb{R}^2 \setminus \{\phi(0)\}}_{\text{connected}}$$

which is a contradiction to the above fact. $\square$

1.3 Pullback and Pushout in Top

Definition 1.16 *Consider the diagram*

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in $\underline{\text{Top}}$. *Then the* pullback *of* f *and* g *is* $X \amalg_Z Y \in \underline{\text{Top}}$ *given by*

$$X \amalg_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y$$

(with subspace topology).

Lemma 1.17 $X \amalg_Z Y$ *has the following universal property:*

$$\begin{tikzcd} & & X \arrow[dr, "f"] & \\ W \arrow[urr, bend left, "\alpha"] \arrow[drr, bend right, "\beta"'] \arrow[r, dashed, "\exists!"', "h"] & X \amalg_Z Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & & Z \\ & & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof h is given by $\{\alpha, \beta\} : W \to X \times Y$ which maps into $X \amalg_Z Y$, because we assumed $f \circ \alpha = g \circ \beta$. $\square$

Note

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & \{\centerdot\} \end{tikzcd}$$

yields $X \amalg_{\{\centerdot\}} Y = X \times Y$ ($\{\centerdot\}$: terminal object in $\underline{\mathrm{Top}}$).

Definition 1.18 *Consider the diagram*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \\ X & \end{tikzcd}$$

in $\underline{\mathrm{Top}}$. *Then the* pushout $X \amalg_Z Y \in \underline{\mathrm{Top}}$ *of* f *and* g *is given by* $X \amalg Y/\sim$ *where* $i_X f(z) \sim i_Y g(z)$ *for all* $z \in Z$.

Lemma 1.19 $X \amalg_Z Y$ *has the following universal property:*

$$\begin{tikzcd} & Y \arrow[dr, "i_Y"] \arrow[drr, bend left] & & \\ Z \arrow[ur, "g"] \arrow[dr, "f"'] & & X \amalg_Z Y \arrow[r, dashed, "\exists!"] & W \\ & X \arrow[ur, "i_X"'] \arrow[urr, bend right] & & \end{tikzcd}$$

Sometimes we write $X \cup_Z Y$ instead of $X \amalg_Z Y$.

Note $\varnothing \overset{\exists!}{\to} X \in \underline{\mathrm{Top}}$: $\varnothing$ is an initial object in $\underline{\mathrm{Top}}$.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \amalg_\varnothing Y \end{tikzcd}$$

so $X \amalg_\varnothing Y = X \amalg Y$.

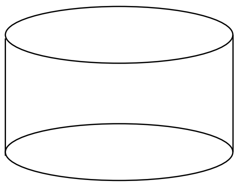


Figure 1: Cylinder on X

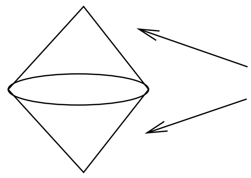


Figure 2: Suspension of X

1.4 Cone and Suspension

Definition 1.20 *Let* $I := [0,1] \in \underline{\mathrm{Top}}$ *be the unit interval,* $X \in \underline{\mathrm{Top}}$*. Then* $X \times I$ *is called the* cylinder on X *(figure 1) and*

$$CX := (X \times I)/(X \times \{1\})$$

the cone on X*.*

Definition 1.21 $\Sigma X := CX \amalg_X CX$ *is the* suspension *of* X*:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

where $i : X \hookrightarrow CX$, $x \mapsto \overline{(x,0)}$ *is the canonical inclusion (mapping points to equivalence classes).*
From figure 2, it follows that $\Sigma X \cong CX/(X \times \{0\})$*.*

Example $\Sigma S^n \cong S^{n+1}$

1.5 Homotopy

Definition 1.22 $f, g : X \to Y$ *in* $\underline{\mathrm{Top}}$ *are called* homotopic*, and we write* $f \simeq g$*, if* $\exists F : X \times I \to Y$ *with* $F(x,0) = f(x)$ *and* $F(x,1) = g(x)$*. We call* F *a* homotopy *from* f *to* g*, and write* $F : f \simeq g$*.*

"$\simeq$" is an equivalence relation on $C(X,Y)$; write

$$[X,Y] := C(X,Y)/\simeq$$

Homotopy is compatible with composition: If

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Z \underset{v}{\overset{u}{\rightrightarrows}} W$$

and $f \simeq g$, $\alpha \simeq \beta$, $u \simeq v$, then:

$$\begin{aligned} \alpha \circ f &\simeq \beta \circ g \\ u \circ \alpha &\simeq v \circ \beta \\ u \circ \alpha \circ f &\simeq v \circ \beta \circ g \end{aligned}$$

so we can define the homotopy category of topological spaces:

Definition 1.23 *$X, Y \in \underline{\mathrm{Top}}$, $f : X \to Y$ a continuous map. If there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$, then f is a* homotopy equivalence*.*
X and Y are called homotopy equivalent *if there is a homotopy equivalence between them.*

Definition 1.24 $\underline{\mathrm{HTop}}$ *is the category consisting of topological spaces as objects and* $\mathrm{mor}(X,Y) := [X,Y]$ *as morphisms. "Isomorphisms" in this category are homotopy equivalences (i.e. $X, Y \in \underline{\mathrm{Top}}$ are "isomorphic" if they are homotopy equivalent).*

Example $\mathbb{R}^n \simeq \mathbb{R}^m$, because $\mathbb{R}^n \simeq \{\centerdot\} \simeq \mathbb{R}^m$. Let:

$$\begin{aligned} F : \mathbb{R}^n \times I &\to \mathbb{R}^n \\ (x,t) &\mapsto tx \end{aligned}$$

then $F(x,1) = \mathrm{id}_{\mathbb{R}^n}(x)$, $F(\cdot,0) = (0 : \mathbb{R}^n \xrightarrow{0} \mathbb{R}^n)$ i.e. $F : 0 \simeq \mathrm{id}_{\mathbb{R}^n}$.
$\mathbb{R}^n \simeq \{\centerdot\}$:

$$f : \mathbb{R}^n \to \{\centerdot\}, \quad g : \{\centerdot\} \to \mathbb{R}^n, \centerdot \mapsto 0$$

$f \circ g = \mathrm{id}_{\{\centerdot\}}$ and $g \circ f = (x \mapsto 0) \simeq \mathrm{id}_{\mathbb{R}^n}$

Definition 1.25 *$X \in \underline{\mathrm{Top}}$ is called* contractible*, if $X \simeq \{\centerdot\}$.*

Example $\varnothing \neq X \in \underline{\mathrm{Top}} \Rightarrow CX \simeq \{\centerdot\}$. Proof:
$(CX \overset{\exists!}{\to} \{\centerdot\} \overset{\text{“cone point”}}{\to} CX) \simeq \mathrm{id}_{CX}$, where the equivalence is induced by:

$$\begin{aligned} \tilde{F} : (X \times I) \times I &\to X \times I \\ ((x,s),t) &\mapsto (x,(1-t)s+t) \end{aligned}$$

Definition 1.26 *$A \overset{i}{\hookrightarrow} X \in \underline{\mathrm{Top}}$ is called a* retract *if: $\exists\, r : X \to A$, s.t. $r \circ i = \mathrm{id}_A$ where i is the inclusion of A in X.*
A retract is called a deformation retract *if it satisfies the additional condition: $i \circ r \simeq \mathrm{id}_X$ with a homotopy $F : X \times I \to X$ satisfying $\forall a \in A, \forall t \in I : F(a,t) = a$.*

Example $\{\text{cone point}\} \subset CX$ is a deformation retract.

Definition 1.27 *Let $f : X \to Y$ be in $\underline{\mathrm{Top}}$.*

$$M_f := \big((X \times I) \amalg Y\big) / \big\langle (x,0) \sim f(x) \big\rangle$$

is called the mapping cylinder *of f.*

Definition 1.28 *Let $f : X \to Y$ be in $\underline{\mathrm{Top}}$.*

$$C_f := M_f / (X \times \{1\})$$

is called the mapping cone *of f.*

Obviously, $Y \overset{\text{can}}{\subset} M_f$ is a deformation retract ($\Rightarrow M_f \simeq Y$).

$$\text{can} : \quad \begin{tikzcd} x \arrow[d, mapsto] \\ (x,1) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\simeq"] & Y \\ M_f \arrow[ur, "\text{retraction } r"'] & \end{tikzcd}$$

The canonical inclusion is a so called “cofibration” (see later).

Note $C_f / Y \cong \Sigma X$

Definition 1.29 *Given $f : X \to Y$, a sequence:*

$$X \overset{f}{\to} Y \to C_f \to \Sigma X \overset{\Sigma f}{\to} \Sigma Y \to C_{\Sigma f} \to \Sigma^2 X \to \cdots$$

is called a mapping cone sequence *(Puppe sequence).*

Definition 1.30 *Let $X, Y \in \underline{\text{Top}}$, and $C(X,Y) := \{X \overset{cont}{\to} Y\}$, then*

$$M(K,U) := \{f \in C(X,Y) \mid f(K) \subset U\}$$

where $K \subset X$ compact and $U \subset Y$ open, defines a subbasis of the compact-open topology *(co-topology) on $C(X,Y)$.*
Notation: $CO(X,Y) \in \underline{\text{Top}}$ denotes $C(X,Y)$ with this topology.

Definition 1.31 *$x_0 \in X$, the map defined by:*

$$\begin{aligned} \mathrm{ev}_{x_0} : C(X,Y) &\to Y \\ f &\mapsto f(x_0) =: \mathrm{ev}_{x_0}(f) \end{aligned}$$

is called the evaluation map*.*

Note ev_{x_0} is continuous. Proof: $U \subset Y$ open $\Rightarrow \mathrm{ev}_{x_0}^{-1}(U) = \{f \in C(U,Y) \mid f(x_0) \in U\} = M(\underbrace{\{x_0\}}_{\text{compact}}, U)$ open in $CO(X,Y)$.

Problem: in sets

$$\{X \times Y \xrightarrow{f} Z\} \overset{\text{bij}}{\leftrightarrow} \left\{ \begin{array}{ll} \check{f} : X & \to \mathrm{maps}(Y,Z) \\ \quad x & \mapsto \check{f}(x) = (y \mapsto f(x,y)) \end{array} \right\}$$

Theorem 1.32 *$X, Y, Z \in \underline{\text{Top}}$, Y locally compact, then there is a canonical isomorphism: $C(X \times Y, Z) \overset{\cong}{\to} C(X, CO(Y,Z))$.*

Example $Y = I = [0,1]$

$$\left\{ \begin{array}{c} X \times I \to Z \\ \text{“homotopy”} \end{array} \right\} \overset{\text{bij}}{\leftrightarrow} \left\{ X \to \underbrace{CO(I,Z)}_{Z^I,\ \text{“path space on } Z\text{”}} \right\}$$

1.6 Pairs of topological spaces

Definition 1.33 *Let $X \in \underline{\text{Top}}$, the category whose objects are pairs (X,A) with $A \subset X$ a subspace, and morphisms $f : (X,A) \to (Y,B)$ with $f : X \to Y \in \underline{\text{Top}}$, $f(A) \subset B$ is called the* category of pairs *($\underline{\text{Top}^2}$).*

Note We have a functor $\underline{\text{Top}} \to \underline{\text{Top}^2}$, given by $X \mapsto (X, \varnothing)$.

Definition 1.34 *$X \in \underline{\mathrm{Top}}^2$ with $A = \{x_0\}$ (the* base-point*) is called a* pointed topological space*, and the category containig these spaces is the* category of pointed topological spaces ($\underline{\mathrm{Top}_\bullet}$). *Morphisms in this category are base-point preserving maps, and homotopies are always assumed to be based (i.e. base-point preserving).*

Note $\underline{\mathrm{Top}_\bullet} \subset \underline{\mathrm{Top}}^2$

Definition 1.35 *If $X, Y \in \underline{\mathrm{Top}_\bullet}$, then $X \simeq_\bullet Y$ denotes a based homotopy equivalence, $\underline{\mathrm{HTop}_\bullet}$ is the associated homotopy category.*

We usually think of $0 \in [0,1]$ to be the base-point of $[0,1] \in \underline{\mathrm{Top}_\bullet}$.

Definition 1.36 (wedge product) *The coproduct (see 1.12) in $\underline{\mathrm{Top}_\bullet}$ is defined as:*

$$X \vee Y := (X \amalg Y)/\langle x_0 \sim y_0 \rangle$$

where x_0, y_0 are the base-points of X, Y, and $\bar{x_0} = \bar{y_0}$ is the base-point of $X \vee Y$.

1.7 Mapping spaces

Let $X, Y \in \underline{\mathrm{Top}_\bullet}$ with base points x_0, y_0, then $X \times Y \in \underline{\mathrm{Top}_\bullet}$ with base point (x_0, y_0). Consider the "forget" functor $X \times Y \to Z$, with $Z \in \underline{\mathrm{Top}}$. As above, $CO(X,Y)$ denotes $C(X,Y)$ with the compact-open topology. We want a correspondence:

$$(f : X \times Y \to Z) \leftrightarrow (\check{f} : X \to CO_\bullet(Y,Z))$$

Definition 1.37

$$CO_\bullet(X,Y) := \{f \in CO(X,Y) \mid f(x_0) = y_0\}$$

with the constant map $c : x \mapsto y_0$ as base-point.

$CO_\bullet(X,Y) \subset CO(X,Y)$ with subspace topology. $\check{f}$ should be based ($x_0 \mapsto c$), i.e.

$$\check{f}(x_0)(y) = f(x_0, y) = z_0$$

$\Rightarrow f$ must map $\{x_0\} \times Y$ to $\{z_0\}$. Similiarly, $\check{f}(x)(y_0) = f(x, y_0) = z_0$. This motivates the following definition.

Definition 1.38 (smash product)

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

Theorem 1.39 *Let $X, Y, Z \in \underline{\mathrm{Top}_\bullet}$, Y locally compact, and define $C_\bullet(X, Y)$ to be the set of pointed maps $X \to Y$. Then*

$$C_\bullet(X \wedge Y, Z) \overset{bij}{\to} C_\bullet(X, CO_\bullet(Y, Z))$$

Example $S^1 \wedge X \overset{\text{can}}{\leftarrow} \Sigma X$ ($SX := S^1 \wedge X$ is called the *reduced suspension* of X). We can set e.g. $Y = S^1$, then

$$C_\bullet(X \wedge S^1, Z) \overset{\text{bij}}{\to} C_\bullet(X, \Omega Z)$$

where ΩZ denotes the *loop space* $CO_\bullet(S^1, Z)$ (which consists of the loops in Z at the base-point z_0).
So we have

$$\underline{\mathrm{Top}_\bullet} \underset{\Omega}{\overset{S}{\rightleftarrows}} \underline{\mathrm{Top}_\bullet}$$

where

$$S(X) = SX = S^1 \wedge X$$
$$\Omega(X) = \Omega X = CO_\bullet(S^1, X)$$

(S left-adjoint to Ω, Ω right-adjoint to S) and we get a natural bijection

$$C_\bullet(SX, Y) \overset{\simeq}{\to} C_\bullet(X, \Omega Y)$$

Furthermore we can pass to the homotopy categories

$$\underline{\mathrm{HTop}_\bullet} \underset{\Omega}{\overset{S}{\rightleftarrows}} \underline{\mathrm{HTop}_\bullet}$$

and get

$$[SX, Y]_\bullet \overset{\text{bij}}{\to} [X, \Omega Y]_\bullet$$

i.e. S, Ω is still a pair of adjoint functors. (see Hatcher, p.530, discussion after Prop.A.14)

1.8 Homotopy groups

Definition 1.40 (fundamental group) *Let $X \in \underline{\mathrm{Top}_\bullet}$, then the* fundamental group *of X is defined as:*

$$\pi_1 X := [S^1, X]_\bullet$$

Definition 1.41 *For $n \geq 2$,*

$$\pi_n X := \pi_1(\Omega^{n-1} X)$$

where $\Omega^i X = \Omega(\Omega^{i-1} X)$ $(i \geq 1)$ and $\Omega^0 X = X$.

Note

$$[S^n, X]_{\bullet} \overset{\text{bij}}{\to} [S^{n-1}, \Omega X]_{\bullet} \overset{\text{bij}}{\to} [S^1, \Omega^{n-1} X]_{\bullet} = \pi_n X$$

Claim $\pi_n X$ is abelian for $n \geq 2$. This follows from

Theorem 1.42 *Let $Y \in \underline{\mathrm{Top}_{\bullet}}$. Then $\pi_1 \Omega Y$ is abelian.*

Proof Let $\mu : \Omega Y \times \Omega Y \to \Omega Y$ be the obvious multiplication of loops (usually written $\mu(\omega, \sigma) = \omega \star \sigma$). $(\Omega Y, \mu)$ is a "group up to homotopy". This means:

i) *associative*: The diagram

$$\begin{tikzcd}
\Omega Y \times \Omega Y \times \Omega Y \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"'] & \Omega Y \times \Omega Y \arrow[d, "\mu"] \\
\Omega Y \times \Omega Y \arrow[r, "\mu"'] & \Omega Y
\end{tikzcd}$$

commutes up to homotopy.

ii) *inverses*: $\exists i : \Omega Y \to \Omega Y$ such that

$$\begin{tikzcd}
\Omega Y \arrow[r, "\{\mathrm{id}, i\}"] \arrow[dr, "\mathrm{const}"'] & \Omega Y \times \Omega Y \arrow[d, "\mu"] & \Omega Y \arrow[l, "\{i, \mathrm{id}\}"'] \arrow[dl, "\mathrm{const}"] \\
& \Omega Y &
\end{tikzcd}$$

commutes up to homotopy.

iii) *identity element*:

$$\begin{tikzcd}
\Omega Y \arrow[r, "\{\mathrm{id}, \mathrm{const}\}"] \arrow[dr, equal] & \Omega Y \times \Omega Y \arrow[d, "\mu"] & \Omega Y \arrow[l, "\{\mathrm{const}, \mathrm{id}\}"'] \arrow[dl, equal] \\
& \Omega Y &
\end{tikzcd}$$

So $[W,\Omega Y]_\bullet$ is a group, induced by μ.

$$\begin{array}{ccccc} [W,\Omega Y]_\bullet \times [W,\Omega Y]_\bullet & \xrightarrow[\text{can}]{\simeq} & [W,\Omega Y \times \Omega Y]_\bullet & \xrightarrow{\mu_\star} & [W,\Omega Y]_\bullet \\ & & [\phi] & \longmapsto & [\mu \circ \phi] \end{array}$$

Now look at $\pi_1\Omega Y = [S^1,\Omega Y]_\bullet$. This group has two group structures: The "π-product" (being a fundamental group $\pi_1(\cdot)$) and the "μ-product" (being a loop space).
Now we have to show that π-product $=$ μ-product, and that the group is commutative.
$\mu : \Omega Y \times \Omega Y \to \Omega Y$ induces a π-homomorphism

$$\pi_1\Omega Y \times \pi_1\Omega Y \xrightarrow{\mu_\star} \pi_1\Omega Y$$

Therefore:

$$\begin{aligned} \mu_\star((\alpha,\beta) \underset{\pi}{+} (\gamma,\delta)) &= \mu_\star(\alpha,\beta) \underset{\pi}{+} \mu_\star(\gamma,\delta) \\ \Leftrightarrow \mu_\star(\alpha \underset{\pi}{+} \gamma, \beta \underset{\pi}{+} \delta) &= \mu_\star(\alpha,\beta) \underset{\pi}{+} \mu_\star(\gamma,\delta) \\ \Leftrightarrow (\alpha \underset{\pi}{+} \gamma) \underset{\mu}{+} (\beta \underset{\pi}{+} \delta) &= (\alpha \underset{\mu}{+} \beta) \underset{\pi}{+} (\gamma \underset{\mu}{+} \delta) \end{aligned}$$

e.g. taking $\gamma = \beta = e$ shows that the group structure is the same:

$$\alpha \underset{\mu}{+} \delta = \alpha \underset{\pi}{+} \delta$$

and taking $\alpha = \delta = e$ shows that the group is abelian:

$$\gamma \underset{\mu}{+} \beta = \beta \underset{\pi}{+} \gamma = \beta \underset{\mu}{+} \gamma$$

□

More generally we could use the same proof to show the

Theorem 1.43 *X an H-space ("Hopf") $\Rightarrow$ $\pi_1 X$ abelian, $d : X \times X \to X$ with 2-sided unit up to homotopy (note: no associativity or inverses required!).*

Corollary 1.44 *G Lie group, $e \in G$ base-point $\Rightarrow$ $\pi_1 G$ abelian.*

1.9 Adjoint Functors

$$\begin{tikzcd} \underline{\mathrm{C}} \arrow[r, shift left, "F"] & \underline{\mathrm{D}} \arrow[l, shift left, "G"] \end{tikzcd}$$

Suppose one has a natural bijection:

$$\mathrm{mor}_{\underline{\mathrm{C}}}(GX, Y) \overset{\text{bij}}{\to} \mathrm{mor}_{\underline{\mathrm{D}}}(X, FY)$$

Then G is called a left-adjoint to F and F is called a right-adjoint to G.
$\Rightarrow$ "G commutes with colim" (e.g. coproducts, pushout); "F commutes with lim" (e.g. products, pullback).

2 CW-Complexes

Definition 2.1 *A CW-structure on $X \in \underline{\mathrm{Top}}$ is a filtration $X_{-1} = \varnothing \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq X$ with:*

1. *$X = \bigcup X_n = \mathrm{colim}_{n \geq 0} X_n$, i.e. $A \subset X$ open $\Leftrightarrow A \cap X^n$ open $\forall n$*
2. *X^n is a push-out of:*

$$\begin{tikzcd} \amalg S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \amalg D^n \arrow[r, "\tilde{f}"] & X^n = X^{n-1} \cup_f \amalg D^n \end{tikzcd}$$

2.1 Facts and definitions

1. X^n is called a n-skeleton, f the *attaching map* for the n-cells.
2. CW-complexes are Hausdorff.
3. $\tilde{f}(D^n) =: \bar{e}^n$ is called a "closed $n-$cell".
4. $\tilde{f}(\mathring{D}^n) =: e^n$ is called an "open $n-$cell".

   Remark e^n is in general *not* open in X.

5. $A \subset X \in \underline{\mathrm{CW}}$, is called a *subcomplex* of X if A is closed and an union of cells of X. (A has to be closed to ensure that it has a proper CW-structure.)
6. By construction: as a *set* $X = \coprod_n \coprod_k e_k^n$

7. $X \in \underline{\mathrm{CW}}$ is called *finite* if it is a "union" (see Hatcher, example 0.6) of finitely many cells. A finite CW-complex is *compact.*

8. $X \in \underline{\mathrm{CW}}$, $C \subset X$ compact $\Rightarrow \exists A$ finite subcomplex of X, with $C \subset A$.

9. Each $X \in \underline{\mathrm{CW}}$ is compactly generated as a space. (Proof: $B \subset X$ closed $\Leftrightarrow \tilde{B} \subset \tilde{f}^{-1}(B)$ closed in $\coprod_x D^n_x \Leftrightarrow \tilde{B} \cap D^n_x$ closed $\forall x, n \Leftrightarrow$ $B \cap \bar{e}^n_x$ closed).

10. $X^0 \subseteq X$ is discrete, i.e. composed of single points.

11. If $X = X^1$ then X is called *graph.*

12. A CW-complex X is connected if and only if it is *path-connected.*

13. $X \in \underline{\mathrm{CW}}$ is called n-dimensional if $X = X^n$

Example $S^n, \mathbb{R}P^n(\mathbb{C}P^n), T^2 = S^1 \times S^1$.
S^n:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "c"] \arrow[d, hook] & \{\bullet\} \arrow[d] \\
D^n \arrow[r] & S^n = D^0 \cup_c D^n
\end{tikzcd}$$

Alternatively: $S^n = D^0 \amalg D^0 \cup D^1 \amalg D^1 \cup \ldots \cup D^n \amalg D^n$, $S^1 = D^0 \amalg D^0 \cup D^1 \amalg D^1$
$\mathbb{R}P^n = S^n / \langle x \sim -x \rangle$:

$$\begin{tikzcd}
S^{n-1} \amalg S^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\Gamma_{\bullet}"] & S^{n-1} \arrow[d, hook] \arrow[loop right, "T"] \\
D^n_+ \amalg D^n_- \arrow[r] & S^n \arrow[loop right, "T"]
\end{tikzcd}$$

where $T : S^n \to S^n, x \mapsto -x$.
One can extend the antipode T to the whole push-out diagram by letting it exchange D^n_+ with D^n_-.
$\mathrm{quot}(\Gamma_{\bullet}) = \Gamma_{\bullet} / \langle x \sim Tx \rangle$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{R}P^{n-1} \arrow[d, hook] \\
D^n \arrow[r] & \mathbb{R}P^n
\end{tikzcd}$$

$\Rightarrow \mathbb{R}P^n = D^0 \cup D^1 \cup \ldots \cup_f D^n$.
$\mathbb{C}P^n$: see above, $\mathbb{C}P^n = D^0 \cup D^2 \cup D^4 \ldots \cup D^{2n}$.

Torus: $T = S^1 \times S^1 = S^1 \vee S^1 \cup_f D^2$

$$\begin{tikzcd}
S^1 \arrow[r, "f"] \arrow[d, hook] & S^1 \vee S^1 \arrow[d, hook] \\
D^2 \arrow[r] & S^1 \vee S^1 \cup_f D^2 =: T
\end{tikzcd}$$

Definition 2.2 $f : X \to Y$*,* $X, Y \in \underline{\mathrm{CW}}$ *is called cellular if:*

$$f(X^n) \subseteq Y^n, \quad \forall n \geq 0$$

Theorem 2.3 (Cellular Approximation Theorem) *Let* $f : X \to Y$*,* $X, Y \in \underline{\mathrm{CW}}$*,* f *continuous, then* f *is homotopic to a cellular map* $g : X \to Y$*.*

Proof (later, simplicial approx.) □

Remark There is a relative version of the cellular approximation theorem: let $f \in \underline{\mathrm{CW}}^2$, $f : (X, A) \to (Y, B)$ ($(X, A) \in \underline{\mathrm{Top}}^2$, where X and $A \subset X$ have a CW-structure) with $f|A : A \to B$ cellular, then there is a cellular map $g : (X, A) \to (Y, B)$ with $f \simeq g$ and $f|A = g|A$.

Corollary 2.4 *For* $0 < k < n$*,* $\pi_k(S^n) = 0$*.*

Proof $\pi_k(S^n) = [S^k, S^n]_\bullet$. Let $[f] \in \pi_k(S^n)$, $f : S^k \to S^n$, replace f by g, $g \simeq f$, and g cellular.
$S^n = D^0 \amalg D^0 \cup \ldots \cup D^n \amalg D^n$

$$\begin{tikzcd}
g : S^k \arrow[r] \arrow[dr, dashed] & (S^n)^k \arrow[r, phantom, "\subsetneq"] & S^n \\
& S^n \setminus \{\mathrm{pt.}\} \arrow[r, phantom, "\simeq"] & \{\bullet\} \arrow[u, hook, "\subseteq"]
\end{tikzcd}$$

$\Rightarrow g \simeq$ const. $\Rightarrow \pi_k(S^n) = 0$. □

Corollary 2.5 X *connected,* $X \in \underline{\mathrm{CW}_\bullet}$*,* $X = \bigcup_{n \geq 0} X^n$

- $k \geq n + 1 \Rightarrow \pi_n X^k \xrightarrow{\cong} \pi_n X$.
- $k = n \Rightarrow \pi_n X^n \twoheadrightarrow \pi_n X$

Proof $[f : S^n \to X] \in [S^n, X]_\bullet$, CW-app. $\Rightarrow \exists g : S^n \to X$, $f \simeq g$, g cellular.
$\Rightarrow$

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "g"'] & X \\ & X^k \arrow[u, hook] \end{tikzcd}$$

$k \geq n \Rightarrow \pi_n X^k \twoheadrightarrow \pi_n X$
If $f \simeq g \in [S^n, X]$ $\exists H : S^n \times I \to X$, $H(\cdot, 0) = f$, $H(\cdot, 1) = g$, $H(x_0, t) = y_0$.
Serie 3, ex.1: $S^n \times I$ is $n+1$-dim. CW-complex. $\overset{\text{CW-appr.}}{\Rightarrow}$ $\exists \tilde{H}$:

$$\begin{tikzcd} S^n \times I \arrow[r, "H"] \arrow[dr, "\tilde{H}"'] & X \\ & X^k \arrow[u, hook] \end{tikzcd} \qquad (k \geq n+1)$$

$f, g \in [S^n, X^k]_\bullet = \pi_n(X^k)$
$f \simeq g \Rightarrow \pi_n(X^k) \to \pi_n(X)$ is injective for $k \geq n+1$. $\square$

Corollary 2.6 *X connected CW-complex ($x_0 \in X$):*

$$\pi_1 X^2 \xrightarrow{\cong} \pi_1 X$$

Definition 2.7 *$A \subset X$, is a* neighbourhood deformation retract *(NDR) if there is an (open) neighbourhood $B \subset X$ of A and $A \subset B$ a deformation retract.*

Lemma 2.8 *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "NDR"'] & Y \arrow[d, hook] \\ X \arrow[r] & Z \end{tikzcd}$$

with f an arbitrary map, be a push-out (in Top*). Then $Y \subset Z$ is a NDR.*

Example

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook, "\text{NDR}"'] & X \arrow[d, hook] \\ D^n \arrow[r] & X \cup_{S^{n-1}} D^n = X \cup_f D^n \end{tikzcd}$$

Corollary 2.9 *$X \in$ CW, $A \subset X$ subcomplex $\Rightarrow A \subset X$ NDR.*

Definition 2.10 *(Summarized from Topology SS 05, which see. Ed.)*
The amalgamated product $G = G_1 *_{G_{12}} G_2$ *is defined by the following push-out in* $\underline{\mathrm{Gr}}$*:*

$$\begin{tikzcd} G_{12} \arrow[r] \arrow[d] & G_1 \arrow[d] \\ G_2 \arrow[r] & G \end{tikzcd}$$

If $G_{12} = 1$*, then* $G = G_1 * G_2$ *is called the* free product*.*

Theorem 2.11 (Classical van Kampen) $X = U \cup V$*,* $X \in \underline{\mathrm{Top}_\bullet}$*,* $U, V \subset X$ *open. If* $U, V, U \cap V$ *path-connected:*

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook] & X \end{tikzcd} \quad \overset{\pi_1}{\rightsquigarrow} \quad \begin{tikzcd} \pi_1(U \cap V) \arrow[r, "\beta"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\Gamma."] & \pi_1(V) \arrow[d] \\ \pi_1(U) \arrow[r] & \pi_1(X) \end{tikzcd}$$

i.e. $\pi_1(X) \cong (\pi_1(V) * \pi_1(U)) \,/\, \langle \alpha x (\beta x)^{-1},\ x \in \pi_1(U \cap V) \rangle$.

There is a more general version of the classical van Kampen theorem, which does not require the involved sets to be open.

Theorem 2.12 (van Kampen for push-outs)

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & V \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$

a push-out in $\underline{\mathrm{Top}_\bullet}$*, with* $U \subset W$ *and* $U \subset V$ *NDRs, and* U, V, W *path-connected, then:*

$$\pi_1 X \text{ is push-out of: } \begin{tikzcd} \pi_1 U \arrow[r] \arrow[d] & \pi_1 V \\ \pi_1 W & \end{tikzcd}$$

Proof Look at:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & V_1 \arrow[r, hook] \arrow[dd, hook] & V \arrow[d, hook] \\ W_1 \arrow[rr] \arrow[d, hook] & & V \cup_U W_1 \arrow[d, hook] \\ W \arrow[r] & W \cup_U V_1 \arrow[r, hook] & X \end{tikzcd}$$

$\square$

Example $X \in \underline{\mathrm{CW}_\bullet}$ connected, $X = A \cup B$, A, B connected subcomplexes. $C := A \cap B$ is then also a subcomplex; assume it is connected. $\Rightarrow C \subset A$ and $C \subset B$ are NDR. Then:

$$\pi_1 X \cong \text{push-out:} \quad \begin{tikzcd} \pi_1 C \arrow[r] \arrow[d] & \pi_1 A \\ \pi_1 B & \end{tikzcd}$$

Corollary 2.13 $X, Y \in \underline{\mathrm{CW}_\bullet} \Rightarrow \pi_1(X \vee Y) \cong \pi_1 X * \pi_1 Y$ *(free product $\equiv$ coproduct in $\underline{\mathrm{Gr}}$)*

Proof

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \pi_1 X \arrow[d] \\ \pi_1 Y \arrow[r] & \pi_1 X * \pi_1 Y \end{tikzcd} \qquad \text{(free product)}$$

$\square$

Example Free group in 2 generators: $\pi_1(S^1 \vee S^1) \cong \pi_1 S^1 * \pi_1 S^1 \cong \mathbb{Z} * \mathbb{Z}$ $(\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z})$

If you choose a base-point of $X \in \underline{\mathrm{CW}_\bullet}$, it should be a 0-cell. Now some CW-Complexes have more than one 0-cell, so you want to find a space which has exactly one 0-cell, e.g. $S^n = D^0 \cup_\phi D^n$, instead of $S^n = D^0 \amalg D^0 \cup D^1 \amalg D^1 \cup \ldots \cup D^n \amalg D^n$.

