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A mapping of a generalisation of a thin film equation on a moving substrate

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Abstract

Lie group analysis is used to determine an invertible transformation between a generalised thin film equation $u_t + (\alpha u^n \partial^m u/\partial x^m)_x = 0$ and a generalised thin film equation on a moving substrate $u_t + (\alpha u^n \partial^m u/\partial x^m - v(t)u)_x = 0$ where $v(t)$ is the substrate velocity. Consequently we obtain new solutions for the motion of a hard contact lens on a thin film of tears where $\alpha = 1/3$, $n = m = 3$.

Keywords: Generalised thin film equation; Lie group analysis; Moving substrate

1. Introduction

The generalisation of the thin film equation

$$\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^3 u}{\partial x^3}\right), \tag{1.1}$$

where n is a constant has been considered by Smyth and Hill [22], Bertozzi et al. [3], Bernis [2], Myers [17] and King and Bowen [14]. King [13] extends the analysis of (1.1) by considering

$$\frac{\partial u}{\partial t} = -\left(u^n \frac{\partial^3 u}{\partial x^3} + \alpha u^{n-1} \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \beta u^{n-2}\left(\frac{\partial u}{\partial x}\right)^3\right) \tag{1.2}$$

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and

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \left|\frac{\partial^3 u}{\partial x^3}\right|^{m-1} \frac{\partial^3 u}{\partial x^3}\right), \tag{1.3}$$

where n, α and β are constants. Eqs. (1.2) and (1.3) can be regarded as generalisations of (1.1). Generalisations of the porous medium equation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right), \tag{1.4}$$

are discussed by Smyth and Hill [22] and King and Bowen [14]. Waiting-time solutions of (1.4) have been obtained by Kath and Cohen [12]. In the papers mentioned above properties and solutions of (1.1)–(1.4) are investigated and discussed. Equations of the form (1.1) are important in the investigation of thin film flows dominated by surface tension. The review by Myers [17] focuses on surface tension effects in thin film flow. The interested reader is also referred to the review by Oron et al. [18] on thin films. Recently, Momoniat et al. [15] have used Lie group analysis to determine new solutions for the surface tension driven spreading of a thin film.

In this paper we are interested not only in generalisations of the thin film equation but also in thin film equations with moving substrates. A particular example we consider has been derived by Moriarty and Terrill [16] to model the motion of hard contact lenses on a thin film of tears above the eye. Their equation is given by

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{3}u^3 \frac{\partial^3 u}{\partial x^3} - v(t)u\right). \tag{1.5}$$

Equation (1.5) is also discussed in the review by Myers [17]. We use Eq. (1.5) as a particular equation to which the results obtained in this paper can be applied. In this paper we use Lie group analysis to determine invertible mappings between equations of the form (1.1) (with $n = 3$) to equations of the form (1.5).

The Lie group technique is concerned with determining transformations of the independent variables t, x and the dependent variable u of the form

$$\bar{t} = \bar{t}(t, x, u, a), \quad \bar{x} = \bar{x}(t, x, u, a), \quad \bar{u} = \bar{u}(t, x, u, a), \tag{1.6}$$

where a is a constant. The transformations (1.6) must leave the equation under consideration form invariant, i.e. after applying the transformations (1.6) to (1.5) we must obtain

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial}{\partial \bar{x}}\left(\frac{1}{3}\bar{u}^3 \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} - v(\bar{t})\bar{u}\right). \tag{1.7}$$

The constant a in (1.6) is assumed to be small, i.e. $a << 1$. A consequence of a small is that the transformations (1.6) can be expanded using Taylor series to form the infinitesimal transformations

$$\bar{t} \approx t + a\xi_1(t, x, u), \quad \bar{x} \approx x + a\xi_2(t, x, u), \quad \bar{u} \approx u + a\eta(t, x, u), \tag{1.8}$$

where

$$\xi_1(\bar{t}, \bar{x}, \bar{u}) = \frac{d\bar{t}}{da}, \quad \xi_2(\bar{t}, \bar{x}, \bar{u}) = \frac{d\bar{x}}{da}, \quad \eta(\bar{t}, \bar{x}, \bar{u}) = \frac{d\bar{u}}{da}. \tag{1.9}$$

The system of ordinary differential equations (1.9) is solved subject to the initial conditions

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}|_{a=0} = x, \quad \bar{u}|_{a=0} = u. \tag{1.10}$$