2.2 HEP: Homotopy Extension Property

Definition 2.14 $(X, A) \in \underline{\mathrm{Top}}^2$. *$A \subset X$ has the* homotopy extension property *(HEP) if for every $f : X \to Y$ and homotopy $F : f|A \simeq g : A \to Y$ we can extend F to $\tilde{F} : X \times I \to Y$ such that*

$$\tilde{F}|(A \times I) = F$$

This is often expressed as a diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \arrow[dr, "f"] & \\ A \times I \arrow[r, hook] \arrow[rr, bend right, "F"'] & X \times I \arrow[r, dotted, "\exists \tilde{F}"] & Y \end{tikzcd}$$

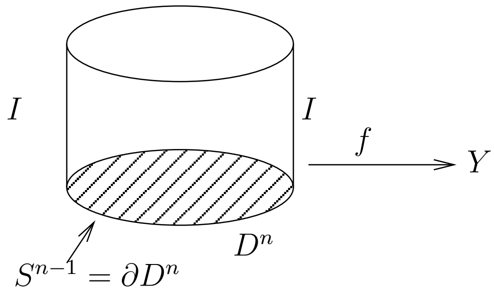


Figure 3: $S^{n-1} \subset D^n$

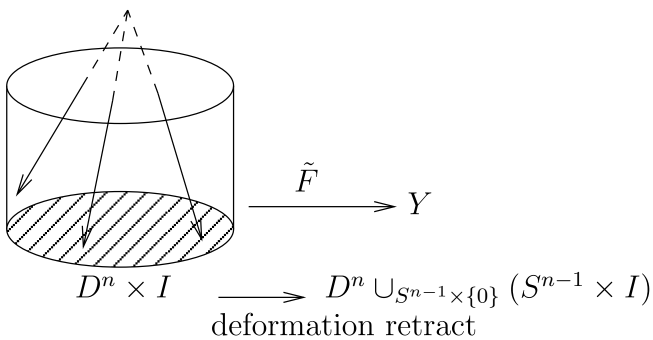


Figure 4: Definition of $\tilde{F}$

Example $S^{n-1} \subset D^n$ has HEP.

Proof Look at figure 3. From

$$\begin{aligned} f :& D^n \to Y \\ f|S^{n-1} :& S^{n-1} \to Y \\ F :& S^{n-1} \times I \to Y \\ F :& f|S^{n-1} \simeq g \end{aligned}$$

we get a map

$$\begin{tikzcd} D^n \cup_{S^{n-1}\times\{0\}} (S^{n-1} \times I) \arrow[r, "\Phi=\langle f,F\rangle"] & Y \\ D^n \times I \arrow[u, "\rho"] \arrow[ur, dotted, "?"', "\tilde{F}"] & \end{tikzcd}$$

so we define $\tilde{F}$ as in figure 4, namely $\tilde{F} := \Phi \circ \rho$.

$\square$

Lemma 2.15

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "has\ HEP"'] & B \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

push-out in $\underline{\text{Top}} \Rightarrow B \subset Y$ *has HEP.*

Definition 2.16

$$X^Y := C(Y, X)$$

Remark

$$(X \times I \to Y) \overset{\text{bij}}{\to} (X \to Y^I)$$

Proof

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, hook] \arrow[rr, bend left, "F \circ f = G|A"] & B \arrow[d, hook] \arrow[r, "F"] & Z^I \\ X \arrow[r] \arrow[rru, bend right, "G \text{ from HEP}"'] & Y \arrow[ru, "\tilde{F} \text{ from p.o.}"] & \end{tikzcd}$$

$\Rightarrow$ get $\tilde{F}$ from push-out property ($\tilde{F}$ induced by $\{\tilde{G}, F\}$). $\square$

Corollary 2.17 $S^{n-1} \subset D^n$ *has HEP in*

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, phantom, "\cap"] & Y \arrow[d, phantom, "\cap"] \\ D^n \arrow[r] & Y \cup_f D^n \end{tikzcd}$$

therefore so does $Y \subset (Y \cup_f D^n)$.

Note HEP is transitive: $U \subset V$ HEP, $V \subset W$ HEP $\Rightarrow U \subset W$ HEP.

Theorem 2.18 $(X, A) \in \underline{\text{CW}}^2 \Rightarrow A \subset X$ *has the HEP.*

Theorem 2.19 $(X, A) \in \underline{\text{CW}}^2$, $A \simeq \bullet$ *(contractible)* $\Rightarrow$

$$pr : X \to X/A$$

is a homotopy equivalence (note that X/A *is a CW-complex, see homework set 3).*

Proof

$$\begin{tikzcd} A \arrow[r, "G"] \arrow[d, hook] & X^I \arrow[d, "ev_0"] \\ X \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "\exists \tilde{G}"] & X \end{tikzcd}$$

corresponds to $\mathrm{id}_A \simeq_G c_{a_0}$ $(G : A \times I \to A)$, so $\exists \tilde{G} : X \times I \to X$ with

$$\begin{aligned} \tilde{G}(x, 0) &= x \\ \tilde{G}(a, 1) &= a_0 \\ \tilde{G}(a, t) &\in A \end{aligned}$$

and therefore $\tilde{G}$ defines a map H by

$$\begin{tikzcd} (X/A) \times I \arrow[r, "H"] & X/A \\ X \times I \arrow[u, "pr"] \arrow[r, "\tilde{G}"'] & X \arrow[u, "pr"'] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\tilde{G}(\cdot,1)"] \arrow[d, "pr_{X/A}"'] & X \arrow[d, "pr_{X/A}"] \\ X/A \arrow[r, "\simeq \mathrm{id}_{X/A}"'] \arrow[ur, dashed, "\tilde{g}"'] & X/A \end{tikzcd}$$

$\Rightarrow$ $\tilde{g}$ and $pr_{X/A}$ are homotopy inverses. $\square$

Definition 2.20 *Every group G can be described by generators g_i and relators r_i. If there are only finitely many of them, as in*

$$G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle$$

then the group is called finitely presented *and G is countable. In this case we can also describe it as*

$$G = (\textit{free group on } (\tilde{g}_1, \ldots, \tilde{g}_n))/(\textit{normal subgroup generated by words } \tilde{r}_i)$$

Example (i) $G = \langle g \mid \rangle \cong \mathbb{Z}$

(ii) $G = \langle g \mid g^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$

(iii) $G = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

Theorem 2.21 *Let $X \in \underline{\mathrm{CW}}$, $A \subset X$ subcomplex, $A \simeq \centerdot$. Then $X \xrightarrow{\simeq} X/A$, and therefore $\pi_1 X \cong \pi_1(X/A)$.*

Example $X \in \underline{\mathrm{CW}_\centerdot} \Rightarrow \Sigma X \simeq S^1 \wedge X =: SX$ ("reduced suspension")

$$\begin{tikzcd} & (S^1 \times X)/(S^1 \vee X) \arrow[d, phantom, "="] \\ \Sigma X \arrow[r, "\simeq"] & S^1 \wedge X \arrow[d, phantom, "\cong"] \\ & \Sigma X/(I \times \{x_0\}) \end{tikzcd}$$

$X \in \underline{\mathrm{CW}_\centerdot}$ connected ($\Rightarrow$ path-connected) $\Rightarrow X_1 \subset X$ connected, i.e. X_1 is a connected graph which contains a maximal subtree $T \subset X_1$. Note that a tree is a *contractible subcomplex* since it may not contain any loops! T also contains all vertices in X_1 (if one is missing, attach it through an edge of choice).
Now suppose we contract T:

$$X \xrightarrow{\simeq} X/T$$
$$\pi_1 X \xrightarrow{\cong} \pi_1(X/T)$$

X/T has just one 0-cell, so it forms a natural base-point!
If we take $Y \in \underline{\mathrm{CW}_\centerdot}$ with $Y_0 = \{\text{base-point}\}$ ($\Rightarrow Y$ connected), then

$$Y_0 \subset Y_1 : \qquad \begin{tikzcd} \amalg_I S^0 \arrow[r] \arrow[d, hook] & Y_0 = \{\centerdot\} \arrow[d, phantom, "\cap"] \\ \amalg_I D^1 \arrow[r] & Y_1 = \bigvee_I S^1 \end{tikzcd}$$

and

$$Y_0 \subset Y_1 \subset Y_2 : \qquad \begin{tikzcd} \amalg S^1 \arrow[r, "\Phi"] \arrow[d, hook] & Y_1 \arrow[d] \\ \amalg D^2 \arrow[r] & Y_2 \end{tikzcd}$$

where Φ is homotopic to a cellular map $\tilde{\Phi}$ ($S^1 = D^0 \cup D^1$ as CW-complex).
Replacing Φ by $\tilde{\Phi}$ yields the push-out

$$\begin{tikzcd} \bigvee S^1 \arrow[r, "\tilde{\Phi}"] \arrow[d, hook] & Y_1 \arrow[d] \\ \bigvee D^2 \arrow[r] & \tilde{Y}_2 \simeq Y_2 \end{tikzcd}$$

($\tilde{Y}_2 \simeq Y_2$ by Hatcher, prop. 0.18, $\tilde{Y}_2$ a CW-complex by exercice 3.5) i.e.

$$\tilde{Y}_2 = (\bigvee_I S^1) \cup_{\tilde{\Phi}} (\bigvee_J D^2)$$

Recall: $X \in \underline{\mathrm{CW}_\bullet}$ connected $\Rightarrow \pi_1 X_2 \xrightarrow{\cong} \pi_1 X$.
$X_2 \supset X_1 \supset T$ maximal subtree:

$$\pi_2 X \cong \pi_2 X_2 \cong \pi_2(X_2/T)$$

(note $X_2/T \simeq (\bigvee_I S^1) \cup (\bigvee_J D^2)$).

Lemma 2.22

$$\pi_1\left((\bigvee_I S^1) \cup_{\tilde{\Phi}} (\bigvee_J D^2)\right) \cong \langle g_\alpha, \alpha \in I \mid r_\beta, \beta \in J\rangle$$

Note $\tilde{\Phi}$ yields maps

$$D^2_\beta \supset S^1_\beta \xrightarrow{\tilde{\Phi}_\beta} \bigvee_I S^1$$

with

$$[\tilde{\Phi}_\beta] \in \pi_1(\bigvee_I S^1) \cong F(I)$$

where $F(I)$ is the free group on I.

Proof Use van Kampen Theorem for CW-complexes:

$$\begin{tikzcd} \bigvee_J S^1 \arrow[r, "\tilde{\Phi}"] \arrow[d, hook] & \bigvee_I S^1 \arrow[d, hook] \\ \bigvee_J D^2 \arrow[r] & (\bigvee_I S^1) \cup_{\tilde{\Phi}} (\bigvee_J D^2) \end{tikzcd}$$

(note that $\bigvee_J D^2$ is contractible). Applying π_1 we can map this into a push-out on $\underline{\mathrm{Gr}}$:

$$\begin{tikzcd} \pi_1(\bigvee_J S^1) \arrow[r, "\tilde{\Phi}_\#"] \arrow[d] & \pi_1(\bigvee_I S^1) \arrow[d] \\ \{1\} \arrow[r] & G \end{tikzcd}$$

$\pi_1(\bigvee_J S^1)$ is the free group on J, and similiarly for I, so we can write

$$\tilde{r}_\beta := \tilde{\Phi}_\#(f_\beta)$$

and get

$$G \cong \langle g_\alpha, \alpha \in I \mid r_\beta, \beta \in J\rangle$$

where r_β corresponds to $\tilde{r}_\beta$. $\square$

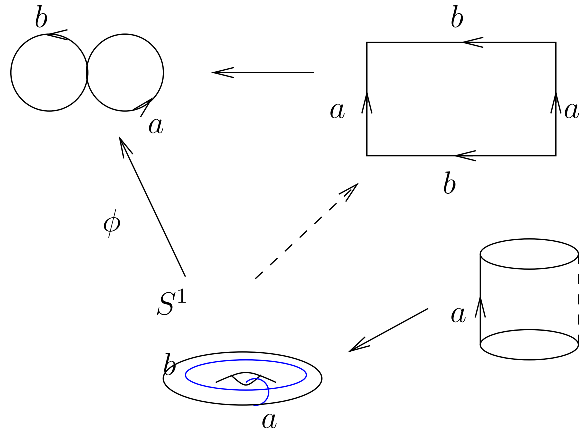


Figure 5: Example 2

Corollary 2.23 *Let $G = \langle g_\alpha, \alpha \in I \mid r_\beta, \beta \in J \rangle$, then there is a "canonical" 2-dimensional CW-complex $X(G)$ with $\pi_1 X(G) \cong G$, namely $X(G) := (\bigvee_I S^1) \cup_\phi (\bigvee_J D^2)$ where ϕ has components $\phi_\beta : S^1 \to \bigvee_I S^1$ corresponding to the r_β's. ($X(G)$ is called the* presentation complex *of G with its presentation).*

Example

1. $G = \mathbb{Z} = \langle g \mid \rangle \Rightarrow X(G) = S^1$ $(\pi_1 S^1 \cong \mathbb{Z})$
2. $G = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle \Rightarrow X(G) = (S^1 \vee S^1) \cup_\phi D^2$, $\phi : S^1 \to S^1 \vee S^1 : [\phi] \in \pi_1(S^1 \vee S^1) \cong \mathbb{Z} \times \mathbb{Z} = \langle \tilde{a} \rangle \times \langle \tilde{b} \rangle$, $[\tilde{\phi}] = \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}$.

   Obviously: $(S^1 \vee S^1) \cup_\phi D^2 \cong S^1 \times S^1$.

$$
\begin{tikzcd}
S^1 \arrow[r, "\phi"] \arrow[d] & S^1 \vee S^1 \arrow[d, hook] \\
D^2 \arrow[r] & (S^1 \vee S^1) \cup_\phi D^2 \cong S^1 \times S^1
\end{tikzcd}
$$

   a push-out, see also figure 5. $(\pi_1(S^1 \times S^1) \cong \pi_1 S^1 \times \pi_1 S^1 \cong \mathbb{Z} \times \mathbb{Z})$.

Definition 2.24 *$X \in \underline{\mathrm{CW}_\bullet}$ is called a $K(G, 1)$, if:*

i. *X connected*

ii. $\pi_1 X \cong G$

iii. $\pi_i X = 0$, $i > 1$

Remark (without proof) Such an X depends up to homotopy only on G.

Actually:

$$\underline{\mathrm{Gr}} \xrightarrow{K(\cdot,1)} \underline{\mathrm{HCW}_{\bullet}}$$

$$\begin{tikzcd} G \arrow[r, mapsto] \arrow[d, "f"] & K(G,1) \arrow[d, "K(f,1)"] \\ H \arrow[r, mapsto] & K(H,1) \end{tikzcd}$$

where $\underline{\mathrm{HCW}_{\bullet}}$ is the homotopy category of pointed CW-complexes.
$K(\cdot,1)$ a functor.
$K(\cdot,1)$: "fully faithful", i.e.

1. $K(G,1) \simeq K(H,1) \Rightarrow G \cong H$
2. $\hom(G,H) \overset{\text{bij.}}{\to} [K(G,1),K(H,1)]_{\bullet}$

Example 1. $K(\mathbb{Z},1) = S^1$ (i.e. $\pi_1 S^1 \cong \mathbb{Z}$, and $\pi_i S^1 = 0\ \forall i > 1$)

$\pi_i S^1 = \{0\}$ for $i > 1$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\text{universal cover}"] \\ S^i \arrow[r, "\phi"] \arrow[ur, dashed] & S^1 \end{tikzcd}$$

$\Rightarrow \phi \simeq \bullet$ since $\mathbb{R} \simeq \{\bullet\}$.

2. $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{R}P^\infty = \bigcup \mathbb{R}P^n$, where $\mathbb{R}P^n$ is the n-skeleton of $\mathbb{R}P^\infty$
$\pi_1 \mathbb{R}P^\infty = \pi_1 \mathbb{R}P^2 = \pi_1(S^1 \cup_\phi D^2) = \langle g \mid g^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $\phi : S^1 \to S^1$ of degree 2.
$i > 1$: $\pi_i \mathbb{R}P^\infty \cong \pi_i(\mathbb{R}P^{i+1}) \cong \pi_i S^{i+1} = \{0\}$ as $i < i+1$.
$S^{i+1} \to \mathbb{R}P^{i+1}$: 2-fold cover

3. Similarly (but harder): $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ i.e. $\pi_i \mathbb{C}P^\infty = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & \text{else} \end{cases}$

3 Homology Theories

Axioms: (S. Eilenberg + N. Steenrod, early 50's)

$$\begin{array}{ccc} \underline{\mathrm{Top}}^2 & \ni & (X,A) \\ \big\downarrow I & & \big\downarrow \\ \underline{\mathrm{Top}}^2 & \ni & (A,\varnothing) \\ \big\uparrow & & \big\uparrow \\ \underline{\mathrm{Top}} & \ni & A \end{array}$$

Definition 3.1 *A* homology theory $\{h_n\}_{n\in\mathbb{Z}}$ *is a family of functors:*

$$h_n : \underline{\mathrm{Top}}^2 \to \underline{\mathrm{Ab}} \quad ((X,A) \mapsto h_n(X,A)); n \in \mathbb{Z}$$

and natural transformations

$$\partial_n : h_n \to h_{n-1} \circ I \quad (h_n(X,A) \xrightarrow{\partial_n} h_{n-1}(A,\varnothing) =: h_{n-1}(A))$$

such that the following axioms hold:

1. $f \simeq g$ $(f,g : (X,A) \to (Y,B)) \Rightarrow h_n f = h_n g$ *("homotopy invariance").*
2. *"Long exact sequence": $(X,A) \in \underline{\mathrm{Top}}^2$. Then there is a natural long exact sequence:*

$$\ldots \to h_n A \to h_n X \to h_n(X,A) \xrightarrow{\partial_n} h_{n-1}A \to \ldots$$

   i.e. $(A,\varnothing) \hookrightarrow (X,\varnothing) \hookrightarrow (X,A)$. *We often write just ∂ for ∂_n.*

3. *"Additivity":*

$$\forall n : h_n(\coprod X_\alpha) \cong \bigoplus\nolimits_\alpha h_n X_\alpha$$

4. *"Excision": $X \supset B \supset A$ such that $\bar{A} \subset \mathring{B}$.*

$$\Rightarrow h_n(X \setminus A, B \setminus A) \xrightarrow{\cong} h_n(X,B)$$

If in addition $h_\star$ satisfies the

5. *"Dimension Axiom": $h_n(\{\centerdot\}) = 0$ if $n \neq 0$.*

then $h_\star$ is called an ordinary *homology theory.*

We write $H_\star$ for a homology theory with

$$h_n(\{\centerdot\}) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0 & \text{else.} \end{cases}$$

Example $X = X_1 \cup X_2$ with $X_i \subset X, i = 1, 2$ open. Consider $X \supset X_2 \supset X_2 \setminus (X_1 \cap X_2)$: $X_2 \setminus X_1 \cap X_2 = X \setminus X_1$ is closed. So

$$X_2 = \mathring{X}_2 \supset X_2 \setminus (X_1 \cap X_2) = \overline{X_2 \setminus (X_1 \cap X_2)}$$

We note

$$X \setminus \underbrace{(X_2 \setminus X_1 \cap X_2)}_{A} = X_1; \quad X_2 \setminus (X_2 \setminus X_1 \cap X_2) = X_1 \cap X_2$$

and by the excision axiom

$$h_n(X_1, X_1 \cap X_2) \xrightarrow{\cong} h_n(X, X_2)$$

Theorem 3.2 (Mayer-Vietoris sequence) *Let $X = X_1 \cup X_2$, $X_i \subset X$ open. Then there is a natural long exact sequence*

$$\ldots \to h_n(X_1 \cap X_2) \xrightarrow{\alpha} h_n(X_1) \oplus h_n(X_2) \xrightarrow{\beta} h_n X \xrightarrow{\partial} h_{n-1}(X_1 \cap X_2) \to \ldots$$

where

$$\alpha(x) = (h_n(j_1)(x), h_n(j_2)(x)),$$

with $j_k : X_1 \cap X_2 \hookrightarrow X_k$, and

$$\beta(y, z) = (h_n(i_1)(y) - h_n(i_2)(z))$$

with $i_k : X_k \hookrightarrow X$.

Proof Look at $(X_1, X_1 \cap X_2)$ and (X, X_2):

$$\begin{tikzcd}
\cdots \arrow[r] & h_n(X_1 \cap X_2) \arrow[r, "\alpha_1"'] \arrow[d, "\alpha_2"'] & h_n(X_1) \arrow[r] \arrow[d] & h_n(X_1, X_1 \cap X_2) \arrow[r, "\partial"] \arrow[d, "\cong"', "\text{excision}"] & h_{n-1}(X_1 \cap X_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & h_n(X_2) \arrow[r] & h_n(X) \arrow[r] & h_n(X, X_2) \arrow[r, "\partial"'] & h_{n-1}(X_2) \arrow[r] & \cdots
\end{tikzcd}$$

a commutative diagram $\Rightarrow$ exactness of MV sequence follows by "diagram chasing".
E.g. exactness of "$\oplus$": We have to prove that $\ker \beta = \operatorname{im} \alpha$.

(i) $\operatorname{im}\alpha \subset \ker\beta$: $x \in h_n(X_1 \cap X_2)$

$$\begin{aligned}\Rightarrow \beta(\alpha(x)) &= \beta(h_n(j_1)(x), h_n(j_2)(x))\\ &= h_n(i_1)h(j_1)(x) - h_n(i_2)h_n(j_2)(x)\\ &= h_n(i_1 \circ j_1)(x) - h_n(i_2 \circ j_2)(x)\\ &= 0\end{aligned}$$

(ii) $\ker\beta \subset \operatorname{im}\alpha$: $x \in h_n(X_1) \oplus h_n(X_2) \xrightarrow{\beta} h_n(X)$. Assume $\beta x = 0$, i.e.

$$\underbrace{h_n(i_1)}_{\alpha_1} x_1 = \underbrace{h_n(i_2)}_{\alpha_2} x_2 =: z \in h_n(X)$$

Now $z \mapsto 0$ in $h_n(X, X_2)$ and therefore (by excision) $x_1 \mapsto 0$, so $\exists \tilde{x}_1$ in $h_n(X_1 \cap X_2)$ such that $\tilde{x}_1 \mapsto x_1$. Suppose $\tilde{x}_1 \mapsto \tilde{x}_2$ in $h_n(X_2)$. We *cannot* conclude $\tilde{x}_2 = x_2$, but we know that $\tilde{x}_2 \mapsto z$, so $\tilde{x}_2 - x_2 \mapsto 0$. Then take $\Delta \in h_{n+1}(X, X_2)$ such that $\Delta \mapsto \tilde{x}_2 - x_2$, and take $\tilde{\Delta} \in h_{n+1}(X_1, X_1 \cap X_2)$ with $\tilde{\Delta} \mapsto \Delta$ (by excision). Now define $\tilde{x}_1' = (\tilde{x}_1 - \operatorname{im}\tilde{\Delta})$, which finally maps to x_1 and x_2 in the respective groups. [You're Not Expected To Understand This. Use a colour pen on the above diagram. Ed.] $\square$

$X = X_1 \cup X_2$, $X_i \subset X$ open. $\Rightarrow X$ is push-out of

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, hook] \arrow[d, hook] & X_1 \\ X_2 & \end{tikzcd}$$

Universal Property:

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r] \arrow[d, hook] & X_1 \arrow[r, "f"] \arrow[d, hook] & Y \\ X_2 \arrow[r] \arrow[urr, bend right, "g"'] & X \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

Theorem 3.3 (Mayer-Vietoris sequence for push-outs)

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

a push-out with $A \subset B$ a NDR, and A closed in B. Then there is a natural long exact sequence (MV-Sequence)

$$\ldots \longrightarrow h_n A \xrightarrow{\alpha} h_n B \oplus h_n C \xrightarrow{\beta} h_n D \xrightarrow{\partial} h_{n-1} A \longrightarrow \ldots$$

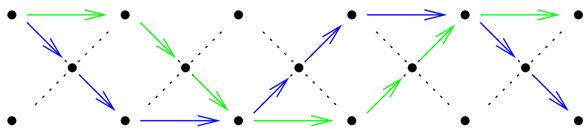


Figure 6: Braid

Proof As before, working with A replaced by a suitable neighbourhood. $\square$

Example $X \in \underline{\mathrm{CW}}$, $X = X_1 \cup X_2$, $X_i \subset X$ subcomplex $\Rightarrow$

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, hook] \arrow[d, hook, "\mathrm{NDR}"'] & X_1 \arrow[d, hook] \\
X_2 \arrow[r] & X
\end{tikzcd}$$

is a push-out with MV-Sequence:

$$\ldots \longrightarrow h_n(X_1 \cap X_2) \longrightarrow h_n X_1 \oplus h_n X_2 \longrightarrow h_n X \xrightarrow{\partial} h_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

Theorem 3.4 *$X \supset B \supset A$, $(B, A) \hookrightarrow (X, A) \hookrightarrow (X, B)$. Then there is a natural long exact sequence* (triple sequence)*:*

$$\ldots \longrightarrow h_n(B, A) \longrightarrow h_n(X, A) \longrightarrow h_n(X, B) \xrightarrow{\partial} h_{n-1}(B, A) \longrightarrow \ldots$$

Proof Uses "Braid Lemma":

Lemma 3.5 (Braid Lemma) *Given a "braid diagram" with four braids, as in figure 6. Assume 3 of them are exact, and the fourth one satisfies* $(\to \cdot \to) = (\xrightarrow{0})$ *Then the fourth one is exact too.*

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & h_n(B,A) \arrow[rr] \arrow[dr] & & h_{n-1}A \arrow[rr] \arrow[dr] & & h_{n-1}X \arrow[rr] \arrow[dr] & & h_{n-1}(X,B) \arrow[dr] & \\
& & h_n(X,A) \arrow[ur] \arrow[dr] & & h_{n-1}B \arrow[dr] & & h_{n-1}(X,A) \arrow[ur] \arrow[drr] & & \\
& & & {} & & {} & & & {}
\end{tikzcd}$$

is a triple sequence $\square$

Theorem 3.6 (relative version of MV) *Let*

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & C \arrow[d] \\
B \arrow[r] & D
\end{tikzcd}$$

be a push-out in $\underline{\text{Top}}$ *with* $A \subset B$ *a NDR and* $A \subset B$ *closed. Take any* $W \subset A$*, then there is a natural long exact sequence:*

$$\cdots \to h_n(A,W) \to h_n(B,W) \oplus h_n(C,W) \to h_n(D,W) \to h_{n-1}(A,W) \to \cdots$$

Proof As before, starting with "Triple sequence". □

Theorem 3.7 (Suspension Theorem) *Let* $x_0 \in X \in \underline{\text{Top}}$*. Then there is a natural isomorphism:*

$$h_n(X,\{x_0\}) \xrightarrow{\cong} h_{n+1}(\Sigma X,\{x_0\})$$

Proof Look at:

$$\begin{tikzcd}
x_0 \in X \arrow[r, hook] \arrow[d, hook, "\text{NDR, } X \subset CX \text{ closed}"'] & CX \ni x_0 \arrow[d, hook] \\
CX \arrow[r] & \Sigma X
\end{tikzcd}$$

apply MV, with $W = \{x_0\}$. Note $CX \simeq \{\centerdot\}$ so:

$$h_n\{x_0\} \xrightarrow{\cong} h_nCX \xrightarrow{0} h_n(CX,\{x_0\}) \xrightarrow{\partial} h_{n-1}\{x_0\} \xrightarrow{\cong} h_{n-1}CX$$

$\Rightarrow h_n(CX,\{x_0\}) = 0 \ \forall n \Rightarrow$ MV:

$$\begin{aligned} \ldots \to h_{n+1}(X,\{x_0\}) \to h_{n+1}(CX,\{x_0\}) \oplus h_{n+1}(CX,\{x_0\}) \to \\ \to h_{n+1}(\Sigma X,\{x_0\}) \xrightarrow{\partial} h_n(X,\{x_0\}) \to \ldots \end{aligned}$$

where $h_{n+1}(\Sigma X,\{x_0\})$ has to be $\cong h_n(X,\{x_0\})$. □

MV for CW-complexes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

$A, B, C \in \underline{\text{CW}}$, f cellular, $A \subset C$ subcomplex $\Rightarrow D \in \underline{\text{CW}}$.

$$\ldots \to h_nA \to h_nB \oplus h_nC \to h_nD \xrightarrow{\partial} h_{n-1}A \to \ldots$$

Definition 3.8 *Let* $h_\star$ *be a homology theory. Then we define*

$$\tilde{h}_n(X) = \ker(h_nX \to h_n\{\centerdot\})$$

for $X \in \underline{\text{Top}}$*. We call* $\tilde{h}_\star X$ *the* reduced homology *of* X*.*

Example Let $X \in \underline{\mathrm{Top}}$ and $x_0 \in X$ (note that $h_n\varnothing = 0$ for all n by additivity).
$\{x_0\} \subset X$ yields, $X \overset{\text{can}}{\to} \{x_0\}$

$$\cdots \longrightarrow h_n\{x_0\} \rightleftarrows h_n(X) \longrightarrow h_n(X,\{x_0\}) \xrightarrow{\partial} h_{n-1}\{x_0\} \rightleftarrows \cdots$$

$\Rightarrow \exists$ a split short exact sequence

$$0 \longrightarrow h_n\{x_0\} \rightleftarrows h_n(X) \longrightarrow h_n(X,\{x_0\}) \longrightarrow 0$$

and therefore

$$h_n(X) \cong h_n(X,\{x_0\}) \oplus h_n\{x_0\}$$
$$\Rightarrow\ \tilde{h}_n(X) \cong h_n(X,\{x_0\})$$

If $X \in \underline{\mathrm{Top}_\bullet}$ with base-point x_0,

$$\tilde{h}_n(X) \underset{\text{can}}{\cong} h_n(X,\{\text{base-point}\})$$

Corollary 3.9 (to MV) *There is a natural "suspension isomorphism"*

$$\tilde{\sigma}_n(X) : \tilde{h}_n(X) \overset{\cong}{\to} \tilde{h}_{n+1}(\Sigma X)$$

$$\begin{tikzcd} a_0 \in & A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ & C \arrow[r] & D \end{tikzcd}$$

as before. MV-sequence "relative to $\{a_0\}$" (in $\underline{\mathrm{CW}_\bullet}$):

$$\cdots \to h_n(A,\{a_0\}) \to h_n(B,\{a_0\}) \oplus h_n(C,\{a_0\}) \to h_n(D,\{a_0\}) \xrightarrow{\partial} \cdots$$

$$\cdots \longrightarrow \tilde{h}_nA \longrightarrow \tilde{h}_nB \oplus \tilde{h}_nC \longrightarrow \tilde{h}_nD \xrightarrow{\partial} \cdots$$

"MV sequence for reduced homology".

Note $\tilde{h}_n\{\bullet\} = 0\ \forall n \Rightarrow$ if X is contractible, then $\tilde{h}_nX = 0\ \forall n$.

Corollary 3.10 $\tilde{h}_nX \cong \tilde{h}_{n+1}\Sigma X$

Proof Look at

$$\begin{tikzcd} x_0 \in & X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{push out}"] & CX \arrow[d] \\ & CX \arrow[r] & \Sigma X \end{tikzcd}$$

MV-sequence yields

$$\ldots \to \tilde{h}_n X \to \underbrace{\tilde{h}_n CX}_{0} \oplus \underbrace{\tilde{h}_n CX}_{0} \to \tilde{h}_n \Sigma X \xrightarrow{\partial} \tilde{h}_{n-1} X \to \underbrace{\ldots}_{0}$$

so ∂ must be an isomorphism. □

Example $\tilde{h}_n S^k \cong \tilde{h}_{n-1} S^{k-1} \cong \ldots \cong \tilde{h}_{n-k} S^0$
$S^k \cong \Sigma S^{k-1}$
But $S^0 \cong \{\centerdot\} \amalg \{\centerdot\}$:

$$h_n S^0 \cong \underbrace{h_n\{\centerdot\}}_{A} \oplus \underbrace{h_n\{\centerdot\}}_{A}$$

and

$$\tilde{h}_n S^0 \cong \ker(A \oplus A \xrightarrow{\phi} A;\ (a,b) \mapsto a+b) \cong A$$

via

$$\begin{aligned} A &\xrightarrow{\cong} \ker(A \oplus A \xrightarrow{\phi} A) \\ x &\mapsto (x, -x) \end{aligned}$$

We conclude that $\tilde{h}_i S^0 \cong h_i\{\centerdot\}$ for all i, and therefore

$$\tilde{h}_n S^k \cong \tilde{h}_{n-k} S^0 \cong h_{n-k}\{\centerdot\}$$

So, if $h_\star$ satisfies the dimension axiom:

$$\tilde{h}_n S^k \cong \begin{cases} h_0\{\centerdot\}, & \text{if } n = k \\ 0 & \text{else.} \end{cases}$$

If $H_\star$ is an "ordinary homology theory with coefficients $\mathbb{Z}$", i.e.

$$H_n\{\centerdot\} \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

then

$$H_n S^k \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = k = 0 \\ \mathbb{Z} & \text{if } n = 0 \text{ or } n = k,\ k > 0 \\ 0 & \text{else} \end{cases}$$

Proof (1) $k = 0$:

$$H_n S^0 \cong H_n\{\centerdot\} \oplus H_n\{\centerdot\} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0 \\ 0 & \text{else.} \end{cases}$$

(2) $k > 0$:

$$H_n S^k \cong \tilde{H}_n S^k \oplus H_n\{\centerdot\}$$
$$H_{n-k}\{\centerdot\} \cong \tilde{H}_{n-k} S^0$$
$$\Rightarrow H_n S^k = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}, & n = k \\ 0 & \text{else.} \end{cases}$$

$\square$

Note In the reduced case this boils down to

$$\tilde{H}_n S^k \cong \begin{cases} \mathbb{Z}, & n = k \\ 0 & \text{else.} \end{cases}$$

because

$$\tilde{H}_n S^k \cong \tilde{H}_{n-k} S^0 \cong H_{n-k}(\{\centerdot\})$$

Corollary 3.11 $H_1 S^1 \cong \mathbb{Z}$

Definition 3.12 *Let θ be a generator of $H_1 S^1$. $f : S^1 \to S^1$ has* $\deg(f) \in \mathbb{Z}$ *the* degree *of f defined by:*

$$(H_1 f)(\theta) = \deg(f) \cdot \theta \in H_1 S^1$$

Lemma 3.13 *Let $f_k : S^1 \to S^1$ be the k-power map:*

$$z \mapsto z^k, \quad z \in S^1 = \{c \in \mathbb{C} \mid |c| = 1\}$$

then $\deg(f_k) = k$.

Proof k=2: $f_2(z) = z^2$ corresponds to:

$$S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{\nabla} S^1$$

where ∇ is a folding map $\langle \mathrm{id}, \mathrm{id} \rangle$.

c induces:

$$\begin{array}{c} C_{\bullet}(S^1 \vee S^1, X) \longrightarrow C_{\bullet}(S^1, X) \\ \cong \\ \Omega X \times \Omega X \xrightarrow{\mu} \Omega X \end{array}$$

thus:

$$H_1(f_2): \; H_1S^1 \overset{H_1c}{\to} \underbrace{H_1(S^1 \vee S^1)}_{\cong H_1S^1 \oplus H_1S^1} \overset{H_1\nabla}{\to} H_1S^1$$

yields:

$$\begin{array}{l} \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \overset{\nabla_*}{\to} \mathbb{Z} \\ 1 \mapsto (s,t) \quad \mapsto s+t \end{array}$$

where s is obtained from:

$$\begin{tikzcd} H_1S^1 \arrow[r] \arrow[dr, "\mathrm{id}"] & H_1(S^1 \vee S^1) \arrow[d, "\mathrm{pr}"] \\ & H_1S^1 \end{tikzcd}$$

where id maps θ to $s \cdot \theta$, therefore $s = 1$, and similarly $t = 1$.

$\Rightarrow H_1(f_2)(\theta) = 2\theta : \deg f_2 = 2$.

□

Remark $f_k : S^1 \to S^1$ yields $\Sigma^{n-1} f_k : S^n \to S^n$ using the suspension isomorphism:

$$H_n(\Sigma^{n-1} f_k) : H_nS^n \to H_nS^n$$

which is a multiplication by k (i.e. $\Sigma^{n-1} f_k : S^n \to S^n$ has degree k)

Lemma 3.14 *$X, Y \in$ Top, then $\tilde{h}_n(X \vee Y) \cong \tilde{h}_nX \oplus \tilde{h}_nY$ if "base-point is good" (i.e. $\{x_0\} \subset X$ and $\{y_0\} \subset Y$ NDR), and $H_n(X \vee Y) \cong H_nX \oplus H_nY$ if $n \neq 0$.*

Proof

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

a push-out. MV:

$$\underbrace{\tilde{h}_n\{\bullet\}}_{0} \to \tilde{h}_nX \oplus \tilde{h}_nY \to \tilde{h}_n(X \vee Y) \overset{\partial}{\to} \underbrace{\tilde{h}_{n-1}\{\bullet\}}_{0}$$

□

Remark CW-complexes are locally contractible, therefore every $x_0 \in X \in$ <u>CW</u> is a "good" base-point.

Definition 3.15 *$k > 0$, $k \in \mathbb{Z}$: $f_k : S^1 \to S^1$, then the* Moore-space of type $(\mathbb{Z}/k\mathbb{Z}, 1)$ *is defined as:*

$$M(\mathbb{Z}/k\mathbb{Z}, 1) := S^1 \cup_{f_k} D^2$$

Lemma 3.16

$$\tilde{H}_i M(\mathbb{Z}/k\mathbb{Z}, 1) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & i = 1 \\ 0 & else \end{cases}$$

or more generally: $H_n X \cong \tilde{H}_n X$ if $n \neq 0$ and:

$$H_n M(\mathbb{Z}/k\mathbb{Z}, 1) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/k\mathbb{Z} & n = 1 \\ 0 & else \end{cases}$$

Proof We have a push-out diagram:

$$\begin{tikzcd} S^1 \arrow[r, "f_k"] \arrow[d, hook] & S^1 \arrow[d] \\ \{\centerdot\} \simeq D^2 \arrow[r] & M(\mathbb{Z}/k\mathbb{Z}, 1) =: M \end{tikzcd}$$

and the MV-sequence yields:

$$\ldots \to \tilde{H}_i S^1 \to \underbrace{\tilde{H}_i D^2}_{0} \oplus \tilde{H}_i S^1 \to \tilde{H}_i M \xrightarrow{\partial} \ldots$$

where $\tilde{H}_i S^1 \to \tilde{H}_i S^1$ has degree k. so:

$$0 \oplus \underbrace{\tilde{H}_2 S^1}_{=0} \to \tilde{H}_2 M \xrightarrow{\partial} \underbrace{\tilde{H}_1 S^1}_{\cong \mathbb{Z}} \rightarrowtail 0 \oplus \underbrace{\tilde{H}_1 S^1}_{\cong \mathbb{Z}} \twoheadrightarrow \tilde{H}_1 M \xrightarrow{\partial} 0$$

$$\cong\mathbb{Z} \xrightarrow{\text{mult. by } k} \cong\mathbb{Z}$$

$\Rightarrow \tilde{H}_1 M = \mathbb{Z}/k\mathbb{Z}$, $\tilde{H}_i M = 0$, $i \neq 1$. $\square$

Corollary 3.17

$$\tilde{H}_i(\Sigma M(\mathbb{Z}/k\mathbb{Z}, 1)) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & i = 2 \\ 0 & else \end{cases}$$

$$\tilde{H}_i(\Sigma^{n-1}(M(\mathbb{Z}/k\mathbb{Z}, 1) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & i = n \\ 0 & else \end{cases}$$

and $M(\mathbb{Z}/2\mathbb{Z}, 1) = S^1 \cup_{f_2} D^2 = \mathbb{R}P^2$.

3.1 Application of MV-sequence

Theorem 3.18 *Let h_* be a homology theory, then:*

$$h_n(S^d \times X) \cong h_n(X) \oplus h_{n-d}(X)$$

Proof Consider the following push-out:

$$\begin{tikzcd}
S^d \times X \arrow[r, hook] \arrow[d, hook, "\text{NDR}"'] & (D^{d+1} \times X) \simeq X \arrow[d, hook] \\
D^{d+1} \times X \arrow[r] & \Sigma S^d \times X
\end{tikzcd}$$

Now form the MV-sequence "mod X" (i.e. $X \subset S^d \times X$, by choosing a base-point for S^d), remember that $\Sigma S^d \cong S^{d+1}$

$$\begin{aligned}
\ldots \to h_{n+1}(D^{d+1} \times X, X) \oplus h_{n+1}(D^{d+1} \times X, X) \to h_{n+1}(S^{d+1} \times X, X) \xrightarrow{\partial} \\
\xrightarrow{\partial} h_n(S^d \times X, X) \to h_n(D^{d+1} \times X, X) \oplus h_n(D^{d+1} \times X, X) \to \ldots
\end{aligned}$$

$\Rightarrow h_{n+1}(S^{d+1} \times X, X) \xrightarrow{\cong} h_n(S^d \times X, X)$, and

$$h_n(S^d \times X, X) \xrightarrow{\cong} h_{n-1}(S^{d-1} \times X, X) \xrightarrow{\cong} \ldots \xrightarrow{\cong} h_{n-d}(S^0 \times X, X)$$

with $S^0 \times X \cong X \amalg X$

$$\begin{tikzcd}
h_{n-d}(X) \arrow[r, tail] & h_{n-d}(S^0 \times X) \arrow[r, two heads] & h_{n-d}(S^0 \times X, X)
\end{tikzcd}$$

$\Rightarrow h_{n-d}(S^0 \times X, X) \cong h_{n-d}(X)$
$X \subset S^d \times X$ yields:

$$\ldots \xrightarrow{0} h_n(X) \to h_n(S^d \times X) \to h_n(S^d \times X, X) \xrightarrow{0} h_{n-1}(X) \to \ldots$$

$\Rightarrow h_n(S^d \times X) \cong h_n(X) \oplus \underbrace{h_n(S^d \times X, X)}_{\cong h_{n-d}(X)}$ ∎

Corollary 3.19 $H_i(\underbrace{S^1 \times \ldots \times S^1}_{k \text{ copies}}) \cong \begin{cases} \mathbb{Z}^{\binom{k}{i}} & 0 \le i \le k \\ 0 & \text{else} \end{cases}$

Remark Recall: $H_i(*) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$

Proof $H_i(S^1 \times \underbrace{S^1 \times \ldots \times S^1}_{k-1 \text{ copies}}) \cong \underbrace{H_i((S^1)^{k-1})}_{\mathbb{Z}^{\binom{k-1}{i}}} \oplus \underbrace{H_{i-1}((S^1)^{k-1})}_{\mathbb{Z}^{\binom{k-1}{i-1}}} \cong \mathbb{Z}^{\binom{k}{i}}$ $\square$

Example $H_i(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & \text{else} \end{cases}$

4 Singular and cellular homology

4.1 Singular homology

We want to construct an ordinary homology theory on $\underline{\text{Top}}^2$.

Definition 4.1 *Standard n-simplex:*

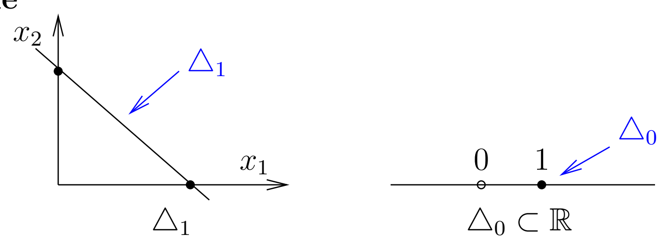
$$\Delta_n := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_j = 1, x_i \geq 0 \, \forall i \right\}$$

Δ_n has $n+1$ "faces" $i_k^n : \Delta_{n-1} \to \Delta_n$ given by:

$$i_k^n(x_1, \ldots, x_n) = \begin{cases} (0, x_1, \ldots, x_n) & k = 1 \\ (x_1, \ldots, 0, x_k, \ldots, x_n) & 1 < k < n+1 \\ (x_1, \ldots, x_n, 0) & k = n+1 \end{cases}$$

(so $1 \leq k \leq n+1$)

Example



Definition 4.2 $X \in \underline{\text{Top}}$*:*

$$C_n^{\text{sing}}(X) := \bigoplus_{\sigma : \Delta_n \to X} \mathbb{Z}_\sigma$$

with $\mathbb{Z}_\sigma \cong \mathbb{Z}$ *(free abelian group, with basis* $\{\sigma : \Delta_n \to X\}$*)*

$\sigma : \Delta_n \to X$ *is called a* singular n*-simplex of* X*, and:*

$$\begin{array}{rl} \partial_n : & C_n^{\text{sing}}(X) \to C_{n-1}^{\text{sing}}(X) \\ & (\sigma : \Delta_n \to X) \mapsto \sum_k (-1)^{k+1} (\Delta_{n-1} \xrightarrow{i_k^n} \Delta_n \xrightarrow{\sigma} X) \end{array}$$

for which we write: $\partial_n \sigma = \sum_k (-1)^{k+1} \sigma \circ i_k^n$

One checks that

$$C_n^{\text{sing}}(X) \xrightarrow{\partial_n} C_{n-1}^{\text{sing}}(X) \xrightarrow{\partial_{n-1}} C_{n-2}^{\text{sing}}(X)$$

is 0, i.e. $\partial_{n-1}\partial_n = 0$:

$$B_{n-1}^{\text{sing}}(X) := \operatorname{im}(\partial_n) \subset \ker \partial_{n-1} =: Z_{n-1}^{\text{sing}}(X)$$

($(n-1)$-cycles) and $B_{n-1}^{\text{sing}}(X)$ ($(n-1)$-boundaries)

$$H_n^{\text{sing}}(X) := Z_n^{\text{sing}}(X)/B_n^{\text{sing}}(X)$$

"n-th singular homology group of X"

$$H_0^{\text{sing}}(X) := C_0^{\text{sing}}(X)/B_0^{\text{sing}}(X)$$

We use the following convention: $C_i^{\text{sing}}(X) = 0$ if $i < 0$, and

$$C_1^{\text{sing}}(X) \xrightarrow{\partial_1} C_0^{\text{sing}}(X) \xrightarrow{\partial_0} C_{-1}^{\text{sing}}(X) = 0$$

$\{C_n^{\text{sing}}(X), \partial_n\}_{n\in\mathbb{Z}}$ is the singular chain complex of X. We usually just write $C_\star^{\text{sing}}(X)$ and we often just write ∂ for ∂_n ($\Rightarrow \partial\partial = 0$).
H_n^{sing} is a functor: Given $f : X \to Y$, we define

$$C_n^{\text{sing}}(f) : C_n^{\text{sing}}(X) \to C_n^{\text{sing}}(Y)$$

by looking at a generator $\sigma : \triangle_n \to X$ of $C_n^{\text{sing}}(X)$:

$$(\sigma : \triangle_n \to X) \;\mapsto\; (\triangle_n \xrightarrow{\sigma} X \xrightarrow{f} Y)$$

so $C_n^{\text{sing}}(f)(\sigma) = f \circ \sigma$, and therefore $C_n^{\text{sing}}(\text{id}) = \text{id}$ and

$$C_n^{\text{sing}}(f \circ g)(\sigma) = (f \circ g) \circ \sigma = f \circ (g \circ \sigma) = \left(C_n^{\text{sing}}(f) \circ C_n^{\text{sing}}(g)\right)(\sigma)$$

Compatibility with "∂": Given $f : X \to Y$, we consider

$$\begin{tikzcd} C_n^{\text{sing}}(X) \arrow[r, "C_n^{\text{sing}} f"] \arrow[d, "\partial"] & C_n^{\text{sing}}(Y) \arrow[d, "\partial"] \\ C_{n-1}^{\text{sing}}(X) \arrow[r, "C_{n-1}^{\text{sing}} f"] & C_{n-1}^{\text{sing}}(Y) \end{tikzcd}$$

This diagram is commutative: Take a generator $\sigma \in C_n^{\text{sing}}(X)$ and compute

$$\partial\left((C_n^{\text{sing}} f)(\sigma)\right) = \partial(f \circ \sigma) = \sum_k (-1)^{k+1} (f \circ \sigma) \circ i_k^n$$

$$(C_{n-1}^{\text{sing}} f)(\partial\sigma) = C_{n-1}^{\text{sing}}(f) \left(\sum_k (-1)^{k+1} \sigma \circ i_k^n \right) = \sum_k (-1)^{k+1} f \circ (\sigma \circ i_k^n)$$

so the two turn out to be the same, therefore

$$C_n^{\text{sing}}(f)\left(Z_n^{\text{sing}}(X)\right) \subset Z_n^{\text{sing}}(Y)$$

$$C_n^{\text{sing}}(f) B_n^{\text{sing}}(X) \subset B_n^{\text{sing}}(Y)$$

Therefore, f induces

$$\begin{tikzcd} B_n^{\text{sing}}(X) \arrow[r, hook] \arrow[d] & Z_n^{\text{sing}}(X) \arrow[r, two heads] \arrow[d] & H_n^{\text{sing}}(X) \arrow[d, dotted, "H_n f"] \\ B_n^{\text{sing}}(Y) \arrow[r, hook] & Z_n^{\text{sing}}(Y) \arrow[r, two heads] & H_n^{\text{sing}}(Y) \end{tikzcd}$$

Definition for H_n^{sing} on $\underline{\text{Top}}^2$: Take $(X, A) \in \underline{\text{Top}}^2$, $A \subset X$, then

$$C_n^{\text{sing}}(A) \subset C_n^{\text{sing}}(X)$$
$$(\sigma : \triangle_n \to A) \mapsto (\sigma : \triangle_n \to A \subset X)$$

$$C_n^{\text{sing}}(X, A) := C_n^{\text{sing}}(X) / C_n^{\text{sing}}(A)$$

and we can define ∂_n as the induced map ∂ from

$$\begin{tikzcd} C_n^{\text{sing}}(X) \arrow[d, "\partial"] & C_n^{\text{sing}}(A) \arrow[l, hook'] \arrow[d, "\partial"] \\ C_{n-1}^{\text{sing}}(X) & C_{n-1}^{\text{sing}}(A) \arrow[l, hook'] \end{tikzcd}$$

which means $C_n^{\text{sing}}(X, A) \xrightarrow{\partial_n} C_{n-1}^{\text{sing}}(X, A)$. Now we can finally write down the

Definition 4.3 *Let* $(X, A) \in \underline{\mathrm{Top}}^2$*; then*

$$H_n^{\mathrm{sing}}(X, A) := \ker\left(C_n^{\mathrm{sing}}(X, A) \xrightarrow{\partial_n} C_{n-1}^{\mathrm{sing}}(X, A)\right)$$
$$/\,\mathrm{im}\left(C_{n+1}^{\mathrm{sing}}(X, A) \xrightarrow{\partial_{n+1}} C_n^{\mathrm{sing}}(X, A)\right)$$

This defines functors $H_n : \underline{\mathrm{Top}}^2 \to \underline{\mathrm{Ab}}$.

We need a natural transformation $H_n^{\mathrm{sing}}(X, A) \xrightarrow{\partial} H_{n-1}^{\mathrm{sing}}(A)$. This "$\partial$" is defined as follows: Take $[z] \in H_n^{\mathrm{sing}}(X, A)$, $z \in C_n^{\mathrm{sing}}(X, A)$. Look at a cycle $\tilde{z} \in C_n^{\mathrm{sing}} X$:

$$\begin{tikzcd}
C_n^{\mathrm{sing}}(A) \arrow[r, hook] \arrow[d] & C_n^{\mathrm{sing}} X \arrow[r, two heads, "\alpha"] \arrow[d, "\partial"] & C_n^{\mathrm{sing}}(X, A) \arrow[d, "\partial"] & \ni & z = \alpha\tilde{z} \arrow[d, mapsto] \\
C_{n-1}^{\mathrm{sing}}(A) \arrow[r, hook] & C_{n-1}^{\mathrm{sing}} X \arrow[r, two heads] & C_{n-1}^{\mathrm{sing}}(X, A) & \ni & 0
\end{tikzcd}$$

$\partial\tilde{z} \in C_{n-1}^{\mathrm{sing}} A \subset C_{n-1}^{\mathrm{sing}}(X)$ is a cycle in $C_{n-1}^{\mathrm{sing}}(A)$, namely $\partial(\partial\tilde{z}) = (\partial\partial)z = 0$. So define:

$$\begin{aligned} H_n(X, A) &\xrightarrow{\partial} H_{n-1}(A) \\ [z] &\mapsto [\partial\tilde{z}] \end{aligned}$$

If we choose another counter image $\tilde{z}' \in C^{\mathrm{sing}}(X)$ of z: $\tilde{z}' - \tilde{z} \in C_n^{\mathrm{sing}}(A)$, so for some $a \in C_n^{\mathrm{sing}}(A)$ we have $\partial\tilde{z}' - \partial\tilde{z} = \partial a \in C_{n-1}^{\mathrm{sing}}(A)$ and therefore $[\partial\tilde{z}'] = [\partial\tilde{z}] \in H_{n-1}^{\mathrm{sing}}(A)$

Theorem 4.4 $(H_*^{\mathrm{sing}}, \partial)$*is a homology theory, satifying the dimension axiom.*

Proof (Sketch)

1. Homotopy Axiom:

   $F : f \simeq g$; $f : X \to Y$, $g : X \to Y \overset{?}{\Rightarrow} H_n f = H_n g : H_n^{\mathrm{sing}} X \to H_n^{\mathrm{sing}} Y$.

   $F : X \times I \to Y$, $F(x, 0) = f(x)$, $F(x, 1) = g(x)$

   $$X \underset{i_1}{\overset{i_0}{\rightrightarrows}} X \times I \xrightarrow{F} Y \qquad Fi_0 = f, Fi_1 = g$$

   $\Rightarrow$ it suffices to check that $H_n^{\mathrm{sing}} i_0 = H_n^{\mathrm{sing}} i_1$, because then:

   $$\begin{aligned} H_n^{\mathrm{sing}} f &= H_n^{\mathrm{sing}}(F \circ i_0) = H_n^{\mathrm{sing}} F \circ H_n^{\mathrm{sing}} i_0 \\ &= H_n^{\mathrm{sing}} F \circ H_n^{\mathrm{sing}} i_1 = H_n^{\mathrm{sing}}(F \circ i_1) = H_n^{\mathrm{sing}}(g) \end{aligned}$$

So we have to consider: $X \overset{i_0}{\underset{i_1}{\rightrightarrows}} X \times I$:

$$C_n^{\text{sing}} i_0, C_n^{\text{sing}} i_1 : C_n^{\text{sing}} X \rightrightarrows C_n^{\text{sing}}(X \times I)$$

$C_*^{\text{sing}} i_0$, $C_*^{\text{sing}} i_1$ are chain homotopic (see chapter 6)$\Rightarrow H_* i_0 = H_* i_1$

2. Long exact sequence axiom:

   $(X, A) \in \underline{\text{Top}}^2$:

$$0 \to C_*^{\text{sing}} A \to C_*^{\text{sing}} X \twoheadrightarrow C_*^{\text{sing}}(X, A) \to 0$$

   short exact sequence of chain complexes. This gives rise to a long exact "homology sequence" (see chapter 6)

$$\ldots \to H_n^{\text{sing}} X \to H_n^{\text{sing}}(X, A) \xrightarrow{\partial} H_{n-1}^{\text{sing}} A \to H_{n-1}^{\text{sing}} X \to \ldots$$

3. Additivity: $C_n^{\text{sing}}(\coprod_{\alpha \in I} X_\alpha) \cong \bigoplus_{\alpha \in I} C^{\text{sing}}(X_\alpha)$. $\Delta_n \xrightarrow{f} \coprod_{\alpha \in I} X_\alpha$, compatible with $\partial \Rightarrow f(\Delta_n) \subset X_\alpha$ for some α (because Δ_n is connected) $\Rightarrow$ induces:

$$H_n^{\text{sing}}\Big(\coprod_{\alpha \in I} X_\alpha\Big) \cong \bigoplus_{\alpha \in I} H_n^{\text{sing}}(X_\alpha)$$

4. Excision: Given $X \supset B \supset A$ with $\bar{A} \subset \mathring{B} \subset X$

   $\overset{?}{\Rightarrow} H_n^{\text{sing}}(X \setminus A, B \setminus A) \xrightarrow{\cong} H_n^{\text{sing}}(X, B)$

   Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be a covering of X with $U_\alpha \subset X$, $\alpha \in I$ with $\bigcup_{\alpha \in I} \mathring{U}_\alpha = X$. Define $C_n^{\mathfrak{U}}(X)$ as subgroup of $C_n^{\text{sing}} X$ generated by the singular n-simplices $f : \Delta_n \to X$ such that $f(\Delta_n) \subset U_\alpha$ for some α ("$\mathfrak{U}$-small simplices"). $\Rightarrow C_*^{\mathfrak{U}}(X) \subset C_*^{\text{sing}}(X)$ is a subcomplex and it induces an isomorphism in homology:

$$\ker(C_n^{\mathfrak{U}} \xrightarrow{\partial} C_{n-1}^{\mathfrak{U}}) / \operatorname{im}(C_{n+1}^{\mathfrak{U}} \to C_n^{\mathfrak{U}}) =: H_n^{\mathfrak{U}} X \xrightarrow{\cong} H_n^{\text{sing}} X$$

   (See Lück p. 29). Idea: for Δ_n "barycentric subdivision": new vertices are barycentres of faces (figure 7). Now take for $\mathfrak{U}$ the cover: $X = X \setminus A \cup B$

$$(X \setminus \bar{A}) = (X \mathring{\setminus} A) \subset (X \setminus A) \Rightarrow X = (X \mathring{\setminus} A) \cup \mathring{B}, \quad \bar{A} \subset \mathring{B} \subset B$$

   the function:

$$C_n^{\text{sing}}(X \setminus A)/C_n^{\text{sing}}(B \setminus A) \to C_n^{\text{sing}} X / C_n^{\text{sing}} B$$

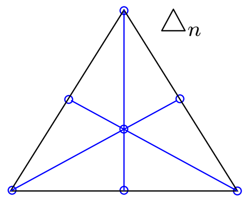


Figure 7: Barycentric subdivision

should induce an isomorphism in homology. Look at:

$$C_n^{\mathfrak{U}}(X) = C_n^{\text{sing}}(X \setminus A) + C_n^{\text{sing}}(B) \subset C_n^{\text{sing}} X$$

$$\begin{aligned} \Rightarrow \quad & C_n^{\text{sing}}(X \setminus A)/C_n^{\text{sing}}(B \setminus A) \cong \\ & \cong \underbrace{\left(C_n^{\text{sing}}(X \setminus A) + C_n^{\text{sing}} B\right)}_{C_n^{\mathfrak{U}}(X)} / \underbrace{C_n^{\text{sing}}(B \setminus A) + C_n^{\text{sing}}(B)}_{C_n^{\text{sing}}(B)} \end{aligned}$$

$\phi : C_n^{\mathfrak{U}}(X)/C_n^{\text{sing}}(B) \to C_n^{\text{sing}}(X)/C_n^{\text{sing}}(B)$ use the following lemma:

Lemma 4.5 *Given a diagram of chain complexes:*

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r] \arrow[d, "\alpha"'] & B_* \arrow[r] \arrow[d, "\beta"'] & C_* \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & D_* \arrow[r] & E_* \arrow[r] & F_* \arrow[r] & 0 \end{tikzcd}$$

if two of $H_*\alpha$, $H_*\beta$, $H_*\gamma$ are an isomorphism, then the third one is too.

Proof

$$\begin{tikzcd} \cdots \arrow[r] & H_n A_* \arrow[r] \arrow[d, "\cong"] & H_n B_* \arrow[r] \arrow[d, "\cong"] & H_n C_* \arrow[r, "\partial"] \arrow[d] & H_{n-1} A_* \arrow[r] & H_{n-1} B_* \arrow[r] & \cdots \\ \cdots \arrow[r] & H_n D_* \arrow[r] & H_n E_* \arrow[r] & H_n F_* \arrow[r, "\partial"] & H_{n-1} D_* \arrow[r] & H_{n-1} E_* \arrow[r] & \cdots \end{tikzcd}$$

□

$$\begin{tikzcd} 0 \arrow[r] & C_*^{\text{sing}} B \arrow[r] \arrow[d, "\text{id}"] & C_*^{\mathfrak{U}} X \arrow[r] \arrow[d, "\text{lemma}"] & C_*^{\mathfrak{U}} X / C_*^{\text{sing}} B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_*^{\text{sing}} B \arrow[r] & C_*^{\text{sing}} X \arrow[r] & C_*^{\text{sing}} X / C_*^{\text{sing}} B \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow H_*\phi$ is an isomorphism.

$\Rightarrow H_*^{\text{sing}}$ is a homology theory.

5. H_*^{sing} satisfies the dimension axiom:

   Claim:
$$H_n^{\text{sing}}(\{*\}) \cong \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

   Indeed:
$$\ldots \to \underbrace{C_n^{\text{sing}}(\{*\})}_{\text{generated by } \sigma_n : \Delta_n \overset{\exists!}{\to} \{*\}} \xrightarrow{\partial} C_{n-1}^{\text{sing}}(\{*\}) \to \ldots \to C_0^{\text{sing}}(\{*\}) \to 0$$

   so $C_*^{\text{sing}}(\{*\})$ looks as follows:
$$\ldots \to \mathbb{Z} = \langle \sigma_n \rangle \xrightarrow{\partial_n} \mathbb{Z} = \langle \sigma_{n-1} \rangle \xrightarrow{\partial_{n-1}} \ldots \to \mathbb{Z} \to 0$$

   with
$$\partial_n \sigma_n = \sum_{1 \leq k \leq n+1} (-1)^{k+1} (\sigma_{n-1}) = \begin{cases} 0 & n+1 \text{ even} \\ \sigma_{n-1} & n+1 \text{ odd} \end{cases}$$

   $C_*^{\text{sing}}(\{*\})$:
$$\cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \xrightarrow{0} \underset{\substack{= \\ \langle \sigma_1 \rangle}}{\mathbb{Z}} \xrightarrow[\partial_1]{0} \underset{\substack{= \\ \langle \sigma_0 \rangle}}{\mathbb{Z}} \xrightarrow[\partial_0]{0} 0$$

   and $H_n^{\text{sing}}(\{*\}) = 0$ for $n > 0$. $H_0^{\text{sing}}(\{*\}) = \underbrace{\ker(\partial_0)}_{\mathbb{Z}} / \underbrace{\operatorname{im}(\partial_1)}_{\{0\}} \cong \mathbb{Z}$

□

Some Applications:

1. $H_n^{\text{sing}}(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$

2. if $k > 0$: $H_n^{\text{sing}}(S^k) \cong \begin{cases} \mathbb{Z} & n = 0 \text{ or } n = k \\ 0 & \text{else} \end{cases}$

Corollary 4.6 (Brouwer fixed point theorem) *Every map $f : D^n \to D^n$ has a fixed point (i.e. an $x \in D^n$ with $f(x) = x$).*

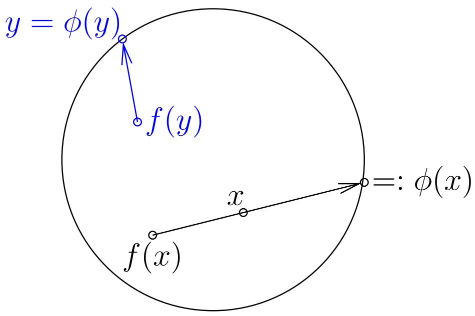


Figure 8: Definition of ϕ

Proof Suppose f has no fixed point. Consider the ray from $f(x)$ to x ($x \in D^n$), and its intersection $\phi(x)$ with $\partial D^n = S^{n-1}$ (figure 8).

$$\phi : \begin{tikzcd} D^n \arrow[r] & S^{n-1} \\ S^{n-1} \arrow[u, hook] \arrow[ur, "\phi|S^{n-1}=\mathrm{id}"'] & \end{tikzcd}$$

Apply H^{sing}_{n-1} (assuming $n > 0$)

$$\begin{tikzcd} H^{\mathrm{sing}}_{n-1} D^n \arrow[r] & H^{\mathrm{sing}}_{n-1} S^{n-1} \\ H^{\mathrm{sing}}_{n-1} S^{n-1} \arrow[u, hook] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

if $n = 1$: $\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[u] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$ if $n > 1$: $\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \\ \mathbb{Z} \arrow[u] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$

in either case, this is a contradiction.