The transformations (1.8) form a one-parameter group of local point transformations if the transformations have the closure property, are associative, there is an identity transformation and an inverse transformation exists. The constant a is the group parameter. The Lie point symmetry generator of the group (1.8) is given by (see e.g. Bluman and Kumei [4])

$$X = \xi^1(t, x, u)\partial_t + \xi^2(t, x, u)\partial_r + \eta(t, x, u)\partial_u, \tag{1.11}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\partial_u = \partial/\partial u$. The use of superscripts (1.11) instead of subscripts will be motivated later. Applying the Lie group technique to an equation with an arbitrary function one obtains a differential equation that the arbitrary function must satisfy. This is known as a group classification. The interested reader is referred to the books by Ovsiannikov [19], Bluman and Kumei [4] and Stephani [24] and the papers by Ibragimov et al. [9], Ibragimov and Torrisi [10], Yürüsoy [25] and Pakdemirli and Sahin [20] for applications of the group classification technique. When one extends the invariance to include v as a dependent variable, then the Lie point symmetries admitted by the system are

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{3}u^3\frac{\partial^3 u}{\partial x^3} - v(t, x)u\right), \tag{1.12}$$

$$\frac{\partial v(t, x)}{\partial x} = 0, \tag{1.13}$$

where $v = v(t, x)$ must be determined. This is known as an equivalence transformation. The interested reader is referred to Ibragimov and Torrisi [11] and Traciná [23] for an application of equivalence transformations. The Lie point symmetry generator (1.11) admitted by a differential equation under consideration can be determined in a systematic way. This is discussed in Section 2.

In this paper, we consider the generalised thin film equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\alpha u^n \frac{\partial^m u}{\partial x^m}\right) = 0 \tag{1.14}$$

and the generalised thin film equation on a moving substrate

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\alpha u^n \frac{\partial^m u}{\partial x^m} - v(t)u\right) = 0 \tag{1.15}$$

where n, m and α are constants. We show how the Lie group technique can be used to determine an invertible transformation between (1.14) and (1.15). A consequence of this transformation is that the solutions of (1.14) can be mapped to solutions of (1.15) and vice versa. We then consider (1.5) as a particular example. We show how the new results obtained by Momoniat et al. [15] can be used to determine new solutions of (1.5).

The paper is divided up as follows. In Section 2 the Lie point symmetries admitted by (1.14) and (1.15) are determined. From these Lie point symmetries a mapping between the two equations is determined. In Section 3 we consider an application of the mapping determined in Section 2. Concluding remarks are made in Section 4.

2. Lie group analysis and mappings

The coefficients ξ^1, ξ^2 and η of the Lie point symmetry generator (1.11) admitted by (1.14) are calculated by solving the determining equation

$$X^{[m+1]}\left(\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\alpha u^n\frac{\partial^m u}{\partial x^m}\right)\right)\Bigg|_{(1.14)}=0. \tag{2.1}$$

The operator $X^{[m+1]}$ is the $(m+1)$-th extension of the operator X given by

$$X^{[m+1]}=X+\zeta^1\partial_{u_t}+\zeta^2\partial_{u_x}+\zeta^{22}\partial_{u_{xx}}+\cdots+\zeta^{i_1i_2\ldots i_{m+1}}\partial_{u_{(m+1)}},\quad i_s=2, \tag{2.2}$$

where

$$u_{(m+1)}=\frac{\partial^{m+1}u}{\partial x^{m+1}}, \tag{2.3}$$

$$\zeta^i=D_i\eta-\left(D_i\xi^j\right)u_j,\quad i=1,2, \tag{2.4}$$

$$\zeta^{i_1i_2\ldots i_k}=D_{i_k}\eta_{i_1i_2\ldots i_{k-1}}-\left(D_{i_k}\xi^j\right)u_{i_1i_2\ldots j},\quad i_s=2, s=1,2,\ldots,m. \tag{2.5}$$

We have used superscripts for the coefficients ξ^1 and ξ^2 so that in the calculation of the extension of the Lie point symmetry generator (1.11) admitted by (1.14) we can use the Einstein convention in (2.4) and (2.5). The operators, D_i, are the operators of total differentiation with

$$D_1=D_t=\partial_t+u_t\partial_u+u_{tt}\partial_{u_t}+u_{tx}\partial_{u_x}+\cdots, \tag{2.6}$$

$$D_2=D_x=\partial_x+u_x\partial_u+u_{tx}\partial_{u_t}+u_{xx}\partial_{u_x}+\cdots. \tag{2.7}$$

The determining equation (2.1) is separated by coefficients of derivatives of u. An overdetermined system of nonlinear partial differential equations for ξ^1, ξ^2 and η is obtained. Lie point symmetries admitted by differential equations can be calculated using computer algebra packages like MathLie [1] and LIE [6,21]. We find that (1.14) admits the generators of Lie point symmetries

$$X_1=\partial_t,\quad X_2=\partial_x,\quad X_3=t\partial_t-\frac{u}{n}\partial_u,\quad X_4=x\partial_x+\frac{m+1}{n}u\partial_u. \tag{2.8}$$