$\square$

Corollary 4.7 (Invariance of dimension) $\mathbb{R}^n \cong \mathbb{R}^m \Rightarrow n = m$

Proof Let:

$$\phi : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$$
$$x_0 \mapsto \phi(x_0)$$

$\Rightarrow$ induces $\mathbb{R}^n \setminus \{x_0\} \xrightarrow{\cong} \mathbb{R}^m \setminus \{\phi(x_0)\}$. But: $\mathbb{R}^n \setminus \{x_0\} \simeq S^{n-1}$ and $\mathbb{R}^m \setminus \{\phi(x_0)\} \simeq S^{m-1}$ imply: $S^{n-1} \simeq S^{m-1}$ and therefore $H^{\mathrm{sing}}_* S^{n-1} \cong H^{\mathrm{sing}}_* S^{m-1} \Rightarrow n = m$. $\square$

Theorem 4.8 (Borsuk-Ulam Theorem) *There is no injective map $S^2 \to \mathbb{R}^2$.*

Note $S^2 \setminus \{x\} \cong \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ via $\mathbb{R}^2 \cong \mathring{D}^2 \subset \mathbb{R}^2$

Proof Suppose $\phi : S^2 \to \mathbb{R}^2$ injective $\Rightarrow \phi(x) \neq \phi(-x)\ \forall x \in S^2$. Let $\psi(x) = \frac{\phi(x)-\phi(-x)}{\|\phi(x)-\phi(-x)\|} \in S^1$

$$\begin{tikzcd} S^2 \arrow[r, "\psi"] \arrow[d, "x\sim -x"'] & S^1 \arrow[d, "y\sim -y"] \\ \mathbb{R}P^2 \arrow[r, dashed, "\exists\bar{\psi}"] & \mathbb{R}P^1 \cong S^1 \end{tikzcd} \tag{D}$$

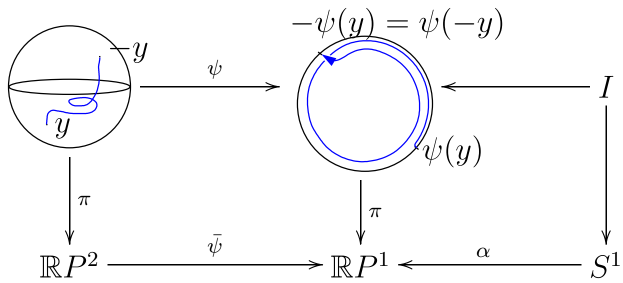
$\bar{\psi}$ induced by ψ because $\psi(A_2 x) = A_1\psi(x)$ with $A_2 : S^2 \to S^2, x \mapsto -x$, $A_1 : S^1 \to S^1, y \mapsto -y$.
Claim: From the diagram (D) we have

$$H_1^{\text{sing}}(\bar{\psi}) : H_1^{\text{sing}}\mathbb{R}P^2 \xrightarrow{\neq 0} H_1^{\text{sing}}\mathbb{R}P^1$$

which is a contradiction because

$$\begin{aligned} &H_1^{\text{sing}}\mathbb{R}P^2 = \mathbb{Z}/2\mathbb{Z} &&(\mathbb{R}P^2 = S^1 \cup_2 e^2) \\ &H_1^{\text{sing}}\mathbb{R}P^1 = \mathbb{Z} &&(\mathbb{R}P^1 \cong S^1) \end{aligned}$$

Proof of the claim: We use the following fact on covering spaces: Let $X \xrightarrow{\pi} Y$ be a covering. For every loop ω with base-point x_0, there is a unique lift $\tilde{\omega}$ for a given initial point $\tilde{x}_0$ over x_0 (i.e. $\pi(\tilde{x}_0) = x_0$). (See Topologie SS 05.) If $w \simeq$ const. (i.e. $[\omega] = 0 \in \pi_1(X, x_0)$) then $\tilde{\omega}$ has to be a loop too (this follows from the homotopy lifting property for $\pi : X \to Y$). So we can look at (D) as



If we take a path $\sigma : I \to S^2$ from y to $-y$, then $\pi\sigma$ is a loop in $\mathbb{R}P^2 \Rightarrow [\pi\sigma] \in \pi_1(\mathbb{R}P^2)$ is not trivial. The loop $[\pi\psi\sigma] \in \pi_1\mathbb{R}P^1$ is $\neq 0 \Rightarrow$ degree of the corresponding map $S^1 \xrightarrow{\alpha} S^1 = \mathbb{R}P^1$ is $\neq 0$

$$\mathbb{Z}/2\mathbb{Z} \cong H_1^{\text{sing}}(\mathbb{R}P^2) \xrightarrow{H_1^{\text{sing}}(\bar{\psi})} H_1^{\text{sing}}(\mathbb{R}P^1) \cong \mathbb{Z}$$

$\Rightarrow H_1^{\text{sing}}(\bar{\psi}) \neq 0.$ □

Remark General Borsuk-Ulam: $S^n \not\hookrightarrow \mathbb{R}^n$

Proof As before

$$\begin{tikzcd} S^n \arrow[r, "\psi"] \arrow[d, "\pi"] & S^{n-1} \arrow[d, "\pi"] \\ \mathbb{R}P^n \arrow[r, "\bar{\psi}"] & \mathbb{R}P^{n-1} \end{tikzcd}$$

However for $n > 2$

$$H_1^{\text{sing}}\mathbb{R}P^n = H_1^{\text{sing}}\mathbb{R}P^{n-1} = \mathbb{Z}/2\mathbb{Z}$$

so we need to show that any map

$$H_1^{\text{sing}}\mathbb{R}P^n \to H_1^{\text{sing}}\mathbb{R}P^{n-1}$$

is 0 (see later). □

Remark Application: For every point x on the earth, let $t(x)$ be the temperature and $p(x)$ the pressure. Then $\exists x_1 \neq x_2$ on the earth with $t(x_1) = t(x_2)$ and $p(x_1) = p(x_2)$, because otherwise we could embed

$$\begin{aligned} S^2 &\hookrightarrow \mathbb{R}^2 \\ x &\mapsto (t(x), p(x)) \end{aligned}$$

which of course is a contradiction.

4.2 Cellular homology

Let $X \in \underline{\text{CW}}$. We want to define an easily computable $H_n^{\text{cell}}X$ such that $H_n^{\text{cell}}X \cong H_n^{\text{sing}}X$.

Theorem 4.9 (MV for CW-complexes: a variation)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] & Y \cup_f X \end{tikzcd}$$

If $(X, A) \in \underline{\text{CW}}^2$, $Y \in \underline{\text{CW}}$, *and* f *cellular* $\Rightarrow Y \cup_f X \in \underline{\text{CW}}$.
Then

$$H_i^{\text{sing}}(X, A) \xrightarrow{\cong} H_i^{\text{sing}}(Y \cup_f X, Y)\ \forall i$$

Proof Look at the MV-sequence "mod A".

$$\begin{tikzcd} & Z(f) \arrow[d, "\simeq"] \\ A \arrow[ur, hook] \arrow[r, "f"'] & Y \end{tikzcd}$$

where $Z(f)$ is the mapping cylinder $\Rightarrow$ we can assume f is injective, mapping homeomorphically onto its image: "$A \subset Y$". So:

$$\dots \underbrace{H_i^{\text{sing}}(A, A)}_{0} \to H_i^{\text{sing}}(X, A) \oplus H_i^{\text{sing}}(Y, A) \xrightarrow{(*)} H_i^{\text{sing}}(Y \cup_f X, A) \xrightarrow{\partial} \underbrace{H_{i-1}^{\text{sing}}(A, A)}_{0}$$

and $A \subset Y \subset (Y \cup_f X)$ yields

$$\xrightarrow{\partial} H_i^{\text{sing}}(Y, A) \xrightarrow{\phi} H_i^{\text{sing}}(Y \cup_f X, A) \to H_i^{\text{sing}}(Y \cup_f X, Y) \xrightarrow{\partial}$$

with ϕ injective from $(*)$, therefore $H_i^{\text{sing}}(Y \cup_f X, Y) \cong \operatorname{coker}(\phi)$ $\square$

Definition 4.10 (Cellular homology) *Let $X \in \underline{\mathrm{CW}}$, $X_0 \subset X_1 \subset \ldots \subset X$. By definition we have a push-out*

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X_{n-1} \arrow[d] \\ \coprod D^n \arrow[r] & X_n \end{tikzcd}$$

and by the above theorem

$$H_i^{\text{sing}}(X_n, X_{n-1}) \cong H_i^{\text{sing}}\left(\coprod D^n, \coprod S^{n-1}\right) \cong \bigoplus\nolimits_I H_i^{\text{sing}}(D^n, S^{n-1})$$

but if we look at the long exact sequence of $D^n \supset S^{n-1}$,

$$H_i^{\text{sing}}(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & i = n \\ 0, & \textit{else.} \end{cases}$$

so we define

$$C_n^{\text{cell}}(X) := H_n^{\text{sing}}(X_n, X_{n-1}) \cong \bigoplus_{\#\ n\text{-cells}} \mathbb{Z}$$

We need to define $\partial_n : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$: $X_{n-2} \subset X_{n-1} \subset X_n$ yields the triple sequence

$$\begin{tikzcd} H_n^{\text{sing}}(X_n, X_{n-2}) \arrow[r] & H_n^{\text{sing}}(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[d, equal] & H_{n-1}^{\text{sing}}(X_{n-1}, X_{n-2}) \arrow[r] \arrow[d, equal] & \cdots \\ & C_n^{\text{cell}} X \arrow[r] & C_{n-1}^{\text{cell}} X & \end{tikzcd}$$

Claim: $C_n^{\mathrm{cell}}X \xrightarrow{\partial_n} C_{n-1}^{\mathrm{cell}}X \xrightarrow{\partial_{n-1}} C_{n-2}^{\mathrm{cell}}X$ *is zero. Indeed*

$$\begin{tikzcd}
H_n^{\mathrm{sing}}(X_n, X_{n-1}) \arrow[rr, "\partial_n"] \arrow[dr, "\partial"'] & & H_{n-1}^{\mathrm{sing}}(X_{n-1}, X_{n-2}) \arrow[rr, "\partial_{n-1}"] \arrow[dr, "\partial"'] & & H_{n-2}^{\mathrm{sing}}(X_{n-2}, X_{n-3}) \\
& H_{n-1}^{\mathrm{sing}}X_{n-1} \arrow[ur] \arrow[rr, "0"] & & H_{n-2}^{\mathrm{sing}}X_{n-2} \arrow[ur] &
\end{tikzcd}$$

$\Rightarrow \partial_{n-1}\partial_n = 0 \Rightarrow$ *define*

$$H_n^{\mathrm{cell}}X := \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

Theorem 4.11 $X \in \underline{\mathrm{CW}} \Rightarrow H_n^{\mathrm{cell}}X \cong H_n^{\mathrm{sing}}X\ \forall n.$

Proof First, we claim: $H_i^{\mathrm{sing}}X_n \xrightarrow{\cong} H_i^{\mathrm{sing}}X$ if $i < n$; and $H_n^{\mathrm{sing}}X_n \xrightarrow{\text{onto}} H_n^{\mathrm{sing}}X$. Indeed, (X_{n+1}, X_n):

$$\ldots \to \underbrace{H_{i+1}^{\mathrm{sing}}(X_{n+1}, X_n)}_{0 \text{ if } i \neq n} \xrightarrow{\partial} H_i^{\mathrm{sing}}(X_n) \to H_i^{\mathrm{sing}}(X_{n+1}) \to \underbrace{H_i^{\mathrm{sing}}(X_{n+1}, X_n)}_{0 \text{ if } i \neq n+1} \xrightarrow{\partial} \ldots$$

so if $i < n$:

$$H_i^{\mathrm{sing}}X_n \xrightarrow{\cong} H_i^{\mathrm{sing}}X_{n+1} \xrightarrow{\cong} H_i^{\mathrm{sing}}X_{n+2} \to \ldots$$

$\Rightarrow$ for X finite dimensional: $H_i^{\mathrm{sing}}(X_n) \xrightarrow{\cong} H_i^{\mathrm{sing}}X$ if $i < n$.
If $i = n$: $H_n^{\mathrm{sing}}X_n \twoheadrightarrow H_n^{\mathrm{sing}}X_{n+1} \xrightarrow{\cong} \ldots$, so $H_n^{\mathrm{sing}}X_n \twoheadrightarrow H_n^{\mathrm{sing}}X$ if X is finite dimensional.
Now consider the diagram

$$\begin{tikzcd}
& & H_n^{\mathrm{sing}}X_n \arrow[r, two heads, "\phi"] \arrow[d, "\beta"] & H_n^{\mathrm{sing}}X & \\
H_{n+1}^{\mathrm{sing}}(X_{n+1}, X_n) \arrow[r, equal] & C_{n+1}^{\mathrm{cell}}X \arrow[ur, "\partial =: \alpha"] \arrow[r, "\partial_{n+1}"] & C_n^{\mathrm{cell}}X \arrow[r, "\partial_n"] \arrow[d, "\partial"] & C_{n-1}^{\mathrm{cell}}X \arrow[r, equal] & H_{n-1}^{\mathrm{sing}}(X_{n-1}, X_{n-2}) \\
& & H_{n-1}^{\mathrm{sing}}X_{n-1} \arrow[ur] & &
\end{tikzcd}$$

Kernel of ϕ: Look at the long exact sequence of (X_{n+1}, X_n):

$$C_{n+1}^{\mathrm{cell}}X = H_{n+1}^{\mathrm{sing}}(X_{n+1}, X_n) \xrightarrow{\alpha} H_n^{\mathrm{sing}}X_n \overset{\phi}{\twoheadrightarrow} H_n^{\mathrm{sing}}X_{n+1} \cong H_n^{\mathrm{sing}}X$$

$\Rightarrow \ker(\phi) = \operatorname{im}(\alpha)$.
Define

$$\begin{aligned} \gamma : H_n^{\text{sing}}(X) &\longrightarrow H_n^{\text{cell}} X = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) \\ x &\longmapsto [\beta(x)] \end{aligned}$$

Then γ bijective follows from the diagram above. $\square$

$X \in \underline{\text{CW}}$: $H_n^{\text{sing}}(X) \xrightarrow{\cong} H_n^{\text{cell}}(X)$
relative groups:

1. $X \in \underline{\text{CW}}$:
$$\tilde{H}_n^{\text{cell}}(X) = \ker(H_n^{\text{cell}}(X) \to H_n^{\text{cell}}(\{*\}))$$
2. $(X, A) \in \underline{\text{CW}}^2$, $A \neq \varnothing$:
$$H_n^{\text{cell}}(X, A) := \tilde{H}_n^{\text{cell}}(X/A) \qquad (H_n^{\text{cell}}(X, \varnothing) := H_n^{\text{cell}}(X))$$

$\Rightarrow H_n^{\text{cell}}(X, A) \cong H_n^{\text{sing}}(X, A)$ because:

$$\left.\begin{tikzcd} A \arrow[r, hook, "\text{NDR}"] \arrow[d, hook, "\text{NDR}"] & X \arrow[d] \\ CA \arrow[r] & X \cup_A CA \end{tikzcd}\right\} \begin{aligned} H_n^{\text{sing}}(X, A) &\cong H_n^{\text{sing}}(X \cup_A CA, CA) \\ &\cong H_n^{\text{sing}}(X/A, \{*\}) \cong \tilde{H}_n^{\text{sing}}(X/A) \end{aligned}$$

$\Rightarrow (X, A) \in \underline{\text{CW}}^2$ yields a long exact sequence:

$$\ldots \to H_n^{\text{cell}} A \to H_n^{\text{cell}} X \to H_n^{\text{cell}}(X, A) \xrightarrow{\partial} H_{n-1}^{\text{cell}} A \to \ldots$$

Final Remarks:

1. $X \in \underline{\text{Top}} \Rightarrow H_0^{\text{sing}}(X) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$, where $\pi_0 X := [\{*\}, X]$

   Note $f, g : \{*\} \to X$ are homotopic if and only if $f(*)$ and $g(*)$ are in the same path component of X: $\pi_0 X \overset{\text{bij.}}{\leftrightarrow} \{\text{path components of } X\}$

   Proof
   $$\ldots \to C_1^{\text{sing}} X \xrightarrow{\partial_1} C_0^{\text{sing}} X \xrightarrow{\partial_0} 0$$
   C_0^{sing} has basis $\sigma : \{0\} = \Delta_0 \to X$
   $\Rightarrow H_0^{\text{sing}}(X) = C_0^{\text{sing}}(X)/\operatorname{im}(\partial_1)$
   $\Rightarrow C_0^{\text{sing}}(X) \ni c = \sum_{x \in X} n_x x$ finite sum $n_x \in \mathbb{Z}$
   $$\begin{aligned} \to C_1^{\text{sing}}(X) &\xrightarrow{\partial_1} C_0^{\text{sing}}(X) \xrightarrow{\partial_0} \\ (\sigma : \Delta_1 \to X) &\mapsto \sigma(0,1) - \sigma(1,0) \end{aligned}$$
   $\Rightarrow C_0^{\text{sing}}(X)/\operatorname{im}(\partial_1) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$ $\square$

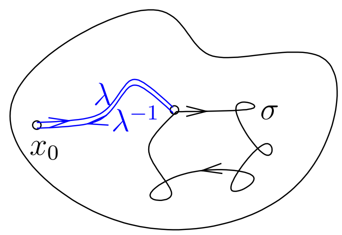


Figure 9: proof

2. $X \in \underline{\mathrm{Top}}_\bullet$, X path connected ($\Rightarrow H_0^{\mathrm{sing}}(X) \cong \mathbb{Z}$) $\Rightarrow$

$$H_1^{\mathrm{sing}}(X) \cong \pi_1(X)/[\pi_1 X, \pi_1 X]$$

with $[\pi_1 X, \pi_1 X]$ the commutator subgroup of $\pi_1(X)$

Proof (Sketch)

Consider the "Hurewicz Homomorphism":

$$\begin{aligned} Hu : \pi_1(X) &\to H_1^{\mathrm{sing}}(X) \\ [S^1 \xrightarrow{f} X]_\bullet &\mapsto H_1^{\mathrm{sing}}(f)(c_1) \quad \text{where } \langle c_1 \rangle = H_1^{\mathrm{sing}} S^1 \end{aligned}$$

claim: Hu induces $\pi_1 X/[\pi_1 X, \pi_1 X] \xrightarrow{\cong} H_1^{\mathrm{sing}}(X)$

- onto:

$$H_1^{\mathrm{sing}}(X) \twoheadleftarrow Z_1^{\mathrm{sing}}(X) = \ker(C_1^{\mathrm{sing}}(X) \xrightarrow{\partial_1} C_0^{\mathrm{sing}}(X)$$

  $\partial_1 \sigma = 0$, with $(\sigma : \Delta_1 \to X) \in C_1^{\mathrm{sing}}(X)$.
  $\partial_1 \sigma = 0 \Rightarrow \sigma$ a loop. "$\sigma \sim$ loop at base point". See figure 9.
  $[\lambda] + [\lambda^{-1}] \in C_1^{\mathrm{sing}}(X)$ is a boundary.
- $\ker(Hu) = [\pi_1 X, \pi_1 X]$: without proof

□

More in general:

Theorem 4.12 (Hurewicz) *Let $X \in \underline{\mathrm{Top}}$ path connected. Then:*

1. $\pi_1 X/[\pi_1 X, \pi_1 X] \xrightarrow{\cong} H_1^{\mathrm{sing}}(X)$
2. *if $\pi_i X = 0$ for $1 \le i < n$ then: $Hu : \pi_n X \xrightarrow{\cong} H_n^{\mathrm{sing}}(X)$*

Example 1. $\pi_1 S^1 \xrightarrow{\cong} H_1^{\mathrm{sing}}(S^1)$

2. $n > 1$: $\pi_n S^n \xrightarrow{\cong} H_n^{\mathrm{sing}} S^n \cong \mathbb{Z}$, $[f : S^n \to S^n]_\bullet \mapsto \deg(f) \cdot c_n$, $H_n^{\mathrm{sing}}(S^n) = \langle c_n \rangle$.

5 Lefschetz Numbers

5.1 Facts from Linear Algebra

Let V, W be finite dimensional $\mathbb{Q}$-vector spaces, and $f : V \to W$ a linear map.

Definition 5.1 *If $V = W$, then $f : V \to V$, thus the trace* $\mathrm{tr}(f) \in \mathbb{Q}$ *is well-defined: Choose a basis $\{e_1, \ldots, e_n\}$ of V, then f can be expressed by an $(n \times n)$-matrix (f_{ij}) with coefficients $f_{ij} \in \mathbb{Q}$. Put*

$$\mathrm{tr}(f) := \sum_{i=1}^{n} f_{ii}$$

Properties of tr:

(1) "Trace property": If

$$V \xrightarrow{f} W \xrightarrow{g} V \xrightarrow{f} W$$

then

$$\mathrm{tr}(g \circ f) = \mathrm{tr}(f \circ g)$$

(Use that for A an $(n \times k)$-matrix, B a $(k \times n)$-matrix, $tr(AB) = tr(BA)$.)

(2) $\mathrm{tr}(\mathrm{id}_V) = \dim_{\mathbb{Q}}(V)$

(3) $\mathrm{tr} : \mathrm{End}_{\mathbb{Q}}(V) \to \mathbb{Q}$ is linear:

$$\begin{aligned} \mathrm{tr}(f + g) &= \mathrm{tr}(f) + \mathrm{tr}(g) \\ \mathrm{tr}(\lambda f) &= \lambda \, \mathrm{tr}(f) \quad (\lambda \in \mathbb{Q}) \end{aligned}$$

(4) Consider the following (commutative) diagram of short exact sequences of $\mathbb{Q}$-vector spaces:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "f"] & W \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & V \arrow[r] & W \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

Then

$$\mathrm{tr}(g) = \mathrm{tr}(f) + \mathrm{tr}(h)$$

Since we can choose any basis of W, choose one by "extending" a basis of $V \Rightarrow$ matrix of g has the form

$$\begin{pmatrix} A_f & * \\ 0 & A_h \end{pmatrix}$$

where A_f and A_h are the matrices of f and h, respectively.

(5) Let A be a finitely generated abelian group, and $f : A \to A$ a homomorphism, then

$$\operatorname{tr}(f \otimes \mathbb{Q}) \in \mathbb{Z}$$
$$(f \otimes \mathbb{Q} : A \otimes \mathbb{Q} \to A \otimes \mathbb{Q})$$

Indeed:

$$\begin{tikzcd} A \arrow[r, "\text{can}"] \arrow[dr] & A \otimes \mathbb{Q} \\ & A/TA \end{tikzcd}$$

where $TA \subset A$ is the torsion subgroup $\Rightarrow A/TA$ is torsion-free $\Rightarrow$ has basis.

Note that if a is a torsion element, i.e. $n \cdot a = 0$, then

$$a \otimes 1 = n \cdot a \otimes \tfrac{1}{n} = 0$$

so tensoring with $\mathbb{Q}$ also "divides out torsion".

$A/TA \cong \mathbb{Z}^n$, $A \otimes \mathbb{Q} \cong \mathbb{Q}^n$:

$$\operatorname{tr}(f \otimes \mathbb{Q}) = \operatorname{tr}(\bar{f} : A/TA \to A/TA)$$

$\bar{f}$ is expressed by a matrix with coefficients in $\mathbb{Z}$ with respect to a basis of A/TA.

Definition 5.2 *Let $X \in \underline{\mathrm{Top}}$ with $H_i^{\mathrm{sing}}(X)$ finitely generated for all i, and 0 for $i \gg 0$. Let $f : X \to X$. Then*

$$L(f) := \sum_i (-1)^i \operatorname{tr}(H_i(f) \otimes \mathbb{Q}) \ \in \mathbb{Z}$$

is called the Lefschetz number *of f.*

$$\Big(H_i(f) \otimes \mathbb{Q} : H_i^{\mathrm{sing}}(X) \otimes \mathbb{Q} \to H_i^{\mathrm{sing}}(X) \otimes \mathbb{Q}\Big)$$

Example $X \simeq \{\centerdot\}$ contractible $\Rightarrow L(f) = 1 \ \forall f : X \to X$ since

$$H_i^{\mathrm{sing}}(X) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "f"] & \{\centerdot\} \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "\simeq"] & \{\centerdot\} \end{tikzcd}$$

is homotopy commutative.

Note X a finite CW-complex $\Rightarrow H_i^{\text{cell}} X$ are finitely generated abelian groups, and $H_i^{\text{cell}} X = 0$ for $i > \dim X \Rightarrow L(f) \in \mathbb{Z}$ is defined for any $f : X \to X$.

Definition 5.3 $f = \mathrm{id} : X \to X$. *Then*

$$\chi(X) = L(f)$$

is called the Euler characteristic *of* X.

Note

$$\chi(X) = \sum (-1)^i \dim_{\mathbb{Q}} \left(H_i^{\text{sing}}(X) \otimes \mathbb{Q} \right)$$

where $\dim_{\mathbb{Q}}(H_i^{\text{sing}}(X) \otimes \mathbb{Q}) =: \beta_i(X)$ is called the i-th *Betti number*.

Theorem 5.4 *Let X be a finite CW-complex. Then*

$$\chi(X) := \sum_i (-1)^i C_i$$

where C_i is the number of i-cells of X.

Proof $H_i^{\text{cell}} X \cong H_i^{\text{sing}} X$, $\forall i$. We need to show that

$$\sum (-1)^i \dim_{\mathbb{Q}} (H_i^{\text{cell}}(X) \otimes \mathbb{Q}) = \sum_i (-1)^i C_i$$

Let:

$$\begin{aligned} C_i &:= C_i^{\text{cell}} X \cong \textstyle\bigoplus_{\#\ i\text{-cells}} \mathbb{Z} \cong \mathbb{Z}^{c_i} \\ Z_i &:= \ker(\partial_i : C_i^{\text{cell}} X \to C_{i-1}^{\text{cell}} X) \\ B_i &:= \operatorname{im}(\partial_{i+1} : C_{i+1}^{\text{cell}} X \to C_i^{\text{cell}} X) \end{aligned}$$

with $H_i := H_i^{\text{cell}} X = Z_i / B_i$
$\Rightarrow$ Have short exact sequences:

$$B_i \hookrightarrow Z_i \twoheadrightarrow H_i \Rightarrow B_i \otimes \mathbb{Q} \hookrightarrow Z_i \otimes \mathbb{Q} \twoheadrightarrow H_i \otimes \mathbb{Q}$$

and

$$Z_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \subset C_{i-1}$$

resp.

$$Z_i \otimes \mathbb{Q} \hookrightarrow C_i \otimes \mathbb{Q} \twoheadrightarrow B_{i-1} \otimes \mathbb{Q}$$

$\Rightarrow \dim_{\mathbb{Q}} Z_i \otimes \mathbb{Q} = \dim_{\mathbb{Q}} B_i \otimes \mathbb{Q} + \dim_{\mathbb{Q}} H_i \otimes \mathbb{Q}$
$\underbrace{\dim_{\mathbb{Q}} C_i \otimes \mathbb{Q}}_{c_i} = \dim_{\mathbb{Q}} Z_i \otimes \mathbb{Q} + \dim_{\mathbb{Q}} B_{i-1} \otimes \mathbb{Q}$
$\Rightarrow$

$$\begin{aligned} &\textstyle\sum_i (-1)^i C_i = \sum (-1)^i (\dim_{\mathbb{Q}} Z_i \otimes \mathbb{Q} + \dim_{\mathbb{Q}} B_{i-1} \otimes \mathbb{Q}) \\ &= \textstyle\sum_i (-1)^i (\dim_{\mathbb{Q}} B_i \otimes \mathbb{Q} + \dim_{\mathbb{Q}} H_i \otimes \mathbb{Q} + \dim_{\mathbb{Q}} B_{i-1} \otimes \mathbb{Q}) \\ &= \textstyle\sum_i (-1)^i \dim_{\mathbb{Q}} H_i \otimes \mathbb{Q} \end{aligned}$$

□

Example $S^n = D^0 \cup D^n$, therefore

$$\chi(S^n) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Application "Euler's Formula"
Take a polyhedral decomposition of S^2, e.g. a cube, a tetrahedron, ..., and write

$$\begin{aligned} v &= (\#\text{ vertices} = 0\text{-cells}) \\ e &= (\#\text{ edges} = 1\text{-cells}) \\ f &= (\#\text{ faces} = 2\text{-cells}) \end{aligned}$$

Then

$$v - e + f = 2 = \chi(S^2)$$

Definition 5.5 *X is an* ENR *(Euclidean neighbourhood retract) if:*

$$\exists \phi : X \xrightarrow{\cong} \phi(X) \subset \mathbb{R}^n$$

such that $\phi(X)$ is a retract of some neighbourhood in $\mathbb{R}^n$. e.g. finite CW-complexes are compact ENR's (see Hatcher)

Note Definition of (finite) simplicial complex should be clear (if not, it's time to go to the library!)

Lemma 5.6 *A compact ENR is a retract of a finite simplicial complex.*

Proof $X \subset \mathbb{R}^n$, X compact, retract of neighbourhood $X \subset N \subset \mathbb{R}^n$, $N \xrightarrow{r} X$; we may assume that N is open in $\mathbb{R}^n$.
Triangulate $\mathbb{R}^n$, such that all simplices are "very small" and we can assume that if a simplex σ of $\mathbb{R}^n$ has $\sigma \cap X \neq \varnothing$, then $\sigma \subset N$.
$\Rightarrow$ choose finite simplicial complex $Y \equiv \{\sigma \subset \mathbb{R}^n \mid \sigma \cap X \neq \varnothing\}$. Then $X \subset Y \subset N$, $Y \xrightarrow{r/Y} X$. $\square$

Theorem 5.7 (Simplicial Approximation Theorem) • *Simplicial map between simplicial complexes*

$$\begin{array}{ccc} f : X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \sigma \text{ (simplex)} & \xmapsto{\text{affine map}} & f(\sigma) \text{ (simplex)} \end{array}$$

- *X simplicial complex, $B(X)$: "barycentric subdivision", B^kX: k-fold barycentric subdivision ($B^0X = X$).*

Theorem 5.8 *Let X, Y be finite simplicial complexes and $f : X \to Y$ any (continuous) map. Then there is a $k \geq 0$ such that: $f : B^kX \to Y$ is homotopic to a simplicial map $g : B^kX \to Y$. (see Hatcher)*

Theorem 5.9 *Let X be a finite simplicial complex, $f : X \to X$ and $\varepsilon > 0$. Then there is a $k \geq 0$ and a simplicial map $g : B^kX \to B^kX$ with $g \simeq f$ and $\|g(x) - f(x)\| < \varepsilon \,\forall\, x$. (see Hatcher)*

Theorem 5.10 (Lefschetz Fixed Point Theorem) *Let $f : X \to X$, X a compact ENR. If $L(f) \neq 0$ then f has a fixed point.*

Proof Choose $X \overset{i}{\hookrightarrow} Y$, $Y \xrightarrow{r} X$, Y finite simplicial complex.
Put $\tilde{f} := i \circ f \circ r : Y \to Y$. Claim: $L(\tilde{f}) = L(f)$.
$H_i\tilde{f} : H_i^{\mathrm{sing}}Y \to H_i^{\mathrm{sing}}Y$ has:

$$\begin{aligned}
\mathrm{tr}(H_i^{\mathrm{sing}}\tilde{f}) &= \mathrm{tr}(H_i^{\mathrm{sing}}(i \circ f \circ r)) \\
&= \mathrm{tr}(H_i^{\mathrm{sing}}(i) \circ (H_i^{\mathrm{sing}}(f) \circ H_i^{\mathrm{sing}}(r))) \\
&= \mathrm{tr}(H_i^{\mathrm{sing}}(f)(\underbrace{H_i^{\mathrm{sing}}(r) \circ H_i^{\mathrm{sing}}(i)}_{id})) = \mathrm{tr}(H_i^{\mathrm{sing}}(f))
\end{aligned}$$

$\Rightarrow L(\tilde{f}) = L(f)$. Moreover:

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\} = \mathrm{Fix}(\tilde{f}) = \{y \in Y \mid \tilde{f}y = y\}$$

Indeed:

1. $x \in \mathrm{Fix}(f) \Rightarrow \tilde{f}(x) = if(\underbrace{rx}_{x}) = f(x) = x$

2. $y \in \mathrm{Fix}(\tilde{f}) \Rightarrow \underbrace{\tilde{f}(y)}_{if(ry)} = y \Rightarrow y \in X \Rightarrow ry = y \Rightarrow f(y) = y \in X.$

$\Rightarrow$ we may assume that X is a finite simplicial complex. Assume that $L(f) \neq 0$ and $\mathrm{Fix}(f) = \varnothing$. We will show that this yields a contradiction:
$f : X \to X$, X with metric $\|\cdot\| \Rightarrow \exists\, m > 0$ such that $\|f(x) - x\| \geq m \,\forall\, x$ because of compactness.
Choose $k \gg 0$ so that $f \simeq g : B^kX \to B^kX$, g simplicial and $\|g(x) - f(x)\| < \frac{m}{2} \Rightarrow \mathrm{Fix}\, g = \varnothing$.

$\Rightarrow$ we can choose k even larger, so that $g(\sigma) \cap \sigma = \varnothing$ for every σ simplex of $B^k X$; g is cellular and induces:

$$C_i^{\text{cell}} g : C_i^{\text{cell}}(B^k X) \to C_i^{\text{cell}} B^k X$$

with matrix:

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix}$$

$\Rightarrow \operatorname{tr}(C_i^{\text{cell}}(g)) = 0$
$\Rightarrow \sum(-1)^i \operatorname{tr}(C_i^{\text{cell}}(g)) = 0 = \sum(-1)^i \operatorname{tr}(H_i^{\text{cell}}(g)) = L(g)$.
So $L(f) \neq 0 \Rightarrow f$ has a FP. $\square$

An application of this theorem is this generalization of Brouwer's Fixed Point Theorem:

Theorem 5.11 *Let $f : X \to X$, X compact, contractible ENR. Then f has a fixed point.*

Proof

$$H_i^{\text{sing}}(X) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$$

since $X \simeq_\phi \{\centerdot\}$.

$$\begin{tikzcd} H_0^{\text{sing}} f : H_0^{\text{sing}}(X) \arrow[r] \arrow[d, "\phi_*"', "\cong"] & H_0^{\text{sing}}(X) \arrow[d, "\phi_*"', "\cong"] \\ H_0^{\text{sing}}(\{\centerdot\}) \arrow[r, "\text{id}"] & H_0^{\text{sing}}(\{\centerdot\}) \end{tikzcd}$$

$\Rightarrow L(f) = 1 \neq 0$: f has a fixed point. $\square$

Theorem 5.12 *Let $f : X \to X$ be a simplicial automorphism of a finite simplicial complex. Then*

$$L(f) = \chi(\operatorname{Fix}(f))$$

where $\operatorname{Fix}(f) = \{x \in X \mid f(x) = x\} \subset X$.

Proof Replace X by its second baricentric subdivision $B^2(X) \Rightarrow \operatorname{Fix}(f)$ subcomplex of $B^2(X) \Rightarrow$ if $\sigma \in B^2(X)$ is a k-simplex then either $f|\sigma = \text{id}_\sigma$, or $f(\mathring{\sigma}) \cap \mathring{\sigma} = \varnothing$. Then look at $C_*^{\text{cell}} B^2 X =: C_*$:

$$\begin{tikzcd} C_n \arrow[r, "C_n(f)"] \arrow[d, equal] & C_n \arrow[d, equal] \\ C_n^{\text{cell}}(\operatorname{Fix}(X)) \oplus B \arrow[r, "\phi"'] & C_n(\operatorname{Fix}(X)) \oplus B \end{tikzcd}$$

where ϕ has a matrix of the form

$$\left(\begin{array}{c|ccc} \mathrm{id} & & & \\ \hline & 0 & & \\ & & \ddots & \\ & & & 0 \end{array}\right)$$

thus

$$\operatorname{tr} C_n(f) = (\#\ n\text{-simplices in } \operatorname{Fix}(B^2X))$$

Then on the one hand

$$\Rightarrow \sum_{n=0}^{\dim X} (-1)^n \operatorname{tr} C_n(f) = \sum (-1)^n (\#\ n\text{-simplices of } \operatorname{Fix}(B^2X)) = \chi(\operatorname{Fix}(X))$$

and on the other hand

$$\sum_{n=0}^{\dim X} (-1)^n \operatorname{tr} C_n(f) = \sum (-1)^n \operatorname{tr}(H_n^{\mathrm{cell}} f) = L(f)$$

$\square$

Example Let $\phi : S^n \to S^n$, $n > 0$, the reflection on the equator.

$$\begin{aligned} L(\phi) &= \underbrace{\operatorname{tr}(H_0^{\mathrm{cell}}\phi)}_{1} + (-1)^n \operatorname{tr} H_n^{\mathrm{cell}}(\phi) = \chi(\operatorname{Fix}(\phi)) = \chi(S^{n-1}) \\ &= 1 + (-1)^{n-1} \cdot 1 \end{aligned}$$

$$H_n^{\mathrm{cell}}(\phi) : \qquad \begin{array}{ccc} H_n^{\mathrm{cell}} S^n & \longrightarrow & H_n^{\mathrm{cell}} S^n = \langle c_n \rangle \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

$\deg \phi$ defined by $H_n^{\mathrm{cell}}(\phi)(c_n) = \deg \phi \cdot c_n$

$$\Rightarrow \deg(\phi) = -1\ \forall n$$

6 Universal Coefficient Theorem

6.1 Remarks concerning the tensor product

A, B abelian groups. Then

$$A \otimes B := \bigoplus_{(a,b) \in A \times B} \mathbb{Z}_{(a,b)} / R$$

where $\mathbb{Z}_{(a,b)} = \mathbb{Z}$ with generator $1_{(a,b)}$ and R the subgroup generated by the elements of the forms

$$\begin{aligned} &1_{(a'+a'',b)} - 1_{(a',b)} - 1_{(a'',b)}, \text{ and} \\ &1_{(a,b'+b'')} - 1_{(a,b')} - 1_{(a,b'')} \end{aligned}$$

There is a canonical map (not a homomorphism!)

$$\begin{aligned} A \times B &\longrightarrow A \otimes B \\ (a,b) &\longmapsto \overline{1_{(a,b)}} =: a \otimes b \end{aligned}$$

Universal property of $A \otimes B$:
Note that $A \times B \overset{\text{can}}{\to} A \otimes B$, $(a,b) \mapsto a \otimes b$ is biadditive:

$$\begin{aligned} \text{can}(a'+a'',b) &= \text{can}(a',b) + \text{can}(a'',b) \\ \text{can}(a,b'+b'') &= \text{can}(a,b') + \text{can}(a,b'') \end{aligned}$$

It follows that

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "\text{can}"'] & C \\ A \otimes B \arrow[ur, dotted, "\exists!\, \tilde{f}"'] & \end{tikzcd}$$

($\tilde{f}$ is defined by $\tilde{f}(a \otimes b) := f(a,b)$, which is well defined as $f(a'+a'',b) = f(a',b) + f(a'',b)$ etc.)

Example $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$. Check the universal property:

$$\begin{tikzcd} (m,n) \arrow[d, mapsto] & \mathbb{Z} \times \mathbb{Z} \arrow[r, "f"] \arrow[d, "\text{can}"'] & C \\ m \cdot n & \mathbb{Z} & \end{tikzcd}$$

The map $(m,n) \mapsto m \cdot n$ is biadditive because of the distributive law. Since $f(m,n) = m \cdot f(1,n) = mnf(1,1)$, f is determined uniquely by $f(1,1)$ and we can define $\tilde{f}$ as $\tilde{f}(k) = k \cdot \tilde{f}(1) = k \cdot f(1,1)$.
Similiarly,

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$$

Functoriality: Let $f : A \to C$, $g : B \to D$.

$$\begin{tikzcd} A \times B \arrow[r, "\text{biadd.}"] \arrow[d] & C \otimes D \\ A \otimes B \arrow[ur, dotted, "\exists!\, f \otimes g"'] & \end{tikzcd}$$

where $(f \otimes g)(a \otimes b) := f(a) \otimes g(b)$.

$$A \otimes - : \qquad \underline{\text{Ab}} \longrightarrow \underline{\text{Ab}}$$

$$\begin{tikzcd} B \arrow[r, mapsto] \arrow[d, "g"'] & A \otimes B \arrow[d, "A\otimes g := \mathrm{id}_A \otimes g"] \\ D \arrow[r, mapsto] & A \otimes D \end{tikzcd}$$

similiarly for $- \otimes B$. Note $A \otimes B \xrightarrow{\cong} B \otimes A$.
Generalization: $M \in \underline{\text{Mod-}\Lambda}$ (right Λ-modules), $N \in \underline{\Lambda\text{-Mod}}$ (left Λ-modules). $M \otimes_\Lambda N$ an abelian group:

$$M \otimes_\Lambda N = M \otimes N / \langle m\lambda \otimes n - m \otimes \lambda n \mid \lambda \in \Lambda \rangle$$

Case where Λ is commutative: $M, N \in \underline{\Lambda\text{-Mod}}$ thinking of M as a right-module by $m\lambda := \lambda m$, $\lambda \in \Lambda$, $m \in M$. Then $M \otimes_\Lambda N$ (note $(\lambda m) \otimes_\Lambda n = m \otimes_\Lambda (\lambda n)$) has a Λ-module structure by:

$$\lambda(x \otimes_\Lambda y) := (\lambda x) \otimes_\Lambda y$$

Example $\Lambda \otimes_\Lambda \Lambda \cong \Lambda$

Let $\phi : \Lambda \to \Gamma$ be a ring homomorphism, in particular $\phi(1_\Lambda) = 1_\Gamma$.

$$\begin{aligned} \underline{\Lambda\text{-Mod}} &\xrightarrow{\phi_*} \underline{\Gamma\text{-Mod}} \\ M &\longmapsto \Gamma \otimes_\Lambda M \quad (\gamma\phi(\lambda) \otimes_\Lambda m = \gamma \otimes \lambda m) \end{aligned}$$

Γ a Λ right module via

$$\gamma \cdot \lambda := \gamma \cdot \phi(\lambda),\ \lambda \in \Lambda,\ \gamma \in \Gamma$$

Example $M = \Lambda$:

$$\begin{tikzcd} \Lambda \arrow[r] & \Gamma \otimes_\Lambda \Lambda \arrow[r, "\cong"] & \Gamma \\ & \gamma \otimes_\Lambda \lambda \arrow[r, mapsto] & \gamma\phi(\lambda) \end{tikzcd}$$

as Γ left module.

Tensor products commute with $\oplus$: Let A_α a family of abelian groups, $\alpha \in I$. Then

$$\Big(\bigoplus_I A_\alpha\Big) \otimes B \cong \bigoplus_I (A_\alpha \otimes B)$$

Proof $i_\alpha : A_\alpha \to \bigoplus_I A_\alpha$

$$\Rightarrow i_\alpha \otimes B : A_\alpha \otimes B \to \Big(\bigoplus_I A_\alpha\Big) \otimes B$$

which defines

$$\bigoplus_I (A_\alpha \otimes B) \stackrel{\text{can}}{\to} \Big(\bigoplus_I A_\alpha\Big) \otimes B$$

Define "inverse" by using the biadditive map

$$\Big(\bigoplus A_\alpha\Big) \times B \stackrel{\Phi}{\to} \bigoplus_I (A_\alpha \otimes B)$$

by $\Phi|(A_\alpha \times B) : (a_\alpha, b) \mapsto a_\alpha \otimes b$.
So Φ induces

$$\tilde{\Phi} : \Big(\bigoplus A_\alpha\Big) \otimes B \to \bigoplus (A_\alpha \otimes B)$$

which is inverse to "can". $\square$

Definition 6.1 *$M \in \underline{\Lambda\text{-Mod}}$ is called* free*, if*

$$M \cong \bigoplus_{\alpha \in I} \Lambda_\alpha, \ \Lambda_\alpha := \Lambda$$

or equivalently, M has a basis $\{m_\alpha\}_{\alpha \in I}$ i.e. every $m \in M$ can be expressed as a finite sum $m = \sum \lambda_\alpha m_\alpha$ in a unique way.

Note $\Lambda = K$ a field $\Rightarrow$ all K-modules are free (every K-vector space has a basis).
$\Lambda = \mathbb{Z}$: $\underline{\mathbb{Z}\text{-Mod}} = \underline{\text{Ab}}$, free $\mathbb{Z}$-module $\equiv$ free abelian groups.

Definition 6.2 *$P \in \underline{\Lambda\text{-Mod}}$ is* projective $:\Leftrightarrow \exists Q \in \underline{\Lambda\text{-Mod}}$ *with $P \oplus Q$ a free Λ-module.*

Note $\Lambda = K$ a field $\Rightarrow$ all Λ-modules are projective.
$\Lambda = \mathbb{Z}$: projective $\mathbb{Z}$-modules $\equiv$ free abelian groups.

Example $\mathbb{Z}/2\mathbb{Z}$ is a non-free, projective $\mathbb{Z}/6\mathbb{Z}$-module.

Definition 6.3 *A* chain complex (C_*, ∂) *consists of modules $C_i \in \underline{\Lambda\text{-Mod}}$ connected by morphisms $\partial_i \in \underline{\Lambda\text{-Mod}}$ $(i \geq 0)$:*

$$\ldots \to C_{i+1} \to C_i \stackrel{\partial_i}{\to} C_{i-1} \to \ldots \to C_1 \stackrel{\partial_1}{\to} C_0 \stackrel{\partial_0}{\to} 0$$

such that $\partial_{i-1}\partial_i = 0 (\equiv \partial^2 = 0)$ $(\Leftrightarrow \operatorname{im} \partial_i \subseteq \ker \partial_{i-1})$

Definition 6.4 *Homology of* (C_n, ∂_n)*:*

$$H_n(C_*) := \ker \partial_n / \operatorname{im} \partial_{n+1} \quad (n \geq 0)$$

Definition 6.5 *A morphism of chain complexes,* $f_* : C_* \to D_*$ *is a family of* Λ*-linear maps* $f_i : C_i \to D_i$ $(i \geq 0)$*, such that:* $\partial_i f_i = f_{i-1}\partial_i$*,* $i \geq 1$*.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial"] \arrow[d, "f_{i+1}"'] & C_i \arrow[r, "\partial"] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r, "\partial"'] & D_i \arrow[r, "\partial"'] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark f_* induces a map of homology groups, i.e. $H_*(f) : H_*C_* \to H_*D_*$

Definition 6.6 f_**,* g_**:* $C_* \to D_*$ *are called* chain homotopic *if* $\exists\{h_n : C_n \to D_{n+1} \mid n \geq 0\}$ *such that* $f - g = \partial h + h\partial$ *(*$f_n - g_n = \partial_{n+1}h_n + h_{n-1}\partial_n$*)*
Notation: $f \simeq g$

Lemma 6.7 $f_*, g_* : C_* \to D_*$*,* $f \simeq g \Rightarrow H_*(f) = H_*(g)$

Proof $[x] \in H_nC_*$, $x \in \ker \partial_n$, $(H_nf)([x]) = [f(x)]$, $H_ng([x]) = [gx] \in H_nD_*$

$$\begin{aligned}(H_nf - H_ng)[x] &= [f(x) - g(x)] = [\partial_{n-1}h_n + h_{n-1}\partial_n x] \\ &= \underbrace{[\partial hx]}_{=0} + \underbrace{[h\partial x]}_{=0} = 0\end{aligned}$$

$\Rightarrow H_nf = H_ng$, $\forall n \geq 0$ $\square$

Definition 6.8 *Let* $M \in \underline{\Lambda\text{-Mod}}$*. A* projective resolution *is a chain complex* P_* *such that:*

$$\ldots \to P_i \to \ldots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0 \qquad M = P_0/\operatorname{im}\partial_1$$

is exact*, and each* P_i*,* $i \geq 0$ *is a projective* Λ*-module.*

Lemma 6.9 *Every* Λ*-module* M *admits a projective resolution*

$$F_*(M) \twoheadrightarrow M$$

(canonical free resolution)

Proof

$$F_0(M) = \bigoplus_{\alpha \in M} \Lambda_\alpha \overset{\phi_0}{\twoheadrightarrow} M$$
$$1_\alpha \mapsto \alpha$$

$$\begin{tikzcd}
F_1(M) \arrow[r, "\phi_1"] \arrow[d] & F_0(M) \arrow[r, two heads, "\phi_0"] & M \\
\ker \phi_0 \arrow[ur, hook] & &
\end{tikzcd}$$

$F_1(M) = F_0(\ker \phi_0)$, then the claim follows inductively $\square$

We need an equivalent definition of projective modules:

Lemma 6.10 *$P \in \underline{\Lambda\text{-Mod}}$ proj. $\Leftrightarrow \forall g : N \twoheadrightarrow M$, $\forall P \xrightarrow{f} M$ $\exists \tilde{f} : P \to N$ such that $g \circ \tilde{f} = f$ i.e.*

$$\begin{tikzcd}
 & N \arrow[d, two heads, "g"] \\
P \arrow[ur, dotted, "\exists \tilde{f}"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Proof "$\Rightarrow$" This is obvious for free Λ-modules, thus choose $Q \in \underline{\Lambda\text{-Mod}}$ such that $P \oplus Q = F$ ($\equiv$ free module)

$$\begin{tikzcd}
 & & N \arrow[d, two heads] \\
F = P \oplus Q \arrow[urr, dashed, "\tilde{\lambda}"] \arrow[r, two heads, shift left] & P \arrow[l, shift left, "i_p"] \arrow[r] & M
\end{tikzcd}
\qquad \tilde{\lambda} \circ i_p = \tilde{f}$$

"$\Leftarrow$" Let $F_0(P) = \bigoplus_{\alpha \in P} \Lambda_\alpha$

$$\begin{tikzcd}
 & F_0(P) \arrow[d, two heads, "\pi"] \\
P \arrow[ur, dashed, "\exists j"] \arrow[r, "\mathrm{id}"'] & P
\end{tikzcd}$$

$\square$

Theorem 6.11
- *Let $P_* \twoheadrightarrow M$ be a* projective chain complex *(i.e. P_i is projective Λ-mod. and $\partial^2 = 0$)*
- *Let $0 \leftarrow R_* \twoheadrightarrow N$ be a* resolution *(i.e. $\ker \partial = \operatorname{im} \partial$).*

Assume a map $M \xrightarrow{\phi} N$. Then there is a map of chain complexes:

$$\begin{tikzcd} P_* \arrow[r, two heads] \arrow[d, "\phi_*"] & M \arrow[d, "\phi"] \\ R_* \arrow[r, two heads] & N \end{tikzcd}$$

This map ϕ_* is unique up to homotopy.

Proof First we prove existence of $\phi_* : P_* \to R_*$ (use definition of projective module):

$$\begin{tikzcd} P_1 \arrow[r, "\partial_1^P"] \arrow[d, dashed, "\phi_1"] \arrow[dd, dashed, bend right] & P_0 \arrow[r, two heads, "\partial_0^P"] \arrow[d, dashed, "\phi_0"] & M \arrow[d, "\phi"] \\ R_1 \arrow[r, "\partial_1^R"] \arrow[d] & R_0 \arrow[r, two heads, "\partial_0^R"] & N \\ \ker \partial_0^R \arrow[ur, hook] & & \end{tikzcd}$$

Check: $\operatorname{im}(\phi_0 \partial_1^P) \subset \ker \partial_0^R \Rightarrow \exists \phi_1$. The rest follows by induction.
Next we want to show that ϕ_* is unique up to homotopy. Let σ_* be another "lifting" of ϕ, i.e.

$$\begin{tikzcd} P_* \arrow[r, two heads] \arrow[d, shift right, "\phi_*"'] \arrow[d, shift left, "\sigma_*"] & M \arrow[d, "\phi"] \\ R_* \arrow[r, two heads] & N \end{tikzcd}$$

want to show that $\exists \{h_n : P_n \to R_{n+1}, n \geq 0\}$ such that $\phi - \sigma = \partial h + h\partial$:

$$\begin{tikzcd} \cdots \arrow[r] & P_{i+1} \arrow[r, "\partial_{i+1}"] & P_i \arrow[r, "\partial_i"] \arrow[dl, dashed, "h_i"'] \arrow[d, shift right, "\phi_i"'] \arrow[d, shift left, "\sigma_i"] & P_{i-1} \arrow[r] \arrow[dl, dashed, "h_{i-1}"] & \cdots \\ \cdots \arrow[r] & R_{i+1} \arrow[r, "\partial_{i+1}"] & R_i \arrow[r, "\partial_i"] & R_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

(Proof by induction)
so by induction we have: $\partial_i h_{i-1} + h_{i-2}\partial_{i-1} = \phi_{i-1} - \sigma_{i-1}$
We want to "solve" the equation for h_i:

$$\begin{aligned} &\partial_{i+1} h_i + h_{i-1}\partial_i = \phi_i - \sigma_i \\ \Leftrightarrow\ &\partial_{i+1} h_i = \underbrace{\phi_i - \sigma_i - h_{i-1}\partial_i}_{\text{this maps to } \ker(\partial_i : R_i \to R_{i-1})\ (*)} \end{aligned}$$

Proof of (*): $x \in R_i$

$$\partial_i(\phi_i - \sigma_i - h_{i-1}\partial_i)(x) = \partial_i\phi_i(x) - \partial_i\sigma_i(x) - \underbrace{\partial_i h_{i-1}}_{\phi_{i-1}-\sigma_{i-1}-h_{i-2}\partial_{i-1}} \partial_i(x)$$

$$= \partial_i\phi_i(x) - \partial_i\sigma_i(x) - \phi_{i-1}\partial_i(x) + \sigma_{i-1}\partial_i(x) + \underbrace{h_{i-2}\underbrace{\partial_{i-1}\partial_i}_{=0} x}_{=0}$$

$$= 0 \quad \text{since } \phi_*, \sigma_* \text{ are chain maps}$$

The lifting property of projective modules shows that:

$$\begin{tikzcd}
P_{i+1} \arrow[r] & P_i \arrow[r, "\partial_i"] \arrow[dl, dashed, "\exists h_i"'] \arrow[d, shift right, "\phi_i"'] \arrow[d, shift left, "\sigma_i"] \arrow[ddl, dashed, two heads] & \cdots \arrow[dl, dashed, "h_{i-1}"] \\
R_{i+1} \arrow[r] \arrow[d, two heads] & R_i \arrow[r, "\partial_i"] & \cdots \\
\ker \partial_i \arrow[ur] & &
\end{tikzcd}$$

$\Rightarrow \partial_{i+1}h_i = \phi_i - \sigma_i - h_{i-1}\partial_i \Leftrightarrow \partial_{i+1}h_i + h_{i-1}\partial_i = \phi_i - \sigma_i$ $\square$

Corollary 6.12 *Let $P_*^{(i)} \twoheadrightarrow M$, $i = 1, 2$ be two projective resolutions of M. Then $P_*^{(1)}$ and $P_*^{(2)}$ are chain homotopy equivalent, i.e.*

$$\begin{aligned} \exists j_1 :\ & P_*^{(1)} \to P_*^{(2)} \\ \exists j_2 :\ & P_*^{(2)} \to P_*^{(1)} \end{aligned}$$

such that $j_1 \circ j_2 \simeq \mathrm{id}$, $j_2 \circ j_1 \simeq \mathrm{id}$.

Proof (Use theorem above)

$$\begin{tikzcd}
P_*^{(1)} \arrow[r, two heads] \arrow[d, "j_1"] & M \arrow[d, "\mathrm{id}"] \\
P_*^{(2)} \arrow[r, two heads] \arrow[d, "j_2"] & M \arrow[d, "\mathrm{id}"] \\
P_*^{(1)} \arrow[r] & M
\end{tikzcd}$$

but

$$\begin{tikzcd}
P_*^{(1)} \arrow[r] \arrow[d, "\mathrm{id}"] & M \arrow[d, "\mathrm{id}"] \\
P_*^{(1)} \arrow[r] & M
\end{tikzcd}$$

is also a lifting. By uniqueness we get $j_2 \circ j_1 \simeq \mathrm{id}$. Analog for $j_1 \circ j_2 \simeq \mathrm{id}$ $\square$

For simplicity assume Λ is a commutative ring.

Definition 6.13 *Let $M, N \in \underline{\Lambda\text{-Mod}}$ (in general $M \in \underline{\text{Mod-}\Lambda}$, $N \in \underline{\Lambda\text{-Mod}}$). For $i \geq 0$,*

$$\mathrm{Tor}_i^\Lambda(M,N) := H_i(F_*(M) \otimes_\Lambda N)$$

Note

$$\ldots \to F_i(M) \otimes_\Lambda N \overset{\partial_i \otimes \mathrm{id}_N}{\longrightarrow} F_{i-1}(M) \otimes_\Lambda N \to \ldots$$

$\Rightarrow F_*(M) \otimes_\Lambda N$ is a chain complex.

It is important to see that Tor_i^Λ does not depend on the choice of the projective resolution $(F_*(M))$.

Lemma 6.14 *Let $P_* \twoheadrightarrow M$ be any projective resolution, then*

$$\mathrm{Tor}_i^\Lambda(M,N) \cong H_i(P_* \otimes_\Lambda N)$$

Proof

$$\begin{tikzcd}
F_*(M) \arrow[r, two heads] \arrow[d] & M \arrow[d, "\mathrm{id}"] \\
P_* \arrow[r, two heads] \arrow[d] & M \arrow[d, "\mathrm{id}"] \\
F_*(M) \arrow[r, two heads] & M
\end{tikzcd}$$

$\Rightarrow$ "$F_*(M) \simeq P_*$". $- \otimes_\Lambda N$ preserves the homotopy since $- \otimes_\Lambda N$ is additive (i.e. $(f+g) \otimes_\Lambda N = f \otimes_\Lambda N + g \otimes_\Lambda N$).
$f \simeq g$, $\exists h : g - f = \partial h + h\partial$

$$\begin{aligned} g \otimes N - f \otimes N &= (g-f) \otimes N = (\partial h + h\partial) \otimes N \\ &= \partial h \otimes N + h\partial \otimes N = (\partial \otimes N)(h \otimes N) + (h \otimes N)(\partial \otimes N) \end{aligned}$$

$F_*(M) \simeq P_* \Rightarrow F_*(M) \otimes_\Lambda N \simeq P_* \otimes_\Lambda N \Rightarrow$

$$\mathrm{Tor}_*^\Lambda(M,N) = H_*(F_*(M) \otimes_\Lambda N) \cong H_*(P_* \otimes_\Lambda N)$$

$\square$

Lemma 6.15 *The functor $- \otimes_\Lambda N : \underline{\text{Mod-}\Lambda} \to \underline{\text{Ab}}$ is* right exact, *i.e. if $U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ is exact, then*

$$U \otimes_\Lambda N \to V \otimes_\Lambda N \to W \otimes_\Lambda N \to 0$$

is exact.

Proof $W \otimes_\Lambda N$ is generated by elements $w \otimes n = \beta\tilde{v} \otimes m = (\beta \otimes \mathrm{id})(\tilde{v} \otimes n)$
$\Rightarrow \beta \otimes \mathrm{id}$ is surjective.
Obviously $\operatorname{im} \alpha \otimes \mathrm{id} \subseteq \ker(\beta \otimes \mathrm{id})$. Want to prove that $\ker(\beta \otimes \mathrm{id}) = \operatorname{im}(\alpha \otimes \mathrm{id})$.
For that we construct an inverse map to

$$V \otimes_\Lambda N / \operatorname{im}(\alpha \otimes \mathrm{id}) \overset{\pi}{\twoheadrightarrow} V \otimes_\Lambda N / \ker(\beta \otimes \mathrm{id}) \cong W \otimes_\Lambda N$$

Construct γ as

$$\begin{aligned} \gamma : W \times N &\to V \otimes_\Lambda N / \operatorname{im}(\alpha \otimes \mathrm{id}) \\ (w, n) &\mapsto \overline{\tilde{v} \otimes n} \end{aligned}$$

γ is well-defined: Let $\hat{v}$, $\beta\hat{v} = n$.

$$\tilde{v} \otimes n - \hat{v} \otimes n = (\tilde{v} - \hat{v}) \otimes n = \alpha u \otimes n = (\alpha \otimes \mathrm{id})(u \otimes n)$$

The following is easily checked:

- γ is bilinear $\Rightarrow$

$$\gamma : W \otimes_\Lambda N \to V \otimes_\Lambda N / \operatorname{im}(\alpha \otimes \mathrm{id})$$

- $\gamma \circ \pi = \mathrm{id}$
- $\pi \circ \gamma = \mathrm{id}$

$\Rightarrow \ker(\beta \otimes \mathrm{id}) = \operatorname{im}(\alpha \otimes \mathrm{id})$ $\square$

Corollary 6.16 *There is a natural isomorphism*

$$\mathrm{Tor}_0^\Lambda(M, N) \cong M \otimes_\Lambda N$$

Proof $- \otimes_\Lambda N$ is right exact.

$$\begin{tikzcd} F_1 \arrow[r] & F_0 \arrow[r, two heads] \arrow[d] & M \arrow[r] & 0 \arrow[r, squiggly, "- \otimes_\Lambda N"] & F_1 \otimes_\Lambda N \arrow[r, "\partial_1 \otimes N"] & F_0 \otimes_\Lambda N \arrow[r, "\partial_0 \otimes N"] \arrow[d] & M \otimes_\Lambda N \arrow[r] & 0 \\ & 0 & & & & 0 & & \end{tikzcd}$$

thus

$$\mathrm{Tor}_0(M, N) = F_0 \otimes N / \operatorname{im}(\partial_1 \otimes N) \cong M \otimes_\Lambda N$$

because $\operatorname{im}(\partial_1 \otimes N) = \ker(\partial_0 \otimes N)$ from right exactness. $\square$

Example (Group Homology)
For some group G, define

$$\mathbb{Z}G = \left\{ \sum n_g g \,\middle|\, g \in G \right\}$$

Let $M \in \underline{\mathbb{Z}\text{G-Mod}}$ and $\mathbb{Z} \in \underline{\text{Mod-}\mathbb{Z}\text{G}}$ with trivial G-action (i.e. $m \cdot g = m$, $m \in \mathbb{Z}$, $g \in G$ linearly extended).
$\mathbb{Z}G$ is a ring:

$$\left(\sum n_g g\right) \cdot \left(\sum m_k k\right) = \sum n_g m_k gk$$

As an abelian group:

$$\mathbb{Z}G = \bigoplus_G \mathbb{Z}$$

If $P_* \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$H_i(G; M) := \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M)$$

$$\left(H_*(S, M) \cong H_*^{\text{sing}}(K(S, 1), M)\right)$$

$$H_0(G; M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M = M / \langle m - gm \mid g \in G \rangle$$

[FIXME: Konfusion zum Jahreswechsel]
Then $H_i((P_*M) \otimes_\Lambda N) \cong \operatorname{Tor}_i^\Lambda(M, N) := H_i(F_*(M) \otimes_\Lambda N)$, $F_*M \twoheadrightarrow M$ ($F_0 M = \bigoplus_M \Lambda$ etc.) "canonical free resolution"
special case: "Homology groups of G with coefficients in M"
$H_i(G; M) := \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M)$, where G is a group, M left $\mathbb{Z}G$-module.

$$H_i(G; -) : \underline{\mathbb{Z}\text{G-Mod}} \to \underline{\text{Ab}} \qquad i \in \mathbb{Z}$$

$- \otimes_\Lambda N$ is right-exact. ($0 \to M' \to M \to M'' \to 0$ exact, then $M' \otimes_\Lambda N \to M \otimes_\Lambda N \to M'' \otimes_\Lambda N \to 0$ is exact)
$\Rightarrow \operatorname{Tor}_0^\Lambda(M, N) \cong M \otimes_\Lambda N$ (e.g. $H_0(G; M) \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M \cong M_G := M/\langle m - gm \rangle$, with $m \in M$,$g \in G$)
Case $\Lambda = \mathbb{Z}$: $\underline{\Lambda\text{-Mod}} = \underline{\text{Ab}} = \underline{\text{Mod-}\Lambda}$

Lemma 6.17 *A, $B \in \underline{\text{Ab}} \Rightarrow \operatorname{Tor}_i^{\mathbb{Z}}(A, B) = 0$, $i > 1$.*
(We write $\operatorname{Tor}(A, B)$ for $\operatorname{Tor}_1^{\mathbb{Z}}(A, B)$, and $\operatorname{Tor}_0^{\mathbb{Z}}(A, B) \cong A \otimes_{\mathbb{Z}} B$)

Proof 1. A free abelian group, $A \xrightarrow{\text{id}\,(\cong)} A$ is a proj. resolution

$$\Rightarrow H_i(P_*(A) \otimes_{\mathbb{Z}} B) = 0,\ i > 0$$

$$\Rightarrow \operatorname{Tor}_i^{\mathbb{Z}}(A, B) = 0 \text{ for } i > 0$$

2. A arbitrary abelian group:

$$P_1(A) := K \hookrightarrow F_0 A \overset{\varepsilon}{\twoheadrightarrow} A \quad \text{proj. res of } A$$

$P_i(A) = 0, i > 1$, $K := \ker \varepsilon$ free abelian group (subgroup of free abelian group).

$\Rightarrow \underbrace{H_i(P_*(A) \otimes_{\mathbb{Z}} B)}_{\cong \operatorname{Tor}_i^{\mathbb{Z}}(A,B)} = 0$ for $i > 1$.

$\square$

Remark Also true for modules over a PID.