In a similar way we find that (1.15) admits the generators of Lie point symmetries

$$\begin{aligned}&Y_1=\partial_t-v(t)\partial_x,\quad Y_2=\partial_x,\quad Y_3=t\partial_t-tv(t)\partial_x-\frac{u}{n}\partial_u,\\&Y_4=\left(x+\int v(t)\mathrm{d}t\right)\partial_x+\frac{m+1}{n}u\partial_u.\end{aligned} \tag{2.9}$$

Note that instead of obtaining an equation for the arbitrary function $v(t)$ as one would expect when using Lie group analysis, the arbitrary function is part of the symmetries admitted by (1.15). The interested reader is referred to the books by Bluman and Kumei [4], Ibragimov [7, 8] and Ovsiannikov [19] for more information on the application of modern group analysis to differential equations.

The Lie algebra formed by the generators of symmetries (2.8) admitted by (1.14) is given by

$$\begin{array}{c|cccc} & X_1 & X_2 & X_3 & X_4 \\ \hline X_1 & 0 & 0 & -X_1 & 0 \\ X_2 & 0 & 0 & 0 & -X_2 \\ X_3 & X_1 & 0 & 0 & 0 \\ X_4 & 0 & X_2 & 0 & 0 \end{array} \tag{2.10}$$

The Lie algebra formed by the generators of symmetries (2.9) admitted by (1.15) is given by

$$\begin{array}{c|cccc} & Y_1 & Y_2 & Y_3 & Y_4 \\ \hline Y_1 & 0 & 0 & -Y_1 & 0 \\ Y_2 & 0 & 0 & 0 & -Y_2 \\ Y_3 & Y_1 & 0 & 0 & 0 \\ Y_4 & 0 & Y_2 & 0 & 0 \end{array} \tag{2.11}$$

The algebras (2.10) and (2.11) have the same structure constants. This suggests that a transformation may exist between (1.14) and (1.15) (see Ovsiannikov [19], Bluman and Kumei [4] and Stephani [24]). We look for invertible transformations of the form

$$t = \phi_1(\bar{t}, \bar{x}, \bar{u}), \quad x = \phi_2(\bar{t}, \bar{x}, \bar{u}), \quad u = \phi_3(\bar{t}, \bar{x}, \bar{u}). \tag{2.12}$$

We solve the system of twelve equations (see Bluman and Kumei [4] and Stephani [24])

$$Y_i = X_i(t)\partial_t + X_i(x)\partial_x + X_i(u)\partial_u \tag{2.13}$$

to find that

$$t = \bar{t}, \quad x = \bar{x} - \int^{\bar{t}} v(s)\mathrm{d}s, \quad u = \bar{u}. \tag{2.14}$$

Using the transformations (2.14) we can transform (1.14) into

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(\alpha \bar{u}^n \frac{\partial^m \bar{u}}{\partial \bar{x}^m} - v(\bar{t})\bar{u}\right) = 0. \tag{2.15}$$

We use the overbars in (2.14) and consequently (2.15) to distinguish the transformation from (1.14) to (2.15) from the transformations from (2.15) to (1.14). Any solution of (1.14) can be transformed into a solution of (2.15) and vice versa using (2.14).

3. A particular example

In this section we consider the case when $\alpha = 1/3$ and $m = n = 3$. Eqs. (1.14) and (1.15) then simplify to

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{3}u^3 \frac{\partial^3 u}{\partial x^3}\right), \tag{3.1}$$

and

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{3}u^3 \frac{\partial^3 u}{\partial x^3} - v(t)u\right). \tag{3.2}$$

We provide a brief derivation of (3.2) from thin film theory to show where the substrate velocity $v(t)$ is relevant. We also show how $v(t)$ can be thought of as the slip velocity of the thin film.

The thin film approximations to the Navier–Stokes and continuity equations in two dimensions are given by (see e.g. [17,15])

$$\frac{\partial p}{\partial x} = \frac{\partial^2 v_x}{\partial y^2}, \tag{3.3}$$

$$\frac{\partial p}{\partial y} = 0, \tag{3.4}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{3.5}$$

The system (3.3)–(3.5) is solved subject to the following boundary conditions. On the boundary $y = 0$

$$v_x(x, 0, t) = -v(t), \tag{3.6a}$$

$$v_y(x, 0, t) = 0. \tag{3.6b}$$

The boundary condition (3.6a) has been used by Moriarty and Terrill [16] in the modelling of the motion of a hard contact lens on a thin film of tears. Chung [5] uses a similar boundary condition when expressing the system (3.3)–(3.6) in two-dimensional and cylindrical polar coordinates for which the arbitrary velocity at the base represents the slip velocity of the thin film. Boundary condition (3.6b) implies that the substrate is impermeable or that cavities do not form in the film. On the free surface $y = u(t, x)$,

$$p(x, u, t) = -\frac{\partial^2 u}{\partial x^2}, \tag{3.7a}$$

$$\frac{\partial v_x}{\partial y} = 0. \tag{3.7b}$$

Boundary condition (3.7a) implies that the normal stress jump at the free surface is proportional to the curvature while (3.7b) implies that there is zero shear on the free surface. We also have the condition that a particle on the free surface must remain on the free surface for the duration of the motion. Therefore

$$v_y = \frac{\partial u}{\partial t} + v_x \frac{\partial u}{\partial x}. \tag{3.8}$$

From (3.5) we have that

$$v_y = -\int_0^u \frac{\partial v_x}{\partial x} \mathrm{d}y. \tag{3.9}$$

Combining (3.9) with (3.8) we find that

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left[\int_0^u v_x \mathrm{d}y\right]. \tag{3.10}$$

From (3.4) with (3.7a) we have that

$$p(x, y, t) = -\frac{\partial^2 u}{\partial x^2}. \tag{3.11}$$

Substituting (3.11) into (3.3) and imposing (3.6a) and (3.7b) we find that

$$v_x(x, y, t) = \left(yu - \frac{1}{2}y^2\right)\frac{\partial^3 u}{\partial x^3} - v(t). \tag{3.12}$$

Substituting (3.12) into (3.9) we obtain (3.2).

Eq. (3.1) admits the well known waiting-time solution (see e.g. Smyth and Hill [22] and Momoniat et al. [15])

$$u(t, x) = \left(\frac{\kappa(x + \beta_1)^4}{(t + \beta_2)}\right)^{1/3}, \tag{3.13}$$

where $\kappa = -81/56$ and β_1 and β_2 are constants. Using the transformations (2.14) and suppressing the overbars, we find that (3.2) admits the solution

$$u(t, x) = \left(\frac{\kappa(x + \int^t v(s)\mathrm{d}s + \beta_1)^4}{(t + \beta_2)}\right)^{1/3}. \tag{3.14}$$

Also, from Momoniat et al. [15], Eq. (3.1) admits the travelling wave solution

$$x(\tau) + Vt = ac_1 \int \tau^{-1} Z^{-2}(\tau)\mathrm{d}\tau + c_3, \tag{3.15}$$

$$u(\tau) = \left(\mp\frac{9a^3 c}{4}\right)^{1/3} c_1 \tau^{-2/3} Z^{-2}(\tau), \tag{3.16}$$

$$Z(\tau) = \begin{cases} c_1 J_{1/3}(\tau) + c_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ c_1 I_{1/3}(\tau) + c_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases} \tag{3.17}$$

where J, Y, I and K are Bessel functions and a, c_1, c_2 and c_3 are constants. V is the wave velocity. Using the transformations (2.14) we find that (3.2) admits the travelling wave solution

$$x(\tau) + \int^t v(s)\mathrm{d}s + Vt = ac_1 \int \tau^{-1} Z^{-2}(\tau)\mathrm{d}\tau + c_3, \tag{3.18}$$

which is coupled with (3.16).

4. Concluding remarks

The results obtained in this paper are both novel and useful. We have determined an invertible transformation (2.14) that transforms (1.14) into (1.15). The invertible transformation (2.14) also transforms solutions of (1.14) into solutions of (1.15) and vice versa. We then considered the particular example of the motion of hard contact lenses on a thin film of tears. Here the transformation (2.14) transforms (3.1) into (3.2). Similarly solutions of (3.1) can be transformed into solutions of (3.2). We showed how the waiting-time and travelling wave solutions admitted by (3.1) are transformed into solutions of (3.2). We also note that the solutions obtained by Smyth and Hill [22], Bertozzi et al. [3], Bernis [2], Myers [17] and King and Bowen [14] to equations of the form (1.14) when $m = 3$ can also be mapped to solutions of (1.15) for $m = 3$, thereby greatly increasing the number of known solutions of (1.15).

The construction of mappings from one differential equation to another is well known in Lie group analysis (see e.g. Ovsiannikov [19], Bluman and Kumei [4] and Stephani [24]). The difficulty is in finding transformations that lead to a deeper insight into the physical problem at hand. An example of such a transformation is the Hopf–Cole transformation from the heat equation to Burgers' equation. The results obtained in this paper give a deeper insight into the behaviour of generalised thin film equations (1.14) and (1.15) that have application in the flow of thin Newtonian and non-Newtonian fluids. So, while it may always be possible to find transformations of one equation to another using Lie groups, the value of the transformation comes from the physical insights it provides.

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