Exercise $\operatorname{Tor}(A, B)$ is a torsion group ($A, B \in \underline{\text{Ab}}$)

6.2 Computation of Tor-groups

We would like to show:

$$\operatorname{Tor}_i^{\Lambda}(\operatorname{dirlim}_I M_\alpha, \operatorname{dirlim}_J N_\beta) \cong \operatorname{dirlim}_{I \times J} \operatorname{Tor}_i^{\Lambda}(M_\alpha, N_\beta)$$

Direct limit (of groups, modules, sets, etc.):

- basic example: $M = \bigcup_{\alpha \in I} M_\alpha$, $M_\alpha, M \in \underline{\Lambda\text{-Mod}}$ s.t. I partially ordered "index" set: PO-set, with:

  $$\alpha \leq \beta \Leftrightarrow M_\alpha \subset M_\beta \subset M$$

  "directed" i.e. if $\alpha, \beta \in I$ then $\exists \gamma \in I$ with $\alpha \leq \gamma, \beta \leq \gamma$ (so $M_\alpha \subset M$, $M_\beta \subset M$ satisfy $M_\alpha \subset M_\gamma, M_\beta \subset M_\gamma$)

- Example: $M \in \underline{\Lambda\text{-Mod}}$ with $\{M_\alpha\}$ the family of finitely generated submodules of M, then $M \cong \operatorname{dirlim} M_\alpha$

- **Definition 6.18** *I PO-set, directed $\Rightarrow$ defines a category $\underline{I}$, with objects the elements $\alpha \in I$, and morphisms:*

  $$\operatorname{mor}(\alpha, \beta) = \begin{cases} \varnothing & \text{if } \alpha \not\leq \beta \\ \text{one morphism} & \text{if } \alpha \leq \beta \end{cases}$$

  ($\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$)

- A functor $F : \underline{\mathrm{I}} \to \underline{\mathrm{C}}$ defines a "directed family" $\{F(\alpha)\}_{\alpha \in I}$ in $\underline{\mathrm{C}}$. (e.g. $\underline{\mathrm{C}} = \underline{\mathrm{Sets}}, \underline{\mathrm{Gr}}, \underline{\Lambda\text{-}\mathrm{Mod}}, \underline{\mathrm{Ab}}$).

  so if $\alpha \leq \beta$: $F(\alpha) \overset{f_{\alpha\beta}}{\to} F(\beta)$

$$\begin{tikzcd} F(\sigma) \arrow[dr, "f_{\sigma\omega}"] & \\ & F(\omega) \\ F(\tau) \arrow[ur, "f_{\tau\omega}"'] & \end{tikzcd} \quad \text{for } \sigma, \tau \in I,\ \sigma \leq \omega, \tau \leq \omega$$

$F : \underline{\mathrm{I}} \to \underline{\mathrm{C}}$
$\operatorname{colim} F \in \underline{\mathrm{C}}$ is an object of $\underline{\mathrm{C}}$, together with morphisms $\phi_\alpha : F(\alpha) \to \operatorname{colim} F$, $\alpha \in \underline{\mathrm{I}}$, with the universal property expressed by the diagram

$$\begin{tikzcd} F(\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "f_{\alpha\beta}"'] \arrow[rr, bend left, "\psi_\alpha"] & \operatorname{colim} F \arrow[r, dotted, "\exists!"] & X \in \underline{\mathrm{C}} \\ F(\beta) \arrow[ur, "\phi_\beta"] \arrow[urr, "\psi_\beta"'] & & \\ F(\gamma) \arrow[uur, "\phi_\gamma"] \arrow[uurr, "\psi_\gamma"'] & & \\ \vdots & & \end{tikzcd}$$

so $\operatorname{colim} F$ (together with ϕ_α's) is unique up to a canonical isomorphism, if it exists.
Case of $\underline{\mathrm{C}} = \underline{\Lambda\text{-}\mathrm{Mod}}$ (or $\underline{\mathrm{Sets}}$, or $\underline{\mathrm{Gr}}$):
$F : \underline{\mathrm{I}} \to \underline{\Lambda\text{-}\mathrm{Mod}}$ a directed family of Λ-moduls. Put

$$\operatorname*{dirlim}_I F(\alpha) := \coprod F(\alpha)/\sim$$

(disjoint union!) with $x_\alpha \sim y_\beta$ for $x_\alpha \in F(\alpha)$, $y_\beta \in F(\beta)$ if $\exists \gamma$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$ and $f_{\alpha\gamma}(x_\alpha) = f_{\beta\gamma}(y_\beta)$.

$$\begin{tikzcd} x_\alpha \in F(\alpha) \arrow[dr] & \\ & F(\gamma) \ni f_{\alpha\gamma}(x_\alpha) = f_{\alpha\beta}(y_\beta) \\ y_\beta \in F(\beta) \arrow[ur] & \end{tikzcd}$$

$\Rightarrow$ $\operatorname{dirlim} F(\alpha)$ has a natural Λ-modul structure. We have canonical maps $F(\alpha) \overset{\phi_\alpha}{\to} \operatorname{dirlim} F(\alpha) \Rightarrow \{\operatorname{dirlim}_I F(\alpha), \phi_\alpha\}$ is "$\operatorname{colim} F$".

Note The universal property of colim then means:

$$\operatorname{Hom}_\Lambda(\operatorname{dirlim} F(\alpha), N) \overset{\cong}{\to} \operatorname{invlim} \operatorname{Hom}_\Lambda(F(\alpha), N)$$

Lemma 6.19 dirlim *is an exact functor on* $\underline{\Lambda\text{-Mod}}$ *(or* $\underline{\text{Gr}}$*), meaning the following: Let*

$$0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$$

be a family of short exact sequences in $\underline{\Lambda\text{-Mod}}$*,* $\alpha \in I$ *(directed PO-set). Assume that if* $\alpha \leq \beta$*, we have*

$$\begin{tikzcd} 0 \arrow[r] & A_\alpha \arrow[r] \arrow[d, "a_{\alpha\beta}"'] & B_\alpha \arrow[r] \arrow[d, "b_{\alpha\beta}"'] & C_\alpha \arrow[r] \arrow[d, "c_{\alpha\beta}"'] & 0 \\ 0 \arrow[r] & A_\beta \arrow[r] & B_\beta \arrow[r] & C_\beta \arrow[r] & 0 \end{tikzcd}$$

commutative.
Then

$$0 \to \operatorname*{dirlim}_I A_\alpha \to \operatorname*{dirlim}_I B_\alpha \to \operatorname*{dirlim}_I C_\alpha \to 0$$

is exact.

Use this to check

$$\mathrm{Tor}_i^\Lambda(\operatorname*{dirlim}_I M_\alpha, \operatorname*{dirlim}_J N_\beta) \cong \operatorname*{dirlim}_{I\times J} \mathrm{Tor}_i^\Lambda(M_\alpha, N_\beta)$$

6.3 Long exact Tor-sequences

Theorem 6.20 *Let* $0 \to A \to B \to C \to 0$ *and* $0 \to U \to V \to W \to 0$ *be short exact sequences in* $\underline{\text{Mod-}\Lambda}$*, resp.* $\underline{\Lambda\text{-Mod}}$*, and take* $X \in \underline{\text{Mod-}\Lambda}$*,* $Y \in \underline{\Lambda\text{-Mod}}$*. Then there are natural exact sequences*

$$\cdots \to \mathrm{Tor}_i^\Lambda(A, Y) \to \mathrm{Tor}_i^\Lambda(B, Y) \to \mathrm{Tor}_i^\Lambda(C, Y) \xrightarrow{\partial} \mathrm{Tor}_{i-1}^\Lambda(A, Y) \to \cdots$$

$$\cdots \to A \otimes_\Lambda Y \to B \otimes_\Lambda Y \to C \otimes_\Lambda Y \to 0$$

and

$$\cdots \to \mathrm{Tor}_i^\Lambda(X, U) \to \mathrm{Tor}_i^\Lambda(X, V) \to \mathrm{Tor}_i^\Lambda(X, W) \xrightarrow{\partial} \mathrm{Tor}_{i-1}^\Lambda(X, U) \to \cdots$$

$$\cdots \to X \otimes_\Lambda U \to X \otimes_\Lambda V \to X \otimes_\Lambda W \to 0$$

Note $\mathrm{Tor}_i^\Lambda(X, U) \cong H_i(X \otimes_\Lambda P_*U)$, where P_*U is a projective resolution of U.

Proof (1) Given $0 \to C_* \to D_* \to E_* \to 0$ a short exact sequence of chain complexes. Then one gets a long *exact* sequence

$$\cdots \to H_i(C_*) \to H_i(D_*) \to H_i(E_*) \xrightarrow{\partial} H_{i-1}(C_*) \to \cdots$$

where ∂ is defined as follows:

$$\begin{tikzcd} 0 \arrow[r] & C_i \arrow[r] & D_i \arrow[r] & E_i \arrow[r] & 0 \\ & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\ & & \hat{x}_i \arrow[r, mapsto] \arrow[d, mapsto] & \tilde{x}_i & \\ & & \partial\hat{x}_i \arrow[r, mapsto] & 0 \in E_{i-1} & \end{tikzcd}$$

$\Rightarrow \partial\hat{x}_i \in C_{i-1}$ a cycle: $\partial(\partial\hat{x}_i) = 0$.

(2)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\ 0 \arrow[r] & P_*A \arrow[r] \arrow[u, two heads] & P_*B \arrow[r] \arrow[u, two heads, dotted] & P_*C \arrow[r] \arrow[u, two heads] & 0 \end{tikzcd}$$

Take $P_iB := P_iA \oplus P_iC$ (see next time).

(next time, with different notation...)

We want to "replace" $0 \to U \to V \to W \to 0$ by a short exact sequence of projective resolutions:

$$0 \to P_*U \to P_*V \to P_*W \to 0$$

this is how:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] & V \arrow[r] & W \arrow[r] & 0 \\ 0 \arrow[r] & P_*U \arrow[r, hook] \arrow[u, two heads] & P_*V \arrow[r, two heads] \arrow[u, two heads, dotted] & P_*W \arrow[r] \arrow[u, two heads] & 0 \end{tikzcd}$$

choose P_*U and P_*W, put $P_iV := P_iU \oplus P_iW$ (which is projective).

induction:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, hook] & V \arrow[r, two heads, "\pi"] & W \arrow[r] & 0 \\ 0 \arrow[r] & P_0U \arrow[r] \arrow[u, two heads, "\varepsilon_U"] & (x,y) \in P_0V \arrow[r] \arrow[u, two heads, dotted, "\varepsilon_V"'] & y \in P_0W \arrow[r] \arrow[u, two heads, "\varepsilon_W"'] \arrow[ul, "\exists\phi"'] & 0 \end{tikzcd}$$

$\exists\phi$ s.t. $\pi\phi = \varepsilon_W$, since P_0W is proj.

$\varepsilon_V(x,y) := \varepsilon_U(x) + \phi(y) \in V$

continue:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & V \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & P_0U \arrow[r] \arrow[u, two heads] & P_0V \arrow[r] \arrow[u, two heads] & P_0W \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & \ker\varepsilon_U \arrow[r] \arrow[u, hook] & \ker\varepsilon_V \arrow[r] \arrow[u, hook] & \ker\varepsilon_W \arrow[r] \arrow[u, hook] & 0 \\
 & P_1U \arrow[r, hook] \arrow[u, two heads] & P_1V \arrow[r, two heads] \arrow[u, two heads, dotted] \arrow[uu, bend left] & P_1W \arrow[u, two heads] &
\end{tikzcd}$$

exact
(serpent lemma)

$\Rightarrow$ get:

$$0 \to P_*U \to P_*V \to P_*W \to 0$$

short exact sequence of resolutions; from $X \otimes_\Lambda -$

$$0 \to X \otimes_\Lambda P_*U \to X \otimes_\Lambda P_*V \to X \otimes_\Lambda P_*W \to 0$$

is exact ($P_iU \to P_iV$ has splitting $P_iV \to P_iU$)

Take long exact sequence: "Tor-sequence"

$\square$

Example Tor $\in$ Ab.
Take $\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ here $\mathrm{Tor}_i^{\mathbb{Z}} \equiv 0$, $i \geq 2$; $\mathrm{Tor}_1^{\mathbb{Z}} = \mathrm{Tor}$, $\mathrm{Tor}_0^{\mathbb{Z}} =$ "$\otimes_{\mathbb{Z}}$" $\forall\, A \in$ Ab:

$$0 \to \mathrm{Tor}(\mathbb{Z}, A) \to \mathrm{Tor}(\mathbb{Q}, A) \to \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$$
$$\xrightarrow{\partial} \mathbb{Z} \otimes_{\mathbb{Z}} A \to \mathbb{Q} \otimes_{\mathbb{Z}} A \to (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} A \to 0$$

Claim: $\mathbb{Q} \cong \mathrm{dirlim}_{\alpha\in\mathbb{N}}\, \mathbb{Z}_\alpha$, $\mathbb{Z}_\alpha = \mathbb{Z}$ where $\{\mathbb{Z}_\alpha\}_{\alpha\in\mathbb{N}}$ is the following directed system:
$\mathbb{N}$ PO set: divisibility: $\alpha \leq \beta \Leftrightarrow \alpha|\beta \Rightarrow$ directed PO set

Note $A \subset \mathbb{Q}$ finitely generated subgroup, is either $\cong \mathbb{Z}$ or 0.

Proof

$$\begin{tikzcd}
 & & 1 \arrow[dr, mapsto] \arrow[dddddll, mapsto, bend right] & \\
 & & \in & \frac{1}{\alpha} \\
 & & \mathbb{Z}_\alpha \arrow[ddd] \arrow[dr, hook] & \in \\
 & & & \mathbb{Q} \\
 & & & \in \\
\frac{\beta}{\alpha} & \in & \mathbb{Z}_\beta \arrow[uur, hook] & \frac{1}{\beta} \\
 & & \in & \\
 & & 1 \arrow[uur, mapsto] &
\end{tikzcd}$$

$\Rightarrow$ check now that $\mathbb{Q}$ has universal property of $\operatorname{dirlim}_{\mathbb{N}} \mathbb{Z}_\alpha$. $\square$

$$\Rightarrow \quad \mathrm{Tor}(\mathbb{Q}, A) \cong \operatorname*{dirlim}_{\mathbb{N}} \mathrm{Tor}(\mathbb{Z}_\alpha, A) = 0$$

$$(\Rightarrow \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A) \cong \ker(A \to A \otimes_{\mathbb{Z}} \mathbb{Q},\ a \mapsto a \otimes 1))$$

$$(\Rightarrow \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A) \cong TA \subset A)$$

Note $F \in \underline{\Lambda\text{-Mod}}$ free $\Rightarrow \mathrm{Tor}_i^\Lambda(-, F) \equiv 0$, $i > 0$
$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Q}, -) \equiv 0$, $i > 0$ but $\mathbb{Q} \in \underline{\mathrm{Ab}}$ *not* free.

Definition 6.21 *$M \in \underline{\Lambda\text{-Mod}}$ is called* flat*, if*

$$\begin{aligned} - \otimes_\Lambda M : \underline{\text{Mod-}\Lambda} &\to \underline{\mathrm{Ab}} \\ N &\mapsto N \otimes_\Lambda M \end{aligned}$$

is exact, i.e. if $0 \to N_1 \to N_2 \to N_3 \to 0$ short exact in $\underline{\text{Mod-}\Lambda}$, then $0 \to N_1 \otimes_\Lambda M \to N_2 \otimes_\Lambda M \to N_3 \otimes_\Lambda M \to 0$ short exact.

Theorem 6.22 *$M \in \underline{\Lambda\text{-Mod}}$ is flat $\Leftrightarrow \mathrm{Tor}_i^\Lambda(-, M) = 0$ $\forall i > 0$.*

Proof $\mathrm{Tor}_1^\Lambda(-, M) = 0 \Rightarrow M$ flat follows from long exact Tor sequence.
Claim: M flat $\Rightarrow \mathrm{Tor}_i^\Lambda(-, M) = 0$ $\forall i > 0$.
Look at $0 \to \Omega N \to F_0 N \twoheadrightarrow N \to 0$:

$$\cdots \to \underbrace{\mathrm{Tor}_1^\Lambda(F_0 N, M)}_{0} \to Tor_1^\Lambda(N, M) \to \underbrace{\Omega N \otimes_\Lambda M \to F_0 N \otimes_\Lambda M \twoheadrightarrow N \otimes M}_{\text{short exact}}$$

since M flat.
Thus M flat $\Rightarrow \mathrm{Tor}_1^\Lambda(-, M) \equiv 0 \Rightarrow$ (need to show) $\mathrm{Tor}_i^\Lambda(-, M) \equiv 0$ $\forall i > 1$.
$N \in \underline{\text{Mod-}\Lambda}$: $0 \to \Omega N \to F_0 N \twoheadrightarrow N \to 0$. Long exact Tor sequence (for $j \geq 2$):

$$0 \to \mathrm{Tor}_j^\Lambda(N, M) \xrightarrow{\partial} \mathrm{Tor}_{j-1}^\Lambda(\Omega N, M) \to 0$$

("dimension shifting": $\forall N \in \underline{\text{Mod-}\Lambda}$, $\forall M \in \underline{\Lambda\text{-Mod}}$:

$$\mathrm{Tor}_j^\Lambda(N, M) \cong Tor_{j-1}^\Lambda(\Omega N, M)$$

for $j \geq 2$.) $\square$

What abelian groups are flat?

Lemma 6.23 *$A \in \underline{\mathrm{Ab}}$ flat $\Leftrightarrow$ A torsion-free.*

Proof If $x \in B$ has order $n > 0$,

$$0 \to \mathbb{Z} \overset{n}{\hookrightarrow} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \qquad /B \otimes_{\mathbb{Z}} -$$

$$0 \to B \xrightarrow{n} B \to B \otimes \mathbb{Z}/n\mathbb{Z} \to 0$$

not exact since $x \in \ker(B \xrightarrow{n} B)$.
$A \in \underline{\mathrm{Ab}}$ torsion-free $\Rightarrow A = \operatorname{dirlim} A_\alpha$, $A_\alpha \subset A$ free abelian, finitely generated $\Rightarrow \mathrm{Tor}_i^{\mathbb{Z}}(A, -) \equiv 0$, $i > 0 \Rightarrow A$ flat. $\square$

Application: Homology with coefficients

C_* a chain complex in $\underline{\text{Mod-}\Lambda}$, $M \in \underline{\Lambda\text{-Mod}}$:

$$H_i(C_*; M) := H_i(C_* \otimes_\Lambda M)$$

e.g. $X \in \underline{\mathrm{Top}}$: $C_* = C_*^{\mathrm{sing}}(X)$,

$$H_i^{\mathrm{sing}}(X; A) := H_i(C_*^{\mathrm{sing}}(X); A) = H_i(C_*^{\mathrm{sing}}(X) \otimes_{\mathbb{Z}} A)$$

$A \in \underline{\mathrm{Ab}}$: "singular homology groups of X with coefficients in A."
$A = K$ a field: $C_*^{\mathrm{sing}}(X) \otimes_{\mathbb{Z}} K$ K-vector space $\Rightarrow H_*^{\mathrm{sing}}(X; K)$ K-vector spaces.

$$H_i^{\mathrm{sing}}(X; \mathbb{Z}) := H_i^{\mathrm{sing}}(C_*^{\mathrm{sing}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}) \cong H_i(C_*^{\mathrm{sing}}(X)) = H_i^{\mathrm{sing}}(X)$$

Theorem 6.24 (Universal Coefficient Theorem) *Let C_* be a flat chain complex in $\underline{\text{Mod-}\Lambda}$, and let $M \in \underline{\Lambda\text{-Mod}}$ such that $\mathrm{Tor}_i^\Lambda(-, M) \equiv 0$ for $i > 1$ (i.e. ΩM is flat). Then there is a natural short exact sequence:*

$$0 \to H_i(C_*) \otimes_\Lambda M \to H_i(C_* \otimes_\Lambda M) \to \mathrm{Tor}_1^\Lambda(H_{i-1}(C_*), M) \to 0$$

Proof 1. M flat.

Look at

$$C_* : \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots$$

$C_i \supset Z_i = \ker \partial_i$: cycles; $C_{i-1} \supset B_{i-1} = \operatorname{im} \partial_i$: boundaries. Thus

$$0 \to Z_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \to 0$$
$$0 \to B_i \hookrightarrow Z_i \twoheadrightarrow H_i \to 0$$

Tensoring with M:

$$\begin{array}{ll} 0 \to Z_i \otimes_\Lambda M \to C_i \otimes M \to B_{i-1} \otimes_\Lambda M \to 0 & \text{exact} \\ 0 \to B_i \otimes_\Lambda M \to Z_i \otimes M \to H_i \otimes_\Lambda M \to 0 & \text{exact} \end{array}$$

$$\Rightarrow \quad H_i(C_* \otimes_\Lambda M) = Z_i \otimes_\Lambda M / B_i \otimes_\Lambda M \cong (H_i C_*) \otimes_\Lambda M$$

2. General case:

   Look at:

$$0 \to \Omega M \to \underbrace{F_0 M}_{\text{free} \Rightarrow \text{flat}} \to M \to 0$$

   $\mathrm{Tor}_i^\Lambda(-, M) \xrightarrow[\partial]{\cong} \mathrm{Tor}_{i-1}^\Lambda(-, \Omega M)$, $i \geq 2 \Rightarrow \Omega M$ flat since $\mathrm{Tor}_1^\Lambda(-, \Omega M) = 0$.

   Look at:

$$0 \to C_* \otimes_\Lambda \Omega M \to C_* \otimes_\Lambda F_0 M \to C_* \otimes_\Lambda M \to 0$$

   is a short exact sequence (because C_* is flat) and yields a long exact sequence in homology:

$$\begin{tikzcd} \cdots \arrow[r] & H_i(C_* \otimes_\Lambda \Omega M) \arrow[r, "\alpha"] \arrow[d, "\cong"] & H_i(C_* \otimes_\Lambda F_0 M) \arrow[r] \arrow[d, "\cong"] & H_i(C_* \otimes_\Lambda M) \arrow[r, "\partial"] & {} \\ & (H_i(C_*)) \otimes_\Lambda \Omega M \arrow[r, "\tilde{\alpha}"] & (H_i(C_*)) \otimes_\Lambda F_0 M & & \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\partial"] & H_{i-1}(C_* \otimes_\Lambda \Omega M) \arrow[r, "\beta"] \arrow[d, "\cong"] & \cdots \arrow[d, "\cong"] \\ & H_{i-1}(C_*) \otimes_\Lambda \Omega M \arrow[r, "\tilde{\beta}"] & H_{i-1}(C_*) \otimes_\Lambda F_0 M \end{tikzcd}$$

   (isos by case 1) $\Rightarrow$ get short exact sequence:

$$0 \to \mathrm{coker}(\alpha) \to H_i(C_* \otimes_\Lambda M) \to \ker \beta \to 0$$

   where $\mathrm{coker}(\alpha) \cong \mathrm{coker}\, \tilde{\alpha} \cong H_i(C_*) \otimes_\Lambda M$ (right-exactness of $- \otimes_\Lambda M$) and $\ker \beta \cong \ker \tilde{\beta} \cong \mathrm{Tor}_1^\Lambda(H_{i-1} C_*, M)$.

$\square$

Example Λ a PID (principal ideal domain)
$\Rightarrow \mathrm{Tor}_2(\cdot, \cdot) \equiv 0$
$\Rightarrow$ Get UCT for *any* M

Example $X \in \underline{\text{Top}}$, $A \in \underline{\text{Ab}}$; then one defines:

$$H_i^{\text{sing}}(X;A) = H_i(C_*^{\text{sing}}(X) \otimes_{\mathbb{Z}} A) \quad (H_i^{\text{sing}}(X;\mathbb{Z}) =: H_i^{\text{sing}}X)$$

$$\Rightarrow \boxed{0 \to H_i^{\text{sing}}(X) \otimes_{\mathbb{Z}} A \to H_i^{\text{sing}}(X;A) \to \text{Tor}(H_{i-1}^{\text{sing}}(X), A) \to 0 \qquad \text{UCT}}$$

Example "Homology of groups"
For M a $\mathbb{Z}G$-module we defined

$$H_i(G;M) := H_i(P_*(G) \otimes_{\mathbb{Z}G} M)$$

where P_*G is a projective resolution of $\mathbb{Z}$ considered as trivial $\mathbb{Z}G$-module.

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \twoheadrightarrow \mathbb{Z}$$

M: in general so that $\text{Tor}_i^{\mathbb{Z}G}(-,M) \not\equiv 0$, $i \geq 2$. Example of flat ΩM: Take $M = (\mathbb{Z}/n\mathbb{Z})[G] \Rightarrow$ have short exact sequence

$$0 \to \mathbb{Z}[G] \xrightarrow{n} \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}[G]$$

$\Rightarrow H_*(P_* \otimes M)$ fits into short exact sequence

$$0 \to H_i(P_*) \otimes_{\mathbb{Z}G} M \to H_i(P_* \otimes_{\mathbb{Z}G} M) \to \text{Tor}_1^{\mathbb{Z}G}(H_{i-1}(P_*), M) \to 0$$

where $H_i(P_*) = 0$ for $i > 0$, so $H_i(G;M) = 0$ for $i > 1$ and

$$H_1(G;M) \cong \text{Tor}_1^{\mathbb{Z}G}(\underbrace{H_0P_*}_{\mathbb{Z}}, M) = H_1(G;M)$$

$$H_0(G;M) \cong \underbrace{H_0(P_*)}_{\mathbb{Z}} \otimes_{\mathbb{Z}G} M \cong M_G \qquad (\text{"coinvariants"})$$

7 Künneth Formula

What is $H_*^{\text{sing}}(X \times Y)$ in terms of $H_*^{\text{sing}}(X)$ and $H_*^{\text{sing}}(Y)$?
$\rightsquigarrow$ study $H_*(C_* \otimes_\Lambda D_*)$ (where C_*, D_* complexes in $\underline{\text{Mod-}\Lambda}$, $\underline{\Lambda\text{-Mod}}$, respectively).

Definition 7.1 (Tensor Product of Chain Complexes) *Let (C_*, ∂_C), (D_*, ∂_D) two chain complexes.*
Then $(C_* \otimes_\Lambda D_*, \partial)$ denotes the chain complex with

$$(C_* \otimes_\Lambda D_*)_i := \bigoplus_{k+\ell=i} (C_k \otimes_\Lambda D_\ell)$$

For $x \otimes_\Lambda y \in C_k \otimes_\Lambda D_\ell$ put

$$\partial(x \otimes_\Lambda y) = (\partial x) \otimes_\Lambda y + (-1)^k x \otimes_\Lambda \partial y$$

$\Rightarrow \partial\partial = 0$*; $H_i(C_* \otimes_\Lambda D_*) =$ "ker / im".*

$\Rightarrow \exists$ natural map $H_k(C_*) \otimes_\Lambda H_\ell(D_*) \to H_{k+\ell}(C_* \otimes_\Lambda D_*)$ defined in the obvious way:

$$\begin{array}{ccccc} H_k(C_*) & \otimes_\Lambda & H_\ell(D_*) & \xrightarrow{\mu} & H_{k+\ell}(C_* \otimes_\Lambda D_*) \\ \in & & \in & & \\ a & & b & & \\ \uparrow & & \uparrow & & \\ \tilde{a} \in C_k & & \tilde{b} \in D_\ell & & \end{array}$$

$$\mu(a \otimes_\Lambda b) := [\tilde{a} \otimes_\Lambda \tilde{b}]$$

Claim: $\tilde{a} \otimes_\Lambda \tilde{b} \in C_* \otimes_\Lambda D_*$ a cycle. Look at

$$\partial(\tilde{a} \otimes_\Lambda \tilde{b}) = \partial_C \tilde{a} \otimes_\Lambda \tilde{b} + (-1)^{|a|} \tilde{a} \otimes_\Lambda \partial_D \tilde{b} = 0$$

For α, β boundaries, $\alpha = \partial_C \tilde{\alpha}$, $\beta = \partial_D \tilde{\beta}$:

$$\begin{aligned} (\tilde{a} + \alpha) \otimes_\Lambda (\tilde{b} + \beta) &= \tilde{a} \otimes_\Lambda \tilde{b} + \alpha \otimes_\Lambda \tilde{b} + \tilde{a} \otimes_\Lambda \beta + \alpha \otimes_\Lambda \beta \\ &= \tilde{a} \otimes_\Lambda \tilde{b} + \partial(\tilde{\alpha} \otimes_\Lambda \tilde{b}) + (-1)^{|\beta|} \partial(\tilde{a} \otimes_\Lambda \tilde{\beta}) + \partial(\tilde{\alpha} \otimes_\Lambda \tilde{\beta}) \end{aligned}$$

where all but the first term are boundaries. $\Rightarrow$ get map

$$\mu_n : \bigoplus_{k+\ell=n} (H_k(C_*) \otimes_\Lambda H_\ell(D_*)) \to H_n(C_* \otimes_\Lambda D_*)$$

Now the optimist would assume μ_n is an isomorphism. This would be too simple, but is not too far off, as the Künneth formula shows:

Theorem 7.2 (Künneth Formula) *Let C_*, D_* be flat complexes and assume $\mathrm{Tor}_2^\Lambda(-,-) \equiv 0$ (e.g. Λ a PID). Then there is a natural short exact sequence:*

$$\begin{aligned} 0 \to \bigoplus_{i+j=n} (H_i(C_*) \otimes_\Lambda H_j(D_*)) &\xrightarrow{\mu_n} H_n(C_* \otimes_\Lambda D_*) \\ \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^\Lambda(H_i C_*, H_j D_*) &\to 0 \end{aligned}$$

Proof Look at $B_i \subset Z_i \subset D_i$, boundaries and cycles for D_*. $B_* \subset Z_* \subset D_*$ where B_* and Z_* are subcomplexes (with $\partial \equiv 0$). $Z_*/B_* = H_*(D_*)$ and:

$$D_i \overset{\partial}{\twoheadrightarrow} B_{i-1} \subset D_{i-1}, \quad B_{i-1} =: (\Sigma B_*)_i$$

(so $H_j(\Sigma C_*) = H_{j-1}(C_*)$) yielding a map:

$$D_* \twoheadrightarrow \Sigma B_*$$

of chain complexes.
$\Rightarrow$ have a short exact sequence of chain complexes:

$$\begin{array}{cc} 0 \to Z_* \to D_* \to \Sigma B_* \to 0 & /C_* \otimes_\Lambda - \\ 0 \to C_* \otimes_\Lambda Z_* \to C_* \otimes_\Lambda D_* \to C_* \otimes_\Lambda \Sigma B_* \to 0 & \end{array}$$

is exact:

$$0 \to C_i \otimes_\Lambda Z_j \to C_i \otimes_\Lambda D_j \to C_i \otimes_\Lambda B_{j-1} \to 0 \qquad (C_i \text{ is flat})$$

(with $0 \to Z_j \to D_i \to B_{j-1} \to 0$ short exact.)
apply H_* to get a long exact sequence:

$$\begin{gathered} \ldots \to H_{i+1}(C_* \otimes \Sigma B_*) \xrightarrow{\partial}_\alpha H_i(C_* \otimes_\Lambda Z_*) \to H_i(C_* \otimes_\Lambda D_*) \to \\ \to H_i(C_* \otimes_\Lambda \Sigma B_*) \xrightarrow{\partial}_\beta H_{i-1}(C_* \otimes_\Lambda Z_*) \to \ldots \end{gathered}$$

$\Rightarrow$

$$0 \to \operatorname{coker} \alpha \to H_i(C_* \otimes_\Lambda D_*) \to \ker \beta \to 0$$

is exact.
$\operatorname{coker} \alpha$:

$$H_{i+1}(C_* \otimes_\Lambda \underbrace{\Sigma B_*}_{\partial=0}) \xrightarrow{\alpha} H_i(C_* \otimes_\Lambda \underbrace{Z_*}_{\partial=0})$$

$\to$ look at $C_* \otimes_\Lambda (\Sigma B_k)$. Idea: Because $\delta \equiv 0$ for the complex ΣB_* we have some "additivity" and we can look at $C_* \otimes_\Lambda (\Sigma B_k)$. We want to apply the UCT.
Claim: B_k, Z_k are flat.
By assumption: $B_i \subset Z_i \subset D_i$ flat $\forall\, i \Rightarrow Z_i$, B_i flat as $\mathrm{Tor}_2^\Lambda = 0$.
Namely: $\mathrm{Tor}_2^\Lambda = 0 \Rightarrow \mathrm{Tor}_1^\Lambda(X, -)$, $\mathrm{Tor}_1^\Lambda(-, Y)$ are left exact (from long Tor sequence). $\Rightarrow$ submodules of flat modules are flat in this case.
(A flat, $B \subset A \Rightarrow \forall\, C$: $\mathrm{Tor}_1^\Lambda(B, C) \hookrightarrow \underbrace{\mathrm{Tor}_1^\Lambda(A, C)}_{=0} \Rightarrow \mathrm{Tor}_1^\Lambda(B, -) = 0$: B flat.)
back to $\operatorname{coker} \alpha$ (use the UCT):

$$\begin{array}{ccc} H_{i-1}(C_* \otimes_\Lambda \Sigma B_*) & \xrightarrow{\quad\alpha\quad} & H_i(C_* \otimes_\Lambda Z_*) \\ \uparrow \cong & & \uparrow \cong \\ \bigoplus_{k+l=i+1} H_k(C_*) \otimes_\Lambda \Sigma B_l & \longrightarrow & \bigoplus_{s+t=i} H_s(C_*) \otimes_\Lambda Z_t \\ \shortparallel & & \\ \bigoplus_{k+m=i} H_k(C_*) \otimes_\Lambda B_m & & \end{array}$$

$$0 \to B_m \hookrightarrow Z_m \to H_m(D_*) \to 0 \qquad /H_k(C_*) \otimes_\Lambda -$$

short exact.

$$\ldots H_k(C_*) \otimes_\Lambda B_m \to H_k(C_*) \otimes_\Lambda Z_m \twoheadrightarrow \underbrace{H_k(C_*) \otimes_\Lambda H_m(D_*)}_{\Rightarrow \text{coker}\,\alpha \cong \bigoplus_{k+m=i} H_k(C_*) \otimes H_m(D_*)} \to 0$$

exact.
Look at

$$0 \to \mathrm{Tor}_1^\Lambda(H_k(C_*), H_m(D_*)) \to H_k(C_*) \otimes_\Lambda B_m \to \ldots$$

and compute $\ker \beta$ by a similar argument as above. $\square$

Applied to singular homology, one gets:

Theorem 7.3 (Künneth Formula) $X, Y \in \underline{\text{Top}}$. *Then there is a natural short exact sequence:*

$$\begin{aligned} 0 \to \bigoplus_{i+j=n} (H_i^{\text{sing}} X \otimes H_j^{\text{sing}} Y) \to H_n^{\text{sing}}(X \times Y) \\ \to \bigoplus_{s+t=n-1} \mathrm{Tor}(H_s^{\text{sing}} X, H_t^{\text{sing}} Y) \to 0 \end{aligned}$$

(without proof: the sequence is split!)

Proof Apply KF for $\Lambda = \mathbb{Z}$ and $C_*^{\text{sing}} X \otimes_\mathbb{Z} C_*^{\text{sing}} Y$ to compute $H_i(C_*^{\text{sing}} X \otimes_\mathbb{Z} C_*^{\text{sing}} Y)$. Then we get ($C_*^{\text{sing}} X, C_*^{\text{sing}} Y$ are $\mathbb{Z}$-flat, and $\mathrm{Tor}_2^\mathbb{Z} = 0$):

$$\begin{aligned} 0 \to \bigoplus_{i+j=n} \underbrace{H_i(C_*^{\text{sing}} X)}_{H_i^{\text{sing}} X} \otimes_\mathbb{Z} \underbrace{H_j(C_*^{\text{sing}} Y)}_{H_j^{\text{sing}} Y} \to \underbrace{H_n(C_*^{\text{sing}} X \otimes_\mathbb{Z} C_*^{\text{sing}} Y)}_{\stackrel{?}{=} H_n^{\text{sing}}(X \times Y)} \\ \to \mathrm{Tor}(H_s(C_*^{\text{sing}} X), H_t(C_*^{\text{sing}} Y)) \end{aligned}$$

$\exists$ map of chain complexes

$$\begin{aligned} C_*^{\text{sing}} X \otimes C_*^{\text{sing}} Y &\to C_*^{\text{sing}}(X \times Y) \\ a \otimes b &\mapsto \lambda(a \otimes b) \end{aligned}$$

$\Rightarrow$ chain homotopy equivalence (Eilenberg-Zilber theorem). Namely:

$$\left.\begin{array}{ll} a: & \triangle_n \to X \\ b: & \triangle_m \to Y \end{array}\right\} a \times b : \triangle_n \times \triangle_m \to X \times Y$$

"subdivide" prism $\triangle_n \times \triangle_m$ into $(n+m)$-simplices. $\square$

Theorem 7.4 (KF for group homology) *G group: $H_iG := H_i(G;\mathbb{Z}) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$*
G, H groups $\Rightarrow \exists$ natural short exact sequence:

$$0 \to \bigoplus_{i+j=n} H_iG \otimes H_jH \to H_n(G\times H) \to \bigoplus_{s+t=n-1} \mathrm{Tor}(H_sG, H_tH) \to 0$$

(without proof: the sequence is split!)

Proof Take $X = K(G,1)$, a CW-complex with

$$\pi_i X = \begin{cases} G & i=1 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow \tilde{X}$ is contractible ($\tilde{X}$ CW-complex with $\pi_i\tilde{X} = 0\,\forall i \Rightarrow$ (Whitehead) $\tilde{X}$ contractible)
$\Rightarrow C_*^{\text{sing}}\tilde{X}$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$: $C_i^{\text{sing}}\tilde{X}$ free/$\mathbb{Z}$, basis $\triangle_i \xrightarrow{\phi} \tilde{X} \circlearrowleft G$.
$\Rightarrow$

$$H_i(C_*^{\text{sing}}\tilde{X} \otimes_{\mathbb{Z}G} \mathbb{Z}) \cong \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = H_iG \cong H_i^{\text{sing}}X$$

so take $X = K(G,1)$, $Y = K(H,1) \Rightarrow X\times Y$ has

$$\pi_i(X\times Y) \cong \begin{cases} G\times H, & i=1 \\ 0, & \text{else} \end{cases}$$

$\Rightarrow K(G,1)\times K(H,1) \simeq K(G\times H,1)$
$\Rightarrow H_iH \cong H_i^{\text{sing}}Y$
$H_i(G\times H) \cong H_i^{\text{sing}}(X\times Y)$
KF for $X\times Y$ yields KF for $G\times H$

$\square$

8 Geometric Realization Functor

$\exists$ functor

$$\underline{\mathrm{Top}} \xrightarrow{\Gamma} \underline{\mathrm{CW}} \subset \underline{\mathrm{Top}}$$

$$X \longmapsto \Gamma X$$

together with a natural onto map $\varepsilon_X : \Gamma X \to X$

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\varepsilon_X} & X \\ {\scriptstyle\Gamma f}\downarrow & & \downarrow{\scriptstyle f} \\ \Gamma Y & \xrightarrow[\varepsilon_Y]{} & Y \end{array}$$

where Γf is cellular (always a commutative diagram) such that

(1) $X \in \underline{\mathrm{CW}} \Rightarrow \varepsilon_X : \Gamma X \xrightarrow{\simeq} X$

(2) ε_X induces $H_*^{\mathrm{sing}}\Gamma X \xrightarrow{\cong} H_*^{\mathrm{sing}} X$

(3) ε_X induces $\pi_i(\Gamma X, w) \xrightarrow{\cong} \pi_i(X, \varepsilon_X w)\ \forall i, \forall w$

Definition 8.1 *$f : X \to Y$ in $\underline{\mathrm{Top}}$ is called a* weak homotopy equivalence*, if f induces $\pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, f(x_0))\ \forall x_0 \in X, \forall i$.*
(Also for $i = 0$: $[S^0, X]_\bullet \cong [\{x_0\}, X] \Rightarrow f$ induces bijection of path components of X and Y)

Example

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \Gamma X \arrow[u, "\varepsilon_X"] \arrow[r, "\Gamma f"] & \Gamma Y \arrow[u, "\varepsilon_Y"'] \end{tikzcd}$$

with f weak homotopy equivalence: ε_X and ε_Y are weak homotopy equivalences (WHE) by (3) $\Rightarrow \Gamma f$ a WHE too $\Rightarrow$ (by Whitehead) Γf a homotopy equivalence.

Consider $K_* = \{K_n\}_{n \geq 0}$ simplicial set consisting of:

1. Sets K_n, $n \geq 0$ (*n-simplices*)
2. Face operations, degeneracy operators

   $d_i : K_n \to K_{n-1}$, $0 \leq i \leq n$, (faces)
   $s_i : K_n \to K_{n+1}$, $0 \leq i \leq n$, (degeneracies)

   satisfying certain relations, motivated as follows:

   Example $K_* = \Delta_*$ "simplicial complex of $X \in \underline{\mathrm{Top}}$" with $S_i X := \{\Delta_i \xrightarrow{f} X \mid f \text{ continuous}\}$ where $\mathbb{R}^{i+1} \supset \Delta_i = (t_0, \ldots, t_i), \sum t_j = 1$ is the standard i-simplex.

   $$\begin{aligned} S_i X &\xrightarrow{d_j} S_{i-1} X \\ (f : \Delta_i \to X) &\mapsto (\Delta_{i-1} \overset{\delta_j}{\hookrightarrow} \Delta_i \xrightarrow{f} X) \\ & \quad (t_0, \ldots, t_{i-1}) \mapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}) \end{aligned}$$

   $f \mapsto d_j f := f \circ \delta_j$. And:

   $$\begin{aligned} S_i X &\xrightarrow{s_j} S_{i+1} X \\ (\Delta_i \xrightarrow{f} X) &\mapsto (\Delta_{i+1} \xrightarrow{\sigma_j} \Delta_i \xrightarrow{f} X) \\ & \quad (t_0, \ldots, t_{i+1}) \mapsto (t_0, \ldots, t_{j-1}, t_j + t_{j+1}, \ldots, t_{i+1}) \end{aligned}$$

All the relations between the d's and s's in S_*X are taken to be relations in the general K_*.
K_* has a "geometric realization" given by:

$$|K_*| := \coprod_{n\geq 0} K_n \times \Delta_n / \sim \quad \in \underline{\mathrm{CW}}$$

where K_n is a discrete topological space, and Δ_n has the usual topology. $\sim$ is generated by:

$$\begin{array}{ccc} (a,x) & \sim & (d_i a, y) \\ a \in K_n, x \in \Delta_n & & d_i a \in K_{n-1} \\ x = \delta_i y & & y \in \Delta_{n-1} \end{array}$$

$(f, \sigma_j z) \sim (s_j f, z)$

Definition 8.2 $X \in \underline{\mathrm{Top}}$: $\Gamma X := |S_* X| \in \underline{\mathrm{CW}}$.

one checks: $C^{\mathrm{cell}}_x(\Gamma X) \stackrel{\phi}{\twoheadleftarrow} C^{\mathrm{sing}}_n X \supset D^{\mathrm{sing}}_n X$, with $D^{\mathrm{sing}}_n X$ $(= \ker\phi)$ generated by degerate singular simplices. $D^{\mathrm{sing}}_x X \subset C^{\mathrm{sing}}_* X$, $D^{\mathrm{sing}}_x X$ being a subcomplex (contractible chain complex).
$\Rightarrow \phi$ induces an isomorphism:

$$\begin{tikzcd} H^{\mathrm{sing}}_* X \arrow[r, "\cong"] & H^{\mathrm{cell}}_*(\Gamma X) \\ H^{\mathrm{sing}}_*(\Gamma X) \arrow[u, "\exists \text{ natural iso } \gamma_*"] \arrow[ur, phantom, "\cong"] & \end{tikzcd}$$

$$\begin{array}{rrl} \gamma : & \Gamma X & \to X \\ & [(a,x)] & \mapsto a(x) \end{array}$$

continuous surjection, with $a : \Delta_n \to X : a \in S_n X, x \in \Delta_n$.
$\Gamma : \underline{\mathrm{Top}} \to \underline{\mathrm{CW}}$ is a functor

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "\lambda"'] & \Gamma X \arrow[d, dotted, "\Gamma\lambda"] & = & \coprod (S_n X) \times \Delta_n/\sim \arrow[d, dotted] & \ni & {[(a,x)]} \arrow[d, mapsto] & a : \Delta_n \to X, x \in \Delta_n \\ Y \arrow[r, mapsto] & \Gamma Y & = & \coprod (S_n Y) \times \Delta_n/\sim & \ni & {[(\lambda a, x)]} & \end{tikzcd}$$

Theorem 8.3 (Basic Theorem) *For all $X \in \underline{\mathrm{Top}}$, $\gamma : \Gamma X \to X, \omega \mapsto \gamma\omega$ induces $\gamma_* : \pi_i(\Gamma X, \omega) \xrightarrow{\cong} \pi_i(X, \gamma\omega), \forall \omega \in \Gamma X$.*

Example $X = \{\centerdot\}$:

$$C_*^{\text{sing}}\{\centerdot\} : \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \to \cdots \to \mathbb{Z}$$

$s_n\{\centerdot\} = \{\triangle_n \xrightarrow{\exists!} \{\centerdot\}\}$
$C_*^{\text{sing}}\{\centerdot\} \supset D_*^{\text{sing}}\{\centerdot\} \cong 0_* \Rightarrow$

$$C_n^{\text{sing}}/D_n^{\text{sing}} = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

$$H_i(C_*^{\text{sing}}/D_*^{\text{sing}}) = \begin{cases} 0 & i > 0 \\ \mathbb{Z} & i = 0 \end{cases}$$

Eilenberg Mac Lane spaces

π a discrete group.
$B_*\pi$ simplicial set with $B_n\pi := (\pi)^n \xrightarrow{d_i} B_{n-1}\pi$ with

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 0 < i < n \\ (g_1, \ldots, g_n) & i \geq n \end{cases}$$

$$B_n\pi \xrightarrow{s_i} B_{n+1}\pi$$
$$(g_1, \ldots, g_n) \mapsto (g_1, \ldots, 1, \ldots, g_n)$$

where 1 is at position $i + 1$.

Definition 8.4 $K(\pi, 1) := |B_*\pi|$ *(connected and has only one* 0*-cell which serves as base-point.)*

Theorem 8.5
$$\pi_i(K(\pi, 1)) \cong \begin{cases} \pi & i = 1 \\ 0 & \text{else} \end{cases}$$

Remark If $X, Y \in \underline{\text{CW}}$ with $\pi_j X \cong \pi_j Y = 0$, $j \neq n$, and $\pi_n X \cong \pi_n Y$, then $X \simeq Y$ (we write $K(\pi, n)$ for such an X, $\pi \cong \pi_n(K(\pi, n))$).

$\Rightarrow$ we get a functor

$$K(\cdot,1): \qquad \underline{\mathrm{Gr}} \longrightarrow \underline{\mathrm{CW}_\bullet}$$

$$\begin{tikzcd} \pi \arrow[r, mapsto] \arrow[d, "f"'] & K(\pi,1) \arrow[d, "K(f{,}1)"] \\ G \arrow[r, mapsto] & K(G,1) \end{tikzcd}$$

with $\pi_1(K(\pi,1)) \cong \pi$.

$$\begin{tikzcd} K(\pi,1) \arrow[r, mapsto] \arrow[d, "\phi\in\underline{\mathrm{CW}_\bullet}"] & \pi_1(K(\pi,1)) \arrow[d, "\phi_*=\pi_1(\phi)"] \\ K(G,1) \arrow[r, mapsto] & \pi_1(K(G,1)) \end{tikzcd}$$

and $\pi_1(K(f,1))$ is f (up to natural equivalence). Every $\phi : K(\pi,1) \to K(G,1)$ is, up to homotopy, of the form $K(f,1)$:

$$\pi_1 : [K(\pi,1), K(G,1)]_\bullet \overset{\text{bij}}{\to} \mathrm{Hom}(\pi, G)$$

(without proof).
If π is an abelian group $\Rightarrow B_*\pi$ is a *simplicial group*:

$$B_n\pi := (\pi)^n, \quad \mu : \pi \times \pi \to \pi$$

(μ is a homomorphism $\Rightarrow \pi$ abelian). $\Rightarrow K(\pi,1)$ a *topological* group.
Now take $G \in \underline{\mathrm{Top}}$ a topological group. B_*G becomes a simplicial, topological group, i.e.

$$B_nG := (G)^n \in \underline{\mathrm{Top}}$$

d_i, s_i: continuous group homomorphisms. Define

$$|B_*G| := \coprod_{\substack{y\in B_n \\ n\geq 0}} y \times \triangle_n/\!\sim =: BG$$

This is called the *classifying space* of G. If G is an abelian topological group, then so is BG.

Note $G = \pi$ discrete $\Rightarrow BG = K(\pi,1)$ (not a group unless G abelian).
$G \in \underline{\mathrm{Top}}$ a topological group and abelian $\Rightarrow BG \in \underline{\mathrm{Top}}$ an abelian topological group and $\pi_iBG \cong \pi_{i-1}G$ (G not necessarily connected: $\pi_0G \cong \pi_1BG$).

G topological abelian $\Rightarrow BG$ topological abelian $\Rightarrow B(BG) =: B^2G, \dots, B^nG$ all topological abelian groups.
$G = \pi$ discrete abelian group:

$$BG = K(\pi, 1) \mapsto B(BG) = K(\pi, 2), \dots, B^nG = K(\pi, n)$$

G topological group $\rightsquigarrow |B_*G| := BG \in \underline{\mathrm{Top}}$ ($\in \underline{\mathrm{CW}}$ if G discrete) such that $\pi_i BG \cong \pi_{i-1}G$ for $i \geq 1$.
If G discrete, then

$$\pi_i BG = \begin{cases} \pi_0 G = G & i = 1 \\ 0 & i > 1 \end{cases}$$

and we write $K(G, 1) := BG$.

Remark G topological group: Define E_*G with $E_nG := (G)^{n+1}$ and "suitable" d and s. E_nG has G-action by

$$(g_1, \dots, g_{n+1}) \cdot g = (g_1, \dots, g_{n+1}, g)$$

$$E_nG \twoheadrightarrow (E_nG)/G =: B_nG$$

$EG := |E_*G|$ with $(EG)/G \xrightarrow{\cong} BG$ free G-space, and even $EG \xrightarrow{\text{proj}} BG$ fibration with fiber G (principal G-bimodule) and $EG \simeq \{\centerdot\}$: "$G \to EG \to BG$" $\rightsquigarrow$ long exact homotopy sequence

$$\pi_j G \to \underbrace{\pi_j EG}_{0} \to \pi_j BG \xrightarrow{\partial} \pi_{j-1} G \to \underbrace{\pi_j EG}_{0} \to \dots$$

$G = A$ abelian, discrete: $BA = K(A, 1)$ topological abelian group $\Rightarrow B(K(A, 1)) = BBA =: B^2A = K(A, 2)$ topological abelian group, etc. $\Rightarrow K(A, n) := B^nA$

$$\begin{aligned} T : \underline{\mathrm{Top}} &\to \underline{\mathrm{CW}} \\ X &\mapsto TX \end{aligned}$$

and $\gamma(X) : \Gamma X \to X$.
Take $W \in \underline{\mathrm{CW}}$:

$$\begin{tikzcd} W \arrow[r, "f"] & X \\ \Gamma W \arrow[u, "\gamma(W)"] \arrow[r, "\Gamma f"'] & \Gamma X \arrow[u, "\gamma(X)"'] \end{tikzcd}$$

$\gamma(X)$ is an isomorphism in π_*, and it turns out $\gamma(W)$ is a homotopy equivalence.

$$[W, \Gamma X] \xrightarrow{\gamma_*} [W, X] = [i(W), X]$$

$$\pi_i(\Gamma X), W) \xrightarrow{\cong} \pi_i(X, \gamma(W)) \quad \forall W$$

$\Rightarrow$ $\underline{\text{HTop}} \underset{i}{\overset{\Gamma}{\rightleftarrows}} \underline{\text{HCW}}$ are adjoints on the homotopy categories $\underline{\text{HTop}}$, $\underline{\text{HCW}}$. Γ turns weak homotopy equivalences into homotopy equivalences.

Remarks concerning cohomology

h^* cohomology theory with h^i contravariant (on $\underline{\text{Top}}^2$). Most axioms directly correspond to homology, except additivity where we have

$$\begin{array}{ccc} h^i\left(\coprod_{\alpha\in I}(X_\alpha, A_\alpha)\right) & \xrightarrow{\cong} & \prod_I h^i(X_\alpha, A_\alpha) \\ \big\downarrow {\scriptstyle pr_\alpha^*} & & \\ h_i(X_\alpha, A_\alpha) & & \end{array}$$

where pr_α^* is induced by inclusions $(X_\alpha, A_\alpha) \hookrightarrow \coprod(X_\alpha, A_\alpha)$.

Example Singular cohomology with coefficients in $A \in \underline{\text{Ab}}$: Put

$$C^i_{\text{sing}}(X; A) := \text{Hom}_{\mathbb{Z}}(C_i^{\text{sing}} X, A) \in \underline{\text{Ab}}$$

the "singular cochains". The boundary ∂ of $C_*^{\text{sing}} X$ induces "coboundary" δ in $C_{\text{sing}}(X; A)$ yielding a cochain complex $(C^*_{\text{sing}}(X; A), \delta)$, $\delta^i : C^i_{\text{sing}} \to C^{i+1}_{\text{sing}}$, $\delta\delta = 0$.

$$H^i_{\text{sing}}(X; A) := \ker \delta^i / \operatorname{im} \delta^{i-1}$$

i.e. cocycles modulo coboundaries. The dimension axiom becomes

$$H^i_{\text{sing}}(\{\centerdot\}; A) = \begin{cases} A & i = 0 \\ 0 & \text{else} \end{cases}$$

since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \cong A$.
Special case: $A = k$ a field:

$$H_i^{\text{sing}}(x; k) = H_i(\underbrace{C_i^{\text{sing}} X \otimes_{\mathbb{Z}} k}_{k\text{-vector space}}) : \quad k\text{-vector space}$$

and $H^i_{\text{sing}}(X; k) = H^i(C^*_{\text{sing}}(X; k))$

$$\begin{array}{ccc} C^i_{\text{sing}}(X; k) = \text{Hom}_{\mathbb{Z}}(C_i^{\text{sing}} X, k) & \xrightarrow{\theta, \cong} & \text{Hom}_k(C_i^{\text{sing}} X \otimes_{\mathbb{Z}} k, k) : \text{dual } k\text{-VS of } (C_i^{\text{sing}} X \otimes_{\mathbb{Z}} k) \\ f : C_i^{\text{sing}} X \to k & & \theta f : C_i^{\text{sing}} X \otimes_{\mathbb{Z}} k \to k, (a \otimes \lambda) \mapsto \lambda f(a) (k\text{-field}) \end{array}$$

$\mathrm{Hom}_k(C_x^{\mathrm{sing}} X \otimes_{\mathbb{Z}} k, k)$: cochain complex of k-VS. $\Rightarrow H^i_{\mathrm{sing}}(X;k) \cong \mathrm{Hom}_k(H_i^{\mathrm{sing}}(X,k),k)$ dual VS.

Theorem 8.6 *$X \in \underline{\mathrm{Top}}$: $H^*_{\mathrm{sing}}(X) := H^*_{\mathrm{sing}}(X;\mathbb{Z})$ is in a natural way a gradient ring (commutative in the graded sense). Moreover k field $\Rightarrow H^*_{\mathrm{sing}}(X;k)$ is a graded k-algebra.*

- graded ring:

$$\begin{array}{rcl} H^i(X) \times H^j(X) & \overset{\text{biadditive}}{\to} & H^{i+j}(X) \\ (x,y) & \mapsto & x \cup y \text{ ``cup product''} \end{array}$$

  (x,y) have degree: $|x| = i$, $|y| = j$ and $1 \in H^0_{\mathrm{sing}} X$.

- graded commutative:

  $x \cup y = (-1)^{|x||y|}(y \cup x)$

  $x \cup 1 = 1 \cup x = x$, $\forall\, x$

- the definition of "$\cup$":

  - external product:

$$\begin{array}{c} H^i_{\mathrm{sing}} X \times H^i_{\mathrm{sing}} Y \to H^{i+j}_{\mathrm{sing}}(X \times Y) \\ (a,b) \mapsto a \times b \end{array} \qquad \text{with } i+j=n$$

    a represented by $\tilde{a} : C_i^{\mathrm{sing}} X \to \mathbb{Z}$
    b represented by $\tilde{b} : C_j^{\mathrm{sing}} Y \to \mathbb{Z}$
    $\tilde{a} \otimes \tilde{b} : \bigoplus_{s+t=n}(C_s^{\mathrm{sing}} X \otimes C_t^{\mathrm{sing}} Y) \supset C_i^{\mathrm{sing}} \otimes C_j^{\mathrm{sing}} Y \to \mathbb{Z}$

$$\begin{tikzcd} \bigoplus_{s+t=n}(C_s^{\mathrm{sing}} X \otimes C_t^{\mathrm{sing}} Y) \arrow[r] \arrow[d] & \mathbb{Z} \\ C_n^{\mathrm{sing}}(X \times Y) & \end{tikzcd}$$

    yielding a chain ($\simeq$) equiv.

    $\tilde{a} \otimes \tilde{b}$ yields a cocycle, hence:

$$[\tilde{a} \otimes \tilde{b}] \in H^n(X \times Y)$$

  - take $X = Y$:

$$\begin{tikzcd} \bigoplus_{i+j=n}(H^i_{\mathrm{sing}} X \times H^j_{\mathrm{sing}} X) \arrow[r] \arrow[rr, bend right, "\text{this defines graded ring structure}"'] & H^n_{\mathrm{sing}}(X \times Y) \arrow[r, "\Delta^*"] & H^n_{\mathrm{sing}} X \end{tikzcd}$$

    where $\Delta X \to X \times X$, $t \mapsto (t,t)$ is the diagonal embedding.

Example 1. $n > 0$: $H^*_{\text{sing}} S^n$ has:

$$H^i_{\text{sing}} S^n = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases}$$

$$\left.\begin{matrix} 1 \in H^0_{\text{sing}} S^n \\ \langle x \rangle H^n_{\text{sing}} S^n \end{matrix}\right\} H^*_{\text{sing}} S^n \cong \mathbb{Z}[x]/\langle x^2 \rangle$$

Fact:

$$H_i^{\text{sing}}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, i \text{ even} \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H^i_{\text{sing}}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, i \text{ even} \\ 0 & \text{else} \end{cases}$$

Fact: $H_i^{\text{sing}}(X)$ free abelian $\forall i \Rightarrow H^i_{\text{sing}}(X) \cong \text{Hom}_{\mathbb{Z}}(H_i^{\text{sing}}(X), \mathbb{Z})$
Fact: $H^*_{\text{sing}}(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1} \rangle$
$\langle x \rangle = H^2_{\text{sing}}(\mathbb{C}P^n, 2)$, $\langle x^n \rangle = H^{2n}(\mathbb{C}P^n)$
$n \geq 1$: $\mathbb{C}P^1 = S^2$ and $\underbrace{H^*_{\text{sing}}(\mathbb{C}P^\infty) = \mathbb{Z}[x]}_{|x|=2}$

8.1 Hopf-Invariant

In previous sections we have discussed the homotopy groups for Spheres in the cases:

$$\begin{aligned} \pi_n S^n &\cong \mathbb{Z} \qquad n \geq 1 \\ \pi_k S^n &= 0 \qquad k < n \end{aligned}$$

What happens when $k > n$?
First it is *almost* always finite (Serre).

Theorem 8.7 *$\pi_k S^n$, $k > n$ is infinite $\Leftrightarrow$ n even and $k = 2n - 1$.*

Hopf:

$$S^{2n-1} \xrightarrow{\phi} S^n \to \underbrace{S^n \cup_\phi e^{2n}}_{C(\phi)}$$

$S^n = \{.\} \cup e^n$

$$\Rightarrow H_i^{\text{sing}}(S^n \cup e^{2n}) \cong \begin{cases} \mathbb{Z} & i = n, i = 2n, i = 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H^i_{\text{sing}}(S^n \cup e^{2n}) \cong \begin{cases} \mathbb{Z} & i = n, i = 2n, i = 0 \\ 0 & \text{else} \end{cases}$$

$H^n_{\text{sing}}(S^n \cup e^{2n}) = \langle x \rangle \cong \mathbb{Z}$, $H^{2n}_{\text{sing}}(S^n \cup e^{2n}) = \langle y \rangle \cong \mathbb{Z}$.
Fix x and y as follows: $S^n \to C(\phi)$ canonical inclusion, induces:

$$\begin{array}{rcl} H^n_{\text{sing}}(C(\phi)) & \xrightarrow{\cong} & H^n_{\text{sing}}(S^n) = \langle [S^k] \rangle \\ x & \longmapsto & [S^n] \end{array}$$

where $[\cdot]$ is the "orientation class".

$$S^n \hookrightarrow C(\phi) \to C(\phi)/S^n \overset{\text{can},\cong}{\to} S^{2n}$$

$$\begin{array}{l} H^{2n}_{\text{sing}}(S^{2n}) \overset{\cong}{\to} H^{2n}_{\text{sing}}(C(\phi)) \\ \qquad [S^2n] \mapsto y \qquad \text{(choose } y \text{ this way)} \end{array}$$

$Y \overset{\text{const},\phi}{\to} Y$: $C(\phi) \cong Y \vee (\Sigma X)$
So: $S^{2n-1} \overset{\phi}{\to} S^n$, $\phi \simeq * \Rightarrow C(\phi) \simeq S^n \vee S^{2n}$.
$S^{2n-1} \overset{\phi}{\to} S^n$ arbitrary: $x \in H^n_{\text{sing}}(C(\phi)) \Rightarrow x^2 \in H^{2n}_{\text{sing}}(C(\phi)) = \langle y \rangle \Rightarrow \exists\, H(\phi) \in \mathbb{Z}$ s.t. $x^2 = H(\phi) \cdot y$. $H(\phi)$: *Hopf-Invariant* of ϕ.
For instance: $\phi \simeq * \Rightarrow$ there is a θ:

$$S^n \vee S^{2n} \simeq C(\phi) \overset{\theta}{\to} S^n$$

inducing:

$$\begin{array}{c} w \mapsto x \\ H^n_{\text{sing}} S^n \overset{\cong}{\to} H^n_{\text{sing}} C(\phi) \\ w^2 \mapsto x^2 \end{array}$$

where $w^2 = 0 \Rightarrow x^2 = 0 \Rightarrow H(\text{const}) = 0$.
$H(\phi)$ is a homotopy invariant of ϕ, and:

$$[\phi] \in \pi_{2n-1}(S^n) \overset{H}{\to} \mathbb{Z}$$

is a group homomorphism.
n odd $\Rightarrow H : \pi_{2n-1}S^n \to \mathbb{Z}$ is the 0 map. Why? $x^2 = H(\phi)y$, $x \in H^n_{\text{sing}}$ for n odd: $x^2 = -x^2 \Rightarrow x^2 = 0 \Rightarrow H(\phi) = 0 \forall\, \phi$.

Exercise n even $\Rightarrow H : \pi_{2n-1}S^n \to \mathbb{Z}$ is $\neq 0$. Therefore: $\pi_{2n-1}S^n \cong \mathbb{Z} \oplus ?$
(See problem set 12: $S^n \times S^n = (S^n \vee S^n) \cup_\psi e^{2n}$

$$\begin{tikzcd} \psi : s^{2n-1} \arrow[r] \arrow[dr, dashed, "\phi = \nabla \circ \psi"'] & S^n \vee S^n \arrow[d, "\nabla\text{:folding map}"] \\ & S^n \end{tikzcd}$$

n even $\Rightarrow H(\phi) = 2$.

Hopf-Invariant-One-Problem:
For which n does there exist a map $\phi : S^{2n-1} \to S^n$ of Hopf-Invariant 1?

Theorem 8.8 (Adams) $H(\phi) = 1 \Rightarrow n = 2, 4$ *or* 8.

9 Theorems of Hurewicz and Whitehead

Definition 9.1 $X \in \underline{\mathrm{Top}_\bullet}$ *is* n-connected, *if* $\pi_i X = 0$ *for* $i \leq n$.

$\pi_0 X = [S^0, X]_\bullet$: set of path components of X.

Example

1. X 0-connected $\Leftrightarrow$ X path connected.
2. X 1-connected $\Leftrightarrow$ X path connected, $\pi_1 X = 0$

Reminder:

Definition 9.2 (Hurewicz homomorphism) $X \in \underline{\mathrm{Top}}$,

$$\begin{array}{ccc} \pi_i X & \xrightarrow{Hu} & H_i^{\mathrm{sing}} X \\ [f] & \longmapsto & f_*[S^n] \end{array}$$

$f : S^1 \to X$, $f_* : H_1^{\mathrm{sing}} S^i \to H_i^{\mathrm{sing}} X$.

Theorem 9.3 (Hurewicz) *$X \in \underline{\mathrm{Top}_\bullet}$, X 0-connected, then:*

1.

$$\begin{tikzcd} \pi_1 X \arrow[r, two heads] \arrow[dr, two heads] & H_1^{\mathrm{sing}} X \\ & \pi_1 X/[\pi_1 X, \pi_1 X] \arrow[u, "\cong"'] \end{tikzcd}$$

2. X *1-connected* $\Rightarrow$ $H_1^{\mathrm{sing}} X = 0$ *(by 1)*

and if X is n-connected, $n > 0$ then:

$$\pi_{n+1} X \xrightarrow[\cong]{Hu} H_{n+1}^{\mathrm{sing}} X$$

Corollary 9.4 *Suppose $\pi_i X = 0$ for $1 \leq i < n$, $n > 0$, X 0-connected, then:* $H_i X = 0$, $i < n$.
Conversely, if X is 1-connected and $H_j^{\mathrm{sing}} X = 0$ for $j < m$ then $\pi_j X = 0$ for $j < m$.

Example $X = S^k$ is $(k-1)$-connected:

$$\left.\begin{array}{r}\pi_i S^k = 0, i < k\\ H_i S^k = 0, i < k\end{array}\right\} \pi_k S^k \xrightarrow{\cong} H_k^{\text{sing}} S^k \cong \mathbb{Z}$$

There is also a relative version of Hurewicz:
$(X, A) \in \underline{\mathrm{Top}}^2_{\bullet}$: $x_0 \in A \subset X$. $\pi_n(X, A)$: set of pointed homotopy classes of "diagrams":

$$\begin{tikzcd} D^n \arrow[r] & X \\ \partial D^n = S^{n-1} \arrow[u, phantom, "\cup"] \arrow[r] & A \arrow[u, phantom, "\cup"] \end{tikzcd}$$

- has a natural group structure for $n \geq 2$.
- we have a long exact homotopy sequence (if $x_0 \in A$ is a global base-point, i.e. $\{x_0\} \subset A$, $\{x_0\} \subset X$ has HEP):

$$\ldots \xrightarrow{\partial} \pi_n A \to \quad \pi_n X \quad \to \quad \pi_n(X, A) \quad \xrightarrow{\partial} \pi_{n-1} A \to \ldots$$

$$(f : S^n \to X) \mapsto \left(\begin{tikzcd} D^n \arrow[r, "\tilde{f}"] & X \\ S^{n-1} \arrow[u, phantom, "\cup"] \arrow[r] & A \arrow[u, phantom, "\cup"] \end{tikzcd}\right)$$

Note

$$(f : S^n \to X) \leftrightarrow \begin{tikzcd} D^n \arrow[r, "\tilde{f}"] & X \\ S^{n-1} \arrow[u, phantom, "\cup"] \arrow[r] & \{x_0\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\left(\begin{tikzcd} D^n \arrow[r] & X \\ S^{n-1} \arrow[u, phantom, "\cup"] \arrow[r, "\phi"'] & A \arrow[u, phantom, "\cup"] \end{tikzcd}\right) \overset{\partial}{\mapsto} [\phi] \in \pi_{n+1} A$$

Theorem 9.5 (Relative version of Hurewicz) *$(X, A) \in \underline{\mathrm{Top}}_{\bullet}$, A, X 1-connected (with good $x_0 \in A \subset X$). Then the first non-vanishing homotopy group of (X, A) is isomorphic to the first non-vanishing homology group of (X, A)*

Corollary 9.6 *Given $f : X \to Y$ with $\pi_i X \xrightarrow{\cong} \pi_i Y$, $i \leq n$ (both 0-connected), then $H_i X \xrightarrow{\cong} H_i Y$, $\in \leq n-1$. Conversely, if X, Y are 1-connected and $H_j X \xrightarrow{\cong} H_j X$, $j \leq n$ then $\pi_j X \xrightarrow{\cong} \pi_j Y$, $j \leq n-1$*

Proof Idea: Replace f by an inclusion:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, hook] & Y \\ & Z(f) \arrow[u, "\cong"] \end{tikzcd}$$

□

Corollary 9.7 *$X, Y \in \underline{\mathrm{Top}_\bullet}$, both 1-connected. $f : X \to Y$, then the following are equivalent:*

1. $\pi_i X \xrightarrow{\cong} \pi_i Y$, $\forall i$
2. $H_i X \xrightarrow{\cong} H_i Y$, $\forall i$

Definition 9.8 *$f : X \to Y$ in $\underline{\mathrm{Top}}$ is called a weak homotopy equivalence, if:*

$$\pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, f(x_0))$$

$\forall\, x_0 \in X, \forall i$.

Theorem 9.9 (Whitehead) *$f : X \to Y$ in $\underline{\mathrm{CW}}$. Then f is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Corollary 9.10 *$f : X \to Y$ in $\underline{\mathrm{CW}_\bullet}$ both 1-connected. Then f is a homotopy equivalence if and only if:*

$$H_i^{\mathrm{cell}} X \xrightarrow{\cong} H_i^{\mathrm{cell}} Y, \forall i$$

($X \in \underline{\mathrm{Top}}$: $H_i^{\mathrm{sing}} \Gamma X \xrightarrow{\cong} H_i^{\mathrm{sing}} X$, $\forall i$)

10 Spectra

$\underline{\mathrm{CW}_\bullet}$, $\underline{\mathrm{CW}_\bullet}^2$

$$[Z, CO_\bullet(X, Y)]_\bullet \cong [Z \wedge X, Y]_\bullet [Z, \underbrace{\Gamma(CO_\bullet(X, Y))}_{F(X,Y)}]_\bullet$$

$F(X, -)$ is right adjoint to $X \wedge -$.
$\Omega_{\underline{\mathrm{CW}}} X := F(S^1, X)$

Lemma 10.1 *$(X, A) \in \underline{\mathrm{CW}_\bullet}^2$, $Z \in \underline{\mathrm{CW}_\bullet}$. One has an exact sequence of sets*

$$[X/A, Z] \xrightarrow{\alpha} [X, Z] \xrightarrow{\beta} [A, Z]$$

i.e. $\beta(f) = \text{const.} \Leftrightarrow f \in \mathrm{im}(\alpha)$

Proof i) $f \in \mathrm{im}\, \alpha$:

$$\begin{tikzcd} X/A \arrow[dr, "\exists g"] & \\ X \arrow[u] \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes up to homology $\Rightarrow$ $f|A \simeq \text{const.}$

ii) $f : X \to Z$ such that $f|A \simeq$ const.. $\exists f' \cong f$ with $f'|A =$ const. $\Rightarrow$ $\bar{f} : X/A \to Z$ such that $\alpha[\bar{f}] = [f]$.

$\square$

We want $[-, Z]$ to be groups, so choose $Z = \Omega_{\underline{\mathrm{CW}}} Y$ (abelian groups: $Z = \Omega^2_{\underline{\mathrm{CW}}} Y$).
Want "long exact sequences": Use Puppe sequence for $A \subset X \in \underline{\mathrm{CW}}$.

$$\begin{tikzcd}
A \arrow[r, "i"] & X \arrow[r] \arrow[dr, hook] & X/A \arrow[r] & \Sigma A \arrow[r] \arrow[d, equal] & \Sigma X \arrow[r] & \Sigma X/A \\
& & C(i) \arrow[u, leftarrow, "\cong"] \arrow[r, "\mathrm{can}"'] & C(i)/X \arrow[r] & \cdots \arrow[r] & \Sigma^i
\end{tikzcd}$$

This yields a long exact sequence

$$\ldots \underbrace{\to [\Sigma A, Z] \to [\Sigma X/A, Z]}_{\text{groups}} \underbrace{\to [\Sigma A, Z] \to [X/A, Z] \to [X, Z] \to [A, Z]}_{\text{abelian groups}}$$

$$[V_\alpha X_\alpha, Z] \cong \prod_\alpha [X_\alpha, Z]$$

Upshot $[-, Z]$ could look like a cohomology theory.

Definition 10.2

$$\begin{aligned} \underline{\underline{T}} = \{ & T_i,\ i \in \mathbb{N}, \\ & \sigma_i : \Sigma T_i \to T_{i+1} \} \end{aligned}$$

is called a pre-spectrum. *If the adjoints $T_i \to \Omega T_{i+1}$ are weak equivalences, $\underline{\underline{T}}$ is called an* Ω-spectrum. *If $T_i \xrightarrow{\cong} \Omega T_{i+1}$ are homeomorphisms, $\underline{\underline{T}}$ is called a* spectrum.

Homology groups of $\underline{\underline{T}}$: There are maps

$$\pi_{i+k} T_i \to \pi_{i+k+1} T_{i+1}$$

given by:

$$[\Sigma^{i+k}, T_i] \xrightarrow{\Sigma} [S^{i+k+1}, \Sigma T_i] \xrightarrow{\sigma_*} [S^{i+k+1}, T_{i+1}]$$

$$\pi_k T := \operatorname*{colim}_{i \geq |k|} \pi_{i+k} T_i$$

(note that this makes sense for $k < 0$!)
There is a functor ("spectrification") which turns any pre-spectrum into a spectrum, without changing the homology groups.

Example 1. $\underline{\underline{S}}$ sphere spectrum: $T_i = S^i$, $\sigma S^{i+1} \xrightarrow{=} S^{i+1}$ ($\sigma = \mathrm{id}$). This is a pre-spectrum. (Take spectrification for a spectrum.) $\pi_k \underline{\underline{S}} = \pi_k^{st} S^0$, so $\pi_k \underline{\underline{S}} = 0$ for $k < 0 \Rightarrow \underline{\underline{S}}$ is called a connective spectrum.

2. Bott spectrum:

$$T_{2i} = BU \times \mathbb{Z} = B(\operatorname*{colim}_{n \geq 1} U(n))$$

$$T_{2i+1} = U = \operatorname*{colim}_{n \geq 1} U(n)$$

Theorem 10.3 (Bott periodicity)

(a) $BU \times \mathbb{Z} \xrightarrow{\cong} \Omega U$

(b) $U \xrightarrow{\cong} \Omega(BU \times \mathbb{Z})$

$\Rightarrow$ (structure maps) $\Sigma T_0 \to T_1$ comes from (a), $\Sigma T_1 \to T_0$ from (b). This defines a spectrum (modulo spectrification) and is denoted $\underline{\underline{BU}} = \underline{\underline{T}}$. Specifically:

$$\pi_k \underline{\underline{BU}} = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z} & k \text{ even} \end{cases}$$

If $\underline{\underline{T}}$ is a (pre-)spectrum, $X \wedge \underline{\underline{T}}$ (i.e. $(X \wedge \underline{\underline{T}})_i = X \wedge T_i$) is a (pre-)spectrum in the obvious way.

Definition 10.4 (Homology theory)

$$H_i(X, \underline{\underline{T}}) = \pi_i(X_+ \wedge \underline{\underline{T}})$$

where $X_+ = X \amalg \{\centerdot\}$ is X with an added artificial base point.

($X \in \underline{\mathrm{CW}}$)
On pairs X, A:

$$\begin{aligned} A \neq \varnothing: &\qquad H(X, A, \underline{\underline{T}}) = \ker\big(H(X/A, \underline{\underline{T}}) \to H(\{\centerdot\}, \underline{\underline{T}})\big) \\ A = \varnothing: &\qquad H(X, A, \underline{\underline{T}}) = H(X, \underline{\underline{T}}) \end{aligned}$$

Note $H_i(\{\centerdot\}, \underline{\underline{T}}) = \pi_i(\underline{\underline{T}})$

Example 1. $H_i(X, \underline{\underline{S}}) \cong \pi_i^{st}(X_+)$

2. $H_i(X, \underline{\underline{BU}}) = \pi_i(X_+ \wedge \underline{\underline{BU}}) =: K_i(X)$, the *K-homology*:

$$K_i(\{\centerdot\}) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} & i \text{ even} \end{cases}$$

One can define cohomology: $Z \in \underline{\mathrm{CW}_\centerdot}$, "function spectrum" $F(X, \underline{\underline{T}})$ (i.e. $F(X, \underline{\underline{T}})_i = F(X, T_i)$).

Definition 10.5 (Cohomology theory)

$$H^i(X, \underline{\underline{T}}) := \pi_{-1}\left(F(X_+, \underline{\underline{T}})\right)$$

Example 1. $H^i(X, \underline{\underline{BU}}) = \pi_i\left(F(X_+, \underline{\underline{BU}})\right)$

$$[S^{i+k}, F(X_+, \underline{\underline{BU}}_k)] \cong [S^{-i+k} \wedge X_+, \underline{\underline{BU}}_k] \cong [X_+, \Omega^{-i+k}\underline{\underline{BU}}_k]$$
$$= \begin{cases} [X_+, BU \times \mathbb{Z}] & i \text{ even} \\ [X_+, U] & i \text{ odd} \end{cases}$$

$\Rightarrow$ (Bott periodicity)

$$H^i(X, \underline{\underline{BU}}) = \begin{cases} [X, BU \times \mathbb{Z}] & i \text{ even} \\ [X, U] & i \text{ odd} \end{cases}$$

where $[X, BU \times \mathbb{Z}] = K^0X = K_0(C(X))$ (later).

2. Eilenberg-McLane-spectrum $\underline{\underline{HG}}$, G an abelian group:

$$\underline{\underline{HG}}_k := K(G, k)$$
$$\pi_{n+1}(X) \cong \pi_n(\Omega X)$$

$\sigma : \Sigma K(G, k) \to K(G, k+1)$ come from weak equivalences $K(G, k) \xrightarrow{\simeq} \Omega K(G, k+1)$. If necessary take spectrification.

$$H_i(\{\centerdot\}, \underline{\underline{HG}}) = \pi_i(\underline{\underline{HG}})$$

compute $\pi_i(\underline{\underline{HG}})$: (sketch)

$$\pi_{i+k}(K(G, k)) \xrightarrow{\Sigma} \pi_{i+k+1}(\Sigma K(G, k)) \xrightarrow{\sigma_*} \pi_{i+k+1(K(G,k+1))}$$

where σ_* is iso for $k \gg i \Rightarrow \Sigma$ is iso $\Rightarrow$

$$H_i(\{\centerdot\}, \underline{\underline{HG}}) = \begin{cases} G & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Using "uniqueness result" for ordinary homology it then follows that one has a natural isomorphism ($X \in \underline{\mathrm{CW}}$):

$$H_i^{\mathrm{sing}}(X; G) \cong H_i^{\mathrm{cell}}(X; G) \cong H_i(X, \underline{\underline{HG}})$$

The advantage of working with spectra $\underline{\underline{T}}$ is that cohomology takes a simple form

$$H^i(X,\underline{\underline{T}}) = \pi_{-i}(F(X_+,\underline{\underline{T}})) \cong \operatorname*{colim}_k \pi_{-i+k}(F(X_+,\underline{\underline{T}}_k))$$

$$\begin{tikzcd}
{[S^{-i+k}, F(X_+,T_k)]} \arrow[r, "\Sigma"] \arrow[d, "\cong"] \arrow[dr, dashed, "\cong"] & {[S^{-i+k+1}\wedge X_+, \Sigma T_k]} \arrow[d, "\sigma_*"] \\
{[S^{-i+k}\wedge X_+, T_k]} \arrow[r, "\cong"'] & {[S^{-i+k+1}\wedge X_+, T_k]}
\end{tikzcd}$$

so for $k = i$

$$\begin{aligned}
H^i(X,\underline{\underline{T}}) &= \pi_{-i+k}(F(X_+,T_k)) = \pi_0(F(X_+,T_k)) = [S^0, F(X_+,T_k)]_. \\
&\cong [S^0 \wedge X_+, T_k]_. = [X, T_k]_. \\
&= [X, T_i]_.
\end{aligned}$$

Thus for $\underline{\underline{T}} = \underline{\underline{HG}}$:

Theorem 10.6 *For $X \in \underline{\mathrm{CW}}$ one has a natural isomorphism*

$$H^i_{\mathrm{sing}}(X;G) \cong [X, K(G,i)]$$

Corollary 10.7

$$\begin{aligned}
H^1_{\mathrm{sing}}(X,\mathbb{Z}) &= [X, S^1] \\
H^2_{\mathrm{sing}}(X,\mathbb{Z}) &= [X, \mathbb{C}P^\infty]
\end{aligned}$$

Morphisms in the category of spectra: $\underline{\underline{T}} \xrightarrow{\underline{\underline{f}}} \underline{\underline{S}}$ with $\underline{\underline{f}} \equiv f_i : T_i \to S_i$ such that

$$\begin{tikzcd}
\Sigma T_i \arrow[r, "\sigma_{\underline{\underline{T}}}"] \arrow[d, "\Sigma f_i"'] & T_{i+1} \arrow[d, "f_{i+1}"] \\
\Sigma S_i \arrow[r, "\sigma_{\underline{\underline{S}}}"'] & S_{i+1}
\end{tikzcd}$$

commutes.

Spectra: generalized topological spaces

$$\underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Top}_.} \xrightarrow{\Sigma^\infty} \underline{\mathrm{Spectra}}$$

$$Y \longmapsto Y_+$$

$$X \longmapsto \Sigma^\infty X$$

$\Sigma^\infty X$ is the *suspension spectrum*.
Prespectrum $\underline{\underline{T}}$ with $T_i = \Sigma^i X$ becomes a spectrum by "spectrification": $\Sigma T_i \to T_{i+1}$, $\widetilde{T_i} \to \Omega T_{i+1}$, $\Sigma(\Sigma^i X) \to \Sigma^{i+1} X$.
$\underline{\underline{T}}$ a spectrum: defines a homology (and cohomology) theory on $\underline{\text{CW}}$ (or on $\underline{\text{Top}}$ via the geometric realization functor $\underline{\text{Top}} \xrightarrow{\Lambda} \underline{\text{CW}}$).
One puts:

$$h_i(X;\underline{\underline{T}}) := \pi_i(X_+ \wedge \underline{\underline{T}}) \cong \operatorname*{dirlim}_k \pi_{i+k}(X_+ \wedge T_k)$$

$$h_i(\{\centerdot\};\underline{\underline{T}}) := \operatorname*{dirlim}_k \pi_{i+k}(T_k) = \pi_i(\underline{\underline{T}})$$

which can be $\neq 0$ for $i \in \mathbb{Z}$ (even $i < 0$).

$$\begin{aligned} h^i(X;\underline{\underline{T}}) &= \pi_{-i}(F(X_+,\underline{\underline{T}})) = \operatorname*{dirlim}_{k\geq i} \pi_{-i+k}(F(X_+,T_k)) = \operatorname*{dirlim}_{k\geq i}[S^{-i+k}, F(X_+,T_k)]_{\centerdot} \\ &\cong \operatorname*{dirlim}_{k\geq i}[S^{-i+k}\wedge X_+, T]_{\centerdot} \cong \operatorname*{dirlim}_{k\geq i}[X_k, \Omega^{-i+k}T_k] \end{aligned}$$

Example 1. $\underline{\underline{KA}}$ (A abelian group) the "Eilenberg-MacLane spectrum".

$$(\underline{\underline{KA}})_k = K(A,k) \simeq \Omega K(A,k+1)$$

has property that

$$h^i(X;\underline{\underline{KA}}) = [X, K(A,i)] \cong H^i(X;A)$$

for $X \in \underline{\text{CW}}$. $h^i(X;\underline{\underline{KA}}) = \Rightarrow H^i$ "representable". Furthermore

$$h_i(X;\underline{\underline{KA}}) = \operatorname*{dirlim}_k \pi_{i+k}(X_+ \wedge K(A,k)) \cong H_i(X;A)$$

For example, $K(\mathbb{Z},1) \simeq S^i$, $K(\mathbb{Z},2) = BS^1 \simeq \mathbb{C}P^\infty \Rightarrow$

$$\begin{aligned} H^1(X;\mathbb{Z}) &\cong [X,S^1] \\ H^2(X;\mathbb{Z}) &\cong [X,K(\mathbb{Z},2)] = [X,\mathbb{C}P^\infty] \end{aligned}$$

K-Theory: "Bott spectrum" $\underline{\underline{BU}}$

$$(\underline{\underline{BU}})_k = \begin{cases} BU \times \mathbb{Z} & k \text{ even} \\ U & k \text{ odd} \end{cases}$$

where

$$\begin{aligned} BU &:= \operatorname{dirlim} BU(u) \\ U &:= \operatorname{dirlim} U(u) \end{aligned}$$

so $\Omega BU \simeq U$, $\Omega U \simeq BU \times \mathbb{Z}$

$$\pi_1(U(u)) \cong \mathbb{Z} \quad n \geq 1$$

$U(1) = S^1 \Rightarrow \pi_0(\Omega U) \simeq \pi)$, $U \cong \mathbb{Z}$. So

$$h^i(X; \underline{\underline{BU}}) \cong \begin{cases} [X, BU \times \mathbb{Z}] & i \text{ even} \\ [X, U] & i \text{ odd} \end{cases}$$

Define $K^i(X) := h^i(X; \underline{\underline{BU}})$, similiarly $K_i(X)$. Here:

$$K^i(\{\centerdot\}) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

since U connected.

Vector bundles

$X \in \underline{\text{Top}}$. A vector bundle over X is an onto map

$$\pi : E \to X$$

such that

1. $\pi^{-1}(x) \cong \mathbb{C}^n$ (homeomorphic) $\forall x \in X$
2. "local triviality": $\forall x \in X$ $\exists$nbhd $U \subset X$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\exists\phi", "\cong"'] \arrow[d] & U \times \mathbb{C}^n \arrow[dl, "pr_U"] \\ U & \end{tikzcd}$$

   commutes and ϕ is a linear isomorphism on fibers

$$\phi : \pi^{-1}(u_0) \xrightarrow{\cong} pr_U^{-1}(u_0)$$

$\xi \ \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ VB and $\eta : \begin{smallmatrix} F \\ \downarrow \\ X \end{smallmatrix}$ too:

$$\xi \cong \eta \Leftrightarrow \begin{tikzcd} E \arrow[r, "\exists \text{ hom and linear iso on fibres}"] \arrow[dr] & F \arrow[d] \\ & X \end{tikzcd}$$

we write iso(ξ) for the iso-class of ξ. $\xi : \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ is called trivial (of dim n) if $\xi \cong \theta_n$, $\theta_n : \begin{smallmatrix} X\times\mathbb{C}^n \\ \downarrow \\ X \end{smallmatrix}$ $\Rightarrow$ X connceted, $X \neq \varnothing$ then VB $\xi : \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ has well-definied dimension.

Definition 10.8 $\operatorname{Vect}_n X$*: set of iso-classes of* $\mathbb{C}^n$*-Bundles / X.*

$$\begin{tikzcd} E_1 \arrow[dr, "\pi_1", "\xi_1"'] & & E_2 \arrow[dl, "\pi_2"', "\xi_2"] \\ & X & \end{tikzcd} \qquad \xi_1 \oplus \xi_2 : \begin{array}{l} E := \{(u,v) \in E_1 \times E_2 \mid \pi_1 u = \pi_2 v\} \\ \big\downarrow \pi \\ X \ni x_0 : \pi^{-1}(x_0) \cong \pi_1^{-1}(x_0) \oplus \pi_2^{-1}(x_0) \end{array}$$

$\oplus$ *yields:* $\operatorname{Vect}_n X \times Vect_m X \to Vect_{n+m} X$*, and:*

$$\left.\begin{array}{r} \operatorname{Vect}_n X \to \operatorname{Vect}_{n+1} X \\ \operatorname{iso}(\xi) \mapsto \operatorname{iso}(\xi \oplus \theta_1) \end{array}\right\} \operatorname{Vect} X := \operatorname*{dirlim}_{n \geq 0} \operatorname{Vect}_n X$$

$\Rightarrow$ $[\xi \oplus \theta_n] = [\xi] \in \operatorname{Vect} X$ is a commutative semi-gp with identity, $[\xi]$ is represented by:

$$\begin{array}{ccc} \operatorname{iso}(\xi) & \in & \operatorname{Vect}_n X \\ \downarrow & & \downarrow \\ \operatorname{iso}(\xi \oplus \theta_m) & & \operatorname{Vect}_{n+m} X \end{array}$$

with: $[\xi] + [\eta] := [\xi \oplus \eta]$, $[\theta_n] =$ "0": $[\xi] + [\theta_n] = [\xi]$.

Theorem 10.9 X *compact* $\Rightarrow$ $\operatorname{Vect}(X)$ *is a group.*

Proof uses: $\xi : \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ a $\mathbb{C}^n$- bundle $\Rightarrow$ $\exists$ some m and $\eta : \begin{smallmatrix} F \\ \downarrow \\ X \end{smallmatrix}$ $\mathbb{C}^m$-bundle s.t. $\xi \oplus \eta \cong \theta_{n+m}$ etc. $\square$

10.1 Universal $\mathbb{C}^n$-Bundle

- Grassmannian $G_{n,k}$ of n-dim linear subspaces in $\mathbb{C}^{n+k}$.

  $E_n \overset{\pi}{\twoheadrightarrow} G_{n,k}$ canonical $\mathbb{C}^n$-bundle.

  Case $n = 1$: $G_{1,k}$: 1-dim subspaces of $\mathbb{C}^{1+k} \supset S^{2k+2}$

  $$x_0 \in \mathbb{C}P^k \overset{\pi}{\leftarrow} E \supset \pi_1(X_0) \cong \mathbb{C}S^{2k+1} \twoheadrightarrow \mathbb{C}P^k = S^{2k+1}/S^1$$

  canonical line bundle

$G_{n,k} \subset G_{n,k+1} \subset \ldots \subset \underbrace{\bigcup_{k\geq 0}}_{\simeq BU(n)} G_{n,k} =: G_n$ infinite Grassmannian of $\mathbb{C}^n$-planes (i.e. $\Omega G_n \simeq U(n)$).

$\Rightarrow \mathbb{C}P^\infty$ has $\Omega\mathbb{C}P^\infty \simeq S^1$. $S^1 \to (*) \to \mathbb{C}P^\infty$.

$\Rightarrow \exists$ canonical $\mathbb{C}^n$-bundle $E(n) \to BU(n)$ the "universal" $\mathbb{C}^n$-bundle.

- $X \xrightarrow{f} BU(n)$ produces $\mathbb{C}^n$-bundle $f^*E(n) \twoheadrightarrow X$ via pull-back:

$$\begin{tikzcd}
f^*E(n) \arrow[rr] \arrow[d, "\phi"] & & E(n) \arrow[d, "\mathcal{X}_n:\ \text{universal } \mathbb{C}^n\text{-bundle}"] \\
x_0 \in X \arrow[rr, "f"] & & BU(n) \ \text{"classifying space for } \mathbb{C}^n\text{-bundles"}
\end{tikzcd}$$

$f^*E(n) = \{(x,y) \mid f(x) = \pi(y)\} \subset X \times E(n)$

$\Rightarrow \phi^{-1}(x_0) \cong \pi^{-1}f(x_0)$ "$\mathbb{C}$"

iso class of $f^*E(n)$ depends only on homotopy class of f, therefore:

Theorem 10.10 *Let X be a CW-complex, then:*

$$[X, BU(n)] \to \operatorname{Vect}_n X$$

is a bijection.

Example "Chern-Classes"

Let $\xi : E \to X$, $\mathbb{C}^n$-bundle over CW-complex X. Thus $\exists!\, f_\xi : X \to BU(n)$ such that: $\xi \cong f_\xi^*(\mathcal{X}_n)$.

$$H^* f_\xi : \underbrace{H^*BU(n)}_{\mathbb{Z}[c_1,\ldots,c_n]} \to H^*X$$

with $\mathbb{Z}[c_1,\ldots,c_n]$ the polynomial ring in $c_1 \ldots c_n$, where $c_k \in H^{2k}BU(n)$ (c_i universal chen classes)

$$c_i(\xi) := H^{2i}(f)(c_i) \in H^{2i}X$$

easy: ξ trivial bundle $\Rightarrow c_i(\xi) = 0 \forall\, i$.

$$\begin{array}{c} \mathrm{Vect}\,X = \mathrm{dirlim}_n \mathrm{Vect}_n\,X \\ \Big\downarrow{\scriptstyle \text{can (bij. for } X \textit{ finite} \text{ CW)}} \\ [X, BU] \end{array}$$

Recall:

$$\begin{array}{l} K^0X \cong [X, BU \times \mathbb{Z}] \cong [X, BU] \times [X, \mathbb{Z}] \\ \Big\downarrow{\scriptstyle \dim} \\ [X, \mathbb{Z}] \end{array}$$

with $x_0 \in X$, X connected

$$\begin{tikzcd} \{x_0\} \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "\mathrm{pr}"] & \{x_0\} \end{tikzcd} \;:\; \begin{tikzcd} K^0\{x_0\} \arrow[r, "\mathrm{pr}"] \arrow[rr, bend right, "\mathrm{id}"'] & K^0X \arrow[r, "i^*"] & K^0\{x_0\} \end{tikzcd}$$

$\Rightarrow$

$$K^0X \cong \underbrace{\tilde{K}^0}_{\ker i^* \text{ or } \operatorname{coker} pr^*} X \oplus \mathbb{Z}$$

with $\tilde{K}^0X \cong [X, BU] \overset{X \text{ finite connected}}{\cong} \mathrm{Vect}\,X$

Remark X finite CW. K^0X = "Grothendieck group of complex VB / X"

Definition 10.11 *Grothendieck group:* $\coprod_{n \geq 0} \mathrm{Vect}_n\,X =: S$ *commutative semi-group.*

$a, b \in S$: $a \in \mathrm{Vect}_n\,X$, $b \in \mathrm{Vect}_m\,X$, $a + b$: iso-class of $\xi(a) \oplus \xi(b)$ where $[\xi(a)] = a$, $[\xi(b)] = b$. $a + b \in \mathrm{Vect}_{n+m}\,X$.
$Gr(S)$: Grothendieck group of S, e.g. $S(\mathbb{N}) \cong \mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$
with $\sim$: $(u, v) \sim (x, y) \Leftrightarrow u - v = x - y$, $u + y = x + v$.
$\rightsquigarrow$ general definition:

$$Gr(S) = S \times S/\sim$$

$s_1, s_2) \sim (t_1, t_2) \Leftrightarrow s_1 + t_2 + w = t_1 + s_2 + w$ for some w.
$\Rightarrow$ Gr is a group, with component-wise addition and 0: $x \in S : (x, x)$ a representative of 0. Inverse: $(s_1, s_2) : (s_2, s_1)$.
One checks: X finite CW $\Rightarrow K^0X \cong Gr(\coprod \mathrm{Vect}_n\,X)$.

Theorem 10.12 $S^{4m+1} \xrightarrow{f} S^{2m}$*,* $m \geq 1$*,* $H(f) = 1 \Rightarrow m = 1, 2, 4$*.*

The proof relies on "Adams-Operations" $\psi^k : K^0X \to K^0X$ (X finite CW). ψ^k, $k \in \mathbb{Z}$ additive, $\psi^1 = \mathrm{id}$, $\psi^k\psi^\ell = \psi^\ell\psi^k\ \forall k, \ell$. K^0X is a ring with multiplication defined as the tensor product $- \otimes -$ of vector bundles: p prime $\Rightarrow$

$$\psi^p x \equiv x^p \mod p\ x \in K^0X$$

$S^{2m} = \langle x_m \rangle$, $\tilde{K}^0(S^{2m}) \cong \mathbb{Z} \Rightarrow \psi^k(x_m) = k^m x_m$.

Proof $S^{4m-1} \xrightarrow{f} S^{2m}$ yields $X(f) = S^{2m} \cup_f e^{4m} \Rightarrow \tilde{K}^0(S^{2m} \cup_f e^{4m}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$$\begin{tikzcd} S^{2m} \cup_f e^{4m} \arrow[r, "pr"] & S^{4m} \\ S^{2m} \arrow[u, "ind"'] & \end{tikzcd}$$

$$\begin{tikzcd} \tilde{K}^0(X(f)) \arrow[d] & \tilde{K}^0(S^{4m}) = \langle x_{2m} \rangle \arrow[l, "pr^*"'] \\ \tilde{K}^0(S^{2m}) = \langle x_m \rangle & \end{tikzcd}$$

and $\exists\ \tilde{x}_m, \tilde{x}_{2m} \in \tilde{K}^0(X(f))$: $\tilde{K}^0(X(f)) = \langle \tilde{x}_m \rangle \oplus \langle \tilde{x}_{2m} \rangle$, ψ^k's "natural" $\Rightarrow$

$$\psi^k(\tilde{x}_{2m}) = k^{2m}\tilde{x}_{2m}$$
$$\psi^k(\tilde{x}_m) = \alpha\tilde{x}_m + \beta\tilde{x}_{2m}$$

where $\alpha = k^m$, $\beta = \beta(k) \in \mathbb{Z}$.
Now:

$$\begin{aligned}\psi^2(\psi(3(\tilde{x}_m)) &= \psi^2(3^m\tilde{x}_m + \beta(3)\tilde{x}_{2m} = 3^m\psi^2(\tilde{x}_m) + \beta(3)\psi^2(\tilde{x}_{2m}) \\ &= 3^m \cdot 2^m\tilde{x}_m + 3^m\beta(2)\tilde{x}_{2m} + \beta(3)2^{2m}\tilde{x}_{2m}\end{aligned}$$

$$\begin{aligned}\psi^3(\psi^2(\tilde{x}_m)) &= \psi^3(2^m\tilde{x}_m + \beta(2)\tilde{x}_{2m}) \\ &= 3^m \cdot 2^m\tilde{x}_m + 2^m\beta(3)\tilde{x}_{2m}\beta(2)3^{2m}\tilde{x}_{2m}\end{aligned}$$

so

$$3^m\beta(2)(3^m - 1)\tilde{x}_{2m} = 2^m\beta(3)(2^m - 1)\tilde{x}_{2m}$$

where $\tilde{x}_{2m}$ can be canceled.

$$\psi^2\tilde{x}_m = 2^m\tilde{x}_m + \beta(2)\tilde{x}_{2m} \equiv \tilde{x}_m^2 \mod 2$$

$H(f) = 1 \Rightarrow \tilde{x}_m^2 = H(f)\tilde{x}_{2m}$

$$\begin{aligned}\tilde{x}_m^2 &\equiv \beta(2)\tilde{x}_{2m} \mod 2\\ &\equiv H(f)\tilde{x}_{2m}\end{aligned}$$

$\Rightarrow \beta(2)$ odd since $H(f)$ odd $\Rightarrow 2^m \mid 3^m - 1$ to which the only solutions are $m = 1, 2, 4$ (exercise!). □

Application: A finite dimensional division algebra over $\mathbb{R}$ ((non)-commutative field). Then $\dim_n A = 1, 2, 4$ or 8.

Proof $A = \mathbb{R}^n$:

$$\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^n \setminus \{0\} \xrightarrow{\mu} \mathbb{R}^n \setminus \{0\}$$
$$S^{n-1} \times S^{n-1} \xrightarrow{\bar{\mu}} S^{n-1}$$

(using $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ has bidegree $(1,1)$) where μ has no 0-divisors.

Hopf: $S^k \times S^k \xrightarrow{\phi} S^k$ of bidegree (p,q), k odd $\rightsquigarrow$ "Hopf-construction" $\tilde{\phi} : S^{2k+1} \to S^{k+1}$ of $H(\tilde{\phi}) = pq$. Thus $\mathbb{R}^n \cong A$ division algebra over $\mathbb{R} \Rightarrow \exists S^{2n-1} \xrightarrow{\lambda} S^n$ of Hopf invariant 1 ($\Rightarrow n$ even) $\Rightarrow$ (Adams) $n = 2, 4$ or 8, e.g.

$n = 2$: $\mathbb{C}$
$n = 4$: $\mathbb{H}$
$n = 8$: Cayley numbers

□

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Zur Prüfung

- Die Sprache wird Deutsch sein (ev. auch Englisch, falls der Student das möchte)
- Zusammenhänge sind wesentlich wichtiger als viele Details.
- Übungen: wichtig
- Spectra sind nicht unwichtig, aber sie wurden eher als Ausblick behandelt, dementsprechend werden sie sicherlich nicht das Schwergewicht der Prüfung bilden.